

Lifting G -irreducible but GL_n -reducible Galois representations

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In recent work, the authors proved a general result on lifting G -irreducible odd Galois representations $\mathrm{Gal}(\overline{F}/F) \rightarrow G(\overline{\mathbb{F}}_\ell)$, with F a totally real number field and G a reductive group, to geometric ℓ -adic representations. In this note we take G to be a classical group and construct many examples of G -irreducible representations to which these new lifting methods apply, but to which the lifting methods currently provided by potential automorphy theorems do not.

1. Introduction

Let G be a smooth group scheme over \mathbb{Z}_ℓ such that G^0 is a split connected reductive group scheme, and G/G^0 is finite of order prime to ℓ . Let F be a number field with algebraic closure \overline{F} and absolute Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$, and let $\bar{\rho}: \Gamma_F \rightarrow G(\overline{\mathbb{F}}_\ell)$ be a continuous homomorphism. The question of whether $\bar{\rho}$ admits a geometric (or, more ambitiously, automorphic or motivic) lift $\rho: \Gamma_F \rightarrow G(\overline{\mathbb{Z}}_\ell)$ has attracted a great deal of interest at least since Serre formulated his modularity conjecture in the case $F = \mathbb{Q}$, $G = \mathrm{GL}_2$, and $\bar{\rho}$ absolutely irreducible and odd (see Definition 1.1 below). There are essentially two types of methods, first studied for $G = \mathrm{GL}_2$ and F totally real, for proving such lifting theorems, a purely Galois-theoretic approach developed by Ramakrishna ([15], [16]), and an approach developed in [11] and [10] making crucial use of potential automorphy ([21]). Much further work on both of these methods (and their descendants) has led to the papers [7], for the Galois-theoretic method, and [1], for the automorphic methods, which more or less represent the state-of-the-art (see also [5] for a refinement of the local hypotheses in [1]).

In the present paper, we will discuss examples when G is a classical group, and where the Galois-theoretic methods of [7] apply, but where current potential automorphy methods do not. The basic feature here constraining the range of applicability of potential automorphy methods is the distinction between $\bar{\rho}$ that are G -irreducible and $\bar{\rho}$ that remain irreducible after composition with a faithful representation $G \rightarrow \mathrm{GL}_N$: recall that $\bar{\rho}$ is G -irreducible if its image is contained in no proper parabolic subgroup of $G(\overline{\mathbb{F}}_\ell)$.

We first carry out our analysis for examples (§3 and §4) whose construction is rather straightforward, but which we hope serves to convince the reader of the ubiquity of the phenomenon we are highlighting; in §3 we simply sum two-dimensional representations, while in §4 we explain a quite general but “soft” approach to constructing examples using Moret-Bailly’s result on the “potential inverse Galois problem with local conditions” ([13]). The bulk of the paper treats a more elaborate example (§5-§A) arising from orthogonal representations $\Gamma_{\mathbb{Q}} \rightarrow \mathrm{SO}_N(\mathbb{F}_\ell)$ constructed in Zywina’s work on the inverse Galois problem ([23]).

Whether one approaches lifting problems via potential automorphy or just via Galois deformation theory, a critical hypothesis is the following generalization of Serre’s “oddness” condition:

Definition 1.1. We say $\bar{\rho}: \Gamma_F \rightarrow G(\overline{\mathbb{F}}_\ell)$ is odd if for all $v \mid \infty$,

$$h^0(\Gamma_{F_v}, \bar{\rho}(\mathfrak{g}^{\mathrm{der}})) = \dim(\mathrm{Flag}_{G^0}),$$

where $\bar{\rho}(\mathfrak{g}^{\mathrm{der}})$ is the Lie algebra of the derived group G^{der} of G^0 , equipped with the action of Γ_F via the composite $\mathrm{Ad} \circ \bar{\rho}$, and Flag_{G^0} is the flag variety of G^0 .

In the examples of §3 and §4, this oddness condition will follow immediately from the construction. Most of our technical work is to establish the oddness condition in Zywina’s examples; we deduce it from a topological calculation of the action of complex conjugation on (a piece of) the cohomology of a real elliptic surface. We hope these calculations may be of some independent interest.

We state the main theorem of this note (this is Theorem 6.1 below).

Theorem 1.2. *Let $N \geq 6$ be an even integer, and let $\ell \gg_N 0$ be a sufficiently large prime.*

- 1) *There is a totally real field F which is solvable over \mathbb{Q} and infinitely many non-isomorphic Galois representations*

$$\bar{\rho}: \Gamma_F \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$$

such that $\bar{\rho}$ is irreducible as an SO_{N+1} -valued representation, but reducible as a GL_{N+1} -valued representation, and $\bar{\rho}$ admits a geometric lift $\rho: \Gamma_F \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ with Zariski-dense image.

- 2) *Moreover, if we assume that representations $\Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$ admit Hodge–Tate regular de Rham lifts $\Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$, then in part (1) we may take $F = \mathbb{Q}$.*

Let us explain the restriction in this theorem that gives us only a conditional result when $F = \mathbb{Q}$. In addition to oddness, the other requirement for producing lifts (via either of the methods described above) of residual representations $\bar{\rho}: \Gamma_F \rightarrow G(\overline{\mathbb{F}}_\ell)$ is that $\bar{\rho}|_{\Gamma_{F_v}}$ should have lifts at each place v of F —of course, this is obvious unless $\bar{\rho}$ is ramified at v —and that when v lies above ℓ these lifts may be taken to be de Rham and Hodge–Tate regular. It is perhaps to be expected that this is no condition at all, but it is not known in this generality, and unfortunately we have nothing to add at present to this question. In our examples, the most serious obstacle is the existence of Hodge–Tate regular lifts when $v|\ell$, and we will mostly avoid this by producing examples over some number field F allowing us to rig the local behavior of our $\bar{\rho}$. That said, in contrast to the examples based on the potential inverse Galois problem, in the theorem above arising from Zywinia’s examples we could take $F = \mathbb{Q}$, if we knew that representations $\Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$ admit a Hodge–Tate regular de Rham lift. Recently the techniques for producing such lifts have greatly advanced: work of Emerton–Gee ([6]) addresses this question with GL_{N+1} in place of SO_{N+1} , and others are at work extending the methods of [6] to other groups. In the examples arising from Moret–Bailly’s theorem (§4), we cannot necessarily ensure that F/\mathbb{Q} is solvable as in Theorem 1.2.

We make some further remarks on the relation of our examples to the automorphic theory. In §4 and §5, the representations we construct have the form $\bar{\rho} = \bar{\theta} \oplus 1$, where N is an even integer, for some $\bar{\theta}: \Gamma_F \rightarrow \mathrm{SO}_N(\overline{\mathbb{F}}_\ell)$ with “large” image, so $\bar{\rho}$ is valued in $\mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$ and is SO_{N+1} -irreducible but GL_{N+1} -reducible. We note that our examples suggest, on the automorphic side, the existence of congruences between certain non-endoscopic cusp forms on $\mathrm{Sp}_N(\mathbb{A}_F)$ and certain endoscopic forms associated to its (split) endoscopic group SO_N . Moreover, when $N \equiv 2 \pmod{4}$, $\bar{\theta}$ itself is not odd and cannot

be lifted using the methods of [7], [1], or indeed of any hoped-for extension of potential automorphy methods. Indeed, we expect $\bar{\theta}$ will only have geometric $\mathrm{SO}_N(\overline{\mathbb{Z}}_\ell)$ -lifts, if any, with some multiplicity greater than 1 in its Hodge–Tate weights (as GL_N -valued representation), and heuristically one expects its deformation ring (parametrizing de Rham lifts with fixed Hodge–Tate data) to be zero-dimensional. In many cases we prove that $\bar{\theta}$ indeed has no regular lift in Proposition 4.4 (see also Remark 4.3).

It remains an interesting open problem to develop potential automorphy methods, and corresponding lifting theorems, in a more group-intrinsic manner, the difference between G -irreducibility and GL_n irreducibility explored in this paper providing one motivation for doing so. To do this would require further understanding of Langlands functoriality, e.g., between automorphic representations of inner forms of GSpin_{2n+1} and GL_{2n} , including compatibility with local Langlands at ramified primes. Moreover, allowing representations that are G -irreducible but GL_n -reducible would force the Taylor–Wiles method to grapple with contributions from endoscopic groups where, interestingly, automorphic representations on an endoscopic group that are not discrete series at infinity can intervene (again see Remark 4.3).

2. A theorem from [7]

We recall a weakening of the main theorem of [7] (the full result yields more precise conclusions about the local restrictions and the image of the lift), which will be our input for lifting residual representations $\bar{\rho}$ once we have established some of their local and global properties:

Theorem 2.1 (See Theorem A of [7]). *Let $\ell \gg_G 0$ be a prime. Let F be a totally real field, and let $\bar{\rho}: \Gamma_F \rightarrow G(\overline{\mathbb{F}}_\ell)$ be a continuous representation unramified outside a finite set of finite places S containing the places above ℓ . Let \tilde{F} denote the smallest extension of F such that $\bar{\rho}(\Gamma_{\tilde{F}})$ is contained in $G^0(\overline{\mathbb{F}}_\ell)$, and assume that $[\tilde{F}(\zeta_\ell): \tilde{F}]$ is strictly greater than an integer a_G depending only on the root datum of G (see [7, Lemma A.6]). Fix a geometric lift $\mu: \Gamma_F \rightarrow G/G^{\mathrm{der}}(\overline{\mathbb{Z}}_\ell)$ of $\bar{\mu} := \bar{\rho} \pmod{G^{\mathrm{der}}}$, and assume that $\bar{\rho}$ satisfies the following:*

- $\bar{\rho}$ is odd.
- $\bar{\rho}|_{\Gamma_{\tilde{F}(\zeta_\ell)}}$ is absolutely irreducible.
- For all $v \in S$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a continuous lift $\rho_v: \Gamma_{F_v} \rightarrow G(\overline{\mathbb{Z}}_\ell)$ such that $\rho_v \pmod{G^{\mathrm{der}}} = \mu|_{\Gamma_{F_v}}$; and that for $v \nmid \ell$ this lift may be chosen to be de

Rham and regular in the sense that the associated Hodge–Tate cocharacters are regular.

