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An estimate on energy of min-max Seiberg-Witten Floer generators

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In [1], Cristofaro-Gardiner, Hutchings and Ramos proved that embedded contact homology (ECH) capacities of a 4-dimensional symplectic manifold can recover the volume. In particular, a certain sequence of ratios constructed from ECH capacities, indexed by positive integers, was shown to converge to the volume in the index $k \to +\infty$ limit. There were two main steps in [1] to proving this theorem: The first step used estimates for the energy of min-max Seiberg-Witten Floer generators to see that the $k \to +\infty$ limit of the ratios was a lower bound for the volume. The second step used embedded balls in a certain symplectic four manifold to prove that the $k \to \infty$ limit of the ratios was an upper bound.

Stronger estimates on the energy of min-max Seiberg-Witten Floer generators are derived in this paper that give an effective bound for finite index k on the norm of the difference between the ECH ratio at index k and the volume. This bound implies directly (by taking $k \to \infty$) the theorem in [1] that ECH capacities recover volume.

Section 1 introduces the prior knowledge and the main theorem. Section 2 and Section 3 prove the main theorem. Section 4 is an addendum that talks about the Seiberg-Witten Floer min-max generators.

1. Introduction

1.1. The main result

According to [3], a 4-dimensional Liouville domain is a symplectic 4-dimensional manifold with a contact 3-manifold as boundary, such that the symplectic form is exact, and the domain itself is a weakly exact symplectic cobordism of its boundary to the empty set.

Supposing that (X,ω) is a 4-dimensional Liouville domain, then its ECH capacities as defined in [3] are a certain nondecreasing set of numbers

$$0 \leq c_1(X,\omega) \leq c_2(X,\omega) \leq \cdots \leq c_m(X,\omega) \leq \cdots \leq +\infty.$$

that can be canonically associated to the pair (X,ω). Subsection 1.2 says more about these numbers.

ECH capacities are useful in the study of 4-dimensional symplectic geometry, because they give obstructions on symplectic embedding problems. One particular example is, ECH capacities give sharp obstructions for symplectic embedding problems between two 4-dimensional ellipsoids (see [3]).

An amazing fact about ECH capacities is that, ECH capacities can recover the volume (Theorem 1.1 in [1]). To be precise,

$$\lim_{m\to+\infty} c_m(X,\omega)^2 = 4m\mathrm{Vol}(X,\omega),$$

where

$$\mathrm{Vol}(X,\omega) = \frac{1}{2}\int_X \omega\wedge\omega.$$

The main result of this paper is an estimate on the above convergence:

Theorem 1.

$$\left|\frac{c_m(X,\omega)^2}{4k} - \mathrm{Vol}(X,\omega)\right| = O(m^{-\frac{1}{126}}).$$

Remark 2. Gardiner and Savale very recently (as this paper was being revised for publication) derived a stronger version of Theorem 1. See their preprint [2].

1.2. Preliminaries on ECH and ECH capacities

The definition of ECH capacities relies on a concept called "embedded contact homology" ("ECH" for short) in 3-dimensional contact geometry, whose definition can be found in [3].

By way of a summary: For the purpose of this paper, embedded contact homology $ECH(Y,\lambda,\Gamma)$ is a certain $\mathbb{Z}/2\mathbb{Z}$ module that is assigned to any given a 3-dimensional compact, oriented contact manifold Y with a contact 1-form λ, and a given homology class $\Gamma\in H_1(Y)$. It is not necessary to recall the full definition of ECH here, but the following features:

(1) Suppose ξ is the contact structure of λ, that is, the kernal of λ in the tangent bundle of M. Notice ξ is a 2-dimensional real bundle with a

canonical orientation given by $d\lambda$, so $c_1(\xi)$ is defined (because any almost complex structure on ξ that is compatible with the orientation gives the same $c_(\xi)$). If $c_1(\xi)+2PD(\Gamma)$ is torsion in $H^2(Y,\mathbb{Z})$, where $PD(\Gamma)$ is the Poincare dual of Γ, then any two elements in $ECH(Y,\lambda,\Gamma)$ have a relative $\mathbb{Z}$ degree. Moreover, if a fiducial element of $ECH(Y,\lambda,\Gamma)$ is chosen to have degree 0, which is always assumed in this paper, then each element of $ECH(Y,\lambda,\Gamma)$ has an absolulte $\mathbb{Z}$ grading.

(2) Each pair (λ,Γ) on Y induces a Spin$^{\mathbb{C}}$ structure $s(\lambda,\Gamma)$ of Y. And $ECH(Y,\lambda,\Gamma)$ depends only on the topology of Y and the Spin$^{\mathbb{C}}$ structure $s(\lambda,\Gamma)$ up to isomorphism. So ECH is a topological invariant.

(3) Each element $\sigma\in ECH(Y,\lambda,\Gamma)$ has a min-max action, written as $c_{\sigma(Y,\lambda)}$, whose definition can be found in Section 1 of [1] (where the same notation as $c_{\sigma(Y,\lambda)}$ is used).

The ECH capacities of a Liouville domain (X,ω) are defined to be the min-max actions of certain ECH classes of its contact boundary (Y,λ). To be precise, suppose $\mathrm{Vol}(X,\omega)<\infty$, then each $c_k(X,\omega)$ is the min-max action of some certain ECH class in $ECH^{2k}(Y,\lambda,0)$, where $2k$ is the degree of the certain ECH class, and where the fiducial class $\{\emptyset\}$ is set to have degree 0. More details can be found in [3].

In particular, Theorem 1 is a special case of the following Theorem 3. (The reader can compare this to a similar argument in [1], whose Theorem 1.1 was a special case of its Theorem 1.2).

Theorem 3. *Let (Y,λ) be a closed three contact manifold and let $\Gamma\in H_1(Y)$. Let ξ be the contact structure determined by λ. Suppose that $c_1(\xi)+2PD(\Gamma)$ is torsion in $H^2(Y,\mathbb{Z})$. Moreover, suppose $\{\sigma_m\}(m=0,1,2,3,\ldots)$ is a sequence of nonzero classes in $ECH(Y,\xi,\Gamma)$ whose mth term has a degree $k_m\in\mathbb{Z}$ and $\lim\limits_{m\to+\infty}k_m=+\infty$. (Here k_m can be defined as an integer because $c_1(\xi)+2PD(\Gamma)$ is torsion). Then*

$$\left|\frac{c_{\sigma_m}(Y,\lambda)^2}{2k_m}-\mathit{Vol(Y)}\right|=O(k_m^{-\frac{1}{126}}).$$

The Vol(Y) above is defined to be

$$\mathrm{Vol(Y)}=\frac{1}{2}\int_Y\lambda\wedge d\lambda.$$

Notice the definition of volume here is different from [1], whose volume is twice of here. This is because the author wants to keep consistence with [6].

