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The crepant transformation conjecture implies the monodromy conjecture

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In this note we prove that the crepant transformation conjecture for a crepant birational transformation of Lawrence toric DM stacks studied in [10] implies the monodromy conjecture for the associated wall crossing of the symplectic resolutions of hypertoric stacks, due to Braverman, Maulik and Okounkov.

1. Introduction

Let X_1, X_2 be two birationally equivalent smooth symplectic Deligne–Mumford (DM) stacks, which are symplectic resolutions of a singular symplectic stack. Suppose by mirror symmetry they correspond to the two large radius points $0, \infty$ in the compactified Kähler moduli space $\mathcal{M}$. The derived categories of X_1 and X_2 are expected to be equivalent:

$$D^b(X_1) \cong D^b(X_2).$$

This equivalence depends on a homotopy class of a path from 0 to ∞ and thus gives a map

$$\rho : \pi_1(\mathcal{M}) \to \mathrm{Aut}(D^b(X_i)).$$

Also this map ρ induces an automorphism of $K^0(X_i)$ in the level of K-theory

$$\rho_K : \pi_1(\mathcal{M}) \to \mathrm{Aut}(K^0(X_i)),$$

where $K^0(X_i)$ are the Grothendieck K-groups of X_i. The monodromy conjecture for symplectic resolutions was formulated by Braverman–Maulik–Okounkov in [5], and it can be stated as follows: the monodromy of the quantum connection $\bigtriangledown$ for X_i is the same as the above automorphism given by the equivalence in the K-theory. In the case of Hilbert scheme of points on the plane, in [3] Bezrukavnikov and Okounkov have proved that the monodromy of the quantum differential equation is isomorphic to ρ_K in the level of K-theory.

On can study the monodromy conjecture for a large class of symplectic DM stacks. For example, the stratified Mukai flops in [13], [6], [20]; and Mukai type flops of Nakajima quiver varieties [22]. In this paper we prove the conjecture for the crepant birational transformation of hypertoric DM stacks given by varying the stability conditions in the GIT construction.

From [10], a single wall crossing of Lawrence toric DM stacks is given by varying stability conditions in the GIT construction of the Lawrence toric DM stacks. There is a one-to-one correspondence between the GIT data of Lawrence toric DM stacks and the extended Lawrence stacky fans in [15]. The wall crossing of hypertoric DM stacks is also given by varying the stability conditions in the GIT construction. Generalizing the construction in [10], we introduce the *extended stacky hyperplane arrangements* and define the hypertoric DM stacks associated with them. It turns out that the hypertoric DM stack associated with the extended stacky hyperplane arrangement is isomorphic to the hypertoric DM stack associated with the underlying stacky hyperplane arrangement, see §2.2, and there is a one-to-one correspondence between the GIT data and the extended stacky hyperplane arrangements.

Let $X_+ \dashrightarrow X_-$ be a crepant birational map between two smooth Lawrence toric DM stacks given by a single wall crossing in [10]. They are derived equivalent, and the equivalence is given by a Fourier–Mukai transformation, see [20] and [9]. In [10], the authors prove that the equivariant Fourier–Mukai transformation on the K-theory matches the analytic continuation of the I-function, hence matches the analytic continuation of the quantum connections which are determined by the I-function. Thus the genus zero crepant transformation conjecture due to Y. Ruan [24] is proved. The wall crossing $X_+ \dashrightarrow X_-$ implies that the associated birational transformation $Y_+ \dashrightarrow Y_-$ for the hypertoric DM stacks is crepant. The derived categories of Y_+ and Y_- are equivalent, and the Fourier–Mukai functor gives such an equivalence. We prove that the Fourier–Mukai transformation matches the analytic continuation of quantum connections of the hypertoric DM stacks, which is induced by the analytic continuation of the associated Lawrence toric DM stacks, see Theorem 4.2.

Crepant birational transformation of hypertoric DM stacks is the local model of Mukai flops or stratified Mukai flops for general symplectic DM stacks, see [2] for the smooth case, and [19] for the stacky case. Let $Y_+ \dashrightarrow Y_-$ be a Mukai flop for symplectic DM stacks $Y_\pm$. The authors in [23], [20] have proved that the bounded derived categories of $Y_\pm$ are equivalent, and the kernel is also given by the Fourier–Mukai transformation. The construction and calculation in this paper will play a role in the study for general Mukai type flops by degeneration method to the local models.

The paper is outlined as follows. In §2 we study the GIT data of hypertoric DM stacks and construct the wall crossing. In §3 we calculate the Fourier–Mukai transformation for the wall crossing of the hypertoric DM stacks; and in §4 we talk about the analytic continuation and prove the monodromy conjecture.

Conventions

In this paper we work entirely algebraically over the field of complex numbers. (Quantum) cohomology and K-theory groups are taken with complex coefficients. We refer to [4] Page 195 for the construction of Gale dual $\beta^\vee : \mathbb{Z}^m \to DG(\beta)$ from $\beta : \mathbb{Z}^m \to N$. We denote by $N \to \overline{N}$ the natural map modulo torsion. For a positive integer m, we use $[m]$ to represent the set $\{1, \ldots, m\}$.

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2. Wall crossing of Hypertoric Deligne–Mumford Stacks

In this section we prove that the single wall crossing of Lawrence toric DM stacks in sense of [10, §5] gives rise to a wall crossing of hypertoric DM stacks.

2.1. Lawrence toric DM stacks and the GIT construction

In this section we use extended stacky fans in [15] to define Lawrence toric DM stacks. As in [10], the I-function of the Lawrence toric DM stack depends on the extended stacky fan, and it captures the information of the twisted sectors of the toric DM stack, which can be seen from the extended stacky fan.

Definition 2.1. An *S-extended Lawrence stacky fan* is a quadruple $\mathbf{\Sigma}_L = (\mathbf{N}_L, \Sigma_L, \beta_L, S)$, where:

- $\mathbf{N}_L$ is a finitely generated abelian group (torsions allowed);
- Σ_L is a rational Lawrence simplicial fan in $\mathbf{N} \otimes \mathbb{R}$ in sense of [14, §4];
- $\beta \colon \mathbb{Z}^N \to \mathbf{N}$ is a homomorphism; we write $b_i = \beta(e_i) \in \mathbf{N}$ for the image of the ith standard basis vector $e_i \in \mathbb{Z}^N$, and write $\bar{b}_i$ for the image of b_i in $\mathbf{N} \otimes \mathbb{R}$;
- $S \subset \{1, \ldots, N\}$ is a subset, such that $N = 2n + |S|$ for a nonnegative integer n.

such that:

- each one-dimensional cone of Σ_L is spanned by $\bar{b}_i$ for a unique $i \in \{1, \ldots, N\} \setminus S$, and each $\bar{b}_i$ with $i \in \{1, \ldots, N\} \setminus S$ spans a one-dimensional cone of Σ_L;
- for $i \in S$, $\bar{b}_i$ lies in the support $|\Sigma_L|$ of the fan.

The vectors b_i for $i \in S$ are called the *extended vectors*. The *Lawrence toric DM stack* associated to an extended Lawrence stacky fan $\mathbf{\Sigma}_L$ depends only on the underlying Lawrence stacky fan and is defined as the quotient stack

$$X_{\mathbf{\Sigma}_L} := [U/K], \quad \text{with} \quad U = \mathbb{C}^{2n} \setminus \mathbb{V}(I_{\Sigma_L}) \times (\mathbb{C}^\times)^{|S|},$$

where I_{Σ_L} is the irrelevant ideal of the Lawrence fan Σ_L and K is a finitely generated abelian group, which acts on $\mathbb{C}^N$ through the data of extended Lawrence stacky fan.

We require that the extended Lawrence stacky fan $\mathbf{\Sigma}_L$ satisfies the following condition:

(C1) the map $\beta \colon \mathbb{Z}^N \to \mathbf{N}$ is surjective.

This condition can be always achieved by adding enough extended vectors.

The Lawrence toric DM stack $X_{\mathbf{\Sigma}_L}$ is semi-projective and has a torus fixed point, see [17]. We explain the GIT construction of $X_{\mathbf{\Sigma}_L}$ from the extended Lawrence stacky fan $\mathbf{\Sigma}_L = (\mathbf{N}, \Sigma_L, \beta_L, S)$ satisfying (C1). We define a free $\mathbb{Z}$-module $\mathbb{L}$ by the exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^N \xrightarrow{\beta} \mathbf{N} \longrightarrow 0 \tag{2.1}$$

and define $K := \mathbb{L} \otimes \mathbb{C}^\times$. The dual of (2.1) is an exact sequence:

$$0 \longrightarrow \mathbf{N}^\vee \longrightarrow (\mathbb{Z}^N)^\vee \xrightarrow{\beta^\vee} \mathbb{L}^\vee \tag{2.2}$$

and we define the character $D_i \in \mathbb{L}^\vee$ of K to be the image of the ith standard basis vector in $(\mathbb{Z}^N)^\vee$ under $\beta^\vee$. We have

$$(\mathbb{Z}^N)^\vee \cong (\mathbb{Z}^n \oplus \mathbb{Z}^n)^\vee \oplus \mathbb{Z}^{|S|}$$

and by reordering

$$\{D_1, \dots, D_N\} = \{D_1, \dots, D_n, -D_1, \dots, -D_n, D_{2n+1}, \dots, D_{2n+|S|}\}.$$

For $I \subset \{1, 2, \dots, N\}$, let σ_I denote the cone in $\mathbf{N} \otimes \mathbb{R}$ generated by $\bar{b}_i$ for $i \in I$. Let $\overline{I}$ be the complement of $I \subset \{1, 2, \cdot, N\}$. Set

$$\mathcal{A}_\theta := \left\{ I \subset \{1, 2, \dots, N\} \mid S \subset I,\ \sigma_{\overline{I}} \text{ is a cone of } \Sigma_L \right\}.$$

to be the collection of *anticones*. The *stability condition* $\theta \in \mathbb{L}^\vee \otimes \mathbb{R}$ is taken to lie in $\bigcap_{I \in \mathcal{A}} \angle_I$, where

$$\angle_I = \left\{ \textstyle\sum_{i \in I} a_i D_i \mid a_i \in \mathbb{R},\ a_i > 0 \right\} \subset \mathbb{L}^\vee \otimes \mathbb{R}.$$

The property of Lawrence toric fan Σ_L ensures that this intersection is non-empty. We understand $\angle_\emptyset = \{0\}$. Let

$$U_\theta = \bigcup_{I \in \mathcal{A}_\theta} (\mathbb{C}^\times)^I \times \mathbb{C}^{\overline{I}},$$

where $(\mathbb{C}^\times)^I \times \mathbb{C}^{\overline{I}} = \{(z_1, \dots, z_N) \in \mathbb{C}^N \mid z_i \neq 0 \text{ for } i \in I\}$. From the property of Lawrence toric fan, the stability condition θ satisfies the following consitions:

(A1) $\{1, 2, \dots, N\} \in \mathcal{A}_\theta$;

(A2) for each $I \in \mathcal{A}_\theta$, the set $\{D_i : i \in I\}$ spans $\mathbb{L}^\vee \otimes \mathbb{R}$ over $\mathbb{R}$.

