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Bernstein theorems for minimal cones with weak stability

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In this paper, we derive some rigidity results for minimal cones under some weak stability conditions both in codimension 1 and higher codimension.

1. Introduction

The classic papers [5] by Simons and [3] by Scheon-Simon-Yau contains proofs of the fact that all n-dimensional minimal stable cones in $\mathbb{R}^{n+1}$ are trivial for $2 \le n \le 6$. The stability condition is $\int_M |A|^2\xi^2 \le \int_M |\nabla_M \xi|^2$ for the second fundamental form of $M^n \subset \mathbb{R}^{n+1}$ and all $\xi \in C_0^1(M)$. On the other hand, the same stability inequality is unavailable for stable minimal cones (also minimal surfaces) of high codimensions. Even for n-dimensional stable normal flat cones - codimensional one surfaces are all automatically normal flat - the stability inequality becomes a weaker one, $\frac{1}{n}\int_M |A|^2\xi^2 \le \int_M |\nabla_M \xi|^2$. In [8] by Xin, [7] by Xin-Yang, and [6] by Smocyk-Wang-Xin, the authors obtained some Bernstein type results for higher codimensional graph-like minimal submanifolds under certain restrictions of Gauss map and/or normal flatness assumptions. For special Lagrangian graphical cones, it is well-known that three dimensional ones are flat (cf. [9]). The flatness of four dimensional ones follows from [2] by Nadirashvili-Vladuct.

In this paper, we study rigidity of minimal cones under a weaker stability condition

$$\alpha \int_M |A|^2\xi^2 \le \int_M |\nabla_M \xi|^2, \tag{1.1}$$

for a dimensional constant $\alpha < 1$ and any $\xi \in C_0^1(M)$. Meanwhile, under a stronger stability condition ($\alpha > 1$), we could reach the rigidity of minimal cones of higher dimensions than the known ones.

Theorem 1.1. *Let $M \subset \mathbb{R}^{n+1}, n \geq 2$ be an n-dimensional minimal cone with $\overline{M} \setminus M = \{0\}$. If (1.1) holds for*

$$\alpha > \frac{n}{2} - \frac{1 + (n-2)\sqrt{n}}{n-1} \tag{1.2}$$

then M must be a hyperplane.

Remark 1.1. Particularly, for dimension $n = 6$, we just need $\alpha > 0.84$. This condition is weaker than $\alpha = 1$ in [3]. And for $n \leq 5$, the corresponding conditions are as follows: (1) if $0.134 < \alpha \leq \frac{1}{3}$, then $2 \leq n \leq 3$; (2) if $\frac{1}{3} < \alpha \leq 0.573$, then $2 \leq n \leq 4$; (3) if $0.573 < \alpha \leq 0.840$, then $2 \leq n \leq 5$. Moreover, if $n \geq 7$, we need $\alpha > 1$.

Theorem 1.2. *Let $M \subset \mathbb{R}^{2n}$, $n \geq 2$ be an n-dimensional special Lagrangian cone with $\overline{M} \setminus M = \{0\}$. If (1.1) holds for*

$$\begin{aligned} \alpha > \max \Bigg\{ &\frac{3}{2}\left[1 - \frac{3}{(n-1)^2}\right], \\ &\frac{3}{4}\left[n + 1 - \frac{3 + \sqrt{3(n-1)^4(2n-1) - 6(n-1)^2(n+1) + 9}}{(n-1)^2}\right] \Bigg\}. \end{aligned} \tag{1.3}$$

then M must be a hyperplane.

Remark 1.2. In particular, (1) if $0.375 < \alpha \leq 1$, then $2 \leq n \leq 3$; (2) if $1 < \alpha \leq 1.219$, then $2 \leq n \leq 4$; (3) if $1.219 < \alpha \leq 1.32$, then $2 \leq n \leq 5$; (4) if $1.32 < \alpha \leq 1.375$, then $2 \leq n \leq 6$.

Theorem 1.3. *Let $M \subset \mathbb{R}^{n+m}$, $n, m \geq 2$ be an n-dimensional minimal cone with $\overline{M} \setminus M = \{0\}$. If (1.1) holds for*

$$\begin{aligned} \alpha > max \Bigg\{ &\frac{3}{2}\left[1 - \frac{2}{m(n-1)}\right], \\ &\frac{3}{4}\left[n - \frac{2 + 2\sqrt{m^2(n-1)^3 - mn(n-1) + 1}}{m(n-1)}\right] \Bigg\}. \end{aligned} \tag{1.4}$$

then M must be a hyperplane.

