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On the Toda systems of VHS type

Chen-Yu Chi

We consider the Toda systems of VHS type with singular sources and provide a criterion for the existence of solutions with prescribed asymptotic behaviour near singularities. We also prove the uniqueness of solution. Our approach uses Simpson's theory of constructing Higgs-Hermitian-Yang-Mills metrics from stability.

1. Introduction

Let M be a compact Riemann surface and $g = ds^2$ be a smooth hermitian metric on M, and denote the Laplacian associated to g by Δ_g and the Gaussian curvature of g by K_g. For $\epsilon = \pm 1$, we consider systems of partial differential equations of the following form

$$\epsilon \begin{pmatrix} \frac{1}{4}\Delta_g u_1 - \frac{K_g}{2} \\ \frac{1}{4}\Delta_g u_2 - \frac{K_g}{2} \\ \vdots \\ \frac{1}{4}\Delta_g u_n - \frac{K_g}{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \begin{pmatrix} e^{u_1} \\ e^{u_2} \\ \vdots \\ e^{u_n} \end{pmatrix} \tag{1.1}$$

on M with finitely many points removed. We call it a Toda system of VHS type or hermitian type according to $\epsilon = 1$ or -1. Here VHS stands for (polarized complex) variation of Hodge structure. The reason of the name will be clear later.

The Toda systems we consider here are usually called type A Toda systems, manifesting its relation to the Lie algebra A_n. One can consider more general types of Toda systems by replacing the Cartan matrix of type A by those of other types. For the case of smooth solutions on M, i.e. $\mu = (0, \dots, 0)$, there have appeared many studies relating Toda systems with harmonic maps and Higgs bundles, for example, [1] and [3]. The case with nontrivial singular sources is more involved and is the subject of the current paper.

Despite the formal similarity, these two types of Toda systems are quite different in nature, both analytically and geometrically. The VHS type is related to the notion of stability and the hermitian type is more related to the consideration of harmonic maps. In this paper, after developing a general geometric formalism of these systems, which is similar for both types, we will focus on Toda systems of VHS type and treat those of hermitian type in another paper.

Definition 1.1. (1) For any n-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of real valued functions on M, we define $S_\mu := \{p \in M : \mu(p) \neq (0, \ldots, 0)\}$. μ is called an assignment of singular strengths if S_μ is finite. We call points of S_μ "punctures" (with respect to μ). (2) For any assignment of singular strengths μ, an n-tuple $u = (u_1, \ldots, u_n) \in \mathcal{C}^\infty(M \setminus S_\mu)$ is called μ-admissible if for every $p \in S_\mu$ there exists a holomorphic chart (U, z) centered at p and functions $v_j \in \mathcal{C}^\infty(U \setminus \{p\})$ which are bounded such that

$$u_j = 2\mu_j(p) \log |z| + v_j$$

on $U \setminus \{p\}$, $j = 1, \ldots, n$. (In this paper, log always means the natural logarithm.) (3) For an assignment of singular strengths μ and any $p \in M$ we let

$$d_{k,p}(\mu) := \frac{-1}{n+1} \sum_{j=1}^{n} (n+1-j)\mu_j(p) + \sum_{j=1}^{k} \mu_j(p) \tag{1.2}$$

and define

$$d_k(\mu) := \sum_{p \in M} d_{k,p}(\mu),$$

$k = 1, \ldots, n$.

Our main result is the following theorem (Theorem 4.2).

Theorem. *The μ-admissible solution to (1.1) is unique if it exists. There exists a μ-admissible solution to (1.1) if and only if*

$$d_{n-m+1}(\mu) + \cdots + d_n(\mu) < m(n-m+1)(\text{genus}(M) - 1),$$

$m = 1, \ldots, n$.

The arrangement of this paper is as follows. In Section 2, we introduce the notion of complex pre-VHS, diagonality, and special linearity. We show that to a Toda system in the above sense there is associated a diagonal special linear complex variations of Hodge structure, whose underlying bundle and polarization is an almost canonically chosen smooth hermitian vector bundle (V, h). If the Gauss-Manin connection preserves the hermitian form $h(C^{-1}\cdot, \cdot)$ (C being the Weil operator), then the Toda system is of VHS type (the traditional polarization); if instead it preserves the hermitian metric h, the Toda system is of hermitian type.

In Section 3, we recall the notion of Higgs bundle and introduce a Higgs bundle E on X related to a system of VHS type. We establish a correspondence between n-tuples of *positive* functions on $M \setminus S_\mu$ whose squares have μ-admissible logarithms and hermitian metrics on E with some type of asymptotic property. The procedure mimics that of establishing a correspondence between complex variations of Hodge structure and system of Hodge bundles as in [5], except that we drop the flatness requirement on curvature. Under this correspondence, n-tuples formed by taking the positive square root of the exponential of μ-admissible solutions to the Toda system correspond to flat metrics.

