Weight modules of quantum Weyl algebras

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We develop a general framework for studying relative weight representations for certain pairs consisting of an associative algebra and a commutative subalgebra. Using these tools we describe projective and simple weight modules for a common localisation of two quantum versions of Weyl algebras, one by Maltsiniotis and the other one by Akhavizadegan and Jordan, for generic values of the deformation parameters.

Introduction and notation

There is a natural interest in the representation theory of Weyl algebras, both of finite and infinite rank, as the simplest deformations of polynomial rings with numerous applications in different areas of mathematics and physics. A systematic study of *weight* representations of Weyl algebras was

undertaken in [BBF] following the results of [Bl], [Ba], [BO], [DGO], [BB] among others. Generalizations of Weyl algebras — infinite rank, generalized, modular, quantum etc. — were studied in many papers, see for example [Ba], [M], [MT], [H], [FGM], [C], [FI], [AJ], [LMZ].

Representations of interest in all papers mentioned above are those on which a certain commutative subalgebra acts in a locally finite way. General *Harish-Chandra* categories of such representations for a pair (R, A) consisting of an associative algebra A and a commutative subalgebra R were introduced and studied in [DFO1]. These categories play a very important role in representation theory, in particular in the study of Gelfand-Tsetlin modules for the Lie algebra gl_n [DFO2], [O].

In this paper we consider a general framework for subcategories of Harish-Chandra categories consisting of *relative weight* representations for a pair (R, A) on which R is diagonalizable.

The pairs (R, A) of our interest are so-called of *strongly-free type*. For such pairs we describe projective and simple relative weight representations in Section 1. This approach allows us to address weight representations of the above mentioned algebras in a systematic way. We hope that the technique developed here will be useful in the study of the representation theory of many types of algebras.

As an application we focus in this paper on the representations of quantum Weyl algebras, motivated by their strong connections with representations of quantum universal enveloping algebras, — see [LZZ], [FKZ] — and quantum affine Lie algebras [FHW]. We consider two versions of quantum Weyl algebras. The first one was introduced by G. Maltsiniotis (see [M]) in his attempt to develop a quantum differential calculus on the standard deformation of the affine space. Though its definition makes it a very natural ring of skew differential operators on the quantum affine space (where classical partial derivatives are replaced by so-called Jackson operators, which are multiplicative analogues) it has the drawback of being a non-simple algebra, hence losing a major property of the classical Weyl algebra. This observation leads to the seek of a simple localisation of this algebra and was the original motivation for the works [AJ] and [J] of M. Akhavizadegan and D. Jordan. In these works, the authors introduced a slightly different deformation of the Weyl algebra, technically easier to handle and then showed that the two quantum analogues actually share a common simple localisation which, in a sense, is a better quantum analogue of the classical Weyl algebra. These algebras depend on an n-tuple of deformation parameters $\bar{q} = (q_1, \ldots, q_n) \in (\Bbbk^*)^n$ and an $n \times n$ skew symmetric matrix $\Lambda = (\lambda_{ij})$ for some n.

Denote by $\mathcal{B}_n^{\bar{q},\Lambda}$ the common simple localisation of both versions of quantum Weyl algebras above. For a certain commutative Laurent polynomial subalgebra R° of $\mathcal{B}_n^{\bar{q},\Lambda}$ the pair $(R^\circ, \mathcal{B}_n^{\bar{q},\Lambda})$ is of strongly-free type. We apply the above results to describe the projective $\mathcal{B}_n^{\bar{q},\Lambda}$-modules in Section 3. In Section 4 we obtain one of our main results: a classification of simple relatively weight $\mathcal{B}_n^{\bar{q},\Lambda}$-modules for generic values of deformation parameters and special matrix Λ with all entries equal to 1 (Theorem 4.2.2). In order to deal with the general matrix Λ we use Zhang twists [Z] and show in Section 5 how the general case can be reduced to the special case when all entries of matrix Λ equal to 1. It leads in Section 6 to our main result - classification and construction of simple weight $\mathcal{B}_n^{\bar{q},\Lambda}$-modules. We note that the case of rank two quantum Weyl algebra and arbitrary deformation parameters was considered in [[H], Theorem 6.14].

We fix a field $\mathbb{k}$. Whenever the symbol $\otimes$ is used, it stands for the tensor product over $\mathbb{k}$ of $\mathbb{k}$-vector spaces, while $M \otimes_A N$ will denote the tensor product of a right module M and a left module N over a ring A and if A is a $\mathbb{k}$-algebra, $\mathrm{Aut}_{\mathbb{k}}(A)$ denotes its group of $\mathbb{k}$-algebra automorphisms.

We will denote by $\mathbb{N}$ the set of non negative integers and by $\mathbb{N}^*$ the subset of positive integers.

1. A general setting for weight categories

The aim of this section is to study the following context, which turns out to appear very often in representation theory. Consider a $\mathbb{k}$-algebra A containing a commutative subalgebra R. One may consider (left) A-modules on which the subalgebra R acts in a semisimple way. That is, A-modules M which can be decomposed as a direct sum of subspaces indexed by characters of R on which any element of R acts by scalar multiplication by its image by the relevant character. Clearly, such *generalised eigenspaces* must be R-submodules and, if the connection between A and R is close enough, such a decomposition can provide a good understanding of the A-module structure of M.

We are particularly interested by the case in which (A, R) is a *strongly-free* extension. Roughly speaking, this means that there is a subset $\mathbf{b}$ of A which is a basis of A both as a right and as a left module with the additional property that commutation relations between R and elements of $\mathbf{b}$ are controlled by automorphisms of the $\mathbb{k}$-algebra R.

In the first subsection, we recall useful results on weight modules over a commutative $\mathbb{k}$-algebra. These are modules on which R acts in a semisimple

way — in the above sense. In the second subsection, given a pair (A, R) consisting of a $\Bbbk$-algebra A and a commutative subalgebra R, we introduce the notion of relative weight A-module, that is A-modules which are weight modules relative to R. In particular, we show that if the extension $R \subseteq A$ is *strongly-free*, then the natural induction from R to A preserves projective modules. This provides a natural process to construct A-modules which are projective in the category of relative weight A-modules as we show in Subsection 3. Subsection 4 then is devoted to the study of the simple objects in the above category. It turns out that they all are quotients of the aforementioned projective objects. The last subsection is of a more technical nature: we focus on tensor products of strongly-free extensions and study how the projective and simple objects discussed above behave in this case.

1.1. Weight modules over commutative algebras

In this subsection, unless otherwise specified, R denotes a commutative $\Bbbk$-algebra.

Definition 1.1.1. A *weight* of R is a morphism of $\Bbbk$-algebras from R to $\Bbbk$. The set of weights of R will be denoted $\widehat{R}$.

Let $\mathrm{Max}(R)$ be the set of maximal ideals of R. There is a map

$$\begin{array}{ccc} \widehat{R} & \longrightarrow & \mathrm{Max}(R) \\ \phi & \mapsto & \ker(\phi). \end{array}$$

The link between a weight of R and its kernel will be useful in the sequel. We make it clear in the following remark.

Remark 1.1.2. Recall the natural morphism of algebras $\Bbbk \longrightarrow R$, $\lambda \mapsto \lambda.1_R$.

1) Notice that, for all $r \in R$ and for all $\phi \in \widehat{R}$, the element $r - \phi(r).1_R$ belongs to $\ker(\phi)$. From this, it follows at once that

$$R = \Bbbk.1_R \oplus \ker(\phi).$$

In particular, the map $\widehat{R} \longrightarrow \mathrm{Max}(R)$ mentioned above is injective. Of course, it does not need be surjective.

2) Let now σ be an automorphism of the $\Bbbk$-algebra R.
   a) It is straightforward to verify that $\ker(\phi) = \sigma(\ker(\phi \circ \sigma))$.
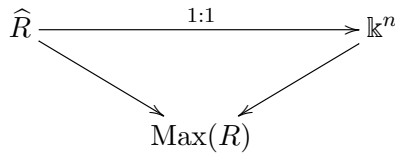
b) It follows, using the first item, that

$$\sigma(\ker(\phi)) = \ker(\phi) \Longleftrightarrow \phi \circ \sigma = \phi.$$

Remark 1.1.3. It is clear that there is a right action of the group $\mathrm{Aut}_{\Bbbk}(R)$ of $\Bbbk$-algebra automorphisms of R on $\widehat{R}$ as follows :

$$\begin{array}{rcl} \widehat{R} \times \mathrm{Aut}_{\Bbbk}(R) & \longrightarrow & \widehat{R} \\ (\phi, \sigma) & \mapsto & \phi \circ \sigma. \end{array}$$

Example 1.1.4. The case where R is a polynomial algebra or a Laurent polynomial algebra will be of particular importance.

