

Virtual Abelian varieties of GL_2 -type

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This paper studies a class of Abelian varieties that are of GL_2 -type and with isogenous classes defined over a number field k . We treat the cases when their endomorphism algebras are either (1) a totally real field K or (2) a totally indefinite quaternion algebra over a totally real field K . Among the isogenous class of such an Abelian variety, we identify one whose Galois conjugates can be described in terms of actions of Atkin-Lehner operators and the class group of K . Thus we deduce that such Abelian varieties are parametrised by finite quotients of certain PEL Shimura varieties. These new families of moduli spaces are further analysed when they are of dimension 2. We provide explicit numerical bounds for when they are surfaces of general type. In addition, for two particular examples, we show that they are both rational surfaces by computing the coordinates of inequivalent elliptic points and studying the intersections of Hirzebruch cycles with exceptional divisors.

Introduction	904
1 Virtual Abelian varieties of GL_2-type	907
2 Moduli space of virtual Abelian varieties	914
3 Classification of Hilbert modular surfaces	925
References	942

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Introduction

In [18], Ribet considered the modularity problem of elliptic curves defined over $\bar{\mathbb{Q}}$ whose $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -conjugates are all isogenous to each other. The case of CM elliptic curves with such a property was first studied by Gross[6] who coined the name, \mathbb{Q} -curve. Ribet showed that given a non-CM \mathbb{Q} -curve C there exists a simple Abelian variety A of GL_2 -type (Definition 1.1) defined over \mathbb{Q} having C as a $\bar{\mathbb{Q}}$ -quotient. Let E denote the endomorphism algebra of A . It must be a totally real number field in this case. Recall that the Tate- ℓ -module $V_\ell A$ of A is free of rank 2 over $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Let λ be a prime of E lying above ℓ and set $V_\lambda A = V_\ell A \otimes_{E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} E_\lambda$. Then the Galois action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on A gives rise to a λ -adic representation of degree 2. The question whether a \mathbb{Q} -curve is modular reduces to showing the modularity of the λ -adic representations associated to A . An affirmative answer is furnished by Serre's conjecture on mod- ℓ -representations of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which was proved by Khare-Wintenberger[13]. Thus Abelian varieties of GL_2 -type, as well as their geometric quotients, are of particular interest in the study of modularity.

Let k be a number field. Instead of \mathbb{Q} -curves, one may as well consider Abelian varieties B over \bar{k} whose $\text{Gal}(\bar{k}/k)$ -conjugates are isogenous to B . They are the object of study in this paper in which we generalise several pieces of related work, in the hope that a version of Serre's conjecture on modularity of mod- ℓ -representations of $\text{Gal}(\bar{k}/k)$ becomes available in the future. Our main objective is to construct moduli spaces for these Abelian varieties. We study Galois orbits of the Abelian varieties and relate them to the orbits under the actions of class group of the centre of endomorphism algebra and Atkin-Lehner operators. Much of the difficulty we encounter arises from having more complex structure of endomorphism algebras and from having fields with non-trivial class groups. We also estimate the Chern numbers of the moduli spaces and compute two examples. This can potentially lead to explicit examples of Abelian varieties which may provide a test ground for the many conjectures involving Abelian varieties, for example, the BSD conjecture. We will now review the literature and further discuss our results.

Whereas all elliptic curves are automatically of $\text{GL}_2(\mathbb{Q})$ -type, there are many more possibilities for the endomorphism algebras of Abelian varieties. We will focus on the sub-maximal case, namely (non-CM) Abelian varieties of GL_2 -type. Even after this restriction there are still two possibilities which are commonly known as the case of real multiplication (RM) and the case

of quaternionic multiplication (QM) (c.f. Proposition 1.8). For ease of exposition, we first define k -virtuality which captures the notion of having isogenous Galois conjugates.

Definition 0.1. An Abelian variety B over \bar{k} is said to be k -virtual if for all $\sigma \in \text{Gal}(\bar{k}/k)$, there exists an isogeny $\mu_\sigma : {}^\sigma B \rightarrow B$ such that for all $\alpha \in \text{End}(B)$, we have $\alpha \circ \mu_\sigma = \mu_\sigma \circ {}^\sigma \alpha$.

For non-CM elliptic curves, the requirement on compatibility with endomorphism ring is vacuous and in fact, \mathbb{Q} -curve is a short hand for \mathbb{Q} -virtual elliptic curve. The departure from the more traditional nomenclature is to clarify that the Abelian varieties are only ‘virtually’ defined over k rather than truly defined over k .

In [15], Pyle extended the result on relation between \mathbb{Q} -virtual elliptic curves and Abelian varieties over \mathbb{Q} of GL_2 -type to that on relation between \mathbb{Q} -virtual Abelian varieties of GL_2 -type and Abelian varieties over \mathbb{Q} of GL_2 -type and in [7], Guitart generalised the result to that over arbitrary number field, but he only considered geometric quotients of the Ribet-Pyle varieties which have number fields as endomorphism algebra. We first establish an analogous result when the Abelian variety of GL_2 -type has QM. We actually prove the result in a uniform way for both RM and QM case (c.f. Cor. 1.11). This shows that the study of virtual Abelian varieties of GL_2 -type can be transferred to Abelian varieties of GL_2 -type and vice versa.

We also extend Elkies’s work[4] on the construction of moduli spaces of k -virtual elliptic curves to the case of k -virtual Abelian varieties of GL_2 -type. For each prime ℓ of \mathbb{Q} (the endomorphism algebra of a non-CM elliptic curve C), Elkies associated an ℓ -local tree to C where, roughly speaking, the vertices represent isomorphism classes of elliptic curves and the edges represent primitive ℓ -isogenies. Applying a graph theoretic argument, he showed that in the isogenous class of C there exists one elliptic curve C_0 whose Galois conjugates are controlled by a certain level structure on C_0 . The observation is that the Galois orbit of C_0 , which is, a priori, difficult to describe, is actually contained in its Atkin-Lehner orbit. Thus the moduli spaces are Atkin-Lehner quotients of modular curves with certain level structure and the k -rational points give rise to k -virtual elliptic curves. For simple k -virtual Abelian varieties B of GL_2 -type and for a prime λ of the centre K of the endomorphism algebra of B , we can construct a λ -local tree in an analogous way except that the class group of K now plays a subtle role. Again our formulation treats both the RM and QM cases largely uniformly. We show in Theorems 2.19, 2.22 that the moduli spaces are quotients

of PEL Shimura varieties by the group which is an extension of the Atkin-Lehner group by the class group of K and the k -rational points give rise to k -virtual Abelian varieties of GL_2 -type. We note that Guitart and Molina[8] worked out the moduli spaces of virtual QM Abelian surfaces and showed that they are Atkin-Lehner quotients of Shimura curves. As the centre of the endomorphism algebra in their case is \mathbb{Q} which has class number 1, our case is much more complicated.

The moduli spaces of \mathbb{Q} -virtual elliptic curves have been well-studied. Elkies[4] produced some explicit equations for his moduli spaces which are quotients of modular curves. González-Lario[5] classified those that are of genus 0 or 1. Based on their parametrisation, Quer[16] computed explicit equations of some \mathbb{Q} -curves. We attempt to classify our moduli spaces. At this point, we focus on Abelian surfaces. For moduli spaces of k -virtual RM Abelian surfaces of $\mathrm{GL}_2(E)$ -type, the PEL Shimura varieties are disjoint unions of Hilbert modular surfaces. We then go on to analyse the moduli spaces along the line started by Hirzebruch[12] and extended by Hirzebruch-Van de Ven[10] and Hirzebruch-Zagier[11]. A thorough write-up is available in the book of Van der Geer[20]. Our family of Hilbert modular surfaces has not been considered in the literature. In this paper, as a first step, we treat only the case when E has trivial narrow class group and when the level structure is $\mathcal{O}_E/\mathfrak{p}$ for some prime \mathfrak{p} of \mathcal{O}_E , and leave the more technical/interesting cases for the future. Based on the previous results, we are able to estimate the Chern numbers of the desingularisation of our Hilbert modular surfaces and to determine explicit bounds on the discriminant of E and the size of the level structure beyond which the Hilbert modular surfaces are of general type (Theorem 3.23). By Lang's conjecture, we do not expect them to furnish many k -rational points. Thus we turn to examine the Hilbert modular surfaces associated to $E = \mathbb{Q}(\sqrt{5})$ with $\mathfrak{p} = (2)$ and $E = \mathbb{Q}(\sqrt{13})$ with $\mathfrak{p} = (4 + \sqrt{13})$. (See Section 3.4 for precise description of these two Hilbert modular surfaces.) By studying configuration of rational curves coming from desingularisation and Hirzebruch cycles on these two Hilbert modular surfaces, we conclude that they are both rational surfaces. In the process we have computed the explicit coordinates of the inequivalent elliptic points which for $E = \mathbb{Q}(\sqrt{13})$ should be new. The method is due to Gundlach[9]. However as the discriminant increases, the domain in which one scans for elliptic points grows much larger than a fundamental domain. Thus determining inequivalent ones becomes much harder. Further analysis of the moduli spaces will be part of our future research topic.

The structure of the article is as follows. In Sec. 1, we describe the possible endomorphism algebras for Abelian varieties of GL_2 -type and show

that the geometric factors of simple Abelian varieties over k of GL_2 -type are k -virtual Abelian varieties of GL_2 -type. In Sec. 2 we determine moduli spaces of k -virtual Abelian varieties of GL_2 -type by extending Elkies's local tree constructions. We show that the moduli spaces are quotients of Hilbert modular surfaces or quaternionic Shimura varieties by the extension of Atkin-Lehner group by a class group. In Sec. 3, we analyse the cusp and quotient singularities of the Hilbert modular surfaces in question and estimate their Chern numbers to show that most of them are of general type. Finally we give two examples where the moduli spaces are rational surfaces in Sec. 3.4.

1. Virtual Abelian varieties of GL_2 -type

In this section we define the virtual Abelian varieties of GL_2 -type and deduce some preliminary results. We introduce the notions of GL_2 -type and virtuality separately.

1.1. Endomorphism algebras of Abelian varieties of GL_2 -type

Let k be a field of characteristic 0 and \bar{k} its algebraic closure. In this article, k is most often a number field. Let A be an Abelian variety over k . Write $\mathrm{End}(A)$ for its endomorphism ring. The endomorphisms are required to be defined over k . The ring $\mathrm{End}(A_{\bar{k}})$ consists of all potential endomorphisms of A . The endomorphism algebra $\mathrm{End}^0(A)$ is defined to be $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let E be a number field. We will consider those Abelian varieties A that admit a \mathbb{Q} -algebra embedding $E \hookrightarrow \mathrm{End}^0(A)$.

Definition 1.1. An Abelian variety A defined over k is said to be of GL_2 -type if for some number field E such that $[E : \mathbb{Q}] = \dim A$, there is an embedding of \mathbb{Q} -algebras $E \hookrightarrow \mathrm{End}^0(A)$. If the number field E is specified, we say that A is of $\mathrm{GL}_2(E)$ -type.

We do not require $\mathrm{End}^0(A)$ to be isomorphic to E , as we intend to study moduli spaces of Abelian varieties where Abelian varieties with bigger endomorphism algebras arise naturally and they form special cycles. Now we make a more general definition.

Definition 1.2. An Abelian variety A defined over k is said to be of $\mathrm{GL}_n(D)$ -type if for some division algebra D over \mathbb{Q} such that $[D : \mathbb{Q}] = 2 \dim A/n$, there is an embedding of \mathbb{Q} -algebras $D \hookrightarrow \mathrm{End}^0(A)$.

We note that in this case the Tate module $V_\ell(A)$ is free of rank n over $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. This is the rationale behind the naming.

We would like to focus on the non-CM Abelian varieties. We make precise what we mean by CM. ‘Potentially CM’ is probably more correct, but we opt for a shorter name here.

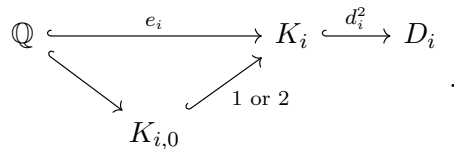
Definition 1.3. An Abelian variety A defined over k is said to be of CM-type if for some CM algebra E such that $[E : \mathbb{Q}] = 2 \dim A$, there is an embedding of \mathbb{Q} -algebras $E \hookrightarrow \text{End}^0(A_{\bar{k}})$.

Remark 1.4. 1) Every elliptic curve is automatically of $\text{GL}_2(\mathbb{Q})$ -type.
 2) Sometimes we simply say that E acts on A when we mean that E acts on A up to isogeny.

The requirement of having a big number field acting on an Abelian variety is very strong. We investigate its implication. Assume that the Abelian variety A is isogenous to

$$\prod_{i=1}^m A_i^{r_i}.$$

where A_i ’s are simple Abelian varieties which are pairwise non-isogenous. Fix a polarisation of A . Then we have the associated Rosati involution on the endomorphism algebra of A . The endomorphism algebra $\text{End}^0(A_i)$ is a division algebra classified by Albert. We refer to the book of Mumford [14] for details. Set $D_i = \text{End}^0(A_i)$ and let K_i denote the centre of D_i and $K_{i,0}$ the set of fixed points of the Rosati involution. Put $e_i = [K_i : \mathbb{Q}]$ and $d_i^2 = [D_i : K_i]$. The degree $[K_i : K_{i,0}]$ is either 1 or 2. Marking the relative degrees on the diagram, we have



Composing the embedding $E \hookrightarrow \text{End}^0(A) \cong \prod_{i=1}^m M_{r_i}(D_i)$ with projection onto each factor $M_{r_i}(D_i)$, we get embeddings $E \hookrightarrow M_{r_i}(D_i)$ for all i . A maximal subfield of $M_{r_i}(D_i)$ has degree $r_i e_i d_i$ over \mathbb{Q} . In addition, the following constraints are in effect: $e_i d_i \mid \dim A_i$ if D_i is of type I, II or III; $e_i d_i^2 \mid 2 \dim A_i$ if D_i is of type IV. The types are as in [14, page 187]. Briefly, an endomorphism algebra of type I is a totally real number field, that of type II is a

totally indefinite quaternion algebra over a totally real number field, that of type III is a totally definite quaternion algebra over a totally real number field and that of type IV is a division algebra over a CM field. Write $\deg E$ for $[E : \mathbb{Q}]$. Note that

$$\begin{aligned} \deg E &= \dim A \geq r_i \dim A_i; \\ r_i e_i d_i &\geq \deg E \end{aligned}$$

for all i . Thus if for any i , $e_i d_i \leq \dim A_i$, we are forced to have $m = 1$ and A is isogenous to $A_1^{r_1}$ with $e_1 d_1 = \dim A_1$. In this case A_1 has action by a field of degree equal to $\dim A_1$. Thus A_1 is a simple Abelian variety of GL_2 -type. Now suppose for all i , $e_i d_i > \dim A_i$. This can happen only when all D_i 's are of type IV with $d_i = 1$ and $e_i = 2 \dim A_i$. In other words all A_i 's have CM. We have

$$2r_i \dim A_i \geq \deg E = \dim A = \sum_{j=1}^m r_j \dim A_j$$

for all i . Thus $r_i \dim A_i \geq r_j \dim A_j$ for all i and j . As a result $r_i \dim A_i = r_j \dim A_j$ for all i and j and $\deg E = m r_1 \dim A_1$. Hence $m \leq 2$. When $m = 1$, A is isogenous to $A_1^{r_1}$ which is a power of a CM Abelian variety and E is not a maximal field acting on A . When $m = 2$, A is isogenous to $A_1^{r_1} \times A_2^{r_2}$ with A_1 and A_2 being CM Abelian varieties such that $r_i \dim A_i = r_j \dim A_j$, E is a maximal field acting on A and furthermore E is a finite field extension of a CM field.