Remark 4. There might be a potential further application of Theorem 3:

In [4], Irie used "ECH min-max actions recover the volume" theory to prove that on compact 3-manifold, Reeb orbits are dense for a generic contact form. As a corollary, on a compact 2-manifold, closed geodesics are dense for a generic metric.

Theorem 1 might carry some hints to a quantitative estimate of the above theorem. For example, one possible conjecture is: Given a 2-dimensional compact manifold with a given metric g, given an $\epsilon > 0$, there exists a metric g_ϵ such that $\|g_\epsilon - g\| \leq \epsilon$ and g_ϵ has a closed geodesic with length at most $C\epsilon^{-\delta}$. Here C and δ should be at least independent with ϵ. Moreover, δ may be also independent with g, and even the 2-dimensional manifold.

To prove Theorem 3, some background knowledge on Seiberg-Witten theory is needed.

1.3. Seiberg-Witten equations

Suppose (Y, λ) is a closed, connected, smooth three manifold with a contact form λ, and is also equipped with a metric satisfying :$|\lambda| = 1$, $|d\lambda| = 2$, and the volume form is $\frac{1}{2}\lambda \wedge d\lambda$. Thus, $*\lambda = \frac{1}{2}d\lambda, *d\lambda = 2\lambda$.

Choose a $\text{Spin}^{\mathbb{C}}$ structure on Y with spinor bundles S. A connection on S compatible with the metric on Y is uniquely determined by its induced connection on $\det S$. Let A denote a connection on $\det S$, and D_A denote its Dirac operator on S. Moreover, choose a fiducial connection A_0 and write $A = A_0 + 2a$.

This paper always assumes $c_1(\det S)$ is torsion.

Let ψ denote a section of S. A pair (a, ψ) is called a "configuration", typically denoted by c. When $\psi = 0$, it is called reducible, otherwise irreducible.

Definition 5. In [6], Taubes considered a perturbed version of Seiberg-Witten equations:

$$*da = r(\psi^+ \iota\psi - i\lambda) + *d\mu + \mathfrak{T}(a, \psi), \tag{6}$$

$$2rD_A\psi = 2rD_{A_0+2a}\psi = \mathfrak{S}(a, \psi). \tag{7}$$

Here μ is a one form, $\mathfrak{T}, \mathfrak{S}$ are perturbations. Moreover, suppose P is the big Banach space of tame perturbations created in [5]. Then $\mathfrak{T}, \mathfrak{S}$ can be chosen to be the gradient of some $g \in P$ with $\|g\|_P$ bounded.

Notation 8. Let $(SW)_{r,e_\mu+g}$ denote the Equations (6), (7).$(SW)_{r,e_\mu}$ and $(SW)_r$ means "$g=0$" and "$g=0,\mu=0$" versions respectively.Let $N_{r,e_\mu+g}$ denote the set of all solutions to $(SW)_{r,e_\mu+g}$.

Definition 9. In the book [5], the "from" version of Seiberg-Witten Floer homology $\widehat{HM}_{-k}(Y)_{r,e_\mu+g}$ is defined (where $-k$ is the degree, the metric g is supposed to be generic). Generators of $\widehat{HM}_{-k}(Y)_{r,e_\mu+g}$ are solutions to $(SW)_{r,e_\mu+g}$. It is not necessary to recall the full definition here. The following statements are what will be needed in this paper:

(1) For any r and μ, $\widehat{HM}_{-k}(Y)_{r,e_\mu+g}$ is only defined for a generic g. However, for different (r,μ,g) and (r',μ',g') where it is defined, there is a canonical isomorphism $T:\widehat{HM}_{-k}(Y)_{r,e_\mu+g}\to\widehat{HM}_{-k}(Y)_{r',e_{\mu'}+g'}$ to identify them, so one can talk about $\widehat{HM}_{-k}(Y)$ without referring to (r,μ,g). This paper will discuss more about the isomorphism T later.

(2) The generators of the complex used to define $\widehat{HM}_{-k}(Y)_{r,e_\mu+g}$ are of two sorts: each reducible solution contributes infinitely but countably many generators with different degrees which are bounded from above but not from below; each irreducible solution contributes only one generator with a unique degree, which is also called the degree of the irreducible solution.

(3) Fix k,μ and suppose $\|g\|_P$ is small and bounded (and g is generic), then for r large enough, all generators are contributed by irreducible solutions (this is proved in [6]).

(4) Fix r,μ,g (g generic), for k large enough, all generators are contributed by reducible solutions (this is because there are only finitely many irreducible solutions for fixed $(SW)_{r,e_\mu+g}$, see [6]).

(5) This paper only considers the mod 2 homologies. And it always regards gauge equivalence configurations as the same thing (for example, a sequence of configurations converges to another means they converge modulo gauge equivalence).

1.4. Functionals $\mathfrak{a}, cs$ and E

Definition 10. Given a configuration $c=(a,\psi)$, its $\mathfrak{a}, cs$ and E are defined as:

$$E=i\int_Y\lambda\wedge(da+*\bar\omega_K),\qquad cs=-\int_Y a\wedge da,$$
$$\mathfrak{a}=\frac{1}{2}(cs-rE)+r\int_Y<D_{A_0+a}\psi,\psi>.$$

$\bar{\omega}_K$ here is a balanced term, whose definition refers to (2.3) of [6] and is omitted here. Since $i\int_Y \lambda \wedge *\bar{\omega}_K$ is just a constant, it is not important when doing estimates in this paper.

Moreover, when μ and $g \in P$ are also chosen, one can define:

$$e_\mu = i\int_Y \mu \wedge da,$$

and

$$\mathfrak{a}_\mu = \mathfrak{a} + e_\mu, \qquad \mathfrak{a}_{\mu,g} = \mathfrak{a} + e_\mu + g.$$

Then $(SW)_{r,e_\mu+g}$ is equivalent to the assertion that $\nabla\mathfrak{a}_{\mu,g}(a,\psi) = 0$.