(A1) ensures that X_θ is non-empty; (A2) ensures that X_θ is a DM stack. Under these assumptions, $\mathcal{A}_\theta$ is closed under enlargement of sets; i.e., if $I \in \mathcal{A}_\theta$ and $I \subset J$ then $J \in \mathcal{A}_\theta$. The Lawrence toric DM stack is the quotient stack $X_{\mathbf{\Sigma}_L} = X_\theta = [U_\theta / K]$. The *GIT data* for $X_{\mathbf{\Sigma}_L}$ consists of

- $K \cong (\mathbb{C}^\times)^r$, a connected torus of rank r;

- $\mathbb{L} = \mathrm{Hom}(\mathbb{C}^\times, K)$, the cocharacter lattice of K;
- $D_1, \ldots, D_n, -D_1, \ldots, -D_n, D_{2n+1}, \ldots, D_{2n+|S|} \in \mathbb{L}^\vee = \mathrm{Hom}(K, \mathbb{C}^\times)$, the characters of K;
- stability condition $\theta \in \mathbb{L}^\vee \otimes \mathbb{R}$.

Conversely, to obtain an extended Lawrence stacky fan from GIT data, consider the exact sequence (2.1). Let $b_i = \beta(e_i) \in \mathbf{N}$ and $\overline{b}_i \in \mathbf{N} \otimes \mathbb{R}$ be as above. The extended Lawrence stacky fan $\mathbf{\Sigma}_\theta = (\mathbf{N}, \Sigma_\theta, \beta_L, S)$ corresponding to our data consists of the group $\mathbf{N}$ and the map β_L defined above, together with a fan Σ_θ in $\mathbf{N} \otimes \mathbb{R}$ and S given by

$$\Sigma_\theta = \{\sigma_I : \overline{I} \in \mathcal{A}_\theta\}, \qquad S = \{i \in \{1, \ldots, N\} : \overline{\{i\}} \notin \mathcal{A}_\theta\}.$$

The quotient construction in [15, §2] coincides with the GIT quotient construction, and therefore X_θ is the Lawrence toric DM stack corresponding to $\mathbf{\Sigma}_\theta$.

2.2. Hypertoric DM stacks and the GIT data

We give the GIT construction of hypertoric DM stacks. The GIT data of the hypertoric DM stack is useful for the construction of the wall crossing. We introduce the notion of *extended stacky hyperplane arrangements* and define the corresponding hypertoric DM stacks. We prove that there is a one-to-one correspondence between the GIT data of the hypertoric DM stacks and the extended stacky hyperplane arrangements, generalizing the idea in §2.1.

Definition 2.2. An *extended stacky hyperplane arrangement* $\mathfrak{A} = (\mathbf{N}, \beta, \theta, S)$ consists of the following data:

- $\mathbf{N}$ is a finitely generated abelian group;
- $\beta : \mathbb{Z}^m \to \mathbf{N}$ is a map. Also we write $b_i = \beta(e_i)$, the image of the standard generator e_i of $\mathbb{Z}^m$;
- $S \subset [m]$ and let $n := m - |S|$. Let $\beta_{\mathrm{red}} : \mathbb{Z}^n \to \mathbf{N}$ determined by $\{b_i | i \in [m] \setminus S\}$ is a map and

  $$\beta^\vee_{\mathrm{red}} : (\mathbb{Z}^n)^\vee \to \mathbb{L}^\vee_{\mathrm{red}}$$

  be the Gale dual of β_{red}. The element $\theta \in \mathbb{L}^\vee_{\mathrm{red}}$ is a generic element in sense of [16, §2].

The data above satisfy the following conditions:

- $\mathfrak{A}_{\mathrm{red}} = (\mathbf{N}, \beta_{\mathrm{red}}, \theta)$ is a stacky hyperplane arrangement in sense of [16, Definition 2.1].

Remark 2.3. $\{b_i | i \in S\}$ are called extended vectors.

We prove that an extended stacky hyperplane arrangement $\mathfrak{A}$ determines an extended Lawrence stacky fan, hence a Lawrence toric DM stack X_θ. The hypertoric DM stack $Y_\theta := Y_{\mathfrak{A}}$ is a closed substack of X_θ. First we write down some diagrams of exact sequences from the extended stacky hyperplane arrangements:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_{\mathrm{red}}} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{N} & \xrightarrow{\cong} & \mathbf{N} & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Taking Gale dual yields:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & (\mathbb{Z}^n)^\vee & \longleftarrow & (\mathbb{Z}^m)^\vee & \longleftarrow & (\mathbb{Z}^{m-n})^\vee & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle \beta^\vee_{\mathrm{red}}} & & \downarrow{\scriptstyle \beta^\vee = (\beta^\vee_{\mathrm{red}}, \beta^\vee_S)} & & \downarrow{\scriptstyle =} & & \\ 0 & \longleftarrow & \mathbb{L}^\vee_{\mathrm{red}} & \longleftarrow & \mathbb{L}^\vee & \longleftarrow & \mathbb{Z}^{m-n} & \longleftarrow & 0. \end{array}$$

Considering the following diagram:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & (\mathbb{Z}^n)^\vee \oplus (\mathbb{Z}^n)^\vee & \longleftarrow & (\mathbb{Z}^n)^\vee \oplus (\mathbb{Z}^n)^\vee \oplus (\mathbb{Z}^{|S|})^\vee & \longleftarrow & (\mathbb{Z}^{m-n})^\vee & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle (\beta^\vee_{\mathrm{red}}, -\beta^\vee_{\mathrm{red}})} & & \downarrow{\scriptstyle (\beta^\vee_{\mathrm{red}}, -\beta^\vee_{\mathrm{red}}, \beta^\vee_S)} & & \downarrow{\scriptstyle =} & & \\ 0 & \longleftarrow & \mathbb{L}^\vee_{\mathrm{red}} & \longleftarrow & \mathbb{L}^\vee & \longleftarrow & \mathbb{Z}^{m-n} & \longleftarrow & 0, \end{array}$$

and taking Gale dual again:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{2n} & \longrightarrow & \mathbb{Z}^{2n+|S|} & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_{L,\mathrm{red}}} & & \downarrow{\scriptstyle \beta_L} & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{N}_L & \xrightarrow{\cong} & \mathbf{N}_L & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

For the map $\beta_{L,\mathrm{red}} : \mathbb{Z}^{2n} \to \mathbf{N}_L$, and the generic element $\theta \in \mathbb{L}^\vee_{\mathrm{red}}$, there is a Lawrence simplicial fan Σ_θ constructed in [16, §2]. Hence we have an extended Lawrence stacky fan

$$\mathbf{\Sigma_L} = (\mathbf{N}_L, \Sigma_\theta, \beta_L, S)$$

in Definition 2.1. The Lawrence toric DM stack

$$X_\theta := X_{\mathbf{\Sigma_L}} = [U_\theta/K]$$

is defined in §2.1. Let $\mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n, u_1, \ldots, u_{|S|}]$ be the coordinate ring of $\mathbb{C}^N = \mathbb{C}^{2n+|S|}$. Let $I_{\beta^\vee}$ be the ideal

$$I_{\beta^\vee} := \left\langle \sum_{i=1}^{n} (\beta^\vee_{\mathrm{red}})^*(x)_i z_i w_i |\ x \in \mathbb{L}_{\mathrm{red}} \right\rangle, \tag{2.3}$$

where $(\beta^\vee_{\mathrm{red}})^*$ is the map $\mathbb{L}_{\mathrm{red}} \to \mathbb{Z}^n$ and $(\beta^\vee_{\mathrm{red}})^*(x)_i$ is the i-th component of the vector $(\beta^\vee_{\mathrm{red}})^*(x)$. Let $V_\theta \subset U_\theta$ be the closed subvariety determined by the ideal in (2.3). The hypertoric DM stack

$$Y_\theta = [V_\theta/K]$$

is a quotient stack.

