Math. Res. Lett.
Volume 26, Number 6, 1759–1787, 2019

Asymptotic Chow stability of toric Del Pezzo surfaces

King-Leung Lee, Zhiyuan Li, Jacob Sturm, and Xiaowei Wang

In this note, we study the effective Chow polystability of toric Del Pezzo surfaces appearing in the moduli space of Kähler-Einstein Fano varieties constructed in [19].

1. Introduction

Since the invention of geometric invariant theory [14] by David Mumford, GIT has been successfully applied to the construction of various kinds of moduli spaces, e.g. moduli spaces of stable vector bundles over a projective curves and of moduli spaces of polarized varieties (X, L). In particular, when X is a canonically polarized manifold, it was shown by Mumford and Gieseker in dimension 1, by Gieseker [7] in dimension 2, and in arbitrary dimensions by Donaldson [5] (making use of the work of Aubin , Yau [1, 32] and Zhang [**?**]) that $(X, L = \mathscr{O}_X(K_X))$ is asymptotically Chow stable (see also [22]). That is, given a smooth canonically polarized variety $(X, \mathscr{O}_X(K_X))$, that there exists an r_0 such that $(X, \mathscr{O}_X(rK_X))$ is Chow stable for any $r \geq r_0$. More generally, if (X, L) is a polarized manifold, GIT also plays a role in the existence of constant scalar curvature Kähler metrics in the class of L (see for example the survey article [23]).

In order to compactify the moduli space it is necessary to include *singular* varieties (e.g. by the stable reduction theorem for curves). In general, it is quite difficult to extend above works to singular varieties, even for the dimension one case (cf.[7, 10]). On the other hand, it was shown in [31], that *asymptotic* Chow stability does not form a proper moduli space in general by exhibiting some explicit punctured families of *canonically polarized* varieties with no asymptotic Chow semistable filling. However, in [11], a proper moduli space of smoothable K-semistable *Fano varieties* is constructed based on the seminal work of [2–4] and [28]. It is natural to ask whether or not the moduli space of $\mathbb{Q}$-Fano varieties can be realized as an asymptotic or effective GIT moduli space at least when the dimension is *small* [1] . The advantage of having such a GIT description is that it helps us effectively compute and understand the geometry of moduli spaces, as exemplified in the work of [19] which is crucially based on explicit GIT constructions. To answer this question, one needs to understand first the dimension two case, in particular those Fano varieties appear in the moduli spaces of K-semistable Del Pezzo surfaces constructed [19]. For smooth Kähler-Einstein Fano manifolds, by Mabuchi's extension [12, 13] of Donaldson's work [5] we know that they are all asymptotic Chow polystable provided their automorphism groups are semi-simple. Unfortunately, it seems quite difficult to extend Donaldson and Mabuchi's approach in [5, 12, 13] to singular Fano varieties, [2] and at least to the best of our knowledge so far, there is *not a single non-smooth* example of a $\mathbb{Q}$-Fano variety whose asymptotic Chow stability is *known*. In this note we want to close this gap by studying the *effective* Chow stability of some singular toric Del Pezzo surfaces. The original motivation is the following question which was asked of us by Odaka and Laza.

Question 1.1. *Is the K-polystable cubic surface* $X := \{xyz = w^3\} \subset \mathbb{P}^3$ *asymptotically Chow stable?*

To state our main result, let

---

[1]We remark that Ono, Sano and Yotsutani succeeded in constructing a seven dimensional toric Fano Kähler-Einstein manifold that is not asymptotic Chow stable in [20]. But that did not rule out the asymptotic GIT approach completely (see Remark 5.5 for more explanation).

[2] Actually, the orbifold version of Donaldson's theorem does not hold when K_X is ample [16, Corollary 3.3].

1) $(X_1 = \mathbb{P}^2/(\mathbb{Z}/9\mathbb{Z}), \mathscr{O}_{X_1}(1) := \mathscr{O}_{X_1}(-3K_{X_1})) \subset (\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(1))$ with the $\mathbb{Z}/9\mathbb{Z} = \langle \xi = \exp 2\pi\sqrt{-1}/9\rangle$-action generated by $\xi \cdot [z_0, z_1, z_2] = [z_0, \xi \cdot z_1, \xi^{-1} \cdot z_2]$.
2) $(X_2 = (\mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}/4\mathbb{Z}), \mathscr{O}_{X_2}(1) := \mathscr{O}_{X_2}(-2K_{X_2})) \subset (\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(1))$, $\mathbb{Z}/4\mathbb{Z} = \langle \xi \rangle$-action generated by

$$\xi \cdot ([z_1, z_2], [w_1, w_2]) = ([\sqrt{-1}z_1, z_2], [-\sqrt{-1}w_1, w_2]).$$

3) $(X_3 = \{xyz = w^3\}, \mathscr{O}_{X_3}(1) := \mathscr{O}_{X_3}(-K_{X_3})) \subset (\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$;
4) $(X_4 = Q_1 \cap Q_2, \mathscr{O}_{X_4}(1) := \mathscr{O}_{X_4}(-K_{X_4})) \subset (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1))$ with

$$\begin{cases} Q_1 : z_0 z_1 + z_2 z_3 + z_4^2 = 0 \\ Q_2 : \lambda z_0 z_1 + \mu z_2 z_3 + z_4^2 = 0 \quad \lambda \neq \mu, \end{cases}$$

It is known that these are the *only* $\mathbb{Q}$-Gorenstein smoothable toric Kähler-Einstein (i.e. K-polystable) Del Pezzo surfaces of deg $= 1, 2, 3, 4$ thanks to the work of [26, Theorem 2.3.3]. In particular, they are parametrized in the proper moduli spaces constructed in [19, Theorem 4.1, 4.2, 4.3, 5.13, 5.28]. Then our main result is the following:

Theorem 1.2. *Let $(X_i, \mathscr{O}_{X_i}(k))$ be on in the list above. Then $(X_i, \mathscr{O}_{X_i}(k))$ is*

(1) *Chow unstable for any $k \geq 1$ when $i = 1$;*

(2) *Chow polystable for $k \geq 2$ when $i = 2$;*

(3) *Chow polystable for $k \geq 1$ when $i = 3, 4$.*

Finally, we want to remark that in general K-stability is better behaved under finite quotients in contrast to Chow stability, as the dependence of k for the normalization of $\mathrm{GL}(N_k + 1)$ to $\mathrm{SL}(N_k + 1)$ is hard to keep track for the projective embedding $X \hookrightarrow \mathbb{P}^{N_k}$ induced by $H^0(\mathscr{O}_X(k))$. That partially explains the reason why the Chow stability does not follow from the K-stability directly even for global finite quotients. In particular, in [16, Corollary 3.3] Odaka constructed examples of canonically polarized surfaces with only orbifold singularities that are *not* asymptotic Chow stable. In the Fano case, the orbifold version of Mabuchi's extension [12, 13] of Donaldson's work also fails in general. For example, we have the following (cf. Section 4):

Example 1.3 (Example 4.2). *Fix an* $3 \leq a \in \mathbb{N}$*, let*

$$(Y = (\mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}/2a\mathbb{Z}), \mathscr{O}_Y(1) = \mathscr{O}_Y(-aK_Y))$$

with the $\mathbb{Z}/2a\mathbb{Z} = \langle \xi_{2a} = \exp \pi\sqrt{-1}/a \rangle$*-action given by*

$$\xi_{2a} \cdot ([z_1, z_2], [w_1, w_2]) = ([\xi_{2a} z_1, z_2], [\xi_{2a}^{-1} w_1, w_2]).$$

Then Y *is an asymptotic Chow* unstable *Fano toric Kähler-Einstein surface with vanishing Chow invariants* (cf. [12, Definition 2.4]).

Our paper is organized as follows: in section two we review some basic facts of GIT, in particular, we reduce the checking of stability to a purely combinatorial problem thanks to the fact that the X_i are toric. In section three, we will carry out the main estimate that is needed for the proof of the last case of Theorem 1.2. In section four we extend the main estimate used in section three and prove the second cases of Theorem 1.2. It turns out this is the most delicate calculation. In the last section, we establish the first case by showing the non-vanishing of the Chow weight of the torus action. We note that examples of asymptotic Chow unstable Fano toric Kähler-Einstein manifolds were first found in [20].

2. Basics on GIT and symplectic quotient

In this section we include a symplectic quotient proof of Kempf's instability result [9, Corollary 4.5], which reduces checking Chow stability for projective varieties to a smaller group provided the variety admits a large symmetry group.

2.1. Kempf's instability theory

Let G be a reductive algebraic group acting on a polarized pair $(Z, \mathscr{O}_Z(1))$, i.e. $\mathscr{O}_Z(1)$ is G-linearzed. If we fix a *maximal compact subgroup* $K < G$ together with a K-invariant Hermitian metric with a positive curvature form ω on L and K-invariant inner product on $\mathfrak{k} := \mathrm{Lie}(K)$, then we obtain a holomorphic Hamiltonian K-action on (Z, ω) with moment map

$$\mu_K : Z \longrightarrow \mathfrak{k}.$$

Let $z \in Z$ be a point with stabilizer $G_z < G$.