In closing, we show that the weak stability condition is necessary certainly in Theorem 1.1 by simple examples. Consider the Simon's type cones

$C_{p,q} := \{(x,y) = (x_1, \ldots, x_p, y_1, \ldots, y_q) \in \mathbb{R}^{p+q} : x_1^2 + \cdots + x_p^2 = y_1^2 + \cdots + y_q^2\}$.
Claim. $C_{p,q}$ is an $n(= p+q-1)$-dimensional minimal cone in $\mathbb{R}^{n+1}$ with coefficient $\alpha < 1$ in (1.1), when $n \leq 6$.

In fact, it's easy to check that the mean curvature $H \equiv 0$, i.e. $C_{p,q}$ is minimal, and the norm of the second fundamental form of $C_{p,q}$ satisfies $|A|^2 = \frac{n-1}{r^2}$. By the stationary assumption of $M(= C_{p,q})$, the first variation satisfies $\int_M div_M X = 0$ (see [4]), for any smooth vector X with compact support in M. Choose $X = \frac{\xi^2}{r^2} x$, where $\xi \in C_0^1(M)$, and $x \in M$. Using Schwarz inequality, we obtain

$$\int_M \frac{(n-2)^2}{4r^2} \xi^2 \leq \int_M |\nabla_M \xi|^2,$$

Note that $|A|^2 = \frac{n-1}{r^2}$, thus

$$\alpha \int_M |A|^2 \xi^2 \leq \int_M |\nabla_M \xi|^2, \text{ where } \alpha = \frac{(n-2)^2}{4(n-1)}.$$

Particularly, we have that: (1) $\alpha = \frac{1}{8}$, if $n = 3$; (2) $\alpha = \frac{1}{3}$, if $n = 4$; (3) $\alpha = \frac{9}{16}$, if $n = 5$; (4) $\alpha = \frac{4}{5}$, if $n = 6$.

2. Proof of codimension 1

In this section, we will prove the codimension 1 case, that is Theorem 1.1.

Suppose that $M \subset \mathbb{R}^{n+1}$ is an n-dimensional minimal embedded submanifold. We denote the tangent bundle of M by TM. Choose a locally defined orthonormal frame in TM, $\tau_1, \ldots, \tau_n$ and a unit normal vector ν of M. Then $\tau_1, \ldots, \tau_n, \nu$ consist a basis of $\mathbb{R}^{n+1}$. The second fundamental form refers to $A = h_{ij} \tau_i \bigotimes \tau_j$, where $h_{ij} = -\langle \nabla_{\tau_i} \nu, \tau_j \rangle, i, j = 1, \ldots, n$. Now the well-known Simons' identity follows (see [3])

$$\Delta_M \left(\frac{1}{2} |A|^2 \right) = h_{ij,k}^2 - |A|^4 + h_{ij} H_{,ij} + H h_{ij} h_{mi} h_{mj}.$$

Furthermore, as M is minimal, the mean curvature vanishes i.e. $H = \sum_{i=1}^n h_{ii} = 0$. Therefore,

$$\Delta_M \left(\frac{1}{2} |A|^2 \right) = |\nabla_M A|^2 - |A|^4. \tag{2.5}$$

Lemma 2.1. *Suppose that $M \subset \mathbb{R}^{n+1}$ is an n-dimensional minimal cone with $\overline{M} \setminus M = \{0\}$, then*

$$|\nabla_M A|^2 \geq \left(1 + \frac{2}{n-1}\right) |\nabla_M |A||^2 + 2\left(1 - \frac{1}{n-1}\right) r^{-2}|A|^2. \tag{2.6}$$

Proof. For a fixed point $x \in M$, we choose a frame in TM, denoted by $\tau_1, \ldots, \tau_n$ where $\tau_n = \frac{x}{|x|}$, and a unit normal vector ν, such that $\Gamma_{ij}^k = 0$, and $h_{ij} = 0, i \neq j$. Then,

$$h_{in} = 0, \quad h_{ij,n} = -r^{-1}h_{ij}, \quad i,j = 1, \ldots, n. \tag{2.7}$$

In view of (2.7), we compute

$$\begin{aligned} |\nabla_M A|^2 = \sum_{i,j,k=1}^{n} h_{ij,k}^2 &\geq \sum_{i,j,k=1}^{n-1} h_{ij,k}^2 + 3\sum_{i,j=1}^{n-1} h_{ij,n}^2 \\ &= \sum_{i,j,k=1}^{n-1} h_{ij,k}^2 + 3r^{-2}\sum_{i,j=1}^{n} h_{ij}^2 \\ &= \sum_{i,j,k=1}^{n-1} h_{ij,k}^2 + 3r^{-2}|A|^2. \end{aligned}$$

and,

$$\begin{aligned} |\nabla_M |A||^2 &= |A|^{-2}\sum_{k=1}^{n}\left(\sum_{i=1}^{n-1} h_{ii}h_{ii,k}\right)^2 \\ &= |A|^{-2}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1} h_{ii}h_{ii,k}\right)^2 + |A|^{-2}\left(\sum_{i=1}^{n-1} h_{ii}h_{ii,n}\right)^2 \\ &= |A|^{-2}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1} h_{ii}h_{ii,k}\right)^2 + r^{-2}|A|^2. \end{aligned}$$