Proposition 2.4. *The hypertoric DM stack $Y_\theta = Y_{\mathfrak{A}}$ is isomorphic to the hypertoric DM stack $Y_{\mathfrak{A}_{red}}$ associated to its underlying stacky hyperplane arrangement $\mathfrak{A}_{red}$ defined in [16, §2].*

Proof. Any extended stacky hyperplane arrangement $\mathfrak{A}$ has an underlying stacky hyperplane arrangement $\mathfrak{A}_{\mathrm{red}}$ by forgetting the extra data S. The hypertoric DM stack is defined by a stacky hyperplane arrangement in [16, §2]. Since for extended stacky fans, the associated toric DM stack is isomorphic to the toric DM stack associated to its underlying stacky fan, see [15], the hypertoric DM stack $Y_\theta = Y_{\mathfrak{A}}$ associated an extended hyperplane arrangement is also isomorphic to the hypertoric DM stack $Y_{\mathfrak{A}_{\mathrm{red}}}$ associated to its underlying stacky hyperplane arrangement. The difference is that there are more extra power of $\mathbb{C}^\times$'s modulo by the same rank of power of $\mathbb{C}^\times$'s. We omit the details. □

Hence the GIT data for hypertoric DM stack Y_θ consists of the following:

- $K \cong (\mathbb{C}^\times)^r$, a connected torus of rank r;
- $\mathbb{L} = \mathrm{Hom}(\mathbb{C}^\times, K)$, the cocharacter lattice of K;

- $D_1, \dots, D_n, -D_1, \dots, -D_n, D_{2n+1}, \dots, D_{2n+|S|} \in \mathbb{L}^\vee = \mathrm{Hom}(K, \mathbb{C}^\times)$, characters of K;
- stability condition $\theta \in \mathbb{L}^\vee \otimes \mathbb{R}$.

Remark 2.5. From a similar argument as in §2.1, given the GIT data of the hypertoric DM stack Y_θ, we can construct an extended stacky hyperplane arrangement $\mathfrak{A} = (\mathbf{N}, \beta, \theta, S)$, and vice versa.

2.3. The wall crossing of hypertoric DM stacks

We prove that the single wall crossing of Lawrence toric DM stacks gives rise to the wall crossing of hypertoric DM stacks.

Recall that the space $\mathbb{L}^\vee \otimes \mathbb{R}$ of stability conditions is divided into chambers by the closures of the sets $\angle_I$, $|I| = r-1$, and the Lawrence toric DM stack X_θ depends on θ only via the chamber containing θ. For any stability condition θ, the set U_θ contains the big torus $T := (\mathbb{C}^\times)^N$. Thus for any two such stability conditions θ_1, θ_2 there is a canonical birational map $X_{\theta_1} \dashrightarrow X_{\theta_2}$, induced by the identity transformation between $T/K \subset X_{\theta_1}$ and $T/K \subset X_{\theta_2}$.

Let C_+, C_- be chambers in $\mathbb{L}^\vee \otimes \mathbb{R}$ that are separated by a hyperplane wall W, so that $W \cap \overline{C_+}$ is a facet of $\overline{C_+}$, $W \cap \overline{C_-}$ a facet of $\overline{C_-}$, and $W \cap \overline{C_+} = W \cap \overline{C_-}$. Choose stability conditions $\theta_+ \in C_+$, $\theta_- \in C_-$ satisfying (A1–A2) and set $U_+ := U_{\theta_+}$, $U_- := U_{\theta_-}$, $X_+ := X_{\theta_+}$, $X_- := X_{\theta_-}$, and

$$\mathcal{A}_\pm := \mathcal{A}_{\theta_\pm} = \big\{ I \subset \{1, 2, \dots, N\} : \theta_\pm \in \angle_I \big\}.$$

Then $C_\pm = \bigcap_{I \in \mathcal{A}_\pm} \angle_I$. Let $\varphi \colon X_+ \dashrightarrow X_-$ be the birational transformation induced by the toric wall-crossing from C_+ to C_-. Let $e \in \mathbb{L}$ denote the *primitive lattice vector* in $W^\perp$ such that e is positive on C_+ and negative on C_-. We fix the notations

- $M_+ := \{ i \in \{1, \dots, N\} \mid D_i \cdot e > 0 \}$,
- $M_- := \{ i \in \{1, \dots, N\} \mid D_i \cdot e < 0 \}$,
- $M_0 := \{ i \in \{1, \dots, N\} \mid D_i \cdot e = 0 \}$.

From our construction of Lawrence toric DM stacks, $|M_+| = |M_-|$.

Choose θ_0 from the relative interior of $W \cap \overline{C_+} = W \cap \overline{C_-}$. The stability condition θ_0 does not satisfy (A1–A2) on the GIT data, but consider

$$\mathcal{A}_0 := \mathcal{A}_{\theta_0} = \{ I \subset \{1, \dots, N\} : \theta_0 \in \angle_I \}$$

and the corresponding toric Artin stack $X_0 := X_{\theta_0} = [U_{\theta_0}/K]$. Here X_0 is not Deligne—Mumford, as the $\mathbb{C}^\times$-subgroup of K corresponding to $e \in \mathbb{L}$ (the defining equation of the wall W) has a fixed point in $U_0 := U_{\theta_0}$. The stack X_0 contains both X_+ and X_- as open substacks. Let $\overline{X}_0$ be the coarse moduli space of the stack X_0.

The canonical line bundles of X_+ and X_- are given by the character $-\sum_{i=1}^{2n} D_i = 0$ of K. This means that Lawrence toric DM stacks are Calabi–Yau. There are canonical blow-down maps $g_\pm \colon X_\pm \to \overline{X}_0$, and $K_{X_\pm} = g_\pm^\star \mathcal{O}_{\overline{X}_0}$. We have a commutative diagram:

(2.4)
$$\begin{array}{ccccc} & & \widetilde{X} & & \\ & \swarrow^{f_+} & & \searrow^{f_-} & \\ X_+ & & \overset{\varphi}{\dashrightarrow} & & X_- \\ & \searrow_{g_+} & & \swarrow_{g_-} & \\ & & \overline{X}_0 & & \end{array}$$

So the birational map φ is *crepant*, since $f_+^\star(K_{X_+}) = f_-^\star(K_{X_-})$ are trivial.

To construct $\widetilde{X}$, consider the action of $K \times \mathbb{C}^\times$ on $\mathbb{C}^{N+1}$ defined by the characters $\widetilde{D}_1, \dots, \widetilde{D}_{N+1}$ of $K \times \mathbb{C}^\times$, where:

$$\widetilde{D}_j = \begin{cases} D_j \oplus 0 & \text{if } j < N+1 \text{ and } D_j \cdot e \leq 0 \\ D_j \oplus (-D_j \cdot e) & \text{if } j < N+1 \text{ and } D_j \cdot e > 0 \\ 0 \oplus 1 & \text{if } j = N+1 \end{cases}$$

Consider the chambers $\widetilde{C}_+$, $\widetilde{C}_-$, and $\widetilde{C}$ in $(\mathbb{L} \oplus \mathbb{Z})^\vee \otimes \mathbb{R}$ that contain, respectively, the stability conditions

$$\widetilde{\theta}_+ = (\theta_+, 1) \qquad \widetilde{\theta}_- = (\theta_-, 1) \qquad \text{and} \qquad \widetilde{\theta} = (\theta_0, -\varepsilon)$$

where ε is a very small positive real number. Let $\widetilde{X}$ denote the toric DM stack defined by the stability condition $\widetilde{\theta}$. We have, by [10, Lemma 6.16], that the toric DM stack corresponding to the chamber $\widetilde{C}_\pm$ is $X_\pm$. Furthermore, there is a commutative diagram as in (2.4), where: $f_\pm \colon \widetilde{X} \to X_\pm$ are toric blow-ups, arising from the wall-crossing from $\widetilde{C}$ to $\widetilde{C}_\pm$.

Now we take into account the ideal (2.3). We have the corresponding hypertoric DM stacks

$$Y_\pm := Y_{\theta_\pm} \subset X_\pm$$

and the hypertoric stack $Y_0 \subset X_0$ determined by the ideal $I_{\beta^\vee}$ in (2.3). Let $\overline{Y}_0$ be the coarse moduli space of Y_0 and let $\widetilde{Y} \subset \widetilde{X}$ be the substack determined by the ideal $I_{\beta^\vee}$ in (2.3).

Proposition 2.6. *We have the following diagram:*

(2.5)
$$
\begin{array}{ccccc}
 & & \widetilde{Y} & & \\
 & \swarrow^{F_+} & & \searrow^{F_-} & \\
Y_+ & & \overset{\phi}{\dashrightarrow} & & Y_- \\
 & \searrow_{G_+} & & \swarrow_{G_-} & \\
 & & \overline{Y}_0 & &
\end{array}
$$

which gives the crepant transformation morphism of hypertoric DM stacks.

Proof. The contractions G_+, G_- are constructed in [18, §4]. The maps $F_\pm$ are induced from the maps $f_\pm$ in (2.4). The birational map ϕ is crepant since $Y_\pm$ are Calabi–Yau stacks. □

Example 2.7. We consider the following GIT data:

- $K \cong \mathbb{C}^\times$;
- $D_1 = (1), D_2 = (2), D_3(= (-1), D_4 = (-2) \in \mathbb{L}^\vee = \mathrm{Hom}(K, \mathbb{C}^\times) = \mathbb{Z}$;
- stability conditions $\theta_+ = (1), \theta_- = (-1) \in \mathbb{L}^\vee \otimes \mathbb{R}$.

We can easily construct the Lawrence toric DM stacks

$$X_\pm = \mathcal{O}_{\mathbb{P}(1,2)}(-1) \oplus \mathcal{O}_{\mathbb{P}(1,2)}(-2).$$

The crepant transformation $\varphi : X_+ \dashrightarrow X_-$ is an Atiyah type flop.

We have the stacky hyperplane arrangements $\mathfrak{A}_\pm = (\mathbf{N}, \beta_\pm, \theta_\pm)$, where $\beta_\pm : \mathbb{Z}^2 \to \mathbf{N}$ is the Gale dual of $\beta_\pm^\vee : \mathbb{Z}^2 \to \mathbb{L}^\vee$ determined by $\{D_1, D_2, D_3, D_4\}$. The corresponding hypertoric DM stacks

$$Y_\pm = T^*\mathbb{P}(1,2).$$

The crepant birational map $\phi : Y_+ \dashrightarrow Y_-$ is a Mukai type flop.