Then there is a lift

$$\begin{array}{ccc} & & G(\overline{\mathbb{Z}}_\ell) \\ & \nearrow \rho & \downarrow \\ \Gamma_F & \xrightarrow{\bar{\rho}} & G(\overline{\mathbb{F}}_\ell) \end{array}$$

of $\bar{\rho}$ satisfying:

- *The projection of ρ to $G/G^{\mathrm{der}}(\overline{\mathbb{Z}}_\ell)$ equals μ .*
- *ρ is unramified outside a finite set of primes, and the restrictions $\rho|_{\Gamma_{F_v}}$ for $v|\ell$ are de Rham and regular, having the same ℓ -adic Hodge type as ρ_v .*
- *The Zariski-closure of the image $\rho(\Gamma_F)$ contains G^{der} .*

3. An elementary example

In this brief section, we consider a relatively simple example where $G = \mathrm{GSp}_{2n}$, defined with respect to the symplectic form $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, where 1_n denotes the $n \times n$ identity matrix. Let $\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_n: \Gamma_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be n Galois representations satisfying

- The $\bar{\rho}_i|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ are irreducible and pairwise non-isomorphic.
- The determinants $\det(\bar{\rho}_i)$ are independent of i .
- The $\bar{\rho}_i$ are all odd, i.e. $\det(\bar{\rho}_i(c)) = -1$ for all i .

Writing $\bar{\rho}_i(g) = \begin{pmatrix} a_i(g) & b_i(g) \\ c_i(g) & d_i(g) \end{pmatrix}$, define

$$\bar{\rho}(g) = \left(\begin{array}{cccc|cccc} a_1(g) & 0 & \cdots & 0 & b_1(g) & 0 & \cdots & 0 \\ 0 & a_2(g) & 0 & \cdots & 0 & b_2(g) & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n(g) & 0 & \cdots & 0 & b_n(g) \\ \hline c_1(g) & 0 & \cdots & 0 & d_1(g) & 0 & \cdots & 0 \\ 0 & c_2(g) & 0 & \cdots & 0 & d_2(g) & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & c_n(g) & 0 & \cdots & 0 & d_n(g) \end{array} \right).$$

The condition that $\det(\bar{\rho}_i)$ be independent of i guarantees that $\bar{\rho}$ is valued in GSp_{2n} and has symplectic multiplier $\det(\bar{\rho}_i)$. Oddness of each $\bar{\rho}_i$ therefore implies oddness of $\bar{\rho}$. Since the $\bar{\rho}_i|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ are irreducible, non-isotropic and pairwise non-isomorphic, the only subspaces invariant under $\bar{\rho}$ are sums of some subset of the $\bar{\rho}_i$. It follows that $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ leaves no (nonzero) isotropic subspace invariant, so does not have image contained in any proper parabolic subgroup of GSp_{2n} . Thus $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is GSp_{2n} -irreducible, while obviously not GL_{2n} -irreducible.

Proposition 3.1. *Assume $\ell \gg_n 0$. With notation as above, $\bar{\rho}$ admits a Hodge–Tate regular de Rham lift $\rho: \Gamma_{\mathbb{Q}} \rightarrow \mathrm{GSp}_{2n}(\bar{\mathbb{Z}}_\ell)$ with Zariski-dense image.*

Proof. From the above discussion, this will now follow from Theorem 2.1 provided we can check the local hypotheses in that theorem. For this, we note that by Lemma 3.2 below each $\bar{\rho}_i|_{\Gamma_{\mathbb{Q}_\ell}}$ admits a potentially crystalline lift $\rho_{i,\ell}$ satisfying:

- $\det(\rho_{i,\ell})$ is independent of i .
- The Hodge–Tate weights of each $\rho_{i,\ell}$ are distinct and also distinct as we vary i .

Let $\mu: \Gamma_{\mathbb{Q}} \rightarrow \bar{\mathbb{Z}}_\ell^\times$ be a global character such that $\mu|_{\Gamma_{\mathbb{Q}_\ell}} = \det(\rho_{i,\ell})$. Such a μ exists either by the local and global Kronecker–Weber theorems or by noting that $\det(\rho_{i,\ell})$ is an integer power of the cyclotomic character multiplied by a finite-order character; this reduces to the case of finite-order characters, where we can realize any finite extension of \mathbb{Q}_ℓ as a completion of a finite extension of \mathbb{Q} . For such a μ , the local hypothesis at ℓ of Theorem 2.1 is satisfied by the choice of similitude character μ and the local lift ρ_ℓ constructed by summing the $\rho_{i,\ell}$ in the same way as $\bar{\rho}$ is defined in terms of the $\bar{\rho}_i$. For primes $p \neq \ell$ where $\bar{\rho}$ ramifies, [3, Theorem 1.1] implies that $\bar{\rho}|_{\Gamma_{\mathbb{Q}_p}}$ admits a lift $\Gamma_{\mathbb{Q}_p} \rightarrow \mathrm{GSp}_{2n}(\bar{\mathbb{Z}}_\ell)$ with similitude character μ . We now conclude by Theorem 2.1. \square

Here is the local lemma used in the proof:

Lemma 3.2. *Let $\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_n: \Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$ be continuous representations with the same determinant $\bar{\tau} = \det(\bar{\rho}_i)$, $i = 1, \dots, n$. Then there exist potentially crystalline lifts $\rho_i: \Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{GL}_2(\bar{\mathbb{Z}}_\ell)$ such that the union of the Hodge–Tate weights of the ρ_i is a set with $2n$ distinct elements, and $\det(\rho_i)$ is independent of i .*

Proof. First we note that it suffices to produce potentially crystalline lifts ρ_i with (all taken together) distinct Hodge–Tate weights and $\det(\rho_i)$ having the same (single) Hodge–Tate weight for all i . Indeed, then each quotient $\det(\rho_i)/\det(\rho_1)$ is a finite-order character valued in a pro- ℓ group (since the reduction mod ℓ is trivial), and so we can extract a square root and twist ρ_i to have the same determinant as ρ_1 (and this finite-order twist does not affect the property of being potentially crystalline). To finish the proof, we apply [14, Théorème 2.5.3, Theorem 2.5.4], which show the following:

- If $\bar{\rho}_i$ is irreducible, then for any choice of Hodge–Tate weights $\{m_{i,1}, m_{i,2}\}$, there exists a potentially crystalline lift ρ_i of $\bar{\rho}_i$ with these weights. We choose these in such a way that $m_{i,1} + m_{i,2}$ is independent of i , but the multi-set $\{m_{i,j}\}_{i,j}$ is in fact a set.
- If $\bar{\rho}_i = \begin{pmatrix} \bar{\chi}_{i,1} & * \\ 0 & \bar{\chi}_{i,2} \end{pmatrix}$ is an extension of characters, then for each i there are potentially crystalline lifts $\chi_{i,j}$ of $\bar{\chi}_{i,j}$ with Hodge–Tate weights summing to any pre-specified value and a potentially crystalline lift of $\bar{\rho}_i$ that is an extension of $\chi_{i,2}$ by $\chi_{i,1}$. (To see this claim requires inspection of the proof of [14, Théorème 2.5.4].) \square

4. Approach via the potential IGP with local conditions

The general approach of the present section is quite flexible and will certainly apply to other Galois images and target groups than those used here; we do not strive for maximal generality. Let $N = 2n \geq 4$ be an even integer, and consider the standard embedding $\mathrm{SO}_N \rightarrow \mathrm{SO}_{N+1}$ of special orthogonal groups over \mathbb{Z}_ℓ (defined, for definiteness, with respect to the symmetric pairings given by the identity matrices¹). Let $\Gamma = \mathrm{SO}_N(\mathbb{F}_\ell)$, and fix the following order two element c_∞ of Γ :

- If $N \equiv 0 \pmod{4}$, then $c_\infty = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ (note that $\det(c_\infty) = (-1)^n = 1$).

¹ SO_N may not be split over \mathbb{Z}_ℓ , but SO_{N+1} is split since $N + 1$ is odd. Many variants on the present construction are possible; in particular, we could take different forms of SO_N .

- If $N \equiv 2 \pmod{4}$, then $c_\infty = \left(\begin{array}{cc|cc} 0_{n-1} & 0 & 1_{n-1} & 0 \\ 0 & -1 & 0 & 0 \\ \hline 1_{n-1} & 0 & 0_{n-1} & 0 \\ 0 & 0 & 0 & -1 \end{array} \right)$
(note that $\det(c_\infty) = (-1)^{n-1} = 1$).

We now circumvent the inverse Galois problem by applying Moret-Bailly's solution ([13]) to the potential inverse Galois problem with local conditions, and we thus produce the examples of this section:

Proposition 4.1. *There exists a totally real field F/\mathbb{Q} and a Galois extension K/F satisfying:*

- 1) *There is an isomorphism $\bar{\theta}: \text{Gal}(K/F) \xrightarrow{\sim} \Gamma$.*
- 2) *K/\mathbb{Q} is linearly disjoint from $\mathbb{Q}(\zeta_\ell)/\mathbb{Q}$.*
- 3) *For all complex conjugations $c_v \in \text{Gal}(K/F)$ with $v|\infty$, $\bar{\theta}(c_v)$ is conjugate to c_∞ .*
- 4) *The prime ℓ splits completely in F/\mathbb{Q} , and for all places $v|\ell$ of F , and all $w|v$ of K , $K_w = F_v$ is the trivial extension.*

Assume now $\ell \gg_N 0$, and define $\bar{\rho} = \bar{\theta} \oplus 1$. Then $\bar{\rho}: \Gamma_F \rightarrow \text{SO}_{N+1}(\mathbb{F}_\ell)$ has a Hodge–Tate regular geometric lift $\rho: \Gamma_F \rightarrow \text{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ with image Zariski-dense in SO_{N+1} .