1.5. Actions on min-max generators

Fix a homology class from $\widehat{HM}_{-k}(Y)$, denoted by $\{\sigma\}$. In [6], the "min-max" generators were defined in $\widehat{HM}_{-k}(Y)_{r,e_\mu}$ for the class $\{\sigma\}$ when μ is generic and r is large.

However, in this paper, **the min-max generator for** $r \geq 0, \mu$**, and** $\{\sigma\}$, denoted as $\hat{c}(r)_\mu$, is slightly different from that in [6], and thus carries more features. Its definition defers to Section 4 (Lemma 54).

Now, suppose $\hat{c}(r)_\mu$ is given, then there is a number $r(k,\sigma)$ related to it:

Definition 11. Suppose $\{\sigma\}$ is fixed and has degree $-k$ with k large, μ is chosen to be generic, then

$$r(k,\sigma) = \inf\{s \geq 1 |\ \hat{c}(r) \text{ is irreducible whenever } r > s\}.$$

Remark 12. (1) In fact, $r(k,\sigma)$ depends on not only k and $\{\sigma\}$, but also on $\hat{c}(r)$ (see Section 4 for details). However, $\hat{c}(r)$ is assumed to be chosen a prior, and is not indicated in the notation $r(k,\sigma)$.

(2) $r(k,\sigma)$ is finite. This will be explained in Subsection 2.3, or see [6].

(3) When $r > r(k,\sigma)$, $\hat{c}(r)$ must be irreducible. When $r \leq r(k,\sigma)$, $\hat{c}(r)$ can be either reducible or irreducible. However, there exists a nondecreasing sequence $0 < s_1 \leq s_2 \leq \cdots$ with $\lim\limits_{j\to\infty} s_j = r(k,\sigma)$ such that $\hat{c}(s_j)$ $(j = 1,2,3,\ldots)$ are all reducible.

Property 13 below lists all the useful features of $\hat{c}(r)_\mu$ in this paper.

Property 13. (1) For any μ, the action $\hat{\mathfrak{a}}(r) = \mathfrak{a}_{r,e_\mu}(\hat{c}(r))$ is a continuous function of $r \geq 1$.

(2) For a generic μ, when $r > r(k,\sigma)$, $\hat{\mathfrak{a}}(r)$ is continuous and piecewise differentiable. Its differential, $\frac{d\hat{\mathfrak{a}}(r)}{dr}$, is equal to $-\frac{1}{2}\hat{E}(r)$, where $\hat{E}(r) = E(\hat{c}(r))$ is only piecewise continuous.

(3) Suppose μ is bounded, degree$\{\sigma\} = -k$ and k is large enough, then $\hat{c}(2)$ is reducible.

(4) Although $\hat{c}(r)$ is called min-max generator for convenience, it might not be an actual min-max component of the homology $\{\sigma\}$. In fact, in [6], Taubes's min-max generator (denoted as $\hat{c}_T(r)$ here) is an actual min-max component of $\{\sigma\}$. However, that $\hat{c}_T(r)$ is only defined when $r \in U$, where U is an open dense subset of $(r(k,\sigma),\infty)$. The crucial relationship between $\hat{c}_T(r)$ and $\hat{c}(r)$ is

$$\lim_{r\in U, r\to\infty} |E(\hat{c}_T(r)) - E(\hat{c}(r))| = 0. \tag{14}$$

All the features above will be illustrated in Section 4.

Remark 15. (1) $\hat{c}(r)$ itself is not uniquely determined. But any choice obeying the requirements (1)-(4) of the definition will suffice.

(2) Even when $r \in U$, $\hat{c}_T(r)$ might be different from $\hat{c}(r)$. (This is because Taubes used some extra r-dependent perturbation of Seiberg-Witten equation to define $\hat{c}_T(r)$, see part (d) of Section 3 of [6].) However their difference is not a trouble because of identity (14).

(3) Although $\hat{\mathfrak{a}}(r)$ is a continuous function of r, $\hat{c}(r)$ might not be continuous. In fact, when μ is generic and $r > r(k,\sigma)$, $\hat{c}(r)$ is only piecewise continuous (see Subsection 4.3 for details).

(4) Usually $\{\sigma\} \in \widehat{HM}_{-k}(Y)$ is chosen a prior and can be omitted in the subsequent arguments.

1.6. The key estimate on the energy

Remark 16. In this paper, C is some big enough positive constant which is independent with r,k, which can have different values in different formulas. Plus, $O(\cdots)$ means absolutely smaller than $C\cdot(\cdots)$.

Granted the definitions in Subsections 1.3 to 1.5, here is the main estimate:

Theorem 17. *Suppose that k is a large enough integer,* degree$\{\sigma\} = -k$*; and that μ is chosen to be generic and have small norm. Then $r(k,\sigma) \geq 2$*

(see Definition 11) and if $r > r(k,\sigma)$, then

$$\left|\frac{\hat{E}(r)^2}{8\pi^2 k} - Vol(Y)\right| = O(k^{-\frac{1}{126}}).$$

The proof of Theorem 17 is in Section 3.

1.7. Relations with ECH capacities

In [1], it implies the following relationship between the energy in Seiberg-Witten-Floer theory and $c_\sigma(Y,\lambda)$ in ECH theory:

$$\lim_{r\in U, r\to+\infty} E(\hat{c}_T(r)) = 2\pi c_\sigma(Y,\lambda).$$

Remark 18. (1) On the left hand side, $\{\sigma\}$ should be understood as a Seiberg-Witten Floer cohomology class, $\hat{c}_T(r)$ should be understood as Taube's min-max generator of Seiberg-Witten Floer cohomology. However, the estimates in paper for Seiberg-Witten Floer homology are still valid for Seiberg-Witten Floer cohomology, so it is safe to use the same notation as for Seiberg-Witten Floer homology elsewhere.

(2) On the right hand side, the ECH class σ is induced by the Seiberg-Witten cohomology class $\{\sigma\}$ by a well-known isomorphism between ECH and Seiberg-Witten cohomology.

Together with Theorem 17 and Formula (14), this relation implies Theorem 3.

Section 2 and Section 3 are the analysis towards the key estimate Theorem 17.