If furthermore we assume that E is stabilised by the Rosati involution on A . Then by positivity of Rosati involution, E is either a totally real field with Rosati involution acting as identity or a CM field with Rosati involution acting as complex conjugation.

We have shown:

Proposition 1.5. *Let A be an Abelian variety of GL_2 -type over k .*

- 1) *If A is not a CM Abelian variety, then A is isogenous to A_1^r where A_1 is a simple Abelian variety of GL_2 -type and $r \in \mathbb{Z}_{>0}$.*
- 2) *If A is a CM Abelian variety, then A is isogenous either to A_i^r where A_i is a simple CM Abelian variety and $r \in \mathbb{Z}_{>0}$ or to $A_1^{r_1} \times A_2^{r_2}$ where A_i is a simple CM Abelian variety and $r_i \in \mathbb{Z}_{>0}$ for $i = 1, 2$ and $r_1 \dim A_1 = r_2 \dim A_2$.*

Remark 1.6. Since obviously E also acts on $A_{\bar{k}}$, we also get a description of the decomposition of A over \bar{k} .

Now we focus on simple Abelian varieties of $\mathrm{GL}_2(E)$ -type. Their endomorphism algebra can be strictly larger than E . Let D denote $\mathrm{End}^0(A)$, K the centre of D and K_0 the set of fixed points in K of the Rosati involution. Put $e = [K : \mathbb{Q}]$, $e_0 = [K_0 : \mathbb{Q}]$ and $d^2 = [D : K]$.

Proposition 1.7. *Let A be a simple Abelian variety of $\mathrm{GL}_2(E)$ -type over k . Let $g = \dim A$. Then the endomorphism algebra of A must be of one of the following forms.*

- 1) $D = K = E$ is a totally real number field.
- 2) D is a division quaternion algebra over a totally real field K with $[K : \mathbb{Q}] = g/2$ and E is a quadratic extension of K contained in D .
- 3) $D = K = E$ is a CM field.
- 4) D is a division quaternion algebra over a CM field K with $[K : \mathbb{Q}] = g/2$ and E is a quadratic field extension of K contained in D .
- 5) $D = K$ is a CM field with $[K : \mathbb{Q}] = 2g$ and E is a subfield of K with $[K : E] = 2$.

Furthermore, if k is algebraically closed, then D cannot be of type III (totally definite quaternion division algebra over a totally real number field).

Proof. If D is of type I, II or III, then we have the constraint $ed|g$. When D is of type I, then $d = 1$ and $e|g$. Thus we must have $e = g$ and $K = E$. When D is of type II or III, then $d = 2$ and $2e|g$. A maximal subfield of D is of degree $2e$. We must have $2e = g$ and E must be a quadratic extension of K . Of course, this can only happen when g is even.

Now assume that D is of type IV. We have the constraint $e_0d^2|g$. A maximal subfield of D is of degree $2e_0d$. Thus $2e_0d \geq g$. We must have $d = 1$ or 2 . When $d = 1$, we deduce from $e_0|g$ and $2e_0 \geq g$ that $e_0 = g/2$ or g . In the former case, we get $D = K = E$ and this can only occur when g is even. In the latter case we get that $D = K$ is a CM field with $[K : \mathbb{Q}] = 2g$ and E is a subfield of K with $[K : E] = 2$. When $d = 2$, we deduce from $4e_0|g$ and $4e_0 \geq g$ that $e_0 = g/4$. This can only occur when $4|g$. In this case, D is a division quaternion algebra over K which is CM with $[K : \mathbb{Q}] = g/2$ and E is a quadratic extension of K contained in D .

When k is of characteristic 0 and is algebraically closed, then we can rule out more possibilities. By [19, Proposition 15], D cannot be of type

III; $\text{End}^0(A)$ is forced to grow larger. In fact, A is isogenous to A_1^2 with A_1 CM. \square

If furthermore we assume $k = \mathbb{C}$, certain analytic representations of D on the Lie algebra of A cannot occur. We summarise the results of [19, Sec. 4]. When D is of type I, II or III, the rational representations of D must contain all of its inequivalent irreducible representations with the same multiplicity. Thus E acts on $\text{Lie}(A)$ via all of its embeddings into \mathbb{C} with each occurring once. Now assume that D of type IV. Then we have

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_d(\mathbb{C}) \times \cdots \times M_d(\mathbb{C})$$

where the product is e_0 -fold or indexed by the e_0 embeddings of K_0 into \mathbb{R} . The e_0 natural projections account for half the number of the inequivalent irreducible representations of D . Denote these by χ_ν for $\nu = 1, \dots, e_0$. Then χ_ν and $\bar{\chi}_\nu$ account for all the inequivalent irreducible representations of D . Let r_ν (resp. s_ν) be the multiplicity of χ_ν (resp. $\bar{\chi}_\nu$) occurring in the analytic representation of D . We note that in our case, $r_\nu + s_\nu = 2g/de$ which is 2 or 1. Then [19, Prop. 14, 18, 19] says that if $\sum r_\nu s_\nu = 0$ or $r_\nu = s_\nu = 1$ for all ν , then cases (3) and (4) in Prop. 1.7 cannot occur.

Assume that A and all of its endomorphisms can be defined over \mathbb{R} . Then the analytic representation of D on $\text{Lie}(A_{\mathbb{R}})$ must be such that $r_\nu = s_\nu = 1$. Then cases (3) and (4) in Prop. 1.7 do not occur. A CM Abelian variety cannot be defined over a totally real number field, so case (5) is not possible for such A .

Assume that each of the embedding of E into \mathbb{C} occurs exactly once in the analytic representation of E on $\text{Lie}(A_{\mathbb{C}})$. Then this also forces that $r_\nu = s_\nu = 1$, ruling out cases (3) and (4). In case (5) which is the case of CM Abelian variety, for each conjugate pair of embedding of K into \mathbb{C} , exactly one of them occurs. In order for each embedding of E to occur, E has to be the totally real subfield K_0 of K .

Summarising the above, we get:

Proposition 1.8. *Let A be a simple complex Abelian variety of $GL_2(E)$ -type over a number field k . Let $g = \dim A$. Assume one of the following.*

- (a) *Each of the embedding of E into \mathbb{C} occurs exactly once in the analytic representation of E on $\text{Lie}(A_{\mathbb{C}})$.*
- (b) *A and all of its endomorphisms can be defined over \mathbb{R} .*

Then we have exactly the following possibilities.

- 1) $D = K = E$ is a totally real number field.
- 2) D is a totally indefinite division quaternion algebra over a totally real number field K with $[K : \mathbb{Q}] = g/2$ and E is a quadratic extension of K contained in D .
- 3) $D = K$ is a CM field with $[K : \mathbb{Q}] = 2g$ and E is the totally real subfield K_0 of K . This case does not occur when we assume (b).

1.2. Virtual Abelian varieties

We give the definition of virtuality first.

Definition 1.9. Let F be a Galois extension of k contained in \bar{k} . An Abelian variety B over F is said to be k -virtual if every element of $\text{End}(B_{\bar{k}})$ can be defined over F and for all $\sigma \in \text{Gal}(F/k)$, there exists an isogeny $\mu_\sigma : {}^\sigma B \rightarrow B$ such that for all $\alpha \in \text{End}^0(B)$, $\alpha \circ \mu_\sigma = \mu_\sigma \circ \sigma\alpha$.

Such Abelian varieties arise, for example, in the following fashion.

Lemma 1.10. Let A be a simple Abelian variety over k such that $A_{\bar{k}}$ is isogenous to B^r where B is a simple Abelian variety over \bar{k} . Then B is a k -virtual Abelian variety.

Proof. Fix an isogeny $f : A_{\bar{k}} \rightarrow B^r$. Let $\sigma \in \text{Gal}(\bar{k}/k)$. Then we have

$${}^\sigma B^r \xleftarrow{\sigma f} {}^\sigma A_{\bar{k}} \xrightarrow{i_\sigma} A_{\bar{k}} \xrightarrow{f} B^r,$$

where i_σ is the canonical isomorphism. Thus by uniqueness of decomposition, ${}^\sigma B$ is isogenous to B . Let $D = \text{End}^0(A)$ and K be the centre of D . Let $D' = \text{End}^0(B)$ and $K' = Z(\text{End}^0(B))$ be its centre. We have the embeddings

$$\begin{array}{ccc} D & \hookrightarrow & M_r(D') \\ \uparrow & & \uparrow \\ K & \hookrightarrow & K' \end{array} .$$

As every endomorphism α in D is defined over k , we have $\sigma\alpha = i_\sigma^{-1} \circ \alpha \circ i_\sigma$. Now let $\alpha \in K'$. This can be viewed as an endomorphism (up to isogeny) of B^r by acting diagonally. We note that $f^{-1} \circ \alpha \circ f$ lies in K , so it is defined

over k . Thus

$$\begin{aligned} \alpha \circ f \circ i_\sigma \circ \sigma f^{-1} &= f \circ (f^{-1} \circ \alpha \circ f) \circ i_\sigma \circ \sigma f^{-1} \\ &= f \circ (i_\sigma \circ \sigma f^{-1} \circ \sigma \alpha \circ \sigma f \circ i_\sigma^{-1}) \circ i_\sigma \circ \sigma f^{-1} \\ &= f \circ i_\sigma \circ \sigma f^{-1} \circ \sigma \alpha. \end{aligned}$$

This means that the following diagram commutes:

$$\begin{array}{ccc} \sigma B^r & \xrightarrow{f \circ i_\sigma \circ \sigma f^{-1}} & B^r \\ \downarrow \sigma \alpha & & \downarrow \alpha \\ \sigma B^r & \xrightarrow{f \circ i_\sigma \circ \sigma f^{-1}} & B^r \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccc} \sigma B & \xrightarrow{\mu_\sigma} & B \\ \downarrow \sigma \alpha & & \downarrow \alpha \\ \sigma B & \xrightarrow{\mu_\sigma} & B \end{array}$$

where μ_σ is an isogeny induced by $f \circ i_\sigma \circ \sigma f^{-1}$. In this sense, μ_σ is K' -equivariant.

Now we augment K' -equivariance to D' -equivariance. We have a morphism of central simple algebras

$$\begin{aligned} D' &\rightarrow D' \\ \alpha &\mapsto \sigma^{-1}(\mu_\sigma^{-1} \circ \alpha \circ \mu_\sigma), \end{aligned}$$

as the condition on μ_σ shows that if $\alpha \in Z(D')$, then $\sigma^{-1}(\mu_\sigma^{-1} \circ \alpha \circ \mu_\sigma) = \alpha$. By Skolem-Noether Theorem, there exists an element $\beta \in D'$ such that $\sigma^{-1}(\mu_\sigma^{-1} \circ \alpha \circ \mu_\sigma) = \beta \circ \alpha \circ \beta^{-1}$ for all $\alpha \in D'$. Thus $\alpha \circ \mu_\sigma \circ \sigma \beta = \mu_\sigma \circ \sigma \beta \circ \sigma \alpha$ for all $\alpha \in D'$. Changing the isogeny μ_σ to $\mu_\sigma \circ \sigma \beta$, we get D' -equivariance. \square

Noting how non-CM Abelian varieties of GL_2 -type decomposes (Prop. 1.5), we get the following:

Corollary 1.11. *Absolutely simple factors of non-CM Abelian varieties of GL_2 -type over k are k -virtual Abelian varieties of GL_2 -type.*

Given a simple k -virtual Abelian variety B over \bar{k} of GL_2 -type, one can construct a simple Abelian variety A of GL_2 -type over k such that it has

B as an absolutely simple factor. This converse problem has been studied in [15, 18] over \mathbb{Q} and in [7] over arbitrary number field k even though the definition of GL_2 -type is more restrictive than here. Their methods generalise easily to the current case. Thus we just record the result.

Proposition 1.12. *Let B be a non-CM k -virtual Abelian variety over \bar{k} of GL_2 -type. Then there exists a non-CM simple Abelian variety A over k of GL_2 -type such that $A_{\bar{k}}$ is isogenous to a power of B .*

2. Moduli space of virtual Abelian varieties

The aim of this section is to determine a moduli space of k -virtual Abelian varieties of $\mathrm{GL}_2(E)$ -type up to isogeny. One key step is the construction of λ -local trees (in the sense of graph theory) for our Abelian varieties where λ is a finite place of K where K denotes the centre of $D = \mathrm{End}^0(A)$. Our construction generalises that of Elkies [4] where he associated certain trees to non-CM elliptic curves. The major difficulty in the case of Abelian varieties comes from the fact that the endomorphism ring is much more complicated. We still manage to produce trees whose vertices are k -virtual Abelian varieties of $\mathrm{GL}_2(E)$ -type up to a certain equivalence relation and whose edges represent simple isogenies. Via graph theoretic properties of the trees, for a given k -virtual Abelian Varieties of $\mathrm{GL}_2(E)$ -type, we can find an isogenous Abelian variety whose Galois orbit is contained in the (generalised) Atkin-Lehner orbit. This makes it possible to represent k -virtual Abelian Varieties by k -points on a quotient of a certain Shimura variety.

2.1. Local trees

After excluding the CM case, there are two cases left for the endomorphism algebra of an Abelian variety of $\mathrm{GL}_2(E)$ -type. One is when $\mathrm{End}^0(A)$ is isomorphic to exactly E and the other is when $\mathrm{End}^0(A)$ is isomorphic to a division quaternion algebra D that contains E (c.f. Prop. 1.7). After changing A to an isogenous Abelian variety, we may assume that $\mathrm{End}(A)$ is isomorphic to \mathcal{O}_E in the former case and that it is isomorphic to a maximal order of D in the latter case. Let K be the centre of D . Fix a maximal order \mathcal{O}_D of D . We use extensively results on maximal orders over complete discrete valuation ring or over Dedekind domain. One good reference is Reiner's book[17].