Our main tool in getting the criterion for existence of solutions is the theory of getting Higgs-Hermitian-Yang-Mills metrics (which are flat in our case) from stability, developed by Hitchin [2] and Simpson [5]. As mentioned earlier, the presence of singularities is the main difficulty to overcome. In order to deal with singularities, use the results of [5] for quasiprojective curves. It should be noted that the correspondence between variation of Hodge structures and Higgs-Hermitian-Yang-Mills metrics for the bundles discussed in this paper is a specialization of that between stable filtered local systems and irreducible tame harmonic bundles on a punctured algebraic curves, as established in [6].

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2. Complex variation of Hodge structure

In this section we relate Toda systems with flat connections. The formalism has appeared in studies of Toda systems on a region of $\mathbf{R}^2$, for example, in [3]. The vector bundles underlying the flat connections in these classical situations are mainly trivial bundles. We propose a simple globalization as the preparation for further development.

Let X be a complex manifold and denote the sheaf of germs of smooth (p,q)-forms by $\mathcal{A}_X^{p,q}$. ($\mathcal{A}_X = \mathcal{A}_X^0 := \mathcal{A}_X^{0,0}$.)

Definition 2.1. (1) A complex pre-variation of Hodge structure (complex pre-VHS for short) $(\{V^{r,s}\}, \nabla)$ on X of weight an integer n consists of

(i) smooth complex vector bundles $V^{r,s}$ over X, $r, s \in \mathbf{Z}$, $r+s=n$, with $V^{r,s}=0$ for all but finitely many (r,s) and

(ii) a connection $\nabla : \mathcal{A}_X^0(V) \to \mathcal{A}_X^1(V)$ on $V := \oplus_{r+s=n} V^{r,s}$ which only has components of *total* degree $(1,0)$ and $(0,1)$. In other words, $\nabla = \oplus_{r,s} \nabla^{r,s}$ with

$$\nabla^{r,s} : \mathcal{A}_X^0(V^{r,s}) \to \mathcal{A}_X^{1,0}(V^{r-1,s+1}) \oplus \mathcal{A}_X^{1,0}(V^{r,s}) \oplus \mathcal{A}_X^{0,1}(V^{r,s}) \oplus \mathcal{A}_X^{0,1}(V^{r+1,s-1})$$

for each pair (r,s). If we write $\nabla = \theta + \nabla^{\mathrm{Hodge}} + \theta'$ where

$$\nabla^{\mathrm{Hodge}} = \oplus_{r+s=n} \{\nabla^{\mathrm{Hodge},r,s} : \mathcal{A}_X^0(V^{r,s}) \longrightarrow \mathcal{A}_X^{1,0}(V^{r,s}) \oplus \mathcal{A}_X^{0,1}(V^{r,s})\},$$

$$\theta := \oplus_{r+s=n} \{\theta^{r,s} : \mathcal{A}_X^0(V^{r+1,s-1}) \longrightarrow \mathcal{A}_X^{1,0}(V^{r,s})\},$$

and

$$\theta' := \oplus_{r+s=n} \{\theta'^{r,s} \mathcal{A}_X^0(V^{r-1,s+1}) \longrightarrow \mathcal{A}_X^{0,1}(V^{r,s})\},$$

then it is clear that ∇^{Hodge} is also a connection and θ and θ' are $\mathcal{A}_X$-linear. θ is called the Hodge field of the complex pre-VHS $(\{V^{r,s}\}, \nabla)$.

(2) A complex pre-VHS $(\{V^{r,s}\}, \nabla)$ over X is called diagonal if the curvature F^{∇} of ∇ is diagonally valued, i.e.

$$F^{\nabla}(Y_1, Y_2)(V_x^{r,s}) \subset V_x^{r,s}, \text{ for all } (r,s), Y_1, Y_2 \in T_x X, x \in X.$$

(3) A complex pre-VHS $(\{V^{r,s}\}, \nabla)$ over X is called a special linear complex pre-VHS (SL-complex pre-VHS for short) if its induced connection on $\det V$ is flat.

(4) A complex pre-VHS $(\{V^{r,s}\}, \nabla)$ is a complex VHS if ∇ is flat.