Let

$$|\nabla_M A|_{n-1}^2 = \sum_{i,j,k=1}^{n-1} h_{ij,k}^2,$$

and

$$|\nabla_M |A||_{n-1}^2 = |A|^{-2}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1} h_{ii}h_{ii,k}\right)^2.$$

Here the subscript $(n-1)$ about $|\nabla_M A|^2$ and $|\nabla_M |A||^2$ means that we compute them in the sense of that M is an $(n-1)$-dimensional submanifold. Thus, the above two estimates show

$$\begin{aligned}&|\nabla_M A|^2 \geq |\nabla_M A|^2_{n-1} + 3r^{-2}|A|^2,\\ &\text{and } |\nabla_M |A||^2 = |\nabla_M |A||^2_{n-1} + r^{-2}|A|^2.\end{aligned} \tag{2.8}$$

In virtue of the minimal assumption of M and (2.7), we have $\sum_{i=1}^{n-1} h_{ii} = -h_{nn} = 0$ which implies M is also an $(n-1)$-dimensional minimal submanifold. Thus

$$|\nabla_M A|^2_{n-1} \geq \left(1 + \frac{2}{n-1}\right) |\nabla_M |A||^2_{n-1}. \tag{2.9}$$

Combing with (2.8) and (2.9) gives (2.6). □

Now we start to prove Theorem 1.1.

Proof. Using (2.5) and (2.6),

$$\begin{aligned}2\left(1 - \frac{1}{n-1}\right) r^{-2}|A|^2 \leq \Delta_M \left(\frac{1}{2}|A|^2\right)\\ - \left(1 + \frac{2}{n-1}\right) |\nabla_M |A||^2 + |A|^4.\end{aligned} \tag{2.10}$$

Multiplying $|A|^{2(\alpha-1)}\xi^2$ with $\xi \in C_0^1(M)$ and integrating over M on both side, and using integration by parts, we arrive at

$$\begin{aligned}&2\left(1 - \frac{1}{n-1}\right) \int_M r^{-2}|A|^{2\alpha}\xi^2\\ &\leq -2\int_M |A|^{2\alpha-1}\xi\langle\nabla_M |A|, \nabla\xi\rangle\\ &\quad - \left(2\alpha - 1 + \frac{2}{n-1}\right) \int_M |\nabla_M |A||^2|A|^{2(\alpha-1)}\xi^2 + \int_M |A|^{2(\alpha+1)}\xi^2.\end{aligned}$$

On the other hand, applying (1.1) with $|A|^\alpha\xi$ in place of ξ gives

$$\begin{aligned}\int_M |A|^{2(1+\alpha)}\xi^2 \leq \alpha \int_M |\nabla_M |A||^2|A|^{2(\alpha-1)}\xi^2\\ + \alpha^{-1}\int_M |A|^{2\alpha}|\nabla\xi|^2 + 2\int_M |A|^{2\alpha-1}\xi\langle\nabla_M |A|, \nabla\xi\rangle.\end{aligned}$$

We continue

$$
\begin{aligned}
&2\left(1-\frac{1}{n-1}\right)\int_M r^{-2}|A|^{2\alpha}\xi^2\\
&\leq -\left(\alpha-1+\frac{2}{n-1}\right)\int_M |\nabla_M|A||^2|A|^{2(\alpha-1)}\xi^2+\frac{1}{\alpha}\int_M |A|^{2\alpha}|\nabla\xi|^2.
\end{aligned}
$$

Since $\alpha \geq 1-\frac{2}{n-1}$ (by (1.2)), then

$$
2\left(1-\frac{1}{n-1}\right)\int_M r^{-2}|A|^{2\alpha}\xi^2 \leq \frac{1}{\alpha}\int_M |A|^{2\alpha}|\nabla\xi|^2. \tag{2.11}
$$

We claim that (2.11) holds for any $\xi \in Lip(M)$ with

$$
\int_M r^{-2}|A|^{2\alpha}\xi^2 < \infty. \tag{2.12}
$$

In fact, substituting ξ by $\xi\eta_\epsilon$ in (2.11) and letting $\epsilon \to 0$ and using (2.12), we can conclude. Here, η_ϵ satisfies $\eta_\epsilon(x) = \eta_\epsilon(|x|) \in [0,1], |\nabla\eta_\epsilon|(x) \leq \frac{2}{|x|}, \eta_\epsilon(x) \equiv 1,$ for $|x| \in (\epsilon^{-1}, \epsilon),$ and $\eta_\epsilon(|x|) \equiv 0,$ for $|x| < \frac{\epsilon}{2},$ or $|x| > \frac{2}{\epsilon}.$