To unify the construction for the two cases of endomorphism algebras, set

$$\begin{aligned} S = \mathcal{O}_E \quad \text{and} \quad R = \mathcal{O}_E, \\ \text{or} \quad S = \mathcal{O}_D \quad \text{and} \quad R = \mathcal{O}_K. \end{aligned}$$

Let $\mathcal{A}(S)$ be the category where the objects are Abelian varieties A of GL_2 -type such that $\text{End}(A) \cong S$ and the morphisms are S -linear isogenies. As usual, let $T_\ell A$ denote the Tate- ℓ -module associated to an Abelian variety A . Let λ be a prime of R . Write R_ℓ for $R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, R_λ for the completion of R at λ and $\mathcal{O}_{D,\lambda}$ for $\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\lambda}$. Let ϖ_λ be a uniformiser of λ . To avoid confusion, sometimes we write $\varpi_{E,\lambda}$ (resp. $\varpi_{K,\lambda}$) to indicate which field we are working with. When λ ramifies in D , set $\varpi_{D,\lambda}$ be a uniformiser of the prime ideal of $\mathcal{O}_{D,\lambda}$, i.e., $\varpi_{D,\lambda}^2 = u\varpi_{K,\lambda}$ for some $u \in \mathcal{O}_{D,\lambda}^\times$. Set $T_\lambda A = T_\ell A \otimes_{R_\ell} R_\lambda$. This is a free $\mathcal{O}_{E,\lambda}$ -module (resp. free left- $\mathcal{O}_{D,\lambda}$ -module) of rank 2 (resp. 1).

We construct a graph out of $\mathcal{A}(S)$ as follows. Fix a prime λ of R . The vertices are equivalence classes of Abelian varieties in $\mathcal{A}(S)$. We say the Abelian varieties A and B are equivalent if there exists a morphism $f : A \rightarrow B$ such that the image of $T_\lambda f : T_\lambda A \rightarrow T_\lambda B$ is $\varpi_\lambda^n T_\lambda B$ for some $n \in \mathbb{Z}_{\geq 0}$. Write $[A]_\lambda$ for the equivalence class of A . Let $d = \text{deg } f$. Then we have a morphism $g : B \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow [d]_A & \swarrow g \\ & & A \end{array} .$$

This induces the commutative diagram for Tate modules:

$$\begin{array}{ccc} T_\lambda A & \xrightarrow{T_\lambda f} & T_\lambda B \\ & \searrow d & \swarrow T_\lambda g \\ & & T_\lambda A \end{array} .$$

The image of d is of the form $\varpi_\lambda^{n'} T_\lambda A$ for some integer $n' \geq 0$. Thus $T_\lambda g$ has image $\varpi_\lambda^{n'-n} T_\lambda A$, which shows that the equivalence relation is well-defined. We note that the Abelian varieties $A/A[\mathfrak{a}]$ for \mathfrak{a} running over all ideals of R correspond to the same vertex in the graph for each λ .

Next we define the edges of the graph. For $r \in \mathbb{Z}_{>0}$, set M_r to be

- 1) (Case E) $\mathcal{O}_{E,\lambda} / \varpi_{E,\lambda}^r \mathcal{O}_{E,\lambda}$ if $S = \mathcal{O}_E$;

- 2) (Case D_{ram}) $\mathcal{O}_{D,\lambda}/\varpi_{D,\lambda}\mathcal{O}_{D,\lambda}$, if $S = \mathcal{O}_D$ and λ ramifies in D ;
- 3) (Case D_{split}) $(\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}^r\mathcal{O}_{K,\lambda})^2$, if $S = \mathcal{O}_D$ and λ splits in D .

When $r = 1$, then we are quotienting out the maximal left ideal in all three cases. In case D_{ram} , r can only take the value 1. Connect two vertices $[A]_\lambda$ and $[B]_\lambda$ if there exist representatives A for $[A]_\lambda$ and B for $[B]_\lambda$ and a morphism $f : A \rightarrow B$ such that $\text{coker}(T_\lambda f)$ is isomorphic to M_1 . We show that the edge is bidirectional. Let d be the degree of f . Then we get a morphism $g : B \rightarrow A$ such that $[d]_A = g \circ f$. Then the image of d is of the form $\varpi_\lambda^{n'} T_\lambda A$ for some integer $n' > 0$. Assume we are in Case E . Then for some choice of $\mathcal{O}_{E,\lambda}$ -basis $\{e_1, e_2\}$ of $T_\lambda A$, the image of $T_\lambda g$ is $\varpi_{E,\lambda}^{n'-1} \mathcal{O}_{E,\lambda} e_1 \oplus \varpi_{E,\lambda}^{n'} \mathcal{O}_{E,\lambda} e_2$. Thus g factors as the composition of the natural projection from B to $B/B[\lambda^{n'-1}]$ and a morphism $g' : B/B[\lambda^{n'-1}] \rightarrow A$ with $\text{coker}(T_\lambda g') \cong \mathcal{O}_{E,\lambda}/\lambda$. Assume that we are in Case D_{ram} . Then the image of $T_\lambda g$ is $\varpi_{D,\lambda} \varpi_{K,\lambda}^{n'-1} T_\lambda A$. Thus g factors as the composition of the natural projection from B to $B/B[\lambda^{n'-1}]$ and a morphism $g' : B/B[\lambda^{n'-1}] \rightarrow A$ with $\text{coker}(T_\lambda g') \cong \mathcal{O}_{D,\lambda}/\varpi_{D,\lambda} \mathcal{O}_{D,\lambda}$. Assume that we are in Case D_{split} . Fix an isomorphism $\mathcal{O}_{D,\lambda} \cong M_2(\mathcal{O}_{K,\lambda})$. Then for some choice of generators $e_1, e_2 \in T_\lambda A$, image of $T_\lambda g$ is $\varpi_{K,\lambda}^{n'-1} M_2(\mathcal{O}_{K,\lambda}) e_1 \oplus \varpi_{K,\lambda}^{n'} M_2(\mathcal{O}_{K,\lambda}) e_2$. Thus g factors as the composition of the natural projection from B to $B/B[\lambda^{n'-1}]$ and a morphism $g' : B/B[\lambda^{n'-1}] \rightarrow A$ with $\text{coker}(T_\lambda g') \cong (\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda} \mathcal{O}_{K,\lambda})^2$.

Lemma 2.1. *Assume that Abelian varieties A and B represent the same vertex. Then there does not exist a morphism $f : A \rightarrow B$ with $\text{coker}(T_\lambda f)$ isomorphic to M_1 . In particular, there is no loop in the graph.*

Proof. Assume that there exists such an f . Since A and B represent the same vertex, there exists a morphism $g : B \rightarrow A$ such that $T_\lambda g$ has image $\varpi_\lambda^n T_\lambda A$ for some $n \in \mathbb{Z}_{\geq 0}$. As the composite $g \circ f$ lies in the centre of $\text{End}(A)$, $T_\lambda g \circ T_\lambda f$ has image of the form $\varpi_\lambda^{n'} T_\lambda A$ for some $n' \in \mathbb{Z}_{\geq 0}$. This is not possible if f is such that $\text{coker}(T_\lambda f)$ is isomorphic to M_1 . □

At this point we see that in case D_{ram} each connected component of the graph consists of 2 vertices connected by an edge and hence is a tree. Recall that in graph theory, a walk is an alternating sequence of vertices and edges and a path is a walk in which all edges are distinct and all vertices (except possibly the first and the last) are distinct.

Lemma 2.2. *Exclude Case D_{ram} . Assume that two vertices V_0 and V_n can be connected by a path of length n via vertices V_1, V_2, \dots and V_{n-1} . Then there exists representatives A_i for V_i for $i = 0, \dots, n$ and morphisms*

$f_i : A_{i-1} \rightarrow A_i$ for $i = 1, \dots, n$ such that $\text{coker}(T_\lambda f_i) \cong M_1$ for $i = 1, \dots, n$ and that $\text{coker}(T_\lambda(f_n \circ \dots \circ f_1)) \cong M_n$.

Proof. When $n = 1$, the statement is true by construction. Assume the statement holds for all paths with lengths less than n . We get a morphism

$$g : A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n$$

with $\text{coker}(f_i) \cong M_1$ for $i = 2, \dots, n$ and $\text{coker}(T_\lambda g) \cong M_{n-1}$ and a morphism $f_1 : A_0 \rightarrow A_1$ with $\text{coker}(T_\lambda f)$ isomorphic to M_1 . Consider the cokernel of the composite $T_\lambda(g \circ f_1)$. We separate the two cases.

Assume that we are in Case E . Then $\text{coker}(T_\lambda(g \circ f_1))$ is an extension of $\text{coker}(T_\lambda g) \cong \mathcal{O}_{E,\lambda}/\varpi_{E,\lambda}^{n-1}\mathcal{O}_{E,\lambda}$ by $\text{coker}(T_\lambda f) \cong \mathcal{O}_{E,\lambda}/\varpi_{E,\lambda}\mathcal{O}_{E,\lambda}$ as $\mathcal{O}_{E,\lambda}$ -modules. Thus it is isomorphic to

$$\mathcal{O}_{E,\lambda}/\varpi_{E,\lambda}^{n-1}\mathcal{O}_{E,\lambda} \oplus \mathcal{O}_{E,\lambda}/\varpi_{E,\lambda}\mathcal{O}_{E,\lambda} \quad \text{or} \quad \mathcal{O}_{E,\lambda}/\varpi_{E,\lambda}^n\mathcal{O}_{E,\lambda}.$$

In the former case, we can see that $\text{coker}(T_\lambda(f_2 \circ f_1))$ must be isomorphic to $(\mathcal{O}_{E,\lambda}/\varpi_{E,\lambda}\mathcal{O}_{E,\lambda})^2$. This means that A_0 and A_2 represent the same vertex.

Assume that we are in Case D_{split} . Then $\text{coker}(T_\lambda(g \circ f_1))$ is an extension of $\text{coker}(T_\lambda g) \cong (\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}^{n-1}\mathcal{O}_{K,\lambda})^2$ by $\text{coker}(T_\lambda f) \cong (\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}\mathcal{O}_{K,\lambda})^2$ as left $M_2(\mathcal{O}_{K,\lambda})$ -modules. If we consider extensions as $\mathcal{O}_{K,\lambda}$ -modules, then there are 3 possibilities:

$$\begin{aligned} &(\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}\mathcal{O}_{K,\lambda} \oplus \mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}^{n-1}\mathcal{O}_{K,\lambda})^2, (\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}^n\mathcal{O}_{K,\lambda})^2, \\ &(\mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}\mathcal{O}_{K,\lambda} \oplus \mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}^{n-1}\mathcal{O}_{K,\lambda}) \oplus \mathcal{O}_{K,\lambda}/\varpi_{K,\lambda}^n\mathcal{O}_{K,\lambda}. \end{aligned}$$

The third one does not occur in the list of left $M_2(\mathcal{O}_{K,\lambda})$ -module extensions. Similar to the arguments in Case E , the first one will force A_0 and A_2 to represent the same vertex, leading to a contradiction.

Thus in both cases we are led to the conclusion that $\text{coker}(T_\lambda(g \circ f_1)) \cong M_n$. □

Proposition 2.3. *Each connected component of the graph is a tree.*

Proof. We need to show that there is no cycle. Assume that there is a cycle of length n from the vertex V to V . By Lemma 2.2, there exist A and B in the equivalence class V and a morphism $f : A \rightarrow B$ such that $\text{coker}(T_\lambda f) \cong M_n$. This is not possible by Lemma 2.1. Thus there can be no cycle. □

Definition 2.4. Let λ be a prime of R . Each connected component of the graph constructed above is called a λ -local tree. Let A be an Abelian variety

in $\mathcal{A}(S)$. The λ -local tree containing the vertex $[A]_\lambda$ is called the λ -local tree associated to A .

If $A \in \mathcal{A}(S)$ is a k -virtual Abelian variety, then every vertex of the λ -local tree associated to A is an equivalence class of a k -virtual Abelian variety and the Galois group $\text{Gal}(\bar{k}/k)$ acts on the tree.

Definition 2.5. Let A be a k -virtual Abelian variety in $\mathcal{A}(S)$. Set $O_\lambda(A)$ to be the $\text{Gal}(\bar{k}/k)$ -orbit $\{[\sigma A]_\lambda \mid \sigma \in \text{Gal}(\bar{k}/k)\}$ contained in the λ -local tree.

A priori, the Galois orbit $O_\lambda(A)$ associated to a k -virtual Abelian variety A is hard to describe. However for some special vertices in the tree the Galois orbit is essentially contained in the Atkin-Lehner orbit which we will describe below. We recall a definition from graph theory.

Definition 2.6. For a finite subset U of vertices of a tree, the centre of U is defined to be the central edge or central vertex on any one of the longest paths connecting two vertices in U .

Remark 2.7. There are possibly multiple longest paths, but they give the same centre. Thus the centre is well-defined.

Definition 2.8. The λ -centre associated to a k -virtual Abelian variety $A \in \mathcal{A}(S)$ is defined to be the centre of the Galois orbit $O_\lambda(A)$ in the λ -local tree.

Since the Galois orbit $O_\lambda(A)$ is Galois stable, we have:

Proposition 2.9. *The λ -centre of k -virtual Abelian variety $A \in \mathcal{A}(S)$ is fixed under the action of $\text{Gal}(\bar{k}/k)$. (If the centre is an edge it can possibly be flipped.) Furthermore the vertices in the Galois orbit $O_\lambda(A)$ are at the same distance to (the nearer vertex of) the λ -centre.*

We consider the λ -centres that are central edges.

Proposition 2.10. *The set of central edges associated to a k -virtual Abelian variety $A \in \mathcal{A}(S)$ is an S -linear isogeny invariant. Thus it is an invariant for the λ -local tree.*

Proof. Suppose the λ -centre associated to A is an edge. Let $B \in \mathcal{A}(S)$ be an Abelian variety that is S -linearly isogenous to A . We need to show that the λ -centre of B is the same edge.

First we note that there exists an element in $\text{Gal}(\bar{k}/k)$ that exchanges the two vertices connected by the central edge. Otherwise all Galois conjugates of A would be on one side of the edge, contrary to the fact that this edge is central. Once we have an edge which is flipped under Galois action there can be no fixed vertices or other fixed edges in the tree. Since the λ -centre associated to B is fixed under Galois action, it must be the same edge that is the λ -centre for A . \square

Remark 2.11. Central vertices are not necessarily isogeny invariants. For example we can take an Abelian variety $A \in \mathcal{A}(\mathcal{O}_E)$ over k and take $B = A/C$ where C is a k -subgroup of A isomorphic to \mathcal{O}_E/λ . Then obviously the central vertices, $[A]_\lambda$ and $[B]_\lambda$, are not the same vertex by construction.

Definition 2.12. Let $A \in \mathcal{A}(\mathcal{O}_E)$ be a k -virtual Abelian variety. Set $\Sigma(A)$ to be the set of primes λ of \mathcal{O}_E such that the λ -centre of A is an edge.