Definition 2.2. Let $(\{V^{r,s}\}, \nabla)$ be a complex pre-VHS of weight n. A Hodge-polarization (resp. hermitian-polarization) consists of $\{h^{r,s}\}$ where $h^{r,s}$ is a smooth hermitian metric on $V^{r,s}$ for each (r,s) such that if $h := \oplus_{r+s=n} h^{r,s}$ then $h(C^{-1}\cdot, \cdot)$ (resp. $h(\cdot,\cdot)$) is preserved by ∇ where C is the Weil operator defined by

$$C|_{V^{r,s}} := i^{r-s}\mathrm{id}_{V^{r,s}} = i^n(-1)^s \mathrm{id}_{V^{r,s}}.$$

More precisely, this means that

$$X\langle \sigma, \tau\rangle = \langle \nabla_X \sigma, \tau\rangle + \langle \sigma, \nabla_{\overline{X}}\tau\rangle,$$

for all $X \in T_xX$, $\sigma, \tau \in \mathcal{A}^0_X(V)_x$, and $\langle\cdot,\cdot\rangle = h(C^{-1}\cdot,\cdot)$ (resp. $h(\cdot,\cdot)$). If this is the case, we say that $(\{V^{r,s}\}, \nabla)$ is Hodge(resp. hermitian)-polarized by $\{h^{r,s}\}$ (or h for short). It is not hard to see that $\{h^{r,s}\}$ is a Hodge-polarization (resp. hermitian-polarization) if and only if ∇^{Hodge} preserves h and $\theta' = \theta^*$ (resp. $-\theta^*$), where

$$\theta^* : \mathcal{A}^0_X(V) \longrightarrow \mathcal{A}^{0,1}_X(V)$$

is the adjoint of θ with respect to h.

Now we specialize to the situation related to Toda systems. We consider (X, g), a (possibly noncompact) Riemann surface with a smooth hermitian metric. In the following we fix a smooth line bundle L on X such that

$$L^{\otimes(n+1)} = K_X^{\otimes\frac{n(n+1)}{2}} \tag{2.1}$$

and define

$$V^{n-k,k} := \begin{cases} L\otimes K_X^{\otimes -k}, & \text{if } k = 0,\dots,n. \\ 0, & \text{otherwise} \end{cases} \quad \text{and } V := \oplus_k V^{n-k,k}. \tag{2.2}$$

Finally, let

$$h_g = \oplus_k h^{n-k,k} \text{ be the metric on } V \text{ induced by } g \text{ via (2.1).} \tag{2.3}$$

Note that $\det V$ is trivial.

Remark 2.1. In this situation, we have a bijection between the set of "potential Hodge field" and $\mathcal{C}^\infty(X)^n$ since

$$\mathcal{A}_X^{1,0}\big(\mathrm{Hom}(V^{n-k+1,k-1}, V^{n-k,k})\big) \simeq \mathcal{A}_X^0.$$

More precisely, let $\varphi = \lambda dz$ $(\lambda > 0)$ be a unitary frame of K_X and e_L be a unitary frame of L such that $e_L^{n+1} = \varphi^{\frac{n(n+1)}{2}}$ via $L^{n+1} \simeq K_X^{\frac{n(n+1)}{2}}$. Then $e_k := e_L \otimes \varphi^{-k}$ is a unitary frame of $V^{k,n-k}$ and the correspondence is given by

$$\{\theta^{n-k,k} = b_k \varphi \otimes e_k \otimes e_{k-1}^*\}_{k=0,\dots,n} \longleftrightarrow (b_1, \dots, b_n) \in \mathcal{C}^\infty(X)^n.$$

Therefore, when talking about the Hodge field of a complex pre-VHS whose underlying bundles are of the form in (2.2) (which is the only kind of complex pre-VHS of interest in the following), we will use $(b_1, \dots, b_n)$ and θ interchangeably.

A natural question is: for a given $(b_1, \dots, b_n) \in \mathcal{C}^\infty(X)^n$, can one find connections $\nabla^{\mathrm{Hodge},n-k,k}$ on $V^{n-k,k}$, $k = 0, \dots, n$, such that $\nabla := \theta + \nabla^{\mathrm{Hodge}} + \theta^*$ defines a complex VHS pcolarized by h_g?

Proposition 2.1. *We have a bijection between the set*

$$\{(b_1, \dots, b_n) \in \mathcal{C}^\infty(X)^n : \textit{All } b_k \textit{ are nonvanishing}\}$$

and the set of all diagonal SL-complex pre-VHSs which are underlied by $\{V^{r,s}\}$ defined in (2.2) and Hodge(resp. hermitian)-polarized by h_g with all $\theta^{r,s}$ nonvanishing. Under this bijection, those n-tuples $(b_1, \dots, b_n)$ with $(\log |b_1|^2, \dots, \log |b_n|^2)$ solving (1.1) with $\epsilon = 1$(resp. -1) correspond exactly to complex VHSs.