2. Some preliminary estimates

2.1. Estimates from Taubes

From [6] and [1], many inequalities are obtained to be used. They are stated in the following:

Lemma 19. *(1) Suppose (a,ψ) is a solution to $(SW)_{r,e_\mu+g}$ (g is generic, μ and g are bounded), then*

$$E \leq r\mathrm{Vol}(Y) + C. \tag{20}$$

(2) If (a,ψ) in (1) is irreducible and suppose its E, r has a positive lower bound, then

$$|cs + 2e_\mu + 2g - 4\pi^2 k| \leq Cr^{\frac{31}{16}}, \tag{21}$$

$$|cs + 2e_\mu + 2g| \leq Cr^{\frac{2}{3}}E^{\frac{4}{3}}. \tag{22}$$

(3) If (a,ψ) in (1) is reducible, and r has a positive lower bound, then

$$cs + 2e_\mu + 2g = \frac{1}{2}r^2\mathrm{Vol}(Y) + O(r), \tag{23}$$

$$E = r\mathrm{Vol}(Y) + O(1), \tag{24}$$

$$\mathfrak{a} = -\frac{1}{4}r^2\mathrm{Vol}(Y) + O(r). \tag{25}$$

Moreover, a corollary can be derived from the above estimates which will be used later:

Corollary 26. *c is a irreducible solution to $(SW)_{r,e_\mu+g}$ with $r \geq 1$ and μ, g bounded, then*

$$\mathfrak{a}_{r,e_\mu+g}(c) > 2\pi^2 k - \frac{1}{2}r^2\mathrm{Vol(Y)} - Cr^{\frac{31}{16}}. \tag{27}$$

Proof. This is a corollary directly from (21),(24) and the fact that, for a solution, $\mathfrak{a}_{r,e_\mu+g}(c) = \frac{1}{2}(cs - rE) + e_\mu + g$. □

2.2. A lower bound of $r(k,\sigma)$

Lemma 28. *When μ is bounded, and when k is large, then $r(k,\sigma) \geq 2$.*

Proof. This is simply because of the bullet (3) in Property 13. □

Theorem 29. *Here is an estimate on $r(k,\sigma)$:*

$$r(k,\sigma)^2 \geq \frac{8\pi^2 k}{\mathrm{Vol(Y)}} - Ck^{\frac{32}{33}}.$$

Proof. If $r^2 < \frac{8\pi^2 k}{\mathrm{Vol(Y)}} - Ck^{\frac{32}{33}}$, let c_{irr}, c_{red} be any irreducible and reducible solutions to $(SW)_{r,e_\mu}$ respectively.

From (27) one gets

$$\begin{aligned}\mathfrak{a}_{r,e_\mu}(c_{irr}) &> 2\pi^2 k - \frac{r^2}{2}\mathrm{Vol(Y)} - Cr^{\frac{31}{16}} \\ &> -\frac{r^2}{4}\mathrm{Vol(Y)} + Ck^{\frac{33}{34}} + O(r^{\frac{31}{16}}) \geq -\frac{r^2}{4}\mathrm{Vol(Y)} + Ck^{\frac{33}{34}}.\end{aligned}$$

The last step is because $k = O(r^2)$, so $r^{\frac{31}{16}} = o(k^{\frac{33}{34}})$.

However, from (25) one gets

$$\mathfrak{a}_{r,e_\mu}(c_{red}) < -\frac{1}{4}r^2\mathrm{Vol(Y)} + Cr < -\frac{r^2}{4}\mathrm{Vol(Y)} + Ck^{\frac{33}{34}} < \mathfrak{a}_{r,e_\mu}(c_{irr}).$$

Notice $\hat{\mathfrak{a}}(r)$ is continuous w.r.t. r, so when $1 < r^2 < \frac{8\pi^2 k}{\mathrm{Vol(Y)}} - Ck^{\frac{32}{33}}$, $\hat{c}(r)$ cannot shift between reducible and irreducible (since there is always a positive gap between their actions).

Since $\hat{c}(2)$ is reducible, so $\hat{c}(r)$ must be reducible as long as$r^2 < \frac{8\pi^2 k}{\mathrm{Vol(Y)}} - Ck^{\frac{32}{33}}$, which implies $r(k,\sigma)^2 \geq \frac{8\pi^2 k}{\mathrm{Vol(Y)}} - Ck^{\frac{32}{33}}$. □

Remark 30. Before moving to the upper bound of $r(k,\sigma)$, the author wants to introduce a "informal proof" of Theorem 29, which is incorrect but carries some hints and clarifications on what to expect:

Since $r(k,\sigma)$ is the borderline between where $\hat{c}(r)$ to be irreducible and reducible, it should satisfy all of (1) (2) (3) in Lemma 19, which implies

$$\begin{aligned}cs(\hat{c}(r(k,\sigma))) + 2e_\mu(\hat{c}(r(k,\sigma))) &= \frac{1}{2}r(k,\sigma)^2\mathrm{Vol(Y)} + O(r(k,\sigma)) \\ &= 8\pi^2 k + O(r(k,\sigma)^{\frac{31}{16}}).\end{aligned}$$

Thus it seems that $r(k,\sigma)^2 = \frac{8\pi^2 k}{\mathrm{Vol(Y)}} + O(k^{\frac{31}{32}})$.

This above argument is invalid because $cs(\hat{c}(r(k,\sigma))) + 2e_\mu(\hat{c}(r(k,\sigma)))$ is not continuous in general, and also because the spectral flow estimate, i.e., (2) of the Lemma 19 (also see [6])) is not satisfied by generators contributed from reducible solutions, even near the borderline. However, one can still say something about the degree of reducible generators from the spectral flow, which will imply an upper bound of $r(k,\sigma)$.

2.3. An upper bound of $r(k,\sigma)$

Theorem 31. *Suppose that $a+g$ is used to replace a for some generic, small normed $g \in P$. If $a+g$ is used to defined the Seiberg-Witten equations (which will henceforth be assumed), then*

$$r(k,\sigma)^2 \leq \frac{8\pi^2 k}{\mathrm{Vol(Y)}} + Ck^{\frac{31}{32}}.$$

Proof. Based on [6] , each reducible generator of the "from" version of the Seiberg-Witten Floer complex corresponds to an eigenvector of the Dirac operator $D_{A-ir\lambda+2\mu}$ with negative eigenvalue. The degree of such a generator differs by a constant (independent of the eigenvector, eigenvalue and r) from -2 times the sum of two numbers, $\mathfrak{X}$ and $\mathfrak{Y}$. These are defined as follows: The number $\mathfrak{X}$ is the number of negative eigenvalues above the eigenvalue of the given eigenvector. Meanwhile, the number $\mathfrak{Y}$ is the spectral flow for the family $D_{A-is\lambda+2\mu}, s \in [0,r]$.