There is an analogous definition for k -virtual Abelian varieties in $\mathcal{A}(\mathcal{O}_D)$. We note that for each prime λ of \mathcal{O}_K , there exists a unique prime ideal $\tilde{\lambda}$ of \mathcal{O}_D that lies above λ [17, Theorem 22.4]. The λ -central edge associated to A determines a maximal left ideal \mathcal{M}_λ of \mathcal{O}_D that belongs to $\tilde{\lambda}$, in the sense that $\tilde{\lambda}$ is the annihilator of $\mathcal{O}_D/\mathcal{M}_\lambda$ in \mathcal{O}_D [17, Theorem 22.15].

Definition 2.13. Let $A \in \mathcal{A}(\mathcal{O}_D)$ be a k -virtual Abelian variety. Set $\Sigma(A)$ to be the set of maximal left ideals \mathcal{M}_λ of \mathcal{O}_D determined by the λ -central edges of A .

Remark 2.14. To unify the notation, we also write \mathcal{M}_λ for λ in case $A \in \mathcal{A}(\mathcal{O}_E)$.

Lemma 2.15. *The set $\Sigma(A)$ is a finite set and for almost all λ 's, $[A]_\lambda$ is its own λ -centre.*

Proof. The Abelian varieties $^\sigma A$ for σ in $\text{Gal}(\bar{k}/k)$ end up in the same equivalence class as A , as long as λ does not divide the degree of the isogenies μ_σ 's between the Galois conjugates. Thus there are only finitely many λ 's such that the λ -centre associated to A can be an edge. \square

For each $\mathcal{M}_\lambda \in \Sigma(A)$ we choose one of the vertices V_λ on the central edge and for each $\mathcal{M}_\lambda \notin \Sigma(A)$ we just use the central vertex V_λ . The path connecting $[A]_\lambda$ to V_λ describes the ' λ '-part of an isogeny. Thus the chosen vertices give rise to an Abelian variety A_0 isogenous to A . As the vertices are

equivalence classes of Abelian varieties, we cannot distinguish the Abelian varieties A_0 from $A_0/A_0[\mathfrak{a}]$ for integral ideals \mathfrak{a} of R , or in terms of Serre tensor (c.f. for example [2, Sec. 1.7.4]), from $A_0 \otimes_R \mathfrak{a}$ for fractional ideals \mathfrak{a} of R . These Abelian varieties are exactly the ones that correspond to the vertex $[A_0]_\lambda$ for each λ . We note that if \mathfrak{a} is principal, then $A_0 \otimes_R \mathfrak{a}$ is isomorphic to A_0 . Thus the set of these vertices V_λ determines an Abelian variety up to the action of the class group $\text{Cl}(R)$.

Now let A_0 be an Abelian variety such that $[A_0]_\lambda = V_\lambda$ for all prime λ of R . The finitely many central edges determine a level structure on A_0 . This is an embedding

$$\bigoplus_{\mathcal{M}_\lambda \in \Sigma(A_0)} S/\mathcal{M}_\lambda \hookrightarrow A_0.$$

For $A_0 \in \mathcal{A}(\mathcal{O}_E)$, the left-hand side is isomorphic to $\mathcal{O}_E / \bigcap_{\lambda \in \Sigma(A_0)} \lambda$ or $\mathcal{O}_E / \prod_{\lambda \in \Sigma(A_0)} \lambda$. For $A_0 \in \mathcal{A}(\mathcal{O}_D)$, the left-hand side is isomorphic to $\mathcal{O}_D / \bigcap_{\mathcal{M}_\lambda \in \Sigma(A_0)} \mathcal{M}_\lambda$, since the \mathcal{M}_λ 's belong to distinct primes of \mathcal{O}_K . By Prop. 2.9, the Galois conjugates of A_0 must correspond to the central vertices or the vertices on the central edges. In other words, isogenies from the Galois conjugates to A_0 are controlled by the level structure. Thus we have shown:

Theorem 2.16. *For every k -virtual Abelian variety $A \in \mathcal{A}(S)$, there exists a k -virtual Abelian variety $A_0 \in \mathcal{A}(S)$ which is S -linearly isogenous to A and a level structure*

$$\eta : S / \bigcap_{\mathcal{M}_\lambda \in \Sigma(A_0)} \mathcal{M}_\lambda \hookrightarrow A_0$$

such that for all $\sigma \in \text{Gal}(\bar{k}/k)$, there exist some ideal \mathcal{I} of R and an S -linearly isogeny $A_0 \rightarrow {}^\sigma A_0 \otimes_R \mathcal{I}$ with kernel contained in the image of η .

2.2. Shimura Varieties of PEL type

The analysis in the previous subsection leads us to consider PEL Shimura varieties that classify Abelian varieties A with endomorphism ring that contains S and with level structure $S/\mathcal{N} \hookrightarrow A$ where \mathcal{N} is a full left ideal of S that is square-free, in the sense that \mathcal{N}_λ is either S_λ or a maximal ideal of S_λ . We separate the discussion into two cases.

2.2.1. Abelian varieties with \mathcal{O}_E -action. Consider Abelian varieties of dimension $[E : \mathbb{Q}]$ with \mathcal{O}_E -action. The moduli spaces of Abelian varieties of dimension $[E : \mathbb{Q}]$ with endomorphism algebra E and a prescribed analytic representation of E were studied by Shimura[19]. We restrict to the case

where E is totally real. This is not a restriction if we are in the situation (a) of Prop. 1.8. Then the PEL Shimura varieties are Hilbert modular varieties that classify Abelian varieties A with real multiplication E and with level structure $\mathcal{O}_E/\mathfrak{n} \hookrightarrow A$ where \mathfrak{n} is a square-free ideal of \mathcal{O}_E . We describe more precisely the moduli problem below.

For a fractional ideal \mathfrak{a} of \mathcal{O}_E , let $GL(\mathcal{O}_E \oplus \mathfrak{a})$ denote the subgroup of $GL_2(E)$ that stabilises the lattice $\mathcal{O}_E \oplus \mathfrak{a}$ (with action on the right). Let A be an Abelian variety such that there is an embedding $\iota : \mathcal{O}_E \rightarrow \text{End}(A)$. Let $NS(A)$ denote the Néron-Severi group of A . Let t_x denote the translation by x map for A . Then We have an embedding

$$NS(A) \rightarrow \text{Hom}(A, \hat{A})$$

$$\mathcal{L} \mapsto \phi_{\mathcal{L}} : x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

with image being the set of the symmetric elements in $\text{Hom}(A, \hat{A})$. Set $NS_E(A)$ to be the set of $\mathcal{L} \in NS(A)$ such that $\phi_{\mathcal{L}} \circ \alpha = \hat{\alpha} \circ \phi_{\mathcal{L}}$ for all $\alpha \in \mathcal{O}_E$ with $\hat{\alpha}$ being the dual endomorphism $\hat{A} \rightarrow \hat{A}$. The action of \mathcal{O}_E on $\text{Hom}(A, \hat{A})$ induces an action of \mathcal{O}_E on $NS_E(A)$, making it into an \mathcal{O}_E -module of rank 1. In other words, $NS_E(A)$ is isomorphic to a fractional ideal of \mathcal{O}_E .

Let \mathfrak{c} run over a set of representatives of the narrow class group $Cl^+(E)$ of E . For each embedding $\iota : E \hookrightarrow \mathbb{R}$, we fix an ordering of $\mathfrak{c} \otimes_{\iota} \mathbb{R}$ and thus get a notion of positivity on \mathfrak{c} . We consider Abelian variety A of dimension $g := [E : \mathbb{Q}]$ with

- $\iota : \mathcal{O}_E \hookrightarrow \text{End}(A)$ such that the induced action of E on $\text{Lie}(A)_{\mathbb{C}}$ is given by the g embeddings of E into \mathbb{C} ,
- a weak polarisation $NS_E(A) \xrightarrow{\cong} \mathfrak{c}$ that maps polarisations to positive elements in \mathfrak{c} ,
- a level structure $\eta : \mathcal{O}_E/\mathfrak{n} \hookrightarrow A$.

The isomorphism classes of such complex Abelian varieties are parametrised by the complex points of the Hilbert modular variety

$$Y_0(\mathfrak{n})(\mathbb{C}) = GL_2(E) \backslash (\mathcal{H}^{\pm})^g \times GL_2(\mathbb{A}_{E,f})/U_0(\mathfrak{n}).$$

where the compact open subgroup $U_0(\mathfrak{n})$ of $GL_2(\mathbb{A}_{E,f})$ is defined to be the product of $GL(\mathcal{O}_{E,\lambda} \oplus \mathcal{O}_{E,\lambda}) \cap GL(\mathcal{O}_{E,\lambda} \oplus \mathfrak{n}^{-1}\mathcal{O}_{E,\lambda})$ over all finite places λ

of E . Let U_∞ be the stabiliser of $(i, \dots, i) \in \mathcal{H}^+$. The determinant map

$$\mathrm{GL}_2(E) \backslash \mathrm{GL}_2(\mathbb{A}_E) / U_\infty U_0(\mathfrak{n}) \rightarrow E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_{E,\infty}^+ \prod_{\lambda \nmid \infty} \mathcal{O}_{E,\lambda}^\times.$$

shows that $Y_0(\mathfrak{n})(\mathbb{C})$ has $\# \mathrm{Cl}^+(E)$ connected components.

Now we bring the local trees into play. As $A \otimes_{\mathcal{O}_E} \mathcal{I}$'s for $\mathcal{I} \in \mathrm{Cl}(E)$ correspond to the same vertex in each λ -local tree, we need to consider the action of $\mathrm{Cl}(E)$ on $Y_0(\mathfrak{n})$. The action of $\mathrm{Cl}(E)$ on the isogeny $A \rightarrow A'$ with $A, A' \in \mathcal{A}(E)$ and kernel isomorphic to $\mathcal{O}_E/\mathfrak{n}$ is given by

$$(A \rightarrow A') \mapsto (A \otimes_{\mathcal{O}_E} \mathcal{I} \rightarrow A' \otimes_{\mathcal{O}_E} \mathcal{I}).$$

This changes the polarisation module \mathfrak{c} to $\mathfrak{c}\mathcal{I}^{-2}$. We also need to consider the flipping of central edges in λ -local trees. For each $\lambda|\mathfrak{n}$ we get an action:

$$w_\lambda : (A \xrightarrow{f} A') \mapsto (A/\eta(\mathcal{O}_E/\lambda) \rightarrow A'/f(A[\lambda])).$$

As another application of w_λ gives $(A/A[\lambda] \rightarrow A'/A'[\lambda])$, we see that it is an involution on $\mathrm{Cl}(E) \backslash Y_0(\mathfrak{n})$.

Definition 2.17. Define the extended Atkin-Lehner group to be the group generated by $\mathrm{Cl}(E)$ and w_λ 's for $\lambda|\mathfrak{n}$ and denote it by $\tilde{W}_{E,\mathfrak{n}}$. Set $Y_0^+(\mathfrak{n}) = \tilde{W}_{E,\mathfrak{n}} \backslash Y_0(\mathfrak{n})$.

With this, we can paraphrase Theorem 2.16 above:

Theorem 2.18. *Let $A \in \mathcal{A}(\mathcal{O}_E)$ be a k -virtual Abelian variety. Then there exists an Abelian variety A_0 that is \mathcal{O}_E -linearly isogenous to A such that the Galois orbit $\mathrm{Gal}(\bar{k}/k).A_0$ is contained in the extended Atkin-Lehner orbit $\tilde{W}_{E,\mathfrak{n}}.A_0$.*

Thus by construction, A_0 is a k -rational point on $Y_0^+(\mathfrak{n})$. On the other hand take a k -rational point of $Y_0^+(\mathfrak{n})$ and we get a set Σ of Abelian varieties in $Y_0(\mathfrak{n})$ that lie above it. They are isogenous to each other. Take any one of them, say A_0 . Then its $\mathrm{Gal}(\bar{k}/k)$ -conjugates are still in the set Σ and they are \mathcal{O}_E -linearly isogenous to A_0 by construction. This means that A_0 is a k -virtual Abelian variety of $\mathrm{GL}_2(E)$ -type, even though it may possibly have strictly larger endomorphism algebra than E .

We have shown

Theorem 2.19. *Every k -point on the Hilbert modular variety $Y_0^+(\mathfrak{n})$ gives rise to a k -virtual Abelian variety of $GL_2(E)$ -type. Conversely for any k -virtual Abelian variety A of $GL_2(E)$ -type with endomorphism algebra isomorphic to E there exists a k -virtual Abelian variety of $GL_2(E)$ -type A_0 that is E -linearly isogenous to A such that it corresponds to a k -rational point on $Y_0^+(\mathfrak{n}(A))$ with $\mathfrak{n}(A) = \prod_{\lambda \in \Sigma(A)} \lambda$.*

2.2.2. Abelian varieties with \mathcal{O}_D -action. Next consider Abelian varieties of dimension $[D : \mathbb{Q}]/2$ with \mathcal{O}_D -action where \mathcal{O}_D is a maximal order of \mathcal{O}_D . In the situation of Prop. 1.8, D must be a totally indefinite quaternion algebra over a totally real field K . This is the case we will pursue. We consider the PEL Shimura varieties that parametrise Abelian varieties with \mathcal{O}_D -action with level structure $\mathcal{O}_D/\mathcal{N}$ where \mathcal{N} is a full left ideal of S that is square-free, in the sense that \mathcal{N}_λ is either $\mathcal{O}_{D,\lambda}$ or a maximal ideal of $\mathcal{O}_{D,\lambda}$. Fix a positive anti-involution \dagger on D that fixes K element-wise. Let A be an Abelian variety with $\iota : \mathcal{O}_D \rightarrow \text{End}(A)$. We consider the subset $\text{NS}_D(A)$ of $\text{NS}(A)$ that is compatible with D along with the anti-involution. In other words, $\text{NS}_D(A)$ consists of $\mathcal{L} \in \text{NS}(A)$ such that $\widehat{\iota(\alpha)} \circ \phi_{\mathcal{L}} = \phi_{\mathcal{L}} \circ \iota(\alpha^\dagger)$. As $\text{NS}(A)$ embeds into the symmetric part of $\text{Hom}(A, \hat{A})$, we see that $\text{NS}_D(A)$ is an \mathcal{O}_K -module of rank 1. Let \mathfrak{c} run over a set of representatives of the narrow class group $\text{Cl}^+(K)$ of K . As in the previous case we have a notion of positivity on \mathfrak{c} . Thus we consider Abelian varieties A of dimension $g := 2[K : \mathbb{Q}]$ with

- $\iota : \mathcal{O}_D \hookrightarrow \text{End}(A)$ such that the action of D on $\text{Lie}(A)_{\mathbb{C}}$ is the direct sum of the standard representation composed with the $g/2$ embedding of D into $M_2(\mathbb{C})$ respectively;
- a weak polarisation $\text{NS}_D(A) \xrightarrow{\cong} \mathfrak{c}$ that maps polarisations to positive elements in \mathfrak{c} ,
- a level structure $\eta : \mathcal{O}_D/\mathcal{N} \rightarrow A$.

