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Nonvanishing of Hecke L–functions and Bloch-Kato p-Selmer groups

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The canonical Hecke characters in the sense of Rohrlich form a set of algebraic Hecke characters with important arithmetic properties. In this paper, we prove that for an asymptotic density of 100% of imaginary quadratic fields K within certain general families, the number of canonical Hecke characters of K whose L–function has a nonvanishing central value is $\gg |\mathrm{disc}(K)|^{\delta}$ for some absolute constant $\delta > 0$. We then prove an analogous density result for the number of canonical Hecke characters of K whose associated Bloch-Kato p-Selmer group is finite. Among other things, our proofs rely on recent work of Ellenberg, Pierce, and Wood on bounds for torsion in class groups, and a careful study of the main conjecture of Iwasawa theory for imaginary quadratic fields.

1. Introduction and statement of results

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \mod 4$. Let $\mathcal{O}_K$ be the ring of integers and let

$\varepsilon(n) = (-D/n) = (n/D)$ be the Kronecker symbol. We view ε as a quadratic character of $(\mathcal{O}_K/\sqrt{-D}\mathcal{O}_K)^\times$ via the isomorphism

$$\mathbb{Z}/D\mathbb{Z} \cong \mathcal{O}_K/\sqrt{-D}\mathcal{O}_K.$$

A *canonical Hecke character* of K is a Hecke character ψ_k of conductor $\sqrt{-D}\mathcal{O}_K$ satisfying

$$\psi_k(\alpha\mathcal{O}_K) = \varepsilon(\alpha)\alpha^{2k-1} \quad \text{for} \quad (\alpha\mathcal{O}_K, \sqrt{-D}\mathcal{O}_K) = 1, \quad k \in \mathbb{Z}^+ \tag{1.1}$$

(see [26]). The number of such characters equals the class number $h(-D)$ of K.

The canonical Hecke characters are of great arithmetic interest. For example, Gross [14] proved that given a canonical Hecke character $\psi = \psi_1$ of weight $k = 1$, the Hecke character $\chi_H := \psi \circ N_{H/K}$ of the Hilbert class field H of K corresponds to a unique (up to H-isogeny) elliptic $\mathbb{Q}$-curve $A(D)$ defined over H whose L–function factors as

$$L(A(D), s) = L(\chi_H, s)L(\overline{\chi_H}, s) = \prod_{\psi} L(\psi, s)L(\overline{\psi}, s).$$

Gross conjectured that

$$\operatorname{rank}(A(D)(H)) = \begin{cases} 0, & D \equiv 7 \mod 8 \\ 2h(-D), & D \equiv 3 \mod 8. \end{cases}$$

Because this conjecture predicts an *exact* formula for the rank, the curves $A(D)$ form an important test case for the Birch and Swinnerton-Dyer conjecture. Gross' conjecture is known as a consequence of the works of Gross [14], Rohrlich [25, 26], Montgomery and Rohrlich [21], and Miller and Yang [20].

More recently, the canonical Hecke characters have played a significant role in the works of Bertolini, Darmon, and Prasanna [2], [3], [4] on Chow-Heegner points and p-adic L–functions.

Analytic results. In this paper we will actually consider quadratic twists of the characters ψ_k. Let $d \equiv 1 \mod 4$ be a squarefree integer relatively prime to D. Then $\chi_d := (d/N_{K/\mathbb{Q}}(\cdot))$ is a primitive Hecke character of K of conductor $d\mathcal{O}_K$. Define the quadratic twist of ψ_k by

$$\psi_{d,k} := \chi_d\psi_k.$$

Then $\psi_{d,k}$ is a canonical Hecke character of K of conductor $d\sqrt{-D}\mathcal{O}_K$. To ease notation, we write $\psi = \psi_{d,k}$.

Let $\Psi_{d,k}(D)$ be the set of all such canonical Hecke characters ψ. Then $\#\Psi_{d,k}(D) = h(-D)$, and if we fix any such character ψ_0, then

$$\Psi_{d,k}(D) = \{\psi_0\xi : \ \xi \in \widehat{\mathrm{Cl}(K)}\}$$

where $\mathrm{Cl}(K)$ is the ideal class group of K.

The L–function of ψ is defined by

$$L(\psi, s) := \sum_{\mathfrak{a}} \psi(\mathfrak{a})N(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > k + \frac{1}{2}$$

where the sum is over nonzero integral ideals $\mathfrak{a}$ of K. The L–function $L(\psi, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2k - s$ with central value $L(\psi, k)$ and root number

$$W(\psi) = (-1)^{k-1}\mathrm{sign}(d)(-1)^{\frac{D+1}{4}}. \tag{1.2}$$

We denote the number of nonvanishing central values corresponding to characters in the family $\Psi_{d,k}(D)$ by

$$\mathrm{NV}_{d,k}(D) := \#\{\psi \in \Psi_{d,k}(D) : \ L(\psi, k) \neq 0\}.$$

The nonvanishing of the central values $L(\psi, k)$ has been studied extensively when $(2k-1, h(-D)) = 1$ (see e.g. [25], [26], [21], [35], [24], [20], [17]). In particular, when $(2k-1, h(-D)) = 1$, if $L(\psi, k) \neq 0$ for some character $\psi \in \Psi_{d,k}(D)$, then it follows from work of Rohrlich [27, Theorem 1] and Shimura [32] that *all* of the central values are nonvanishing, that is,

$$\mathrm{NV}_{d,k}(D) = h(-D).$$

On the other hand, if $(2k-1, h(-D)) \neq 1$, the existence of one nonvanishing central value no longer implies that all of the central values are nonvanishing. It is then of interest to study how $\mathrm{NV}_{d,k}(D)$ grows as $D \to \infty$.

Let $E/\mathbb{Q}$ be a number field of discriminant D_E and degree n. Moreover, let $\mathrm{Cl}_\ell(E)$ be the ℓ-torsion subgroup of the ideal class group $\mathrm{Cl}(E)$ of E. Assuming the Generalized Riemann Hypothesis (GRH), Ellenberg and

Venkatesh [11] proved the following non-trivial bound:

$$\#\mathrm{Cl}_\ell(E) \ll_{n,\varepsilon} |D_E|^{\frac{1}{2}-\frac{1}{2\ell(n-1)}+\varepsilon}. \tag{1.3}$$

The third author [19] combined this bound with analytic and algebraic methods to prove (assuming GRH) that

$$\mathrm{NV}_{d,k}(D) \gg_\varepsilon D^{\frac{1}{2(2k-1)}-\varepsilon}. \tag{1.4}$$

In a beautiful recent paper, Ellenberg, Pierce, and Wood [12] combined results in [11] with a new probabilistic sieve method which they call the "Chebyshev sieve" to prove that the bound (1.3) holds unconditionally, up to an exceptional set of discriminants with natural density zero.[1] In particular, the GRH assumption was removed. These results were further strengthened and refined in the subsequent work of Pierce, Turnage-Butterbaugh, and Wood [22].

Armed with these new developments, we will prove an asymptotic formula with a power-saving error term for the number of discriminants with $D \leq X$ for which (1.4) holds unconditionally. In particular, we will prove that (1.4) holds unconditionally for an asymptotic density of 100% of imaginary quadratic fields within certain general families.

In order to state our results more precisely, we fix the following assumptions and notation.

Fix a pair (d,k) such that $\mathrm{sign}(d) = (-1)^{k-1}$. Let $\mathcal{S}_{d,k}$ be the set of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ such that $D \equiv 7 \mod 8$, all prime divisors of d split in K, and D is either a prime number or coprime to $2k-1$. For $X > 0$, define the following subsets of $\mathcal{S}_{d,k}$:

$$\mathcal{S}_{d,k}(X) := \{K \in \mathcal{S}_{d,k} : D \leq X\},$$

and for fixed $\varepsilon > 0$,

$$\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X) := \{K \in \mathcal{S}_{d,k}(X) : \ \mathrm{NV}_{d,k}(D) \gg_\varepsilon D^{\frac{1}{2(2k-1)}-\varepsilon}\}.$$

Remark 1.1. The conditions on D in the definition of $\mathcal{S}_{d,k}$ are technical conditions needed for the proofs. For example, the congruence $D \equiv 7 \mod 8$ ensures that the root number $W(\psi) = 1$ for all $\psi \in \Psi_{d,k}(D)$, and the splitting condition ensures that Heegner points of discriminant $-D$ exist on the modular curve $X_0(4d^2)$.

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[1] Strictly speaking, there are also some very mild restrictions on the range of ℓ when $n = 4, 5$, and one must assume that $\mathrm{Gal}(E^c/\mathbb{Q})$ is non-D_4 when $n = 4$.

Our first result is the following asymptotic formula with a power-saving error term.