1) Let $R = \Bbbk[z_1, \ldots, z_n]$, with $n \in \mathbb{N}^*$. There is a commutative diagram

where the horizontal map sends $\phi \in \widehat{R}$ to the n-tuple $(\phi(z_1), \ldots, \phi(z_n))$, while the left hand side map sends $\phi \in \widehat{R}$ to its kernel and the right hand side map sends $(\alpha_1, \ldots, \alpha_n)$ to $\langle z_1 - \alpha_1, \ldots, z_n - \alpha_n \rangle$.
Clearly, the horizontal map is bijective, while the other two are injective (see Remark 1.1.2).

2) There is a similar diagram for the Laurent polynomial algebra in the indeterminates $z_1, \ldots, z_n$, $n \in \mathbb{N}^*$, replacing $\Bbbk$ by $\Bbbk^*$.

Definition 1.1.5. Let M be an R-module. Given $\phi \in \widehat{R}$, we say that $m \in M$ is an *element of weight ϕ* if, for all $r \in R$, $r.m = \phi(r)m$. We say that an element $m \in M$ is a *weight element* if there exists $\phi \in \widehat{R}$ such that m is an element of weight ϕ. Further, for $\phi \in \widehat{R}$, we set

$$M(\phi) = \{m \in M \,|\, r.m = \phi(r)m, \, \forall r \in R\}.$$

Remark 1.1.6. Let $f : M \longrightarrow N$ be a morphism of R-modules. For all $\phi \in \widehat{R}$, $f(M(\phi)) \subseteq N(\phi)$.

Proposition 1.1.7. *Let M be an R-module.*

1) For all $\phi \in \widehat{R}$, $M(\phi)$ is an R-submodule of M.

2) *Let $n \in \mathbb{N}^*$. If $\phi_1, \ldots, \phi_n$ are pairwise distinct elements of $\widehat{R}$ and, for $1 \leq i \leq n$, $m_i \in M(\phi_i)$, then $\sum_{1 \leq i \leq n} m_i = 0$ if and only if $m_i = 0$, for all $1 \leq i \leq n$. That is, the sum $\sum_{\phi \in \widehat{R}} M(\phi)$ is direct.*

Proof. The proof of the first item is straightforward. Let $n \in \mathbb{N}^*$. We prove the second statement by induction on n, the case $n = 1$ being trivial. Assume the results holds for some $n \in \mathbb{N}^*$. Suppose now that $\phi_1, \ldots, \phi_{n+1}$ are pairwise distinct elements of $\widehat{R}$ and, for $1 \leq i \leq n+1$, consider elements $m_i \in M(\phi_i)$ such that $\sum_{1 \leq i \leq n+1} m_i = 0$. For all $r \in R$, we have that

$$0 = r.\left(\sum_{1 \leq i \leq n+1} m_i \right) = \sum_{1 \leq i \leq n+1} r.m_i = \sum_{1 \leq i \leq n+1} \phi_i(r).m_i.$$

It follows that, for all $r \in R$,

$$\sum_{1 \leq i \leq n} (\phi_{n+1}(r) - \phi_i(r)).m_i = 0.$$

The inductive hypothesis yields that, for all $r \in R$ and $1 \leq i \leq n$, $(\phi_{n+1}(r) - \phi_i(r)).m_i = 0$. But, the ϕ_i's being pairwise distinct, we have $m_1 = \cdots = m_n = 0$ and therefore $m_{n+1} = 0$. This finishes the induction. The result is thus proved. $\square$

Definition 1.1.8. Let M be an R-module. For $\phi \in \widehat{R}$, we will call $M(\phi)$ the *weight subspace* of M of weight ϕ. Further, we say that M is a *weight module* provided it is the (direct) sum of its weight subspaces

$$M = \bigoplus_{\phi \in \widehat{R}} M(\phi)$$

We denote by $R - \mathsf{wMod}$ the full subcategory of $R - \mathsf{Mod}$ whose objects are the weight R-modules.

Proposition 1.1.9. *Let M be an R-module and let L be an R-submodule of M.*

1) *For all $\phi \in \widehat{R}$, $L(\phi) = L \cap M(\phi)$.*

2) *If M is a weight module, then so is L.*

Proof. The first item is obvious. Next, suppose m is a non-zero element of L. There exists $n \in \mathbb{N}^*$ and pairwise distinct elements $\phi_1, \ldots, \phi_n$ of $\widehat{R}$ such

that

$$(1) \qquad\qquad m = \sum_{1 \leq i \leq n} m_i$$

with $m_i \in M(\phi_i) \setminus \{0\}$ for all $1 \leq i \leq n$. So,

$$(2) \qquad \forall r \in R, \qquad r.m - \phi_n(r)m = \sum_{1 \leq i \leq n} (\phi_i(r) - \phi_n(r))m_i \in L.$$

Further, if $n > 1$, then for all $1 \leq i \leq n - 1$, we may choose $r_i \in R$ such that $\phi_i(r_i) \neq \phi_n(r_i)$, and hence $r_i.m - \phi_n(r_i)m \in L \setminus \{0\}$ by Proposition 1.1.7.

We are now ready to prove the second item. Suppose L is not a weight submodule. This means that there exists a non-zero element in L such that at least one of its weight summands is not in L. Among such elements, let us choose m with a minimal number n of non-zero weight summands. Write m as in (1) in such a way that $m_n \notin L$. Notice in addition that we must have $n > 1$. Due to the minimality of n, the argument above based on (2) shows that, for all $1 \leq i \leq n - 1$, $m_i \in L$. It follows that $m_n \in L$. This is a contradiction. Hence L must be a weight submodule. $\qquad\square$

Next, we consider the case where R is a tensor product of commutative algebras, that is $R = R_1 \otimes \cdots \otimes R_s$ for some $s \in \mathbb{N}^*$, where $R_1, \ldots, R_s$ are commutative $\Bbbk$-algebras.

Proposition 1.1.10. *Let $s \in \mathbb{N}^*$, let $R_1, \ldots, R_s$ be commutative $\Bbbk$-algebras and consider the $\Bbbk$-algebra $R = R_1 \otimes \cdots \otimes R_s$.*

1) For $1 \leq i \leq s$, let $\phi_i \in \widehat{R_i}$. The $\Bbbk$-linear map

$$\phi_1 \cdots \phi_s \quad : \qquad\qquad R \quad \longrightarrow \quad \Bbbk$$
$$x_1 \otimes \cdots \otimes x_s \quad \mapsto \quad \phi_1(x_1) \cdots \phi_s(x_s)$$

is a morphism of $\Bbbk$-algebras. Furthermore, we have that

$$\ker(\phi_1 \cdots \phi_s) = W$$

where

$$W = \ker(\phi_1) \otimes R_2 \otimes \cdots \otimes R_s + R_1 \otimes \ker(\phi_2) \otimes \cdots \otimes R_s$$
$$+ \cdots + R_1 \otimes \cdots \otimes R_{s-1} \otimes \ker(\phi_s).$$

2) The map

$$\begin{array}{ccc} \widehat{R_1} \times \cdots \times \widehat{R_s} & \longrightarrow & \widehat{R} \\ (\phi_1, \ldots, \phi_s) & \mapsto & \phi_1 \cdots \phi_s \end{array}$$

is bijective.

Proof. In order to prove the first statement, set $\phi = \phi_1 \cdots \phi_s$. It is easy to see that $W \subseteq \ker(\phi)$. According to Remark 1.1.2, given $1 \leq i \leq s$, $R_i = \Bbbk.1_{R_i} \oplus \ker(\phi_i)$. Hence, $R = \Bbbk.1_R \oplus W$. It follows that if r is an element of R, then there exist $\lambda \in \Bbbk$ and $w \in W$ such that $r = \lambda.1_R + w$, and, since $W \subseteq \ker(\phi)$, we have $\lambda = \phi(r)$. It is now clear that if $r \in \ker(\phi)$, then $r \in W$. The previous point gives rise to a map

$$\begin{array}{ccc} \widehat{R_1} \times \cdots \times \widehat{R_s} & \longrightarrow & \widehat{R} \\ (\phi_1, \ldots, \phi_s) & \mapsto & \phi_1 \cdots \phi_s \end{array}.$$

Now, for $1 \leq i \leq s$, identify R_i with its canonical image in R under the injective algebra morphism $r \mapsto 1 \otimes \cdots \otimes 1 \otimes r \otimes 1 \otimes \cdots \otimes 1$ (r in *ith* position). We may define the map

$$\begin{array}{ccc} \widehat{R} & \longrightarrow & \widehat{R_1} \times \cdots \times \widehat{R_s} \\ \phi & \mapsto & (\phi_{|R_1}, \ldots, \phi_{|R_s}). \end{array}$$

The two precedent maps are inverse to each other, so that they are bijective. $\square$

1.2. Relative weight modules

In this subsection, we fix a pair (R, A) where A is a $\Bbbk$-algebra and R a commutative $\Bbbk$-subalgebra of A.