Let G be the algebraic group over K determined by $G(K) = D^\times$. The isomorphism classes of such Abelian varieties are parametrised by the \mathbb{C} -points of the quaternionic Shimura variety

$$Sh_0(\mathcal{N})(\mathbb{C}) := G(K) \backslash (\mathcal{H}^\pm)^g \times G(\mathbb{A}_{K,f})^\times / U_0(\mathcal{N})$$

where $U_0(\mathcal{N})$ is the product of $U_0(\mathcal{N})_\lambda$ over finite places λ of K defined as follows. When λ ramifies in D , set $U_0(\mathcal{N})_\lambda$ to be $\mathcal{O}_{D,\lambda}^\times$; when λ splits in D and $\mathcal{O}_D/\lambda\mathcal{O}_D$ is not a composition factor of $\mathcal{O}_D/\mathcal{N}$, set $U_0(\mathcal{N})_\lambda$ to

be $\mathrm{GL}_2(\mathcal{O}_{K,\lambda})$; when λ splits in D and $\mathcal{O}_D/\lambda\mathcal{O}_D$ is a composition factor of $\mathcal{O}_D/\mathcal{N}$, set $U_0(\mathcal{N})_\lambda$ to be subgroup of $\mathrm{GL}_2(\mathcal{O}_{K,\lambda})$ with lower-left element in $\varpi_{K,\lambda}\mathcal{O}_{K,\lambda}$. The reduced norm map

$$G(K) \backslash G(\mathbb{A}_K)/U_\infty U_0(\mathcal{N}) \rightarrow K^\times \backslash \mathbb{A}_K^\times/\mathbb{A}_{K,\infty}^+ \prod_{\lambda \nmid \infty} \mathcal{O}_{K,\lambda}^\times.$$

is surjective by Eichler’s theorem and this shows that $Sh_0(\mathcal{N})(\mathbb{C})$ has $\# \mathrm{Cl}^+(K)$ connected components.

Now we bring the local trees into play. As $A \otimes_{\mathcal{O}_K} \mathcal{I}$ ’s for $\mathcal{I} \in \mathrm{Cl}(K)$ correspond to the same vertex in each λ -local tree, we need to consider the action of $\mathrm{Cl}(K)$ on $Sh_0(\mathcal{N})$. The action of $\mathrm{Cl}(K)$ on the isogeny $A \rightarrow A'$ with kernel isomorphic to $\mathcal{O}_E/\mathcal{N}$ is given by

$$(A \rightarrow A') \mapsto (A \otimes_{\mathcal{O}_K} \mathcal{I} \rightarrow A' \otimes_{\mathcal{O}_K} \mathcal{I}).$$

This changes the polarisation module \mathfrak{c} to $\mathfrak{c}\mathcal{I}^{-4}$. We also need to consider the flipping of central edges in λ -local trees. For each λ in the support of $\mathcal{O}_D/\mathcal{N}$, we get an action:

$$w_\lambda : (A \xrightarrow{f} A') \mapsto (A/\eta(\mathcal{O}_D/\mathcal{M}_\lambda) \rightarrow A'/f(A[\lambda])).$$

where \mathcal{M}_λ is the ‘ λ -part’ of \mathcal{N} such that $\mathcal{O}_D/\mathcal{M}_\lambda \cong \mathcal{O}_{D,\lambda}/\mathcal{N}_\lambda$. As another application of w_λ gives $(A/A[\lambda] \rightarrow A'/A'[\lambda])$, we see that it is an involution on $\mathrm{Cl}(K) \backslash Sh_0(\mathcal{N})$.

Definition 2.20. Define the extended Atkin-Lehner group to be the group generated by $\mathrm{Cl}(K)$ and w_λ ’s for λ in the support of $\mathcal{O}_D/\mathcal{N}$ and denote it by $\tilde{W}_\mathcal{N}$. Set $Sh^+(\mathcal{N}) = \tilde{W}_\mathcal{N} \backslash Sh_0(\mathcal{N})$.

The analogous theorems for the quaternionic case are as follows.

Theorem 2.21. *Let $A \in \mathcal{A}(\mathcal{O}_D)$ be a k -virtual Abelian variety. Then there exists an Abelian variety A_0 that is \mathcal{O}_D -linearly isogenous to A such that the Galois orbit $\mathrm{Gal}(\bar{k}/k).A_0$ is contained in the extended Atkin-Lehner orbit $\tilde{W}_\mathcal{N}.A_0$.*

Theorem 2.22. *Every k -point on the quaternionic Shimura variety $Sh^+(\mathcal{N})$ gives rise to a k -virtual Abelian variety of $\mathrm{GL}_1(D)$ -type. Conversely for any k -virtual Abelian variety A of $\mathrm{GL}_1(D)$ -type with endomorphism algebra isomorphic to D , there exists a k -virtual Abelian variety A_0 of $\mathrm{GL}_1(D)$ -type*

that is D -linearly isogenous to A such that it corresponds to a k -rational point on $Sh^+(\mathcal{N}(A))$ with $\mathcal{N}(A) = \bigcap_{\mathcal{M} \in \Sigma(A)} \mathcal{M}$.

3. Classification of Hilbert modular surfaces

We would like to apply the Enriques-Kodaira classification to our moduli spaces of k -virtual Abelian varieties when they are Hilbert modular surfaces. The main reference is van der Geer's book[20]. See also the many works of Hirzebruch and his joint works with Van de Ven or Zagier on Hilbert modular surfaces that date before it, for example, [10–12]. According to Lang's conjecture, we do not expect to have many rational points on varieties of general type. Thus such classification will give us some rough idea where k -virtual Abelian varieties can be found in more abundance. More detailed analysis of the Hilbert modular varieties and the quaternionic Shimura varieties will be part of our future work.

In this section, we will focus on the case where E is a real quadratic field with narrow class number 1 and study the Hilbert modular surfaces $Y_0^+(\mathfrak{p})$ where \mathfrak{p} is a prime ideal of E . This assumption is always in effect. To avoid too much repetition, we will omit it from our statements. We keep the notation from Sec. 2.

First we note some implications of the assumption that $|Cl^+(E)| = 1$. Suppose $E = \mathbb{Q}(\sqrt{D})$ where D is the discriminant. Then D is necessarily either a prime congruent to 1 modulo 4 or $D = 8$. The torsion-free part of the group of units \mathcal{O}_E^\times is generated by an element with norm equal to -1 . Thus in our case $PSL_2(\mathcal{O}_E) = PGL_2^+(\mathcal{O}_E)$. Let $\Gamma_0^E(\mathfrak{p})$ denote the subgroup of elements in $PSL_2(\mathcal{O}_E)$ whose lower-left entry is congruent to 0 modulo \mathfrak{p} . Since, a fortiori, the class group of E is trivial, the group $\tilde{W}_{E,\mathfrak{p}}$ in Definition 2.17 is a group of order 2. For the sake of brevity, we will denote it by W . More precisely, let $\mathfrak{p} = (\varpi_{\mathfrak{p}})$ with $\varpi_{\mathfrak{p}}$ chosen to be totally positive. Then W is generated by the involution on \mathcal{H}^2 given by the action of the element $w_{\mathfrak{p}} = \begin{pmatrix} 0 & 1 \\ -\varpi_{\mathfrak{p}} & 0 \end{pmatrix}$. Hence $Y_0^+(\mathfrak{p})$ is isomorphic to $W\Gamma_0^E(\mathfrak{p}) \backslash \mathcal{H}^2$.

Let $\bar{Y}_0^+(\mathfrak{p})$ be the compactification of $Y_0^+(\mathfrak{p})$ which is given by $W\Gamma_0^E(\mathfrak{p}) \backslash \mathcal{H}^2 \cup \mathbb{P}^1(E)$. Let $X_0^+(\mathfrak{p})$ denote the minimal desingularisation of $\bar{Y}_0^+(\mathfrak{p})$. The book of van der Geer[20] on Hilbert modular surfaces does not consider level structure, so it does not cover our case. However we do rely heavily on its techniques. We are able to show that most surfaces in question are of general type. We will also give some examples of surfaces that are not of general type. First we review how one resolves singularities on the surfaces and then estimate the Chern numbers.

3.1. Cusp singularities

For $\Gamma_0^E(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$ there are two inequivalent cusps 0 and ∞ . They are identified via the Atkin-Lehner operator $w_{\mathfrak{p}}$. The isotropy group of the unique inequivalent cusp ∞ in $WT_0^E(\mathfrak{p})$ is equal to that in $\mathrm{PSL}_2(\mathcal{O}_E)$, as $WT_0^E(\mathfrak{p})$ contains all of those elements in $\mathrm{PSL}_2(\mathcal{O}_E)$ that are of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Thus the type of the cusp singularity is the same as that for $\mathrm{PSL}_2(\mathcal{O}_E)$ and the isotropy group is equal to

$$\begin{aligned}
 (3.1) \quad & \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & \epsilon^{-1} \end{pmatrix} \in \mathrm{SL}_2(E) : \epsilon \in \mathcal{O}_E^\times, \mu \in \mathcal{O}_E \right\} / \{\pm I\} \\
 & \cong \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2^+(E) : \epsilon \in \mathcal{O}_E^{\times+}, \mu \in \mathcal{O}_E \right\} \\
 & \cong \mathcal{O}_E \rtimes \mathcal{O}_E^{\times+}.
 \end{aligned}$$

By [20, Chapter II] we have the minimal resolution of singularity resulting from toroidal embedding and the exceptional divisor consists of a cycle of \mathbb{P}^1 's or of one rational curve with one ordinary double point.

Definition 3.1. Let C_1, \dots, C_m be rational curves on a non-singular surface and let b_1, \dots, b_m be integers. If $m \geq 2$, we require that C_1, \dots, C_m are non-singular. If $m = 1$, we require that C_1 is a rational curve with one ordinary double point. Set $C_0 = C_m$. We say C_1, \dots, C_m form a cycle of type $[b_1, \dots, b_m]^\circ$ if the following hold.

- 1) When $m \geq 3$, the intersection number $C_i.C_j$ is equal to 1, if $|i - j| = 1$, to $-b_i$ if $i = j$ and to 0 otherwise;
- 2) When $m = 2$, the intersection number $C_i.C_j$ is equal to 2, if $|i - j| = 1$, to $-b_i$ for $i = j$;
- 3) When $m = 1$, the intersection number $C_1.C_1$ is equal to $-b_1 + 2$.

3.2. Elliptic fixed points

Now consider the inequivalent elliptic fixed points of $WT_0^E(\mathfrak{p})$ on \mathcal{H}^2 . More generally we consider the elliptic fixed points of $\mathrm{PGL}_2^+(E)$. Suppose $z = (z_1, z_2)$ is fixed by $\alpha = (\alpha_1, \alpha_2)$ in the image of $\mathrm{PGL}_2^+(E)$ in $\mathrm{PGL}_2^+(\mathbb{R})^2$. Then

$$\alpha_j.z_j = \frac{a_j z_j + b_j}{c_j z_j + d_j} = z_j$$

for $j = 1$ or 2 where $\alpha_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$. Solving the equation we get

$$(3.2) \quad z_j = \frac{a_j - d_j}{2c_j} + \frac{1}{2|c_j|} \sqrt{\text{tr}^2(\alpha_j) - 4 \det(\alpha_j)}.$$

Transform z_j to 0 via the Möbius transformation $\zeta_j \mapsto \frac{\zeta_j - z_j}{\zeta_j - \bar{z}_j}$ of \mathbb{C} . Then the isotropy group of z_j acts as rotation around 0 on each factor \mathbb{C} . The action of α_j transfers to that of $\gamma_j \alpha_j \gamma_j^{-1}$ where $\gamma_j = \begin{pmatrix} 1 & -z_j \\ 0 & -\bar{z}_j \end{pmatrix}$. A little computation shows that

$$\gamma_j \alpha_j \gamma_j^{-1} = (z_j - \bar{z}_j)^{-1} \begin{pmatrix} -a_j \bar{z}_j + cz_j \bar{z}_j + d_j z_j - b_j & 0 \\ 0 & a_j z_j - c_j z_j \bar{z}_j - d_j \bar{z}_j + b_j \end{pmatrix}.$$

Using the equation that z_j satisfies we get that the above is equal to

$$(z_j - \bar{z}_j)^{-1} \begin{pmatrix} (a_j - c_j z_j)(z_j - \bar{z}_j) & 0 \\ 0 & (a_j - c_j \bar{z}_j)(z_j - \bar{z}_j) \end{pmatrix}.$$

Thus the rotation angle is twice the argument θ_j of $a_j - c_j z_j$ which satisfies

$$(3.3) \quad \cos \theta_j = \frac{\text{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}}, \quad c_j \sin \theta_j < 0.$$

The isotropy group of an elliptic point is cyclic.

Definition 3.2. We say that the quotient singularity at $(z_1, z_2) \in \mathcal{H}^2$ is of type $(n; a, b)$ if after transferring (z_1, z_2) to $(0, 0)$ as above, a generator of the isotropy group acts as $(w_1, w_2) \mapsto (\zeta_n^a w_1, \zeta_n^b w_2)$ where ζ_n is a primitive n -th root of 1 .

Remark 3.3. Of course, some types are equivalent. We may require that at least one of a and b is coprime to n . When a is coprime to n , we may require that a is equal to 1 by changing the chosen primitive n -th root of 1 . In fact, in the situation we encounter later, both a and b will be coprime to n . Then the quotient singularity is an isolated singularity. It is shown in [20, Section 6, Chapter II] that the exceptional divisor in the resolution of cyclic quotient singularity is a chain of \mathbb{P}^1 's. The table on page 65 of [20] gives some explicit examples.

Definition 3.4. Let C_1, \dots, C_d be non-singular rational curves on a surface S . Assume that $C_i^2 = -c_i$ for $i = 1, \dots, d$, $C_{i-1} \cdot C_i = 1$ for $i = 2, \dots, d$ and that the rest of the intersection numbers involving these non-singular

rational curves are 0. Then we say that C_1, \dots, C_d form a chain of type $[c_1, \dots, c_d]$.

Remark 3.5. With this definition we can be more precise about the exceptional divisor coming from the resolution of cyclic quotient singularity. For cyclic quotient singularity of type $(n; 1, 1)$, the exceptional divisor is of type $[n]$; for quotient singularity of type $(n; 1, -1)$, the exceptional divisor is of type $[2, \dots, 2]$ where 2 appears $n - 1$ times.

Definition 3.6. Let Γ be a discrete subgroup of $\text{PGL}_2^+(\mathbb{R})^2$. Let $a_n^\pm(\Gamma)$ denote the number of Γ -inequivalent elliptic points of type $(n; 1, \pm 1)$. When $n = 2$, simply set $a_2(\Gamma) = a_2^+(\Gamma) = a_2^-(\Gamma)$.