Definition 2.1. We say a G-orbit $G \cdot z \in Z$ is *G-extremal* with respect to the G-action on $(Z, \mathscr{O}_Z(1))$ if and only if there is a maximal compact subgroup $K < G$ together with a K-invariant ω such that the resulting moment map $\mu_K(z)$ satisfies: $\mu_K(z) \in \mathfrak{k}_z$, the stabilizer of z in $\mathfrak{k}$. [3] We say z is *G-polystable* if there is a maximal compact subgroup $K < G$ such that $\mu_{\mathfrak{k}}(z) = 0$.

Now we give a simple and symplectic quotient proof of a slight improvement of Kempf's Instability Theorem [9, Corollary 4.5].

Theorem 2.2. *Fix $z \in Z$, let $G_0 < G_z$ be a* reductive *subgroup. Then $G \cdot z$ is an G-*extremal (resp. poly-stable) *orbit if and only if $C(G_0) \cdot z$ is $C(G_0)$-*extremal (resp. polystable) *with respect to the $C(G_0)$-action, induced by the embedding $i : C(G_0) \hookrightarrow G$ on $(Z, \mathscr{O}_Z(1))$, where $C(G_0) < G$ is the* centralizer *of G_0 in G.*

Proof. Let us fix a maximal compact subgroup $K < G$ such that $(K_0)^{\mathbb{C}} = G_0$ with $K_0 := K \cap G_0$. We define

$$K_H := C(K_0) = \{g \in K \mid \mathrm{Ad}_g h = h, \forall h \in K_0\} < K,$$

the *centralizer* of K_0 in K and $H := K_H^{\mathbb{C}}$. Suppose $H \cdot z$ is H-extremal (resp. polystable). Then there exists $h \in H$ such that

$$\mu_{K_H}(h \cdot z) = i^* \mu_K(h \cdot z) \in \mathfrak{k}_H \cap \mathfrak{k}_{h \cdot z}, (\textit{resp.} = 0 \text{ if } z \text{ is } H\text{-polystable})$$

where $i^* : \mathfrak{k} \to \mathfrak{k}_H$ be the *orthogonal projection* onto $\mathfrak{k}_H$ with respect to a Ad_K-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$.

Since $h \in H = C(G_0)$, we have $\mathrm{Ad}_h G_0 = G_0 < G_{h \cdot z}$. [4] Without loss of generality we may assume that $h = e$, the identity (i.e. replace $h \cdot z$ by z from the beginning). Then

$$(1) \qquad i^* \mu_K(z) \in \mathfrak{k}_H \cap \mathfrak{k}_z, (\textit{resp.}\ \mu_K(z) \perp \mathfrak{k}_H \text{ if } z \text{ is } H\text{-polystable}),$$

where $\mathfrak{k}_H := \mathrm{Lie}(K_H)$. On the other hand, for any $k \in K_0 < G_0 < G_z$ we have

$$\mu_K(z) = \mu_K(k \cdot z) = \mathrm{Ad}_k \mu_K(z),$$

---

[3]It is not hard to see that like the G-polystability, $G \cdot z$ being G-extremal is a purely algebraic property.

[4]Notice that this in particular implies that $H = C(G_0)$, which is a priori defined via z, is indeed independent of the choice of representative $z' \in H \cdot z$.

from which we deduce that $\mu_K(z) \in \mathfrak{c}(K_0) = \mathfrak{k}_H$. This combined with (1) implies that

$$\mu_K(z) \in \mathfrak{k}_z, (\textit{resp.} = 0 \text{ if } z \text{ is } H\text{-polystable})$$

i.e. z is G-extremal (*resp.* G-polystable).

Conversely, suppose $G \cdot z$ is extremal then there is a maximal $K < G$ together with a K-invariant metric ω such that $\|\mu_K(z)\| = \min_{G\cdot z} \|\mu_K\|$ by [15, Theorem 6.2] and $\mu_K(z) \in \mathfrak{c}(K_z) \subset \mathfrak{k}_H$ by [30, Theorem 10], where $K_z := G_z \cap K$ and $\mathfrak{c}(K_z)$ is the Lie algebra of the centralizer of $C(K_z) < K$. These imply that

$$\|\mu_K(z)\| = \min_{G\cdot z} \|\mu_K\| = \min_{H\cdot z} \|\mu_{K_H}\| = \|\mu_{K_H}(z)\|,$$

that is $C(G_0) \cdot z$ is extremal, and hence our proof is completed. □

Corollary 2.3. *We have $z \in Z$ is G-semistable if and only if z is $C(G_0)$-semistable.*

Proof. By our assumption z is H-semistable with $H = C(G_0)$, so there is a

$$z_0 \in \overline{H \cdot z} \subset \overline{G \cdot z} \subset Z$$

such that z_0 is H-polystable. By Theorem 2.2, we know z_0 is G-polystable which completes the proof. □

2.2. Toric varieties

Let $\triangle \subset \mathbb{R}^n$ be a *convex polytope* and we define the *cone* $\mathrm{PL}(\triangle; k)$ in $C^0(k\triangle, \mathbb{R})$, the space of continuous functions on $k\triangle$ as follows. Let $\phi : k\triangle \cap \mathbb{Z}^n \to \mathbb{R}$ be any non-negative function and define:

$$\mathrm{graph}_\phi := \mathrm{Conv}\left\{ \bigcup_{x \in k\triangle \cap \mathbb{Z}^n} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} | \, t \le \phi(x)\} \right\}$$

the *convex hull* of the set $\bigcup_{x\in k\triangle\cap\mathbb{Z}^n} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} | \, t \le \phi(x)\}$.

Definition 2.4. Let $\triangle \subset \mathbb{R}^n$ be any convex polytope. We define

1) A function $C^0(k\triangle,\mathbb{R}) \ni f_\phi : k\triangle \to \mathbb{R}$ is said to be *associated* to a $\phi : k\triangle \cap \mathbb{Z}^n \to \mathbb{R}$ if

$$f_\phi(x) := \max\{t \mid (x,t) \in \mathrm{graph}_\phi\} : k\triangle \longrightarrow \mathbb{R}.$$

2) We define the *cone* $\mathrm{PL}(\triangle; k)$ as follows.

$$\mathrm{PL}(\triangle; k) := \{f_\phi \mid \phi : k\triangle \cap \mathbb{Z}^n \to \mathbb{R}\} \subset C^0(k\triangle,\mathbb{R}). \tag{2}$$

Now to apply Theorem 2.2 to our situation, let $(X_\triangle, L_\triangle)$ be any polarized toric variety (*not necessarily smooth*) with moment polytope $\triangle$. Let $\mathrm{Aut}(X_\triangle)$ denote the *automorphism* of the pair $(X_\triangle, L_\triangle)$. Then $T = (\mathbb{C}^\times)^n < \mathrm{Aut}(X_\triangle)$ is a maximal torus.

Definition 2.5. Let $(X_\triangle, L_\triangle)$ be a polarized toric variety with moment polytope $\triangle$. We define the *Weyl group* $W_\triangle := N(T)/T$ with

$$T = (\mathbb{C}^\times)^n < N(T) := \{g \in \mathrm{Aut}(X_\triangle, L_\triangle) \mid g \cdot T \cdot g^{-1} = T\} < \mathrm{Aut}(X_\triangle)$$

the *normalizer* of $T < \mathrm{Aut}(X_\triangle)$. Clearly, $W_\triangle$ acts on $\triangle \subset \mathbb{R}^n \cong \mathfrak{t}$ via the adjoint action.

Consider a projective embedding

$$(X_\triangle, L^k_\triangle) \longrightarrow (\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$$

with

$$N + 1 = \chi_\triangle(k) = \dim H^0(X_\triangle, L^k_\triangle) = |k\triangle \cap \mathbb{Z}^n| \text{ and } \deg X_\triangle = d.$$

Let $\mathbb{P}^{d,n;N} := \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^{N+1})^{\otimes(n+1)}$ and

$$\begin{aligned}\mathrm{Chow}_k(X_\triangle) := \{\,(H_0,\cdots,H_n) \in ((\mathbb{P}^N)^\vee)^{n+1} \mid \\ H_0 \cap \cdots \cap H_n \cap X \neq \varnothing\} \in \mathbb{P}^{d,n;N}\end{aligned}$$

denote the *k-th Chow form* associated to the embedding above. The following result is essentially due to H. Ono [17].