Proof. With respect to the local unitary frames e_k as were used in Remark 2.1,

$$\nabla^{\mathrm{Hodge}} : A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix} \quad \text{and} \quad \theta : \begin{pmatrix} 0 & & & \\ B_1 & 0 & & \\ & \ddots & \ddots & \\ & & B_n & 0 \end{pmatrix}$$

with $\mathrm{tr}A = 0, \overline{A}^t = -A$, and $B_k = b_k\varphi,\ k = 1, \ldots, n$, and

$$\nabla : \begin{pmatrix} A_0 & \epsilon\overline{B}_1 & & & \\ B_1 & A_1 & \epsilon\overline{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{n-1} & A_{n-1} & \epsilon\overline{B}_n \\ & & & B_n & A_n \end{pmatrix}.$$

All entries of the curvature matrix of ∇ which are away from the main, the upper secondary, and the lower secondary diagonals are zero. The lower secondary diagonal terms are

$$dB_k + (A_k - A_{k-1}) \wedge B_k,$$

$k = 1, \ldots, n$. (We define conventionally $B_0 := 0 =: B_{n+1}$). Therefore,

$$(F^\nabla)_{\text{off diag.}} = 0 \text{ if and only if } \overline{\partial} b_k - i b_k \rho^{0,1} = b_k(A^{0,1}_{k-1} - A^{0,1}_k)$$

(where ρ is the (unique) real 1-form such that $d\varphi = -i\rho \wedge \varphi$), $k = 1, \ldots, n$, if and only if

$$A_{k-1} - A_k = \frac{\overline{\partial} b_k}{b_k} - \frac{\partial \overline{b}_k}{\overline{b}_k} - i\rho, \tag{2.4}$$

$k = 1, \ldots, n$. Since $A_0 + \cdots + A_n = 0$, these conditions completely determine A_k in terms of $(b_1, \ldots, b_n)$, provided that all b_k are nonvanishing. More precisely,

$$A^{0,1}_k = \frac{\overline{\partial}[(b_1^n b_2^{n-1} \cdots b_n)^{\frac{1}{n+1}} (b_1 \cdots b_k)^{-1}]}{(b_1^n b_2^{n-1} \cdots b_n)^{\frac{1}{n+1}} (b_1 \cdots b_k)^{-1}} - i\left(\frac{n}{2} - k\right)\rho^{0,1} \tag{2.5}$$

(where the factor $(b_1 \cdots b_k)^{-1} := 1$ when $k = 0$) and $A^{1,0}_k = -\overline{A^{0,1}_k}$. A direct checking shows that the local expression (2.5) give us the expected global connection $\nabla^{\text{Hodge},n-k,k}$ on the bundle $V^{n-k,k}$. This shows the first part of the proposition.

The diagonal terms of F^∇ are

$$dA_k + (B_k \wedge \overline{B}_k - B_{k+1} \wedge \overline{B}_{k+1}),$$

$k = 0, \ldots, n$. By (2.4), $F^\nabla = 0$ if and only if

$$\partial\overline{\partial} \log |b_k|^2 - i\ d\rho = (-|b_{k-1}|^2 + 2|b_k|^2 - |b_{k+1}|^2)\varphi \wedge \overline{\varphi},$$

i.e.

$$\frac{1}{4}\Delta_g \log|b_k|^2 - \frac{K_g}{2} = -|b_{k-1}|^2 + 2|b_k|^2 - |b_{k+1}|^2,$$

$k = 1, \ldots, n$. Here we used that $d\rho = -\frac{i}{2}K_g\varphi \wedge \overline{\varphi}$. It simply means that

$$(u_1, \ldots, u_n) := (\log|b_1|^2, \ldots, \log|b_n|^2)$$

is a solution to (1.1). □

3. The associated Higgs bundles

We recall the definition in [5]. Let X be a complex manifold.

Definition 3.1. A Higgs bundle (E, Φ) consists of

(i) a holomorphic vector bundle E over X and

(ii) a holomorphic $\mathrm{End}(E)$-valued $(1,0)$-form Φ, called the Higgs field.

Suppose that H is a smooth hermitian metric on E. We will denote the Chern connection on E associated to H by ∇^H and denote the adjoint of Φ with respect to H by Φ^*. Then $\nabla := \Phi + \nabla^H + \Phi^*$ is a connection on E as well.

In the rest of this section we let (M, g) be a compact Riemann surface equipped with a smooth hermitian metric, $\mu = (\mu_1, \ldots, \mu_n)$ an assignment of singular strengths on M, and $X := M \setminus S_\mu$. We also fix a holomorphic spinor bundle $K_M^{\frac{1}{2}}$ on M in our discussion and let $L := (K_M^{\frac{1}{2}})^{\otimes n}$.

Definition 3.2. We define

$$\widetilde{E} := \oplus_k \widetilde{E}^{n-k,k} \text{ and } \widetilde{\Phi} := \oplus_k \widetilde{\Phi}_k$$

where

$$\widetilde{E}^{n-k,k} := L \otimes K_M^{-k}$$

and

$$\Phi_k \longleftrightarrow 1 \in \Gamma\big(M, K_M \otimes \mathrm{Hom}(L \otimes K_M^{-k+1}, L \otimes K_M^{-k})\big),$$

$k = 1, \ldots, n$. Then $(\widetilde{E}, \widetilde{\Phi})$ is a Higgs bundle on M. $(E, \Phi = \oplus_k \Phi_k) := (\widetilde{E}, \widetilde{\Phi})|_X$ is called the associated Higgs bundle on X.