3. The Fourier–Mukai transformation

3.1. The $\mathbb{T} := T \times \mathbb{C}^\times$-action on $X_\pm$ and $Y_\pm$

From the construction of Lawrence toric DM stacks $X_\pm$ and hypertoric DM stacks $Y_\pm$ in §2, there is a torus $\mathbb{T} := T \times \mathbb{C}^\times$ action, where $T = (\mathbb{C}^\times)^m$. Since $U_\pm \subset \mathbb{C}^{2n} \times \mathbb{C}^{|S|}$, the action T acts on $U_\pm$ by the standard action which is given by

$$\begin{aligned}&(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_m)(z_1, \ldots, z_n, w_1, \ldots, w_n, u_1, \ldots, u_{|S|})\\&= (\lambda_1 z_1, \ldots, \lambda_n z_n, \lambda_1^{-1} w_1, \ldots, \lambda_n^{-1} w_n, \lambda_{n+1} u_1, \ldots, \lambda_m u_{|S|});\end{aligned}$$

the extra $\mathbb{C}^\times$ acts by scaling the fibre of $T^*\mathbb{C}^n$. We consider the $\mathbb{T}$-action on $\widetilde{X}$ induced from the inclusion $\mathbb{T} = (\mathbb{C}^\times)^{m+1} \times \{1\} \subset \mathbb{T} \times \mathbb{C}^\times$ and the $\mathbb{T} \times \mathbb{C}^\times$ action on $\widetilde{X}$.

The $\mathbb{T}$-fixed points on $X_\pm$ and $Y_\pm$ are isolated, and in one-to-one correspondence with the minimal anticones $\delta_\pm \in \mathcal{A}_\pm$. Using the correspondence between anticones and cones in the Lawrence toric fan $\Sigma_\pm$, the torus fixed points are given by top dimensional cones in the fan. The torus fixed points all lie in the core of $X_\pm$ and $Y_\pm$.

3.2. The $\mathbb{T}$-equivariant K-theory of $X_\pm$, $Y_\pm$ and the Fourier–Mukai transformation

3.2.1. Equivariant K-theory. Let $K^0_{\mathbb{T}}(X_\pm)$, $K^0_{\mathbb{T}}(Y_\pm)$ be the $\mathbb{T}$-equivariant Grothendieck K-groups of coherent sheaves. They are modules over $K^0_{\mathbb{T}}(pt)$, the ring $\mathbb{Z}[\mathbb{T}]$ of regular functions on the torus $\mathbb{T}$.

For the Lawrence toric DM stack X_θ for a stability condition $\theta \in \mathbb{L}^\vee \otimes \mathbb{R}$, the $\mathbb{T}$-equivariant divisor $\{z_i = 0\}$, $\{w_i = 0\}$ or $\{u_j\} = 0$ for $1 \leq i \leq n$, $1 \leq j \leq |S|$ on X_θ determine $\mathbb{T}$-equivariant line bundles R_i over X_θ for $1 \leq i \leq N$. We denote by R_i the equivariant classes of such line bundles. Similar construction works for the toric DM stack $\widetilde{X}$. We fix notation for $\mathbb{T}$-equivariant line bundles for $X_\pm$ and $\widetilde{X}$.

$$R_1^\pm, \ldots, R_N^\pm \in K^0_{\mathbb{T}}(X_\pm),$$

and

$$\widetilde{R}_1, \ldots, \widetilde{R}_N, \widetilde{R}_{N+1} \in K^0_{\mathbb{T}}(\widetilde{X}).$$

Fix notations of their inverses:

$$S_j^+ := (R_j^+)^{-1}, \qquad S_j^- := (R_j^-)^{-1}, \qquad \widetilde{S}_j = \widetilde{R}_j^{-1}.$$

Let $\hbar$ be the trivial line bundle on which the extra $\mathbb{C}^\times$ factor acts by its basic representation.

Let X_θ be a Lawrence toric DM stack corresponding to a stability condition θ. Each character $p \in \mathrm{Hom}(K, \mathbb{C}^\times) = \mathbb{L}^\vee$ defines a line bundle $L(p) \to X_\theta$:

$$L(p) = U_\theta \times \mathbb{C}/(z,v) \simeq (g \cdot z, p(g) \cdot v), g \in K.$$

The line bundle $L(p)$ is equipped with the $\mathbb{T}$-action $[z,v] \mapsto [t \cdot z, v], t \in \mathbb{T}$ as in [10, §6.3.2]. So it defines an element in $K^0_{\mathbb{T}}(X_\theta)$. The line bundles $R_i^\pm$ are:

$$R_i^\pm = L_\pm(D_i) \otimes e^{\lambda_i}, 1 \le i \le n,$$
$$R_i^\pm = L_\pm(D_i) \otimes e^{\lambda-\lambda_i}, n+1 \le i \le 2n,$$

and

$$R_i^\pm = L_\pm(D_i) \otimes e^{\lambda_i}, 2n+1 \le i \le N.$$

where $e^{\lambda_i} \in \mathbb{C}[\mathbb{T}]$ stands for the irreducible $\mathbb{T}$-representation given by the i-th projection $\mathbb{T} \to \mathbb{C}^\times$. Here $\lambda_i, \lambda \in R_{\mathbb{T}} = H^\bullet_{\mathbb{T}}(pt, \mathbb{C}) = \mathbb{C}[\lambda_1, \dots, \lambda_m, \lambda]$ are the equivariant first Chern class of the irreducible $\mathbb{T}$-representation given by the i-th projection $\mathbb{T} \to \mathbb{C}^\times$. Since $\mathbb{T} = (\mathbb{C}^\times)^m \times \mathbb{C}^\times = (\mathbb{C}^\times)^{m+1}$, we use λ to represent the $m+1$-equivariant parameter λ_{m+1}. See [10, §6.3.2] for more details.

For $(p,n) \in \mathbb{L}^\vee \oplus \mathbb{Z}$, the $\mathbb{T}$-equivariant line bundle $L(p,n) \to \widetilde{X}$ is similarly defined, and we have:

$$\widetilde{R}_i = L(\widetilde{D}_i) \otimes e^{\lambda_i}, 1 \le i \le N$$

as above, and

$$\widetilde{R}_{m+1} = L(\widetilde{D}_{N+1}) = L(0,1).$$

As in [10, §6.3.2], the classes $L_\pm(p), (p \in \mathbb{L}^\vee)$ generate the equivariant K-group $K^0_{\mathbb{T}}(X_\pm)$, and the classes $\{L(p,n) \mid (p,n) \in \mathbb{L}^\vee \oplus \mathbb{Z}\}$ generate the equivariant K-group $K^0_{\mathbb{T}}(\widetilde{X})$.

3.2.2. Localized K-theory basis. Let $\delta_- \in \mathcal{A}_-$ be a minimal anticone, x_{δ_-} be the $\mathbb{T}$-fixed point on X_-, and

$$i_{\delta_-} : x_{\delta_-} \to X_-$$

be the inclusion. Denote by G_{δ_-} the isotropy group of x_{δ_-}. Then $x_{\delta_-} = BG_{\delta_-}$. From [10, §6.3.2],

$$i^\star_{\delta_-} R_i = 1, \text{ for } i \in \delta_-.$$

A localized basis of $K^0_{\mathbb{T}}(X_-)$, after inverting the non-zero elements of $\mathbb{Z}[\mathbb{T}]$, is given by:

$$\{(i_{\delta_-})_\star \varrho : \varrho \text{ an irreducible representation of } G_{\delta_-}, \delta_- \in \mathcal{A}_- \text{ a minimal anticone}\}. \tag{3.1}$$

By Koszul resolution the structure sheaf $\mathcal{O}_{x_{\delta_-}}$ is given by

$$e_{\delta_-} := \prod_{i \notin \delta_-} (1 - S_i^-).$$

We specify the $\mathbb{T}$-linearization on $(i_{\delta_-})_\star \varrho$. Choosing a lift $\hat{\varrho} \in \mathrm{Hom}(K, \mathbb{C}^\times) = \mathbb{L}^\vee$ for each G_{δ_-}-representation $\varrho : G_{\delta_-} \to \mathbb{C}^\times$, then

$$e_{\delta_-,\varrho} := L_-(\hat{\varrho}) \cdot \prod_{i \notin \delta_-} (1 - S_i^-).$$

Similarly, $\{e_{\delta_+,\varrho}\}$ is a basis for the localized $\mathbb{T}$-equivariant K-theory $K^0_{\mathbb{T}}(X_+)$.

3.2.3. Fourier–Mukai transformation. In [10, §6.3.3], the authors calculate the Fourier–Mukai transformation for the localized $\mathbb{T}$-equivariant K-theory $K^0_{\pm}(X_\pm)$.

Proposition 3.1 ([10], Theorem 6.19).