Any A-module M is an R-module by restriction of scalars. Hence, if $\phi \in \widehat{R}$, we may speak of elements of M of weight ϕ.

Let $A - \mathsf{Mod}$ be the category of left A-modules. We denote by $A - \mathsf{wMod}$ the full subcategory of $A - \mathsf{Mod}$ whose objects are weight R-modules. We call such modules *relative weight* modules. We have the following restriction of scalars functors

$$A - \mathsf{Mod} \xrightarrow{\quad\text{Res}\quad} R - \mathsf{Mod}$$

$$\Big\uparrow {\scriptstyle can.} \qquad\qquad\qquad \Big\uparrow {\scriptstyle can.}$$

$$A - \mathsf{wMod} \xrightarrow{\quad\text{Res}\quad} R - \mathsf{wMod}$$

Of course, there is also an induction functor $\underline{\mathrm{Ind}} = A \otimes_R -$ from $R - \mathsf{Mod}$ to $A - \mathsf{Mod}$. In order to be able to induce weight modules from R to A, we need the pair (R, A) to satisfy some additional hypotheses.

Definition 1.2.1. We say that the pair (R, A) is of *strongly-free type* if there exists a subset $\mathbf{b}$ of A such that:

1) $1 \in \mathbf{b}$;

2) $\mathbf{b}$ is a basis of A as a left R-module;

3) $\forall b \in \mathbf{b}$, there exists an algebra automorphism σ_b of R such that, $\forall r \in R$, $rb = b\sigma_b(r)$.

Remark 1.2.2. If the pair (R, A) is of strongly-free type, then, there exists a subset of A which is an R-basis of A both as left and as right module over R.

Lemma 1.2.3. *Assume the pair (R, A) is of strongly-free type. Let $\phi \in \widehat{R}$ and suppose M is an R-module. Then,*

1) *for all $m \in M(\phi)$, for all $b \in \mathbf{b}$, $b \otimes_R m \in (\underline{\mathrm{Ind}}(M))(\phi \circ \sigma_b)$;*

2) *if M is a weight R-module, then $\underline{\mathrm{Ind}}(M)$ is an object of $A - \mathsf{wMod}$.*

Proof. The proof of the first item is straightforward. The second one follows from the first since $\mathbf{b}$ generates A as left R-module. $\qquad\square$

As a consequence of Lemma 1.2.3, there is a commutative diagram for the induction functors:

$$
\begin{array}{ccc}
A - \mathsf{Mod} & \xleftarrow{\quad\underline{\mathrm{Ind}}\quad} & R - \mathsf{Mod} \\
\uparrow{\scriptstyle can.} & & \uparrow{\scriptstyle can.} \\
A - \mathsf{wMod} & \xleftarrow{\quad\underline{\mathrm{Ind}}\quad} & R - \mathsf{wMod}
\end{array}
$$

Corollary 1.2.4. *Assume the pair (R, A) is of strongly-free type. The pair $(\underline{\mathrm{Ind}}, \underline{\mathrm{Res}})$ is an adjoint pair of functors between $A - \mathsf{wMod}$ and $R - \mathsf{wMod}$. In particular, if M is a relative weight A-module and N is a weight R-module, then there is an isomorphism of abelian groups as follows:*

$$
\mathsf{Hom}_{A-\mathsf{wMod}}(\underline{\mathrm{Ind}}(N), M) \cong \mathsf{Hom}_{R-\mathsf{wMod}}(N, \underline{\mathrm{Res}}(M)).
$$

Proof. It is well known that the pair $(\underline{\mathrm{Ind}}, \underline{\mathrm{Res}})$ is an adjoint pair of functors between $A - \mathsf{Mod}$ and $R - \mathsf{Mod}$. The proof follows immediately from this fact. $\square$

Corollary 1.2.5. *Assume the pair (R, A) is of strongly-free type. The functor $\underline{\mathrm{Ind}} : R - \mathsf{wMod} \longrightarrow A - \mathsf{wMod}$ preserves projective objects.*

Proof. This is an immediate consequence of Corollary 1.2.4 since $\underline{\mathrm{Res}}$ is exact. $\square$

1.3. A class of projective weight A-modules

Let R be a commutative $\Bbbk$-algebra. Consider $\phi \in \widehat{R}$. The R-module $R/\ker(\phi)$ is clearly an object of $R - \mathsf{wMod}$. More precisely, we have that any element of this module is of weight ϕ, that is $R/\ker(\phi) = (R/\ker(\phi))(\phi)$. In the sequel, we will denote it by $\Bbbk_\phi$.

Lemma 1.3.1. *Given $\phi \in \widehat{R}$, the module $\Bbbk_\phi$ is a projective object in the category $R - \mathsf{wMod}$.*

Proof. Consider the diagram

$$
\begin{array}{c}
\Bbbk_\phi \\
\downarrow f \\
N_1 \xrightarrow{\ \pi\ } N_2 \longrightarrow 0
\end{array}
$$

in $R - \mathsf{wMod}$ with exact horizontal row. Using Prop. 1.1.7, we get that there is an element z of weight ϕ in N_1 such that $\pi(z) = f(\overline{1_R})$. Since $\Bbbk_\phi$ is a one dimensional $\Bbbk$-vector space, there is a unique $\Bbbk$-linear map $g : \Bbbk_\phi \longrightarrow N_1$ sending $\overline{1_R}$ to z. Thus, the diagram

$$
\begin{array}{c}
\Bbbk_\phi \\
{}^{g}\swarrow \quad \downarrow f \\
N_1 \xrightarrow{\ \pi\ } N_2 \longrightarrow 0
\end{array}
$$

is commutative. It is easy to see that g is actually a map in $R - \mathsf{wMod}$, since z has weight ϕ. (Here, we use Remark 1.1.2, point 1.) Hence, $\Bbbk_\phi$ is projective. $\qquad\square$

For the rest of this section, let (R, A) be a pair as in the introduction to section 1.2, assume (R, A) is of strongly-free type and let $\mathbf{b}$ be a basis of the R-module A as in Definition 1.2.1.

Let $\phi \in \widehat{R}$. Set $P_\phi = \underline{\mathrm{Ind}}(\Bbbk_\phi)$. We already know that P_ϕ is an object of $A - \mathsf{wMod}$.

Corollary 1.3.2. *For any $\phi \in \widehat{R}$, the module P_ϕ is a projective object in the category $A - \mathsf{wMod}$.*

Proof. This is immediate by Lemma 1.3.1 and Corollary 1.2.5. $\qquad\square$

Now we collect some properties of the A-modules P_ϕ, $\phi \in \widehat{R}$, that will be useful later.

Remark 1.3.3 (Structure of P_ϕ, $\phi \in \widehat{R}$). Recall that we assume (R, A) is of strongly-free type and let $\mathbf{b}$ be a basis of the R-module A as in Definition 1.2.1. Fix $\phi \in \widehat{R}$. For all $r \in R$, we put $\bar{r} = r + \ker(\phi) \in \Bbbk_\phi$.

1) Fix $b \in \mathbf{b}$. We have that $Rb = bR \subseteq A$. Further, Rb is a direct summand of A both as a left and as a right R-submodule. Now, consider

the following subset of P_ϕ:

$$b \otimes_R \mathbb{k}_\phi := \{b \otimes_R \bar{r}, \, r \in R\} \subseteq P_\phi.$$

Clearly, $b \otimes_R \mathbb{k}_\phi$ is the left R-submodule of P_ϕ generated by $b \otimes_R \bar{1}$ as well as the $\mathbb{k}$-subspace of P_ϕ generated by $b \otimes_R \bar{1}$ (see Remark 1.1.2). On the other hand, it is the image of the natural left R-linear map

$$bR \otimes_R \mathbb{k}_\phi \xrightarrow{\subseteq \otimes_R \mathrm{id}} P_\phi.$$

But, bR being a direct summand of A as a right R-submodule, the above map must be injective. Hence, it identifies $bR \otimes_R \mathbb{k}_\phi$ and $b \otimes_R \mathbb{k}_\phi$ as left R-modules.