(The reason for the use here of a generic g to perturb $\mathfrak{a}$ is that, the above argument requires the Dirac operator to have spectrum with multiplicity 1 for each eigenvalue, see [6] for details.)

Thus, the degree of a reducible generator (when $r \geq 1$) is

$$-k = -2\mathfrak{X} - 2\mathfrak{Y} + C \leq -2\mathfrak{Y} + C = -\frac{1}{8\pi^2} r^2 \mathrm{Vol(Y)} + O(r^{\frac{31}{16}}).$$

Thus $-k \leq -\frac{1}{8\pi^2} r^2 \mathrm{Vol(Y)} + Cr^{\frac{31}{16}}$, whenever $\hat{c}(r)$ is reducible.

Thus $\frac{1}{8\pi^2} r(k,\sigma)^2 \mathrm{Vol(Y)} \leq k + Cr^{\frac{31}{16}}$, which implies Theorem 31. □

Combining Theorem 29 and Theorem 31 together, here comes the final conclusion about $r(k,\sigma)$:

Theorem 32.

$$r(k,\sigma)^2 = \frac{8\pi^2 k}{\mathrm{Vol(Y)}} + O(k^{\frac{32}{33}}).$$

And also,

$$\hat{\mathfrak{a}}(r(k,\sigma)) = -\frac{1}{4} r(k,\sigma)^2 \mathrm{Vol(Y)} + O(r(k,\sigma)) = -2\pi^2 k + O(k^{\frac{32}{33}}).$$

The last step is because $r(k,\sigma) = O(k^{\frac{1}{2}}) \leq O(k^{\frac{32}{33}})$.

3. The crucial estimate

In this section, the goal is to prove Theorem 17.

3.1. Differential equations

Lemma 33. *Suppose μ is generic and bounded.Let*

$$y_1 = \frac{\hat{cs}(r) - 4\pi^2 k + 2\hat{e}_\mu(r)}{r} \quad \textit{and} \quad y_2 = \frac{\hat{cs}(r) + 2\hat{e}_\mu(r)}{r},$$

When $r > r(k,\sigma)$, the functions $E - y_1$ and $E - y_2$ are continuous, piecewise differentiable; and where they are differentiable, they satisfy the following equation:

$$\frac{d(\hat{E} - y_i)}{dr} = \frac{y_i}{r}, \quad i = 1, 2. \tag{34}$$

Proof. From the bullet (2) of Property 13, one knows

$$\frac{d\hat{\mathfrak{a}}(r)}{dr} = -\frac{1}{2}\hat{E}(r). \tag{35}$$

Also notice $y_1 = \frac{-2\hat{a} - 4\pi^2 k}{r} + \hat{E}$, $y_2 = -\frac{2\hat{a}}{r} + \hat{E}$ by definition. Differentiate these formulas using the formula in (35) for derivatives of $\hat{\mathfrak{a}}$, and one gets (34). □

Lemma 36. *Here are the estimates on initial values:*

$$I_1 := (\hat{E} - y_1)(r(k,\sigma)) = r(k,\sigma)\mathrm{Vol}(\mathrm{Y}) + O(r(k,\sigma)^{\frac{31}{33}}), \tag{37}$$

$$I_2 := (\hat{E} - y_2)(r(k,\sigma)) = \frac{1}{2}r(k,\sigma)\mathrm{Vol}(\mathrm{Y}) + O(1). \tag{38}$$

Proof. From Theorem 32, one gets

$$(\hat{E} - y_2)(r(k,\sigma)) = \frac{-2\hat{\mathfrak{a}}(r(k,\sigma))}{r(k,\sigma)} = \frac{1}{2}r(k,\sigma)\mathrm{Vol}(\mathrm{Y}) + O(1).$$

Also,

$$\begin{aligned}(\hat{E}-y_1)(r(k,\sigma)) &= \frac{-2\hat{\mathfrak{a}}(r(k,\sigma))-4\pi^2 k}{r(k,\sigma)}\\ &= r(k,\sigma)\mathrm{Vol(Y)}+O(1)+\frac{O(k^{\frac{32}{33}})}{r(k,\sigma)}\\ &= r(k,\sigma)\mathrm{Vol(Y)}+O(r(k,\sigma)^{\frac{31}{33}}).\end{aligned}$$

The last step is because $k = O(r(k,\sigma)^2)$. □

There is one more estimate to exhibit before moving on:

Lemma 39. *For any $r \geq r(k,\sigma)$,*

$$|y_1| \leq Cr^{\frac{15}{16}}, \tag{40}$$

$$|y_2| \leq Cr^{-\frac{1}{3}}\hat{E}(r)^{\frac{4}{3}}. \tag{41}$$

Proof. (40) is directly from the definition of y_1 and (21).

(41) is from the definition of y_2 and (22) (notice $\hat{E}$ is bounded from below is because $\{\sigma\}$ is nontrivial, see [6] for a similar argument). □

3.2. Integrals and asymptotic comparison estimates

Lemma 42. *Suppose $r \geq r(k,\sigma)$,*

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol(Y)} + O(r^{\frac{31}{33}}). \tag{43}$$

Proof. Plug (40) into (34), one gets

$$|(\hat{E}(r)-y_1(r))-I_1)| \leq \int_{r(k,\sigma)}^{r} Cs^{-\frac{1}{16}}ds = O(r^{\frac{15}{16}}),$$

combining with (37) and (40) again one gets

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol(Y)}+O(r(k,\sigma)^{\frac{31}{33}})+O(r^{\frac{15}{16}}) = r(k,\sigma)\mathrm{Vol(Y)}+O(r^{\frac{31}{33}}).$$

The last step is because $r \geq r(k,\sigma)$ and $\frac{31}{33} > \frac{15}{16}$. □

Lemma 44. *Suppose*

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r^{\delta} + r(k,\sigma)^{\epsilon}) \tag{45}$$

with δ, ϵ *constrained as follows: First* $0 < \delta < 1$*, and* $\delta \neq \frac{1}{4}$*). Second,* $\frac{4\delta}{3\delta+1} \leq \epsilon < 1$*. Then* $\hat{E}(r) = r(k,\sigma)$ *obeys the stronger bound:*

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r^{\frac{4}{3}\delta - \frac{1}{3}} + r(k,\sigma)^{\epsilon}).$$

Proof. Plugging (45) into (41), one gets

$$\begin{aligned} |y_2| &\leq Cr^{-\frac{1}{3}}(r(k,\sigma)\mathrm{Vol}(Y) + O(r^{\delta}) + O(r(k,\sigma)^{\epsilon}))^{\frac{4}{3}} \\ &= O(r^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r^{\frac{4}{3}\delta - \frac{1}{3}}). \end{aligned} \tag{46}$$