1) If $\delta_- \in \mathcal{A}_-$ is a minimal anticone such that $\delta_- \in \mathcal{A}_+$, then

$$\mathbb{FM}(e_{\delta_-,\rho}) = e_{\delta_+,\rho},$$

where $\delta_+ = \delta_-$ is the same anticone, but taken as in $\mathcal{A}_+$;

2) If $\delta_- \in \mathcal{A}_-$, but $\delta_- \notin \mathcal{A}_+$, then $\mathbb{FM}(e_{\delta_-,\varrho})$ is equal to

$$\frac{1}{l}\sum_{t\in\mathcal{T}}\left(\frac{1-S^+_{j_-}}{1-t^{-1}}\cdot L_+(\hat{\varrho})t^{\hat{\varrho}\cdot e}\cdot\prod_{\substack{j\notin\delta_-\\ D_j\cdot e<0}}(1-S^+_j)\cdot\prod_{\substack{i\notin\delta_-\\ D_i\cdot e\geq 0}}\left(1-t^{-D_i\cdot e}S^+_i\right)\right)$$

where j_- is the unique element of δ_- such that $D_{j_-}\cdot e<0$, $l=-D_{j_-}\cdot e$ and

$$\mathcal{T}=\left\{\zeta\cdot(R^+_{j_-})^{1/l}:\zeta\in\mu_l\right\}.$$

3.3. The $\mathbb{T}$-equivariant K-theory $K^0_{\mathbb{T}}(Y_\pm)$ of hypertoric DM stacks

Let

$$\iota_\pm : Y_\pm \hookrightarrow X_\pm$$

be the inclusion of hypertoric DM stacks $Y_\pm$ to their associated Lawrence toric DM stacks $X_\pm$.

Lemma 3.2. *The pullback*

$$\iota^\star_\pm : K^0_{\mathbb{T}}(X_\pm)\overset{\cong}{\to}K^0_{\mathbb{T}}(Y_\pm)$$

is an isomorphism on the $\mathbb{T}$-equivariant K-theory.

Proof. This is from [12, Theorem 5.4]. On the other hand, one can directly calculate that the equivariant K-theory groups $K^0_{\mathbb{T}}(Y_\pm)$ are also generated by line bundles $R^\pm_1,\dots,R^\pm_N$ modulo the same relations as in the Lawrence toric DM stack case. □

Remark 3.3. For the Lawrence toric DM stacks $X_\pm$,

$$K^0_{\mathbb{T}}(X_\pm)\cong\frac{\mathbb{C}[(R^\pm_1)^{\pm1},\dots,(R^\pm_N)^{\pm1},\hbar^{\pm1}]}{I+J},$$

where

$$I=\{(1-R^\pm_{i_1})\cdots(1-R^\pm_{i_k})\mid\overline{\{i_1,\dots,i_k\}}\notin\mathcal{A}_\pm\}$$

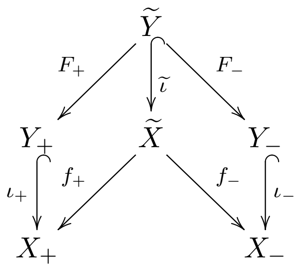
is the ideal generated by the products for subsets $\{i_1,\dots,i_k\}$ not lying in the set of anticones; and

$$J=\{R^\pm_i-(\hbar\cdot(R^\pm_{n+i})^{-1})\mid 1\leq i\leq n\}$$

is the ideal generated by the relation of the line bundles $R^\pm_i$ and $R^\pm_{n+i}$.

To study the Fourier–Mukai transformation of the crepant birational map $\phi : Y_+ \dashrightarrow Y_-$, we set up the following diagram:

(3.2)



We denote by $\Phi := \mathbb{FM}$ the equivariant Fourier–Mukai transformation for $X_\pm$. The Fourier–Mukai transformation

$$\Psi : K^0_{\mathbb{T}}(Y_-) \to K^0_{\mathbb{T}}(Y_+)$$

is given by:

$$E \mapsto \Psi(F) = (F_+)_\star F_-^\star(E).$$

Proposition 3.4. *There is a commutative diagram of $\mathbb{T}$-equivariant K-theory groups:*

$$\begin{array}{ccc} K^0_{\mathbb{T}}(X_-) & \xrightarrow{\Phi} & K^0_{\mathbb{T}}(X_+) \\ \downarrow{\scriptstyle \iota_-^\star} & & \downarrow{\scriptstyle \iota_+^\star} \\ K^0_{\mathbb{T}}(Y_-) & \xrightarrow{\Psi} & K^0_{\mathbb{T}}(Y_+) \end{array}$$

which implies that Ψ is an isomorphism on K-theory groups.

Proof. By Proposition 3.2, the pullbacks $\iota_\pm^\star$ are isomorphisms. Then we directly check the commutative diagram using (3.2): for any element $E \in K^0_{\mathbb{T}}(X_-)$,

$$\begin{aligned} \iota_+^\star \circ \Phi(E) &= \iota_+^\star \circ ((f_+)_\star f_-^\star(E)) \\ &= (F_+)_\star \circ \widetilde{\iota}^\star \circ f_-^\star(E) \\ &= (F_+)_\star \circ F_-^\star \circ \iota_-^\star(E) \\ &= \Psi \circ \iota_-^\star(E). \end{aligned}$$

Here we use the equality

$$(F_+)_\star \circ \widetilde{\iota}^\star = \iota_+^\star \circ (f_+)_\star,$$

which is from the Tor-independent result of $\widetilde{X}, Y_+$ over X_+, since $Y_+ \subset X_+, \widetilde{Y} \subset \widetilde{X}$ are defined by the same ideal (2.3) and the pullback along f_+ of the Koszul resolution of $\mathcal{O}_{Y_+}$ in X_+ gives the Koszul resolution of $\mathcal{O}_{\widetilde{Y}}$ in $\widetilde{X}$. □

The torus $\mathbb{T}$-fixed points of $Y_\pm$ are the same as the torus $\mathbb{T}$-fixed points of $X_\pm$, which all lie in the core. The localized K-theory basis of $K^0_{\mathbb{T}}(Y_\pm)$ are also generated by the minimal anticones $\delta_\pm \in \mathcal{A}_\pm$. Let $\delta_- \in \mathcal{A}_-$ be a minimal anticone. Then

$$i_{\delta_-} : x_{\delta_-} \hookrightarrow Y_- \subset X_-$$

is the inclusion of the fixed point x_{δ_-}. For each $p \in \mathbb{L}^\vee$, it also defines a line bundle

$$L_-^{Y_-}(p) = V_- \times \mathbb{C}/K,$$

which is the pullback $\iota_-^\star L_-(p)$ of $L_-(p)$ on X_-. From now on we denote by $L_-(p)$ the line bundle on Y_- determined by $p \in \mathbb{L}^\vee$.

Set

$$e^{Y_-}_{\delta_-,\varrho} := L_-(\hat{\varrho}) \cdot \prod_{i \notin \delta_-} (1 - S_i^-),$$

where $\hat{\varrho} \in \mathbb{L}^\vee$ is the lift of ϱ. Then $\{e^{Y_-}_{\delta_-,\varrho}\}$ is a basis for the localized $\mathbb{T}$-equivariant K-theory of Y_-. Similarly we have a localized $\mathbb{T}$-equivariant K-theory basis $\{e^{Y_+}_{\delta_+,\varrho}\}$ of Y_+.

Proposition 3.5 (Fourier–Mukai transformation for hypertoric DM stacks).

1) If $\delta_- \in \mathcal{A}_-$ is a minimal anticone such that $\delta_- \in \mathcal{A}_+$, then

$$\Psi(e^{Y_-}_{\delta_-,\rho}) = e^{Y_+}_{\delta_+,\rho},$$

where $\delta_+ = \delta_-$ is the same anticone, but taken as in $\mathcal{A}_+$;

2) If $\delta_- \in \mathcal{A}_-$, but $\delta_- \notin \mathcal{A}_+$, then $\Psi(e^{Y_-}_{\delta_-,\varrho})$ is equal to

$$\frac{1}{l}\sum_{t \in \mathcal{T}} \left(\frac{1 - S^+_{j_-}}{1 - t^{-1}} \cdot L_+(\hat{\varrho}) t^{\hat{\varrho}\cdot e} \cdot \prod_{\substack{j \notin \delta_- \\ D_j \cdot e < 0}} (1 - S_j^+) \cdot \prod_{\substack{i \notin \delta_- \\ D_i \cdot e \geq 0}} \left(1 - t^{-D_i \cdot e} S_i^+\right) \right)$$

where j_- is the unique element of δ_- such that $D_{j_-} \cdot e < 0$, $l = -D_{j_-} \cdot e$ and

$$\mathcal{T} = \left\{ \zeta \cdot (R^+_{j_-})^{1/l} : \zeta \in \mu_l \right\}.$$

Proof. This result is from Proposition 3.4, and the Fourier–Mukai transformation formula Φ in Proposition 3.1. □

4. Analytic continuation of the quantum connection

In this section we prove that the Fourier–Mukai transformation $\Psi : K^0_{\mathbb{T}}(Y_-) \to K^0_{\mathbb{T}}(Y_+)$ matches the analytic continuation of quantum connections for $Y_\pm$, hence the monodromy conjecture.

4.1. Equivariant quantum cohomology

This section serves as a general introduction to equivariant quantum cohomology. We fix a smooth DM stack X with the torus $\mathbb{T}$-action.

4.1.1. The $\mathbb{T}$-equivariant quantum cohomology. The moduli stack $\overline{\mathcal{M}}_{0,n}(X, d)$ of degree $d \in H_2(X, \mathbb{Q})$ twisted stable maps to X carries a $\mathbb{T}$-action, and a virtual fundamental cycle $[\overline{\mathcal{M}}_{0,n}(X, d)]^{\mathrm{virt}} \in H_{*,\mathbb{T}}(\overline{\mathcal{M}}_{0,n}(X, d))$. There are $\mathbb{T}$-equivariant evaluation maps[1]:

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X, d) \to IX$$

to the inertia stack IX of X for $1 \leq i \leq n$, see [8], [1].

