Math. Res. Lett.
Volume 26, Number 6, 1603–1627, 2019

The Kähler-Ricci flow on pseudoconvex domains

Huabin Ge, Aijin Lin, and Liangming Shen

We establish the existence of the Kähler-Ricci flow on pseudoconvex domains with general initial metrics without curvature bounds. We could show that the evolving metric is simultaneously complete, and the corresponding normalized Kähler-Ricci flow converges to the complete Kähler-Einstein metric, which generalizes Topping's simultaneously complete Ricci flow on surfaces to high dimensional case.

1. Introduction

In [7, 8, 19–21] Giesen and Topping studied the Ricci flow on surfaces assuming the initial metric is incomplete or has unbounded curvature. Based on Hamilton's work on surfaces [9], they used different settings with respect to different conformal structures. To deal with the bad initial condition, they considered an approximated sequence of the initial metrics and established the compactness of the sequence of approximated flows using Perelman's pseudolocality theorem [13]. In [21] Topping pointed out that in high dimensional case, specifically for Kähler manifolds, one possible approach is to study the complex Monge-Ampère flow associated to the Kähler-Ricci flow. In this paper we will consider the Kähler-Ricci flow on pseudoconvex domains as a natural generalization.

As a fundamental topic in complex analysis and complex geometry, there have been plenty of works on pseudoconvex domains in different aspects. For example, Fefferman considered the Bergman kernel and related it to the complex Monge-Ampère equations [6]. In [5] Cheng-Yau began to study the existence of Kähler-Einstein metrics with negative Ricci curvature on general pseudoconvex manifolds. Furthermore Mok and Yau studied the completeness of the Kähler-Einstein metrics on bounded domains of holomorphy and gave some curvature criterions related to those domains [12].

Similar to compact Kähler manifolds, the Kähler-Ricci flow can be used to study complete Kähler metrics on noncompact Kähler manifolds. In [2], Chau showed that the normalized Kähler-Ricci flow converges to the complete Kähler-Einstein metric provided that the initial metric ω_0 is smooth, complete and satisfies that $Ric(\omega_0) + \omega_0 = \sqrt{-1}\partial\bar{\partial} f$ for some smooth and bounded function f. Later this condition was weakened by Lott-Zhang [11]. Those results hold on pseudoconvex manifolds by [5, 12].

In this paper, we will establish the existence and characterize the long time behavior of the Kähler-Ricci flow with weaker initial metrics. Similar to Giesen and Topping, we can show that the solution to the Kähler-Ricci flow will be simultaneously complete. First we consider a strictly pseudoconvex domain Ω with smooth boundary, i.e., there exists a smooth strictly plurisubharmonic function φ defined on $\Omega \subset \mathbb{C}^n$ such that $\partial\Omega = \{\varphi = 0\}$ is smooth. Thus the smooth positive $(1,1)$-form $\sqrt{-1}\partial\bar{\partial}\varphi$ is an incomplete smooth Kähler form on Ω. Furthermore, by [5] there exists a complete metric defined by $-\sqrt{-1}\partial\bar{\partial}\log(-\varphi)$ which is asymptotically hyperbolic near the boundary. Suppose our initial metric is incomplete, we have the following result:

Theorem 1.1. *Given an incomplete Kähler metric ω_0 on the strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary which is defined by $\partial\Omega = \{\varphi = 0\}$, if the curvature tensors of ω_0 have uniformly bounded covariant derivatives, the solution to the Kähler-Ricci flow*

$$\begin{cases} \frac{\partial}{\partial t}\omega = -Ric(\omega) \\ \omega(0) = \omega_0 \end{cases} \tag{1.1}$$

exists on $\Omega \times [0, \infty)$ and the solution $\omega(t)$ is complete when $t > 0$. Moreover, the corresponding solution to the normalized Kähler-Ricci flow converges to the complete Kähler-Einstein metric as $t \to \infty$.

To establish the flow solution which is simultaneously complete, we will transform the equation (1.1) to the complex Monge-Ampère flow. One natural problem is the behavior of the solution at the initial time. We will adapt similar approximating method in [3] to overcome this difficulty. After that we will derive suitable a priori estimates to establish the existence and the limit behavior of this flow.

However this method cannot be applied to general pseudoconvex domains or pseudoconvex manifolds directly as we do not have such complete

Kähler metrics with the form $-\sqrt{-1}\partial\bar{\partial}\log(-\varphi)$. One possible idea is to construct an approximation sequence of solutions on exhausting strictly pseudoconvex domains with smooth boundaries, as Cheng-Yau did for Kähler-Einstein metric [5]. The main difficulty of this approach is that the Kähler-Ricci flow solutions could not be locally uniformly controlled from above by the local Poincaré metric, which is different from the case of Kähler-Einstein metrics. Thus we could not obtain the compactness of the solutions on exhausting domains as [5]. However, if there is one complete background metric with strictly negative Ricci curvature and uniformly bounded curvature tensor, we can still derive a solution to the Kähler-Ricci flow using a similar method to Theorem 1.1. In fact this background metric enables us to consider not only pseudoconvex manifolds. More precisely, we have

Theorem 1.2. *Given a Kähler manifold M with a noncomplete Kähler metric ω_0, whose curvature tensors have uniformly bounded covariant derivatives. If there exists a complete Kähler metric ω_M such that $Ric(\omega_M) \leq -C_0\omega_M$ for some constant $C_0 > 0$ and the curvature tensors of ω_M have uniformly bounded covariant derivatives, then there exists a long time solution $\omega(t)$ to the Kähler-Ricci flow* (1.1) *with the initial metric ω_0 which is simultaneously complete as soon as $t > 0$. Furthermore, the normalized Kähler-Ricci flow converges to the Kähler-Einstein metric in Cheng-Yau [5] and the solution $\tilde{\omega}(t)$ is simultaneously complete as soon as $t > 0$.*

This work could be considered as a higher dimensional generalization of Topping's simultaneously complete surface flow. However until now, we cannot prove the uniqueness of the Kähler-Ricci flow solution. We hope in the future we can establish some uniqueness results under some constraints. Moreover we hope to generalize our result to general pseudoconvex manifolds without the completeness of the background metric with negative Ricci curvature. Finally, we hope this work on pseudoconvex domains could help us to establish the existence of Kähler-Ricci flow on more general noncompact Kähler manifolds or Stein manifolds, which could be another approach to understanding the structures of such manifolds.

2. The parabolic Omori-Yau maximal principle and parabolic Schwarz Lemma

In this section we will introduce basic tools in the analysis of complete metrics. One basic principle is the parabolic Omori-Yau maximal principle under the settings of complete metrics. The elliptic version was used in [5, 12] while the parabolic version was used in [2, 3, 11, 15]. See Proposition 1.6 of [5] for a detailed proof. Here we cite the Omori-Yau maximal principles in the following which will be used in this paper:

Proposition 2.1. *Suppose there exists a bounded curvature solution $\omega(t)$ to the Kähler-Ricci flow on $M \times [0,T]$ such that $C^{-1}\omega_0 \leq \omega(t) \leq C\omega_0$ where ω_0 is the initial complete metric and C is a positive constant. Then if a smooth function $\psi(x,t)$ is bounded from above on $M \times [0,T]$,*

1) *for each time $t \in [0,T]$ there exist a sequence of points x_k such that*

$$\psi(x_k,t) \to \sup_{M\times\{t\}} \psi,\ |\nabla\psi(x,t)|_t \leq \frac{1}{k},\ \sqrt{-1}\partial\bar{\partial}\psi(x_k,t) \leq \frac{1}{k}\omega(x_k,t);$$

2) *there exist a sequence of points x_k and time $\bar{t} \in [0,T]$ such that*

$$\psi(x_k,\bar{t}) \to \sup\psi,\ |\nabla\psi(x,\bar{t})|_{\bar{t}} \leq \frac{1}{k},\ \sqrt{-1}\partial\bar{\partial}\psi(x_k,\bar{t}) \leq \frac{1}{k}\omega(x_k,\bar{t}),$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)\psi(x_k,\bar{t}) \geq -\frac{1}{k},$$

where $\sup\psi$ *denotes the superimum over* $M \times [0,T]$.