Choose $r_0 \geq r(k,\sigma)$, when $r \geq r_0$, plug the above inequality into (34),

$$\begin{aligned} &|(\hat{E}(r) - y_2(r)) - (\hat{E}(r_0) - y_2(r_0))| \\ &\leq C\int_{r_0}^{r} s^{-\frac{4}{3}}r(k,\sigma)^{\frac{4}{3}} + s^{\frac{4}{3}\delta - \frac{4}{3}}ds \\ &= O(r^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r_0^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r^{\frac{4}{3}\delta - \frac{1}{3}} + r_0^{\frac{4}{3}\delta - \frac{1}{3}}). \end{aligned} \tag{47}$$

Thus, using (46) again on $y_2(r)$ and $y_2(r_0)$ on the left hand side above, it implies

$$\begin{aligned} &|\hat{E}(r) - \hat{E}(r_0)| \\ &= O(r^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r_0^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r^{\frac{4}{3}\delta - \frac{1}{3}} + r_0^{\frac{4}{3}\delta - \frac{1}{3}} + r^{\frac{4}{3}\delta - \frac{1}{3}} + r_0^{\frac{4}{3}\delta - \frac{1}{3}}) \\ &= O(r^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r_0^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r^{\frac{4}{3}\delta - \frac{1}{3}} + r_0^{\frac{4}{3}\delta - \frac{1}{3}}). \end{aligned} \tag{48}$$

Remember $r \geq r_0 \geq r(k,\sigma) \geq 2$, so whether or not $\frac{4}{3}\delta - \frac{1}{3}$ is positive, there is always $r_0^{\frac{4}{3}\delta - \frac{1}{3}} = O(r^{\frac{4}{3}\delta - \frac{1}{3}} + r(k,\sigma)^{\epsilon})$, and $r^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} \leq r_0^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}}$.

Together with (45),

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r_0^{\delta} + r(k,\sigma)^{\epsilon} + r_0^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} + r^{\frac{4}{3}\delta - \frac{1}{3}}).$$

The above inequality is also true when $r(k,\sigma) \leq r \leq r_0$ directly by (45).

So by choosing $r_0 = r(k,\sigma)^{\frac{4}{3\delta+1}}$ so that $r_0^{\delta} = r_0^{-\frac{1}{3}}r(k,\sigma)^{\frac{4}{3}} = r(k,\sigma)^{\frac{4\delta}{3\delta+1}} = O(r(k,\sigma)^{\epsilon})$, one has $\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r(k,\sigma)^{\epsilon} + r^{\frac{4}{3}\delta - \frac{1}{3}})$. $\square$

Lemma 49. *Same condition as Lemma 44 , but the result is*

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r(k,\sigma)^{\epsilon}).$$

Proof. Starting with any δ, iterating Lemma 44, by replacing δ with $\frac{4}{3}\delta - \frac{1}{3} = 1 - \frac{4}{3}(1-\delta)$ finite many times, and increase a little bit if it touches $\frac{1}{4}$, until it is below 0, so the corresponding term can be bounded, and can be absorbed into $O(r(k,\sigma)^{\epsilon})$. Finally it becomes,

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r(k,\sigma)^{\epsilon}).$$

□

3.3. Proof of Theorem 17

An appeal to Lemma 49 can be made starting from $\delta = \frac{31}{33}, \epsilon = \frac{4\delta}{3\delta+1} = \frac{62}{63}$ because of (43). This appeal leads to the bound:

$$\hat{E}(r) = r(k,\sigma)\mathrm{Vol}(Y) + O(r(k,\sigma)^{\frac{62}{63}}).$$

So

$$\hat{E}(r)^2 = r(k,\sigma)^2\mathrm{Vol}(Y)^2 + O(r(k,\sigma)^{\frac{125}{63}}).$$

So using Theorem 32 again,

$$\hat{E}(r)^2 - 8\pi^2 k\mathrm{Vol}(Y) = O(r(k,\sigma)^{\frac{125}{63}} + k^{\frac{32}{33}}) = O(k^{\frac{125}{126}}).$$

The last step is because $r(k,\sigma) = O(k^{\frac{1}{2}})$, so $O(r(k,\sigma)^{\frac{125}{63}} + k^{\frac{32}{33}}) = O(k^{\frac{125}{126}} + k^{\frac{32}{33}}) = O(k^{\frac{125}{126}})$.

Finally, the above is equivalent to

$$\left|\frac{\hat{E}(r)^2}{8\pi^2 k} - \mathrm{Vol}(Y)\right| = O(k^{-\frac{1}{126}}).$$

4. Existence of min-max generators

This section gives the construction of min-max generators.

4.1. Construction of $\hat{\mathfrak{a}}(r)$ for any μ

Definition 50. Fix r and μ, for any integer $m > 1$, choose $g_m \in P$ with $\|g_m\|_P < \frac{1}{m}$ and generic (so $\widehat{HM}_k(Y)_{r,e_\mu+g_m}$ is well-defined). Let

$$\hat{\mathfrak{a}}(r)_{e_\mu+g_m} = \min\{\max\{\mathfrak{a}_{r,e_\mu+g_m}(c) \mid c \text{ is a generator of } \sigma\} \mid \sigma \text{ is a representative of} \{\sigma\}\}.$$

Furthermore, let $\hat{\mathfrak{a}}(r) = \lim_{m\to+\infty} \hat{\mathfrak{a}}(r)_{e_\mu+g_m}$, then one gets:

Lemma 51. *$\hat{\mathfrak{a}}(r)$ doesn't depend on g_m.*

Proof. Suppose there are two different ways of choosing g_m, denoted as g_m and g'_m separately. Connect them via a generic path $g(s) \in P$, $-\infty < s < +\infty$ which is defined so that $g(s) = g_m + e_\mu$ where $s < -1$, and $g(s) = g'_m + e_\mu$ where $s > +1$. The path can also be chosen to obey the bound $\|\frac{dg}{ds}\| \leq \frac{4}{m}$. (The notation here uses $\|\cdot\|$ to denote the P norm defined in [5].)