Proof. Existence of K , F , and $\bar{\theta}$ follow immediately from [13]. To finish the proof, we must verify that the hypotheses of Theorem 2.1 are satisfied for $\bar{\rho} = \bar{\theta} \oplus 1$. A maximal (proper) parabolic subgroup of SO_{N+1} is the stabilizer of an isotropic subspace $W \subset \overline{\mathbb{F}}_\ell^{N+1}$. Since the image of $\bar{\theta}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is $\text{SO}_N(\mathbb{F}_\ell)$, which acts absolutely irreducibly in its standard N -dimensional representation for $\ell \gg_N 0$, $\bar{\rho}$ stabilizes exactly two proper subspaces of $\overline{\mathbb{F}}_\ell^{N+1}$, namely the space of $\bar{\theta}$ and the complementary line. Clearly neither of these subspaces is isotropic, so in all cases $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_\ell)}}$ is SO_{N+1} -absolutely irreducible.

Booher's result ([3, Theorem 1.1]) shows that for v not above ℓ at which $\bar{\rho}$ is ramified, $\bar{\rho}|_{\Gamma_{F_v}}$ has a lift to $\Gamma_{F_v} \rightarrow \text{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$. (Note that Booher's result shows a GO_{N+1} -deformation ring with fixed orthogonal multiplier is formally smooth of suitably large dimension; we fix the multiplier to be trivial to produce an $\text{O}_{N+1} = \text{SO}_{N+1} \times \{\pm 1\}$ lift and then it is clear that the image must lie in SO_{N+1} .) Since the extensions K_w/F_v are trivial for $v|\ell$, $\bar{\rho}|_{\Gamma_{F_v}}$ has a crystalline Hodge–Tate regular lift by Lemma 4.2. To conclude, we check that $\bar{\rho}$ is odd. Indeed, since the adjoint representation of SO_{N+1} is

isomorphic to the second exterior power of the standard representation, we find

$$\mathrm{Tr}(\bar{\rho}(c)|_{\mathfrak{so}_{N+1}}) = \frac{\mathrm{Tr}(\bar{\rho}(c))^2 - \mathrm{Tr}(\bar{\rho}(c^2))}{2} = \frac{1 - (N+1)}{2} = -\mathrm{rk}(\mathrm{SO}_{N+1}).$$

This is equivalent to the oddness condition $\dim(\mathfrak{so}_{N+1}^{\mathrm{Ad}(\bar{\rho}(c))=1}) = \dim \mathrm{Flag}_{\mathrm{SO}_{N+1}}$ assumed in Theorem 2.1. \square

Lemma 4.2. *Let K/\mathbb{Q}_p be a finite extension, G a split reductive group over \mathbb{Z}_p and $\bar{\rho} : \Gamma_K \rightarrow G(\mathbb{F}_p)$ the trivial representation. Then $\bar{\rho}$ has a Hodge–Tate regular crystalline lift $\rho : \Gamma_K \rightarrow G(\mathbb{Z}_p)$.*

Proof. The p -adic cyclotomic character $\kappa_p : \Gamma_K \rightarrow \mathbb{Z}_p^*$ is crystalline. Therefore, so is $(\kappa_p)^{p-1}$ and it is moreover trivial modulo p . If $\chi : \mathbb{G}_m \rightarrow G$ is any regular cocharacter, then we may clearly take ρ to be $\chi \circ (\kappa_p)^{p-1}$. \square

Remark 4.3. Perhaps the most interesting case here is when $N \equiv 2 \pmod{4}$. Then the representations $\bar{\theta}$ are not odd, and indeed cannot be odd since -1 does not belong to the Weyl group of SO_N for any $N \equiv 2 \pmod{4}$.² Existing lifting techniques cannot lift them to geometric SO_N -representations, and we expect that no such lifts that are Hodge–Tate regular as GL_N -representations can exist. See Proposition 4.4 below for a proof of this in some cases, but we now explain a general heuristic. Such a lift (necessarily GL_N -irreducible) would conjecturally arise from an automorphic representation π of (split) SO_N/\mathbb{Q} with cuspidal transfer to GL_N . The archimedean L-parameter $\mathrm{rec}_{\pi_\infty} : W_{\mathbb{R}} \rightarrow \mathrm{SO}_N(\mathbb{C})$ would then (by archimedean purity for GL_N) restrict to $\mathbb{C}^\times \subset W_{\mathbb{R}}$ as $\mathrm{rec}_{\pi_\infty}(z) = z^\mu \cdot \bar{z}^{-\mu}$ for some cocharacter μ of the diagonal torus $T \subset \mathrm{SO}_N$. Let $\{e_i^*\}_{i=1}^n$ be the standard basis of $X_*(T)$. Then the most regular situation that can arise has $\mathrm{rec}_{\pi_\infty}(j)$ equal to the element c_∞ above, and $\mu = \sum_{i=1}^n p_i e_i^*$ with p_1, \dots, p_{n-1} distinct, and $p_n = 0$ (using the Weil group relation $jzj^{-1} = \bar{z}$). Such an L-parameter is in fact SO_N -regular, but it is clearly not GL_N -regular. We note that these “most regular” lifts that might be possible would be the ℓ -adic representations associated to cuspidal automorphic representations on (suitable forms of) SO_N that are non-degenerate limits of discrete series at archimedean places. See [8] for important recent progress on the problem of associating Galois representations to automorphic representations that are non-degenerate limits of discrete series at infinity.

²We learned from Wushi Goldring the observation that an SO_N -representation that is not odd can transfer to an odd SO_{N+1} -representation.

We can unconditionally rule out “many” candidates for regular lifts of $\bar{\theta}$ using potential automorphy theorems:

Proposition 4.4. *Let $\rho: \Gamma_{\mathbb{Q}} \rightarrow \mathrm{O}_N(\overline{\mathbb{Z}}_{\ell})$ for $\ell > 2(2N + 1)$ be a continuous representation satisfying:*

- 1) ρ is unramified outside a finite set of primes.
- 2) $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_{\ell})}}$ is GL_N -irreducible.
- 3) $\rho|_{\Gamma_{\mathbb{Q}_{\ell}}}$ is as GL_N -representation potentially diagonalizable in the sense of [1, §1.4] and has distinct Hodge–Tate weights.

Then $\mathrm{Tr}(\rho(c)) = 0$. In particular, when $N \equiv 2 \pmod{4}$, ρ cannot factor through SO_N .

Proof. In this proof we freely use the terminology and notation of [1]; we argue as in, for instance, [1, Proposition 3.3.1], making use of Harris’s tensor product trick. Choose a quadratic imaginary field K/\mathbb{Q} linearly disjoint from $\mathbb{Q}(\bar{\rho}, \zeta_{\ell})/\mathbb{Q}$, and let $\psi: \Gamma_K \rightarrow \overline{\mathbb{Z}}_{\ell}^{\times}$ be a geometric (hence potentially crystalline and potentially diagonalizable) character such that $r := \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}}}(\psi)$ satisfies:

- $r^* \cong r \otimes \mu$, for some character μ with $\mu(c) = -1$ (this imposes no condition on ψ).
- $r|_{\Gamma_{\mathbb{Q}_{\ell}}}$ is Hodge–Tate regular, with Hodge numbers $\{m, 0\}$ having the property that no two Hodge numbers of ρ differ by $|m|$.
- $(\bar{\rho} \otimes \bar{r})|_{\Gamma_{\mathbb{Q}(\zeta_{\ell})}}$ is irreducible.

To produce such a ψ , we can apply [1, Lemma A.2.5] as in [1, Proposition 4.1.1]. Namely, let m be some integer such that no two Hodge numbers of ρ differ by m , let q be a rational prime split in $K(\zeta_{\ell})$ at which $\bar{\rho}$ is unramified, let $\bar{\chi}_0: \Gamma_{\mathbb{Q}} \rightarrow \overline{\mathbb{F}}_{\ell}^{\times}$ be any finite-order character ramified at q , with Teichmüller lift $\chi_0: \Gamma_{\mathbb{Q}} \rightarrow \overline{\mathbb{Z}}_{\ell}^{\times}$, and let $\chi = \kappa_{\ell}^m \chi_0$, where κ_{ℓ} denotes the ℓ -adic cyclotomic character. At the two places v and \bar{v} of K above ℓ , define $\psi_v = \chi|_{\Gamma_{K_v}}$ and $\psi_{\bar{v}} = 1$ (as characters of Γ_{K_v} and $\Gamma_{K_{\bar{v}}}$, respectively). At the two places v_q and \bar{v}_q of K above q , define $\psi_{v_q} = \chi|_{\Gamma_{K_{v_q}}}$ and $\psi_{\bar{v}_q} = 1$. Letting $S = \{v, \bar{v}, v_q, \bar{v}_q\}$, we have that for each $w \in S$, $\psi_w \psi_w^c|_{I_{K_w}} = \chi|_{I_{K_w}}$, so [1, Lemma A.2.5] implies there exists a character $\psi: \Gamma_K \rightarrow \overline{\mathbb{Z}}_{\ell}^{\times}$ such that $\psi\psi^c = \chi|_{\Gamma_K}$ and $\psi|_{I_{K_w}} = \psi_w|_{I_{K_w}}$ for all $w \in S$. The resulting $r = \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}}}(\psi)$ satisfies the three desired properties: the first two are clear, and we now check the irreducibility. By assumption, $\bar{\rho}|_{\Gamma_{\mathbb{Q}(\zeta_{\ell})}}$ is irreducible, so if $(\bar{\rho} \otimes \bar{r})|_{\Gamma_{\mathbb{Q}(\zeta_{\ell})}}$

failed to be irreducible, it would have a constituent \overline{W} restricting to either $\bar{\rho}\bar{\psi}|_{\Gamma_{K(\zeta_\ell)}}$ or $\bar{\rho}\bar{\psi}^c|_{\Gamma_{K(\zeta_\ell)}}$. Since $\bar{\rho}\bar{\psi}|_{\Gamma_{K(\zeta_\ell)}}$ is ramified at (places above) v_q but not at (places above) \bar{v}_q , neither $\bar{\rho}\bar{\psi}|_{\Gamma_{K(\zeta_\ell)}}$ nor $\bar{\rho}\bar{\psi}^c|_{\Gamma_{K(\zeta_\ell)}}$ can extend to $\Gamma_{\mathbb{Q}(\zeta_\ell)}$, establishing the claim.