By Proposition 2.1, we could show the parabolic Schwarz Lemma as follows:

Proposition 2.2. *Suppose there exists a bounded curvature solution $\omega(t)$ to the Kähler-Ricci flow on $M \times [0,T]$ such that $C^{-1}\omega_0 \leq \omega(t) \leq C\omega_0$ where ω_0 is the initial complete metric and C is a positive constant. If there exists a Kähler metric ω on M such that its Ricci curvature is negative and bounded above by $-C'$, then it holds that*

$$\omega(t)^n \geq (C't)^n\omega^n.$$

Proof. We adapt the proof in [12, 24] to the parabolic case. As $\omega(t)$ solves the Kähler-Ricci flow, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\frac{\omega^n}{\omega(t)^n} = R(t) - R(t) + tr_{\omega(t)}Ric(\omega) \leq -C'n\left(\frac{\omega^n}{\omega(t)^n}\right)^{1/n},$$

where $R(t)$ denotes the scalar curvature of $\omega(t)$ and the last inequality follows from the mean value inequality. Set $u := \frac{\omega^n}{\omega(t)^n}$, it follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right) u \leq -C' n u^{\frac{n+1}{n}}.$$

Thus we have that

$$\left(\frac{\partial}{\partial t} - \Delta\right) ((u+c)^{-1/n} - C't) \geq -\frac{1}{n}\left(1 + \frac{1}{n}\right) \frac{|\nabla u|^2}{(u+c)^{2+1/n}}.$$

By Proposition 2.1 for any t there are sequence of points (x_k, t) such that $u(x_k, t) \to \sup u$ and

$$\frac{\partial}{\partial t}((u+c)^{-1/n} - C't)(x_k, t) \geq -\frac{C''}{k}$$

where C'' depends on c. Let $k \to \infty$ it follows that $\sup_M((u+c)^{-1/n} - C't)$ is non-decreasing in t. Finally let $c \to 0$ the proposition follows. □

For future purposes, we introduce two important inequalities as following, which proof are quite standard, see for example, [2, 16]:

Lemma 2.3. *Suppose there exists a solution $\omega(t)$ to the Kähler-Ricci flow on $M \times [T', T]$, and there exist two Kähler metrics ω_1 with lower bisectional curvature bound $-k_1$ and ω_2 with upper bisectional curvature bound k_2, then it holds that*

(i) (Aubin-Yau)

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log tr_{\omega_1}\omega(t) \leq k_1 tr_{\omega(t)}\omega_1; \tag{2.1}$$

(ii) (Chern-Lu)

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log tr_{\omega(t)}\omega_2 \leq k_2 tr_{\omega(t)}\omega_2. \tag{2.2}$$

3. Kähler-Ricci flow on strictly pseudoconvex domains with incomplete initial metric

In this section we will prove Theorem 1.1. First we briefly recall the coordinate system introduced in [5]. As $\partial\Omega$ is smooth, for any point on the boundary there is a constant δ_0 such that the ball with radius δ_0 and tangential to the $\partial\Omega$ at that point lies inside Ω. Without loss of generality we can assume $\delta_0 = 1$. For $z_0 \in \Omega \subset \mathbb{C}^n$, If the Euclidean distance $d(z_0, \partial\Omega)$ is greater than a constant, e.g., assuming to be 1, obviously we could choose the unit ball around z_0 and standard Euclidean holomorphic coordinates as its local coordinate chart. If $d(z_0, \partial\Omega) < 1$, choose the ball B containing z_0 and touching the boundary with radius 1, and assume the coordinates of the center of the ball and z_0 are $(0, 0, \dots, 0)$ and $(\eta, 0, \dots, 0)$ with $\eta < 1$. Then there is a biholomorphic map Φ_{z_0} which sends $(\eta, 0, \dots, 0)$ to $(0, 0, \dots, 0)$. Then we can use the standard Euclidean coordinates $v_1, \dots, v_n$ on $\Phi_{z_0}(B)$ as the coordinates on B. By the estimates in [5] the differential $d\Phi_{z_0}$ is uniformly bounded. Finally we can cover Ω by those balls associated with the holomorphic coordinates assigned as above, which can be denoted as $\{(V, (v_1, \dots, v_n))\}$. Furthermore, under such coordinate charts, for any positive integer l we say a metric $g_{i\bar{j}}$ has bounded geometry on Ω if $g_{i\bar{j}} \in C^l$, $c^{-1}\delta_{ij} \le g_{i\bar{j}} \le c\delta_{ij}$ for some constant $c > 1$, and there exist constants $A_1, \dots, A_l$ such that for any multi-indices α, β with $|\alpha| + |\beta| \le l$, we have

$$\left| \frac{\partial^{|\alpha|+|\beta|}}{\partial v_\alpha \partial \bar{v}_\beta} g_{i\bar{j}} \right| \le A_{|\alpha|+|\beta|}.$$

Next, consider the metric $\bar{\omega} = -\sqrt{-1}\partial\bar{\partial}\log(-\varphi)$ where φ is the smooth defining function of the domain Ω such that the boundary $\partial\Omega = \{\varphi = 0\}$. By the computations in [5], under the coordinate system above, we can see that the metric tensor of $\bar{\omega}$ is

$$\bar{g}_{i\bar{j}} = \frac{\varphi_i \varphi_{\bar{j}}}{\varphi^2} - \frac{\varphi_{i\bar{j}}}{\varphi},$$

which is complete in the direction orthogonal to $\partial\Omega$. Moreover the bisectional curvature of $\bar{\omega}$ is asymptotically -1 near $\partial\Omega$. Note that as φ is smooth, it was shown in [5] that $\bar{\omega}$ has bounded geometry of any order l. Now consider an incomplete metric ω_0 whose covariant derivatives of its curvature tensors are uniformly bounded. we want to verify that $\omega := \omega_0 + \bar{\omega}$ has the same asymptotical behavior with $\bar{\omega}$ near $\partial\Omega$. Note that we have

$$g_{i\bar{j}} = g_{0i\bar{j}} - \frac{\varphi_{i\bar{j}}}{\varphi} + \frac{\varphi_i \varphi_{\bar{j}}}{\varphi^2},$$

which is obviously a complete metric and asymptotical to $\bar{\omega}$ as g_0 is not complete. Now set the Hermitian metric $\tilde{g}_{i\bar{j}} = \varphi_{i\bar{j}} - \varphi g_{0i\bar{j}}$, we have

$$g^{i\bar{j}} = (-\varphi)\tilde{g}^{i\bar{l}}\left(\delta_{jl} - \frac{\tilde{g}^{k\bar{j}}\varphi_k\varphi_{\bar{l}}}{|\nabla\varphi|^2_{\tilde{g}} - \varphi}\right) = (-\varphi)\left(\tilde{g}^{i\bar{j}} - \frac{\varphi^i\varphi^{\bar{j}}}{|\nabla\varphi|^2_{\tilde{g}} - \varphi}\right),$$