Now we restrict to the elliptic fixed points of $\Gamma_0^E(\mathfrak{p})$. For there to be any, we need $\text{tr}^2(\alpha_j) < 4 \det(\alpha_j) = 4$ for $j = 1, 2$ for some $\alpha \in \Gamma_0^E(\mathfrak{p})$. For varying discriminant D of the real quadratic field E , the only possible values $\text{tr}(\alpha_j)$ can assume are:

$$0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, (1 \pm \sqrt{5})/2.$$

Then from the expression (3.3) for $\cos(\theta_j)$, we get:

Lemma 3.7. *When the discriminant D is greater than 12, the elliptic elements of $\Gamma_0^E(\mathfrak{p})$ can only be of order 2 or 3.*

3.3. Estimation of Chern numbers

Let $X_0(\mathfrak{p})$ be the minimal desingularisation of $\Gamma_0^E(\mathfrak{p}) \backslash \mathcal{H}^2 \cup \mathbb{P}^1(E)$. Then it is simply-connected since there is no non-trivial Hilbert modular form of weight $(2, 0)$ or $(0, 2)$ [20, Lemma 6.3]. The Atkin-Lehner operator w extends to an involution on $X_0(\mathfrak{p})$ which has at least one fixed point. Thus we see that the quotient $X_0^+(\mathfrak{p})$ is simply-connected. Equivalently $X_0^+(\mathfrak{p})$ is a surface with vanishing irregularity. We rely on the table [1, Table 10, Page 244] which gives the Enriques-Kodaira classification for minimal surfaces. For easier reference, we record in Table 1 the rows where the first Betti number b_1 can possibly be zero. Let c_i be the i -th Chern class. The Chern class $c_i(S)$ of a surface S is the Chern class of the tangent bundle. Let χ denote the Euler characteristic and p_a denote the arithmetic genus. It is not known if $X_0^+(\mathfrak{p})$ is a minimal surface. As blowing down an exceptional curve increases $c_1^2(X_0^+(\mathfrak{p}))$ by 1 and leaves $\chi = (c_1^2(X_0^+(\mathfrak{p})) + c_2(X_0^+(\mathfrak{p}))) / 12$ invariant, we have the following criterion.

Class of Surface	Kodaira Dimension	c_1^2	c_2
minimal rational surface	$-\infty$	8 or 9	4 or 3
Enriques surface	0	0	12
K3 surface	0	0	24
minimal honestly elliptic surface	1	0	≥ 0
minimal surface of general type	2	> 0	> 0

Table 1: Classification of surfaces.

Proposition 3.8. *Let S be a nonsingular algebraic surface with vanishing irregularity. If $\chi > 1$ and $c_1^2(S) > 0$, then S is of general type.*

Now we will estimate the Chern numbers of $X_0^+(\mathfrak{p})$. We begin by defining the local Chern cycle.

Definition 3.9. Let S be a normal surface with isolated singular points and let S' be its desingularisation. Suppose p is a singular point on S and the irreducible curves C_1, \dots, C_m on S' form the resolution of p . Then the local Chern cycle of p is defined to be the unique divisor $Z = \sum_{i=1}^m a_i C_i$ with rational numbers a_i such that the adjunction formula holds:

$$Z.C_i - C_i.C_i = 2 - 2p_a(C_i).$$

Remark 3.10. We can be precise about what the exceptional divisors are for cyclic quotient singularities and cusp singularities.

For cyclic quotient singularity of type $(n; 1, 1)$, the exceptional divisor is of type $[n]$ and consists of one non-singular rational curve C_1 ; the local Chern cycle is $(1 - 2/n)C_1$. For quotient singularity of type $(n; 1, -1)$, the exceptional divisor consists of a chain of non-singular rational curves C_1, \dots, C_{n-1} of type $[2, \dots, 2]$; the local Chern cycle is 0.

For cusp singularity, the exceptional divisor consists of a cycle of rational curves C_1, \dots, C_m of type $[b_0, \dots, b_m]^\circ$ for some integer $m \geq 1$; the local Chern cycle is $\sum_{i=1}^m C_i$.

As we will make frequent comparison to the surfaces associated to the full Hilbert modular group $PSL_2(\mathcal{O}_E)$, we set up some notation to facilitate the analysis. Let $\Gamma \subset PGL_2^+(\mathbb{R})^2$ be commensurable with $PSL_2(\mathcal{O}_E)$. Set Y_Γ to be the quotient $\Gamma \backslash \mathcal{H}^2$ and let X_Γ be the minimal desingularisation of $\Gamma \backslash \mathcal{H}^2$. In this notation our Hilbert modular surface $X_0^+(\mathfrak{p})$ is $X_{W\Gamma_0^E(\mathfrak{p})}$. As

is computed on page 64 of [20] we have the following with a slight change of notation:

Theorem 3.11. *The Chern numbers for X_Γ are given as follows:*

$$(3.4) \quad c_1^2(X_\Gamma) = 2 \operatorname{vol}(\Gamma \backslash \mathcal{H}^2) + c + \sum a(\Gamma; n; a, b)c(n; a, b),$$

$$(3.5) \quad c_2(X_\Gamma) = \operatorname{vol}(\Gamma \backslash \mathcal{H}^2) + l + \sum a(\Gamma; n; a, b) \left(l(n; a, b) + \frac{n-1}{n} \right)$$

where $a(\Gamma; n; a, b)$ is the number of quotient singularity of $\Gamma \backslash \mathcal{H}^2$ of type $(n; a, b)$; for a quotient singularity of type $(n; a, b)$, $c(n; a, b)$ is the self-intersection number of the local Chern cycle, $l(n; a, b)$ is the number of curves in the resolution; c is the sum of the self-intersection number of the local Chern cycles of cusp singularities and l is sum of number of curves in the resolution of cusps.

We also record a theorem of Siegel on the volume of Hilbert modular varieties. See [20, Theorem IV.1.1].

Theorem 3.12. *Let E be a totally real field of degree n over \mathbb{Q} . Let ω be the invariant volume form on \mathcal{H}^n :*

$$(-1)^n \frac{1}{(2\pi)^n} \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \dots \wedge \frac{dx_n \wedge dy_n}{y_n^2}.$$

Then

$$(3.6) \quad \operatorname{vol}(\operatorname{PSL}_2(\mathcal{O}_E) \backslash \mathcal{H}^2) := \int_{\operatorname{PSL}_2(\mathcal{O}_E) \backslash \mathcal{H}^2} \omega = 2\zeta_E(-1).$$

Now we will estimate the Chern numbers under the assumption that $D > 12$. This ensures that we only have elliptic points of type $(2; 1, 1)$ or $(3; 1, \pm 1)$ for $\Gamma_0^E(\mathfrak{p})$ and hence only elliptic points of type $(2; 1, 1)$, $(3; 1, \pm 1)$, $(4; 1, \pm 1)$ or $(6; 1, \pm 1)$ for $W\Gamma_0^E(\mathfrak{p})$. From [20, II. 6] as summarised in Remark 3.5, we know how the elliptic points are resolved and can compute the self-intersection number of local Chern cycles. Thus after we plug in the values,

equation (3.4) reads

$$\begin{aligned}
 c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) &= \frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}_E) : \Gamma_0^E(\mathfrak{p})]4\zeta_E(-1) + c - \frac{1}{3}a_3^+(W\Gamma_0^E(\mathfrak{p})) \\
 &\quad - a_4^+(W\Gamma_0^E(\mathfrak{p})) - \frac{8}{3}a_6^+(W\Gamma_0^E(\mathfrak{p})); \\
 c_2(X_{W\Gamma_0^E(\mathfrak{p})}) &= \frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}_E) : \Gamma_0^E(\mathfrak{p})]2\zeta_E(-1) + l \\
 (3.7) \quad &+ \left(1 + \frac{1}{2}\right) a_2(W\Gamma_0^E(\mathfrak{p})) + \left(1 + \frac{2}{3}\right) a_3^+(W\Gamma_0^E(\mathfrak{p})) \\
 &+ \left(2 + \frac{2}{3}\right) a_3^-(W\Gamma_0^E(\mathfrak{p})) + \left(1 + \frac{3}{4}\right) a_4^+(W\Gamma_0^E(\mathfrak{p})) \\
 &+ \left(3 + \frac{3}{4}\right) a_4^-(W\Gamma_0^E(\mathfrak{p})) + \left(1 + \frac{5}{6}\right) a_6^+(W\Gamma_0^E(\mathfrak{p})) \\
 &+ \left(5 + \frac{5}{6}\right) a_6^-(W\Gamma_0^E(\mathfrak{p})).
 \end{aligned}$$

First we estimate $c_2(X_{W\Gamma_0^E(\mathfrak{p})})$.

Lemma 3.13 ([20, Section VII.5, eq. (1)]). *For all fundamental discriminant D , $\zeta_E(-1) > \frac{D^{3/2}}{360}$.*

As $a_2, a_3^\pm, a_4^\pm, a_6^\pm$ and l are non-negative, we get

Proposition 3.14. $c_2(X_{W\Gamma_0^E(\mathfrak{p})}) > (\mathbf{N} \mathfrak{p} + 1) \frac{D^{3/2}}{360}$.

Now we estimate $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})})$. The self-intersection number c of the local Chern cycle at the cusp is equal to that for $\mathrm{PSL}_2(\mathcal{O}_E)$ as the isotropy group for the unique cusp in $W\Gamma_0^E(\mathfrak{p})$ is the same as that in $\mathrm{PSL}_2(\mathcal{O}_E)$. Thus we use the results from [20] directly. There the quantity c is shown to be equal to the negative of the length of the cycle in [20, eq. (7), II.5] and the length satisfies the inequality below [20, eq. (2), VII.5]:

$$(3.8) \quad c > -\frac{1}{2} \sum_{|x| < \sqrt{D}} \sigma_0\left(\frac{D-x^2}{4}\right) := -\frac{1}{2} \sum_{x^2 < D, x^2 \equiv D \pmod{4}} \sum_{a > 0, a | \frac{D-x^2}{4}} 1.$$

Combining the inequality with that in [20, Lemma VII.5.3], we get the following lemma. Note that the condition that E has narrow class number 1 is in effect.

Lemma 3.15. *The self-intersection number c of the local Chern cycle of the cusp singularity satisfies*

$$(3.9) \quad c > -\frac{1}{2}D^{1/2} \left(\frac{3}{2\pi^2} \log^2(D) + 1.05 \log(D) \right).$$

Then we estimate a_2 and a_3^\pm .

Definition 3.16. Let $h(D)$ denote the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$ where D is a fundamental discriminant.

Lemma 3.17 ([20, Lemma VII.5.2]). *If $-\Delta < -4$ is a fundamental discriminant then $h(-\Delta) \leq \frac{\sqrt{\Delta}}{\pi} \log \Delta$.*

The following is from [20, page 17]. The other cases listed there are ruled out because E is assumed to have the narrow class number 1.

Lemma 3.18. *If $D > 12$, then*

$$(3.10) \quad \begin{aligned} a_2(\mathrm{PSL}_2(\mathcal{O}_E)) &= h(-4D) \\ a_3^+(\mathrm{PSL}_2(\mathcal{O}_E)) &= \frac{1}{2}h(-3D). \end{aligned}$$

Combining the above two lemmas we get:

Lemma 3.19. *If $D > 12$, then*

$$(3.11) \quad \begin{aligned} a_2(\mathrm{PSL}_2(\mathcal{O}_E)) &\leq \frac{\sqrt{4D}}{\pi} \log(4D) \\ a_3^+(\mathrm{PSL}_2(\mathcal{O}_E)) &\leq \frac{\sqrt{3D}}{2\pi} \log(3D). \end{aligned}$$

Now we put in level structure.

Lemma 3.20. *If $D > 12$ then*

$$(3.12) \quad \begin{aligned} a_2(\Gamma_0^E(\mathfrak{p})) &\leq \frac{3\sqrt{4D}}{\pi} \log(4D) \\ a_3^+(\Gamma_0^E(\mathfrak{p})) &\leq \frac{3\sqrt{3D}}{2\pi} \log(3D). \end{aligned}$$

Proof. Let z be an elliptic point of $\mathrm{PSL}_2(\mathcal{O}_E)$ with isotropy group generated by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have coset decomposition of $\mathrm{PSL}_2(\mathcal{O}_E) = \cup_\alpha \Gamma_0^E(\mathfrak{p})\delta_\alpha \cup$

$\Gamma_0^E(\mathfrak{p})\delta_\infty$, where $\delta_\alpha = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ with $\alpha \in \mathcal{O}_E$ running through a set of representatives of $\mathcal{O}_E/\mathfrak{p}$ and $\delta_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This elliptic point corresponds to several $\Gamma_0^E(\mathfrak{p})$ -inequivalent points: $\delta_\alpha z$'s and $\delta_\infty z$. All elliptic points for $\Gamma_0^E(\mathfrak{p})$ must be one of those. To see which ones of $\delta_\alpha z$'s are elliptic points for $\Gamma_0^E(\mathfrak{p})$, we just need to check if $\delta_\alpha g \delta_\alpha^{-1}$ is in $\Gamma_0^E(\mathfrak{p})$, since we are dealing with elliptic points of type $(2; 1, 1)$ and $(3; 1, \pm 1)$ only. This is equivalent to checking if $c - (d - a)\alpha - b\alpha^2$ is in \mathfrak{p} . In $\mathbb{F}_\mathfrak{p}$, the equation $c - (d - a)\alpha - b\alpha^2 = 0$ has at most two solutions unless $c, a - d, b \in \mathfrak{p}$. Now we claim that it is not possible to have $c, a - d, b \in \mathfrak{p}$. If the claim holds then only two of $\delta_\alpha z$'s can be elliptic points for $\Gamma_0^E(\mathfrak{p})$. Adding in the point $\delta_\infty z$, we get at most three elliptic points for $\Gamma_0^E(\mathfrak{p})$ lying over z . Thus the number of elliptic points of a given type can increase to at most threefold that for $PSL_2(\mathcal{O}_E)$. Combining with the inequalities of Lemma 3.19, we get the inequalities in the statement of this lemma.