Consider the SW trajectories on $Y \times \mathbb{R}$ using perturbation $g(s)$. The corresponding instantons on $Y \times \mathbb{R}$ give an isomorphism

$$T : \widehat{HM}_{-k}(SW)_{r,g_m+e_\mu} \to \widehat{HM}_{-k}(SW)_{r,g'_m+e_\mu}.$$

To be precise, T is the map $\hat{m}$ defined in Definition 25.3.4 of the book [5], evaluating at the cohomology class "1" of the blown-up configuration space of $Y \times \mathbb{R}$. The above T is a prior only an homomorphism from $\widehat{HM}_\bullet(SW)_{r,g_m+e_\mu}$ to $\widehat{HM}_\bullet(SW)_{r,g'_m+e_\mu}$. (See Theorem 23.1.5 and its corollary in the book [5].) Here, $HM_\bullet$ stands for the negative completion of the homology, in the sense of Definition 3.1.3 of the book [5]. (This notation is not important in this paper.)

However, in the special case as above, T is an isomorphism and keeps the degree $-k$. This is because here $c_1(S)$ is torsion and the perturbation is balanced, and the cobordism $Y \times \mathbb{R}$ is a cylinder. For a generic $g(s)$, the above T counts the instantons on $Y \times \mathbb{R}$ in four different ways (see Definition 25.3.3 of the book [5], where T has four components which form a 2×2 matrix. The four components are correspondence to : (1) irreducible to irreducible, (2)irreducible to reducible, (3)reducible to irreducible, (4) reducible to reducible respectively). Carefully checking them, one finds that in each component, T only counts the (possibly broken) instantons on $Y \times \mathbb{R}$ which connects elements in $\widehat{HM}_\bullet(SW)_{r,g_m+e_\mu}$ to $\widehat{HM}_\bullet(SW)_{r,g'_m+e_\mu}$ with the the same degree.

Now let c, c' be solutions to $(SW)_{r,g_m+e_\mu}, (SW)_{r,g'_m+e_\mu}$ both of degree -k, and with c' being a component of Tc. Then, there is at least one instanton trajectory (or possibly a broken one) connecting c to c'. Here an instanton means a family of configurations parametrized by the coordinate s for $\mathbb{R}$ obeying the following conditions: First, the $s \to -\infty$ limit should be c and the $s \to \infty$ limit should be gauge equivalence with c' (still denoted as c'). Second, the s-dependent family of configuration should obey the equation:

$$\frac{d}{ds}c(s) = -\nabla(\mathfrak{a}_r + g(s)).$$

Although the definition of T (see $\hat{m}$ in the Definition 25.3.4 of the book [5]) used the blown-up configuration space, the instanton trajectory used here is only its projection to the configuration space without blown up. Granted above, then

$$\begin{aligned} \mathfrak{a}_{r,g'_m+e_\mu}(c') - \mathfrak{a}_{r,g_m+e_\mu}(c) &= \int_{-\infty}^{+\infty} \frac{d}{ds}(\mathfrak{a}_r(c(s)) + g(s)(c(s)))ds \\ &= \int_{-\infty}^{+\infty} (\nabla(\mathfrak{a}_r + g(s)) \cdot \frac{dc(s)}{ds} + \frac{dg(s)}{ds}(c(s)))ds \\ &= \int_{-\infty}^{+\infty} (-\|\nabla(\mathfrak{a}_r + g(s))\|^2 + \frac{dg(s)}{ds}(c(s))))ds \\ &\leq \int_{-\infty}^{+\infty} \frac{dg(s)}{ds}(c(s))ds \\ &\leq \int_{-1}^{1} \frac{4}{m}\|c(s)\|ds \leq \frac{C_r}{m}. \end{aligned} \tag{52}$$

Here C_r is some constant independent with m, g_m and g'_m.

Suppose $\hat{\sigma}_{r,g_m+e_\mu}$ is a representative of $\{\sigma\}$ and $\hat{c}_{r,g_m+e_\mu}$ is a component of $\hat{\sigma}_{r,g_m+e_\mu}$ which achieves the min-max of action, i.e.,

$$\mathfrak{a}_{r,g_m+e_\mu}(\hat{c}_{r,g_m+e_\mu}) = \hat{\mathfrak{a}}(r)_{g_m+e_\mu}.$$

Let c be any component of $\hat{\sigma}_{r,g_m+e_\mu}$, then by definition,

$$\mathfrak{a}_{r,g_m+e_\mu}(c) \leq \mathfrak{a}_{r,g_m+e_\mu}(\hat{c}_{r,g_m+e_\mu}).$$

Let c' be any component of Tc, one gets , by the above lemma,

$$\mathfrak{a}_{r,g'_m+e_\mu}(c') \leq \mathfrak{a}_{r,g_m+e_\mu}(c) + \frac{C_r}{m} \leq \mathfrak{a}_{r,g_m+e_\mu}(\hat{c}_{r,g_m+e_\mu}) + \frac{C_r}{m}.$$

Since the above is true for any component of $T\hat{\sigma}_{r,g_m+e_\mu}$, which is a representative of $\{\sigma\}$ in the g'_m version of SW homology, one gets:

$$\hat{\mathfrak{a}}(r)_{g'_m+e_\mu} \leq \hat{\mathfrak{a}}(r)_{g_m+e_\mu} + \frac{C_r}{m}.$$

Similarly,

$$\hat{\mathfrak{a}}(r)_{g_m+e_\mu} \leq \hat{\mathfrak{a}}(r)_{g'_m+e_\mu} + \frac{C_r}{m}.$$

So $|\hat{\mathfrak{a}}(r)_{g'_m+e_\mu} - \hat{\mathfrak{a}}(r)_{g_m+e_\mu}| \leq \frac{C_r}{m}$. Let $m \to +\infty$, it implies lemma 51. □

4.2. Continuity of $\hat{\mathfrak{a}}(r)$

Theorem 53. *The $\hat{\mathfrak{a}}(r)$ defined in last subsection is continuous along r.*

Proof. Fix r_0, suppose g_m is chosen as in last subsection for r_0.