Consider then the representation $\rho' := \rho \otimes r: \Gamma_{\mathbb{Q}} \rightarrow \mathrm{GL}_{2N}(\overline{\mathbb{Z}}_\ell)$. Since ρ is orthogonal, and r is symplectic, $\rho': \Gamma_{\mathbb{Q}} \rightarrow \mathrm{GSp}_{2N}(\overline{\mathbb{Z}}_\ell)$ is symplectic. Moreover, $(\rho')^* \cong \rho' \otimes \mu$, and thus we see that ρ' is totally odd polarizable. Moreover, $\rho'|_{\Gamma_{\mathbb{Q}_\ell}}$ is still potentially diagonalizable with distinct Hodge–Tate weights. Thus we can apply the potential automorphy theorem of [1, Corollary 4.5.2] to conclude that (ρ', μ) is potentially automorphic. Now we apply [1, Lemma 2.2.4]: in the notation of *loc. cit.*, $(\rho' = \rho \otimes r \cong \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}}}(\rho|_{\Gamma_K} \otimes \psi), \mu)$ is potentially automorphic and polarized, so $(\rho|_{\Gamma_K} \otimes \psi, \mu)$ is also potentially automorphic and polarized. By [1, Lemma 2.2.1, Lemma 2.2.2], $(\rho, 1)$ is potentially automorphic, i.e. there exists a regular algebraic, polarized cuspidal automorphic representation (π, χ) of $\mathrm{GL}_N(\mathbb{A}_{\mathbb{Q}})$ such that $(\rho, 1) \cong (r_{\ell, \iota}(\pi), \kappa_\ell^{1-N} r_{\ell, \iota}(\chi))$, and $r_{\ell, \iota}(\pi)$ is the automorphic Galois representation associated to π (see [1, Theorem 2.1.1]; note that in their normalization, $r_{\ell, \iota}(\pi)$ is the Galois representation whose local restrictions correspond under local Langlands to $\pi \otimes |\cdot|^{\frac{1-N}{2}}$). It follows that $\chi = |\cdot|^{N-1}$, and $\pi \otimes |\cdot|^{\frac{1-N}{2}}$ is the (on the nose) self-dual regular L-algebraic cuspidal automorphic representation corresponding to ρ under the local Langlands correspondence. Applying [20, Theorem A] to $\pi \otimes |\cdot|^{\frac{1-N}{2}}$, we find that $|\mathrm{Tr}(\rho(c))| \leq 1$, and thus (N is even) $\mathrm{Tr}(\rho(c)) = 0$. \square

Remark 4.5. The main limitation in this result is the potential diagonalizability assumption. But in a compatible system of Hodge–Tate regular automorphic Galois representations, almost all members will be potentially diagonalizable by the theory of Fontaine–Laffaille. Calegari ([4]) has proven a stronger result that a geometric and Hodge–Tate regular $\Gamma_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Z}}_\ell)$ must be odd, without a potential diagonalizability hypothesis.

5. Zywina’s examples

5.1. A review of Zywina’s work

In this section we recall the construction of Zywina ([23]) that in many cases realizes the simple groups of orthogonal type over \mathbb{F}_ℓ as Galois groups of regular extensions of $\mathbb{Q}(t)$, and in particular as Galois groups over \mathbb{Q} (in infinitely many ways). Some of the finer points—namely, the simplicity itself—of this construction will not matter for our present purposes: rather,

we want to use this construction as a source of Galois groups over \mathbb{Q} isomorphic to large subgroups of even orthogonal groups, and we recall only what is necessary for our purposes.

Fix an even integer $N \geq 6$. Let $R = \mathbb{Z}[S^{-1}]$, for a finite set of primes S that may be enlarged as the construction proceeds. Zywinia first considers polynomials $a_2(t), a_4(t), a_6(t) \in R[t]$, such that the discriminant $\Delta(t)$ of the Weierstrass equation $y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$ is non-zero, and the j -invariant is non-constant, and builds a family of quadratic twists of this Weierstrass equation. This leads to the following construction of a rank N orthogonal local system on an open subset of \mathbb{P}_R^1 , which we rapidly summarize:

- Let $A = R[u, \Delta(u)^{-1}]$ and set $M = \operatorname{Spec}(A)$.
- Let $j: U \rightarrow \mathbb{P}_M^1$ be the inclusion of the open M -subscheme defined by $U = \operatorname{Spec}(A[t, (t-u)^{-1}, \Delta(t)^{-1}])$.
- Let $E \rightarrow U$ be the elliptic curve defined by the Weierstrass equation

$$(t-u)y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

- Let $\pi: \mathbb{P}_M^1 \rightarrow M$ be the structure morphism, and define

$$\mathcal{G} = R^1\pi_*(j_*E[\ell]),$$

where $E[\ell]$ is the local system (of \mathbb{F}_ℓ -modules) on U defined by the ℓ -torsion subgroup scheme of E . The sheaf \mathcal{G} on M is clearly constructible, and Zywinia shows ([23, Lemma 3.3]) that it is lisse after suitable enlargement of S . Specializing to a point m of M over a finite field k yields an elliptic curve E_m over the global field $k(t)$, and Zywinia shows the rank of \mathcal{G} is independent of the specialization and can be computed explicitly in terms of bad reduction invariants of E_m ([23, Proposition 2.8, Lemma 3.1]); precisely, the rank N of \mathcal{G} is $-4 + \deg(\operatorname{cond}(E_m))$, where $\deg(\operatorname{cond}(E_m))$ is the degree of the conductor of E_m . By a careful choice of the $a_i(t)$ and case-by-case calculation with Tate's algorithm (in [23, §6]), Zywinia produces examples with our desired even rank $N \geq 6$. Poincaré duality provides an orthogonal pairing $\mathcal{G} \times \mathcal{G} \rightarrow \mathbb{F}_\ell$, and so given a geometric generic point $\bar{\xi}$ of M we obtain a representation

$$\theta_\ell: \pi_1(M, \bar{\xi}) \rightarrow \operatorname{O}(\mathcal{G}_{\bar{\xi}}).$$

In fact Zywina must, in order to realize the simple groups as Galois groups, work with the pull-back $h^*(\mathcal{G})$ of this local system along a suitable finite étale cover $h: W \rightarrow M$, where W is again an open subscheme of \mathbb{P}_R^1 (see the proof of [23, Theorem 4.1]). He denotes by

$$\vartheta_\ell: \pi_1(W) \rightarrow \mathrm{O}(V_\ell)$$

the representation associated to $h^*(\mathcal{G})$ (omitting reference to the choice of compatible base-point).

Recall (see [23, §1.1]) that the spinor norm is a homomorphism $\mathrm{sp}: \mathrm{O}(V_\ell) \rightarrow \mathbb{F}_\ell^\times / (\mathbb{F}_\ell^\times)^2$, and let $\Omega(V_\ell) \subset \mathrm{SO}(V_\ell)$ be the subgroup of elements with trivial spinor norm. When the discriminant $\mathrm{disc}(V_\ell) := \mathrm{sp}(-1)$ is the trivial coset, -1 belongs to $\Omega(V_\ell)$. We will use the following consequence of Zywina's main theorem and arguments:

Proposition 5.1. *Let $N \geq 6$ be an even integer, and let $\ell \geq 5$ be a prime. Then the elliptic curve $E \rightarrow U$ may be chosen to ensure that the quadratic space V_ℓ has discriminant $(\mathbb{F}_\ell^\times)^2$, and that there are infinitely many $w_i \in \mathbb{Q}$ such that the specializations ϑ_{ℓ, w_i} (defined up to conjugation)*

$$\Gamma_{\mathbb{Q}} \xrightarrow{w_i} \pi_1(W) \xrightarrow{\vartheta_\ell} \mathrm{O}(V_\ell)$$

are non-isomorphic and satisfy $\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}}) = \Omega(V_\ell)$.

When $N \equiv 6 \pmod{8}$, we also consider the same family $E \rightarrow U$ and the pull-back along a different $h: W \rightarrow M$, and the above holds except with the conclusion that $\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}})$ contains $\Omega(V_\ell)$ with index 2, and $\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}(i)})$ equals $\Omega(V_\ell)$. We will refer to the two examples when $N \equiv 6 \pmod{8}$ as Case 6_Ω and Case 6_{O} .