In addition, there are obvious $\mathbb{k}$-linear maps as follows:

$$(3) \qquad \begin{array}{ccc} \mathbb{k}_\phi & \longrightarrow & bR \otimes_R \mathbb{k}_\phi \\ \bar{s} & \mapsto & b \otimes_R \bar{s} \end{array} , \qquad \begin{array}{ccc} bR \otimes_R \mathbb{k}_\phi & \longrightarrow & \mathbb{k}_\phi \\ br \otimes_R \bar{s} & \mapsto & r\bar{s} \end{array}$$

which are inverse to each other. As a consequence, $bR \otimes_R \mathbb{k}_\phi$ is a one dimensional $\mathbb{k}$-vector space with basis $b \otimes_R \bar{1}$.

At this stage, it is interesting to notice that the maps (3) are not left R-linear. However, if we denote by ${}^{\sigma_b}\mathbb{k}_\phi$ the left R module obtained by twisting the left action of R on $\mathbb{k}_\phi$ by the automorphism σ_b, then the maps (3) induce a left R-linear isomorphism

$$bR \otimes_R \mathbb{k}_\phi \longrightarrow {}^{\sigma_b}\mathbb{k}_\phi.$$

2) We are now in position to give a very explicit description of P_ϕ. Indeed, by standard results, we have an isomorphism of left R-modules as follows:

$$P_\phi = A \otimes_R \mathbb{k}_\phi = \left(\bigoplus_{b \in \mathbf{b}} bR \right) \otimes_R \mathbb{k}_\phi \cong \bigoplus_{b \in \mathbf{b}} bR \otimes_R \mathbb{k}_\phi.$$

Thus, taking into account the first point, we end up with a $\mathbb{k}$-vector space isomorphism

$$P_\phi \cong \bigoplus_{b \in \mathbf{b}} \mathbb{k}.b \otimes_R \bar{1},$$

where, for all $b \in \mathbf{b}$, $\dim_\mathbb{k}(\mathbb{k}.b \otimes_R \bar{1}) = 1$. In particular, P_ϕ is non-zero and $\{b \otimes_R \bar{1}, b \in \mathbf{b}\}$ is a basis of P_ϕ as a $\mathbb{k}$-vector space.

3) The weight structure of P_ϕ, however, is not yet completely clear. Indeed, we have that,

$$\forall b \in \mathbf{b}, \quad \Bbbk.b \otimes_R \bar{1} \subseteq (P_\phi)(\phi \circ \sigma_b).$$

But, the above inclusion may be strict. Actually, it is the case if and only if there exists $b' \in \mathbf{b}$, $b \neq b'$, such that $\phi \circ \sigma_b = \phi \circ \sigma_{b'}$. We will come back to this problem in Remark 1.4.4

1.4. Simple objects in $A - \mathsf{wMod}$

All along this section, we fix a pair (R, A) as in the introduction to Section 1.2, assume (R, A) is of strongly-free type and let $\mathbf{b}$ be a basis of the R-module A as in Definition 1.2.1. Our aim is to study the simple objects of $A - \mathsf{wMod}$.

Definition 1.4.1. An object S in $A - \mathsf{wMod}$ is *simple* provided it is non-zero and it has no non-trivial weight A-submodule.

Remark 1.4.2. By Proposition 1.1.9, an object M in $A - \mathsf{wMod}$ is simple if and only if it is simple as an object of $A - \mathsf{Mod}$.

The following result shows that any simple object in $A - \mathsf{wMod}$ arises as a simple quotient of some P_ϕ, with $\phi \in \widehat{R}$.

Proposition 1.4.3. Given a simple object S of $A - \mathsf{wMod}$, there exists $\phi \in \widehat{R}$ such that S is isomorphic in $A - \mathsf{wMod}$ to a simple quotient of P_ϕ.

Proof. Let S be a simple object of $A - \mathsf{wMod}$. Since S is not zero, there exists $\phi \in \widehat{R}$ such that $S(\phi) \neq 0$. Let x be a non-zero element in $S(\phi)$. There exists a non-zero R-linear map $R \longrightarrow S$ such that $1 \mapsto x \in S(\phi)$. But, since $x \in S(\phi)$, the above map induces a non-zero R-linear map $\Bbbk_\phi \longrightarrow S$. This shows that $\mathsf{Hom}_{R-\mathsf{wMod}}(\Bbbk_\phi, S) \neq 0$. It follows from Corollary 1.2.4 that $\mathsf{Hom}_{A-\mathsf{wMod}}(P_\phi, S) \neq 0$. Hence, there exists a map $\varphi : P_\phi \longrightarrow S$ in $A - \mathsf{wMod}$ which is non-zero. But, of course, the image of φ is a non-zero subobject of S, so φ must be surjective. The rest is clear. $\qquad\square$

Proposition 1.4.3 shows the importance of understanding the non-zero simple quotients of the modules P_ϕ. The following remark deals with the easiest case.

Remark 1.4.4. Recall the hypotheses of the beginning of this section. Notice that, clearly, $\sigma_1 = \mathrm{id}$. Fix $\phi \in \widehat{R}$. Assume, further, that the map $\mathbf{b} \longrightarrow \widehat{R}$, given by $b \mapsto \phi \circ \sigma_b$ is injective (see Remark 1.3.3). It follows from Remark 1.3.3 that the weight subspace of P_ϕ of weight ϕ is:

$$P_\phi(\phi) = \Bbbk.1 \otimes_R (1 + \ker(\phi)).$$

Let M be a strict weight A-submodule of P_ϕ. Since P_ϕ is generated by its weight subspace $P_\phi(\phi)$ of weight ϕ, and since the latter is a one dimensional $\Bbbk$-vector space, we must have $P_\phi(\phi) \cap M = \{0\}$, which leads to:

$$M \subseteq \bigoplus_{\psi \neq \phi} P_\phi(\psi).$$

It follows that the sum of all strict submodules of P_ϕ in the category $A -$ wMod is again a strict submodule. That is, there is a strict submodule of P_ϕ in $A -$ wMod maximum among strict submodules of P_ϕ in $A -$ wMod. We denote this maximum submodule by N_ϕ. Of course, the corresponding quotient

$$S_\phi := P_\phi/N_\phi$$

is a simple object in $A -$ wMod and it is the unique simple quotient of P_ϕ in $A -$ wMod. Note that the above applies to P_ϕ seen as an object of $A -$ Mod, by Proposition 1.1.9.

1.5. Tensor product of strongly-free extensions

Let $n \in \mathbb{N}^*$. For each i, $1 \leq i \leq n$, let (R_i, A_i) be a pair of strongly-free type and let $\mathbf{b}_i$ be a subset of A_i as in Definition 1.2.1. Let

$$R = \bigotimes_{1 \leq i \leq n} R_i, \qquad A = \bigotimes_{1 \leq i \leq n} A_i$$

and identify R canonically with its image in A. (In the sequel, we will make extensive use of Proposition 1.1.10.)

Suppose in addition that, for $1 \leq i \leq n$, M_i is an object of $A_i -$ wMod. Clearly, $\bigotimes_{1 \leq i \leq n} M_i$ is an object of $A -$ wMod.

Lemma 1.5.1. *The pair (R, A) is of strongly-free type.*

Proof. It is clear that the subset $\mathbf{b} = \bigotimes_{1 \leq i \leq n} \mathbf{b}_i$ of A satisfies the conditions of Definition 1.2.1. $\square$

Remark 1.5.2. Retain the notation above and, for all $1 \leq i \leq n$, let ϕ_i be an element of $\widehat{R_i}$. Set $\phi = \phi_1 \cdots \phi_n$. Recall the description of $\ker(\phi)$ given in Proposition 1.1.10. There is an obvious surjective morphism

$$R \longrightarrow \bigotimes_{1 \leq i \leq n} \Bbbk_{\phi_i}$$

in $R - \mathsf{Mod}$ such that its kernel contains $\ker(\phi)$. Hence, there is a surjective morphism in $R - \mathsf{Mod}$:

$$\Bbbk_\phi \longrightarrow \bigotimes_{1 \leq i \leq n} \Bbbk_{\phi_i}$$

which must be an isomorphism since its source and target are both one dimensional vector spaces over $\Bbbk$.