Theorem 1.2. *Given the prime factorizations* $d = \prod_{i=1}^{m} p_i$ *and* $2k-1 = \prod_{j=1}^{n} q_j^{a_j}$*, define the constant*

$$\delta(d,k) := \frac{1}{2^m \pi^2} \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}} \right) \prod_{j=1}^{n} \left(\frac{1}{1+q_j^{-1}} \right).$$

Then we have

$$\#\mathcal{S}_{d,k,\varepsilon}^{\mathrm{NV}}(X) = \delta(d,k)X + O_{d,k}(X^{1/2}) \tag{1.5}$$

as $X \to \infty$.

Corollary 1.3. *We have*

$$\frac{\#\mathcal{S}_{d,k,\varepsilon}^{\mathrm{NV}}(X)}{\#\mathcal{S}_{d,k}(X)} = 1 + O_{d,k}(X^{-1/2}) \tag{1.6}$$

as $X \to \infty$*. In particular, the bound (1.4) holds for 100% of imaginary quadratic fields* $K \in \mathcal{S}_{d,k}$.

As a crucial step in the proof of Theorem 1.2, we must produce at least one nonvanishing central value *without* assuming that $(2k-1, h(-D)) = 1$. In the following theorem, which is of independent interest, we give an effective criterion for the existence of such a nonvanishing central value.

Theorem 1.4. *Fix a pair* (d,k) *such that* $d \equiv 1 \mod 4$ *is squarefree and* $\mathrm{sign}(d) = (-1)^{k-1}$*. Let* $D \equiv 7 \mod 8$ *be such that all prime divisors of* d *split in* $K = \mathbb{Q}(\sqrt{-D})$*. Then if*

$$D > 64d^4(k+1)^4,$$

there exists at least one character $\psi \in \Psi_{d,k}(D)$ *such that* $L(\psi, k) \neq 0$.

Arithmetic results. We now turn to a discussion of our arithmetic results. Let ψ be an algebraic Hecke character of K of conductor $\mathfrak{f}$ and infinity type $(2k-1, 0)$. Let p be a prime number not dividing $\mathfrak{f}$. The Hecke character ψ corresponds to a p-adic Galois representation $A(\psi)$. Here we will study the Bloch-Kato p-Selmer group $\mathrm{Sel}_p(A(\psi)/K)$ associated to the

Galois representation $A(\psi)$. For the precise definitions of these objects, see Section 5.

Note that when $k = 1$ (so that ψ has infinity type $(1, 0)$), the Bloch-Kato p-Selmer group $\mathrm{Sel}_p(A(\psi)/K)$ equals the usual p-Selmer group $\mathrm{Sel}_p(B_\psi/K)$ of the CM abelian variety B_ψ associated to the Hecke character ψ. It is known that if $\mathrm{Sel}_p(B_\psi/K)$ is finite, then the Mordell-Weil group $B_\psi(K)$ is finite. For a very nice discussion of how to construct the abelian variety B_ψ from ψ, and the relationship between B_ψ and the canonical elliptic $\mathbb{Q}$-curve $A(D)$ associated to ψ discussed above, see the work of Bertolini, Darmon, and Prasanna [2, Sections 2B and 3F].

We will prove the following theorem which relates the nonvanishing of the central value $L(\overline{\psi}, k)$ to the order of the Bloch-Kato p-Selmer group associated to $A(\psi)$.

Theorem 1.5. *Let ψ be an algebraic Hecke character of K of conductor $\mathfrak{f}$ and infinity type $(2k-1, 0)$. Let p be a prime number which splits in K and does not divide $\mathfrak{f}$. Then*

$$L(\overline{\psi}, k) \neq 0 \quad \Longrightarrow \quad \#\mathrm{Sel}_p(A(\psi)/K) < \infty.$$

A result similar to Theorem 1.5 was proved in [15] under the additional condition that p does not divide the degree $[K(\mathfrak{f}) : K]$, where $K(\mathfrak{f})$ denotes the ray class field of conductor $\mathfrak{f}$. We next explain why is it *absolutely crucial* for our applications that this condition be removed. First, observe that the degree is given by (see e.g. [8, Corollary 3.2.4])

$$[K(\mathfrak{f}) : K] = h(-D) \cdot \frac{\phi(\mathfrak{f})}{2},$$

where

$$\phi(\mathfrak{f}) := N_{K/\mathbb{Q}}(\mathfrak{f}) \prod_{\mathfrak{p} | \mathfrak{f}} \left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})}\right)$$

is the generalized Euler ϕ-function for K. Since 2 divides $\phi(\mathfrak{f})$, we know that if p does not divide $[K(\mathfrak{f}) : K]$, then p does not divide the class number $h(-D)$. Now, to prove our density result for nonvanishing central values in Theorem 1.2, we must count imaginary quadratic fields of bounded discriminant with prescribed local conditions. It is very difficult to incorporate the condition $p \nmid h(-D)$ into this counting problem. Cohen and Lenstra [9] have conjectured that for each odd prime p, the probability that $p \nmid h(-D)$ is

$$\prod_{k=1}^{\infty}(1-p^{-k}).$$

See Kohnen and Ono [16] for results in this direction. In the recent work [34], Wiles proved the existence of imaginary quadratic fields with prescribed local conditions whose class numbers are indivisible by a given prime. Beckwith [1] recently proved an effective version of Wiles' theorem. It is conceivable that these results can be used to construct explicit sequences of discriminants which satisfy the local conditions in our counting problem, and whose class numbers are indivisible by p. However, in Theorem 1.5 we instead remove the indivisibility condition altogether by a careful study of the main conjecture of Iwasawa theory for imaginary quadratic fields (see [30, 31, 33]).

We denote the number of characters in the family $\Psi_{d,k}(D)$ whose associated Bloch-Kato p-Selmer group is finite by

$$\mathrm{FS}_{d,k,p}(D) := \#\{\psi \in \Psi_{d,k}(D) :\ \#\mathrm{Sel}_p(A(\psi)/K) < \infty\}.$$

For fixed $\varepsilon > 0$, define the following subset of $\mathcal{S}_{d,k}(X)$:

$$\mathcal{S}^{\mathrm{FS}}_{d,k,p,\varepsilon}(X) := \{K \in \mathcal{S}_{d,k}(X) : \mathrm{FS}_{d,k,p}(D) \gg_\varepsilon D^{\frac{1}{2(2k-1)}-\varepsilon}\}.$$

We have the following density results for finiteness of the Bloch-Kato p-Selmer groups associated to the Galois representations $A(\psi)$.

Theorem 1.6. *Let p be a prime number which splits in $K \in \mathcal{S}_{d,k}$ and does not divide $d\sqrt{-D}\mathcal{O}_K$. Then we have*

$$\#\mathcal{S}^{\mathrm{FS}}_{d,k,p}(X) = \delta(d,k)X + O_{d,k}(X^{1/2})$$

and

$$\frac{\#\mathcal{S}^{\mathrm{FS}}_{d,k,p,\varepsilon}(X)}{\#\mathcal{S}_{d,k}(X)} = 1 + O_{d,k}(X^{-1/2})$$

as $X \to \infty$. In particular, 100% of imaginary quadratic fields $K \in \mathcal{S}_{d,k}$ satisfy the bound

$$\mathrm{FS}_{d,k,p}(D) \gg_\varepsilon D^{\frac{1}{2(2k-1)}-\varepsilon}.$$

Proof. One can use the condition (1.1) to show that a canonical Hecke character $\psi \in \Psi_{d,k}(D)$ satisfies $\overline{\psi(\mathfrak{a})} = \psi(\overline{\mathfrak{a}})$ for integral ideals $\mathfrak{a}$ of K which are prime to $d\sqrt{-D}\mathcal{O}_K$ (see [28]). Since the conductor of ψ is $\mathfrak{f} = d\sqrt{-D}\mathcal{O}_K$, it follows that $\overline{\psi(\mathfrak{a})} = \psi(\overline{\mathfrak{a}})$ for *all* integral ideals $\mathfrak{a}$ of K, and thus $L(\overline{\psi}, s) = L(\psi, s)$. The result now follows immediately from Theorem 1.2, Corollary 1.3, and Theorem 1.5. □

The paper is organized as follows. In Section 2.1 we prove a nonvanishing theorem for half-integral weight theta functions which will be used in the proof of Theorem 1.4. In Section 3 we prove Theorem 1.4. In Section 4 we prove Theorem 1.2 and Corollary 1.3. Finally, in Section 5 we prove Theorem 1.5.

2. Nonvanishing of half-integral weight theta functions

Fix a pair (d, ℓ) where $d \equiv 1 \mod 4$ is a squarefree integer and $\ell \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer such that $\operatorname{sign}(d) = (-1)^\ell$. Define the theta function

$$\theta_{d,\ell}(z) := (2y)^{-\ell/2} \sum_{(n,d)=1} \left(\frac{d}{n}\right) H_\ell(n\sqrt{2y}) e(n^2 z), \quad z = x + iy \in \mathbb{H}$$

where $e(z) := e^{2\pi i z}$ and $H_\ell(x)$ is the degree ℓ Hermite polynomial

$$H_\ell(x) := \frac{1}{(\sqrt{8\pi})^\ell} \sum_{j=0}^{\lfloor \ell/2 \rfloor} \frac{\ell!}{j!(\ell-2j)!} (-1)^j (\sqrt{8\pi}x)^{\ell-2j}.$$

The theta series $\theta_{d,\ell}(z)$ is a weight $\ell + 1/2$ modular form for $\Gamma_0(4d^2)$.