Given $\gamma_1, \dots, \gamma_n \in H^*_{\mathrm{CR},\mathbb{T}}(X)$, we consider the following genus 0 $\mathbb{T}$-equivariant Gromov–Witten invariant:

$$\langle \gamma_1, \dots, \gamma_n \rangle^X_{0,n,d} = \int^{\mathbb{T}}_{[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{virt}}} \prod_i \mathrm{ev}_i^\star \gamma_i.$$

The moduli stack $\overline{\mathcal{M}}_{0,n}(X, d)$ has components indexed by the components of the inertia stack IX. We write

$$IX = \bigsqcup_{f \in \mathsf{B}} X_f$$

for the decomposition of IX into connected components, where B is the index set. Then the component $\overline{\mathcal{M}}_{0,n}(X, d)^{f_1,\dots,f_n}$ is the one which under evaluation

[1] We ignore the issue of trivializing the marked gerbes in our moduli problem. A detailed discussion on this can be found in [1].

maps ev_i, the images lie in the component X_{f_i}. The virtual dimension of $\overline{\mathcal{M}}_{0,n}(X,d)^{f_1,\dots,f_n}$ is:

$$-K_X \cdot d + \dim(X) + n - 3 - \sum_i \mathrm{age}(X_{f_i}). \tag{4.1}$$

If X is not compact (like our Lawrence and hypertoric DM stacks), then the moduli stack $\overline{\mathcal{M}}_{0,n}(X,d)$ is non-compact. There is a $\mathbb{T}$-action on $\overline{\mathcal{M}}_{0,n}(X,d)$. Assume that the $\mathbb{T}$-fixed locus $\overline{\mathcal{M}}_{0,n}(X,d)^{\mathbb{T}}$ is compact, then $\mathbb{T}$-equivariant GW invariants can be defined in the same way, replacing equivariant integration by equivariant residues.

Let $\mathrm{NE}(X) \subset H_2(X,\mathbb{R})$ be the cone generated by classes of effective curves and set

$$\mathrm{NE}(X)_{\mathbb{Z}} := \{d \in H_2(X,\mathbb{Z}) : d \in \mathrm{NE}(X)\}.$$

Let $R_{\mathbb{T}} := H^*_{\mathbb{T}}(pt)$ and $R_{\mathbb{T}}[\![Q]\!]$ the formal power series ring

$$R_{\mathbb{T}}[\![Q]\!] = \left\{ \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} a_d Q^d : a_d \in R \right\}$$

so that Q is a so-called *Novikov variable* (see e.g. [21, III 5.2.1]). For $\gamma_i, \gamma_j, t \in H^*_{\mathrm{CR},\mathbb{T}}(X)$, the big $\mathbb{T}$-equivariant quantum product is defined by:

$$(\gamma_i \star_t \gamma_j, \gamma_k) = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} \sum_{n \geq 0} Q^d \langle \gamma_i, \gamma_j, \underbrace{t, \dots, t}_{n}, \gamma_k \rangle^X_{0,n+3,d} \tag{4.2}$$

The small $\mathbb{T}$-equivariant quantum product is defined by putting $n = 0$:

$$(\gamma_i \star_{\mathrm{sm}} \gamma_j, \gamma_k) = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} Q^d \langle \gamma_i, \gamma_j, \gamma_k \rangle^X_{0,3,d} \tag{4.3}$$

or

$$\gamma_i \star_{\mathrm{sm}} \gamma_j = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}}} Q^d \cdot \mathrm{inv}^\star \cdot \mathrm{ev}_{3,\star}(\mathrm{ev}_1^\star(\gamma_i)\, \mathrm{ev}_2^\star(\gamma_j) \cap [\overline{\mathcal{M}}_{0,3}(X,d)]^{\mathrm{virt}})$$

where $\mathrm{inv} : IX \to IX$ denotes the involution sending $(x,g) \mapsto (x, g^{-1})$, for $x \in X, g \in \mathrm{Aut}(x)$. The big quantum product satisfies the associativity property and makes $H^*_{\mathrm{CR},\mathbb{T}}(X) \otimes R_{\mathbb{T}}[\![Q]\!]$ a ring, which is called the equivariant quantum cohomology ring.

We briefly review the Givental's formalism about the orbifold Gromov–Witten invariants in terms of the Lagrangian cone in certain symplectic vector space, see [10]. Let

$$\mathcal{H}(X) := H^*_{\mathrm{CR},\mathbb{T}}(X,\mathbb{C}) \otimes_{R_\mathbb{T}} S_\mathbb{T}(\!(z^{-1})\!)[\![Q]\!],$$

equipped the non-degenerate $S_\mathbb{T}[\![Q]\!]$-bilinear symplectic form

$$\Omega(f,g) := \mathrm{Res}_{z=0}(f(-z), g(z))_{\mathrm{CR}} dz,$$

where $(-,-)_{\mathrm{CR}}$ is the orbifold Poincaré pairing. Here $S_\mathbb{T}$ is the localization ring of $R_\mathbb{T}$ with respect to the multiplicative set of nonzero homogeneous elements in $R_\mathbb{T}$. Let

$$\mathcal{H}_+ := H^*_{\mathrm{CR},\mathbb{T}}(X) \otimes_{R_\mathbb{T}} S_\mathbb{T}[z][\![Q]\!]; \quad \mathcal{H}_- := z^{-1} H^*_{\mathrm{CR},\mathbb{T}}(X) \otimes_{R_\mathbb{T}} S_\mathbb{T}[z^{-1}][\![Q]\!].$$

Then $\mathcal{H}(X) = \mathcal{H}_+ \oplus \mathcal{H}_-$ and one can think of $\mathcal{H}(X) = T^*(\mathcal{H}_+)$. The genus zero descendant Gromov–Witten potential is a formal function $\mathcal{F}^0_X : (\mathcal{H}_+, -z) \to S_\mathbb{T}[\![Q]\!]$ defined on the formal neighbourhood of $-z$ in $\mathcal{H}_+$ and taking values in $S_\mathbb{T}[\![Q]\!]$:

$$\mathcal{F}^0_X(-z1 + \mathbf{t}(z)) = \sum_{d \in \mathrm{NE}(X)_\mathbb{Z}} \sum_{n=0}^{\infty} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle^X_{0,n,d}.$$

Here $\mathbf{t}(z) = \sum_{n=0}^{\infty} t_n z^n$ with $t_n \in H^*_{\mathrm{CR},\mathbb{T}}(X) \otimes_{R_\mathbb{T}} S_\mathbb{T}[\![Q]\!]$.

Givental's Lagrangian cone $\mathcal{L}_X$ is the graph of the differential $\mathcal{F}^0_{\mathcal{X}}$, more explicitly,

$$\mathcal{L}_X := \{(p,q) \in \mathcal{H}_- \oplus \mathcal{H}_+ \mid p = d_q \mathcal{F}^0_X\} \subset \mathcal{H}.$$

Tautological equations for genus 0 Gromov–Witten invariants imply that $\mathcal{L}_X$ is a cone ruled by a *finite dimensional* family of affine subspaces. A particularly important finite-dimensional slice of $\mathcal{L}_X$ is the *J-function*:

$$J_X(t,z) = 1z + t + \sum_{n,d} \sum_{\alpha} \frac{Q^d}{n!} \left\langle t, \ldots, t, \frac{\phi_\alpha}{z - \bar{\psi}} \right\rangle_{0,n+1,d} \phi^\alpha,$$

where $\{\phi_\alpha\}, \{\phi^\alpha\} \subset H^*_{\mathrm{CR},\mathbb{T}}(X)$ are additive bases dual to each other under $(-,-)_{\mathrm{CR}}$.

4.1.2. Quantum connection. We fix a homogeneous basis

$$\phi_0, \dots, \phi_R$$

for the $\mathbb{T}$-equivariant Chen–Ruan cohomology $H^*_{\mathrm{CR},\mathbb{T}}(X_\pm) \cong H^*_{\mathrm{CR},\mathbb{T}}(Y_\pm)$. Let

$$\tau^0, \dots, \tau^R$$

be the corresponding dual co-ordinates. The equivariant quantum connection is the operator

$$\nabla_i : H^*_{\mathrm{CR},\mathbb{T}}(X)[z] \otimes R_{\mathbb{T}}[\![Q]\!][\![\tau^0, \dots, \tau^R]\!] \to H^*_{\mathrm{CR},\mathbb{T}}(X)[z] \otimes R_{\mathbb{T}}[\![Q]\!][\![\tau^0, \dots, \tau^R]\!]$$

defined by:

$$\nabla_i = \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i \star -),$$

where $\phi \star -$ stands for the big quantum product.

We return to the Lawrence toric DM stacks $X_\pm$, and the hypertoric DM stacks $Y_\pm$. By [18, Proposition 6.4], the pullback $\iota^\star_\pm$ equate Gromov–Witten invariants of $X_\pm$ and $Y_\pm$. It follows that for $\phi \in H^*_{\mathrm{CR},\mathbb{T}}(X_\pm)$, we have an equality of operators:

$$\phi \star_t = \iota^\star_\pm(\phi) \star_{\iota^\star_\pm t}.$$

So

$$\iota^\star_\pm \nabla_i^{X_\pm} = \nabla_i^{Y_\pm}.$$

4.2. The Analytic continuation

4.2.1. I-functions of $X_\pm$ and $Y_\pm$. Recall that in [10, §5.3], the global extended Kähler moduli space is a universal cover of space $\mathcal{M}$, where $\mathcal{M}$ is the secondary toric variety of the GKZ-fan on $\mathbb{L}^\vee \otimes \mathbb{R}$. The GKZ-fan is given by the chamber structures on $\mathbb{L}^\vee \otimes \mathbb{R}$. The rank $\mathrm{rk}(\mathbb{L}^\vee) = r$.

Our wall $W = \overline{C}_+ \cap \overline{C}_-$, where $C_\pm$ are cones of $\mathbb{L}^\vee \otimes \mathbb{R}$. The two torus fixed points P_+ and P_- of $\mathcal{M}$ corresponding to C_+ and C_- are called the large radius limit points. The two toric DM stacks $X_\pm$ are called the mirrors corresponding to these two points. The hypertoric DM stacks $Y_\pm$ have the same LG-mirrors as $X_\pm$, and the toric variety $\mathcal{M}$ is also the global extended Kähler moduli space corresponding to $Y_\pm$.