The following theorem provides a description of the projective objects in $A - \mathsf{wMod}$ as tensor products of projectives.

Theorem 1.5.3. *Using the previous notation, let ϕ_i be an element of $\widehat{R_i}$ for all $1 \leq i \leq n$, and put $\phi = \phi_1 \cdots \phi_n$. There is an isomorphism*

$$P_\phi \cong \bigotimes_{1 \leq i \leq n} P_{\phi_i}$$

in $A - \mathsf{wMod}$.

Proof. It is enough to show that there is an isomorphism in $A - \mathsf{Mod}$:

$$P_\phi \cong \bigotimes_{1 \leq i \leq n} P_{\phi_i}.$$

This is a standard fact; indeed, by classical results, we have an isomorphism of $\Bbbk$-vector spaces

$$\bigotimes_{1 \leq i \leq n} P_{\phi_i} = \bigotimes_{1 \leq i \leq n} (A_i \otimes_{R_i} \Bbbk_{\phi_i}) \cong \left(\bigotimes_{1 \leq i \leq n} A_i \right) \otimes_R \left(\bigotimes_{1 \leq i \leq n} \Bbbk_{\phi_i} \right)$$

$$\cong \left(\bigotimes_{1 \leq i \leq n} A_i \right) \otimes_R \Bbbk_\phi \cong A \otimes_R \Bbbk_\phi = P_\phi$$

where the first isomorphism is given by

$$\otimes_{1 \leq i \leq n} (a_i \otimes_{R_i} k_i) \mapsto (\otimes_{1 \leq i \leq n} a_i) \otimes_R (\otimes_{1 \leq i \leq n} k_i)$$

and the second isomorphism is given by Remark 1.5.2.   □

Now we analyse simple modules in $A - \mathsf{wMod}$.

Lemma 1.5.4. *If, for $1 \leq i \leq n$, S_i is a simple object in $A_i - \mathsf{wMod}$ such that $\dim_{\Bbbk} S_i(\phi) \leq 1$ for any $\phi \in \widehat{R_i}$, then $S_1 \otimes \cdots \otimes S_n$ is a simple object in $A - \mathsf{wMod}$.*

Proof. Suppose that $\dim_{\Bbbk}(S_i)(\phi) \leq 1$ for any $\phi \in \widehat{R_i}$, $i = 1, \ldots, n$. It follows from Proposition 1.1.10 that the weight spaces of the A-module $S_1 \otimes \cdots \otimes S_n$ are also at most one dimensional $\Bbbk$-vector spaces, generated by elementary tensors. Now, let W be a non-zero (weight) submodule of $S_1 \otimes \cdots \otimes S_n$. Then, W contains a non-zero weight element which, by the above, must be an elementary tensor. Let $w = w_1 \otimes \cdots \otimes w_n$ be such an element. For all i such that $1 \leq i \leq n$, w_i is a non-zero element of S_i which, hence, generates the A_i-module S_i. Therefore, w generates the A-module $S_1 \otimes \cdots \otimes S_n$ and $W = S_1 \otimes \cdots \otimes S_n$. The statement is proved.   □

Remark 1.5.5. If we suppose that $\mathrm{End}_{A_i}(S) = \Bbbk$ for any simple object S in $A_i - \mathsf{wMod}$, $i = 1, \ldots, n-1$, then for any collection of simple objects S_i in $A_i - \mathsf{wMod}$, $i = 1, \ldots, n$, the A-module $S_1 \otimes \cdots \otimes S_n$ is simple. Indeed, under this hypothesis, we have that A_i is Schurian and tensor-simple [B], which implies the statement.

Let S be a simple object in $A - \mathsf{wMod}$. By Proposition 1.4.3 there exists $\phi \in \widehat{R}$ such that S is a quotient of P_ϕ, that is $S \simeq P_\phi/N$. We have the following result.

Corollary 1.5.6. *Let $\phi \in \widehat{R}$ be such that all weight subspaces of P_ϕ have dimension at most 1. For $i = 1, \ldots, n$, let $\phi_i \in \widehat{R_i}$ be such that $\phi = \phi_1 \cdots \phi_n$.*

1) *The map $\mathbf{b} \to \widehat{R}$ given by $b \mapsto \phi \circ \sigma_b$ is injective. Further, the module P_ϕ has a unique maximal strict submodule N_ϕ and unique simple quotient $S_\phi = P_\phi/N_\phi$.*

2) *For all $1 \leq i \leq n$, The map $\mathbf{b}_i \to \widehat{R_i}$ given by $b \mapsto \phi_i \circ \sigma_b$ is injective. Further, the module P_{ϕ_i} has a unique maximal strict submodule N_{ϕ_i} and unique simple quotient $S_{\phi_i} = P_{\phi_i}/N_{\phi_i}$.*

3) *There is an isomorphism*

$$S_\phi \cong S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$$

in $A - \mathsf{wMod}$.

Proof. We start by recalling that such ϕ_i's exist due to Proposition 1.1.10.

1) This follows immediately from Remarks 1.3.3 and 1.4.4.

2) We know from Theorem 1.5.3 that there is an isomorphism $\iota : P_\phi \to P_{\phi_1} \otimes \cdots \otimes P_{\phi_n}$. Hence, since all weight spaces of P_ϕ have dimension 1, by Proposition 1.1.10, the same must hold for P_{ϕ_i}, for $i = 1, \ldots, n$. The statement follows as in the first point.

3) Observe that for all i, S_{ϕ_i} must also have weight spaces of dimension at most one. Thus, we are in the hypotheses of the previous lemma; so that $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ is a simple object in $A - \mathsf{wMod}$.
Now, consider the following submodule of $P_{\phi_1} \otimes \cdots \otimes P_{\phi_n}$:

$$X = N_{\phi_1} \otimes P_{\phi_2} \otimes \cdots \otimes P_{\phi_n} + P_{\phi_1} \otimes N_{\phi_2} \otimes \cdots \otimes P_{\phi_n}$$
$$+ \cdots + P_{\phi_1} \otimes \cdots \otimes P_{\phi_{n-1}} \otimes N_{\phi_n}.$$

We have an obvious surjective map

$$P_\phi \simeq P_{\phi_1} \otimes \cdots \otimes P_{\phi_n} \twoheadrightarrow S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$$

whose kernel contains $\iota^{-1}(X)$, hence a surjective map

$$P_\phi / \iota^{-1}(X) \twoheadrightarrow S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}.$$

We now proceed to show that the latter must be an isomorphism. Indeed, taking Proposition 1.1.10 into account, the hypotheses on the dimensions of weight spaces prove that the weights of $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ are the products $\psi_1 \cdots \psi_n$, where ψ_i is a weight of P_{ϕ_i} that is not a weight of N_{ϕ_i}. Analogously, the weights of X are the products $\psi_1 \cdots \psi_n$, where ψ_i is a weight of P_{ϕ_i} and, for some $j \in \{1, \ldots, n\}$, ψ_j is a weight of N_{ϕ_j}.
It follows that both $P_\phi / \iota^{-1}(X)$ and $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ have the same set of weights. Since, in addition, these modules have one dimensional weight spaces, the surjective map must be an isomorphism. But, as quoted above, $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ is simple, which implies $N_\phi = \iota^{-1}(X)$. This proves the statement. $\qquad\square$

2. Quantum Weyl algebras

In this section we will consider two versions of quantum Weyl algebras. For a short account on the description of these quantum Weyl algebras as rings of skew differential operators, see Section 6. In the first subsection, we recall Maltsiniotis' quantum Weyl algebra. In the second we introduce the version of Akhavizadegan and Jordan. In the last subsection, we recall the isomorphism between the convenient localisations of these algebras.

For this, we need to introduce quantum integers. The reader is referred to [AJ] and [J] for a detailed exposition. Some complements may be found in [R1].

Let $q \in \Bbbk^*$. For any integer $i \in \mathbb{N}^*$, we let

$$(i)_q = 1 + q + \cdots + q^{i-1},$$

and extend this definition by $(0)_q = 0$. Further, for $j \in \mathbb{Z} \setminus \mathbb{N}$, we set

$$(j)_q = -q^j(-j)_q = -\left(q^j + q^{j+1} + \cdots + q^{-1}\right).$$

These scalars will be called *quantum integers associated to q* or, for simplicity, *q-integers*.

Remark 2.1.   1) If $q = 1$, then, for all $i \in \mathbb{Z}$, $(i)_q = i$.