To prove Theorem 1.4, we will need the following effective zero-free region for $\theta_{d,\ell}(z)$ which is of independent interest.

Proposition 2.1. *If $y = \operatorname{Im}(z) > (\ell+2)^2$, then $\theta_{d,\ell}(z) \neq 0$.*

The following inequalities will be used in the proof of Proposition 2.1.

Lemma 2.2. *For $x > \ell$ we have*

$$\left(\frac{8\pi - 2}{8\pi - 1}\right) x^\ell \leq H_\ell(x) \leq x^\ell.$$

Proof. First write

$$H_\ell(x) = \sum_{j=0}^{\lfloor \ell/2 \rfloor} \frac{\ell!}{j!(\ell-2j)!}(-1)^j \frac{x^{\ell-2j}}{(8\pi)^j} = x^\ell - \frac{\ell!}{(\ell-2)!8\pi} x^{\ell-2} + x^\ell \sum_{j=2}^{\lfloor \ell/2 \rfloor} c_{\ell,j},$$

where

$$c_{\ell,j} := \frac{\ell!}{j!(\ell-2j)!} \frac{(-1)^j}{(8\pi)^j x^{2j}}.$$

Now, for $x \geq \ell$ we have the bound

$$\left| \frac{c_{\ell,j+1}}{c_{\ell,j}} \right| = \frac{(\ell-2j)(\ell-2j-1)}{(j+1)8\pi x^2} \leq \frac{\ell^2}{8\pi x^2} \leq \frac{1}{8\pi}.$$

Then it follows that

$$\begin{aligned}
H_\ell(x) &\leq x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} + \sum_{j=2}^{\infty} |c_{\ell,j}| \right] \\
&\leq x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} + \frac{\ell!}{(\ell-2)!8\pi x^2} \sum_{j=1}^{\infty} \left(\frac{1}{8\pi} \right)^j \right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} + \frac{\ell!}{(\ell-2)!8\pi x^2} \left(\frac{1}{8\pi - 1} \right) \right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} \left(\frac{8\pi - 2}{8\pi - 1} \right) \right] \\
&\leq x^\ell.
\end{aligned}$$

On the other hand, arguing similarly with the reverse triangle inequality, for $x \geq \ell$ we have

$$\begin{aligned}
H_\ell(x) &\geq x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} - \sum_{j=2}^{\infty} |c_{\ell,j}|\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} - \frac{\ell!}{(\ell-2)!8\pi x^2} \sum_{j=1}^{\infty} \left(\frac{1}{8\pi}\right)^j\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} \sum_{j=0}^{\infty} \left(\frac{1}{8\pi}\right)^j\right] \\
&= x^\ell \left[1 - \frac{\ell!}{(\ell-2)!8\pi x^2} \left(\frac{8\pi}{8\pi - 1}\right)\right] \\
&= x^\ell \left[1 - \frac{\ell(\ell-1)}{(8\pi-1)x^2}\right] \\
&\geq x^\ell \left[1 - \frac{\ell^2}{(8\pi-1)x^2}\right] \\
&\geq x^\ell \left[1 - \frac{1}{(8\pi-1)}\right] \\
&= \left(\frac{8\pi-2}{8\pi-1}\right) x^\ell.
\end{aligned}$$

□

Lemma 2.3. *If* $t > (\ell+2)^2$ *then*

$$t - \frac{\ell}{4\pi}\log(\pi t) > \frac{\ell+1}{\pi}\log(2) \tag{2.1}$$

and

$$t - \frac{\ell}{12\pi}\log(t) > \frac{3\ell+2}{12\pi}\log(2). \tag{2.2}$$

Proof. We first consider the inequality (2.1). Clearly, we have

$$t - \frac{\ell}{4\pi}\log(\pi t) > \frac{\ell+1}{\pi}\log(2) \quad \Longleftrightarrow \quad 4\pi t - \ell\log(16\pi t) > 4\log 2.$$

Moreover, the function

$$g_\ell(t) := 4\pi t - \ell\log(16\pi t)$$

is strictly increasing for $t > \ell/4\pi$. Hence, if we assume that $t > (\ell+2)^2 > \ell/4\pi$, then we have

$$\begin{aligned} g_\ell(t) > g_\ell((\ell+2)^2) &= 4\pi(\ell+2)^2 - \ell\log(16\pi(\ell+2)^2) \\ &= 16\pi + \ell(16\pi - \log 16\pi) + \ell(4\pi\ell - 2\log(\ell+2)) \\ &> 4\log 2. \end{aligned}$$

On the other hand, the inequality (2.2) follows from (2.1) since

$$t - \frac{\ell}{12\pi}\log(t) > t - \frac{\ell}{4\pi}\log(\pi t) > \frac{\ell+1}{\pi}\log(2) > \frac{3\ell+2}{12\pi}\log(2).$$

□

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Using the definitions of $H_\ell(x)$ and the Kronecker symbol (d/n), along with the condition $\operatorname{sign}(d) = (-1)^\ell$, for $n \neq 0$ we have

$$\begin{aligned} \left(\frac{d}{-n}\right) H_\ell(-n\sqrt{2y}) &= \operatorname{sign}(d)\left(\frac{d}{n}\right)(-1)^\ell H_\ell(n\sqrt{2y}) \\ &= (-1)^{2\ell}\left(\frac{d}{n}\right) H_\ell(n\sqrt{2y}) \\ &= \left(\frac{d}{n}\right) H_\ell(n\sqrt{2y}). \end{aligned}$$

Then the theta series can be written as

$$\theta_{d,\ell}(z) = (2y)^{-\ell/2}\left[\left(\frac{d}{0}\right) H_\ell(0) + 2\sum_{n=1}^{\infty}\left(\frac{d}{n}\right) H_\ell(n\sqrt{2y})e(n^2 z)\right]. \tag{2.3}$$

From here forward we assume that $y > (\ell+2)^2$. We will consider the cases $d = 1$ and $d \neq 1$ separately.

Case 1 ($d = 1$): If $d = 1$, then

$$\left|\left(\frac{1}{0}\right) H_\ell(0)\right| = \frac{\ell!}{(8\pi)^{\ell/2}\,(\ell/2)!}.$$

Therefore, by (2.3) if

$$\sum_{n=1}^{\infty}|H_\ell(n\sqrt{2y})e(n^2 z)| < \frac{\ell!}{2(8\pi)^{\ell/2}\,(\ell/2)!},$$

then the reverse triangle inequality implies that $\theta_{1,\ell}(z) \neq 0$.

Now, consider the function

$$f_\ell(t) := \frac{\log(2^{t-1}t^\ell)}{2\pi(t^2-1)}.$$

Since $f_\ell(t)$ is strictly decreasing for $t > 1$, we have

$$y > (\ell+2)^2 > f_\ell(2) \geq f_\ell(n), \quad n \geq 2.$$

Note that the inequality $y > f_\ell(n)$ is equivalent to

$$\frac{n^\ell}{e^{2\pi(n^2-1)y}} < 2^{1-n}, \quad n \geq 2. \tag{2.4}$$

Then we can now estimate the series as

$$\begin{aligned}
\sum_{n=1}^{\infty} |H_\ell(n\sqrt{2y})e(n^2z)| &\leq (2y)^{\ell/2} \sum_{n=1}^{\infty} \frac{n^\ell}{e^{2\pi n^2 y}} \\
&= (2y)^{\ell/2} \frac{1}{e^{2\pi y}} \sum_{n=1}^{\infty} \frac{n^\ell}{e^{2\pi(n^2-1)y}} \\
&\leq (2y)^{\ell/2} \frac{1}{e^{2\pi y}} \sum_{n=1}^{\infty} 2^{1-n} \\
&= (2y)^{\ell/2} \frac{2}{e^{2\pi y}} \\
&< \frac{1}{2(8\pi)^{\ell/2}} \\
&\leq \frac{\ell!}{2(8\pi)^{\ell/2}\,(\ell/2)!},
\end{aligned}$$

where the first inequality follows from the upper bound in Lemma 2.2 (since $y > (\ell+2)^2$ we have $n\sqrt{2y} \geq \ell$ for all $n \geq 1$), the second inequality follows from (2.4), and a short calculation shows that the third inequality is equivalent to inequality (2.1) of Lemma 2.3. This proves Case 1.