We fix the notations

$$S_\pm = \{i \in \{1, \dots, m\} : \overline{\{i\}} \notin \mathcal{A}_\pm\}; \quad S_0 := S_+ \cap S_-.$$

The secondary toric variety $\mathcal{M}$ is covered by two open charts

$$\mathrm{Spec}(\mathbb{C}[C_+^\vee \cap \mathbb{L}]); \quad \mathrm{Spec}(\mathbb{C}[C_-^\vee \cap \mathbb{L}]) \tag{4.4}$$

which glue along $\mathrm{Spec}(\mathbb{C}[C_W^\vee \cap \mathbb{L}])$, where C_W is the relative interior of $\overline{C_W} := W \cap \overline{C_+} = W \cap \overline{C_-}$. The toric variety $\mathcal{M}$ may be singular.

Let

$$\ell_\pm := \dim(H^2(X_\pm, \mathbb{R})) \cong \dim(H^2(Y_\pm, \mathbb{R})) = r - \#(S_\pm).$$

Consider the subsets $\mathbb{K}_\pm \subset \mathbb{L} \otimes \mathbb{Q}$ by:

$$\mathbb{K}_\pm = \{f \in \mathbb{L} \otimes \mathbb{Q} : \{i \in [m] : D_i \cdot f \in \mathbb{Z}\} \in \mathcal{A}_\pm\}.$$

Define $\widetilde{\mathbb{L}}_+$ (respectively $\widetilde{\mathbb{L}}_-$) to be the free $\mathbb{Z}$-submodules of $\mathbb{L} \otimes \mathbb{Q}$ generated by $\mathbb{K}_+$ (respectively $\mathbb{K}_-$). Note that $\widetilde{\mathbb{L}}_\pm$ are overlattices of $\mathbb{L}$. We have $H_2(X_\pm, \mathbb{R}) \cap \widetilde{\mathbb{L}}_\pm = H_2(|X_\pm|, \mathbb{Z})$. From [10, §5.3], we can choose integral bases:

$$\{p_1^+, \dots, p_{l_+}^+\} \cup \{D_j : j \in S_+\} \subset \widetilde{\mathbb{L}}_+^\vee$$

and

$$\{p_1^-, \dots, p_{l_+}^-\} \cup \{D_j : j \in S_-\} \subset \widetilde{\mathbb{L}}_-^\vee$$

of $\widetilde{\mathbb{L}}_\pm^\vee$ such that

1) $p_1^+, \dots, p_{l_+}^+$ lie in the nef cone $\overline{C'_+} \subset H^2(X_+; \mathbb{R})$;
2) $p_1^-, \dots, p_{l_-}^-$ lie in the nef cone $\overline{C'_-} \subset H^2(X_-; \mathbb{R})$;
3) $p_i^+ = p_i^- \in \overline{C'_W}$ for $i = 1, \dots, l$,

where $l = r - 1 - \#(S_0)$ and

$$C'_\pm \subset C_\pm = C'_\pm \times \sum_{j \in S_\pm} \mathbb{R}_{>0} D_j$$
$$C'_W \subset C_W = C'_W \times \sum_{j \in S_0} \mathbb{R}_{>0} D_j$$

as in [10, §5.2]. Then these bases give co-ordinates on the toric charts (4.4). For $d \in \mathbb{L}$, write y^d for the corresponding element in the group ring $\mathbb{C}[\mathbb{L}]$.

The homomorphisms

$$\mathbb{C}[C_+^\vee \cap \mathbb{L}] \hookrightarrow \mathbb{C}[y_1, \ldots, y_{l_+}, \{x_j : j \in S_+\}], \quad \mathsf{y}^d \mapsto \prod_{i=1}^{l_+} y_i^{p_i^+ \cdot d} \cdot \prod_{j \in S_+} x_j^{D_j \cdot d}$$
$$\mathbb{C}[C_-^\vee \cap \mathbb{L}] \hookrightarrow \mathbb{C}[\widetilde{y}_1, \ldots, \widetilde{y}_{l_-}, \{\widetilde{x}_j : j \in S_-\}], \quad \widetilde{\mathsf{y}}^d \mapsto \prod_{i=1}^{l_-} \widetilde{y}_i^{p_i^- \cdot d} \cdot \prod_{j \in S_-} \widetilde{x}_j^{D_j \cdot d}$$

define the two smooth co-ordinate charts

$$(y_i, x_j : 1 \le i \le l_+, j \in S_+) \quad (\widetilde{y}_i, \widetilde{x}_j : 1 \le i \le l_-, j \in S_-)$$

which are resolutions of (respectively) $\mathrm{Spec}(\mathbb{C}[C_+^\vee \cap \mathbb{L}])$ and $\mathrm{Spec}(\mathbb{C}[C_-^\vee \cap \mathbb{L}])$. We reorder the bases

$$\{p_1^+, \ldots, p_{\ell_+}^+\} \cup \{D_j : j \in S_+\} = \{\mathsf{p}_1^+, \ldots, \mathsf{p}_{r-1}^+, \mathsf{p}_r^+\}$$
$$\{p_1^-, \ldots, p_{\ell_-}^-\} \cup \{D_j : j \in S_-\} = \{\mathsf{p}_1^-, \ldots, \mathsf{p}_{r-1}^-, \mathsf{p}_r^-\}$$

in such a way that $\mathsf{p}_i^+ = \mathsf{p}_i^- \in W$ for $i = 1, \ldots, r-1$ and $\mathsf{p}_r^\pm$ be the unique vector that does not lie on the wall W. Let

$$\{y_i, x_j : 1 \le i \le \ell_+, j \in S_+\} = \{\mathsf{y}_1, \ldots, \mathsf{y}_r\}$$
$$\{\widetilde{y}_i, \widetilde{x}_j : 1 \le i \le \ell_+, j \in S_-\} = \{\widetilde{\mathsf{y}}_1, \ldots, \widetilde{\mathsf{y}}_r\}$$

be the corresponding reordering coordinates of $\mathcal{M}$. Then

$$\widetilde{\mathsf{y}}_i = \begin{cases} \mathsf{y}_i \cdot \mathsf{y}_r^{c_i}, & 1 \le i \le r-1; \\ \mathsf{y}_r^{-c}, & i = r \end{cases}$$

for some $c_i \in \mathbb{Q}, c \in \mathbb{Q}_{>0}$.

For $d \in \mathbb{L} \otimes \mathbb{Q}$, we write

$$d = \bar{d} + \sum_{j \in S_\pm} (D_j.d)\xi_j$$

where $\bar{d}$ is the $H_2(X_\pm, \mathbb{R})$-component of d, and $\xi_j \in \mathbb{L} \otimes \mathbb{Q}$ such that

$$\mathbb{L} \otimes \mathbb{R} \cong H_2(X_\pm; \mathbb{R}) \oplus \bigoplus_{j \in S_\pm} \mathbb{R}\xi_j$$

and $\cap_{j\in S_\pm}\mathrm{Ker}(\xi_j)\cong H^2(X_\pm;\mathbb{R})$. The $H^*_{\mathrm{CR},\mathbb{T}}(X_\pm)$-valued hypergeometric series $I_\pm(\mathsf{y},z)\in H^*_{\mathrm{CR},\mathbb{T}}(X_\pm)\otimes_{R_\mathbb{T}} R_\mathbb{T}((z^{-1}))[\![Q,\sigma_\pm,x]\!]$ is:

$$I_\pm(\mathsf{y},z):=e^{\sigma_\pm/z}\sum_{d\in\mathbb{K}_\pm}\mathsf{y}^d\left(\prod_{j=1}^{N}\frac{\prod_{a:\langle a\rangle=\langle D_j\cdot d\rangle,a\le 0}(u_j+az)}{\prod_{a:\langle a\rangle=\langle D_j\cdot d\rangle,a\le D_j\cdot d}(u_j+az)}\right)\mathbb{1}_{[-d]} \tag{4.5}$$

where

- $\sigma_\pm=\theta_\pm(\sum_{i=1}^r\mathsf{p}_i^\pm\log(\mathsf{y}_i)+c_0(\lambda))$
  $=\sum_{i=1}^{\ell_\pm}\theta_\pm(p_i^\pm)\log y_i-\sum_{j\in S_\pm}\lambda_j\cdot\log x_j+c_0(\lambda)$, where

  $$\theta_\pm:\mathbb{L}^\vee\otimes\mathbb{C}\to H^2_\mathbb{T}(X_\pm,\mathbb{C});\quad \theta_\pm(D_i)=u_i-\lambda_i,$$

  and u_i is the cohomology class in $H^2_\mathbb{T}(X_\pm,\mathbb{C})$ Poincaré dual to the divisor classes $\{z_i=0\}$, $\{w_i\}=0$, or $\{u_j=0\}$. Note that $u_j=0$ if $j\in S_\pm$.
- $\mathsf{y}^d=\mathsf{y}_1^{\mathsf{p}_1^\pm\cdot d}\cdots\mathsf{y}_r^{\mathsf{p}_r^\pm\cdot d}=\prod_{i=1}^{\ell_\pm}y_i^{p_i^\pm\cdot d}\prod_{j\in S_\pm}x_j^{D_j\cdot d}$.

The I-functions $I_\pm(\mathsf{y},z)$ lie on the Givental's Lagrangian cone $\mathcal{L}_{X_\pm}$ determined by genus zero Gromov–Witten invariants. From [10], the I-functions $I_\pm(\mathsf{y},z)$ are analytic in the last variable y_r and we do analytic continuation in terms of y_r.