It remains to prove our claim. Assume otherwise, i.e., $c, a - d, b \in \mathfrak{p}$. Since g is elliptic and $D > 12$, we conclude from Sec. 3.2 that $a + d$ can only take the values $0, \pm 1$. From $ad - bc = 1$, we find that $a^2 \equiv 1 \pmod{\mathfrak{p}}$. Thus $a \equiv \pm 1 \pmod{\mathfrak{p}}$. We may change the matrix that represents g so that $a \equiv 1 \pmod{\mathfrak{p}}$. Thus also $d \equiv 1 \pmod{\mathfrak{p}}$. Using $ad - bc = 1$ again, we must have $a + d \equiv 2 \pmod{\mathfrak{p}^2}$. Since the value of $a + d$ is 0 or ± 1 , \mathfrak{p}^2 divides (2) or (3), but (2) and (3) do not ramify in E as $D > 12$. We get a contradiction. \square

Lemma 3.21. *Suppose $D > 12$. Then*

1) $a_4^+(W\Gamma_0^E(\mathfrak{p})) = 0$ unless (2) is inert in \mathcal{O}_E and $\mathfrak{p} = (2)$;

2) when (2) is inert in \mathcal{O}_E and $\mathfrak{p} = (2)$,

$$a_4^+(W\Gamma_0^E(\mathfrak{p})) \leq \frac{3\sqrt{4D}}{\pi} \log(4D);$$

3) $a_6^+(W\Gamma_0^E(\mathfrak{p})) = 0$ and

$$\frac{1}{3}a_3^+(W\Gamma_0^E(\mathfrak{p})) \leq \frac{\sqrt{3D}}{4\pi} \log(3D)$$

unless (3) is inert in \mathcal{O}_E and $\mathfrak{p} = (3)$;

4) when (3) is inert in \mathcal{O}_E and $\mathfrak{p} = (3)$,

$$\frac{1}{3}a_3^+(W\Gamma_0^E(\mathfrak{p})) + \frac{8}{3}a_6^+(W\Gamma_0^E(\mathfrak{p})) \leq \frac{4\sqrt{3D}}{\pi} \log(3D).$$

Proof. We check the rotation factor (3.3)

$$\cos \theta_j = \frac{\text{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}}$$

associated to an elliptic element $\alpha \in W\Gamma_0^E(\mathfrak{p})$. In order to have a point with isotropy group of order 4 in $W\Gamma_0^E(\mathfrak{p})$ we must have $\cos \theta_j = \pm \frac{\sqrt{2}}{2}$. As $D > 12$, this can only happen when $\mathfrak{p} = (2)$ and $\det(\alpha_j) = \varpi_{\mathfrak{p}}$ modulo squares in \mathcal{O}_E . In order to have a point with isotropy group of order 6 in $W\Gamma_0^E(\mathfrak{p})$ we must have $\cos \theta_j = \pm \frac{\sqrt{3}}{2}$. As $D > 12$, this can only happen when $\mathfrak{p} = (3)$ and $\det(\alpha_j) = \varpi_{\mathfrak{p}}$ modulo squares in \mathcal{O}_E .

The Atkin-Lehner operator w exchanges some of the $\Gamma_0^E(\mathfrak{p})$ -inequivalent $(3; 1, 1)$ -points which result in $(3; 1, 1)$ -points for $W\Gamma_0^E(\mathfrak{p})$ and fixes the rest of the points which result in $(6; 1, 1)$ -points for $W\Gamma_0^E(\mathfrak{p})$. Thus we get

$$(3.13) \quad 2a_3^+(W\Gamma_0^E(\mathfrak{p})) + a_6^+(W\Gamma_0^E(\mathfrak{p})) = a_3^+(\Gamma_0^E(\mathfrak{p})).$$

It is easy to see that

$$\frac{1}{3}a_3^+(W\Gamma_0^E(\mathfrak{p})) + \frac{8}{3}a_6^+(W\Gamma_0^E(\mathfrak{p})) \leq \frac{8}{3}a_3^+(\Gamma_0^E(\mathfrak{p})).$$

Combining with Lemma 3.20, we get our estimate.

The Atkin-Lehner operator exchanges some of the $\Gamma_0^E(\mathfrak{p})$ -inequivalent $(2; 1, 1)$ -points which result in $(2; 1, 1)$ -points for $W\Gamma_0^E(\mathfrak{p})$ and fixes the rest of the points which result in $(4; 1, 1)$ -points for $W\Gamma_0^E(\mathfrak{p})$. All $(4; 1, 1)$ -points for $W\Gamma_0^E(\mathfrak{p})$ arise in this way, but we may get extra $(2; 1, 1)$ -points $W\Gamma_0^E(\mathfrak{p})$ not arising in this way. Thus we have

$$(3.14) \quad a_4^+(W\Gamma_0^E(\mathfrak{p})) \leq a_2^+(\Gamma_0^E(\mathfrak{p})).$$

Combining with Lemma 3.20, we get our estimate. □

Combining all these inequalities (Lemmas 3.13, 3.15, 3.21) we finally arrive at an estimate for $c_1^2(X_0^+(\mathfrak{p}))$.

Proposition 3.22. *Suppose $D > 12$. Then*

$$(3.15) \quad c_1^2(X_0^+(\mathfrak{p})) > (\mathbf{N} \mathfrak{p} + 1) \frac{D^{3/2}}{180} - \frac{1}{2} D^{1/2} \left(\frac{3}{2\pi^2} \log^2 D + 1.05 \log D \right) \\ - \begin{cases} \frac{1}{4\pi} \sqrt{3D} \log(3D) & \text{if } \mathfrak{p} \neq (3) \\ \frac{4}{\pi} \sqrt{3D} \log(3D) & \text{if } \mathfrak{p} = (3) \end{cases} \\ - \begin{cases} 0 & \text{if } \mathfrak{p} \neq (2) \\ \frac{3}{\pi} \sqrt{4D} \log(4D) & \text{if } \mathfrak{p} = (2). \end{cases}$$

Now that we have inequalities for c_1^2 and c_2 of the Hilbert modular surfaces, we can check for what values of D and n these are of general type. For a given D , we may bound c more precisely by using (3.8).

Theorem 3.23. *Suppose $D > 12$ and $Cl^+(\mathbb{Q}(\sqrt{D})) = 1$. Set $n = \mathbf{N} \mathfrak{p} + 1$. Then the Hilbert modular surface $X_0^+(\mathfrak{p})$ is of general type if D or n is sufficiently large or more precisely if the following conditions on D and n are satisfied:*

$D \geq 853$ or $D = 313, 337, 353, 409, 433, 449, 457, 521, 569, 593, 601, 617, 641, 653, 661, 673, 677, 701, 709, 757, 769, 773, 797, 809, 821, 829$	<i>no constraint on n</i>
$D = 241$	$n > 3$
$D = 193$	$n > 3$
$D = 157, 181, 277, 349, 373, 397, 421, 541, 613$	$\mathfrak{p} \neq (2)$
$D = 233, 281$	$\mathfrak{p} \neq (3)$
$D = 149, 173, 197, 269, 293, 317, 389, 461, 509, 557$	$\mathfrak{p} \neq (2), (3)$
$D = 137$	$n > 3$ and $\mathfrak{p} \neq (3)$
$D = 113$	$n > 4$ and $\mathfrak{p} \neq (3)$
$D = 109$	$n > 4$ and $\mathfrak{p} \neq (2)$
$D = 101$	$n > 3$ and $\mathfrak{p} \neq (2), (3)$
$D = 97$	$n > 6$
$D = 89$	$n > 5$ and $\mathfrak{p} \neq (3)$
$D = 73$	$n > 7$
$D = 61$	$n > 6$

$D = 53$	$n > 7$ and $\mathfrak{p} \neq (3)$
$D = 41$	$n > 12$
$D = 37$	$n > 12$
$D = 29$	$n > 15$
$D = 17$	$n > 32$
$D = 13$	$n > 41$

Proof. We note that $n = \mathbf{N} \mathfrak{p} + 1 \geq 3$. By Prop. 3.14, as long as $D > 127$, $c_2(X_{W\Gamma_0^E(\mathfrak{p})}) > 12$. Next we give a rough estimate for D so that $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) > 0$ by using the inequality (3.15). When $\mathfrak{p} \neq (2)$ or (3) , as long as $D > 414$, we have $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) > 0$. When $\mathfrak{p} = (2)$, as long as $D > 849$, we have $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) > 0$. When $\mathfrak{p} = (3)$, as long as $D > 384$, we have $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) > 0$. The numerical computation was done in SageMath[3]. We also used it to produce a list of discriminants of real quadratic fields with narrow class number 1. Here is the list up to 853 which is the smallest one that is greater than 849:

- 5, 8, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137, 149, 157, 173, 181, 193, 197, 233, 241, 269, 277, 281, 293, 313, 317, 337, 349, 353, 373, 389, 397, 409, 421, 433, 449, 457, 461, 509, 521, 541, 557, 569, 593, 601, 613, 617, 641, 653, 661, 673, 677, 701, 709, 757, 769, 773, 797, 809, 821, 829, 853.

Thus for $D \geq 853$ and any \mathfrak{p} we always have $c_2(X_{W\Gamma_0^E(\mathfrak{p})}) > 12$ and $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) > 0$. By Prop. 3.8, these are surfaces of general type.

Next we compute for each of the discriminant D in the list, a sufficient condition on n (or on \mathfrak{p}) so that $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) + c_2(X_{W\Gamma_0^E(\mathfrak{p})}) > 12$ and $c_1^2(X_{W\Gamma_0^E(\mathfrak{p})}) > 0$ are satisfied. We may use the sharper bound (3.8) for the intersection number c of local Chern cycles. We note that the formula (3.15) for estimating c_1^2 branches when we have $\mathfrak{p} = (2), (3)$. Under the constraint of our theorem, (2) is split if and only if $D \equiv 1 \pmod{8}$, (2) is inert if and only if $D \equiv 5 \pmod{8}$; (3) is split if and only if $D \equiv 1 \pmod{3}$ and (3) is inert if and only if $D \equiv 2 \pmod{3}$. For those D 's with inert primes (2) or (3), we compute the values of c_1^2 and c_2 to check if we get surfaces of general type or not. The numerical results are summarised in the table. \square

Remark 3.24. We are providing a sufficient condition for the Hilbert modular surface $X_0^+(\mathfrak{p}) = X_{W\Gamma_0^E(\mathfrak{p})}$ to be of general type. More precise analysis is

needed to determine the exact type of a given $X_0^+(\mathfrak{p})$. We give some examples in the Sec. 3.4.

3.4. Examples

The two examples give rational surfaces.

3.4.1. $D = 5$. Let $E = \mathbb{Q}(\sqrt{5})$. The Galois conjugation is denoted by $*$. By the algorithm[20, eq. (3) on page 38], the cusp resolution at infinity of $\bar{Y}_0^+(\mathfrak{p})$ is a cycle of type $[3]^o$, i.e., a rational curve with an ordinary double point and with self-intersection number -1 . To find the $W\Gamma_0^E(\mathfrak{p})$ -inequivalent elliptic points, we first consider the $PSL_2(\mathcal{O}_E)$ -inequivalent elliptic points which were worked out in [9, Satz 1]. Let $\varepsilon = (1 + \sqrt{5})/2$ and $\varepsilon^* = (1 - \sqrt{5})/2$. We list the type of the elliptic point and a generator of the isotropy group:

$$\begin{array}{ll} (2; 1, 1), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (2; 1, 1), \begin{pmatrix} 0 & -\varepsilon^* \\ -\varepsilon & 0 \end{pmatrix} \\ (3; 1, 1), \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} & (3; 1, -1), \begin{pmatrix} 0 & -\varepsilon^* \\ -\varepsilon & 1 \end{pmatrix} \\ (5; 1, 3), \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} & (5; 1, 2), \begin{pmatrix} 1 & -\varepsilon^* \\ \varepsilon^* & -\varepsilon^* \end{pmatrix}. \end{array}$$

We have coset decomposition $PSL_2(\mathcal{O}_E) = \cup_{\alpha} \Gamma_0^E(\mathfrak{p})\delta_{\alpha} \cup \Gamma_0^E(\mathfrak{p})\delta_{\infty}$ where $\delta_{\alpha} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ with $\alpha \in \mathcal{O}_E$ running through a set of representatives of $\mathcal{O}_E/\mathfrak{p}$ and $\delta_{\infty} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let z be an elliptic point for $PSL_2(\mathcal{O}_E)$ and $\gamma \in PSL_2(\mathcal{O}_E)$ be a generator of the isotropy group. Then $\delta_{\alpha}.z$ (resp. $\delta_{\infty}.z$) is an elliptic points for $\Gamma_0^E(\mathfrak{p})$ if and only if $\delta_{\alpha}\gamma\delta_{\alpha}^{-1}$ (resp. $\delta_{\infty}\gamma\delta_{\infty}^{-1}$) lies in $\Gamma_0^E(\mathfrak{p})$. Write γ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is easy to check that $\delta_{\alpha}\gamma\delta_{\alpha}^{-1} \in \Gamma_0^E(\mathfrak{p})$ if and only if $b\alpha^2 + (d - a)\alpha - c \in \mathfrak{p}$ and that $\delta_{\infty}\gamma\delta_{\infty}^{-1} \in \Gamma_0^E(\mathfrak{p})$ if and only if $b \in \mathfrak{p}$. Once we get the $\Gamma_0^E(\mathfrak{p})$ -inequivalent elliptic points, we need to check how the Atkin-Lehner operator w acts on them. We will work this out with a more specific \mathfrak{p} .

$\mathfrak{p} = (2)$. The Hilbert modular surface $X_0^+(\mathfrak{p})$ is a rational surface. We explain below. We get the following inequivalent elliptic points for $\Gamma_0^E(\mathfrak{p})$. Instead of writing out their coordinates, we write down the type and a generator of the isotropy subgroup of $\Gamma_0^E(\mathfrak{p})$ that fixes each elliptic point.

The coordinates can be recovered by (3.2).

$$\begin{aligned}
 (2; 1, 1), \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} & \qquad (2; 1, 1), \begin{pmatrix} -1 & -\varepsilon^* \\ -2\varepsilon & 1 \end{pmatrix} \\
 (3; 1, 1), \begin{pmatrix} -\varepsilon & 1 \\ -2(1 + \varepsilon) & 1 + \varepsilon \end{pmatrix} & \qquad (3; 1, 1), \begin{pmatrix} -\varepsilon^* & 1 \\ -2(1 + \varepsilon^*) & 1 + \varepsilon^* \end{pmatrix} \\
 (3; 1, -1), \begin{pmatrix} \varepsilon^* & -\varepsilon^* \\ -2\varepsilon & \varepsilon \end{pmatrix} & \qquad (3; 1, -1), \begin{pmatrix} 1 + \varepsilon^* & -\varepsilon^* \\ 2\varepsilon^* & -\varepsilon^* \end{pmatrix}.
 \end{aligned}$$

It can be checked directly that the Atkin-Lehner operator w fixes the two $(2; 1, 1)$ -points respectively. Since there cannot exist elliptic points of type $(6; 1, \pm 1)$ for $W\Gamma_0^E(\mathfrak{p})$, we see that w must exchange the two $(3; 1, 1)$ - (resp. $(3; 1, -1)$ -) points. We get one $(4; 1, 1)$ -, one $(4; 1, -1)$ -, one $(3; 1, 1)$ -, one $(3; 1, -1)$ - and possibly some new $(2; 1, 1)$ -points.