where $\varphi^i = \tilde{g}^{i\bar{l}}\varphi_{\bar{l}}$. By direct but complicated computations, we have

$$\begin{aligned}\Gamma^k_{ij} = &\frac{\varphi_i\delta_{jk} + \varphi_j\delta_{ik}}{-\varphi} + \frac{\varphi_{ij}\varphi^k}{|\nabla\varphi|^2_{\tilde{g}} - \varphi}\\ &+ \left(\tilde{g}^{k\bar{l}} - \frac{\varphi^k\varphi^{\bar{l}}}{|\nabla\varphi|^2_{\tilde{g}} - \varphi}\right)(-\varphi\partial_i g_{0j\bar{l}} - \varphi_i g_{0j\bar{l}} - g_j\delta_{0i\bar{l}} + \varphi_{j\bar{l}i}),\end{aligned}$$

and

$$\begin{aligned}R_{i\bar{j}k\bar{l}} = &-(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) - \frac{1}{\varphi}\left(\tilde{g}_{p\bar{l}}R^p_{i\bar{j}k}(\varphi_{i\bar{j}}) - \frac{\varphi_{,ik}\varphi_{,\bar{j}\bar{l}}}{|\nabla\varphi|^2_{\tilde{g}} - \varphi}\right)\\ &- g_{0k\bar{l},i\bar{j}} + (g_{0i\bar{j}}g_{0k\bar{l}} + g_{0i\bar{l}}g_{0k\bar{j}})\\ &- \frac{1}{\varphi}(\varphi_{i\bar{j}}g_{0k\bar{l}} + \varphi_{i\bar{l}}g_{0k\bar{j}} + \varphi_{k\bar{l}}g_{0i\bar{j}} + \varphi_{k\bar{j}}g_{0i\bar{l}})\\ &- \frac{1}{\varphi}\left(\varphi_i g_{0k\bar{l},\bar{j}} + \varphi_{\bar{j}}g_{0k\bar{l},i} + \varphi_k g_{0i\bar{l},\bar{j}} + \varphi_{\bar{l}}g_{0k\bar{j},i}\right)\\ &+ g^{p\bar{q}}\Bigg(g_{0p\bar{l},\bar{j}}g_{0k\bar{q},l} + g_{0p\bar{l},\bar{j}}\left(\frac{\varphi_{\bar{q}}\varphi_{,ik}}{\varphi^2} + \frac{\varphi_i}{\varphi}g_{0k\bar{q}} + \frac{\varphi_k}{\varphi}g_{0i\bar{q}}\right)\\ &\qquad + g_{0k\bar{q},i}\left(\frac{\varphi_p\varphi_{,\bar{j}\bar{l}}}{\varphi^2} + \frac{\varphi_{\bar{l}}}{\varphi}g_{0p\bar{j}} + \frac{\varphi_{\bar{j}}}{\varphi}g_{0p\bar{l}}\right)\Bigg)\\ &+ \frac{g^{p\bar{q}}}{\varphi^2}\left(\varphi_i\varphi_{\bar{j}}g_{0k\bar{q}}g_{0p\bar{l}} + \varphi_i\varphi_{\bar{l}}g_{0k\bar{q}}g_{0p\bar{j}} + \varphi_k\varphi_{\bar{j}}g_{0i\bar{q}}g_{0p\bar{l}} + \varphi_k\varphi_{\bar{l}}g_{0i\bar{q}}g_{0p\bar{j}}\right)\\ &+ \frac{g^{p\bar{q}}}{\varphi^3}\left(\varphi_p\varphi_{,\bar{j}\bar{l}}(g_{0i\bar{q}}\varphi_k + g_{0k\bar{q}}\varphi_i) + \varphi_{\bar{q}}\varphi_{,ik}(g_{0p\bar{j}}\varphi_{\bar{l}} + g_{0p\bar{l}}\varphi_{\bar{j}})\right),\end{aligned} \tag{3.1}$$

where all the covariant derivatives are with respect to the Kähler metric $\sqrt{-1}\partial\bar{\partial}\varphi$ and $R^p_{i\bar{j}k}(\varphi_{i\bar{j}})$ also denotes the curvature tensor of $\sqrt{-1}\partial\bar{\partial}\varphi$. By the assumption, φ is smooth thus the metric $\sqrt{-1}\partial\bar{\partial}\varphi$ has bounded geometry of all orders. As the incomplete metric g_0 has uniformly bounded covariant derivatives of its curvature tensor, the computations above implies that the metric $g_{i\bar{j}}$ has bounded geometry of all orders on Ω. Furthermore, as $g^{p\bar{q}} = O(|\varphi|)$, it turns out that

$$R_{i\bar{j}k\bar{l}} = -(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) + O\left(\frac{1}{\varphi^2}\right).$$

As $g_{i\bar{j}} = O(\frac{1}{\varphi^2})$, it follows that

$$|R_{i\bar{j}k\bar{l}} + (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})|_g \to 0$$

near $\partial\Omega$ which implies that the bisectional curvature of $\omega = \omega_0 + \bar{\omega}$ is asymptotically -1. By induction we can show that all its covariant derivatives are asymptotically to 0, which are the same to the case in [5].

This observation enables us to establish the short time existence of Kähler-Ricci flow on pseudoconvex domains with initial metric $\omega = \omega_0 + \bar{\omega}$ by Shi's existence theorems [14, 15]. In case that the initial metric ω_0 is incomplete, we will derive uniform estimates for the initial metric $\omega_\epsilon = \omega_0 + \epsilon\bar{\omega}$ and take the limit as $\epsilon \to 0$. To realize this approximation, we will transform the original Kähler-Ricci flow (1.1) to complex Monge-Ampère flow equation as following.

Recall the computation in [5], still under the same local coordinates, we have

$$Ric(\bar{\omega}) + (n+1)\bar{\omega} = -\sqrt{-1}\partial\bar{\partial}\log(det(\varphi_{i\bar{j}})(|d\varphi|^2_\varphi - \varphi)),$$

where $|d\varphi|_\varphi$ is with respect to the Kähler metric $\sqrt{-1}\partial\bar{\partial}\varphi$. For simplicity we set

$$f := \log(det(\varphi_{i\bar{j}})(|d\varphi|^2_\varphi - \varphi))$$

which is smooth with uniformly bounded derivatives of all orders. Now write

$$\omega_t := \omega_0 + (n+1)t\bar{\omega}, \tag{3.2}$$

which is obviously positive for all time and equivalent to the complete metric $\bar{\omega}$ for all positive time. Assume $\omega(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ satisfies Kähler-Ricci flow (1.1), it holds that

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\frac{\partial}{\partial t}u =& -(n+1)\omega - Ric(\omega(t)) = Ric(\bar{\omega}) - Ric(\omega(t)) + \sqrt{-1}\partial\bar{\partial}f\\ =&\sqrt{-1}\partial\bar{\partial}\log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\bar{\omega}^n} + \sqrt{-1}\partial\bar{\partial}f,\end{aligned}$$

then we could derive a complex Monge-Ampère flow equation

$$(3.3) \qquad \begin{cases} \frac{\partial}{\partial t}u = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\bar{\omega}^n} + f \\ u(0) = 0 \end{cases}$$

whose solution will give rise to a solution to Kähler-Ricci flow (1.1), although the converse may not be true. The main problem of equation (3.3) is the degeneracy of ω_t at $t = 0$. To overcome this point we adapt the approximation procedure in [3] which deals with the similar problem in the situation of cusp type metrics.