Definition 3.3. A smooth hermitian metric H on E is called of type μ if $H = \oplus_k H_k$ for some smooth hermitian metrics H_k on $\widetilde{E}^{n-k,k}|_X$ such that for any $p \in M$ and any sufficiently small coordinate chart (U, z) centered at p, the function $|z|^{2d_{k,p}(\mu)}$ (where $d_{k,p}(\mu)$ is the number defined by (1.2)) and the expression of H_k with respect to any local trivialization of $\widetilde{E}^{n-k,k}$ are mutually bounded multiplicatively.

Proposition 3.1. *There is a bijection between*

$$\{(b_1, \dots, b_n) \in \mathcal{C}^\infty(X)^n | b_k > 0 \textit{ with } (\log b_1^2, \dots, \log b_n^2) \ \mu\textit{-admissible}\}$$

and

$$\{\textit{smooth hermitian metrics } H \textit{ on } E \textit{ of type } \mu \textit{ with } \det H = 1\}.$$

Under this bijection, those n-tuples $(b_1, \dots, b_n)$ with $(\log b_1^2, \dots, \log b_n^2)$ solving (1.1) correspond exactly to metrics H with $\Phi + \nabla^H + \Phi^*$ flat.

Proof. We make the following choices:

(a) Fix an atlas of holomorphic charts $\{(U_\alpha, z_\alpha)\}$ on each of whose member $K_M^{\frac{1}{2}}$ is trivialized by a holomorphic frame σ_α with $\sigma_\alpha^{\otimes 2} = dz_\alpha$. If $\sigma_\beta = \psi_{\alpha\beta}\sigma_\alpha$ on $U_\alpha \cap U_\beta$, then

$$\psi_{\alpha\beta}^2 = J_{\alpha\beta} := \frac{dz_\beta}{dz_\alpha}. \tag{3.1}$$

(b) When writing $g = \varphi_\alpha \cdot \overline{\varphi}_\alpha$ with a nonvanishing 1-form φ_α in a coordinate chart (U_α, z_α), we always choose φ_α to be the unique positive multiple of dz_α, i.e. $\varphi_\alpha = \lambda_\alpha dz_\alpha$, $\lambda_\alpha > 0$. We have

$$\frac{\lambda_\alpha}{\lambda_\beta} = |J_{\alpha\beta}| \tag{3.2}$$

on $U_\alpha \cap U_\beta$. In addition, if $d\varphi_\alpha = -i\rho_\alpha \wedge \varphi_\alpha$ for a real 1-form ρ_α, then

$$\frac{\overline{\partial}\lambda_\alpha}{\lambda_\alpha} = -i\rho_\alpha^{0,1}. \tag{3.3}$$

We start with $(b_1, \dots, b_n)$ first. By Proposition 2.1, we have the unique diagonal SL-complex pre-VHS $(V = \oplus_k V^{n-k,k}, \nabla)$ with Hodge field $(b_1, \dots, b_n)$ which is polarized by the metric h_g (2.3). (Recall (2.1) and (2.2).) Since X is a Riemann surface, ∇ defines a holomorphic structure on V. We will show that V is naturally isomorphic to E, the underlying bundle of the associated Higgs bundle, as holomorphic vector bundles. (In general $V^{n-k,k}$ is the restriction of $\widetilde{E}^{n-k,k}$ only as a smooth vector bundle but not as a holomorphic vector bundle.)

In each coordinate chart (U_α, z_α) we select the unitary frame $(e_k)_\alpha := (\sigma_\alpha/|\sigma_\alpha|)^{\otimes n} \otimes \varphi_\alpha^{\otimes -k}$ for $\widetilde{E}^{n-k,k}$, $k = 0, \ldots, n$. We need to get a holomorphic frame $(e'_k)_\alpha = (\delta_k)_\alpha (e_k)_\alpha$ on $U_\alpha \setminus S$ for each α. This means $(\nabla^{\mathrm{Hodge}})^{0,1}(e'_k)_\alpha = 0$ or, equivalently, (by (2.5) and (3.3))

$$\frac{\overline{\partial}(\delta_k)_\alpha}{(\delta_k)_\alpha} + \frac{\overline{\partial}[(b_1^n b_2^{n-1} \cdots b_n)^{\frac{1}{n+1}} (b_1 \cdots b_k)^{-1}]}{(b_1^n b_2^{n-1} \cdots b_n)^{\frac{1}{n+1}} (b_1 \cdots b_k)^{-1}} + \frac{\overline{\partial} \lambda_\alpha^{\frac{n}{2}-k}}{\lambda_\alpha^{\frac{n}{2}-k}} = 0.$$