Case 2 ($d \neq 1$): Since $d \neq 1$, we have $(d/0) = 0$, and (2.3) can be written as

$$\theta_{d,\ell}(z) = 2(2y)^{-\ell/2}\left[H_\ell(\sqrt{2y})e(z) + \sum_{n=2}^{\infty}\left(\frac{d}{n}\right) H_\ell(n\sqrt{2y})e(n^2z)\right].$$

Therefore, if

$$\sum_{n=2}^{\infty} |H_\ell(n\sqrt{2y})e(n^2 z)| < |H_\ell(\sqrt{2y})e(z)|,$$

then the reverse triangle inequality implies that $\theta_{d,\ell}(z) \neq 0$.

Now, we have $H_0(\sqrt{2y}) = 1$ and $H_1(\sqrt{2y}) = \sqrt{2y} > 1$. Moreover, if $\ell \geq 2$ then by the lower bound in Lemma 2.2 we have

$$H_\ell(\sqrt{2y}) \geq \frac{8\pi - 2}{8\pi - 1}(2y)^{\ell/2} > 1.$$

Hence it suffices to show that

$$\sum_{n=2}^{\infty} |H_\ell(n\sqrt{2y})e(n^2 z)| < |e(z)| = \frac{1}{e^{2\pi y}}.$$

A modification of the argument in Case 1 shows that

$$\sum_{n=2}^{\infty} |H_\ell(n\sqrt{2y})e(n^2 z)| \leq (2y)^{\ell/2}\frac{2^{\ell+1}}{e^{8\pi y}} < \frac{1}{e^{2\pi y}},$$

where a short calculation shows that the second inequality is equivalent to inequality (2.2) of Lemma 2.3. This proves Case 2. □

3. Proof of Theorem 1.4

To prove Theorem 1.4, we proceed via a generalization and refinement of the approach in [6], which builds on the works [15, 18, 19]. Roughly speaking, the key idea is to exploit the position of the Heegner points in the cusp at infinity of the modular curve $X_0(4d^2)$.

Fix a pair (d, k) where $d \equiv 1 \mod 4$ is a squarefree integer and $k \in \mathbb{Z}^+$ is a positive integer such that $\operatorname{sign}(d) = (-1)^{k-1}$. Consider the C^∞ function $F_{d,k} : \mathbb{H} \to \mathbb{R}_{\geq 0}$ defined by

$$F_{d,k}(z) := \operatorname{Im}(z)^{k-\frac{1}{2}}|\theta_{d,k-1}(z)|^2.$$

Since $\theta_{d,k-1}(z)$ is a weight $k - 1/2$ modular form for $\Gamma_0(4d^2)$, the function $F_{d,k}(z)$ is $\Gamma_0(4d^2)$-invariant.

Let $D \equiv 7 \mod 8$ be a positive integer such that all prime divisors of d split in $K = \mathbb{Q}(\sqrt{-D})$ (the so-called Heegner hypothesis). Then Heegner points of discriminant $-D$ exist on the modular curve $X_0(4d^2) = \Gamma_0(4d^2)\backslash\mathbb{H}$. In particular, fix a solution $r \mod 8d^2$ of $x^2 \equiv -D \mod 16d^2$. Then for each ideal class $\mathcal{A}$ of K, we can choose a primitive integral ideal $\mathfrak{a} \in \mathcal{A}$ such that

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\left(\frac{b+\sqrt{-D}}{2}\right), \quad a = N_{K/\mathbb{Q}}(\mathfrak{a}) \equiv 0 \mod 4d^2, \quad b \in \mathbb{Z},$$

where $b \equiv r \mod 8d^2$ and $b^2 \equiv -D \mod 4a$. Then

$$\tau_{[\mathfrak{a}]}^{(r)} = \frac{-b+\sqrt{-D}}{2a} \in \mathbb{H}$$

defines a Heegner point on $X_0(4d^2)$ which depends only on the ideal class $[\mathfrak{a}]$ and on $r \mod 8d^2$.

Define the $\mathrm{Cl}(K)$-orbit of Heegner points

$$\mathcal{O}_{D,4d^2,r} := \{\tau_{[\mathfrak{a}]}^{(r)} : [\mathfrak{a}] \in \mathrm{Cl}(K)\}.$$

Then by [15, Theorem 3.5], we have the following formula for the average of the central values:

$$\frac{1}{h(-D)} \sum_{\psi \in \Psi_{d,k}(D)} L(\psi, k) = c(k) \frac{\pi}{\sqrt{D}} \sum_{[\mathfrak{a}] \in \mathrm{Cl}(K)} F_{d,k}(\tau_{[\mathfrak{a}]}^{(r)}), \tag{3.1}$$

where $c(k) := 2(8\pi)^{k-1}/(k-1)!$. This formula is independent of the choice of $r \mod 8d^2$.

Now, pick any primitive integral ideal $\mathfrak{n} \subset \mathcal{O}_K$ of norm $N_{K/\mathbb{Q}}(\mathfrak{n}) = 4d^2$ (by the Heegner hypothesis, there are $2^{t(4d^2)}$ such ideals, where $t(4d^2)$ is the number of prime divisors of $4d^2$). Consider the Heegner point corresponding to the class of $\mathfrak{n}$,

$$\tau_{[\mathfrak{n}]}^{(r)} = \frac{-b+\sqrt{-D}}{8d^2}.$$

Since $F_{d,k}(z) \geq 0$, we may discard the terms with $[\mathfrak{a}] \neq [\mathfrak{n}]$ in (3.1) to get the lower bound

$$\sum_{\psi \in \Psi_{d,k}(D)} L(\psi, k) \geq \pi c(k) \frac{h(-D)}{\sqrt{D}} F_{d,k}(\tau_{[\mathfrak{n}]}^{(r)}). \tag{3.2}$$

By Proposition 2.1, if $\mathrm{Im}(z) > (k+1)^2$ then $F_{d,k}(z) > 0$. Moreover, we have

$$\mathrm{Im}(\tau_{[\mathfrak{n}]}^{(r)}) = \frac{\sqrt{D}}{8d^2} > (k+1)^2 \quad \Longleftrightarrow \quad D > 64d^4(k+1)^4.$$

Hence, if $D > 64d^4(k+1)^4$ then

$$F_{d,k}(\tau_{[\mathfrak{n}]}^{(r)}) > 0.$$

It follows from (3.2) that

$$\sum_{\psi \in \Psi_{d,k}(D)} L(\psi, k) > 0,$$

and so there must exist at least one character $\psi \in \Psi_{d,k}(D)$ such that $L(\psi, k) \neq 0$. This completes the proof of Theorem 1.4. $\square$

4. Proofs of Theorem 1.2 and Corollary 1.3

In this section, we prove Theorem 1.2 and Corollary 1.3.

For convenience, we recall the setup from the introduction. Fix a pair (d, k) such that $\mathrm{sign}(d) = (-1)^{k-1}$. Let $\mathcal{S}_{d,k}$ be the set of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ such that $D \equiv 7 \mod 8$, all prime divisors of d split in K, and D is either a prime number or coprime to $2k-1$. For $X > 0$, define the following subsets of $\mathcal{S}_{d,k}$:

$$\mathcal{S}_{d,k}(X) := \{K \in \mathcal{S}_{d,k} : D \leq X\},$$

and for fixed $\varepsilon > 0$,

$$\mathcal{S}_{d,k,\varepsilon}^{\mathrm{NV}}(X) := \{K \in \mathcal{S}_{d,k}(X) : \ \mathrm{NV}_{d,k}(D) \gg_{\varepsilon} D^{\frac{1}{2(2k-1)} - \varepsilon}\}.$$

In addition, we will need the subset

$$\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X) := \{K \in \mathcal{S}_{d,k}(X) : \ \#\mathrm{Cl}_{2k-1}(K) \ll_{\varepsilon} D^{\frac{1}{2} - \frac{1}{2(2k-1)} + \varepsilon}\}.$$

We first give asymptotic formulas with power-saving error terms for $\#\mathcal{S}_{d,k}(X)$ and $\#\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X)$.

Proposition 4.1. *Given the prime factorizations* $d = \prod_{i=1}^{m} p_i$ *and*

$$2k - 1 = \prod_{j=1}^{n} q_j^{a_j},$$

define the constant

$$\delta(d, k) := \frac{1}{2^m \pi^2} \prod_{i=1}^{m} \left(\frac{1}{1 + p_i^{-1}} \right) \prod_{j=1}^{n} \left(\frac{1}{1 + q_j^{-1}} \right).$$

Then we have

$$\#\mathcal{S}_{d,k}(X) = \delta(d, k)X + O_k \left(2d \left(\prod_{j=1}^{n} q_j \right) X^{1/2} \right)$$

as $X \to \infty$*. Moreover, we have*

$$\#\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X) = \delta(d, k)X + O_{d,k}(X^{1/2})$$

as $X \to \infty$*.*

Proof. First, we decompose the set $\mathcal{S}_{d,k}(X)$ into the disjoint union

$$\mathcal{S}_{d,k}(X) = \mathcal{S}_{d,k}^{1}(X) \sqcup \{K \in \mathcal{S}_{d,k}(X) : D \text{ is prime and } (D, 2k-1) \neq 1\}$$

where

$$\mathcal{S}_{d,k}^{1}(X) := \{K \in \mathcal{S}_{d,k}(X) : (D, 2k-1) = 1\}.$$

The second set in this decomposition is bounded by the number $t(2k-1; X)$ of prime divisors $p \leq X$ of $2k - 1$. Clearly, we have

$$t(2k - 1; X) = O(k),$$

and thus

$$\#\mathcal{S}_{d,k}(X) = \#\mathcal{S}_{d,k}^{1}(X) + O(k).$$

We will need the following result of Ellenberg, Pierce, and Wood [12, Proposition A.1], which counts quadratic number fields of bounded discriminant with prescribed local conditions.