Take $\xi = r^{1-\frac{n}{2}+\alpha+\epsilon}r_1^{-2\epsilon}$ in (2.11), where $r_1 = max\{1, r\}$ and let $\theta = \int_\Sigma |A|^{2\alpha}$, where $\Sigma = M \cap S^{n-1}$, we obtain

$$
2\left(1-\frac{1}{n-1}\right)\theta\int_0^\infty r^{n-3-2\alpha}\xi^2 dr \leq \frac{\theta}{\alpha}\int_0^\infty r^{n-1-2\alpha}|\nabla\xi|^2 dr. \tag{2.13}
$$

After an easy computation, we get

$$
\begin{aligned}
&2\left(1-\frac{1}{n-1}\right)\theta\left[\int_0^1 r^{-1+2\epsilon}dr+\int_1^\infty r^{-1-2\epsilon}dr\right],\\
&\leq \frac{\theta}{\alpha}\left[\left(1-\frac{n}{2}+\alpha+\epsilon\right)^2\int_0^1 r^{-1+2\epsilon}dr+\left(1-\frac{n}{2}+\alpha-\epsilon\right)^2\int_1^\infty r^{-1-2\epsilon}dr\right].
\end{aligned}
$$

If $\theta \equiv 0$, i.e. $|A| \equiv 0$, then we can conclude. Otherwise, when

$$
\frac{n^2-n-2-2(n-2)\sqrt{n}}{2(n-1)} < \alpha < \frac{n^2-n-2+2(n-2)\sqrt{n}}{2(n-1)},
$$

we can choose a suitable $\epsilon > 0$, such that

$$
\left(1-\frac{n}{2}+\alpha\pm\epsilon\right)^2 < 2\alpha\left(1-\frac{1}{n-1}\right).
$$

There is a contradiction to (2.16). For $\alpha \geq \frac{n^2+n-2+2(n-2)\sqrt{n}}{2(n-1)}$, clearly (1.1) holds for some α_0, with $\frac{n^2-n-2-2(n-2)\sqrt{n}}{2(n-1)} < \alpha_0 < \frac{n^2-n-2+2(n-2)\sqrt{n}}{2(n-1)}$. By the above discussion, we can conclude. □

3. Proof of special Lagrangian submanifolds

Let $M \subset \mathbb{R}^n \times \mathbb{R}^n,\ n \geq 2$ be an n-dimensional special Lagrangian submanifold. TM and NM refer to the tangent and normal bundle of M respectively. Take an orthonormal frame in TM, denoted by $\tau_1, \ldots, \tau_n$. Clearly, $J\tau_1, \ldots, J\tau_n$ is a frame in NM, and $\tau_1, \ldots, \tau_n, J\tau_1, \ldots, J\tau_n$ consist a basis of $\mathbb{R}^{2n}$, where J is the complex structure in $\mathbb{C}^n = \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n$, satisfying $J\partial_x = \partial_y$ and $J\partial_y = -\partial_x$. The second fundamental form of M is defined as follows:

$$\begin{gathered}
\langle B(\tau_i, \tau_j), J\tau_k \rangle = \langle A^{J\tau_k}(\tau_i), \tau_j \rangle, h_{ijk} = \langle \nabla_{\tau_i}\tau_j, J\tau_k \rangle, \\
B(\tau_i, \tau_j) = h_{ijk} J\tau_k, 1 \leq i, j, k \leq n, \\
(\nabla \tau_l B)(\tau_i, \tau_j) = h_{ijk,l} J\tau_k,\ 1 \leq i, j, l \leq n.
\end{gathered}$$

Then we know all the subscripts of h are symmetric and the Codazzi equations are as below

$$h_{ijk,l} = h_{ijl,k}, \quad i, j, k, l = 1, \ldots, n. \tag{3.14}$$

Cause M is minimal, the mean curvature satisfies

$$H^i = \sum_{j=1}^{n} h_{ijj} = 0, \quad i = 1, \ldots, n. \tag{3.15}$$

Lemma 3.1. *Let M be an n-dimensional special Lagrangian submanifold in $\mathbb{R}^{2n}$, then*

$$|\nabla_M A|^2 \geq \left(1 + \frac{3}{n^2}\right) |\nabla_M |A||^2. \tag{3.16}$$

Proof. Choose an orthonormal frame in TM, $\tau_1, \ldots, \tau_n$. Again, we see that $\tau_1, \ldots, \tau_n, J\tau_1, \ldots, J\tau_n$ is a basis of $\mathbb{R}^{2n}$. Denote $A^s = A^{J\tau_s}$, then $|A^s|^2 =$