Notice$(SW)_{r_0+\epsilon,e_\mu+e_{\frac{1}{2}\epsilon\lambda}+g_m}$ is the same equation as $(SW)_{r_0,e_\mu+g_m}$ for any $\epsilon \in \mathbb{R}$, and when $|\epsilon|$ is small (say, when $0 \leq |\epsilon| < \delta(r_0, m)$), $\|g_m + e_{\frac{1}{2}\epsilon\lambda}\|_P < \frac{1}{m}$ still holds true. Thus $g_m + e_{\frac{1}{2}\epsilon\lambda}$ can also play the role of "g_m" with $r = r_0 + \epsilon$ when ϵ is small, and they have the same action on min-max generators. Thus, as long as $|\epsilon| < \delta(r_0, m)$, $|\hat{\mathfrak{a}}(r_0)_{g_m+e_\mu} - \hat{\mathfrak{a}}(r_0 + \epsilon)| \leq \frac{C_{r_0+\epsilon}}{m}$. Moreover, remember $|\hat{\mathfrak{a}}(r_0)_{g_m+e_\mu} - \hat{\mathfrak{a}}(r_0)| \leq \frac{C_{r_0}}{m}$, thus $|\hat{\mathfrak{a}}(r_0) - \hat{\mathfrak{a}}(r_0 + \epsilon)| \leq \frac{C_{r_0}+C_{r_0+\epsilon}}{m}$.

Notice C_r is bounded nearby r_0 (for $|\epsilon| < \delta(r_0, m)$), so $m \to \infty$ implies $\frac{C_{r_0}+C_{r_0+\epsilon}}{m}$ can be arbitrarily small, which implies $\hat{\mathfrak{a}}$ is continuous. □

It is always possible to construct $\hat{c}(r)$ in $(SW)_{r,e_\mu}$, if the only requirement is to have an action equal to $\hat{\mathfrak{a}}(r)$. This is the following lemma:

Lemma 54. *For each r, one can choose a solution of $(SW)_{r,e_\mu}$, denoted by $\hat{c}(r)_\mu$(or $\hat{c}(r)$, $\hat{c}$ for short) , such that $\mathfrak{a}_{r,e_\mu}(\hat{c}(r)) = \hat{\mathfrak{a}}(r)$. (The way to choose may not be unique.) $\hat{c}(r)$ is called the **min-max generator**.*

Proof. By a standard compactness argument of Seiberg-Witten equation, $\hat{c}_{r,g_m+e_\mu}$ has a convergent subsequence (modulo gauge equivalence)(see [5]). Just simply choose a limit of such subsequence. □

Moreover,

Theorem 55. *$\hat{c}(r)$ satisfies the formula (14), i.e.,*

$$\lim_{r\in U, r\to\infty} |E(\hat{c}_T(r)) - E(\hat{c}(r))| = 0.$$

Here, $U, \hat{c}_T(r)$ has the same meaning as in (4) of Property 13.

Proof. For $r \in U$, the Seiberg-Witten equations for $\hat{c}_T(r)$ and $\hat{c}(r)$ differ by only a small normed r-dependent tame perturbation, represented by $p(r) \in P$ (see part (d) of Section 3 in [6] for details). Moreover, $p(r)$ can be chosen so that $\|p(r)\|_P < \frac{1}{\lceil C_r \rceil + 1}$, where C_r is defined in the proof of Theorem 51, $\lceil C_r \rceil$ is the smallest integer above C_r. Thus, fix an r, $p(r)$ can play the role of g_m in Definition 50 with $m = \lceil C_r \rceil$. Since $\hat{c}_T(r)$ is an actual min-max component of the homology class $\{\sigma\}$ (see [6]), so by the proof of Theorem 51, $|\hat{a}_T(r) - \hat{a}(r)| \leq \frac{C_r}{\lceil C_r \rceil} \leq 1$.

From the inequalities in Section 2.1 and the definition of actions, it is not hard to see, in any case

$$\frac{-2a}{r} = E + O(r^{-\frac{1}{3}} E^{\frac{4}{3}}).$$

Since $E(\hat{c}(r))$ is bounded by Theorem 17, so

$$\lim_{r \in U, r \to \infty} \left| E(\hat{c}(r)) + \frac{2\hat{a}(r)}{r} \right| = 0.$$

Similarly, $E(\hat{c}_T(r))$ is also bounded (see [6]), so

$$\lim_{r \in U, r \to \infty} \left| E(\hat{c}_T(r)) + \frac{2\hat{a}_T(r)}{r} \right| = 0.$$

Together with $|\hat{a}_T(r) - \hat{a}(r)| \leq \dfrac{C_r}{\lceil C_r \rceil} \leq 1$, one gets

$$\lim_{r \in U, r \to \infty} |E(\hat{c}_T(r)) - E(\hat{c}(r))| = 0.$$

□

Notice, the $\hat{c}(r)$ constructed above might not be piecewise continuous when $r > r(k, \sigma)$. The r-dependent choices are made in the next section (after choosing generic μ) so that the resulting family (parametrized by r) is piecewise continuous when $r > r(k, \sigma)$.

4.3. Piecewise continuity of $\hat{c}(r)$ for generic μ

Reference [6] proved that if μ is generic, then there is a discrete subset in $[2, \infty)$, denoted by $\{p_1, p_2, \ldots\}$, with the following significance: If r is not in this set, then the irreducible solutions of $(SW)_{r, e_\mu}$ are distinguished by the values of their actions. Second, for any $i \in \{1, 2, \ldots\}$, the irreducible solutions of $(SW)_{r, e_\mu}$ for values of r in the interval (p_i, p_{i+1}) can be identified so as to define continuous and piecewise differentiable families of configurations.

Now choose a family $\hat{c}(r)$ in the manner explained previously. This defines the number $r(k,\sigma)$ as in Definition 11. If $r > r(k,\sigma)$ and if r is in some interval (p_i, p_{i+1}) for $i \in \{1, 2, \ldots\}$, then, because the min-max action $\hat{\mathfrak{a}}$ varies continuously, it follows from the remarks of the preceding paragraph that $\hat{c}(r)$ will vary continuously and piecewise differentiably with r for $r \in (p_i, p_{i+1})$.

With the preceding understood, consider next:

Lemma 56. *When $r \in (p_i, p_{i+1})$ and when $r > r(k,\sigma)$, then*

$$\frac{d\hat{\mathfrak{a}}(r)}{dr} = -\frac{1}{2}E(\hat{c}).$$

Proof. This is just because $\hat{c}(r)$ are continuous solutions, thus

$$\frac{d\hat{\mathfrak{a}}(r)}{dr} = \left(\frac{d}{dr}\mathfrak{a}_{r,e_\mu}\right)(\hat{c}(r)) + \left\langle \nabla\mathfrak{a}_{r,e_\mu}, \frac{d}{dr}\hat{c}(r)\right\rangle = -\frac{1}{2}E(\hat{c}(r)).$$

□

The proof is almost done. Only the third property in Theorem 13 needs to be checked. But this property is just a corollary of (4) in Definition 9.

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