Proposition 4.2. *Let K be a quadratic extension of $\mathbb{Q}$ and P be a finite set of rational primes. For each $p \in P$ choose a splitting type in K and assign a corresponding density as follows:*

$$\begin{aligned} \delta_p &:= \frac{1}{2}(1+p^{-1})^{-1}, && \text{if } p \text{ is split or inert in } K; \\ \delta_p &:= (1+p)^{-1}, && \text{if } p \text{ is ramified in } K. \end{aligned}$$

Let $e = \prod_{p \in P} p$ and $\delta_e = \prod_{p\in P} \delta_p$. Let $N_2^{\pm}(X;P)$ count the number of real (respectively imaginary) quadratic extensions K of $\mathbb{Q}$ with fundamental discriminant $|D_K| \leq X$ such that each $p \in P$ has splitting type in K chosen as above. Then we have

$$N_2^{\pm}(X;P) = \delta_e \left(\frac{1}{3} + \frac{1}{6}\right) \frac{1}{\zeta(2)} X + O(eX^{1/2}).$$

We want to use Proposition 4.2 to count $\mathcal{S}^1_{d,k}(X)$. To do this, we must further decompose this set into subsets satisfying appropriate local conditions.

Recall that $\mathcal{S}^1_{d,k}(X)$ consists of those imaginary quadratic fields K such that $D \leq X$, $D \equiv 7 \mod 8$, all prime divisors of d split in K, and $(D, 2k-1) = 1$. Note that

$$D \equiv 7 \mod 8 \iff 2 \text{ splits in } K$$

and

$$(D, 2k-1) = 1 \iff \text{every prime divisor of } 2k-1 \text{ is split or inert in } K.$$

Now, consider the prime factorizations

$$d = \prod_{i=1}^{m} p_i \quad \text{and} \quad 2k - 1 = \prod_{j=1}^{n} q_j^{a_j}$$

(recall that d is squarefree), and define the following sets of prime numbers:

$$P_d := \{2, p_1, \ldots, p_m\} \quad \text{and} \quad Q_k := \{q_1, \ldots, q_n\}.$$

Let $K \in \mathcal{S}^1_{d,k}(X)$. Then a prime $q_j \in Q_k$ is split or inert in K. For $j \in \{1, \ldots, n\}$, define

$$h_j := \left(\frac{-D}{q_j}\right) = \begin{cases} 1, & \text{if } q_j \in Q_k \text{ is split in } K, \\ -1, & \text{if } q_j \in Q_k \text{ is inert in } K \end{cases}$$

where $(-D/\cdot)$ is the Kronecker symbol. Then there is a one-to-one correspondence between vectors $\mathbf{h} := (h_1, \ldots, h_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ and choices of splitting types in K for primes in the set Q_k.

Let $\mathcal{S}_{d,k,\mathbf{h}}(X)$ denote the subset of fields $K \in \mathcal{S}^1_{d,k}(X)$ such that the primes in P_d split in K, and the primes in Q_k decompose in K according to the splitting type corresponding to $\mathbf{h}$. Then we have the disjoint union

$$\mathcal{S}^1_{d,k}(X) = \bigsqcup_{\mathbf{h} \in (\mathbb{Z}/2\mathbb{Z})^n} \mathcal{S}_{d,k,\mathbf{h}}(X).$$

Define the set of prime numbers $R_{d,k} := P_d \cup Q_k$. Then each vector $\mathbf{h} \in (\mathbb{Z}/2\mathbb{Z})^n$ uniquely determines a set of densities

$$D_{d,k,\mathbf{h}} := \{\delta_p : \ p \in R_{d,k}\}$$

where the densities δ_p for $p \in Q_k$ are assigned according to the splitting type corresponding to $\mathbf{h}$. However, since each prime $p \in R_{d,k}$ is split or inert in K, the densities δ_p are equal (recall the definition of δ_p in Proposition 4.2). Hence by Proposition 4.2, for each $\mathbf{h}$ we have the asymptotic formula

$$\begin{aligned}
&\#\mathcal{S}_{d,k,\mathbf{h}}(X) = N_2^-(X; R_{d,k}) \\
&= \frac{1}{2^{m+1}} \left(\frac{1}{1+2^{-1}}\right) \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}}\right) \frac{1}{2^n} \prod_{j=1}^{n} \left(\frac{1}{1+q_j^{-1}}\right) \left(\frac{1}{3} + \frac{1}{6}\right) \frac{1}{\zeta(2)} X \\
&\quad + O\left(2d\left(\prod_{j=1}^{n} q_j\right) X^{1/2}\right) \\
&= \frac{1}{2^{m+n}\pi^2} \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}}\right) \prod_{j=1}^{n} \left(\frac{1}{1+q_j^{-1}}\right) X + O\left(2d\left(\prod_{j=1}^{n} q_j\right) X^{1/2}\right),
\end{aligned}$$

where we used $\zeta(2) = \pi^2/6$. Hence we have

$$\begin{aligned}
&\#\mathcal{S}^1_{d,k}(X) = \sum_{\mathbf{h} \in (\mathbb{Z}/2\mathbb{Z})^n} \#\mathcal{S}_{d,k,\mathbf{h}}(X) \\
&= \frac{1}{2^m\pi^2} \prod_{i=1}^{m} \left(\frac{1}{1+p_i^{-1}}\right) \prod_{j=1}^{n} \left(\frac{1}{1+q_j^{-1}}\right) X + O\left(2d\left(\prod_{j=1}^{n} q_j\right) X^{1/2}\right).
\end{aligned}$$

Combining the preceding results, we get the asymptotic formula

$$\text{(4.1)}\quad \#\mathcal{S}_{d,k}(X) = \frac{1}{2^m\pi^2}\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\prod_{j=1}^{n}\left(\frac{1}{1+q_j^{-1}}\right)X + O_k\left(2d\left(\prod_{j=1}^{n} q_j\right)X^{1/2}\right).$$

Next, define the set

$$\mathcal{S}_{d,k,\varepsilon}^{\neg\mathrm{Tor}}(X) := S_{d,k}(X) \setminus S_{d,k,\varepsilon}^{\mathrm{Tor}}(X)$$

and write

$$\#\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X) = \#\mathcal{S}_{d,k}(X) - \#\mathcal{S}_{d,k,\varepsilon}^{\neg\mathrm{Tor}}(X). \tag{4.2}$$

As a consequence of [22, Remark 3.5 and Theorem 7.2] (which, in particular, improves the bound on the exceptional set of discriminants in [12, Theorem 1.1]), we have

$$\begin{aligned}\text{(4.3)}\quad \#\mathcal{S}_{d,k,\varepsilon}^{\neg\mathrm{Tor}}(X) &\ll \#\{\text{quadratic fields } K/\mathbb{Q} \text{ with } |D_K| \leq X \\ &\qquad\qquad \text{which fail to satisfy } \#\mathrm{Cl}_{2k-1}(K) \ll_\varepsilon D^{\frac{1}{2}-\frac{1}{2(2k-1)}+\varepsilon}\} \\ &\ll_{k,\varepsilon} X^{\varepsilon}.\end{aligned}$$

Finally, combining (4.2) with (4.1) and (4.3) gives the asymptotic formula

$$\#\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X) = \frac{1}{2^m\pi^2}\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\prod_{j=1}^{n}\left(\frac{1}{1+q_j^{-1}}\right)X + O_{d,k}(X^{1/2}).$$

□

Proofs of Theorem 1.2 and Corollary 1.3. Let

$$C_{d,k} := 64d^4(k+1)^4$$

be the constant appearing in Theorem 1.4. Then for $X \gg C_{d,k}$, we decompose the set $\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X)$ into the disjoint union

$$\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X) = \mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(C_{d,k}) \sqcup \mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor},1}(X)$$

where

$$\mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor},1}(X) := \{K \in \mathcal{S}_{d,k,\varepsilon}^{\mathrm{Tor}}(X) : D \geq C_{d,k}\}.$$

Lemma 4.3. *We have* $\mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X) \subset \mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X)$.