$\sum_{i,j} h_{sij}^2$, and $|\nabla_M|A^s||^2 = |A^s|^{-2}\sum_k \left(\sum_{i,j} h_{sij}h_{sij,k}\right)^2$, $1 \le s \le n$. Without loss of generality, we assume that

$$|\nabla_M|A^1||^2 = \max_{1\le s\le n} |\nabla_M|A^s||^2.$$

Furthermore, by choosing a suitable frame $\tau_1, \ldots, \tau_n$ in TM, we suppose that

$$h_{1ij} = 0,\ i \ne j, \tag{3.17}$$

By the above assumptions, we compute

$$\begin{aligned}
|\nabla_M|A||^2 &= |A|^{-2}\sum_{k=1}^n \left(\sum_{s,i,j=1}^n h_{sij}h_{sij,k}\right)^2 \\
&\le n\frac{\sum_{s=1}^n \sum_{k=1}^n \left(\sum_{i,j=1}^n h_{sij}h_{sij,k}\right)^2}{\sum_{s=1}^n |A^s|^2} \\
&\le n|A^1|^{-2}\sum_{k=1}^n \left(\sum_{i,j=1}^n h_{1ij}h_{1ij,k}\right)^2 \\
&= n|A^1|^{-2}\sum_{k=1}^n \left(\sum_{i=1}^n h_{1ii}h_{1ii,k}\right)^2 \\
&\le n\sum_{i,k} h_{1ii,k}^2 = n\sum_{i=1}^n h_{1ii,i}^2 + n\sum_{i\ne k} h_{1ii,k}^2 \\
&= n\sum_{i=1}^n \left(\sum_{j\ne i} h_{1jj,i}\right)^2 + n\sum_{i\ne k} h_{1ii,k}^2 \\
&\le n(n-1)\sum_{i\ne j} h_{1jj,i}^2 + n\sum_{i\ne k} h_{1ii,k}^2 = n^2\sum_{i\ne k} h_{1ii,k}^2,
\end{aligned}$$

here we also used (3.14) and (3.15).

On the other hand, by Schwarz inequality and (3.17), we obtain

$$|\nabla_M|A||^2 = |A|^{-2}\sum_{k=1}^n \left(\sum_{s,i,j=1}^n h_{sij}h_{sij,k}\right)^2 \le \sum_{k=1}^n \sum_{s,i,j=2}^n h_{sij,k}^2 + 3\sum_{i,k=1}^n h_{1ii,k}^2,$$

Finally, we arrive at

$$|\nabla_M A|^2 - |\nabla_M |A||^2 \geq 3\sum_k \sum_{i>1} h_{11i,k}^2 + 3\sum_k \sum_{i\neq j>1} h_{1ij,k}^2 \geq \frac{3}{2}\sum_{i\neq j,k} h_{1ij,k}^2$$
$$\geq \frac{3}{2}\left(\sum_{i\neq j} h_{1ij,i}^2 + \sum_{i\neq j} h_{1ij,j}^2\right) = 3\sum_{i\neq k} h_{1ii,k}^2 \geq \frac{3}{n^2}|\nabla_M |A||^2.$$

□

Lemma 3.2. *Suppose that $M \subset \mathbb{R}^{2n}$, $n \geq 2$ is an n-dimensional special Lagrangian cone with $\overline{M} \setminus M = \{0\}$, then*

$$|\nabla_M A|^2 \geq \left(1 + \frac{3}{(n-1)^2}\right)|\nabla_M |A||^2 + 3\left(1 - \frac{1}{(n-1)^2}\right) r^{-2}|A|^2. \tag{3.18}$$

Proof. For a fixed point $x \in M$, we choose a locally orthonormal frame in TM, denoted by $\tau_1, \ldots, \tau_n$ where $\tau_n = \frac{x}{|x|}$. Again, $\tau_1, \ldots, \tau_n, J\tau_1, \ldots, J\tau_n$ consist a basis of $\mathbb{R}^{2n}$, and

$$h_{ijn} = 0, \quad h_{ijk,n} = -r^{-1}h_{ijk}, \quad i,j,k = 1, \ldots, n. \tag{3.19}$$

Using (3.19), we compute

$$|\nabla_M A|^2 \geq \sum_{i,j,k,l=1}^{n-1} h_{ijk,l}^2 + 4\sum_{i,j,k=1}^{n-1} h_{ijk,n}^2 = \sum_{i,j,k,l=1}^{n-1} h_{ijk,l}^2 + 4r^{-2}\sum_{i,j,k=1}^{n-1} h_{ijk}^2$$
$$= \sum_{i,j,k,l=1}^{n-1} h_{ijk,l}^2 + 4r^{-2}|A|^2.$$