Set $\omega_{t,\epsilon} = \omega_0 + (n+1)(t+\epsilon)\bar{\omega}$ and $\omega_\epsilon(t) = \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}u_\epsilon$, we consider the following approximating Monge-Ampère flow equation:

$$(3.4) \qquad \begin{cases} \frac{\partial}{\partial t}u_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n}{\bar{\omega}^n} + f \\ u_\epsilon(0) = 0 \end{cases}$$

By the arguments above for each $\epsilon > 0$ we know that (3.4) has a complete solution with bounded curvature on $[0, T_{\max})$ where $T_{\max}$ denotes the maximal existence time of the Kähler-Ricci flow solution. Moreover by existence results of corresponding Monge-Ampère flow equation (see e.g., Lemma 4.1 of [2] or Theorem 4.1 of [11]), as the given background metric $\omega_{t,\epsilon}$ is positive and preserves the bounded geometry for any positive t and ϵ, we have $T_{\max} = \infty$. Furthermore, the potential solution u_ϵ is uniformly bounded which only depends on T and ϵ for $t \leq T$. We will apply Omori-Yau maximal principle in Proposition 2.1 to derive uniformly a priori estimates and establish the existence of long time solution. First, we have

Lemma 3.1. *There exists a constant $C(t) > 0$ only depending on t such that for any $t > 0$*

$$(3.5) \qquad |u_\epsilon(\cdot, t)| \leq C(t).$$

Proof. By the argument above we only need to derive uniform a priori estimates for u_ϵ. We take $\psi_\epsilon := u_\epsilon - Ct$ for some undetermined constant C as [3]. For $T < T_{max}$ fix $\bar{t} \in [0, T]$ such that $\max \psi_\epsilon(\cdot, \bar{t})$ attains the maximum of ψ_ϵ on $\Omega \times [0, T]$ and without loss of generality we can assume $\bar{t} > 0$. If there exists a point $p \in \Omega$ such that $\psi(p, \bar{t})$ attains this maximum, apply classical maximal principle to (3.4) at this point, it follows that

$$\frac{\partial}{\partial t}\psi_\epsilon \leq \log \frac{\omega_{t,\epsilon}^n}{\bar{\omega}^n} + f - C \leq -1 \tag{3.6}$$

for some sufficiently large C which is independent of ϵ. However this contradicts with the maximality of $\psi_\epsilon(p,\bar{t})$ in the assumption unless $\bar{t}=0$. Thus we conclude that

$$\sup u_\epsilon(\cdot,t) \leq Ct.$$

In general case, since we have bounded curvature solution as the argument before, by Omori-Yau maximal principle in Proposition 2.1 there exist a sequence of points $p_k \in \Omega$ such that $\psi_\epsilon(p_k,\bar{t}) \to \sup \psi$ on $\Omega \times [0,T]$, and $\sqrt{-1}\partial\bar{\partial}\psi_\epsilon(p_k,\bar{t}) \leq \frac{\omega_{\bar{t},\epsilon}}{k}$. Apply this to (3.4) again we conclude that $\frac{\partial}{\partial t}\psi_\epsilon(p_k,\bar{t}) \leq -1$ for constant C independent of ϵ and all sufficiently large k. On the other hand by the existence of bounded curvature solution we conclude that $\partial_t^2\psi_\epsilon(p_k,\bar{t})$ are uniformly bounded independent of k. These facts contradict the assumption that $\psi_\epsilon(p_k,\bar{t}) \to \sup\psi$ unless $\bar{t}=0$ and we conclude the upper bound for $u_\epsilon(\cdot,t)$ as well.

Using almost the same argument we could derive the uniform lower bound for $u_\epsilon(\cdot,t)$ and conclude this lemma. □

We will apply Omori-Yau maximal principle several times in this work. For simplicity we will always assume the maximum could be attained otherwise just apply similar argument to Lemma 3.1. Now we will give the uniform estimates for the time derivative $\dot{u}_\epsilon$:

Lemma 3.2. *There exist two bounded constants $C_1(t), C_2(t) > 0$ only depending on $C(t)$ in* Lemma 3.1 *such that*

$$n\log t - C_1(t) \leq \dot{u}_\epsilon(\cdot,t) \leq \frac{C_2(t)}{t} + n. \tag{3.7}$$

Proof. We will adapt the inequalities in [3, 17] to this lemma. For the upper bound, we have the following inequality

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_\epsilon\right)(t\dot{u}_\epsilon - u_\epsilon - nt) &= t\left(\frac{\partial}{\partial t} - \Delta_\epsilon\right)\dot{u}_\epsilon + \Delta_\epsilon u_\epsilon - n \\ &= t(n+1)tr_{\omega_\epsilon}\bar{\omega} - tr_{\omega_\epsilon}\omega_t = -tr_\omega\omega_{0,\epsilon} < 0,\end{aligned}$$

where Δ_ϵ is the Laplacian with respect to $\omega_\epsilon(t)$. Apply Omori-Yau maximal principle combined with Lemma 3.1, we conclude the uniformly upper bound estimate. For the lower bound, consider the following inequality

$$\left(\frac{\partial}{\partial t} - \Delta_\epsilon\right)(\dot{u}_\epsilon - n\log t) = (n+1)tr_{\omega_\epsilon}\bar{\omega} - \frac{n}{t} \geq C\left(\frac{\bar{\omega}^n}{\omega_\epsilon^n}\right)^{1/n} - \frac{n}{t}.$$

Observe that $\dot{u}_\epsilon - n\log t$ tends to infinity uniformly at $t = 0$, apply Omori-Yau maximal principle again and still assume the minimum could be attained (otherwise choose a minimizing sequence) at $(p, \bar{t})$ for some $\bar{t} > 0$, it holds that

$$\dot{u}_\epsilon(p, \bar{t}) = \log \frac{\omega_\epsilon^n}{\bar{\omega}^n}(p, \bar{t}) - f(p) \geq n\log \bar{t} - C',$$

thus it holds that $\dot{u}_\epsilon \geq n\log t - C_1(t)$ which is independent of ϵ. □

Base on the lemmas above we could derive a Laplacian estimate:

Lemma 3.3. *There exist two constants $C_3(t), C_4(t) > 0$ independent of ϵ such that*

$$C_3(t)\bar{\omega} \leq \omega_\epsilon(t) \leq C_4(t)\bar{\omega}. \tag{3.8}$$

Proof. By (3.1) or the curvature formula in [5] the bisectional curvature of $\bar{\omega}$ is bounded from above by a constant $k \geq 0$. Then apply Chern-Lu Inequality (2.2), we have

$$\left(\frac{\partial}{\partial t} - \Delta_\epsilon\right)(\log tr_{\omega_\epsilon}\bar{\omega} - A\dot{u}_\epsilon + B\log t) \leq (k - A(n+1))tr_{\omega_\epsilon}\bar{\omega} + \frac{B}{t}.$$

First we choose A such that $k - A(n+1) = -1$, then consider that at $t = 0$, it holds that

$$tr_{\omega_\epsilon}\bar{\omega} \leq \frac{1}{\epsilon}, \quad \dot{u}_\epsilon \geq n\log(n+1)\epsilon,$$

thus $\log tr_{\omega_\epsilon}\bar{\omega} - A\dot{u}_\epsilon + B\log t$ tends to $-\infty$ and the maximum could only be attained for $t > 0$. As we argued before we still assume this maximum is attained at $(p,\bar{t})$ then it holds that $tr_{\omega_\epsilon}\bar{\omega}(p,\bar{t}) \leq \frac{B}{t}$, which implies

$$\log tr_{\omega_\epsilon}\bar{\omega} - A\dot{u}_\epsilon + B\log t \leq (B-1-n)\log t + AC_1(t) + \log B < \infty$$

as long as $B \geq n+1$. Thus choose $B \geq n+1$, $\log tr_{\omega_\epsilon}\bar{\omega} - A\dot{u}_\epsilon + B\log t$ is uniformly bounded from above which implies

$$tr_{\omega_\epsilon}\bar{\omega} \leq e^{\frac{C}{t}}$$

by Lemma 3.2. Next consider the equation (3.4) we have

$$\omega_\epsilon^n = \bar{\omega}^n e^{\dot{u}_\epsilon - f} \leq \bar{\omega}^n e^{\frac{C_2(t)}{t} + C'},$$

which concludes the proof. □

As [2, 17] we can proceed to derive higher order estimates as following:

Lemma 3.4. *There exists a constant $C_5(t)$ such that*

$$|\nabla_{\bar{\omega}}\omega_\epsilon|^2_{\omega_\epsilon} \leq e^{\frac{C_5(t)}{t}},$$

and a constant $C_{t_1,t_2,k} > 0$ for $0 < t_1 < t_2 < T_{max}$ such that

$$|u_\epsilon|_{C^k([t_1,t_2]\times\Omega,\bar{\omega})} \leq C_{t_1,t_2,k}. \tag{3.9}$$

Given the Lemmas above, by the compactness theorem, we conclude that there exist a sequence of numbers $\epsilon_k \searrow 0$ such that u_{ϵ_k} converge to a solution u satisfying (3.3) in the space of $C^0([0,T_{max})\times\Omega) \cap C^\infty((0,T_{max})\times\Omega)$. Moreover by the Laplacian estimate in 3.3 the corresponding flow solution will be complete as long as $t > 0$. By use of maximal principle again we establish the uniqueness of the solution to (3.3):

Proposition 3.5. *There exists a unique solution u satisfying* (3.3) *in the space of* $C^0([0,T_{max})\times\Omega) \cap C^\infty((0,T_{max})\times\Omega)$.

Proof. We only need to prove the uniqueness. Suppose we have two solutions u_1, u_2 satisfying (3.3), then write $v = u_1 - u_2$ and $\omega_2 = \omega_t + \sqrt{-1}\partial\bar{\partial}u_1 = \omega_0 + (n+1)t\bar{\omega} + \sqrt{-1}\partial\bar{\partial}u_2$, we have the following equation for v :

$$\begin{cases} \frac{\partial}{\partial t}v = \log\frac{(\omega_2+\sqrt{-1}\partial\bar{\partial}v)^n}{\omega_2^n} \\ v(0) = 0 \end{cases}$$

v is also a bounded function. If the maximum of $v(t)$ is attained for each t obviously we have $v \leq 0$. For the general case, we apply Omori-Yau maximal principle again that for each $t > 0$ there exists a sequence of points p_k such that $v(p_k, t) \to \sup v(\cdot, t)$ and $\sqrt{-1}\partial\bar{\partial}v(p_k, t) \leq \frac{1}{k}\omega_2$ as $\omega_2(t)$ is a bounded curvature metric for $t > 0$. Thus it holds that

$$\frac{\partial}{\partial t} \sup v(\cdot, t) \leq \limsup_k \log\left(1 + \frac{1}{k}\right)^n = 0$$

in the sense of distribution, which implies $u_1 \leq u_2$. Similarly we have $u_1 \geq u_2$ and the uniqueness is approved. □

Consider Lemma 3.1 again, we have a more precise estimate for u_ϵ essentially by maximal principle:

$$\begin{aligned} \int_0^t n\log(n+1)(s+\epsilon)ds + t\inf_\Omega f &\leq u_\epsilon(t) \\ \leq \int_0^t n\log(c+(n+1)(s+\epsilon))ds &+ t\sup_\Omega f, \end{aligned} \tag{3.10}$$

where c is a constant such that $\omega_0 \leq c\bar{\omega}$ on Ω. Note that this bound is uniformly bounded on any bounded time interval $[0, T]$ which essentially implies the uniformly bounded high order estimates for u_ϵ until $t = T_{max}$. This essentially implies that derivatives of u of all orders at $t = T_{max}$ is uniformly bounded. By the short time existence, this flow could be extended to another time interval, and by (3.10) again, the solution could be extended to infinite time:

Theorem 3.6. *There exists a unique solution u to the equation* (3.3) *in the space of $C^0([0, +\infty) \times \Omega) \cap C^\infty((0, +\infty) \times \Omega)$. This solution gives rise to a family of Kähler metrics $\omega(t)$ which satisfy the Kähler-Ricci flow* (1.1) *on* $[0, +\infty) \times \Omega$ *and are simultaneously complete as soon as* $t > 0$.

Remark 3.7. *Theorem 3.6 shows the proof the first part of Theorem 1.1. However we note that this only generates a solution to the Kähler-Ricci flow* (1.1) *from the unique solution to the complex Monge-Ampère flow* (3.3) *but the Kähler-Ricci flow solution itself may not be unique even in the class of simultaneously complete flow solutions.*

Now we begin the proof the second part of Theorem 1.1. We consider the following equation of normalized Kähler-Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\omega} = -Ric(\tilde{\omega}) - (n+1)\tilde{\omega} \\ \tilde{\omega}(0) = \tilde{\omega}_0 \end{cases} \tag{3.11}$$

Namely the solution of (3.11) could be derived from the unnormalized Kähler-Ricci flow (1.1) by

$$\tilde{\omega}(t) = e^{-(n+1)t}\omega\left(\frac{e^{(n+1)t}-1}{n+1}\right), \tag{3.12}$$

thus the long time existence of the solution to (1.1) in Theorem 3.6 also work for (3.11). However to prove the convergence result, we still need to analyse the behavior of the solution to the corresponding complex Monge-Ampère flow equation. Similar to [16, 18] we set

$$\tilde{\omega}_t = e^{-(n+1)t}\omega_0 + (1-e^{-(n+1)t})\bar{\omega},$$

and $\tilde{\omega}(t) = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{u}$, thus we have the following complex Monge-Ampère flow equation:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u} = \log\frac{(\tilde{\omega}_t+\sqrt{-1}\partial\bar{\partial}\tilde{u})^n}{\bar{\omega}^n} - (n+1)\tilde{u} + f \\ \tilde{u}(0) = 0 \end{cases} \tag{3.13}$$

By the relation (3.12) we have a unique long time solution $\tilde{u}(t)$ to (3.13) in the space of $C^0([0,+\infty)\times\Omega)\cap C^\infty((0,+\infty)\times\Omega)$. The remaining issue is to investigate the limit behavior of the solution $\tilde{u}(t)$ as long as t tends to infinity. Still by Omori-Yau maximal principle, we have the following C^0-estimate for $\tilde{u}$:

Lemma 3.8. *There exists a uniform constant $C > 0$ such that*

$$|\tilde{u}(t)| \leq C \tag{3.14}$$

on $[0,+\infty)\times\Omega$.