We simply take

$$(\delta_k)_\alpha := (b_1^n b_2^{n-1} \cdots b_n)^{\frac{-1}{n+1}} b_1 \cdots b_k \lambda_\alpha^{k-\frac{n}{2}}.$$

Then

$$\frac{(e'_k)_\beta}{(e'_k)_\alpha} = \left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^{k-\frac{n}{2}} \left(\frac{\sigma_\beta}{\sigma_\alpha}\right)^n \left|\frac{\sigma_\beta}{\sigma_\alpha}\right|^{-n} \left(\frac{\varphi_\beta}{\varphi_\alpha}\right)^{-k} = \psi_{\alpha\beta}^n J_{\alpha\beta}^{-k}$$

by (3.1) and (3.2), which is exactly the transition function of the *holomorphic* vector bundle $\widetilde{E}^{n-k,k}|_X = E^{n-k,k}$. More precisely,

$$(e'_k)_\alpha \longleftrightarrow \sigma_\alpha^{\otimes n} \otimes dz_\alpha^{\otimes -k}$$

This shows that V (with the holomorphic structure obtained above) is isomorphic to E. We define a smooth metric H_k on $E_k \simeq V^{n-k,k}$ by setting on $U_\alpha \setminus S_\mu$

$$|(e'_k)_\alpha|^2_{H_k} = (H_k)_\alpha := |(\delta_k)_\alpha|^2,$$

which is of type μ by the assumption on $(b_1, \ldots, b_n)$.

By Definition 3.2,

$$\Phi_k = dz_\alpha \otimes (\sigma_\alpha^{\otimes n} \otimes dz_\alpha^{\otimes -k}) \otimes (\sigma_\alpha^{\otimes n} \otimes dz_\alpha^{\otimes -k+1})^{\otimes -1},$$

which corresponds to $dz_\alpha \otimes (e'_k)_\alpha \otimes (e'_{k-1})_\alpha^{\otimes -1}$. Therefore

$$|\Phi_k|_{g^* \otimes H_k \otimes H^*_{k-1}} = \frac{1}{\lambda_\alpha} \frac{|(\delta_k)_\alpha|}{|(\delta_{k-1})_\alpha|} = b_k. \tag{3.4}$$

It is direct to see that formula (3.4) actually gives us the recipe of going backwards from a metric H on $(E, \Phi = \oplus_k \Phi_k)$ of type μ with $\det H = 1$ to an n-tuple $(b_1, \ldots, b_n)$ of positive functions on X. It remains to check that $(\log b_1^2, \ldots, \log b_n^2)$ defined by (3.4) is μ-admissible if $H = \oplus_k H_k$ is of type μ. Note that $\widetilde{\Phi}_k$ is holomorphic on M and $\Phi_k = \widetilde{\Phi}_k|X$. Since H is of type μ, for any $p \in M$ and a sufficiently small coordinate chart (U, z) centered at p, there exist bounded functions $v_k \in \mathcal{C}^\infty(U \setminus \{p\})$ such that

$$\log b_k^2 = 2 \log |z|^{d_{k,p}(\mu) - d_{k-1,p}(\mu)} + v_k = 2\mu_k(p) \log |z| + v_k,$$

$k = 1, \ldots, n$, i.e. $(\log b_1^2, \ldots, \log b_n^2)$ is of type μ.

As for the last part of the proposition, simply note that V and E are isomorphic over X and the connection ∇ on V coincides with $\Phi + \nabla^H + \Phi^*$ on E under this identification. The proof is then completed by Proposition 2.1. $\square$

By Proposition 3.1, finding μ-admissible solutions to Toda systems of VHS type is equivalent to finding hermitian metrics H of type μ on E with $\det H = 1$ and with $\nabla = \Phi + \nabla^H + \Phi^*$ flat. In the next section we will obtain a criterion for the existence of such kind of metrics.

4. Stability and Higgs-Hermitian-Yang-Mills metrics

In this section we provide the criterion for the existence of solutions to Toda systems of VHS type for compact Riemann surfaces and prove their uniqueness. Our main ingredient is Simpson's theory of constructing Higgs-Hermitian-Yang-Mills metrics on Higgs bundles from stability [5]. Let (X, g) be a Kähler manifold satisfying suitable assumptions (cf. [5], Section 2, Assumptions 1,2 and 3), which will be fulfilled in our situation (cf. [5], Propositons 2.2 or 2.4). We use Λ_g to denote the adjoint operator of the Lefschetz operator $L = \omega\wedge$ on spaces of bundle-valued differential forms, where ω is the Kähler form. When acting on a bundle-valued differential 2-form η it is characterized by the property

$$\Lambda_g \eta \, \frac{\omega^n}{n!} = \eta \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Definition 4.1. ([5], p.877, 878)

(1) A sub-Higgs sheaf of a Higgs bundle (E, Φ) is an analytic subsheaf $\mathcal{V} \subset \mathcal{O}(E)$ such that $\Phi : \mathcal{V} \longrightarrow \mathcal{O}(K_X) \otimes \mathcal{V}$. (If $\mathcal{V}$ is saturated, outside a set of codimension 2 it is the coherent sheaf associated to a subbundle of E.)