Theorem 2.6 (Theorem 1.1, [17]). *Let $(X_\triangle, L_\triangle)$ be a polarized toric variety (not necessarily smooth) with moment polytope $\triangle \subset \mathbb{R}^n$. For a fixed positive integer k,* $\mathrm{Chow}_k(X_\triangle)$ *of* $(X_\triangle, L^k_\triangle) \subset (\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$ *is* polystable *with respect to the action of the* subgroup of diagonal matrices *in* $\mathrm{SL}(N+1)$ *if and only if*

$$\frac{1}{\mathrm{vol}(k\triangle)}\int_{k\triangle} g - \frac{1}{\chi_\triangle(k)}\sum_{x\in\triangle\cap\mathbb{Z}^n} g(x) \geq 0, \tag{3}$$

for any $g \in \mathrm{PL}(\triangle;k)$ *with equality if and only if* g *is affine.*

Now let $(Z, \mathscr{O}_Z(1)) = (\mathbb{P}^{d,n;N}, \mathscr{O}_{\mathbb{P}^{d,n;N}}(1))$, $G = \mathrm{SL}(N+1)$ and $G_0 = N(T) < G_{\mathrm{Chow}_k(X_\triangle)} = \mathrm{Aut}(X_\triangle)$. Then the centralizer $C(G_0) < \mathrm{SL}(N+1)$ is contained in a *maximal torus* (e.g. the subgroup of diagonal matrices) of $\mathrm{SL}(N+1)$. In particular, Theorem 2.6 together with Theorem 2.2 then imply the following

Corollary 2.7. *Let* $(X_\triangle, L_\triangle)$ *be a polarized toric variety with moment polytope* $\triangle$ *as above and* $W = W_\triangle$ *be the Weyl group. Then for any* $k \in \mathbb{N}$, $(X_\triangle, L_\triangle^k)$ *is Chow polystable (i.e.* $\mathrm{Chow}_k(X_\triangle) \in \mathbb{P}^{d,n;N}$ *is GIT polystable with respect to the* $\mathrm{SL}(N+1)$*-action on* $(\mathbb{P}^{d,n;N}, \mathscr{O}_{\mathbb{P}^{d,n;N}}(1))$*) if and only if* (3) *holds for any*

$$g \in \mathrm{PL}(\triangle;k)^W = \{g \in \mathrm{PL}(\triangle;k) \mid g(w\cdot x) = g(x),\ \forall w \in W\},$$

with equality if and only if g *is affine.*

Theorem 2.6 was originally proved in [17] for *integral* Delzant polytope by applying the machinery developed by Gelfand-Kapranov-Zelevinsky in [8]. Here for reader's convenience, we give a slightly simpler and more direct proof.

Proof. of Theorem 2.6. Without loss of generality, we may assume $L_\triangle$ is very ample and $k = 1$. Also since the left hand side of (3) is invariant under adding a constants, we may assume $g \geq 0$.

Let $(\mathcal{X}, \mathscr{L}) \to \mathbb{P}^1$ be any T-equivariant test configuration of $(X_\triangle, L_\triangle)$. In particular, $\mathcal{X}$ is a $n+1$-dimensional toric variety. Let

$$\triangle_g := \{(x,y) \in (\triangle\cap\mathbb{Z}^n)\times\mathbb{R}_{\geq 0} \mid 0 \leq y \leq g(x)\} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$$

be the moment polytope of $\mathcal{X}$, where g is a non-negative rational piecewise-linear concave function defined over $\triangle$. Then we have

$$\mathrm{vol}(\triangle_g) = \int_\triangle g(x)dx \text{ and } \chi_{\triangle_g}(1) - \chi_\triangle(1) = \sum_{x\in\triangle\cap\mathbb{Z}^n} g(x). \tag{4}$$

By the proof of [6, Proposition 4.2.1], we know the *weight* of the $\mathbb{C}^\times$-action on $\wedge^{\chi_\triangle(m)} H^0(\mathcal{X}_0, \mathscr{L}^m|_{\mathcal{X}_0})$ is given by

$$w_m = \chi_{\triangle_g}(m) - \chi_\triangle(m) \tag{5}$$

with asymptotic expansions (cf. [6, Propostion 4.1.3 and equation (4.2.2)])

$$\begin{aligned} &\chi_\triangle(m) = m^n \mathrm{vol}(\triangle) + O(m^{n-1}) \quad \text{and} \\ &\chi_{\triangle_g}(m) = m^{n+1}\mathrm{vol}(\triangle_g) + O(m^n). \end{aligned} \tag{6}$$

On the other hand, the *Chow weight* for the degeneration $(\mathcal{X}, \mathscr{L}) \to \mathbb{P}^1$ is given by the *normalized leading coefficient (n.l.c)* of the top degree term $\frac{m^{n+1}}{(n+1)!}$ in the degree $n+1$ polynomial of m:

$$w_m - m\chi_\triangle(m)\frac{w_1}{\chi_\triangle(1)},$$

where the second term is added in order to *normalize* the $\mathbb{C}^\times$-action on $H^0(\mathcal{X}_0, \mathscr{L}|_{\mathcal{X}_0})$ to be *special linear* (cf.[25, Theorem 3.9 and equation (3.8)]). Then by (5) we obtain

$$\begin{aligned} &w_m - m\chi_\triangle(m)\frac{w_1}{\chi_\triangle(1)} \\ &= \chi_{\triangle_g}(m) - \chi_\triangle(m) - m\chi_\triangle(m)\frac{\chi_{\triangle_g}(1) - \chi_\triangle(1)}{\chi_\triangle(1)} \\ &= m^{n+1}\mathrm{vol}(\triangle_g) - m^{n+1}\mathrm{vol}(\triangle)\frac{\chi_{\triangle_g}(1) - \chi_\triangle(1)}{\chi_\triangle(1)} + O(m^n) \\ &= m^{n+1}\mathrm{vol}(\triangle)\left(\frac{1}{\mathrm{vol}(\triangle)}\int_\triangle g - \frac{1}{\chi_\triangle(1)}\sum_{x\in\triangle\cap\mathbb{Z}^n} g(x)\right) + O(m^n) \end{aligned}$$

where for the last identity we have used (4). Hence the Chow weight for the T-equivariant test configuration $(\mathcal{X}, \mathscr{L}) \to \mathbb{P}^1$ is precisely

$$(n+1)!\mathrm{vol}(\triangle)\left(\frac{1}{\mathrm{vol}(\triangle)}\int_\triangle g - \frac{1}{\chi_\triangle(1)}\sum_{x\in\triangle\cap\mathbb{Z}^n} g(x)\right),$$

which completes the proof.

Finally, to see the equality case we observe that

$$\left(\frac{1}{\mathrm{vol}(\triangle)}\int_{\triangle} x - \frac{1}{\chi_{\triangle}(1)}\sum_{x\in\triangle\cap\mathbb{Z}^n} x\right) \in \mathbb{R}^n,$$

is precisely the Futaki character on the $\mathrm{Lie}(\mathrm{U}(1)^n) = \mathbb{R}^n$. □

Corollary 2.8 (Corollary 4.7, [17]). *If $(X_{\triangle}, L_{\triangle}^k)$ is Chow semistable for $k \in \mathbb{N}$ then we have the following identity of vectors in $\mathbb{R}^n$.*

$$\frac{1}{\chi_{\triangle}(k)}\sum_{x\in k\triangle\cap\mathbb{Z}^n} x = \frac{1}{\mathrm{vol}(k\triangle)}\int_{k\triangle} x dx. \tag{7}$$

Remark 2.9. *The identity (7) is equivalent to the vanishing of Chow weight for the group $T = (\mathbb{C}^\times)^n < \mathrm{Aut}(X_{\triangle})$. In particular, (7) implies that the left hand side of (3) is invariant under addition of an* affine function *to g.*

Example 2.10. *Let $(X_{\triangle}, L_{\triangle}) = (\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$ then*

$$\begin{aligned}&\frac{1}{\mathrm{vol}([0,k])}\int_0^k g - \frac{1}{\chi_{\triangle}(k)}\sum_{x\in k\triangle\cap\mathbb{Z}} g(x)\\ &= \frac{1}{k}\int_0^k g - \frac{1}{k+1}\sum_{i=0}^{k} g(i) \geq 0, \ \forall g \ \textit{concave}\end{aligned}$$

follows from the fact that for all $g \geq 0$ we have

$$\begin{aligned}&\frac{1}{2}g(0) + g(1) + \cdots + g(k-1) + \frac{1}{2}g(k)\\ &\geq \frac{k}{k+1}\left(g(0) + g(1) + \cdots + g(k-1) + g(k)\right)\end{aligned} \tag{8}$$

3. X_3 and X_4

In this section, we will treat $X_{\triangle_3}$ and $X_{\triangle_4}$ simultaneously since both $\triangle_i, i = 3, 4$ allows a decomposition of $\triangle_i$ with the *same* fundamental domain $\triangle_0$ (cf. Figure 1). Let

1) $(X_{\triangle_3}, L_{\triangle_3}) = (X_3, \mathscr{O}_{X_3}(-K_{X_3})) = \{xyz = w^3\} \subset (\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$.