Proof. When $N \equiv 0, 2, 4 \pmod{8}$, or in Case 6_Ω , the existence of specializations w_i with $\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}}) = \Omega(V_\ell)$ is immediate from the proof of [23, Theorem 1.1] (in §6 of *op. cit.*) and the Hilbert irreducibility theorem ([17, §3.3-3.4]), which produces w_i such that the fixed fields of the ϑ_{ℓ, w_i} are linearly disjoint over \mathbb{Q} . Case 6_{O} arises from an earlier version ([22]) of [23]. Because that version is not destined to be published, we rapidly recall the family E considered and its numerical invariants, which are all computed by Tate's algorithm. We follow the notation of [23]. Writing $N = 8n + 6$, define $h(u) = \frac{u^2+1}{2u}$ and $f(t) = \prod_{i=1}^{4n+1} (t - h(i+1))$, and consider the Weierstrass equation (defining

the polynomials $a_i(t)$ in this case)

$$y^2 = (x - t(t^2 - 2)f(t)) \cdot (x - t(t^2 + 1)f(t)) \cdot (x + t(2t^2 - 1)f(t)).$$

For any sufficiently large prime p and any $w \in W(\mathbb{F}_p)$, the points of bad reduction of the elliptic curve $E_{h(w)}$ over $\mathbb{F}_p(t)$ are $x = 1, -1, 0, \infty$, and any root $x = a$ of $(t - h(w))f(t) \pmod{p}$. The bad reduction invariants are as follows:

x	κ_x	split multiplicative reduction iff	$c_x(E_{h(w)})$
-1	I_2	$-3(1 + h(w))f(-1) \in (\mathbb{F}_p^\times)^2$	2
1	I_2	$-3(1 - h(w))f(1) \in (\mathbb{F}_p^\times)^2$	2
0	I_4^*	\emptyset	c_0
∞	I_4^*	\emptyset	c_0
a	I_0^*	\emptyset	4

(The precise value c_0 does not matter here: it will only matter that these invariants are the same at 0 and ∞ .) From this bad reduction data, one easily computes the other invariants of the curve $E_{h(w)}$ as in [23, §2]; we will invoke these calculations below. The calculations of [22, §6.4] are correct except the root number

$$\begin{aligned} \varepsilon_{E_{h(w)}} &= \left(\frac{-3(1 + h(w))f(-1)}{p} \right) \left(\frac{-3(1 - h(w))f(1)}{p} \right) \\ &\quad \times \left(\frac{-1}{p} \right)^2 \left(\frac{-1}{p} \right)^{\deg(f)+1} = \left(\frac{-1}{p} \right) \end{aligned}$$

(in *loc. cit.*, the last exponent $\deg(f) + 1$ is replaced with $\deg(f)$, so there is an incorrect conclusion that the root number is 1). Using this corrected root number, it follows from [23, Proposition 3.2] that $\det(\vartheta_\ell)(\text{Frob}_w) = \left(\frac{-1}{p} \right)$ for all $p \gg 0$, $w \in W(\mathbb{F}_p)$, and so $\det(\vartheta_\ell): \pi_1(W) \rightarrow \mathbb{F}_\ell^\times$ factors through the non-trivial quadratic character $\pi_1(W) \rightarrow \Gamma_{\mathbb{Q}} \rightarrow \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \rightarrow \mathbb{F}_\ell^\times$ (with the first map induced by the structure morphism). The proof of [23, Theorem 4.1] then still shows that the geometric monodromy group $\vartheta_\ell(\pi_1(W_{\overline{\mathbb{Q}}}))$ contains $\Omega(V_\ell)$, and that $\vartheta_\ell(\pi_1(W_{\mathbb{Q}(i)}))$ is contained in $\Omega(V_\ell)$.

Indeed, first we note that the calculation of $\det(\vartheta_\ell)$ shows $\vartheta_\ell(\pi_1(W_{\mathbb{Q}(i)}))$ is contained in $\text{SO}(V_\ell)$. Next we observe that [23, Theorem 3.4] (originally due to Hall in [9]) shows that the image of $\pi_1(M_{\overline{\mathbb{Q}}})$ contains $\Omega(V_\ell)$ —this is easily read off from the bad reduction invariants tabulated above—and then the observation in the proof of [23, Theorem 4.1] that $\vartheta_\ell(\pi_1(W_{\overline{\mathbb{Q}}}))$ also contains $\Omega(V_\ell)$ continues to apply (namely, that the morphism h has

degree at most four—in fact two in this case—so that a Galois closure of $W_{\overline{\mathbb{Q}}} \rightarrow M_{\overline{\mathbb{Q}}}$ has solvable Galois group, whereas $\Omega(V_\ell)$ is simple). We conclude by noting that $\vartheta_\ell(\pi_1(W_{\mathbb{Q}(i)}))$ is contained in $\Omega(V_\ell) \subset \mathrm{SO}(V_\ell)$: this follows as in [23, Theorem 4.1] from the bad reduction invariants in the table (again, this part of the calculation does not depend on the erroneous root number in [22, §6.4]). Thus $\vartheta_\ell(\pi_1(W_{\overline{\mathbb{Q}}})) = \vartheta_\ell(\pi_1(W_{\mathbb{Q}(i)})) = \Omega(V_\ell)$, and $\vartheta_\ell(\pi_1(W)) \subset \mathrm{O}(V_\ell)$ contains $\Omega(V_\ell)$ with index 2. We can again invoke Hilbert irreducibility to produce the desired u_i : there is a thin set $T \subset W(\mathbb{Q})$ such that for all $w \in W(\mathbb{Q}) \setminus T$, $\vartheta_{\ell,w}$ cuts out a Galois extension of \mathbb{Q} with Galois group isomorphic to $\vartheta_\ell(\pi_1(W))$. Since $W(\mathbb{Q}) \setminus T \subset \mathbb{Q}(i)$ is not a thin subset (see [17, Proposition 3.2.1]), applying Hilbert irreducibility to $\vartheta_\ell|_{\pi_1(W_{\mathbb{Q}(i)})}$ shows that there are infinitely many $w_i \in W(\mathbb{Q}) \setminus T$ such that $\vartheta_{\ell,w_i}(\Gamma_{\mathbb{Q}(i)})$ equals $\Omega(V_\ell)$. The resulting extensions are linearly disjoint over $\mathbb{Q}(i)$. \square

Remark 5.2. We have included the additional example of Case 6_{O} to indicate that our results do not depend on all of the fine details of [23]; one could imagine producing many more examples starting from Hall’s big monodromy result ([23, Theorem 3.4]).

5.2. The trace of complex conjugation

Let $w \in W(\mathbb{R})$ be any real point, giving rise (fixing a \mathbb{C} -valued geometric point \bar{w} over w) to the stalk $h^*(\mathcal{G})_{\bar{w}} \cong \mathcal{G}_{h(\bar{w})} \cong H^1(\mathbb{P}_{\mathbb{C}}^1, j_*E_{h(w)}[\ell])$ (we omit indicating the $h(w)$ -specialization in the notation for this \mathbb{P}^1 and j so as not to burden the notation). A minimal proper regular model $\pi: E_{h(w)} \rightarrow \mathbb{P}_{\mathbb{R}}^1$ of the Weierstrass equation (over $\mathbb{R}(t)$) defines our elliptic surface. The map π is smooth over an open subset $U_{h(w)} \subset \mathbb{P}_{\mathbb{R}}^1$ that depends on w .

In our application of the lifting theorem of [7], we most importantly have to understand the action of complex conjugation in the Galois representations $\vartheta_{\ell,w}$, $w \in W(\mathbb{Q})$. The Betti-étale comparison isomorphism, applied to the cohomology of $E_{h(w)}$, reduces this to a topological computation over \mathbb{C} . In order not to interrupt the flow, we perform this computation in Appendix A, and using this we prove the following:

Lemma 5.3. *If $\ell \gg_N 0$, the trace of $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on the \mathbb{F}_ℓ -vector spaces $H^1(\mathbb{P}_{\mathbb{C}}^1, j_*E_{h(w)}[\ell])$ is given by*

$$(5.1) \quad \mathrm{Tr}(c) = \begin{cases} -2 & \text{if } N \equiv 2 \pmod{8}, \\ 0 & \text{if } N \equiv 4 \pmod{8}, \\ 0 & \text{if } N \equiv 6 \pmod{8} \text{ (Case } 6_{\mathrm{O}}), \\ 2 & \text{if } N \equiv 6 \pmod{8} \text{ (Case } 6_{\mathrm{\Omega}}), \\ 0 & \text{if } N \equiv 0 \pmod{8}. \end{cases}$$

Proof. We will deduce the lemma from Proposition A.6. The Cases (1)–(4) therein correspond to the singular fibres, and their configurations over \mathbb{R} , of the elliptic surfaces considered in [23, §6] and (for Case (3_{O})) [22, §6.4] as follows:

- Case (1) describes the fibres when $N \equiv 2 \pmod{8}$;
- Case (2) when $N \equiv 4 \pmod{8}$;
- Case (3_{O}) when $N \equiv 6 \pmod{8}$, and we are in Case 6_{O} , and Case $(3_{\mathrm{\Omega}})$ when we are in Case $6_{\mathrm{\Omega}}$;
- Case (4) when $N \equiv 0 \pmod{8}$.

This claim about the structure of the singular fibres follows almost immediately from the descriptions in [23, §6]. The one point to note is that Tate’s algorithm shows that Zywinia’s description of when the I_n fibres are split or non-split is also valid over \mathbb{R} (e.g., for $N \equiv 2 \pmod{8}$, the I_1 fibre at ∞ is split if and only if -3 is a square in \mathbb{R} , hence it is non-split).

We now claim that, assuming ℓ is large enough as in [23, §2.5], the traces $\mathrm{Tr}(F_{\infty}|V)$ recorded in Proposition A.6 via analysis of the groups $H^1(\mathbb{P}_{\mathbb{C}}^1, j_* R^1 \pi_* \mathbb{Q}_{\pi^{-1}(U_{h(w)})})$ are the negatives of the traces of $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the \mathbb{F}_{ℓ} -vector spaces $H^1(\mathbb{P}_{\mathbb{C}}^1, j_* E_{h(w)}[\ell])$. The formulae in the lemma then follow from the claim and the formulae in Proposition A.6.