(2) For a saturated subsheaf $\mathcal{V}$ and a smooth metric K on E such that $\sup_X |\Lambda_g F_K^{\text{Higgs}}|_K < \infty$,

$$\deg(\mathcal{V}, K) := i \int_X \operatorname{Tr} \Lambda_g F_K^{\text{Higgs}}.$$

(This is either a real number or $-\infty$ by [5], Lemma 3.2.)

We will need the notion of stability for Higgs bundles.

Definition 4.2. Let (E, Φ) be a Higgs bundle on X and A be a group acting on X by biholomorphic maps preserving the metric g and acting on E compatibly by automorphisms $a : E \longrightarrow E$ preserving the metric K and changing Φ by homotheties $a\Phi a^{-1} = \lambda(a)\Phi$. (E, Φ, K) is stable with respect to the A-action if for every proper saturated sub-Higgs sheaf $\mathcal{V}$ preserved by A,

$$\frac{\deg(\mathcal{V}, K)}{\operatorname{rk}(\mathcal{V})} < \frac{\deg(E, K)}{\operatorname{rk}(E)}.$$

For a Higgs bundle (E, Φ) on X with a smooth hermitian metric H, we denote the curvature of the connection $\Phi + \nabla^H + \Phi^*$ by F_H^{Higgs}.

Definition 4.3. H is a Higgs-Hermitian-Yang-Mills metric if the trace-free part of $\Lambda_g F_H^{\text{Higgs}}$ vanishes.

Theorem 4.1. (Simpson) *If (E, Φ, K) is stable with respect to the A-action, then there exists a smooth A-invariant Higgs-Hermitian-Yang-Mills metric H with H and K mutually bounded,* $\det H = \det K$, *and* $\overline{\partial} h + [\Phi, h] \in L^2_{g,K}$, *where h is the unique endomorphism of E such that*

$$(\cdot, \cdot)_H = (h(\cdot), \cdot)_K.$$

If furthermore $\Phi \wedge \Phi = 0$, the first Chern form $c_1(E, K) = 0$, and $\int_X c_2(E, K) \wedge \omega_g^{n-2} = 0$, then the connection $\nabla = \Phi + \nabla^H + \Phi^*$ is flat. Conversely, if there exists an A-invariant Higgs-Hermitian-Yang-Mills metric, then

$$\frac{\deg(\mathcal{V}, K)}{\operatorname{rk}(\mathcal{V})} \leq \frac{\deg(E, K)}{\operatorname{rk}(E)}$$

for every proper saturated sub-Higgs sheaf $\mathcal{V}$ preserved by A and equality holds only if $E = \mathcal{V} \oplus \mathcal{V}^{\perp}$ is an orthogonal direct sum of Higgs subbundles.

Now we are ready to prove our main result. Let M be a compact Riemann surface and $g = ds^2$ be a smooth hermitian metric on M. Recall that for an assignment of singular strengths μ on M we have defined in the introduction the numbers

$$d_k(\mu) := \sum_{p\in M}\left(\frac{-1}{n+1}\sum_{j=1}^{n}(n+1-j)\mu_j(p) + \sum_{j=1}^{k}\mu_j(p)\right),$$

$k = 1, \ldots, n$.

Theorem 4.2. *There exists a μ-admissible solution to (1.1) if and only if*

$$d_{n-m+1}(\mu) + \cdots + d_n(\mu) < m(n-m+1)(\mathrm{genus}(M)-1), \tag{4.1}$$

$m = 1, \ldots, n$. *In addition, the μ-admissible solution is unique if it exists.*

Proof. Let (E, Φ) be the associated Higgs bundle (Definition 3.2). Let A be the group $U(1)^{\times(n+1)} \cap SL(n+1, \mathbf{C})$ acting on X trivially and on E in the obvious diagonal manner. We prove the statement about uniqueness first. The argument is essentially the same as that in the proof of Lemma 10.9 of [5]. Suppose we have two solutions corresponding to Higgs-Hermitian-Yang-Mills metrics H and H' respectively. Let h be the unique endomorphism of E such that $(\cdot,\cdot)_{H'} = (h(\cdot),\cdot)_H$. Note that h is positive definite self-adjoint and bounded with respect to H. Taking trace of Lemma 3.1 (c) in [5] gives

$$\Delta_d \mathrm{Tr}\ h = 2\Delta_\partial \mathrm{Tr}\ h = -\left|\left(\overline{\partial}h + [\Phi, h]\right)h^{\frac{1}{2}}\right|_H^2 \le 0.$$