Proof. Let ζ_{2k-1} be a primitive $(2k-1)$-st root of unity and define the cyclotomic extension $K(\zeta_{2k-1})$ of K. Then the Galois group

$$G_k := \mathrm{Gal}(\overline{\mathbb{Q}}/K(\zeta_{2k-1}))$$

acts on $\Psi_{d,k}(D)$ by

$$\psi \mapsto \psi^\sigma := \sigma \circ \psi, \quad \sigma \in G_k.$$

For a fixed character $\psi_0 \in \Psi_{d,k}(D)$, we denote the Galois orbit of ψ_0 by

$$\mathcal{O}_{\psi_0} = \{\psi_0^\sigma : \sigma \in G_k\}.$$

Now, if $K \in \mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X)$ we have $D \geq C_{d,k}$. Then by Theorem 1.4, there exists a character $\psi_0 \in \Psi_{d,k}(D)$ such that $L(\psi_0, k) \neq 0$. Also, by work of Shimura [32], for any $\sigma \in G_k$ we have

$$L(\psi_0^\sigma, k) \neq 0 \quad \text{if and only if} \quad L(\psi_0, k) \neq 0.$$

Hence it follows that

$$\mathrm{NV}_{d,k}(D) \geq \#\mathcal{O}_{\psi_0}.$$

On the other hand, by [19, Proposition 1.1], if either $D = p$ is a prime number or $(D, 2k-1) = 1$, then

$$\#\mathcal{O}_{\psi_0} = \frac{h(-D)}{\#\mathrm{Cl}_{2k-1}(K)}.$$

Therefore

$$\mathrm{NV}_{d,k}(D) \geq \frac{h(-D)}{\#\mathrm{Cl}_{2k-1}(K)}.$$

By Siegel's theorem, we have

$$h(-D) \gg_\varepsilon D^{1/2-\varepsilon}.$$

Then since $K \in \mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X)$ satisfies the bound

$$\#\mathrm{Cl}_{2k-1}(K) \ll_\varepsilon D^{\frac{1}{2}-\frac{1}{2(2k-1)}+\varepsilon},$$

we get

$$\mathrm{NV}_{d,k}(D) \gg_{\varepsilon} D^{\frac{1}{2(2k-1)}-2\varepsilon}.$$

After replacing 2ε with ε, it follows that $K \in \mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X)$. $\square$

We now continue the proof of Theorem 1.2. Using Lemma 4.3, we get the decomposition

$$\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X) = \mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X) \sqcup \left(\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X) \setminus \mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X)\right).$$

Now, since

$$\#\mathcal{S}^{\mathrm{Tor}}_{d,k,\varepsilon}(C_{d,k}) \ll_{d,k} 1,$$

by Proposition 4.1 we get

$$\begin{aligned}\#\mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X) &= \#\mathcal{S}^{\mathrm{Tor}}_{d,k,\varepsilon}(X) - \#\mathcal{S}^{\mathrm{Tor}}_{d,k,\varepsilon}(C_{d,k}) \\ &= \frac{1}{2^m\pi^2}\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\prod_{j=1}^{n}\left(\frac{1}{1+q_j^{-1}}\right)X + O_{d,k}(X^{1/2}).\end{aligned}$$

Also, since

$$\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X) \setminus \mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X) \subset \mathcal{S}^{\neg\mathrm{Tor}}_{d,k,\varepsilon}(X) \sqcup \mathcal{S}^{\mathrm{Tor}}_{d,k,\varepsilon}(C_{d,k}),$$

the bound (4.3) gives

$$\#\left(\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X) \setminus \mathcal{S}^{\mathrm{Tor},1}_{d,k,\varepsilon}(X)\right) \le \#\mathcal{S}^{\neg\mathrm{Tor}}_{d,k,\varepsilon}(X) + \#\mathcal{S}^{\mathrm{Tor}}_{d,k,\varepsilon}(C_{d,k}) \ll_{d,k,\varepsilon} X^{\varepsilon}.$$

Hence we get

$$\#\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X) = \frac{1}{2^m\pi^2}\prod_{i=1}^{m}\left(\frac{1}{1+p_i^{-1}}\right)\prod_{j=1}^{n}\left(\frac{1}{1+q_j^{-1}}\right)X + O_{d,k}(X^{1/2}).$$

This proves Theorem 1.2.

Finally, by combining the preceding asymptotic formula with Proposition 4.1, we get

$$\frac{\#\mathcal{S}^{\mathrm{NV}}_{d,k,\varepsilon}(X)}{\#\mathcal{S}_{d,k}(X)} = 1 + O_{d,k}(X^{-1/2}).$$

This proves Corollary 1.3. $\square$

5. Bloch-Kato Selmer groups of Galois representations associated to Hecke characters

Let K be an imaginary quadratic field and p be a prime number which splits in K. Let ψ be a Hecke character of K of infinity type $(2k-1, 0)$ and conductor $\mathfrak{f}$. In this section, we prove Theorem 1.5, which states that if $L(\overline{\psi}, k) \neq 0$, then the Bloch-Kato Selmer group of the p-adic Galois representation associated to ψ is finite. Our main tool is Iwasawa theory, and a critical ingredient is the main conjecture of Iwasawa theory for imaginary quadratic fields due to Rubin ([30], [31]). These works exclude the case where the prime number p divides the degree $[K(\mathfrak{f}) : K]$ of the ray class field $K(\mathfrak{f})$ of conductor $\mathfrak{f}$, which is precisely the case we need in this paper (and is a major difference between this section and [15, Section 2]). To remove this condition, we will make important use of recent work of Vigué [33].

First, we define the Bloch-Kato local conditions.

Definition 5.1 ([5], Section 3). Suppose that F is a finite extension of $\mathbb{Q}_p$. Let V be a finite dimensional F-representation on which $\mathrm{Gal}(\overline{K}/K)$ acts continuously, let T be an $\mathcal{O}_F$-lattice inside V (i.e., a free $\mathcal{O}_F$-module inside V of rank equal to $\dim V$) closed under the action of $\mathrm{Gal}(\overline{K}/K)$, and let A denote V/T.

Suppose that L is a number field, and for a prime v of L, let L_v^{ur} denote the maximal unramified extension of L_v. For every prime v not above p, we let

$$\begin{aligned} H^1_f(L_v, V) &= H^1(L_v^{ur}/L_v, V^{G_{K_v^{ur}}}) = \mathrm{Ker}\left(H^1(L_v, V) \to H^1(L_v^{ur}, V)\right), \\ H^1_f(L_v, A) &= \mathrm{Im}\left(H^1_f(L_v, V) \to H^1(L_v, A)\right). \end{aligned}$$

For a prime v above p, we let

$$\begin{aligned} H^1_f(L_v, V) &= \mathrm{Ker}\left(H^1(L_v, V) \to H^1(L_v, V \otimes B_{cris})\right), \\ H^1_f(L_v, A) &= \mathrm{Im}\left(H^1_f(L_v, V) \to H^1(L_v, A)\right) \end{aligned}$$

(see [13] for the definition and properties of B_{cris}.)

Fix an embedding $\overline{\mathbb{Q}} \to \mathbb{C}_p$. Fixing such an embedding does two things. First, it chooses a prime ideal $\mathfrak{p}$ of K above p. Second, through the embedding (and class field theory), ψ induces a character of the Galois group $\psi : \mathrm{Gal}(\overline{K}/K) \to \mathbb{C}_p^*$ (which we also denote by ψ). In fact, all its images are in a certain finite extension of $\mathbb{Q}_p$, and since the infinity type of ψ is $(2k-1, 0)$, ψ factors through $\mathrm{Gal}(K(\mathfrak{f}\mathfrak{p}^\infty)/K)$.

Definition 5.2. Choose a finite extension F of $\mathbb{Q}_p$ so that all the images of ψ are in F^*, and let $\mathcal{O}_F$ be its ring of integers.

We let $\phi_{cyc} : \mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p^\times$ be the cyclotomic character given as follows. For any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, there is $n_\sigma \in \mathbb{Z}_p^\times$ satisfying $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{n_\sigma}$ for every $n \geq 0$ and any primitive p^n-th root of unity ζ_{p^n}. We let $\phi_{cyc}(\sigma) = n_\sigma$. We note that ϕ_{cyc} can be viewed as a character of the Galois group associated to $N_{K/\mathbb{Q}} : I_K \to \mathbb{Z}$.

We let $V(\psi)$ (resp. $T(\psi)$) be the rank-1 F-representation (resp. $\mathcal{O}_F$-representation) of $\mathrm{Gal}(\overline{K}/K)$ given by

$$\sigma \cdot x = \psi(\sigma)\phi_{cyc}^{-k+1}(\sigma)x$$

for every $x \in V(\psi)$ (resp. $x \in T(\psi)$) and $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

Finally, we let $A(\psi)$ denote $V(\psi)/T(\psi)$. The following definition is essentially due to Bloch and Kato ([5]):

$$\mathrm{Sel}_p(A(\psi)/L) = \mathrm{Ker}\left(H^1(L, A(\psi)) \to \prod_v \frac{H^1(L_v, A(\psi))}{H^1_f(L_v, A(\psi))} \right)$$

where v runs over all primes of L.