2) $(X_{\triangle_4}, L_{\triangle_4}) = (X_4, \mathscr{O}_{X_4}(-K_{X_4})) = Q_1 \cap Q_2 \subset (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1))$ with

$$\begin{cases} Q_1 : z_0 z_1 + z_2 z_3 + z_4^2 = 0 \\ Q_2 : \lambda z_0 z_1 + \mu z_2 z_3 + z_4^2 = 0 \quad \lambda \neq \mu \end{cases} \tag{9}$$

with moment polytope $\triangle_i, i = 3, 4$ given in Figure 1.

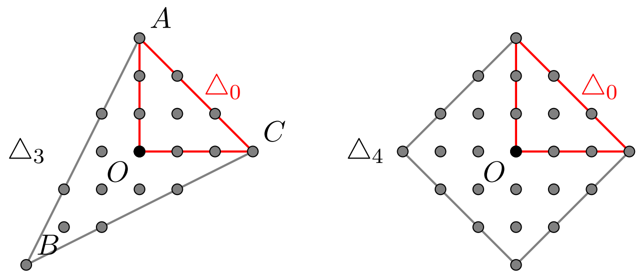

Figure 1: $\triangle_0 \subset \triangle_3$ and $\triangle_0 \subset \triangle_4$.

Notice both $\triangle_i, i = 3, 4$ are invariant under the action of Weyl group $W_i := W_{\triangle_i}, i = 3, 4$ respectively, where

$$W_3 = D_3 = \left\langle \sigma_3 := \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \quad \text{and}$$
$$W_4 = D_4 = \left\langle \sigma_4 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle < \mathrm{GL}(2, \mathbb{Z}).$$

To prove Theorem 1.2, first we establish the necessary condition (7), which is a consequence of the following

Lemma 3.1. *Let μ be any measure defined on $\triangle$ and $\sigma \in \mathrm{SL}(2, \mathbb{R})$ be a element of* order *d satisfying*

1) $\sigma(\triangle) = \triangle$;

2) $\sigma^* d\mu = d\mu$.

Suppose further that $\triangle$ admits a decomposition $\triangle = \bigcup_{i=0}^{d-1} \sigma^k(\triangle_0)$ such that $\sigma^i(\triangle_0^\circ) \cap \sigma^j(\triangle_0^\circ) = \emptyset$ for $i \neq j$, where $\triangle_0^\circ$ denotes the interior of a closed subset $\triangle_0 \subset \triangle$. Then

$$\int_\triangle x d\mu(x) = 0.$$

Proof. By our assumption that $\sigma \in \mathrm{SL}(2,\mathbb{R})$ of order d, we have

$$\sum_{i=0}^{d-1} \sigma^k = 0 \in \mathrm{SL}(2,\mathbb{R}).$$

Hence

$$\begin{aligned}\int_{\triangle} x d\mu(x) &= \sum_{i=0}^{d-1} \int_{\sigma^k(\triangle_0)} x d\mu(x) = \sum_{i=0}^{d-1} \int_{\triangle_0} (x \circ \sigma^k) \cdot (\sigma^k)^* d\mu(x) \\ &= \sum_{i=0}^{d-1} \int_{\triangle_0} (x \circ \sigma^k) \cdot d\mu(x) = \int_{\triangle_0} x \circ (\sum_{i=1}^{d-1} \sigma^k) d\mu(x) = 0\end{aligned}$$

and our proof is completed. □

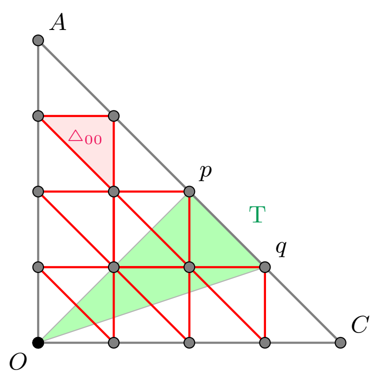

Figure 2: Barycenter division of $k\triangle_0$ and T.

By adding an affine function to g if necessary, Lemma 3.1 and Corollary 2.8 imply that in order to prove case (3) of Theorem 1.2, it suffices to establish the inequality (3) for all g satisfying the following additional property.

Assumption 3.2. *Let $g \in \mathrm{PL}(\triangle_i; k)^{W_i}, i = 3, 4$ satisfying:*

1) $g(0) = \max_{x \in \triangle_i} g(x)$;

2) g vanishes on the vertices of $\triangle_i$.

To achieve this, we will establish the following two key estimates:

- *Trapezoid estimate for* $T = \mathrm{Conv}(O, p, q) \subset \mathbb{R}^2$, the convex hull of (O, p, q).

$$\frac{1}{\mathrm{vol}(T)} \int_T g \geq \frac{g(0) + g(p) + g(q)}{3} \tag{10}$$

with equality if and only if g is affine.

- *Trapezoid estimate for the standard subdivision.*

$$\begin{aligned}\int_{k\triangle} g &\geq \frac{\mathrm{vol}(\triangle_{00})}{3}\left(6 \sum_{x\in(k\triangle_i)^\circ\cap\mathbb{Z}^2} g(x) + 3 \sum_{x\in(k\partial\triangle_i)\cap\mathbb{Z}^2} g(x) - 6\alpha g(0)\right)\\ &= \sum_{x\in(k\triangle_i)^\circ\cap\mathbb{Z}^2} g(x) + \frac{1}{2} \sum_{x\in(k\partial\triangle_i)\cap\mathbb{Z}^2} g(x) - \alpha g(0)\\ &= \sum_{x\in(k\triangle_i)\cap\mathbb{Z}^2} g(x) - \frac{1}{2} \sum_{x\in(k\partial\triangle_i)\cap\mathbb{Z}^2} g(x) - \alpha g(0) \qquad (11)\end{aligned}$$

with equaliy if and only if g is affine, where $\mathrm{vol}(\triangle_{00}) = \frac{1}{2}$ (cf. Figure 2)and $\alpha = \frac{6-\mathrm{ord}(\sigma_i)}{6}, i = 3, 4$.

Proof of Theorem 1.2. To simplify our notation, in the rest of the proof we will use $\triangle$ to denote $\triangle_i,\ i = 3, 4$.

Let us assume the validity of (10) and (11) for the moment and our goal is to prove

$$\frac{1}{\mathrm{vol}(k\triangle)} \int_{k\triangle} g \geq \frac{1}{\chi_\triangle(k)} \sum_{x\in(k\triangle)\cap\mathbb{Z}^n} g(x) \tag{12}$$

for g satisfying Assumption 3.2. By applying the Pick's formula (cf. [21] and [24])

$$\chi_\triangle(k) = |(k\triangle) \cap \mathbb{Z}^2| = \mathrm{vol}(k\triangle) + \frac{b}{2} + 1 \text{ with } b = |(k\partial\triangle) \cap \mathbb{Z}^n| \tag{13}$$

the left hand side of (12) can be written as

$$\left(\frac{1}{\mathrm{vol}(k\triangle)}-\frac{1}{\chi_\triangle(k)}\right)\int_{k\triangle} g+\frac{1}{\chi_\triangle(k)}\int_{k\triangle} g$$

$$\begin{aligned}
(\text{by (11) and (13)}) \geq{}& \frac{\frac{b}{2}+1}{\mathrm{vol}(k\triangle)\cdot\chi_\triangle(k)}\int_{k\triangle} g \\
&+\frac{1}{\chi_\triangle(k)}\left(\sum_{x\in k\triangle^\circ\cap\mathbb{Z}^n} g(x)+\frac{1}{2}\sum_{x\in(\partial k\triangle)\cap\mathbb{Z}^n} g(x)-\alpha g(0)\right) \\
(\text{ by (10)}) \geq{}& \frac{\frac{b}{2}+1}{b\cdot\mathrm{vol}(T)\cdot\chi_\triangle(k)}\cdot\frac{\mathrm{vol}(T)}{3}\left(2\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)+bg(0)\right) \\
&+\frac{1}{\chi_\triangle(k)}\left(\sum_{x\in k\triangle^\circ\cap\mathbb{Z}^n} g(x)+\frac{1}{2}\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)-\alpha g(0)\right) \\
\geq{}& \frac{\frac{b}{2}+1}{3b\cdot\chi_\triangle(k)}\left(2\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)+bg(0)\right) \\
&+\frac{1}{\chi_\triangle(k)}\left(\sum_{x\in k\triangle^\circ\cap\mathbb{Z}^n} g(x)+\frac{1}{2}\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)-\alpha g(0)\right)
\end{aligned}$$

So to prove (12), all we need is

$$\frac{1+b/2}{3b}\left(2\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)+bg(0)\right)\geq\left(\frac{1}{2}\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)+\alpha g(0)\right) \tag{14}$$

which is equivalent to

$$\left(\frac{1+b/2}{3}-\alpha\right)g(0)\geq\left(\frac{1}{2}-\frac{1+b/2}{3b}\cdot 2\right)\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x).$$

Using the fact $g(0)=\max_{x\in\triangle} g(x)\geq\frac{1}{b}\sum_{x\in(k\partial\triangle)\cap\mathbb{Z}^n} g(x)$ and the fact that the coefficient on the right side of above inequality is non-negative , we see that (12) is a consequence of the following:

$$\frac{1}{b}\geq\frac{\frac{1}{2}-\frac{2+b}{3b}}{\frac{2+b}{6}-\alpha} = \frac{b-4}{b^2+(2-6\alpha)b}$$

which is equivalent to $4 \geq 6\alpha - 2$. But this holds as long as $\alpha = \frac{6-\mathrm{ord}(\sigma_i)}{6} \leq 1$ for $i = 1, 2$. This completes the proof of Theorem 1.2 for X_3 and X_4. □

Proof of (10) *and* (11). Inequality (10) follows from the concavity of g and trapezoidal rule. For (11), we triangulate $\triangle_0$ into the union of *basic triangles* $\triangle_{00}$'s as illustrated in Figure 2 and then extend this triangulation to the whole $\triangle_i$ via the Weyl group W_i. Then (11) follows by noticing that

1) each interior lattice points of $\triangle_i^\circ$ that is *not* the point O is the vertex of exactly 6 basic triangles of $\triangle_{00}$;
2) each boundary lattice point of $(\partial\triangle)^\circ$ is is exactly a vertex of 3 basic triangles of $\triangle_{00}$;
3) the point O is the vertex of $\mathrm{ord}(\sigma_i), i = 1, 2$ basic triangles of $\triangle_{00}$ (cf. Figure 2).