To prove the claim, note that [23, Equation (2.1)] shows that there is a short exact sequence³

³Zywinia asserts this when the ground field is $\overline{\mathbb{F}}_p$, but it holds equally well over the complex numbers. It uses the assumption on ℓ and the description of the singular fibers: exactness of $0 \rightarrow j_* T_{\ell}(E) \rightarrow j_* T_{\ell}(E) \rightarrow j_* E[\ell] \rightarrow 0$ is equivalent to there being no ℓ -torsion in $(R^1 j_* T_{\ell}(E))_x = T_{\ell}(E)/(\gamma_x - 1)T_{\ell}(E)$, where γ_x is the local monodromy at the point of bad reduction x . Using the dictionary between types in the Kodaira classification and local monodromy matrices, this follows from the assumption that $\ell \neq 2, 3$ and that ℓ does not divide $\mathrm{ord}_x(j(E))$ when this valuation is negative, i.e. in the multiplicative reduction case. To pass from the above short

$$0 \rightarrow H^1(\mathbb{P}_{\mathbb{C}}^1, j_* T_\ell(E)) \xrightarrow{\cdot \ell} H^1(\mathbb{P}_{\mathbb{C}}^1, j_* T_\ell(E)) \rightarrow H^1(\mathbb{P}_{\mathbb{C}}^1, j_* E[\ell]) \rightarrow 0,$$

so $H^1(\mathbb{P}_{\mathbb{C}}^1, j_* T_\ell(E))$ is a free \mathbb{Z}_ℓ -module of rank equal to the \mathbb{F}_ℓ -dimension of the group $H^1(\mathbb{P}_{\mathbb{C}}^1, j_* E[\ell])$. Since $\ell \neq 2$, the eigenvalues of $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ are the same on each space, and we can therefore compute $\mathrm{Tr}(c|H^1(\mathbb{P}_{\mathbb{C}}^1, j_* E[\ell]))$ by computing $\mathrm{Tr}(c|H^1(\mathbb{P}_{\mathbb{C}}^1, j_* T_\ell(E) \otimes \mathbb{Q}_\ell))$. The $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ module

$$H^1(\mathbb{P}_{\mathbb{C}}^1, j_* T_\ell(E) \otimes \mathbb{Q}_\ell)$$

is isomorphic to $H^1(\mathbb{P}_{\mathbb{C}}^1, j_* R^1 \pi_* \mathbb{Q}_{\ell, \pi^{-1}(U_{h(w)})})^\vee(-1)$ by Verdier duality; dualizing leaves $\mathrm{Tr}(c)$ unchanged, and the -1 Tate twist multiplies $\mathrm{Tr}(c)$ by -1 , proving the claim, hence also the lemma. \square

6. Main theorem

It is now a simple matter to prove our main result:

Theorem 6.1. *Let $N \geq 6$ be an even integer, and let $\ell \gg_N 0$ be a sufficiently large prime.*

- 1) *There is a totally real field F which is solvable over \mathbb{Q} and infinitely many non-isomorphic Galois representations*

$$\bar{\rho}: \Gamma_F \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$$

such that $\bar{\rho}$ is irreducible as an SO_{N+1} -valued representation, but reducible as a GL_{N+1} -valued representation, and $\bar{\rho}$ admits a geometric lift $\rho: \Gamma_F \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$ with Zariski-dense image.

- 2) *Moreover, if we assume that representations $\Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_\ell)$ admit Hodge–Tate regular de Rham lifts $\Gamma_{\mathbb{Q}_\ell} \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_\ell)$, then in part (1) we may take $F = \mathbb{Q}$.*

Proof. Consider one of the specializations $\vartheta_{\ell, w_i}: \Gamma_{\mathbb{Q}} \rightarrow \mathrm{O}(V_\ell)$ from Proposition 5.1, where V_ℓ is a quadratic space over \mathbb{F}_ℓ of rank N and trivial discriminant. We have four possibilities for these representations as in the table below, where we use Lemma 5.3 for the trace computations. In each

exact sequence of sheaves to the desired short exact sequence of cohomology groups follows from the global monodromy calculation [23, Proposition 2.7], which is also valid over \mathbb{C} .

case, using the natural inclusion of SO_N in SO_{N+1} , we define $\bar{\rho}'_{w_i} : \Gamma_{\mathbb{Q}} \rightarrow \mathrm{O}_{N+1}(\overline{\mathbb{F}}_{\ell}) = \mathrm{SO}_{N+1}(\overline{\mathbb{F}}_{\ell}) \times \{\pm 1\}$ (note that $N+1$ is odd) as follows, letting $\delta_{K/\mathbb{Q}}$ be the non-trivial quadratic character of an imaginary quadratic field linearly disjoint from the fixed field of ϑ_{ℓ, w_i} :

$N \pmod{8}$	$\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}})$	$\mathrm{Tr}(\vartheta_{\ell, w_i}(c))$	$\bar{\rho}'_{w_i}$
0	$\Omega(V_{\ell})$	0	$\vartheta_{\ell, w_i} \oplus 1$
2	$\Omega(V_{\ell})$	-2	$\vartheta_{\ell, w_i} \oplus 1$
4	$\Omega(V_{\ell})$	0	$\vartheta_{\ell, w_i} \oplus 1$
6, Case 6Ω	$\Omega(V_{\ell})$	2	$(\delta_{K/\mathbb{Q}} \otimes \vartheta_{\ell, w_i}) \oplus 1$
6, Case $6\mathrm{O}$	$\Omega(V_{\ell}) \subsetneq \vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}}) \subsetneq \mathrm{O}(V_{\ell})$	0	$\vartheta_{\ell, w_i} \oplus 1$

We let $\bar{\rho}_{w_i}$ be the projection to the SO_{N+1} -component; of course, $\bar{\rho}_{w_i} = \bar{\rho}'_{w_i}$ except in Case $6\mathrm{O}$. Each $\bar{\rho}_{w_i}$ is clearly reducible as $\mathrm{GL}_{N+1}(\overline{\mathbb{F}}_{\ell})$ -representation, and we claim that even after restriction to $\Gamma_{\mathbb{Q}(i, \zeta_{\ell})}$ each $\bar{\rho}_{w_i}$ is irreducible as $\mathrm{SO}_{N+1}(\overline{\mathbb{F}}_{\ell})$ -representation. First note that since $\Omega(V_{\ell})/\{\pm 1\}$ is a non-abelian simple group, $\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}(i, \zeta_{\ell})})$ also equals $\Omega(V_{\ell})$. A maximal (proper) parabolic subgroup of SO_{N+1} (as a group over $\overline{\mathbb{F}}_{\ell}$) is the stabilizer of an isotropic subspace $W \subset \overline{\mathbb{F}}_{\ell}^{N+1}$. Since in all cases the image $\vartheta_{\ell, w_i}(\Gamma_{\mathbb{Q}(i, \zeta_{\ell})})$ is $\Omega(V_{\ell})$, $\bar{\rho}_{w_i}$ stabilizes exactly two proper subspaces of $\overline{\mathbb{F}}_{\ell}^{N+1}$, namely V_{ℓ} itself and the complementary line (the standard representation $\Omega(V_{\ell}) \rightarrow \mathrm{SO}(V_{\ell} \otimes \overline{\mathbb{F}}_{\ell})$ is irreducible for $\ell \gg_N 0$). Clearly neither of these subspaces is isotropic, so in all cases $\bar{\rho}_{w_i}|_{\Gamma_{\mathbb{Q}(i, \zeta_{\ell})}}$ is absolutely irreducible as an SO_{N+1} -valued representation.

Each $\bar{\rho}_{w_i}$ is odd by the same calculation as in Proposition 4.1: this is what demands in Case 6Ω incorporating the twist by $\delta_{K/\mathbb{Q}}$.

Finally, to apply Theorem 2.1, we have to check a local lifting hypothesis on $\bar{\rho}_{w_i}$; here is where we will replace \mathbb{Q} by a suitable totally real field. Namely, we do not know at present that any $\bar{\rho}_{w_i}|_{\Gamma_{\mathbb{Q}_{\ell}}}$ admits a Hodge–Tate regular de Rham lift to $\mathrm{SO}_{N+1}(\overline{\mathbb{Z}}_{\ell})$, so we circumvent this problem by passing to a finite extension. For each w_i , $\bar{\rho}_{w_i}|_{\Gamma_{\mathbb{Q}_{\ell}}}$ cuts out a finite extension of \mathbb{Q}_{ℓ} , and as w_i varies these extensions have bounded degree, so their composite L/\mathbb{Q}_{ℓ} is still finite (and solvable). There is a solvable totally real extension F/\mathbb{Q} , linearly disjoint from $\mathbb{Q}(i, \zeta_{\ell})$, such that for all primes v of F above ℓ , the extension F_v/\mathbb{Q}_{ℓ} is isomorphic to L/\mathbb{Q}_{ℓ} . As $\Omega(V_{\ell})$ has no proper abelian quotient, it follows easily that such an F is linearly disjoint from $\mathbb{Q}(\bar{\rho}_{w_i})$ for all w_i .⁴ The required irreducibility (by linear disjointness) and oddness still

⁴A slightly simpler version of this argument would choose the extension F depending on w_i . This would also enable us to avoid invoking Booher’s theorem below, by also choosing $F = F(w_i)$ to trivialize $\bar{\rho}_{w_i}$ at all primes of ramification.

hold for $\bar{\rho}_{w_i}|_{\Gamma_F}$, but now for all places $v|\ell$ of F , $\bar{\rho}_{w_i}|_{\Gamma_{F_v}}$ is trivial, and so $\bar{\rho}_{w_i}|_{\Gamma_{F_v}}$ admits a Hodge–Tate regular crystalline lift $\Gamma_{F_v} \rightarrow \mathrm{SO}_{N+1}(\mathbb{Z}_\ell)$ by Lemma 4.2. At all places v not above ℓ at which $\bar{\rho}_{w_i}$ is ramified, there exists a lift $\Gamma_{F_v} \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{Z}_\ell})$ by [3, Theorem 1.1]. We have therefore satisfied all of the hypotheses of Theorem 2.1, and so for all w_i there are Hodge–Tate regular geometric lifts $\rho_{w_i}: \Gamma_F \rightarrow \mathrm{SO}_{N+1}(\overline{\mathbb{Z}_\ell})$ of $\bar{\rho}_{w_i}$. \square

Remark 6.2. As in Remark 4.3, perhaps the most interesting cases here are when $N \equiv 2 \pmod{8}$ and Case 6_Ω when $N \equiv 6 \pmod{8}$, since then we begin with SO_N -valued representations that are not odd. Again, existing lifting techniques cannot lift these SO_N -representations to Hodge–Tate regular geometric SO_N -representations; of course, replacing $E[\ell]$ by $T_\ell(E)$ provides some geometric lift, but it has only three distinct Hodge–Tate weights, with high multiplicities (since this gives a submodule of $H^2(E, \mathbb{Z}_\ell)$).