Remark 5.3. As observed in the introduction, if $k = 1$ (so that the infinity type of ψ is $(1,0)$), then $\mathrm{Sel}_p(A(\psi)/L)$ equals the usual p-Selmer group $\mathrm{Sel}_p(B_\psi/L)$ of the abelian variety B_ψ associated to ψ, whose finiteness implies the finiteness of the Mordell-Weil group $B_\psi(L)$.

We also define a similar group that we will need for technical reasons.

Definition 5.4. For an (infinite or finite) extension L of K,

$$S(L, A(\psi)) = \mathrm{Ker}\left(H^1(L, A(\psi)) \to \prod_{v \nmid \mathfrak{p}} \frac{H^1(L_v, A(\psi))}{H^1_{ur}(L_v, A(\psi))} \right)$$

where v runs over all primes except for the primes lying above $\mathfrak{p}$.

We now explain the relationship between $\mathrm{Sel}_p(A(\psi)/K)$ and $S(K, A(\psi))$.

Proposition 5.5. *We have*

$$\mathrm{Sel}_p(A(\psi)/K) \subset S(K, A(\psi)).$$

Proof. Recall that ψ factors through $\mathrm{Gal}(K(\mathfrak{f}\mathfrak{p}^\infty)/K)$, and note that $\bar{\mathfrak{p}}$ is unramified over $K(\mathfrak{f}\mathfrak{p}^\infty)/K$. Thus, as a representation of $G_{K_{\bar{\mathfrak{p}}}}$, $V(\psi)$ is unramified. (In the language of p-adic Hodge theory, $V(\psi)$ has Hodge-Tate weight $-k+1$.) Note that

$$D(V(\psi)) = (V(\psi) \otimes B_{cris})^{G_{K_{\mathfrak{p}}}}$$

has a filtration $Fil^0 D(V(\psi))$ given by $B_{dR}^+ \subset B_{dR}$. The short exact sequence

$$\begin{array}{rcl} 0 \to \mathbb{Q}_p \to B_{cris} & \to & B_{cris} \oplus B_{dR}/B_{dR}^+ \to 0 \\ x & \mapsto & ((1-\varphi)x, x \text{ modulo } B_{dR}^+), \end{array}$$

induces a long exact sequence of cohomology groups, which shows that $H_f^1(K_{\bar{\mathfrak{p}}}, V(\psi))$ is the image of $D(V(\psi))/Fil^0 D(V(\psi))$, which is trivial because of the Hodge-Tate weight of $V(\psi)$. Thus, we have

$$H_f^1(K_{\bar{\mathfrak{p}}}, V(\psi)) = 0,$$

which implies that

$$H_f^1(K_{\bar{\mathfrak{p}}}, A(\psi))(=0) \subset H_{ur}^1(K_{\bar{\mathfrak{p}}}, A(\psi)).$$

Together with the fact that the definition of $S(L, A(\psi))$ has no local condition at primes above $\mathfrak{p}$, this shows that $\mathrm{Sel}_p(A(\psi)/K) \subset S(K, A(\psi))$, which proves the claim. $\square$

Remark 5.6. By a more careful argument, one can show that in fact, $\mathrm{Sel}_p(A(\psi)/K)$ is a subgroup of finite index, but this will not be needed.

Definition 5.7. If L is an extension of K, let $M(L)$ be the maximal abelian p-extension of L which is unramified at every prime except the primes above $\mathfrak{p}$, and let $X(L) = \mathrm{Gal}(M(L)/L)$. Similarly, let $H(L)$ be the maximal abelian p-extension of L unramified everywhere, and let $A(L) = \mathrm{Gal}(H(L)/L)$.

In particular, let X_∞ denote $X(K(\mathfrak{f}p^\infty))$, and A_∞ denote $A(K(\mathfrak{f}p^\infty))$.

Following the notation of [31] and [33], we define the following: Let k_∞ be the unique $\mathbb{Z}_p^2$-extension of K. We let $G_\infty = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)$, $G = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/k_\infty)$, and choose $\Gamma(\cong \mathbb{Z}_p^2) \subset G_\infty$ so that $G_\infty \cong \Gamma \times G$.

We define the Iwasawa algebra

$$\Lambda_\infty = \mathcal{O}_F[[G_\infty]] = \varprojlim_F \mathcal{O}_F[\mathrm{Gal}(F/K)]$$

where F runs over all finite extensions of K in $K(\mathfrak{f}p^\infty)$.

For an abelian group $\mathcal{G}$, the Pontryagin dual of $\mathcal{G}$ is $\mathrm{Hom}(\mathcal{G}, \mathbb{Q}/\mathbb{Z})$ (denoted by $\mathcal{G}^\vee$). In particular, if $\mathcal{G}$ is a p-group, $\mathcal{G}^\vee \cong \mathrm{Hom}(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p)$.

Since $\psi\phi_{cyc}^{-k+1}$ factors through $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)$, we have

$$S(K(\mathfrak{f}p^\infty), A(\psi))^\vee \cong X_\infty(\psi^{-1}\phi_{cyc}^{k-1}). \tag{5.1}$$

Theorem 5.8 ([23]). *X_∞ is a Λ_∞-torsion module, and has no non-trivial pseudo-null submodule.*

Proof. The first assertion is [23, II.2 Propositions 20 and 21], and the second is [23, II.2 Theorem 23]. □

Suppose that $\chi : G \to \mathbb{C}_p^\times$ is an (irreducible) character, and let $\mathcal{O}_F[\chi]$ denote

$$\mathcal{O}_F[G]/(\sigma - \chi(\sigma))_{\sigma \in G}.$$

Since we assume $\mathcal{O}_F$ is large enough to contain all the values of χ, $\mathcal{O}_F[\chi] = \mathcal{O}_F$. For a $\mathcal{O}_F[G]$-module Y, we define its χ-quotient by $Y_\chi = \mathcal{O}_F[\chi] \otimes_{\mathcal{O}_F[G]} Y$. (In particular, $\Lambda_{\infty,\chi} \cong \mathcal{O}_F[\chi][[\Gamma]] = \mathcal{O}_F[[\Gamma]]$.)

Note that if Λ is an Iwasawa algebra and X is a finitely generated Λ-torsion module, then X is pseudo-isomorphic to $\prod_i \Lambda/(f_i)$ for some $f_i \in \Lambda$, and we define $\mathrm{char}_\Lambda(X) = (\prod_i f_i)$ (as a principal ideal of Λ). Also note the obvious fact that if Y is a Λ_∞-module, then Y_χ is a $\Lambda_{\infty,\chi}$-module, as well as a Λ_∞-module.

Let $U_\infty = U(K(\mathfrak{f}p^\infty))$, $\overline{\mathcal{E}}_\infty = \overline{\mathcal{E}}(K(\mathfrak{f}p^\infty))$, and $\overline{\mathcal{C}}_\infty = \overline{\mathcal{C}}(K(\mathfrak{f}p^\infty))$ denote the projective limits of the local units congruent to 1 modulo the primes above $\mathfrak{p}$, the completion of global units, and the completion of elliptic units, respectively (see [31, Sections 1 and 4]).

Theorem 5.9 ([33], Theorem 1.1). *For each irreducible character χ of G,*

$$\mathrm{char}_{\Lambda_\infty,\chi}(\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty)_\chi = \mathrm{char}_{\Lambda_\infty,\chi} A_{\infty,\chi}.$$

Through $\mathcal{O}_F[G] \to \prod_\chi \mathcal{O}_F[\chi]$, there are maps $\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty \to \prod_\chi(\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty)_\chi$ and $A_\infty \to \prod_\chi A_{\infty,\chi}$, and since $\mathcal{O}_F[G] \to \prod_\chi \mathcal{O}_F[\chi]$ has a trivial kernel and a cokernel of finite index,

$$\operatorname{char}_{\Lambda_\infty} \overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty = p^\alpha \operatorname{char}_{\Lambda_\infty}\left(\prod_\chi (\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty)_\chi\right)$$
$$\operatorname{char}_{\Lambda_\infty} A_\infty = p^\beta \operatorname{char}_{\Lambda_\infty}\left(\prod_\chi A_{\infty,\chi}\right)$$

for some $\alpha, \beta \in \mathbb{Z}$. Combining Theorem 5.9 with the following exact sequence from class field theory,

$$0 \to \overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty \to U_\infty/\overline{\mathcal{C}}_\infty \to X_\infty \to A_\infty \to 0,$$

we have

$$\operatorname{char}_{\Lambda_\infty} X_\infty = p^\gamma \operatorname{char}_{\Lambda_\infty} U_\infty/\overline{\mathcal{C}}_\infty \tag{5.2}$$

for some $\gamma \in \mathbb{Z}$.

Definition 5.10. Recall that Γ is a subgroup of G_∞ isomorphic to $\mathbb{Z}_p^2$. Let K' be $K \subset K' \subset K(\mathfrak{f}p^\infty)$ so that $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K') \cong \Gamma$.