As mentioned above, Assumption 3 in [5] holds for $(X, g|_X)$, and hence a positive bounded subharmonic function must be harmonic. Therefore,

$$\overline{\partial}h + [\Phi, h] = 0,$$

as is equivalent to saying that h is a holomorphic endomorphism of E commuting with Φ. Since h commutes with the A-action, it acts on E diagonally by multiplication with positive numbers $h_0, \ldots, h_n$. The commutativity of h with Φ implies $h_0 = \cdots = h_n$. Finally, $h_0 \cdots h_n = 1$ since $\det H' = \det H = 1$. This shows that $h = \mathrm{id}_{\mathrm{E}}$ and completes the proof.

We equip E with a metric $K := \oplus_k K_k$ of type μ such that $\det K = 1$. Such a metric can be constructed as follows. Choose a covering of M by coordinate disks (U_α, z_α) and a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. We may assume that

(1) each $s \in S_\mu$ is contained in U_α for exactly one α, denoted as $\alpha(s)$, and $z_{\alpha(s)}(s) = 0$;

(2) for every $s \in S_\mu$, there exists an open neighborhood $W_{\alpha(s)} \subset U_{\alpha(s)}$ of s such that $\rho_{\alpha(s)}|_{W_{\alpha(s)}} \equiv 1$;

(3) every $\widetilde{E}^{n-k,k}$ (Definition 3.2) is trivialized on U_α by a holomorphic local frame $e_{k,\alpha}$.

Let σ be an element in the fibre of E_k above $p \in M \setminus S_\mu$. We write

$$\sigma = \sigma_\alpha e_{k,\alpha}(p)$$

if $p \in U_\alpha$. Recall the numbers $d_{k,s}(\mu)$ defined by (1.2). For $p \in U_\alpha$, we define

$$f_\alpha(p) := \begin{cases} |z_\alpha(p)|^{2d_{k,s}(\mu)}, & \text{if } U_\alpha \cap S_\mu = \{s\}; \\ 1, & \text{otherwise.} \end{cases}$$

The metric K_k on E_k for $k = 1, \ldots, n$ is defined as follows:

$$|\sigma|^2_{K_k} := \sum_{\alpha : p \in U_\alpha} \rho_\alpha(p) f_\alpha(p) |\sigma_\alpha|^2.$$

Finally, we let K_0 on E_0 be defined so that $\det K = 1$ under the canonical isomorphism between $\det E$ and the trivial line bundle. Then $c_1(E, K) = 0$ by the construction of K. $\Phi \wedge \Phi = 0$ and $c_2(E, K) = 0$ automatically on a Riemann surface.

Since dim $M = 1$, proper saturated sub-Higgs sheaves are exactly proper holomorphic subbundles of E preserved by Φ. By the form of Φ (a nilpotent string), it is clear that such kind of subbundles are exactly

$$F^m := E_{n-m+1} \oplus \cdots \oplus E_n,$$

$m = 1, \ldots, n$.

$$\deg(F^m, K) = \sum_{k=n-m+1}^{n} \deg(E_k, K_k).$$

Note that for each k the smooth metric K_k on E_k can be viewed as a singular metric on $\widetilde{E}^{n-k,k}$, whose curvature current represents the first Chern class of $\widetilde{E}^{n-k,k} = L \otimes K_M^{\otimes -k}$. By the Poincaré–Lelong formula,

$$\deg \widetilde{E}^{n-k,k} = \deg(E_k, K_k) - d_k(\mu).$$

Therefore

$$\deg(E_k, K_k) = (\text{genus}(M) - 1)(n - 2k) + d_k(\mu),$$

and hence

$$\deg(F^m, K) = (\text{genus}(M) - 1)m(m - n - 1) + d_{n-m+1}(\mu) + \cdots + d_n(\mu).$$

It is clear that $\deg(E, K) = 0$. If (4.1) holds for $m = 1, \ldots, n$, then (E, Φ, K) is stable. By the first part of Simpson's theorem we obtain a μ-admissible solution to (1.1). Conversely, if there exists such a solution, by the second part of Simpson's theorem, we have

$$d_{n-m+1}(\mu) + \cdots + d_n(\mu) \leq m(n - m + 1)(\text{genus}(M) - 1),$$

$m = 1, \ldots, n$. Actually, all inequalities above are strict since none of the orthogonal complement of F^m is a sub-Higgs sheaf. Therefore (4.1) holds for $m = 1, \ldots, n$. □

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Department of Mathematics, National Taiwan University
No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan
E-mail address: chi@math.ntu.edu.tw, geometrychi@yahoo.com.tw

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