Appendix A. The action of complex conjugation on the cohomology of a real elliptic surface

Recall that for any smooth variety X/\mathbb{R} there is a “transport of structure” isomorphism $F_\infty: H^*(X(\mathbb{C}), \mathbb{Q}) \rightarrow H^*(X(\mathbb{C}), \mathbb{Q})$ induced by the action of complex conjugation on the points of the manifold $X(\mathbb{C})$ and that functoriality of the Betti–étale comparison isomorphism

$$H^*(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H_{\text{ét}}^*(X_{\mathbb{C}}, \mathbb{Q}_\ell),$$

implies that the automorphism F_∞ corresponds to the action of complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the étale cohomology. Moreover, in the Hodge decomposition

$$H^r(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=r} H^{p,q}(X(\mathbb{C})),$$

F_∞ exchanges $H^{p,q}(X(\mathbb{C}))$ and $H^{q,p}(X(\mathbb{C}))$. It follows that when r is odd, $\mathrm{Tr}(c|H_{\text{ét}}^r(X_{\mathbb{C}}, \mathbb{Q}_\ell)) = 0$. For r even, however, the contribution of $H^{\frac{r}{2}, \frac{r}{2}}(X(\mathbb{C}))$ terms can make it a subtle matter to compute the trace. In this section, we address this problem for the middle cohomology of certain elliptic surfaces over \mathbb{R} . We begin with a general lemma:⁵

⁵We thank Prakash Belkale for help in simplifying our original proof of this Lemma.

Lemma A.1. *Let M be a compact differentiable manifold and $f : M \rightarrow M$ a diffeomorphism such that $f^n = \text{Id}_M$ for some positive integer n . Let F be the fixed point locus of f , i.e., the set of points $x \in M$ such that $f(x) = x$. Then*

$$\text{Tr}(f^*|H^*(M, \mathbb{Q})) = \chi(F)$$

where the LHS is the alternating sum of the traces and χ denotes the topological Euler characteristic, i.e., the trace of the identity map.

Proof. By averaging any Riemannian metric on M with respect to the subgroup of $\text{Diff}(M)$ generated by f , we see that M has an f -invariant Riemannian metric g . Using the exponential map with respect to this metric at any point of F , we see that F is a closed submanifold of M (with connected components possibly of varying dimension). For $0 < \epsilon' < \epsilon \ll 1$, let T_ϵ (resp. $T_{\epsilon'}$) be the tubular neighbourhood of F in M of radius ϵ (resp. ϵ') constructed as in [12, Theorem 11.1] using the metric g . Since g is f -invariant, so are these tubular neighbourhoods.

Let A_ϵ be the closure of T_ϵ in M and $B_{\epsilon'} = M \setminus T_{\epsilon'}$. Both of these sets are f -invariant compact manifolds with boundary. Since $B_{\epsilon'} \cap F = \emptyset$, f has no fixed points on $B_{\epsilon'}$. Thus, by the “no fixed points” version of the Lefschetz fixed point theorem, $\text{Tr}(f^*|H^*(B_{\epsilon'}, \mathbb{Q})) = 0$ and $\text{Tr}(f^*|H^*(A_\epsilon \cap B_{\epsilon'}, \mathbb{Q})) = 0$.

We now consider the Mayer–Vietoris sequence for the cover of M given by A_ϵ and $B_{\epsilon'}$. Since f preserves these sets, it induces an endomorphism of this sequence, and exactness implies that the alternating sums of the traces of f^* on the terms of this sequence must be 0. Since the inclusion of F in A_ϵ is a homotopy equivalence, we get that

$$\text{Tr}(f^*|H^*(M, \mathbb{Q})) = \text{Tr}(f^*|H^*(A_\epsilon, \mathbb{Q})) = \chi(F)$$

as claimed. □

Lemma A.2. *Let X be a smooth projective variety over \mathbb{R} . Then*

$$\text{Tr}(F_\infty|H^*(X(\mathbb{C}), \mathbb{Q})) = \chi(X(\mathbb{R}))$$

Proof. This follows immediately from Lemma A.1 since the fixed point locus of F_∞ is precisely $X(\mathbb{R})$. □

We now turn to the study of real elliptic surfaces. The following lemma is standard for elliptic surfaces over \mathbb{C} . Let C/\mathbb{R} be a smooth projective geometrically connected curve.

Lemma A.3. *Let $\pi : E \rightarrow C$ be an elliptic surface over \mathbb{R} . Then*

$$\chi(E(\mathbb{R})) = \sum_{x \in S} \chi(E_x(\mathbb{R})),$$

where $S \subset C(\mathbb{R})$ is the set of real points over which π is not smooth and E_x is the fibre of π over x .

Proof. For a topological space T we denote by $\chi_c(T)$ the Euler characteristic with compact supports. If $U \subset T$ is open and $Z = T \setminus U$, then the long exact sequence of cohomology with compact supports implies that $\chi_c(T) = \chi_c(U) + \chi_c(Z)$ (as long as all the cohomology groups are finite dimensional).

Let $U' = C(\mathbb{R}) \setminus S$. The map $\pi_{\mathbb{R}}$ induced by π from $E(\mathbb{R}) \rightarrow C(\mathbb{R})$ is a proper fibre bundle over each connected component of U' with fibre homeomorphic to a circle, two disjoint circles, or the empty set. The multiplicativity of Euler characteristics (with compact supports) for fibre bundles applied to the connected components of U' then shows that $\chi_c(\pi^{-1}(U')) = 0$. The lemma follows from this, the additivity of χ_c , and the fact that $E(\mathbb{R})$ is compact. \square

The following lemma gives the Euler characteristic of the real points of the types of singular fibres (in the Kodaira classification) of elliptic fibrations with a section over \mathbb{R} that we need. We note that in general this depends on the real structure of the fibre, not just the Kodaira symbol.

Lemma A.4. *In the notation of the Kodaira classification (see, e.g., [19, IV.8]), and allowing all $n \geq 0$ in the I_n^* case, we have:*

$$\begin{aligned} \chi(I_1) &= \begin{cases} -1 & \text{if the fibre is split,} \\ 1 & \text{if the fibre is non-split,} \end{cases} \\ \chi(I_2) &= \begin{cases} -2 & \text{if the fibre is split,} \\ 0 & \text{if the fibre is non-split,} \end{cases} \\ \chi(II) &= 0, \\ \chi(III) &= -1, \\ \chi(I_n^*) &= \begin{cases} -n - 4 & \text{if all components are defined over } \mathbb{R}, \\ -n - 2 & \text{if all but two components are defined over } \mathbb{R}, \end{cases} \\ \chi(III^*) &= -7. \end{aligned}$$

Proof. The lemma follows from [18, Theorem 4.1], where the Euler characteristics of all possible fibres of real elliptic surfaces with a section are computed. \square

Let $W \subset H^2(E(\mathbb{C}), \mathbb{Q})$ be the subspace spanned by the fundamental classes of all irreducible components of all the (complex) fibres of π as well as the fundamental class of a section C of π (defined over \mathbb{R}).

$$(A.1) \quad W = \mathbb{Q}[C] \oplus \mathbb{Q}[E_{sm}] \oplus \bigoplus_{x \in C(\mathbb{C})} \frac{(\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]},$$

where E_x is the scheme theoretic fibre over x , the $E_{x,i}$ are the reduced irreducible components of E_x , E_{sm} is any smooth fibre and $[D]$, for an algebraic 1-cycle D , denotes the fundamental class in $H^2(E(\mathbb{C}), \mathbb{Q})$. The summand corresponding to any irreducible fibre E_x is zero, so the sum is actually finite.

If a complex fibre is not defined over \mathbb{R} , complex conjugation maps it to a distinct fibre, so the trace of F_∞ corresponding to the subspace of W spanned by all the irreducible components of all these fibres is 0. Thus, as far as computing the trace goes, we only need consider E_x for $x \in C(\mathbb{R})$. In this case, the corresponding summand $\frac{(\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]}$ is preserved by F_∞ so it suffices to consider the trace on each such fibre separately. We now make a list of the possibilities in terms of the Kodaira type and the real structure as in Lemma A.4.

Lemma A.5. *For a Kodaira symbol $*$ we denote by $\text{Tr}(*),$ the trace of F_∞ acting on $\frac{\oplus_i \mathbb{Q}[E_{x,i}]}{\mathbb{Q}[E_x]}$, where E_x is a real fibre of type $*$. We have:*

$$\begin{aligned} \text{Tr}(I_1) &= 0, \\ \text{Tr}(I_2) &= -1, \\ \text{Tr}(II) &= 0, \\ \text{Tr}(III) &= -1, \\ \text{Tr}(I_n^*) &= \begin{cases} -n-4 & \text{if all components are defined over } \mathbb{R}, \\ -n-2 & \text{if all but two components are defined over } \mathbb{R}, \end{cases} \\ \text{Tr}(III^*) &= -7. \end{aligned}$$

Proof. The formulae follow from the description of the possible real structures in [18, Theorem 4.1] and the following elementary facts:

- If $E_{x,i}$ is a \mathbb{C} -irreducible component of E_x defined over \mathbb{R} then F_∞ acts on $\mathbb{Q}[E_{x,i}]$ by -1 since complex conjugation reverses orientation.
- If $E_{x,i}$ and $E_{x,j}$ are two distinct \mathbb{C} -irreducible components of E_x which are conjugate, then the trace of F_∞ on $\mathbb{Q}[E_{x,i}] \oplus \mathbb{Q}[E_{x,j}]$ is 0.
- F_∞ acts on $\mathbb{Q}[E_x]$ by -1 . □

The fact that the numbers associated to I_n^* in Lemmas A.4 and A.5 are the same greatly simplifies the computations to follow.