There is K'_∞ satisfying $K' \subset K'_\infty \subset K(\mathfrak{f}p^\infty)$ such that $\mathrm{Gal}(K'_\infty/K') \cong \mathbb{Z}_p$, and every prime above $\mathfrak{p}$ is totally ramified over K'_∞/K' and unramified over $K(\mathfrak{f}p^\infty)/K'_\infty$. Let $\Gamma' = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K'_\infty)(\cong \mathbb{Z}_p)$.

Finally, for any subgroup H of G_∞, let $I(H)$ be the ideal of Λ_∞ generated by $\{\gamma - 1 \mid \gamma \in H\}$. Define

$$\Lambda_H = \Lambda_\infty/I(H) = \mathcal{O}_F[[G_\infty/H]],$$

and for any Λ_∞-module Y, define

$$Y_H = Y/I(H)Y = Y \otimes_{\Lambda_\infty} \Lambda_H.$$

In particular, if $H = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/L)$, we let Λ_L and Y_L denote Λ_H and Y_H, respectively.

Proposition 5.11. *(a) The kernel of the natural map*

$$S(K', A(\psi)) \to S(K(\mathfrak{f}p^\infty), A(\psi))^{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K')}$$

is finite.

(b) The kernel of the natural map

$$S(K, A(\psi)) \to S(K(\mathfrak{f}p^\infty), A(\psi))^{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)}$$

is finite.

Proof. (a) Recall that $\Gamma' = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K'_\infty)$. First, we study the kernel of the natural map

$$S(K'_\infty, A(\psi)) \to S(K(\mathfrak{f}p^\infty), A(\psi))^{\Gamma'}.$$

Considering the definition of $S(L, A(\psi))$, the kernel of this map is contained in

$$\ker\left(H^1(K'_\infty, A(\psi)) \to H^1(K(\mathfrak{f}p^\infty), A(\psi))^{\Gamma'}\right),$$

which is

$$H^1(K(\mathfrak{f}p^\infty)/K'_\infty, A(\psi)^{G_{K(\mathfrak{f}p^\infty)}}) = H^1(\Gamma', A(\psi))$$

by the Hochschild-Serre spectral sequence. Choose a topological generator γ' of Γ'. By a generalization of Tate's lemma on cyclic cohomological groups (see [29, Lemma B.2.8]), we have

$$H^1(\Gamma', A(\psi)) \cong A(\psi)/(\gamma' - 1)A(\psi),$$

which is finite since Γ' acts non-trivially on $A(\psi)$.

Second, and similarly, the kernel of the natural map

$$S(K', A(\psi)) \to S(K'_\infty, A(\psi))^{\mathrm{Gal}(K'_\infty/K')}$$

is contained in $H^1(K'_\infty/K', A(\psi)^{G_{K'_\infty}})$. Since $A(\psi)^{G_{K'_\infty}}$ is finite, this group is finite. Thus we obtain (a).

(b) Since

$$\ker\left(S(K, A(\psi)) \to S(K', A(\psi))^{\mathrm{Gal}(K'/K)}\right)$$

is contained in $H^1(K'/K, A(\psi)^{G_{K'}})$, which is finite, the claim follows from (a). □

Proposition 5.12. *Suppose that $L(\overline{\psi}, k) \neq 0$. Then*

$$\mathrm{char}_{\Lambda_{K'_\infty}}\left(X_\infty(\psi^{-1}\phi_{cyc}^{k-1})_{K'_\infty}\right) = \left(\mathrm{char}_{\Lambda_\infty} X_\infty(\psi^{-1}\phi_{cyc}^{k-1})\right)_{K'_\infty}.$$

Proof. Recall that the evaluation of $f \in \Lambda_\infty$ by a character χ of G_∞ is given by $g \in G_\infty \mapsto \chi(g)$. Suppose that $\mathrm{char}_{\Lambda_\infty} X_\infty = (f)$ and $\mathrm{char}_{\Lambda_\infty} U_\infty/\overline{\mathcal{C}}_\infty = (f')$. By (5.2), $f = p^\gamma \cdot u \cdot f'$ for some $\gamma \in \mathbb{Z}$ and a unit u of Λ_∞. Note that by a generalization of the theorem of Coates and Wiles ([7], [10, Chapter 2, Theorem 4.14]), $\psi\phi_{cyc}^{-k+1}(f') \neq 0$ if and only if $L(\overline{\psi}, k) \neq 0$ (we should note that $\phi_{cyc}^k/\overline{\psi} = \psi\phi_{cyc}^{-k+1}$). Since $\psi\phi_{cyc}^{-k+1}(f') \neq 0$ if and only if $\psi\phi_{cyc}^{-k+1}(f) \neq 0$, we have that $\psi\phi_{cyc}^{-k+1}(f) \neq 0$ if and only if $L(\overline{\psi}, k) \neq 0$.

Let $\mathrm{char}_{\Lambda_\infty} X_\infty(\psi^{-1}\phi_{cyc}^{k-1}) = (g)$. If ι denotes the trivial character, then $(\iota(g)) = (\psi\phi_{cyc}^{-k+1}(f))$ as an ideal of $\mathcal{O}_F$. Thus, if $L(\overline{\psi}, k) \neq 0$, then $g \neq 0$, which also implies that $X_\infty(\psi^{-1}\phi_{cyc}^{k-1})_{K'_\infty}$ is a torsion $\Lambda_{K'_\infty}$-module. Thus, by [31, Lemma 6.2] and Theorem 5.8, we have

$$\mathrm{char}_{\Lambda_{K'_\infty}} \left(X_\infty(\psi^{-1}\phi_{cyc}^{k-1})_{K'_\infty} \right) = \left(\mathrm{char}_{\Lambda_\infty} X_\infty(\psi^{-1}\phi_{cyc}^{k-1}) \right)_{K'_\infty}.$$

□

We are now in a position to prove the following result which implies Theorem 1.5.

Theorem 5.13. *If $L(\overline{\psi}, k) \neq 0$, then $\mathrm{Sel}_p(A(\psi)/K)$ is finite.*

Proof. There is a short exact sequence

$$0 \to C' \to X_\infty(\psi^{-1}\phi_{cyc}^{k-1})_{K'_\infty} \to \prod_i \Lambda_{K'_\infty}/(f_i) \to C \to 0$$

for some $f_i \in \Lambda_{K'_\infty}$ with

$$\left(\prod_i f_i \right) = \left(\mathrm{char}_{\Lambda_\infty} X_\infty(\psi^{-1}\phi_{cyc}^{k-1}) \right)_{K'_\infty}$$

(by Proposition 5.12), and some pseudo-null $\Lambda_{K'_\infty}$-modules C, C' (in this case, it simply means that C, C' are finite modules).

If Y is a Λ_∞-module, $K \subset L \subset K(\mathfrak{f}p^\infty)$, and $H = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/L)$, then recall that Y_L denotes $Y/I(H)Y$. By a long exact sequence of cohomology groups, we can see that $X_\infty(\psi^{-1}\phi_{cyc}^{k-1})_K$ is finite if and only if

$$\prod_i \Lambda_K/(f_i)_K \cong \prod_i \mathcal{O}_F/\left(\iota(f_i)\right)$$

is finite (as above, ι is the trivial character).

It is clear that $\prod_i \mathcal{O}_F/(\iota(f_i))$ is finite if and only if $\iota(\prod f_i) \neq 0$. Also, recall that

$$\left(\prod_i f_i\right) = \left(\mathrm{char}_{\Lambda_\infty} X_\infty(\psi^{-1}\phi_{cyc}^{k-1})\right)_{K'_\infty}.$$

As we saw in the proof of Proposition 5.12 (combined with the above discussion in this proof),

$$\iota\left(\mathrm{char}_{\Lambda_\infty} X_\infty(\psi^{-1}\phi_{cyc}^{k-1})\right) = \psi\phi_{cyc}^{-k+1}\left(\mathrm{char}_{\Lambda_\infty} X_\infty\right) \neq (0)$$

if and only if $L(\overline{\psi}, k) \neq 0$. Thus, by the discussion immediately above and Proposition 5.12, $X_\infty(\psi^{-1}\phi_{cyc}^{k-1})_K$ is finite if and only if $L(\overline{\psi}, k) \neq 0$.

By (5.1),

$$S(K(\mathfrak{f}p^\infty), A(\psi))^{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)}$$

is the Pontryagin dual of $X(K(\mathfrak{f}p^\infty))(\psi^{-1}\phi_{cyc}^{k-1})_K$, and by Proposition 5.11, $S(K, A(\psi))$ is finite if

$$S(K(\mathfrak{f}p^\infty), A(\psi))^{\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)}$$

is finite. Thus, if $L(\overline{\psi}, k) \neq 0$, then $S(K, A(\psi))$ is finite. It follows from Proposition 5.5 that $\mathrm{Sel}_p(A(\psi)/K)$ is finite, which completes the proof. $\square$

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