Proof. By Lemma 3.1 and the relation (3.12) we know that for each $t > 0$ the solution $\tilde{u}(t)$ is bounded by a constant depending on t. For each $t > 0$ we still assume the maximum and minimum can be attained at some points otherwise we use the argument of limiting sequence. Similar to (3.10) we have the following inequality with the same constant c :

$$\begin{aligned} &-(1+e^{-(n+1)t})|f|_{C^0} + e^{-(n+1)t}\int_0^t ne^{(n+1)s}\log(1-e^{-(n+1)s})ds \leq \tilde{u}(t) \\ &\leq (1+e^{-(n+1)t})|f|_{C^0} + e^{-(n+1)t}\int_0^t ne^{(n+1)s}\log(ce^{-(n+1)s}+(1-e^{-(n+1)s}))ds, \end{aligned}$$

thus the uniform estimate for $\tilde{u}(t)$ follows. □

Now we begin to establish the estimate for $\dot{\tilde{u}}$:

Lemma 3.9. *There exist two uniform constants* $C_1, C_2 > 0$ *and* $t_0 > 0$ *such that for all* $t > t_0$,

$$-C_1 \leq \dot{\tilde{u}}(t) \leq C_2 t e^{-(n+1)t}. \tag{3.15}$$

Proof. First by (3.13), we have the evolution equation for $\dot{\tilde{u}}$:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \dot{\tilde{u}} &= tr_{\tilde{\omega}} \dot{\tilde{\omega}}_t - (n+1)\dot{\tilde{u}} \\ &= (n+1)e^{-(n+1)t} tr_{\tilde{\omega}}(\bar{\omega} - \omega_0) - (n+1)\dot{\tilde{u}}. \end{aligned} \tag{3.16}$$

Then we have the following inequality:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)((e^{(n+1)t} - 1)\dot{\tilde{u}} - (n+1)\tilde{u} - n(n+1)t) \\ &= (n+1)\left((1 - e^{-(n+1)t}) tr_{\tilde{\omega}}(\bar{\omega} - \omega_0) - (e^{(n+1)t} - 1)\dot{\tilde{u}} + e^{(n+1)t}\dot{\tilde{u}}\right) \\ &\quad - (n+1)\dot{\tilde{u}} + (n+1)(n - tr_{\tilde{\omega}}\tilde{\omega}_t) - n(n+1) \\ &= -(n+1) tr_{\tilde{\omega}}\omega_0 \leq 0. \end{aligned}$$

As the solution is smooth and with uniformly bounded derivatives for some positive time t then combine Lemma 3.8 and Omori-Yau maximal principle we conclude the upper bound for $\dot{\tilde{u}}$. On the other hand apply (3.16) again, for a large positive T we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)((1 - e^{(n+1)(t-T)})\dot{\tilde{u}} + (n+1)\tilde{u}) \\ &= (n+1)\left((1 - e^{(n+1)(t-T)}) e^{-(n+1)t} tr_{\tilde{\omega}}(\bar{\omega} - \omega_0) - n + tr_{\tilde{\omega}}\tilde{\omega}_t\right) \\ &= (n+1) tr_{\tilde{\omega}}\tilde{\omega}_T - n(n+1). \end{aligned}$$

By the argument of the upper bound of $\dot{\tilde{u}}$ if $t \geq T$ it holds that $(1 - e^{(n+1)(t-T)})\dot{\tilde{u}} + (n+1)\tilde{u}$ is uniformly bounded from below. Now assume that its minimum could be attained at some $t' < T$ and some point p. Then by

Omori-Yau maximal principle at (p, t') it holds that $tr_{\tilde{\omega}}\tilde{\omega}_T \leq n$, which implies that $\tilde{\omega} \geq C\tilde{\omega}_T$. It follows that

$$\begin{aligned}
&(1 - e^{(n+1)(t-T)})\dot{\tilde{u}}(t) + (n+1)\tilde{u}(t) \\
\geq\ &(1 - e^{(n+1)(t'-T)})\dot{\tilde{u}}(p, t') + (n+1)\tilde{u}(p, t') \\
=\ &(1 - e^{(n+1)(t'-T)}) \left(\log \frac{\tilde{\omega}^n}{\bar{\omega}^n} + f - (n+1)\tilde{u}\right)(p, t') + (n+1)\tilde{u}(p, t') \\
\geq\ &(1 - e^{(n+1)(t'-T)}) \log \frac{\tilde{\omega}_T^n}{\bar{\omega}^n} + C \geq C,
\end{aligned}$$

when $t \in [0, T]$. As this normalized flow exists for infinite time, we can choose $T = \infty$, thus the lower bound for $\dot{\tilde{u}}$ follows. □

The Laplacian estimate could be derived from a slight modification of Chern-Lu Inequality:

Lemma 3.10. *There exists a uniform constant $C_3 > 1$ such that for any $t > t_0 > 0$,*

$$C_3^{-1}\bar{\omega} \leq \tilde{\omega}(t) \leq C_3\bar{\omega}. \tag{3.17}$$

Proof. For (2.2) in Proposition 2.3, we have the following modification version in in the settings of normalized Kähler-Ricci flow with same conditions:

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log tr_{\tilde{\omega}(t)}\omega_2 \leq k_2 tr_{\tilde{\omega}(t)}\omega_2 - n - 1. \tag{3.18}$$

Now we have the following inequality:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log tr_{\tilde{\omega}}\bar{\omega} - \frac{k+2}{n+1}(\dot{\tilde{u}} + (n+1)\tilde{u})\right) \\
\leq\ &k tr_{\tilde{\omega}}\bar{\omega} - n - 1 - (k+2)(tr_{\tilde{\omega}}\bar{\omega} - n) \leq -2tr_{\tilde{\omega}}\bar{\omega} + nk + n - 1,
\end{aligned}$$

where k is a positive upper bound of the bisectional curvature of $\bar{\omega}$. Still by Omori-Yau maximal principle and assuming that the maximum could be attained at some point (p, t') for $t > t_0$, it holds that $tr_{\tilde{\omega}}\bar{\omega}(p, t') \leq \frac{nk+n-1}{2}$, then it follows that

$$\begin{aligned}
&\log tr_{\tilde{\omega}}\bar{\omega} - \frac{k+2}{n+1}(\dot{\tilde{u}} + (n+1)\tilde{u}) \\
\leq\ &\left(\log tr_{\tilde{\omega}}\bar{\omega} - \frac{k+2}{n+1}(\dot{\tilde{u}} + (n+1)\tilde{u})\right)(p, t') \leq C,
\end{aligned}$$

by the above two lemmas and then $\bar{\omega} \leq C\tilde{\omega}(t)$. For the other direction, note that from (3.13) we have

$$\log \frac{\tilde{\omega}^n}{\bar{\omega}^n} = \dot{\tilde{u}} + (n+1)\tilde{u} - f \leq C,$$

thus the other side bound and consequently this lemma follow. □

By standard inductions we can establish the uniform high order estimates as t tends to infinity. As it is shown that $\dot{\tilde{u}} \leq C_2 t e^{-(n+1)t}$ we find that $\tilde{u}$ is almost decreasing as t tends to infinity. Considering the uniform boundness of $\tilde{u}$ by Lemma 3.8, $\tilde{u}(t)$ converges to a unique bounded limit function $\tilde{u}_\infty$ in C^0 norm. Moreover this convergence is essentially smooth by the uniform high order estimates. Thus the right hand side of the equation

$$\frac{\partial}{\partial t}\tilde{u} = \log \frac{(\tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{u})^n}{\bar{\omega}^n} - (n+1)\tilde{u} + f$$

uniformly converges to a smooth function on Ω which forces $\dot{\tilde{u}}$ converges to 0 as $\tilde{u}(t)$ converges. It follows that the limit function $\tilde{u}_\infty$ satisfies the complex Monge-Ampère equation:

$$\log \frac{(\bar{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty)^n}{\bar{\omega}^n} = (n+1)\tilde{u}_\infty - f, \tag{3.19}$$

which implies that

$$Ric(\bar{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty) = -(n+1)(\bar{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty),$$

thus we have the following proposition:

Proposition 3.11. *There exist a unique family of solutions $\tilde{u}(t)$ to the normalized complex Monge-Ampère flow* (3.13) *and they converge smoothly to a function $\tilde{u}_\infty$ which solves the complex Monge-Ampère equation* (3.19)*. Consequently there exist a family of solutions $\tilde{\omega}(t)$ (which may not be unique) to the normalized Kähler-Ricci flow* (3.11) *on* $[0, +\infty) \times \Omega$*. Moreover this family of solutions are simultaneously complete and converge to the complete Kähler-Einstein metric $\tilde{\omega}_\infty$ in C^∞ sense.*

Thus we obtain the limit behavior of the normalized simultaneously complete Kähler-Ricci flow on smooth pseudoconvex domains and the proof of Theorem 1.1 is complete.