2) If $q \neq 1$, then,

$$\forall i \in \mathbb{Z} \qquad (i)_q = \frac{1 - q^i}{1 - q}.$$

Recall that an $n \times n$ matrix (λ_{ij}) with entries in $\Bbbk^*$ is called skew-symmetric provided $\lambda_{ii} = 1$ and $\lambda_{ij} = \lambda_{ji}^{-1}$, for all $1 \leq i, j \leq n$.

2.1. The quantum Weyl algebra of G. Maltsiniotis

Definition 2.1.1. *Let $n \in \mathbb{N}^*$. To any n-tuple $\bar{q} = (q_1, \ldots, q_n) \in (\Bbbk^*)^n$ and any $n \times n$ skew symmetric matrix $\Lambda = (\lambda_{ij}) \in M_n(k)$, we associate the $\Bbbk$-algebra $A_n^{\bar{q},\Lambda}$ with generators $y_1, \ldots, y_n, x_1, \ldots, x_n$ and relations:*

$$(4) \quad \forall (i,j) \in \mathbb{N}^2, \ 1 \leq i < j \leq n, \quad \begin{array}{ll} x_i x_j = \lambda_{ij} q_i x_j x_i, & y_i y_j = \lambda_{ij} y_j y_i, \\ x_i y_j = \lambda_{ij}^{-1} y_j x_i, & y_i x_j = \lambda_{ij}^{-1} q_i^{-1} x_j y_i. \end{array}$$

(5) $\forall i \in \mathbb{N},\ 1 \leq i \leq n,\ x_i y_i - q_i y_i x_i = 1 + \sum_{j=1}^{i-1}(q_j - 1)y_j x_j.$

Further, we put $z_0 = 1$ and, for $1 \leq i \leq n$,

$$z_i = x_i y_i - y_i x_i.$$

The family $\{z_1, \ldots, z_n\}$ will play a central role in computations.

Remark 2.1.2. It is easy to see that for all i, $1 \leq i \leq n$:

(6) $z_i = 1 + \sum_{j=1}^{i}(q_j - 1)y_j x_j$ and $z_i = z_{i-1} + (q_i - 1)y_i x_i.$

As a consequence, relations (5) read:

(7) $\forall i, 1 \leq i \leq n,\ \ x_i y_i - q_i y_i x_i = z_{i-1}$

Proposition 2.1.3. *The algebra $A_n^{\bar{q},\Lambda}$ is a noetherian integral domain. In addition, the family $\{y_1^{j_1} x_1^{i_1} \cdots y_n^{j_n} x_n^{i_n},\ (j_1, i_1, \ldots, j_n, i_n) \in \mathbb{N}^{2n}\}$ is a PBW basis of $A_n^{\bar{q},\Lambda}$.*

Proof. See [R1, Remarque 2.1.3] and [AJ, 1.9]. □

Proposition 2.1.4. *For all $1 \leq i \leq n$, z_i is a normal element of $A_n^{\bar{q},\Lambda}$. More precisely, we have:*

- *for all i, j, $1 \leq i < j \leq n$, $z_i x_j = x_j z_i$ and $z_i y_j = y_j z_i$,*
- *for all i, j, $1 \leq j \leq i \leq n$, $z_i x_j = q_j^{-1} x_j z_i$ and $z_i y_j = q_j y_j z_i$,*
- *for all i, j, $1 \leq i, j \leq n$, $z_i z_j = z_j z_i$.*

Proof. See [J, p. 285]. □

As pointed out above, the multiplicative set generated by the elements z_i, $1 \leq i \leq n$, consists of normal elements. It follows that it is an Ore set and that we may form the corresponding localisation.

Definition 2.1.5. We denote by $B_n^{\bar{q},\Lambda}$ the localisation of $A_n^{\bar{q},\Lambda}$ at the multiplicative set generated by the elements z_i, $1 \leq i \leq n$.

Since $A_n^{\bar{q},\Lambda}$ is an integral domain, there is a canonical injective morphism of $\Bbbk$-algebras

$$A_n^{\bar{q},\Lambda} \xrightarrow{can.inj.} B_n^{\bar{q},\Lambda}.$$

In the sequel, we will often identify an element of $A_n^{\bar{q},\Lambda}$ with its image in $B_n^{\bar{q},\Lambda}$ under the above canonical injection.

Denote by R (resp R°) the $\Bbbk$-subalgebra of $A_n^{\bar{q},\Lambda}$ (resp. $B_n^{\bar{q},\Lambda}$) generated by $z_1, \ldots, z_n$ (resp. $z_1, \ldots, z_n$ and their inverses). Hence, R and R° are commutative $\Bbbk$-algebras.

Proposition 2.1.6. *Assume $q_i \neq 1$ for all $1 \leq i \leq n$. The elements $z_1, \ldots, z_n$ are algebraically independent in R. Moreover, the set*

$$\mathbf{b} = \left\{ y_1^{j_1} x_1^{i_1} \cdots y_n^{j_n} x_n^{i_n}, \ (j_1, i_1, \ldots, j_n, i_n) \in \mathbb{N}^{2n}, \ i_k j_k = 0, \ \forall 1 \leq k \leq n \right\}$$

is a basis of $A_n^{\bar{q},\Lambda}$ regarded as a left or as a right R-module. Further, $\mathbf{b}$ is a basis of $B_n^{\bar{q},\Lambda}$ both as a left and as a right R°-module.

Proof. We leave the detailed proof to the interested reader. All the statements follow from a careful use of the above relations and Proposition 2.1.3. $\square$

2.2. The quantum Weyl algebra of M. Akhavizadegan and D. Jordan

Definition 2.2.1. Let $n \in \mathbb{N}^*$. To any n-tuple $\bar{q} = (q_1, \ldots, q_n) \in (\Bbbk^*)^n$ and any $n \times n$ skew symmetric matrix $\Lambda = (\lambda_{ij}) \in M_n(k)$, we associate the $\Bbbk$-algebra $\mathcal{A}_n^{\bar{q},\Lambda}$ with generators $y_1, \ldots, y_n, x_1, \ldots, x_n$ and relations:

$$(8) \qquad \begin{aligned} \forall (i,j) \in \mathbb{N}^2, \ 1 \leq i < j \leq n, \quad & x_i x_j = \lambda_{ij} x_j x_i, \quad & y_i y_j = \lambda_{ij} y_j y_i, \\ & x_i y_j = \lambda_{ij}^{-1} y_j x_i, \quad & y_i x_j = \lambda_{ij}^{-1} x_j y_i. \end{aligned}$$

$$(9) \qquad \forall i \in \mathbb{N}, \ 1 \leq i \leq n, \ x_i y_i - q_i y_i x_i = 1.$$

Let us also define $z_0 = 1$ and, for i such that $1 \leq i \leq n$,

$$z_i = x_i y_i - y_i x_i.$$

As before, these elements will play a crucial role.

Remark 2.2.2. It is easy to see that the four relations in (8) actually hold for all $(i, j) \in \mathbb{N}^2$ such that $1 \leq i \neq j \leq n$ and that:

$$\forall i, \qquad 1 \leq i \leq n, \qquad z_i = 1 + (q_i - 1)y_i x_i.$$

Two results analogous to Propositions 2.1.4 and 2.1.6 are also true in this case.

Proposition 2.2.3. *The algebra $\mathcal{A}_n^{\bar{q},\Lambda}$ is a noetherian integral domain. In addition, the family $\{y_1^{j_1} x_1^{i_1} \cdots y_n^{j_n} x_n^{i_n}, (j_1, i_1, \ldots, j_n, i_n) \in \mathbb{N}^{2n}\}$ is a $\Bbbk$-basis of $\mathcal{A}_n^{\bar{q},\Lambda}$.*

Proof. See [AJ, 1.9].   □

In particular, the family $\{z_1, \ldots, z_n\}$ verifies properties similar to those of the previous subsection.

Proposition 2.2.4. *For all $1 \leq i \leq n$, z_i is a normal element of $\mathcal{A}_n^{\bar{q},\Lambda}$. More precisely, we have:*

- *for all i, j, $1 \leq i \neq j \leq n$, $z_i x_j = x_j z_i$ and $z_i y_j = y_j z_i$,*
- *for all i, $1 \leq i \leq n$, $z_i x_i = q_i^{-1} x_i z_i$ and $z_i y_i = q_i y_i z_i$,*
- *for all i, j, $1 \leq i, j \leq n$, $z_i z_j = z_j z_i$.*

Proof. See [AJ, p. 286].   □

As pointed out before, the multiplicative set generated by all the z_i's consists of normal elements. Hence, it is an Ore set and thus we may form the corresponding localisation.