Our proof of (10) and (11) is thus completed. □

We remark that the proof above actually implies the following (in fact easier) consequence:

Corollary 3.3. *Let* $(X_6 = \mathrm{Bl}_{[1,0,0],[0,1,0],[0,0,1]}\mathbb{P}^2, \mathscr{O}_{X_6}(1) = \mathscr{O}_{X_6}(-K_X))$*, then we have* $(X_6, \mathscr{O}_{X_6}(k))$ *is Chow polystable for all* $k \geq 1$.

Proof. This follows from the fact that $\mathrm{ord}(\sigma) = 6$, hence $\alpha = \frac{6-\mathrm{ord}(\sigma)}{6} = \frac{6-6}{6} = 0 < 1$. The details will be left to the readers. □

Remark 3.4. *Corollary 3.3 follows from the work of Donaldson and Mabuchi [5, 12, 13] for* $k \gg 1$*, but our method gives a sharper result.*

4. X_2

Recall that

$$(X_{\triangle_2}, L_{\triangle_2}) = (X_2 = \mathbb{P}^1 \times \mathbb{P}^1/(\mathbb{Z}/4\mathbb{Z}), \mathscr{O}_{X_2}(-2K_{X_2})) \subset (\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(1))$$

with $\mathbb{Z}/4\mathbb{Z} = \langle\xi\rangle$-action given by

$$\xi \cdot ([z_1, z_2], [w_1, w_2]) = ([\sqrt{-1}z_1, z_2], [-\sqrt{-1}w_1, w_2]).$$

Then the Weyl group $W_2 = W_{\triangle_2} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and

$$\triangle_2 = \mathrm{Conv}\{(-2,0), (2,0), (0,1), (0,-1)\} \text{ (cf. Figure 3)}.$$

It turns out this is the *trickest* case among the $\{X_i\}_{1\leq i\leq 4}$.

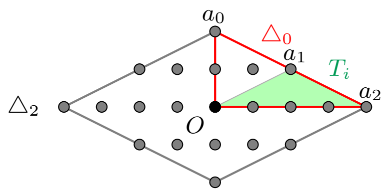

Figure 3: $\triangle_0 \subset k\triangle_2$ with $k = 2$.

For this purpose, we need to extend the main estimate (11) (cf. (16)) used in the last section. Let

$$a_i := (2i, k - i) \in \mathbb{R}^2 \text{ for } 0 \leq i \leq k,$$

denote the *integral points* of the boundary of $k\triangle_0 \subset k\triangle_2$ (cf. Figure 3) and let

$$T_i := \mathrm{Conv}(0, a_i, a_{i+1}), 0 \leq i \leq k-1 \text{ and } b = |\partial(k\triangle_2) \cap \mathbb{Z}^2| \tag{15}$$

Then $\triangle_0 = \bigcup_{i=0}^{k-1} T_i$ and we have the following:

Lemma 4.1. *Let $\delta_k : k\Delta_2 \to \mathbb{R}$ satisfy*

$$\sum_{p\in k\triangle_2} \delta_k(p) = 1$$

Suppose

$$\sum_{p\in(k\triangle_2)\cap\mathbb{Z}^2} g(p) - \int_{k\triangle_2} g(x)dV \leq \frac{1}{2} \sum_{p\in\partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \sum_{p\in k\triangle_2} \delta_k(p)g(p) \tag{16}$$

for all $g \in \mathrm{PL}(\triangle_2; k)^{W_2}$ with equality holding if and only if g is constant. Suppose as well that

$$\frac{(b+2)|W_2|}{2b\cdot \mathrm{vol}(T)} \sum_i \int_{T_i} g \geq \frac{1}{2} \sum_{p\in\partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \sum_{p\in k\triangle_2} \delta_k(p)g(p), \tag{17}$$

for all $g \in \mathrm{PL}(\triangle_2; k)^{W_2}$ *with equality holding if and only if* g *is constant. Then*

$$\frac{1}{\mathrm{vol}(k\triangle_2)} \int_{k\triangle_2} g(x)dV \geq \frac{1}{|(k\triangle_2) \cap \mathbb{Z}^2|} \sum_{p \in (k\triangle_2)\cap\mathbb{Z}^2} g(p), \tag{18}$$
$$|(k\triangle_2) \cap \mathbb{Z}^2| = \chi_{\triangle_2}(k)$$

for all $g \in \mathrm{PL}(\triangle_2; k)^{W_2}$ *with equality holding if and only if* g *is constant.*

Proof. By (16), we deduce that (18) follows from

$$\left(\frac{1}{\mathrm{vol}(k\triangle_2)} - \frac{1}{|(k\triangle_2) \cap \mathbb{Z}^2|}\right) \int_{k\triangle_2} g$$
$$\geq \frac{1}{|(k\triangle_2) \cap \mathbb{Z}^2|} \left(\frac{1}{2} \sum_{p \in \partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \sum_{p \in (k\triangle_2)\cap\mathbb{Z}^2} \delta_k(p) g(p) \right)$$

which is equivalent to

$$\left(\frac{|(k\triangle_2) \cap \mathbb{Z}^2|}{\mathrm{vol}(k\triangle_2)} - 1\right) \int_{k\triangle_2} g \geq \frac{1}{2} \sum_{p \in \partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \sum_{p \in (k\triangle_2)\cap\mathbb{Z}^2} \delta_k(p) g(p). \tag{19}$$

By subdividing $k\triangle_2$ into b triangles as in Figure 3, that is

$$k\triangle_2 = \bigcup_{g \in W_2} g \cdot \triangle_0 \quad \text{and} \quad \triangle_0 = \bigcup_{i=0}^{k-1} T_i$$

then we have

$$\mathrm{vol}(k\triangle_2) = b \sum_i \mathrm{vol}(T_i) = b \cdot \mathrm{vol}(T_0).$$

Using the fact $b = |\partial(k\triangle_2) \cap \mathbb{Z}^2|$ and pluging $g = 1$ into (19) we deduce

$$\left(\frac{|k\triangle_2 \cap \mathbb{Z}^2|}{b \cdot \mathrm{vol}(T)} - 1\right) b \cdot \mathrm{vol}(T) = \frac{1}{2} b + 1.$$

Hence

$$\left(\frac{|k\triangle_2 \cap \mathbb{Z}^2|}{b \cdot \mathrm{vol}(T)} - 1\right) = \frac{b+2}{2b \cdot \mathrm{vol}(T)},$$

our proof is completed by plugging this into (19). □

Proof of Case (2) *of Theorem 1.2.* Our goal is to establish the estimate (16) and (17) for an appropriately chosen δ_k in Lemma 4.1 (cf. (16)).