4. Kähler-Ricci flow on general manifolds with a complete background metric

In this section we will complete the proof of Theorem 1.2. In this situation we will consider $\omega_t := \omega_0 - tRic(\omega_M)$ as the new evolving background metric. As in the beginning of section 3, we could also establish the coordinate system in Definition 1.1 of [5]. Then we need to verify that this background metric has uniformly bounded bisectional curvature for each $t > 0$. Similar to (3.1), we have the following bisectional curvature expansion of $\bar{g} := g_0 + h$ where g_0 and h are the metric tensor of ω_0 and $-tRic(\omega_M)$:

$$\bar{R}_{i\bar{j}k\bar{l}} = -\partial_{k\bar{l}} g_{0i\bar{j}} - \partial_{k\bar{l}} h_{i\bar{j}} + \bar{g}^{p\bar{q}}(\partial_k g_{0i\bar{q}} + \partial_k h_{i\bar{q}})(\partial_{\bar{l}} g_{0p\bar{j}} + \partial_{\bar{l}} h_{p\bar{j}}).$$

As the curvature tensors of ω_0 and ω_M both have uniformly bounded covariant derivatives of all orders, it follows that the bisectional curvature of $\bar{g}$ is uniformly bounded. This enables us to reuse the approximation method in the proof of Theorem 1.1. First set $\omega(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ as the flow solution then the Kähler-Ricci flow equation can be transformed to the following complex Monge-Ampère flow equation:

$$\begin{cases} \frac{\partial}{\partial t}u = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_M^n} \\ u(0) = 0 \end{cases} \tag{4.1}$$

To overcome the difficulty at $t = 0$, similarly, we set

$$\omega_{t,\epsilon} := \omega_0 - (t+\epsilon)Ric(\omega_M)$$

and consider the following approximating flow equation:

$$\begin{cases} \frac{\partial}{\partial t}u_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n}{\omega_M^n} \\ u_\epsilon(0) = 0 \end{cases} \tag{4.2}$$

By the argument above for any closed time interval $[0, T]$ the evolving background metric $\omega_{t,\epsilon}$ has uniformly bounded bisectional curvature, which enables us to apply the Omori-Yau maximal principle again to derive a priori estimates for u_ϵ. As $C_1\omega_M \le -Ric(\omega_M) \le C_2\omega_M$ for two positive constants C_1, C_2 by the assumptions, it follows that

$$C_1(t+\epsilon)\omega_M \le \omega_{t,\epsilon} \le (c' + C_2(t+\epsilon))\omega_M$$

where $c' > 0$ is the constant such that $\omega_0 \le c'\omega_M$ as ω_M is complete. Similar to Lemma 3.1 by applying the Omori-Yau maximal principle we have the following estimate:

$$n\int_0^t \log C_1(s+\epsilon)ds \le u_\epsilon \le n\int_0^t \log(c' + C_2(s+\epsilon))ds \tag{4.3}$$

This estimate also implies that $|u_\epsilon(t)| \le C(t)$ which is independent of ϵ. Next we need to derive a uniform estimate for $\dot{u}_\epsilon$. Similar to Lemma 3.2, we have the following lemma:

Lemma 4.1. *There exist constants $C_1(t), C_2(t) > 0$ which only depend on t such that*

$$n\log t - C_1(t) \le \dot{u}_\epsilon(t) \le \frac{C_2(t)}{t} + n. \tag{4.4}$$

The proof is almost the same to Lemma 3.2 and we skip it. Next we will derive a uniform Laplacian estimate for u_ϵ. As the background metric ω_M is assumed to have uniformly bounded bisectional curvature, similar to Lemma 3.3–3.4 we have the following high order derivative estimates:

Lemma 4.2. *There exist constants $C_3(t), C_4(t), C_5(t) > 0$ independent of ϵ such that*

$$C_3(t)\omega_M \le \omega_\epsilon(t) \le C_4(t)\omega_M, \tag{4.5}$$

$$|\nabla_{\omega_M}\omega_\epsilon|^2_{\omega_\epsilon} \le e^{\frac{C_5(t)}{t}}, \tag{4.6}$$

and a constant $C_{t_1,t_2,k} > 0$ for $0 < t_1 < t_2 < T_{max}$ such that

$$|u_\epsilon|_{C^k([t_1,t_2]\times M,\omega_M)} \le C_{t_1,t_2,k}. \tag{4.7}$$

Thus similarly by the uniform estimates above we derive that there exists a subsequence $\epsilon_k \searrow 0$ such that u_{ϵ_k} converges to a solution solving the original complex Monge-Ampère flow equation (4.1) in the space of $C^0([0,T)\times M) \cap C^\infty((0,T)\times M)$ for any $T > 0$. And by the Laplacian estimate in Lemma 4.2 the flow solution $\omega(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ is complete as soon as $t > 0$. Moreover the uniqueness can also be proved using the same method in Proposition 3.5.

Now we begin to consider the normalized Kähler-Ricci flow on M :

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\omega} = -Ric(\tilde{\omega}) - \tilde{\omega} \\ \tilde{\omega}(0) = \tilde{\omega}_0 \end{cases} \tag{4.8}$$

Similarly the solution to (4.8) can be derived from the solution $\omega(t)$ to the unnormalized Kähler-Ricci flow (1.1) by the equation $\tilde{\omega}(t) = e^{-t}\omega(e^t - 1)$. To characterize the limit behavior of the normalized Kähler-Ricci flow the remaining task is to establish uniform estimates for the solution $\tilde{\omega}(t)$. Now set $\tilde{\omega}_t := e^{-t}\omega_0 + (1 - e^{-t})\omega_M$ and $\tilde{\omega}(t) := \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{u}$ we can derive the following complex Monge-Ampère flow equation:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u} = \log \frac{(\tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{u})^n}{\omega_M^n} - \tilde{u} - f_0 \\ \tilde{u}(0) = 0 \end{cases} \tag{4.9}$$

Use the argument before we can prove that there exists a unique solution $\tilde{u}$ in the space of $C^0([0,T) \times M) \cap C^\infty((0,T) \times M)$ for any $T > 0$. Moreover the metric $\tilde{\omega}(t)$ is simultaneously complete as soon as $t > 0$. To derive a uniform estimate for $\tilde{u}(t)$, instead of directly applying the maximal principle as Lemma 3.8, we will proceed as Theorem 4.1-5.1 in Lott-Zhang [11] (we thank Dr. M.C. Lee for pointing out this approach).

Lemma 4.3. *For $t_0 > 0$, there exists a uniform constant $C > 0$ such that*

$$|\tilde{u}(t)| \leq C, \ |\dot{\tilde{u}}(t)| \leq C \tag{4.10}$$

on $[t_0, +\infty) \times M$.