In view of [18, Proposition 6.4], we may identify[2] the cones $\mathcal{L}_{Y_\pm}$ with the cones $\mathcal{L}_{X_\pm}$. We simply define the I-functions $I^{Y_\pm}(\mathsf{y},z)$ for $Y_\pm$ to be the I-functions $I_\pm(\mathsf{y},z)$. Certainly they determine the cones $\mathcal{L}_{Y_\pm}$.

4.2.2. The analytic continuation. Introduce the following modified Givental's spaces:

$$\widetilde{\mathcal{H}}(X_\pm)=H^*_{\mathrm{CR},\mathbb{T}}(X_\pm)\otimes_{R_\mathbb{T}}[\log z]((z^{-1/k}))$$

and

$$\widetilde{\mathcal{H}}(Y_\pm)=H^*_{\mathrm{CR},\mathbb{T}}(Y_\pm)\otimes_{R_\mathbb{T}}[\log z]((z^{-1/k}))$$

where $k\in\mathbb{N}$ is an integer such that $k\mu^\pm$ have integer eigenvalues for the grading operators $\mu^\pm$ in [10, §2].

---

[2]Since $Y_\pm\subset X_\pm$ is a complete intersection and the normal bundle $N_{Y_\pm/X_\pm}$ is trivial, this is just a simple example of orbifold quantum Riemann–Roch in genus 0 [25].

Proposition 4.1 ([10], §6.2.4). *There is a symplectic transformation*

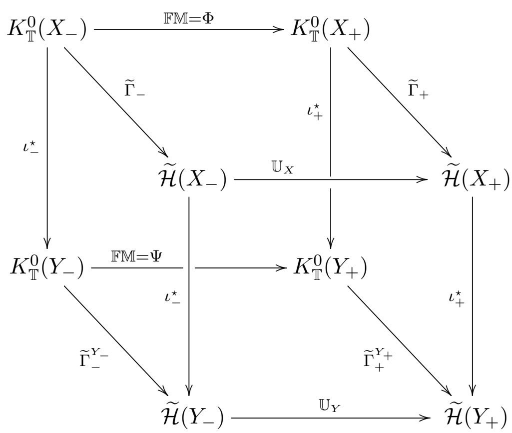
$$\mathbb{U}_X : \widetilde{\mathcal{H}}(X_-) \to \widetilde{\mathcal{H}}(X_+)$$

such that $\mathbb{U}_X(I_-(\mathsf{y}, z)) = I_+(\mathsf{y}, z)$.

Proof. In [10, Theorem 6.13], the authors explicitly calculate the analytic continuation of the H-function, and the analytic continuation of the I-function in [10, §6.2.4]. □

Our main result for $Y_\pm$ is as follows:

Theorem 4.2. *There exists the following diagram:*



where

$$\widetilde{\Gamma}_\pm : K^0_{\mathbb{T}}(X_\pm) \to \widetilde{\mathcal{H}}(X_\pm)$$

is defined by the $\Gamma_\pm(X_\pm)$*-classes in Definition 3.1 of [10], and*

$$\widetilde{\Gamma}^{Y_\pm}_\pm : K^0_{\mathbb{T}}(Y_\pm) \to \widetilde{\mathcal{H}}(Y_\pm)$$

is defined by replacing the $\Gamma_\pm(X_\pm)$*-classes by the* $\Gamma_\pm(Y_\pm)$*-classes. The map*

$$\mathbb{U}_Y : \widetilde{\mathcal{H}}(Y_-) \to \widetilde{\mathcal{H}}(Y_+)$$

is a symplectic transformation on Givental's space for $Y_\pm$*. Moreover,*

1) $\mathbb{U}_Y(I^{Y_-}(\mathsf{y}, z)) = I^{Y_+}(\mathsf{y}, z)$*;*

2) The upper square is a commutative diagram, which implies that the Fourier–Mukai transformation Φ matches the analytic continuation $\mathbb{U}_X$ via the Γ-integral structure;

3) The bottom square is also commutative, which implies that the Fourier–Mukai transformation Ψ matches the analytic continuation $\mathbb{U}_Y$ via the Γ-integral structure.

Proof. The proof is from §4.2.1 and [10, §7.3, §7.4, §7.5]. The difference here is that we can work on $\mathbb{T}$-equivariant K-theory and Chen–Ruan cohomology, not like [10, §7] in the non-equivariant setting, but the calculation is the same. $\square$

Remark 4.3. On the Kähler moduli space $\mathcal{M}$, the Fourier–Mukai transformation Ψ is an equivalence

$$\Psi : D^b(Y_-) \to D^b(Y_+) \quad (K^0_{\mathbb{T}}(Y_-) \stackrel{\cong}{\to} K^0_{\mathbb{T}}(Y_+)).$$

From [10, §6.5], the Fourier–Mukai transformation Ψ corresponds to a path γ from the large radius point of Y_- to the large radius point of Y_+ inside $\mathcal{M}$. Let

$$\Psi' := \mathbb{FM}' = (F_-)_\star F_+^\star : K^0_{\mathbb{T}}(Y_+) \stackrel{\cong}{\to} K^0_{\mathbb{T}}(Y_-)$$

be the Fourier–Mukai transformation on the other side. Then $\Psi' \circ \Psi$ yields a loop in $\pi_1(\mathcal{M})$, which gives rise to an automorphism of the K-theory group by

$$\rho_K : \pi_1(\mathcal{M}) \to \mathrm{Aut}(K^0_{\mathbb{T}}(Y_-)). \tag{4.6}$$

On the other hand, the analytic continuation

$$\mathbb{U}_Y : \widetilde{\mathcal{H}}(Y_-) \to \widetilde{\mathcal{H}}(Y_+)$$

is given along a path γ in Figure 3 of [10], and hence also gives a loop of $\pi_1(\mathcal{M})$ in (4.6). Since the I-function $I_\pm(Y_\pm)$ determines the quantum connection, the above monodromy is the monodromy of the quantum connections. Theorem 4.2 says that these two monodromies are the same, hence the monodromy conjecture in [5].

References

[1] D. Abramovich, T. Graber, and A. Vistoli, *Algebraic orbifold quantum product*, in: Orbifolds in mathematics and physics (Madison,WI,2001), 1–24, Contem. Math. **310**, Amer. Math. Soc., (2002).

[2] N. Addington, W. Donovan, and C. Meachan, *Mukai flops and P-twists*, J. Reine Angew. Math. **748** (2019), 227–240.

[3] R. Bezrukavnikov and A. Okounkov, *Monodromy of the quantum differential equation for the Hilbert scheme of points*, preprint.

[4] L. Borisov, L. Chen, and G. Smith, *The orbifold Chow ring of toric Deligne–Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215.

[5] A. Braverman, D. Maulik, and A. Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. **227** (2011), no. 1, 421–458.

[6] S. Cautis, *Equivalences and stratified flops*, Compositio Math. **148** (2012) no. 1, 185–209.

[7] S. Cautis, *Flops and about: a guide*, EMS Congress Reports (2011), no. 8, 61–101.

[8] W. Chen and Y. Ruan, *Orbifold Gromov–Witten theory*, in: Orbifolds in Mathematics and Physics (Madison, WI, 2001), 25–85, Contem. Math. **310**, Amer. Math. Soc., (2002).

[9] T. Coates, H. Iritani, Y. Jiang, and E. Segal, *K-theoretical and categorical properties of toric Deligne–Mumford stacks*, Pure and Applied Math. Quarterly **11** (2015), no. 2, 239–266.

[10] T. Coates, H. Iritani, and Y. Jiang, *The crepant transformation conjecture for toric complete intersections*, Adv. Math. **329** (2018), 1002–1087.

[11] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng, *A mirror theorem for toric stacks*, Compos. Math. **151** (2015), no. 10, 1878–1912.

[12] D. Edidin, *Strong regular embeddings of Deligne–Mumford stacks and hypertoric geometry*, Michigan Math. J. **65** (2016), no. 2, 389—412.

[13] B. Fu, *Extremal contractions, stratified Mukai flops and Springer maps*, Adv. Math. **213** (2007), 165–182.

[14] T. Hausel and B. Sturmfels, *Toric hyperkähler varieties*, Documenta Mathematica **7** (2002), 495–534.

[15] Y. Jiang, *The orbifold cohomology ring of simplicial toric stack bundles*, Illinois Journal of Mathematics **52** (2008), no. 2, 493–514.

[16] Y. Jiang and H.- H. Tseng, *The orbifold Chow ring of hypertoric Deligne–Mumford stacks*, Journal für die reine und angewandte Mathematik,(Crelle's Journal) **619** (2008), 175–202.

[17] Y. Jiang and H.- H. Tseng, *Note on orbifold Chow ring of semi-projective toric DM stacks*, Communications in Analysis and Geometry **16** (2008), no. 1, 231–250.

[18] Y. Jiang and H.- H. Tseng, *Hypertoric geometry and Gromov–Witten theory*, Journal of Geometry and Physics **126** (2018), 101–126.

[19] Y. Jiang and H.- H. Tseng, *The monodromy conjecture for Mukai type flops*, in preparation.

[20] Y. Kawamata, *D-equivalence and K-equivalence*, J. Diff. Geometry **61** (2002), 147–171.

[21] Y. Manin, Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces, Vol. 47 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, (1999).

[22] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, `arXiv:1211.1287`.

[23] Y. Namikawa, *Mukai flops and derived categories*, J. Renie Angew. Math. **560** (2003), 65–76.

[24] Y. Ruan, *Cohomology ring of crepant resolutions of orbifolds*, in: Gromov–Witten Theory of Spin Curves and Orbifolds, 117–126, Contem. Math. **403**, Amer. Math. Soc., (2006).

[25] H.-H. Tseng, *Orbifold Quantum Riemann–Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), 1–81.

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