Step 1. Establishing (16) *for an appropriately chosen* δ_k *for any* $k \geq 2$. Using the $W_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ symmetry of $k\triangle_2$, it suffices to consider Δ_0 as in Figure 3. Now let us perform the subdivision

$$k\triangle_2 = \mathrm{Conv}\{(\pm 0, k), (\pm 2k, 0)\} = \triangle_{00} \cup \triangle_{01} \text{ (cf. Figure 4)}$$

with

$$\triangle_{00} := \mathrm{Conv}((0,k),(0,0),(k,0)) \text{ and } \triangle_{01} := \mathrm{Conv}((0,k),(k,0),(2k,0)).$$

Clearly, $\triangle_{00}$ is $\mathrm{SL}(2,\mathbb{Z})$ equivalent to $\triangle_{01}$. Next we define a triangulation

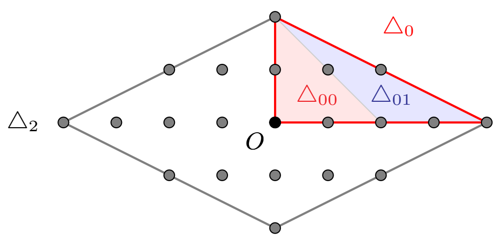

Figure 4: $\triangle_0 \subset k\triangle_2$ with $k = 2$.

of $\triangle$ by introducing a triangulation on $\triangle_0$:

- using the standard triangulation of $\triangle_{00}$ (cf. Figure 2);
- transporting the triangulation of $\triangle_{00}$ to $\triangle_{01}$ via the $\mathrm{SL}(2,\mathbb{Z})$,

Applying (11), we obtain

$$\int_{k\triangle_2} g(x) \geq \sum_{p\in(k\triangle_2)\cap\mathbb{Z}^2} g(p) - \frac{1}{2}\sum_{p\in\partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \frac{g(0,\pm k)}{6} - \frac{g(\pm 2k,0)}{6} - \frac{g(\pm k,0)}{3} - \frac{g(0,0)}{3}$$

where

1) for $g(0,\pm k)$, we have $\frac{1}{6} = \frac{4}{6} - \frac{3}{6}$ since the vertices $(0,\pm k)$ are shared by 4 triangles instead of 3 in the triangulation above.

2) for $g(\pm 2k, 0)$, we have $-\frac{1}{6} = \frac{2}{6} - \frac{3}{6}$ since the vertices $(\pm 2k, 0)$ are shared by 2 triangles instead of 3 in the triangulation above.

3) for $g(\pm k, 0)$ and $g(0, 0)$, we have $-\frac{1}{3} = \frac{4}{6} - \frac{6}{6}$ since the vertices $(\pm k, 0)$ are shared by 4 triangles instead of 6 (as they are boundary point of $\triangle_{00}$ and $\triangle_{01}$ but interior points of $\triangle_2$).

Hence

$$\begin{aligned}
&\sum_{p \in \partial k\triangle_2 \cap \mathbb{Z}^2} g(p) - \int_{k\triangle_2} g(x) \\
&\leq \frac{1}{2} \sum_{p \in \partial(k\triangle_2) \cap \mathbb{Z}^2} g(p) - \frac{1}{6} g(0, \pm k) + \frac{1}{6} g(\pm 2k, 0) + \frac{1}{3}(g(\pm k, 0)) + \frac{1}{3} g(0, 0) \\
&\leq \frac{1}{2} \sum_{p \in \partial(k\triangle_2) \cap \mathbb{Z}^2} g(p) - \frac{1}{6} g(0, \pm k) + \frac{1}{6} g(\pm 2k, 0) + g(0, 0)
\end{aligned}$$

since $g(0, 0) = \max_{k\triangle} g \geq g(\pm k, 0)$ for $g \in \mathrm{PL}(\triangle_2; k)^{W_2}$. Thus we established (16) with $\delta_k : k\triangle_2 \to \mathbb{R}$ defined by

$$\delta_k(p) = \begin{cases} 1 & p = (0, 0) \\ -1/6 & p = (0, \pm k) \\ 1/6 & p = (\pm 2k, 0) \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Step 2, establishing (17). That is, for all $g \in \mathrm{PL}(\triangle_2; k)^{W_2}$ we need to show

$$\frac{b+2}{2b \cdot \mathrm{vol}(T_0)} \sum_{g \in W_2} \sum_i \int_{g \cdot T_i} g \geq \frac{1}{2} \sum_{p \in \partial(k\triangle_2) \cap \mathbb{Z}^2} g(p) + \sum_{p \in k\triangle_2} \delta_k(p) g(p).$$

Let us first consider $T_0 = \mathrm{Conv}((0, 0), (2, k-1), (0, k))$. Applying (10), we have

$$\int_{T_0} g \geq \frac{\mathrm{vol}(T_0)}{3}(g(0) + g(2, k-1) + g(0, k)). \tag{21}$$

By the W_2-symmetry of g, we have $g(-2, k-1) = g(2, k-1)$, this together with the concavity of g imply $g(0, k-1) \geq g(2, k-1)$ and $g(0, k-1) \geq$

$g(0,k)$, so

$$g(0,k-1) \ \geq \ \frac{g(2,k-1)+g(0,k)}{2}. \tag{22}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(T_0)}\int_{T_0} g = \frac{1}{\mathrm{vol}(T_0)}\left(\int_{T_{00}} g + \int_{T_{01}} g\right) \quad \text{(cf. Figure 5)}\\
&\geq \frac{\mathrm{vol}(T_{01})}{\mathrm{vol}(T_0)}\left(\frac{g(0,0)+g(2,k-1)+g(0,k-1)}{3}\right)\\
&\quad + \frac{\mathrm{vol}(T_{00})}{\mathrm{vol}(T_0)}\left(\frac{g(0,k)+g(2,k-1)+g(0,k-1)}{3}\right)\\
&\geq \frac{k-1}{k}\left(\frac{g(0,0)+g(2,k-1)+\frac{g(2,k-1)+g(0,k)}{2}}{3}\right)\\
&\quad + \ \frac{1}{k}\left(\frac{g(0,k)+g(2,k-1)+\frac{g(2,k-1)+g(0,k)}{2}}{3}\right)\\
&= \left(\frac{k-1}{k}\right)\frac{g(0,0)}{3} + \left(\frac{1}{3}+\frac{1}{6}\right)g(2,k-1) + \left(\frac{1}{6}+\frac{1}{3k}\right)g(0,k)\\
&= \left(1-\frac{1}{k}\right)\frac{g(0,0)}{3} + \left(\frac{1}{3}+\frac{1}{6}\right)g(2,k-1) + \left(\frac{1}{3}-\frac{1}{6}+\frac{1}{3k}\right)g(0,k)\\
&= \frac{1}{3}\left(g(0,0)+g(2,k-1)+g(0,k)\right)\\
&\quad - \frac{1}{3k}g(0,0) + \frac{1}{6}g(2,k-1) + \left(-\frac{1}{6}+\frac{1}{3k}\right)g(0,k)
\end{aligned}$$

Combining the estimates with the ones for $T_i, i \neq 0$ based on (10), we obtain

$$\begin{aligned}
\frac{(b+2)|W_2|}{2b\mathrm{vol}(T_0)}\sum_{i=0}^{k-1}\int_{T_i} g \geq \frac{b+2}{2b}&\left(\frac{bg(0)+2\sum_{p\in\partial(k\Delta)} g(p)}{3}\right)\\
+ \frac{b+2}{2b}&\Bigg(-\frac{|W_2|}{3k}g(0) + \frac{1}{6}\cdot g(\pm 2, \pm(k-1))\\
&\quad + \left(\frac{2}{3k}-\frac{1}{3}\right)\left(g(0,k)+g(0,-k)\right)\Bigg)\\
=: \sum_{p\in(k\triangle_2)\cap\mathbb{Z}^2} &\eta(p)g(p)
\end{aligned}$$

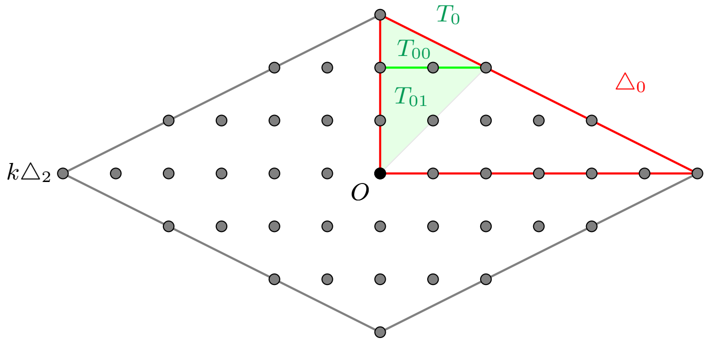

Figure 5: $\triangle_0 \subset k\triangle_2$ with $k=3$.

where $\eta : k\triangle_2 \cap \mathbb{Z}^2 \to \mathbb{R}$ is defined by the right hand side of the above inequality.