Let U be the maximal open subset of $C_{\mathbb{C}}$ over which the map $\pi : E_{\mathbb{C}} \rightarrow C_{\mathbb{C}}$ is smooth and let $j : U(\mathbb{C}) \rightarrow C(\mathbb{C})$ be the inclusion. Let \mathcal{E}^i be the local system on $U(\mathbb{C})$ given by $R^i\pi_*(\mathbb{Q}_{\pi^{-1}(U)})$ and let \mathcal{F} be the constructible sheaf $j_*(\mathcal{E}^1)$. Let $V = H^1(C(\mathbb{C}), \mathcal{F})$. By the decomposition theorem ([2]) and the description of intermediate extension on a smooth curve, we see (noting that the fibres of π are connected) that

$$(A.2) \quad R\pi_*(\mathbb{Q}[2]) \cong j_*\mathcal{E}^0[2] \oplus j_*\mathcal{E}^1[1] \oplus j_*\mathcal{E}^2[0] \oplus \mathcal{P}_{C \setminus U}$$

$$(A.3) \quad \cong \mathbb{Q}[2] \oplus j_*\mathcal{E}^1[1] \oplus \mathbb{Q}[0](-1) \oplus \mathcal{P}_{C \setminus U},$$

where $\mathcal{P}_{C \setminus U}$ is a sheaf supported on $C \setminus U$: it is clear from the decomposition theorem that $\mathcal{P}_{C \setminus U}$ must have punctual support, and to see that it is indeed a sheaf placed in degree zero we note that it must be Verdier self-dual. Its stalks are easily computed using the proper base-change theorem, and we obtain

$$\begin{aligned} H^2(E(\mathbb{C}), \mathbb{Q}) &= H^1(C(\mathbb{C}), j_*\mathcal{E}^1) \oplus H^2(C(\mathbb{C}), \mathbb{Q}) \\ &\oplus H^0(C(\mathbb{C}), \mathbb{Q}(-1)) \oplus \left(\bigoplus_{x \in (C \setminus U)(\mathbb{C})} \frac{(\oplus_i \mathbb{Q}[E_{x,i}])}{\mathbb{Q}[E_x]} \right), \end{aligned}$$

i.e. a direct sum decomposition

$$H^2(E(\mathbb{C}), \mathbb{Q}) \cong V \oplus W.$$

(Properties of the cycle class map imply that the cycle classes of the identity section and the smooth fibre, respectively, account for the terms corresponding to \mathcal{E}^0 and \mathcal{E}^2 .) Furthermore, since U is defined over \mathbb{R} , complex conjugation induces an involution on V (which we also denote by F_∞) and the isomorphism above is equivariant for this action.

Since $\mathrm{Tr}(F_\infty | H^r(E(\mathbb{C}), \mathbb{Q})) = 0$ when r is odd (in fact, for our elliptic surface over \mathbb{P}^1 with non-constant j -invariant these odd cohomology groups

are zero), it is clear that using Lemma A.2 we may compute the trace of F_∞ on V if we know $\chi(E(\mathbb{R}))$ and the trace of F_∞ on W , since the traces on both $H^0(E(\mathbb{C}), \mathbb{Q})$ and $H^4(E(\mathbb{C}), \mathbb{Q})$ are equal to 1.

Proposition A.6. *Let $\pi : E \rightarrow C$ be an elliptic fibration over \mathbb{R} with a section, and assume the singular fibres of π , defined over \mathbb{R} , are of the following forms:*

- 1) *There is one singular fibre of type I_1 that is not split over \mathbb{R} , a singular fibre of type I_2 that is not split over \mathbb{R} , a singular fibre of type III , and all other singular fibres defined over \mathbb{R} are of type I_0^* .*
- 2) *There is one singular fibre of type I_1 that is not split over \mathbb{R} , a singular fibre of type I_1 that is split over \mathbb{R} , two singular fibres of type II , and all other singular fibres defined over \mathbb{R} are of type I_0^* .*
- (3_O) *There is one singular fibre of type I_2 that is split over \mathbb{R} , one singular fibre of type I_2 that is not split over \mathbb{R} , two singular fibres of type I_4^* , and all other singular fibres defined over \mathbb{R} are of type I_0^* .*
- (3_Ω) *There are two singular fibres of type I_2 that are split over \mathbb{R} , two singular fibres of type I_4^* , and all other singular fibres defined over \mathbb{R} are of type I_0^* .*
- (4) *There is one singular fibre of type I_1 that is not split over \mathbb{R} , one singular fibre of type I_2 that is split over \mathbb{R} , one singular fibre of type III^* , and all other singular fibres defined over \mathbb{R} are of type I_0^* .*

Then

$$\mathrm{Tr}(F_\infty|V) = \begin{cases} 2 & \text{in Case (1),} \\ 0 & \text{in Case (2),} \\ 0 & \text{in Case (3}_O\text{),} \\ -2 & \text{in Case (3}_\Omega\text{),} \\ 0 & \text{in Case (4).} \end{cases}$$

Proof. By Lemma A.2 and the preceding discussion,

$$(A.4) \quad \mathrm{Tr}(F_\infty|V) = \chi(E(\mathbb{R})) - (2 + \mathrm{Tr}(F_\infty|W)).$$

We now compute the RHS in each case.

Using Lemma A.3 and the list of singular fibres, we see that in Case (1)

$$\chi(E(\mathbb{R})) = 1 + 0 - 1 - 4a_1 - 2a_2,$$

where a_1 (resp. a_2) is the number of fibres of type I_0^* of the first (resp. second) type. On the other hand, using Lemma A.5 and the list of singular fibres

$$\mathrm{Tr}(F_\infty|W) = -2 + 0 - 1 - 1 - 4a_1 - 2a_2,$$

where the first -2 corresponds to the sum of the traces on a smooth fibre and the section. Inserting these numbers in (A.4) we get that the LHS is 2 as claimed.

By the same method, in Case (2) we have

$$\chi(E(\mathbb{R})) = 1 - 1 + 0 - 4a_1 - 2a_2,$$

and

$$\mathrm{Tr}(F_\infty|W) = -2 + 0 + 0 + 0 - 4a_1 - 2a_2,$$

so in this case $\mathrm{Tr}(F_\infty|V) = 0$ as claimed.

In Case (3_O), we again have cancellation of the contribution from I_0^* and I_n^* fibres, and we find

$$\mathrm{Tr}(F_\infty|V) = -2 + 0 - (2 - 1 - 1 - 1 - 1) = 0.$$

In Case (3_Ω), computing similarly we find $\mathrm{Tr}(F_\infty|V) = -2$.

In Case (4), we likewise find $\mathrm{Tr}(F_\infty|V) = 0$. □

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References

- [1] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609.
- [2] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in: Analysis and Topology on Singular Spaces, I (Luminy, 1981), Vol. 100 of Astérisque, pp. 5–171, Soc. Math. France, Paris (1982).

- [3] J. Booher, *Minimally ramified deformations when $\ell \neq p$* , Compos. Math. **155** (2019), no. 1, 1–37.
- [4] F. Calegari, *Even Galois representations and the Fontaine–Mazur conjecture. II*, J. Amer. Math. Soc. **25** (2012), no. 2, 533–554.
- [5] F. Calegari, M. Emerton, and T. Gee, *Globally realizable components of local deformation rings*, J. Inst. Math. Jussieu (2020), 1–70. DOI: 10.1017/S1474748020000195.
- [6] M. Emerton and T. Gee, *Moduli stacks of étale (ϕ, Γ) -modules and the existence of crystalline lifts*, arXiv:1908.07185, (2019).
- [7] N. Fakhruddin, C. Khare, and S. Patrikis, *Relative deformation theory and lifting irreducible Galois representations*, to appear in Duke Math. J., arXiv:1904.02374, (2019).
- [8] W. Goldring and J.-S. Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations*, Invent. Math. **217** (2019), no. 3, 887–984.
- [9] C. Hall, *Big symplectic or orthogonal monodromy modulo l* , Duke Math. J. **141** (2008), no. 1, 179–203.
- [10] C. Khare, *Serre’s modularity conjecture: the level one case*, Duke Math. J. **134** (2006), no. 3, 557–589.
- [11] C. Khare and J.-P. Wintenberger, *On Serre’s conjecture for 2-dimensional mod p representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$* , Ann. of Math. (2) **169** (2009), no. 1, 229–253.
- [12] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo (1974). Annals of Mathematics Studies, No. 76.
- [13] L. Moret-Bailly, *Extensions de corps globaux à ramification et groupe de Galois donnés*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 6, 273–276.
- [14] A. Muller, *Relèvements cristallins de représentations galoisiennes*, Ph.D. thesis, Université de Strasbourg, (2013).
<https://tel.archives-ouvertes.fr/tel-00873407>.
- [15] R. Ramakrishna, *Lifting Galois representations*, Invent. Math. **138** (1999), no. 3, 537–562.

- [16] R. Ramakrishna *Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur*, Ann. of Math. (2) **156** (2002), no. 1, 115–154.
- [17] J.-P. Serre, Topics in Galois Theory, Vol. 1 of Research Notes in Mathematics, A K Peters, Ltd., Wellesley, MA, second edition (2008). With notes by Henri Darmon.
- [18] R. Silhol, *Real algebraic surfaces with rational or elliptic fiberings*, Math. Z. **186** (1984), no. 4, 465–499.
- [19] J. Silverman, Advanced Topics in The Arithmetic of Elliptic Curves, Springer-Verlag, New York (1994), ISBN 0-387-94328-5.
- [20] O. Taïbi, *Eigenvarieties for classical groups and complex conjugations in Galois representations*, Math. Res. Lett. **23** (2016), no. 4, 1167–1220.
- [21] R. Taylor, *Remarks on a conjecture of Fontaine and Mazur*, J. Inst. Math. Jussieu **1** (2002), no. 1, 125–143.
- [22] D. Zywina, *The inverse Galois problem for orthogonal groups*, arXiv:1409.1151v1, (2014).
- [23] D. Zywina, *The inverse Galois problem for orthogonal groups*, preprint (2014, revised 2019). Available at <http://pi.math.cornell.edu/~zywina/papers/Orthogonal.pdf>.

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