Also,

$$|\nabla_M |A||^2 = |A|^{-2}\sum_{l=1}^{n-1}\left(\sum_{i,j,k=1}^{n-1} h_{ijk}h_{ijk,l}\right)^2 + |A|^{-2}\left(\sum_{i,j,k=1}^{n-1} h_{ijk}h_{ijk,n}\right)^2$$
$$= |A|^{-2}\sum_{l=1}^{n-1}\left(\sum_{i,j,k=1}^{n-1} h_{ijk}h_{ijk,l}\right)^2 + r^{-2}|A|^{-2}\left(\sum_{i,j,k=1}^{n-1} h_{ijk}^2\right)^2$$
$$= |A|^{-2}\sum_{l=1}^{n-1}\left(\sum_{i,j,k=1}^{n-1} h_{ijk}h_{ijk,l}\right)^2 + r^{-2}|A|^2$$

Denote

$$|\nabla_M A|^2_{n-1} = \sum_{i,j,k,l=1}^{n-1} h^2_{ijk,l},$$

and

$$|\nabla_M |A||^2_{n-1} = |A|^{-2} \sum_{l=1}^{n-1} \left(\sum_{i,j,k=1}^{n-1} h_{ijk} h_{ijk,l} \right)^2 .$$

Then the above inequalities imply

$$\begin{aligned} &|\nabla_M A|^2 \geq |\nabla_M A|^2_{n-1} + 4r^{-2}|A|^2, \\ &\text{and } |\nabla_M |A||^2 = |\nabla_M |A||^2_{n-1} + r^{-2}|A|^2. \end{aligned} \tag{3.20}$$

Since M is minimal, we have

$$\sum_{j=1}^{n-1} h_{ijj} = -h_{inn} = 0, \quad i = 1, \ldots, n. \tag{3.21}$$

Hence M is an $(n-1)$-dimensional minimal submanifold. And by Lemma 3.1,

$$|\nabla_M A|^2_{n-1} \geq \left(1 + \frac{3}{(n-1)^2}\right) |\nabla_M |A||^2_{n-1}. \tag{3.22}$$

Our Lemma follows from (3.20) and (3.22). □

Finally, we are ready to prove Theorem 1.2.

Proof. Using Lemma 3.2 and the fact for higher codimensional minimal submanifolds (see [1])

$$\Delta_M \left(\frac{1}{2}|A|^2\right) \geq |\nabla_M A|^2 - \frac{3}{2}|A|^4 \tag{3.23}$$

we obtain

$$\begin{aligned} 3\left(1 - \frac{1}{(n-1)^2}\right) r^{-2}|A|^2 \leq \Delta_M &\left(\frac{1}{2}|A|^2\right) \\ &- \left(1 + \frac{3}{(n-1)^2}\right) |\nabla_M |A||^2 + \frac{3}{2}|A|^4. \end{aligned} \tag{3.24}$$

Multiplying $|A|^{2(\frac{2}{3}\alpha-1)}\xi^2$ with $\xi\in C_0^1(M)$ and integrating over M on both side, and applying integration by parts, we arrive at

$$\begin{aligned}&3\left(1-\frac{1}{(n-1)^2}\right)\int_M r^{-2}|A|^{\frac{4}{3}\alpha}\xi^2\\ \geq&-2\int_M |A|^{\frac{4}{3}\alpha-1}\xi\langle\nabla_M|A|,\nabla\xi\rangle+\frac{3}{2}\int_M|A|^{2(\frac{2}{3}\alpha+1)}\xi^2\\ &-\left(\frac{4}{3}\alpha-1+\frac{3}{(n-1)^2}\right)\int_M|\nabla_M|A||^2|A|^{2(\frac{2}{3}\alpha-1)}\xi^2.\end{aligned}$$

Using (1.1) with $|A|^{\frac{2}{3}\alpha}\xi$ in place of ξ gives

$$\begin{aligned}\int_M|A|^{2(\frac{2}{3}\alpha+1)}\xi^2\leq&\frac{4\alpha}{9}\int_M|\nabla_M|A||^2\,|A|^{2(\frac{2}{3}\alpha-1)}\xi^2\\ &+\frac{1}{\alpha}\int_M|A|^{\frac{4}{3}\alpha}|\nabla\xi|^2+\frac{4}{3}\int_M|A|^{\frac{4\alpha}{3}-1}\xi\langle\nabla_M|A|,\nabla\xi\rangle.\end{aligned}$$