To establish (17), it suffices to show

$$\sum_{p\in k\triangle_2\cap\mathbb{Z}^2} \eta(p)g(p) \geq \frac{1}{2}\sum_{p\in\partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \sum_{p\in k\triangle_2} \delta_k(p)g(p),$$

which is equivalent to

$$(\eta(0)-\widetilde{\delta}_k(0))g(0) \geq \sum_{p\in((k\triangle_2)\cap\mathbb{Z}^2)\setminus\{0\}} (\widetilde{\delta}_k(p)-\eta(p))g(p) \tag{23}$$

with $\widetilde{\delta}_k(p)$ being defined by the following identity

$$\sum_{p\in(k\triangle_2)\cap\mathbb{Z}^2} \widetilde{\delta}_k(p)g(p) = \frac{1}{2}\sum_{p\in\partial(k\triangle_2)\cap\mathbb{Z}^2} g(p) + \sum_{p\in k\triangle_2} \delta_k(p)g(p).$$

As $b=4k$, for $p\neq(0,0)$, $\widetilde{\delta}_k(p)-\eta(p)$ is given by

$$
\begin{aligned}
&(\widetilde{\delta}_k - \eta)(p) \\
&= \begin{cases}
0 & \text{if } 0 \neq p \in k\triangle_2^\circ \\
\frac{1}{2} - \left(\frac{b+2}{2b}\right)\left(\frac{2}{3}\right) & \text{if } p = (\pm 2i, \pm(k-i)), \\
& \quad i \neq 0, 1, k \\
\left(\frac{1}{2} + \frac{1}{6}\right) - \left(\frac{b+2}{2b}\right)\left(\frac{2}{3}\right) & \text{if } p = (\pm 2k, 0) \\
\frac{1}{2} - \left(\frac{b+2}{2b}\right)\left(\frac{2}{3} + \frac{1}{6}\right) & \text{if } p = (\pm 2, \pm(k-1)) \\
\left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{b+2}{2b}\right)\left(\frac{2}{3} + \left(\frac{2}{3k} - \frac{1}{3}\right)\right) & \text{if } p = (0, \pm k).
\end{cases} \\
&= \begin{cases}
0 & \text{if } 0 \neq p \in k\triangle_2^\circ \\
\frac{1}{6} - \frac{1}{6k} & \text{if } p = (\pm 2i, \pm(k-i)), i \neq 0, 1, k \\
\frac{1}{3} - \frac{1}{6k} & \text{if } p = (\pm 2k, 0) \\
\frac{1}{12} - \frac{5}{24k} & \text{if } p = (\pm 2, \pm(k-1)) \\
\frac{1}{6} - \frac{5}{12k} - \frac{1}{6k^2} & \text{if } p = (0, \pm k).
\end{cases}
\end{aligned}
$$

which are *non-negative* when $k \geq 3$. As a consequence, we have

$$
\begin{aligned}
(\eta(0) - \widetilde{\delta}_k(0))g(0) &\geq \sum_{p \in (k\triangle_2) \cap \mathbb{Z}^2} (\widetilde{\delta}_k(p) - \eta(p))g(0) \\
&\geq \sum_{0 \neq p \in (k\triangle_2) \cap \mathbb{Z}^2} (\widetilde{\delta}_k(p) - \eta(p))g(p).
\end{aligned}
$$

with equality if and only if g is constant, and hence (17) is established. For $k \geq 3$, the proof for the case $X_2 = X_{\triangle_2}$ is completed by applying Lemma 4.1.

For $k = 2$, we observe that the estimate (22) can be improved as follows.

$$
g(0,1) \geq \lambda g(2,1) + (1-\lambda)\frac{1}{2}g(0,2) + \frac{1-\lambda}{2}g(0,0),
$$

for any $0 \leq \lambda \leq 1$. By choosing $\lambda = 2/5$ and defining the function $(\widetilde{\delta}_k - \eta)$ accordingly, one can show that it remains non-negative, from which we deduce (17) holds for $k \geq 2$. □

Example 4.2. *Let* $\triangle_a := \mathrm{Conv}\{(0,\pm 1),(\pm a,0)\}$*, then*

$$(X_{\triangle_a} = (\mathbb{P}^1\times\mathbb{P}^1)/(\mathbb{Z}/2a\mathbb{Z}), \mathscr{O}_{X_{\triangle_a}}(1) = \mathscr{O}_{X_{\triangle_a}}(-aK_{X_{\triangle_a}}))$$

with the $\mathbb{Z}/2a\mathbb{Z} = \langle \xi_{2a} = \exp \pi\sqrt{-1}/a\rangle$*-action given by*

$$\xi_{2a}\cdot([z_1,z_2],[w_1,w_2]) = ([\xi_{2a}z_1,z_2],[\xi_{2a}^{-1}w_1,w_2]).$$

Claim 4.3. $X_{\triangle_a}$ *is not Chow stable for any* $k>0$ *when* $a\geq 3$*.*

Proof. For each k, we shall exhibit a function $g_k(x)$ such that the inequality (3) is violated. To do so, let us introduce

$$f_k(x) = \begin{cases} 0 & \text{if } p\neq (0,\pm k) \\ -1 & \text{if } p=(0,\pm k)\end{cases} : (k\triangle_{2a})\cap\mathbb{Z}^2 \longrightarrow \mathbb{R}. \tag{24}$$

and let $g_k(x)\in \mathrm{PL}(\triangle_a;k)^{W_{\triangle_a}}$ be the *minimal piecewise-linear concave* function satisfying $g_k(x)\geq f_k(x)$. This implies that

$$\frac{1}{\mathrm{vol}(k\triangle_a)}\int_{k\triangle_a} g_k = \frac{1}{2ak^2}\int_{k\triangle_a} g_k = \frac{1}{2ak^2}\cdot\frac{-2a}{3} = \frac{-1}{3k^2}$$

as the volume for the simplex with bases $\mathrm{Conv}\{(0,k),(\pm a,k-1)\}$ and height 1 is $\frac{a}{3}$.

On the other hand, by Pick's formula (13), we have

$$|(k\triangle_a)\cap\mathbb{Z}^2| = \mathrm{vol}(k\triangle_a) + \frac{1}{2}|(k\partial\triangle_a)\cap\mathbb{Z}^2| + 1 = 2ak^2 + \frac{4k}{2} + 1,$$

hence

$$\frac{1}{|(k\triangle_a)\cap\mathbb{Z}^2|}\sum_{x\in(k\triangle_a)\cap\mathbb{Z}^2} g_k(x) = \frac{-2}{2ak^2+2k+1}$$

where the numerator -2 is the result of the fact that $g_k=-1$ at the vertices $(0,\pm k)\in k\triangle_{2a}$. So we obtain that for all $k\geq 0$,

$$\begin{aligned}&\frac{1}{\mathrm{vol}(k\triangle_a)}\int_{k\triangle} g_k dx - \frac{1}{|(k\triangle_a)\cap\mathbb{Z}^2|}\sum_{x\in(k\triangle_a)\cap\mathbb{Z}^2} g_k(x) \\ &= \frac{1}{ak^2+k+1/2} - \frac{1}{3k^2} < 0\end{aligned}$$

as long as $a\geq 3$. Our proof is completed by applying Theorem 3. □

Note that the Weyl group for $Y = X_{\triangle_a}$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This implies

$$0 = \frac{1}{\chi_{\triangle_a}(k)} \sum_{x \in (k\triangle_a) \cap \mathbb{Z}^n} x = \frac{1}{\mathrm{vol}(k\triangle_a)} \int_{k\triangle_a} x dx, \tag{25}$$

from which we conclude that Chow invariants are vanishing by Corollary 2.8.

Remark 4.4. *The simplest way to see $X_{\triangle_a}$ is not asymptotic Chow stable is to notice that there are two orbifold points of type $\frac{(1,1)}{2a}$ in $X_{\triangle_a}$ with multiplicity $2a > 3! = 6$ as long as $a > 3$. We also remark that $X_{\triangle_a}$ is not* smoothable *for $a \geq 3$. It would be interesting to find* smoothable *examples as in [16].*

5. X_1

Recall $(X_{\triangle_1}, L_{\triangle_1}) = (X_1 = \mathbb{P}^2/(\mathbb{Z}/9\mathbb{Z}), \mathscr{O}_{X_1}(-3K_{X_1})) \subset (\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(1))$ with the $\mathbb{Z}/9\mathbb{Z} = \langle \xi = \exp 2\pi\sqrt{-1}/9 \rangle$-action generated by

$$\xi \cdot [z_0, z_1, z_2] = [z_0, \xi, z_1, \xi^{-1}, z_2].$$

Then the Weyl group of X_1 is $W_1 = \mathbb{Z}/2$ and

$$\triangle_1 = \mathrm{Conv}\{(1,2), (2,1), (-3,-3)\} \subset \mathbb{R}^2 (\text{cf. Figure 6}).$$

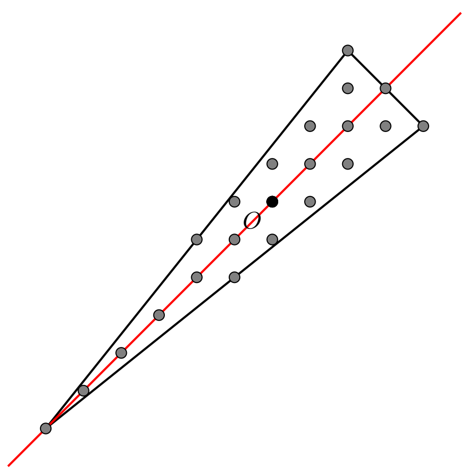

Figure 6: $k\triangle_1$ with $k = 2$.

Theorem 5.1. *X_1 is Chow unstable.*

To see this, first we notice that (10) implies

Lemma 5.2. $\int_{\triangle_1} x dx = (0,0)$.