Proof. It follows from (4.9) that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)\dot{\tilde{u}} &= -\dot{\tilde{u}} - e^{-t}tr_{\tilde{\omega}}(\omega_0 + Ric(\omega_M)), \\ \left(\frac{\partial}{\partial t} - \Delta\right)\ddot{\tilde{u}} &= -\ddot{\tilde{u}} + e^{-t}tr_{\tilde{\omega}}(\omega_0 + Ric(\omega_M)) - \left|\frac{\partial^2\tilde{\omega}}{\partial t^2}\right|^2_{\tilde{\omega}}, \end{aligned}$$

where $\tilde{\omega}$ denotes $\tilde{\omega}(t)$. It follows from the above two equations that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\tilde{u}} + \ddot{\tilde{u}}) = -\left|\frac{\partial^2\tilde{\omega}}{\partial t^2}\right|^2_{\tilde{\omega}} - (\dot{\tilde{u}} + \ddot{\tilde{u}}), \tag{4.11}$$

which implies that

$$\left(\frac{\partial}{\partial t} - \Delta\right) e^t(\dot{\tilde{u}} + \ddot{\tilde{u}}) \leq 0.$$

By the estimates for the solution to the unnormalized Kähler-Ricci flow we know that for $t_0 > 0$ there exists a constant $C' > 0$ depending on t_0 such that

$$|\dot{\tilde{u}}(t_0)| + |\ddot{\tilde{u}}(t_0)| \leq C'.$$

Thus for any $t > t_0$ it follows from the Omori-Yau's maximal principle that

$$e^t(\dot{\tilde{u}} + \ddot{\tilde{u}}) \leq C'e^{t_0} \leq C_1.$$

So we have

$$e^t\dot{\tilde{u}} \leq C_1 t + C_2, \tag{4.12}$$

which implies that

$$|\tilde{u}(t)| \leq C, \ |\dot{\tilde{u}}(t)| \leq C_3.$$

For the lower bound estimates, considering that $Ric(\omega_M) \leq -C\omega_M$, it follows from (4.9) that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{u} + \dot{\tilde{u}}) &= -n - tr_{\tilde{\omega}} Ric(\omega_M) \\
&\geq -n + Ctr_{\tilde{\omega}}\omega_M \geq -n + Cn\left(\frac{\omega_M^n}{\tilde{\omega}^n}\right)^{\frac{1}{n}} \\
&= -n + Cne^{-\frac{\tilde{u}+\dot{\tilde{u}}}{n}}.
\end{aligned}$$

Applying the Omori-Yau maximal principle again and assuming the infimum of $\tilde{u} + \dot{\tilde{u}}$ is attained at (x, t) it follows that

$$-n + Cne^{-\frac{\tilde{u}+\dot{\tilde{u}}}{n}} \leq 0.$$

Thus it follows that

$$\tilde{u} + \dot{\tilde{u}} \geq C_4,$$

which is still true even without the assumption above. Combine the upper bound estimates of $\tilde{u}, \dot{\tilde{u}}$ the lemma follows. □

Similar to Lemma 3.10 we have the following Laplacian estimate for $\tilde{u}$:

Lemma 4.4. *There exists a uniform constant C and $t_0 > 0$ such that for all $t > t_0$,*

$$C^{-1}\omega_M \leq \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{u} \leq C\omega_M. \tag{4.13}$$

By the estimates above we can derive uniform high order estimates from standard iterations. From (4.12) as time tends to infinity $\tilde{u}(t)$ is almost non-increasing. Then it follows from the uniform C^0-estimate (4.10) that $\tilde{u}(t)$ converges to a unique limit $\tilde{u}_\infty$ and moreover this convergence is smooth by the uniform high order estimates. It follows from (4.9) that the limit $\tilde{u}_\infty$ satisfies the following equation:

$$\log \frac{(\omega_M + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty)^n}{\omega_M^n} = \tilde{u}_\infty - f_0, \tag{4.14}$$

which implies that

$$Ric(\omega_M + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty) = -(\omega_M + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty).$$

Thus the limit of the solution to (4.9) is $\tilde{\omega}(\infty) = -Ric(\omega_M) + \sqrt{-1}\partial\bar{\partial}\tilde{u}_\infty$ which satisfies $Ric(\tilde{\omega}(\infty)) = -\tilde{\omega}(\infty)$. This completes the proof of Theorem 1.2. In some sense considering the existence of the complete Kähler-Einstein metric on bounded domains of holomorphy by Cheng-Yau and Mok-Yau [5, 12], we have the following corollary which can be thought as a generalization of Chau's work on bounded domains [2]:

Corollary 4.5. *For any incomplete metric ω_0 with uniformly bounded covariant derivatives there exists a family of solutions to the normalized Kähler-Ricci flow* (4.8) *which is simultaneously complete as soon as $t > 0$ and converges to the complete Kähler-Einstein metric.*

5. Further discussions

By Chau-Li-Tam's work [4], assume the initial metric ω_0 is equivalent to the background metric ω_M in Theorem 1.2. If furthermore ω_0 has bounded curvature or is the limit of a sequence of Kähler metrics with bounded curvature, there exists a solution to the Kähler-Ricci flow (1.1) for infinite time. Considering our result Theorem 1.2, we conjecture that if the initial metric ω_0 with bounded curvature or as the limit of a sequence of Kähler metrics with bounded curvature, and there exists a constant $C > 0$ such that $\omega_0 \leq C\omega_M$ then we have a solution for infinite time. More generally, for those results we still don't know whether we could drop the bounded curvature assumption, as Topping's work on surfaces [19–21].

Next, we still hope to extend Cheng-Yau's approximation method [5] in general pseudoconvex manifolds to the Kähler-Ricci flow case. As we mentioned before the main difficulty is that unlike the Kähler-Einstein case, we cannot use the local background Poincaré metric to control the flow solution from above. One possible idea is to apply Perelman's pseudolocality theorem to derive the local compactness of the Kähler-Ricci flow solutions on the approximating domains. Also it will be wonderful if we can loose the assumptions on background metric, e.g., without the completeness or curvature bound. In fact, we may consider general existence problem of the Kähler-Ricci flow on Stein manifolds by this approach, especially when the initial metric has no curvature bound. In that general Stein manifold we only know that there exists a background metric with non-positive bisectional curvature descending from the standard embedding into the Euclidean space.

Finally we are curious that in what sense our construction of the flow solution is unique. Note that in Lee-Tam's work [10] they can construct complete solutions to Chern-Ricci flow in the same conformal class, which is closer to Topping's construction. It will be interesting to investigate the limit behavior of their approach. We also hope to know among all possible simultaneously complete flow solutions which solution is maximal stretched, as defined by Giesen-Topping [8], and whether it is unique.

Acknowledgements

The three authors would like to express their greatest respect to Professor Gang Tian for constant supports and encouragements. The third author would like to thank Professor Albert Chau and Jingyi Chen for their interest in this work and thank Man-Chun Lee for his advice on improving Theorem 1.2. Finally the authors would like to thank the referee for his beneficial suggestions. The first author is partially supported by the NSFC Grant No. 11871094 and the second author is supported by National Natural Science Foundation of China under Grant No. 11401578 and Scientific Research Program Funds of NUDT under Grant No. ZK18-03-30.

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School of Mathematics, Renmin University of China
Beijing, 100872, P. R. China
E-mail address: hbge@ruc.edu.cn

Dept. of Mathematics, National University of Defense Technology
Changsha, 410073, P. R. China
E-mail address: linaijin@nudt.edu.cn

School of Mathematical Sciences, Beihang University
Beijing, 100191, P. R. China
E-mail address: lmshen1985@gmail.com

Received March 26, 2018