Definition 2.2.5. We denote by $\mathcal{B}_n^{\bar{q},\Lambda}$ the localisation of $\mathcal{A}_n^{\bar{q},\Lambda}$ at the multiplicative set generated by the elements z_i, $1 \leq i \leq n$.

Notice that, again, the fact that $\mathcal{A}_n^{\bar{q},\Lambda}$ is an integral domain implies that the canonical morphism of $\Bbbk$-algebras

$$\mathcal{A}_n^{\bar{q},\Lambda} \stackrel{can.inj.}{\longrightarrow} \mathcal{B}_n^{\bar{q},\Lambda}$$

is injective, and we will identify any element of $\mathcal{A}_n^{\bar{q},\Lambda}$ with its image in $\mathcal{B}_n^{\bar{q},\Lambda}$ under this injection.

Denote by R (resp R°) the $\Bbbk$-subalgebra of $\mathcal{A}_n^{\bar{q},\Lambda}$ (resp. $\mathcal{B}_n^{\bar{q},\Lambda}$) generated by $z_1, \ldots, z_n$ (resp. and their inverses).

Proposition 2.2.6. *Assume $q_i \neq 1$ for all $1 \leq i \leq n$. The elements $z_1, \ldots,$ z_n are algebraically independent in R. Further, the set*

$$\mathbf{b} = \{y_1^{j_1} x_1^{i_1} \cdots y_n^{j_n} x_n^{i_n}, \; (j_1, i_1, \ldots, j_n, i_n) \in \mathbb{N}^{2n}, \; i_k j_k = 0, \; \forall 1 \leq k \leq n\}.$$

is a basis of $\mathcal{A}_n^{\bar{q}, \Lambda}$ regarded as a left and as a right R-module. Moreover, $\mathbf{b}$ is also a basis of $\mathcal{B}_n^{\bar{q}, \Lambda}$ as a left and as a right R°-module.

Proof. It is similar to the proof of Prop. 2.1.6.   □

2.3. A common localisation for quantum Weyl algebras

We finish this section by recalling a result which states that the localisations of both versions of quantum Weyl algebras are isomorphic.

Theorem 2.3.1. *Let $n \in \mathbb{N}^*$. For any n-tuple $\bar{q} = (q_1, \ldots, q_n) \in (\mathbb{k}^*)^n$ and any $n \times n$ skew symmetric matrix $\Lambda = (\lambda_{ij}) \in M_n(\mathbb{k})$, the map defined by:*

$$\begin{array}{rcl}
\theta \; : \; \mathcal{B}_n^{\bar{q}, \Lambda} & \longrightarrow & B_n^{\bar{q}, \Lambda} \\
y_i & \mapsto & y_i \\
x_i & \mapsto & z_{i-1}^{-1} x_i \\
z_i & \mapsto & z_{i-1}^{-1} z_i
\end{array}$$

is an isomorphism of $\mathbb{k}$-algebras.

Proof. See p. 287 of [AJ].   □

3. Weight modules for quantum Weyl algebras

In this section, we fix an n-tuple $\bar{q} = (q_1, \ldots, q_n) \in (\mathbb{k}^*)^n$ and an $n \times n$ skew symmetric matrix $\Lambda = (\lambda_{ij}) \in M_n(\mathbb{k})$ and we associate to this data the algebra $\mathcal{B}_n^{\bar{q}, \Lambda}$ introduced in Section 2. We consider the context of Section 1 for the pair $(R^\circ, \mathcal{B}_n^{\bar{q}, \Lambda})$, where R° is the $\mathbb{k}$-subalgebra of $\mathcal{B}_n^{\bar{q}, \Lambda}$ generated by the z_i's and their inverses. As a consequence of Proposition 2.2.6, R° is a commutative Laurent polynomial $\mathbb{k}$-algebra in the indeterminates $z_1, \ldots, z_n$.

As we will see, the situation will heavily depend on whether some q_i is a root of unity in $\mathbb{k}$ or not. We will say that the n-tuple $\bar{q}$ is *generic* whenever, for all integer i, $1 \leq i \leq n$, q_i is not a root of unity.

In the first subsection, we show that the pair $(R^\circ, \mathcal{B}_n^{\bar{q}, \Lambda})$ is indeed a strongly-free extension. We also consider the action of the canonical generators of $\mathcal{B}_n^{\bar{q}, \Lambda}$ on the basis attached to this extension. From that basis arises in

a natural way an action of the group $\mathbb{Z}^n$ on the weights. We investigate this action in the second subsection and show for example that it is free if and only if no q_i is a root of unity. As we discussed in the first section, each character ϕ of R° gives rise to a projective weight $\mathcal{B}_n^{\bar{q},\Lambda}$-module P_ϕ. The structure of these modules is the subject of the last subsection. We investigate their weights in connection with the above group action, the dimension of their weight spaces, characterise their simplicity and start to discuss whether two such modules are isomorphic or not.

3.1. Basic results on the pair $(R^\circ, \mathcal{B}_n^{\bar{q},\Lambda})$

By Proposition 2.2.6,

$$\mathbf{b} = \left\{ y_1^{j_1} x_1^{i_1} \cdots y_n^{j_n} x_n^{i_n}, \ (j_1, i_1, \ldots, j_n, i_n) \in \mathbb{N}^{2n}, \ i_k j_k = 0, \ \forall k, 1 \le k \le n \right\}.$$

is a basis of $\mathcal{B}_n^{\bar{q},\Lambda}$ regarded either as a left or as a right R°-module. It will be convenient to write this basis in a slightly different way. Let $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. We denote

$$b_k = \prod_{1 \le i \le n} b_{k,i} \qquad \text{where, for } 1 \le i \le n, \qquad b_{k,i} = \left\{ \begin{array}{ll} x_i^{k_i} & \text{if } k_i \ge 0, \\ y_i^{-k_i} & \text{if } k_i \le 0. \end{array} \right.$$

With this notation, we have

$$\mathbf{b} = \{ b_k, \ k \in \mathbb{Z}^n \}.$$

Given $k \in \mathbb{Z}^n$, let σ_k be the $\mathbb{k}$-algebra automorphism of R° such that

$$\forall i, \ 1 \le i \le n, \quad \sigma_k(z_i) = q_i^{-k_i} z_i.$$

From the relations in Proposition 2.2.4, we get that

$$\forall\, k \in \mathbb{Z}^n, \quad \forall r \in R^\circ, \quad r b_k = b_k \sigma_k(r).$$

In the notation of Definition 1.2.1, this means that, for all $k \in \mathbb{Z}^n$, $\sigma_k = \sigma_{b_k}$. These facts afford the following lemma.

Lemma 3.1.1. *The pair $(R^\circ, \mathcal{B}_n^{\bar{q},\Lambda})$ is of strongly-free type.*

We finish this subsection by examining how multiplication by generators modify the elements of $\mathbf{b}$. From now on, we let $\{e_1, \ldots, e_n\}$ stand for the canonical basis of $\mathbb{Z}^n$.

Lemma 3.1.2. *Assume $q_i \neq 1$ for all $1 \leq i \leq n$. Given $k \in \mathbb{Z}^n$ and i such that $1 \leq i \leq n$,*

$$
x_i b_k = \begin{cases} \left(\displaystyle\prod_{1 \leq j < i} \lambda_{ij}^{k_j} \right) b_{k+e_i} & \text{if } k_i \geq 0, \\[3em] \left(\displaystyle\prod_{1 \leq j < i} \lambda_{ij}^{k_j} \right) b_{k+e_i} \dfrac{q_i^{-k_i} z_i - 1}{q_i - 1} & \text{if } k_i < 0 \end{cases}
$$

and

$$
y_i b_k = \begin{cases} \left(\displaystyle\prod_{1 \leq j < i} \lambda_{ij}^{k_j} \right)^{-1} b_{k-e_i} \dfrac{q_i^{-k_i+1} z_i - 1}{q_i - 1} & \text{if } k_i > 0, \\[3em] \left(\displaystyle\prod_{1 \leq j < i} \lambda_{ij}^{k_j} \right)^{-1} b_{k-e_i} & \text{if } k_i \leq 0. \end{cases}
$$

Proof. This is straightforward using relation (8), (9) and the equalities of Remark 2.2.2 and Proposition 2.2.4. $\square$