By the necessity of Chow semistability (7), Theorem 5.1 follows from Lemma 5.2 and the following

Proposition 5.3.

$$\frac{1}{\chi_{\triangle_1}(k)} \sum_{x=(x_1,x_2)\in(k\triangle_1)\cap\mathbb{Z}^2} x = \frac{4\cdot(-k,-k)}{9k^2+3k+2} \neq 0,$$

with $\chi_{\triangle_1}(k) = |(k\triangle_1)\cap\mathbb{Z}^2| = \frac{9k^2+3k+2}{2}$. *In particular, it violates* (7) *and* X_1 *is Chow* unstable *for all* $k \geq 1$.

Proof. By the $W_1 = \mathbb{Z}/2\mathbb{Z}$-symmetry, we have

$$\frac{1}{\chi_{\triangle_1}(k)} \sum_{x\in(k\triangle_1)\cap\mathbb{Z}^2} x = \frac{(1,1)}{\chi_{\triangle_1}(k)} \sum_{x\in(k\triangle_1)\cap\mathbb{Z}^2} x_1 \tag{26}$$

with $x_1 \in \mathbb{R}$ being the first component of $x = (x_1, x_2) \in \mathbb{R}^2$.

Let us define $m := \frac{1}{\chi_{\triangle_1}(k)} \sum_{x\in(k\triangle_1)\cap\mathbb{Z}^2} x_1$. For simplicity, we will only treat the case that k is *even*, [5] then by considering the symmetry about the axis in Figure 4, we obtain

$$\begin{aligned} -m &= \frac{2}{\chi_{\triangle_1}(k)} \left(\sum_{i=1}^{k/2} \frac{9(i-1)(9(i-1)+1)}{2} \right) + \frac{1}{\chi_{\triangle_1}(k)} \frac{\frac{9k}{2}(\frac{9k}{2}+1)}{2} - \frac{3k}{2} \\ &\quad + \frac{2}{\chi_{\triangle_1}(k)} \left(\sum_{i=1}^{k/2} \left(\frac{5+9(i-1)}{2} + \frac{(9(i-1)+4)(9(i-1)+5)}{2} \right) \right) \\ &= \frac{2k}{\chi_{\triangle_1}(k)}. \end{aligned}$$

□

[5]For k *odd*, the derivation is similar and will be left to the readers.

Example 5.4. *For $k = 1$,*

$$\frac{1}{\chi_{\triangle_1}(1)} \sum_{x \in \triangle_1 \cap \mathbb{Z}^2} x = \frac{(-2,-2)}{7}$$

.

Remark 5.5. *We remark that this example as well as the example in [18] have* not *ruled out the possibility of using the asymptotic Chow semistability to compactify the moduli space of Fano varieties contrasting to the case studied in [31], since for those punctured families one might have a limit which is asymptotic Chow polystable and strictly K-semistable simultaneuously.*

Acknowledgements

The second and last author want to express their gratitute to AIM, San Jose for the excellent research environment they provided. The project is originally started during both the second and last authors were participating the AIM workshop on *Stability and moduli spaces* in January, 2017. In particular, we want to thank Radu Laza for proposing Question 1.1 explicitly. We also thank Cristiano Spotti for informing us of the appearance Theorem 2.3.3 in his thesis. The last author wants to thank IHES for the fantastic research environment, where part of the writing was carried out. [6] The work of the last author was partially supported by a Collaboration Grants for Mathematicians from Simons Foundation:281299 and NSF:DMS-1609335.

References

[1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 3, Aiii, A119–A121.

[2] X.-X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.

[6] After the first version of the paper appeared on arXiv, we were informed by Dr. Yotsutani, who has also studied the asymptotic Chow stability of $(X_3, \mathscr{O}_{X_3}(1))$ in Theorem 1.2 in [34]. But his argument seems incomplete as Mabuchi's work does not apply to orbifolds in general (cf. Example 4.2). We thank him for communicating his paper to us.

[3] X.-X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than* 2π, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.

[4] X.-X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches* 2π *and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.

[5] S. Donaldson, *Scalar curvature and projective embeddings, I*, J. Differential Geom. **59** (2001), 479–522.

[6] S. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349.

[7] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. (2) **106** (1977), no. 1, 45–60.

[8] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser Boston, Inc., Boston, MA, (1994).

[9] G. Kempf, *Instability in invariant theory*, Ann. of Math. **108** (1978), no. 2, 299–316.

[10] J. Li and X.-W. Wang, *Hilbert-Mumford criterion for nodal curves*, Compos. Math. **151** (2015), 2076–2130.

[11] C. Li, X.-W. Wang, and C.-Y. Xu, *Degeneration of Fano Kähler-Einstein manifolds*, `arXiv:1411.0761v1`, (2014).

[12] T. Mabuchi, *An obstruction to asymptotic semistability and approximate critical metrics*, Osaka J. Math. **41** (2004), no. 2, 463–472.

[13] T. Mabuchi, *An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. I*, Invent. Math. **159** (2005), no. 2, 225–243.

[14] D. Mumford, J. Fogarty, and F. Kirwan, Geometric Invariant Theory, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], Vol. 34, Springer-Verlag, Berlin, (1994).

[15] L. Ness, *A stratification of the null cone via the moment map*, Amer. J. Math. **106** (1984), no. 6, 1281–1329. With an appendix by David Mumford.

[16] Y. Odaka, *The Calabi conjecture and K-stability*, Int. Math. Res. Not. IMRN **10** (2012), 2272–2288.

[17] H. Ono, *Algebro-geometric semistability of polarized toric manifolds*, Asian J. Math. **17** (2013), 609–616.

[18] H. Ono, Y. Sano, and N. Yotsutani, *An example of an asymptotically Chow unstable manifold with constant scalar curvature*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 4, 1265–1287 (English, with English and French summaries).

[19] Y. Odaka, S. Sun, and C. Spotti, *Compact moduli space of Del Pezzo surfaces and Kähler-Einstein metrics*, J. Differential Geom. **102** (2016), no. 1, 127–172.

[20] H. Ono, Y. Sano, and N. Yotsutani, *An example of an asymptotically Chow unstable manifold with constant scalar curvature*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 4, 1265–1287 (English, with English and French summaries).

[21] G. A. Pick, *Geometrisches zur Zahlentheorie*, Sitzungber Lotos (Prague) **19** (1899), 311–319.

[22] D. H. Phong and J. Sturm, *Scalar curvature, moment maps, and the Deligne pairing*, Amer. J. Math. **126** (2004), no. 3, 693–712.

[23] D. H. Phong and J. Sturm, *Lectures on stability and constant scalar curvature*, Handbook of Geometric Analysis, no. 3, Adv. Lect. Math. (ALM), Vol. 14, Int. Press, Somerville, MA, 2010, pp. 357–436.

[24] H. W. Pullman, *An elementary proof of Pick's theorem*, School Science and Mathematics **79** (1979), 7–12.

[25] J. Ross and R. Thomas, *A study of the Hilbert-Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. **16** (2007), no. 2, 201–255.

[26] C. Spotti, *Degenerations of Kähler-Einstein Fano manifolds*, Ph.D. Thesis, Imperial College **16** (2016), 132 pages. `arXiv:1211.5334`.

[27] C. Spotti, S. Sun, and C.-J. Yao, *Existence and deformations of Kähler-Einstein metrics on smoothable Q-Fano varieties*, Duke Math. J. **165** (2016), no. 16, 3043–3083.

[28] G. Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156.

[29] E. Viehweg, Quasi-projective Moduli for Polarized Manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 30, Springer-Verlag, Berlin, (1995).

[30] X.-W. Wang, *Moment map, Futaki invariant and stability of projective manifolds*, Comm. Anal. Geom. **12** (2004), no. 5, 1009–1037.

[31] X.-W. Wang and C.-Y. Xu, *Nonexistence of aymptotic GIT compactification*, Duke Math. J. **163** (2014), 2217–2241.

[32] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[33] S.-T. Yau, *Open problems in geometry, Differential geometry: partial differential equations on manifolds* (Los Angeles, CA, 1990), Proc. Sympos. Pure Math. **54**, Amer. Math. Soc., Providence, RI, (1993), pp. 1–28.

[34] N. Yotsutani, *Asymptotic Chow semistability implies Ding polystability for Gorenstein toric Fano varieties*, `arXiv:1711.10113`, (2017).

[35] S.-W. Zhang, *Heights and reductions of semi-stable varieties*, Compos. Math. **104** (1996), 77–105.

Department of Mathematics, Sun Yat-Sen University
135 Xingang Xi Road, Guangzhou 510275, China
E-mail address: `lijliang5@mail.sysu.edu.cn`

Shanghai Center for Mathematical Science
Fudan University, 220 Handan Road, Shanghai 200433, China
E-mail address: `zhiyuan.li@fudan.edu.cn`

Department of Mathematics and Computer Science
Rutgers University, Newark NJ 07102-1811, USA
E-mail address: `sturm@rutgers.edu`

Department of Mathematics and Computer Science
Rutgers University, Newark NJ 07102-1811, USA
E-mail address: `xiaowwan@rutgers.edu`

Received February 2, 2018