In conjunction with this, we continue

$$\begin{aligned}&3\left(1-\frac{1}{(n-1)^2}\right)\int_M r^{-2}|A|^{\frac{4}{3}\alpha}\xi^2\\ \leq&-\left(\frac{2}{3}\alpha-1+\frac{3}{(n-1)^2}\right)\int_M|\nabla_M|A||^2|A|^{2(\frac{2}{3}\alpha-1)}\xi^2+\frac{3}{2\alpha}\int_M|A|^{\frac{4\alpha}{3}}|\nabla\xi|^2.\end{aligned}$$

Since $\frac{2}{3}\alpha-1+\frac{3}{(n-1)^2}\geq 0$ (by (1.3)),

$$3\left(1-\frac{1}{(n-1)^2}\right)\int_M r^{-2}|A|^{\frac{4}{3}\alpha}\xi^2\leq\frac{3}{2\alpha}\int_M|A|^{\frac{4}{3}\alpha}|\nabla\xi|^2. \tag{3.25}$$

Similarly, as discussed in the proof of Theorem 1.1, we know that (3.25) holds also for any $\xi\in Lip(M)$ with $\int_M r^{-2}|A|^{\frac{4}{3}\alpha}\xi^2<\infty$.

Choose $\xi=r^{1-\frac{n}{2}+\frac{2}{3}\alpha+\epsilon}r_1^{-2\epsilon}$ in (3.25). Let r_1 and θ be defined in Section 1. Then

$$\left(1-\frac{1}{(n-1)^2}\right)\theta\int_0^\infty r^{n-3-\frac{4}{3}\alpha}\xi^2dr\leq\frac{1}{2\alpha}\theta\int_0^\infty r^{n-1-\frac{4}{3}\alpha}|\nabla\xi|^2dr. \tag{3.26}$$

Again, we only consider the case

$$\frac{3}{4}\left[n+1-\frac{3+\sqrt{3(n-1)^4(2n-1)-6(n-1)^2(n+1)+9}}{(n-1)^2}\right]$$
$$<\alpha<\frac{3}{4}[n+1-\frac{3-\sqrt{3(n-1)^4(2n-1)-6(n-1)^2(n+1)+9}}{(n-1)^2}].$$

Then, we can choose $\epsilon>0$, so that $\left(1-\frac{n}{2}+\frac{2}{3}\alpha\pm\epsilon\right)^2<2\alpha\left(1-\frac{1}{(n-1)^2}\right)$. This leads $|A|\equiv 0$. $\square$

4. Proof of general higher codimension

This section is almost as same as section 3. In this section, $M\subset\mathbb{R}^{n+m},\ m\geq 2$ refers to be an n-dimensional minimal submanifold. TM and NM are same as in section 3. Take a frame in TM, $\tau_1,\ldots,\tau_n$, and a frame in NM, $\nu_1,\ldots,\nu_m$. In the sequel, if without specification, we agree $1\leq i,j,k\leq n,\ 1\leq\alpha,\beta\leq m$ and summation convention. The second fundamental form becomes

$$A=h_{\alpha ij}\tau_i\otimes\tau_j\otimes\nu_\alpha,\ h_{\alpha ij}=\langle\nabla_{\tau_i}\tau_j,\nu_\alpha\rangle,\quad 1\leq i,j\leq n,\ 1\leq\alpha\leq m.$$

Then $h_{\alpha ij}=h_{\alpha ji}$.

Lemma 4.1. *Suppose that $M\subset\mathbb{R}^{n+m},\ n,m\geq 2$ is an n-dimensional minimal cone with $\overline{M}\setminus M=\{0\}$, then*

$$|\nabla_M A|^2\geq\left(1+\frac{2}{m(n-1)}\right)|\nabla_M|A||^2 + 2\left(1-\frac{1}{m(n-1)}\right)r^{-2}|A|^2. \tag{4.27}$$

Proof. For a fixed point $x\in M$, we choose a locally defined orthonormal basis of TM, denoted by $\tau_1,\ldots,\tau_n$ where $\tau_n=\frac{x}{|x|}$ and $\nu_1,\ldots,\nu_m$ a frame in NM. Then