We consider certain Hirzebruch cycles on the Hilbert modular surface. Set

$$(3.16) \qquad \tilde{F}_B = \left\{ (z_1, z_2) \in \mathcal{H}^2 \cup \mathbb{P}^1(E) : (z_2 \quad 1) B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\}$$

where B is a skew-Hermitian matrix in $M_2(E)$, i.e., ${}^tB^* = B$. Let F_B denote the strict transform in $X_0^+(\mathfrak{p})$ of the image of \tilde{F}_B in $Y_0^+(\mathfrak{p})$.

Let $B = \begin{pmatrix} 0 & \sqrt{5}\varepsilon^* \\ \sqrt{5}\varepsilon & 0 \end{pmatrix}$. The $(3; 1, -1)$ -point can be represented by $(-\sqrt{5} + i\sqrt{3})\varepsilon^*/4, (\sqrt{5} + i\sqrt{3})\varepsilon/4$, so obviously it lies on F_B . The $(4; 1, -1)$ -point can be represented by $((1 + i)/(2\varepsilon), (-1 + i)/(-2\varepsilon^*))$. After applying translation by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, we get the $\Gamma_0^E(\mathfrak{p})$ -equivalent elliptic point $((-\sqrt{5} + i)/(2\varepsilon), (-\sqrt{5} + i)/(-2\varepsilon^*))$. Thus we see that the $(4; 1, -1)$ -point also lies on F_B . The stabiliser Γ_B of \tilde{F}_B in $\Gamma_0^E(\mathfrak{p})$ consists of those elements γ such that ${}^t\gamma^*B\gamma = \pm B$. Thus Γ_B is the degree 2 extension of the group

$$(3.17) \qquad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^E(\mathfrak{p}) : a, d \in \mathbb{Z}, c \in 2\varepsilon\sqrt{5}\mathbb{Z}, b \in \varepsilon^*\sqrt{5}\mathbb{Z} \right\}$$

generated by $\begin{pmatrix} \sqrt{5} & 2\varepsilon^* \\ 2\varepsilon & \sqrt{5} \end{pmatrix}$. The stabiliser $\tilde{\Gamma}_B$ of \tilde{F}_B in $W\Gamma_0^E(\mathfrak{p})$ is a degree 2 extension of Γ_B by $\begin{pmatrix} 2 & \varepsilon^*\sqrt{5} \\ 2\varepsilon\sqrt{5} & -4 \end{pmatrix}$. Note that the group (3.17) is isomorphic to $\Gamma_0^{\mathbb{Q}}(10)$ which is the congruence subgroup of $SL_2(\mathbb{Z})$ with lower-left entry congruent to 0 modulo 10. As $\Gamma_0^{\mathbb{Q}}(10\mathbb{Z}) \setminus \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ is isomorphic to \mathbb{P}^1 , the non-singular model of F_B is isomorphic to \mathbb{P}^1 . Let sw denote the involution on $\tilde{Y}_0^+(\mathfrak{p})$ induced by swapping coordinates on \mathcal{H}^2 : $(z_1, z_2) \mapsto (z_2, z_1)$. It can be extended to an involution on $X_0^+(\mathfrak{p})$. If (z_1, z_2) is a point satisfying (3.16)

then (z_2, z_1) is $\Gamma_0^E(\mathfrak{p})$ -equivalent to (z_1, z_2) via $\begin{pmatrix} -\varepsilon^* & \\ & \varepsilon \end{pmatrix}$. This means that sw fixes F_B point-wise. This, in turn, implies that F_B is non-singular. We conclude that F_B is a non-singular rational curve.

Recall that the cusp resolution is formed by gluing copies of \mathbb{C}^2 . Following the method in [20, V.2] we can determine the local equation of F_B on each copy of \mathbb{C}^2 . On the k -th copy of \mathbb{C}^2 the coordinates are related by

$$2\pi iz_j = A_{k-1}^{(j)} \log(u_k) + A_k^{(j)} \log(v_k)$$

where z_j denote the coordinate on the j -th copy of \mathcal{H} for $j = 1, 2$ and (u_k, v_k) denote the coordinates of the k -th copy of \mathbb{C}^2 for $k \in \mathbb{Z}$. For the case at hand, we take $A_0 = 1$ and $A_1 = (3 - \sqrt{5})/2$ which form a Z -basis of \mathcal{O}_E . Other values of A_k 's are omitted. Recall that $a^{(j)}$ denote the image of $a \in E$ via the j -th embedding to \mathbb{R} . Then on the 1-st copy of \mathbb{C}^2 , the equation of F_B becomes $u_1 = v_1$. There is no intersection with coordinate axes in other copies of \mathbb{C}^2 . Thus F_B intersects the cusp resolution at the origin of the 1-st copy of \mathbb{C}^2 which corresponds to the ordinary double point on the cusp resolution. Thus the intersection number of F_B with the cusp resolution is 2. By [20, Corollary 4.1], we get

$$c_1(X_0^+(\mathfrak{p})) \cdot F_B = 2 \text{vol}(\tilde{\Gamma}_B \setminus \mathcal{H}) + \sum_x Z_x \cdot F_B$$

where the sum runs over all singularities x of $Y_0^+(\mathfrak{p})$ and Z_x denotes the local Chern cycle of x . The volume

$$\text{vol}(\tilde{\Gamma}_B \setminus \mathcal{H}) = \frac{1}{4} \text{vol}(\Gamma_0^{\mathbb{Q}}(10) \setminus \mathcal{H}) = \frac{18}{4} \text{vol}(\text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}) = 9\zeta_{\mathbb{Q}}(-1) = -\frac{3}{4}.$$

Here $\zeta_{\mathbb{Q}}$ denotes the Riemann zeta function. The local Chern cycles needed in the computation can be looked up in Remark 3.10. Thus we find

$$c_1(X_0^+(\mathfrak{p})) \cdot F_B = -\frac{3}{2} + \frac{1}{3} \cdot n_3 + \frac{1}{2} \cdot n_4$$

where n_3 is the number of $(3; 1, 1)$ -points that F_B passes through and n_4 is the number of $(4; 1, 1)$ -points that F_B passes through. As intersection numbers are integers, we are force to have $n_3 = 0$ and $n_4 = 1$ and thus $c_1(X_0^+(\mathfrak{p})) \cdot F_B = 1$. By the Adjunction formula, $F_B^2 = -1$. We get a linear configuration of non-singular rational curves with self-intersection numbers $-2, -1, -2$, where the (-2) -curves come from desingularity of the $(3; 1, -1)$ -

and the $(4; 1, -1)$ -points mentioned above. After blowing down F_B we acquire two intersecting (-1) -curves and this shows that the surface $X_0^+(\mathfrak{p})$ is a rational surface by the rationality criterion [20, VII.2.2].

3.4.2. $D = 13$. We adopt essentially the same notation as in the previous example. Now the quadratic field is $E = \mathbb{Q}(\sqrt{13})$. The Galois conjugation is denoted by $*$. We will regard E as a subfield of \mathbb{R} . Set $\varepsilon = (3 + \sqrt{13})/2$ to be a fundamental unit. The cusp resolution at the infinity of $\bar{Y}_0^+(\mathfrak{p})$ is of a configuration of type $[5, 2, 2]^\circ$. We label the non-singular rational curves occurring in the cusp resolution as S_0, S_1 and S_2 . Following the method in [9], we can locate all the $\mathrm{PSL}_2(\mathcal{O}_E)$ -inequivalent elliptic points. We review the process briefly. First we can compute that the y -coordinates of an elliptic fixed point $z = (z_1, z_2)$ in the fundamental domain given as in [9] satisfies

$$1 \leq \left(\frac{17}{8}\right)^2 + (y_1 y_2)^2 + \frac{13}{16}(y_1 y_2),$$

or, in other words,

$$(3.18) \quad y_1 y_2 \geq (-17 + 2\sqrt{94})/16 > 0.149.$$

Consider the elliptic fixed points of order 2. Assume it is fixed by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_E)$. We may assume that $c < 0$. From (3.18) and (3.2), we deduce that $|cc^*| \leq 6$. Thus up to a unit c is either $1, 4 - \sqrt{13}$ or 2 . In the fundamental domain, we have $\varepsilon^{-2} \leq y_1/y_2 < \varepsilon^2$ and (x_1, x_2) lies in the set

$$\mathfrak{P} := \{(u + v\sqrt{13}, u - v\sqrt{13}) \mid -1/2 < u \leq 1/2, -1/4 < v \leq 1/4\}.$$

Thus c can take the following values:

$$(3.19) \quad -1, -\varepsilon, (1 - \sqrt{13})/2, (-5 - \sqrt{13})/2, -2, -2\varepsilon.$$

For each of these values we find all values a and d in \mathcal{O}_E such that $a + d = 0$ and $((a - d)/2c, (a^* - d^*)/2c^*)$ lies in the set \mathfrak{P} . Then we determine the value for $b \in \mathcal{O}_E$ by ensuring the determinant is 1.

Finally we need to check which ones are $\mathrm{PSL}_2(\mathcal{O}_E)$ -conjugate matrices and keep only one of those. We summarise the results below. A set of $\mathrm{PSL}_2(\mathcal{O}_E)$ -inequivalent $(2; 1, 1)$ -points is given by the fixed point of the

following matrices:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\varepsilon^* \\ -\varepsilon & 0 \end{pmatrix}.$$

Next we consider the elliptic fixed points of order 3. We use the same notation as in the case of order 2. Again c can only take the values in (3.19). A similar process produces the matrices whose fixed points form a set of $PSL_2(\mathcal{O}_E)$ -inequivalent points of order 3. We can check which ones are $(3; 1, 1)$ -points and which ones are $(3; 1, -1)$ -points by (3.3). A set of $PSL_2(\mathcal{O}_E)$ -inequivalent $(3; 1, 1)$ -points is given by the fixed point of the following matrices:

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} \varepsilon & 2 \\ -1 - \varepsilon & 1 - \varepsilon \end{pmatrix};$$

a set of $PSL_2(\mathcal{O}_E)$ -inequivalent $(3; 1, -1)$ -points is given by the fixed point of the following matrices:

$$\begin{pmatrix} -\varepsilon & 2(\varepsilon - 1) \\ -\varepsilon & \varepsilon + 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & \varepsilon - 1 \\ \varepsilon^* - 1 & 2 \end{pmatrix}.$$

It is easy to find the $\Gamma_0^E(\mathfrak{p})$ -inequivalent elliptic points from right coset decomposition $PSL_2(\mathcal{O}_E) = \cup_{\alpha} \Gamma_0^E(\mathfrak{p})\delta_{\alpha} \cup \Gamma_0^E(\mathfrak{p})\delta_{\infty}$ where $\delta_{\alpha} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ with $\alpha \in \mathcal{O}_E$ running through a set of representatives of $\mathcal{O}_E/\mathfrak{p}$ and $\delta_{\infty} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$\mathfrak{p} = (4 + \sqrt{13})$. We list the type and one generator of isotropy group for each $\Gamma_0^E(\mathfrak{p})$ -inequivalent elliptic point:

$$\begin{aligned} (3; 1, 1), \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} & \qquad (3; 1, 1), \begin{pmatrix} 1 - \varepsilon^* & 2 \\ -1 - \varepsilon^* & \varepsilon^* \end{pmatrix} \\ (3; 1, -1), \begin{pmatrix} 1 + \varepsilon & \varepsilon \\ -2(\varepsilon - 1) & -\varepsilon \end{pmatrix} & \qquad (3; 1, -1), \begin{pmatrix} 2 & 1 - \varepsilon^* \\ 1 - \varepsilon & -1 \end{pmatrix}. \end{aligned}$$

There is no $(2; 1, 1)$ -point. Since there cannot be any elliptic points with isotropy group of order 6 for $W\Gamma_0^E(\mathfrak{p})$ acting on \mathcal{H}^2 , the Atkin-Lehner operator w must exchange the two $(3; 1, 1)$ -points (resp. $(3; 1, -1)$ -points).

Now consider the curve \tilde{F}_B on $W\Gamma_0^E(\mathfrak{p}) \backslash \mathcal{H}^2$ defined as in (3.16) and set $B = \begin{pmatrix} 0 & 4 - \sqrt{13} \\ -4 - \sqrt{13} & 0 \end{pmatrix}$. Define F_B analogously. The stabiliser $\tilde{\Gamma}_B$ of \tilde{F}_B in $W\Gamma_0^E(\mathfrak{p})$ consists of elements of the form

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^E(\mathfrak{p}) : a, d \in \mathbb{Z}, c \in (4 + \sqrt{13})\mathbb{Z}, b \in (4 - \sqrt{13})\mathbb{Z} \right\}.$$

Thus we find that F_B is birational to $\Gamma_0^{\mathbb{Q}}(3\mathbb{Z}) \backslash \mathcal{H}$ which is of genus 0. The Atkin-Lehner operator sends \tilde{F}_B to $\tilde{F}_{B'}$ with $B' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The latter is

obviously point-wise stable under the swapping operator sw which is the involution on $\tilde{Y}_0^+(\mathfrak{p})$ induced by swapping coordinates on $\mathcal{H}^2: (z_1, z_2) \mapsto (z_2, z_1)$. Thus F_B is a non-singular rational curve. We can compute how F_B intersects with the cusp resolution. Following the notation of [20, V.2], we have $A_{-1} = (5 + \sqrt{13})/2$, $A_0 = 1$, $A_1 = (5 - \sqrt{13})/2$ and $A_2 = 4 - \sqrt{13}$. Then F_B has local equation $u_2 = 1$ on the 2-nd copy of \mathbb{C}^2 and $F_{B'}$ has local equation $u_0 = 1$ in the 0-th copy of \mathbb{C}^2 . Thus the intersection number of F_B with the cusp resolution is 2.

As before we have

$$(3.20) \quad c_1(X_0^+(\mathfrak{p})).F_B = 2 \operatorname{vol}(F'_B) + \sum Z_x.F_B$$

where Z_x is the local Chern cycle at a singular point x . Thus we get

$$\begin{aligned} c_1(X_0^+(\mathfrak{p})).F_B &= 2 \operatorname{vol}(\Gamma_0^{\mathbb{Q}}(3\mathbb{Z}) \setminus \mathcal{H}) + 2 + \frac{1}{3} \cdot n_3 \\ &= -\frac{4}{3} + 2 + \frac{1}{3} \cdot n_3 \end{aligned}$$

with n_3 the number of $(3; 1, 1)$ -points that F_B passes through. As there is just one $(3; 1, 1)$ -point, we are forced to have $n_3 = 1$ and thus $c_1(X_0^+(\mathfrak{p})).F_B = 1$. By Adjunction formula $F_B^2 = -1$. Thus we get a linear configuration of $[-2, -2, -1, -3]$ where the (-2) -curves are S_1 and S_2 from the cusp resolution, the (-1) -curve is F_B and the (-3) -curve is from the resolution of singularity of the $(3; 1, 1)$ -point. After blowing down F_B and S_2 consecutively, we get two intersecting (-1) -curves. Again by the rationality criterion [20, VII.2.2], we conclude that $W\Gamma_0^E((4 + \sqrt{13})) \setminus \mathcal{H}^2$ is a rational surface.

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