$$h_{\alpha in}=0,\quad h_{\alpha ij,n}=-r^{-1}h_{\alpha ij}. \tag{4.28}$$

Using (4.28), we compute

$$
\begin{aligned}
(4.29)\qquad |\nabla_M A|^2 &\geq \sum_\alpha \sum_{i,j,k=1}^{n-1} h^2_{\alpha ij,k} + 3\sum_\alpha \sum_{i,j=1}^{n-1} h^2_{\alpha ij,n} \\
&= \sum_\alpha \sum_{i,j,k=1}^{n-1} h^2_{\alpha ij,k} + 3r^{-2}\sum_\alpha \sum_{i,j=1}^{n-1} h^2_{\alpha ij} \\
&= \sum_\alpha \sum_{i,j,k=1}^{n-1} h^2_{\alpha ij,k} + 3r^{-2}|A|^2.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
|\nabla_M |A||^2 &= |A|^{-2}\sum_{k=1}^{n-1}\left(\sum_\alpha \sum_{i,j=1}^{n-1} h_{\alpha ij}h_{\alpha ij,k}\right)^2 + |A|^{-2}\left(\sum_\alpha \sum_{i,j=1}^{n-1} h_{\alpha ij}h_{\alpha ij,n}\right)^2 \\
&= |A|^{-2}\sum_{k=1}^{n-1}\left(\sum_\alpha \sum_{i,j=1}^{n-1} h_{\alpha ij}h_{\alpha ij,k}\right)^2 + r^{-2}|A|^{-2}\left(\sum_\alpha \sum_{i,j=1}^{n-1} h^2_{\alpha ij}\right)^2 \\
(4.30)\qquad &= |A|^{-2}\sum_{k=1}^{n-1}\left(\sum_\alpha \sum_{i,j=1}^{n-1} h_{\alpha ij}h_{\alpha ij,k}\right)^2 + r^{-2}|A|^2.
\end{aligned}
$$

Let

$$|\nabla_M A|^2_{n-1} = \sum_\alpha \sum_{i,j,k=1}^{n-1} h^2_{\alpha ij,k},$$

and

$$|\nabla_M |A||^2_{n-1} = |A|^{-2}\sum_{l=1}^{n-1}\left(\sum_\alpha \sum_{i,j=1}^{n-1} h_{\alpha ij}h_{\alpha ij,k}\right)^2.$$

Combing with (4.29) and (4.30) gives

$$
\begin{aligned}
(4.31)\qquad |\nabla_M A|^2 &\geq |\nabla_M A|^2_{n-1} + 3r^{-2}|A|^2, \\
\text{and } |\nabla_M |A||^2 &= |\nabla_M |A||^2_{n-1} + r^{-2}|A|^2.
\end{aligned}
$$

Similarly, we see $\sum_{i=1}^{n-1} h_{\alpha ii} = -h_{\alpha nn} = 0,\ \ \alpha = 1, \ldots, m$. Then M is also an $(n-1)$-dimensional minimal submanifold in $\mathbb{R}^{(n-1)+m}$. From [7], we derive

$$(4.32)\qquad |\nabla_M A|^2_{n-1} \geq \left(1 + \frac{2}{m(n-1)}\right)|\nabla_M |A||^2_{n-1}.$$

Our Lemma follows from the (4.31) and (4.32). □

Similarly, we prove Theorem 1.2.

Proof. Using Lemma 4.1 and (3.23), we obtain

$$\begin{aligned} 2\left(1-\frac{1}{m(n-1)}\right) r^{-2}|A|^2 \\ \leq \Delta_M(\frac{1}{2}|A|^2) - \left(1+\frac{2}{m(n-1)}\right)|\nabla_M|A||^2 + \frac{3}{2}|A|^4. \end{aligned} \tag{4.33}$$

Exactly similar to section 3, we derive for any $\xi \in Lip(M)$ with $\int_M r^{-2}|A|^{\frac{4}{3}\alpha}\xi^2 < \infty$

$$\left(1-\frac{1}{m(n-1)}\right)\int_M r^{-2}|A|^{\frac{4}{3}\alpha}\xi^2 \leq \frac{3}{4\alpha}\int_M |A|^{\frac{4}{3}\alpha}|\nabla\xi|^2. \tag{4.34}$$

Choose $\xi = r^{1-\frac{n}{2}+\frac{2}{3}\alpha+\epsilon}r_1^{-2\epsilon}$. Then

$$\left(1-\frac{1}{m(n-1)}\right)\theta\int_0^\infty r^{n-3-\frac{4}{3}\alpha}\xi^2 dr \leq \frac{3}{4\alpha}\theta\int_0^\infty r^{n-1-\frac{4}{3}\alpha}|\nabla\xi|^2 dr. \tag{4.35}$$

Similarly in the proof of Theorem 1.2, we only consider the case

$$\begin{aligned} \frac{3}{4}\left[n - \frac{2+2\sqrt{1+m^2(n-1)^3-mn(n-1)}}{m(n-1)}\right] \\ < \alpha < \frac{3}{4}\left[n - \frac{2-2\sqrt{1+m^2(n-1)^3-mn(n-1)}}{m(n-1)}\right]. \end{aligned}$$

We can choose $\epsilon > 0$, so that $\left(1-\frac{n}{2}+\frac{2}{3}\alpha \pm \epsilon\right)^2 < \frac{4\alpha}{3}\left(1-\frac{1}{m(n-1)}\right)$. Thus $|A| \equiv 0$. □

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