3.2. Action of $\mathbb{Z}^n$ on weights arising from the basis b

Recall from Remark 1.1.3 the right action of $\mathrm{Aut}_{\Bbbk}(R^\circ)$ on $\widehat{R^\circ}$. On the other hand, we have group morphisms as follows

$$
\epsilon : \mathbb{Z}^n \to (\Bbbk^*)^n \qquad (k_1, \ldots, k_n) \mapsto (q_1^{-k_1}, \ldots, q_n^{-k_n})
$$

and

$$
\delta : (\Bbbk^*)^n \to \mathrm{Aut}_{\Bbbk}(R^\circ) \qquad (\lambda_1, \ldots, \lambda_n) \mapsto (z_i \mapsto \lambda_i z_i)_{1 \leq i \leq n}.
$$

Now, composing these maps gives rise to a left action of $(\Bbbk^*)^n$ on $\widehat{R^\circ}$ and a left action of $\mathbb{Z}^n$ on $\widehat{R^\circ}$ given by

$$
(\Bbbk^*)^n \xrightarrow{\delta} \mathrm{Aut}_{\Bbbk}(R^\circ) \longrightarrow \mathfrak{S}(\widehat{R^\circ}) \qquad \text{and}
$$
$$
\mathbb{Z}^n \xrightarrow{\epsilon} (\Bbbk^*)^n \xrightarrow{\delta} \mathrm{Aut}_{\Bbbk}(R^\circ) \longrightarrow \mathfrak{S}(\widehat{R^\circ}),
$$

where $\mathfrak{S}(\widehat{R^\circ})$ is the group of permutations of the set $\widehat{R^\circ}$. Alternatively, the latter may be described as follows:

$$
(10) \qquad \begin{array}{ccc} \mathbb{Z}^n \times \widehat{R^\circ} & \longrightarrow & \widehat{R^\circ} \\ (k, \phi) & \mapsto & \phi \circ \sigma_k. \end{array}
$$

These actions verify some easily proven properties that we list in the following lemma.

Lemma 3.2.1. *Using the notation above, the following holds:*

1) *the left action of $(\Bbbk^*)^n$ on $\widehat{R^\circ}$ is faithful;*

2) *the morphism $\epsilon : \mathbb{Z}^n \longrightarrow (\Bbbk^*)^n$ is injective if and only if $\bar{q}$ is generic;*

3) *the left action of $\mathbb{Z}^n$ on $\widehat{R^\circ}$ is faithful if and only if $\bar{q}$ is generic.*

Fixing $\phi \in \widehat{R^\circ}$, we obtain a map $\mathbb{Z}^n \to \widehat{R^\circ}$.

Lemma 3.2.2. *Let $\phi \in \widehat{R^\circ}$. The map $\mathbb{Z}^n \to \mathbb{Z}^n.\phi \subseteq \widehat{R^\circ}$ sending k to $\phi \circ \sigma_k$ is injective if and only if $\bar{q}$ is generic.*

Proof. Let $k \in \mathbb{Z}^n$. By definition, given i such that $1 \leq i \leq n$, we know that $\phi \circ \sigma_k(z_i) = q_i^{-k_i}\phi(z_i)$. But, z_i being an invertible element of $\mathcal{B}_n^{\bar{q},\Lambda}$, $\phi(z_i)$ is necessarily non-zero. The result follows. $\qquad\square$

Corollary 3.2.3. *The following conditions are equivalent:*

1) *$\bar{q}$ is generic;*

2) *there exists $\phi \in \widehat{R^\circ}$ such that $\mathrm{stab}_{\mathbb{Z}^n}(\phi) = \{0_{\mathbb{Z}^n}\}$;*

3) *for all $\phi \in \widehat{R^\circ}$, $\mathrm{stab}_{\mathbb{Z}^n}(\phi) = \{0_{\mathbb{Z}^n}\}$, in other words the action of $\mathbb{Z}^n$ on $\widehat{R^\circ}$ is free.*

Proof. This is an immediate consequence of Lemma 3.2.2. $\qquad\square$

3.3. Structure of P_ϕ for $\phi \in \widehat{R^\circ}$

Let $\phi \in \widehat{R^\circ}$. Recall that $P_\phi = \mathcal{B}_n^{\bar{q},\Lambda} \otimes_{R^\circ} \Bbbk_\phi$, where $\Bbbk_\phi = R^\circ/\ker(\phi)$. Given $k \in \mathbb{Z}^n$, set

$$v_k = b_k \otimes_R \bar{1}.$$

By Remark 1.3.3, the set $\{v_k, k \in \mathbb{Z}^n\}$, is a $\Bbbk$-basis of P_ϕ and

(11) $$\forall k \in \mathbb{Z}^n, \quad \Bbbk v_k \subseteq P_\phi(\phi \circ \sigma_k).$$

Hence, P_ϕ is a weight module of $\mathcal{B}_n^{\bar{q},\Lambda}$ and the set of weights of P_ϕ is the $\mathbb{Z}^n$-orbit of ϕ, that is $\mathbb{Z}^n.\phi$.

Proposition 3.3.1. *Let* $\phi \in \widehat{R^\circ}$. *If* $\bar{q}$ *is generic, then, for all* $k \in \mathbb{Z}^n$, $\Bbbk v_k = P_\phi(\phi \circ \sigma_k)$. *In particular, the weight decomposition of the weight* $\mathcal{B}_n^{\bar{q},\Lambda}$*-module* P_ϕ *is:*

$$P_\phi = \bigoplus_{k \in \mathbb{Z}^n} \Bbbk v_k = \bigoplus_{k \in \mathbb{Z}^n} P_\phi(\phi \circ \sigma_k).$$

Proof. Given $k \in \mathbb{Z}^n$, we know from the previous discussion that $\Bbbk v_k \subseteq P_\phi(\phi \circ \sigma_k)$. The result is thus an obvious consequence of Lemma 3.2.2. □

Remark 3.3.2. We may actually strengthen Proposition 3.3.1 as follows. Let $\phi \in \widehat{R^\circ}$. The following statements are equivalent:

(i) $\bar{q}$ is generic;

(ii) the map $\mathbf{b} \mapsto \widehat{R^\circ}$, $b \mapsto \phi \circ \sigma_b$ is injective;

(iii) the weight spaces of P_ϕ all have dimension at most one.

This follows from the above discussion taking Lemma 3.2.2 and the inclusion (11) into account.

Remark 3.3.3. Supposing that $\bar{q}$ is generic, the hypotheses of Remark 1.4.4 are fulfilled. Hence, we know that, for all $\phi \in \widehat{R^\circ}$, the module P_ϕ has a maximum strict weight $\mathcal{B}_n^{\bar{q},\Lambda}$-submodule N_ϕ and a unique simple quotient $S_\phi = P_\phi/N_\phi$.

We continue by giving explicit expressions for the action of x_i and y_i on basis vectors of P_ϕ. Recall that $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$.

Lemma 3.3.4. *Assume* $q_i \neq 1$ *for all* $1 \leq i \leq n$. *Let* $\phi \in \widehat{R}$. *Let* $1 \leq i \leq n$ *and let* $k \in \mathbb{Z}^n$:

1) the action of x_i *on* v_k *is given by* :

$$(12) \qquad x_i.v_k = \begin{cases} \left(\displaystyle\prod_{1 \leq j < i} \lambda_{ij}^{k_j} \right) v_{k+e_i} & \text{if } k_i \geq 0, \\[2em] \dfrac{q_i^{-k_i}\phi(z_i) - 1}{q_i - 1} \left(\displaystyle\prod_{1 \leq j < i} \lambda_{ij}^{k_j} \right) v_{k+e_i} & \text{if } k_i < 0; \end{cases}$$

2) *the action of y_i on v_k is given by :*

$$(13) \qquad y_i.v_k = \begin{cases} \dfrac{q_i^{-k_i+1}\phi(z_i) - 1}{q_i - 1} \left(\displaystyle\prod_{1 \le j < i} \lambda_{ij}^{k_j} \right)^{-1} v_{k-e_i} & \text{if } k_i > 0, \\[4ex] \left(\displaystyle\prod_{1 \le j < i} \lambda_{ij}^{k_j} \right)^{-1} v_{k-e_i} & \text{if } k_i \le 0. \end{cases}$$

Proof. It is an immediate consequence of Lemma 3.1.2. $\qquad\qquad\square$

It is clear after Lemma 3.3.4 that a key feature of P_ϕ depends on whether the scalars appearing in relations (12) and (13) may vanish or not, which motivates the next definition.

Definition 3.3.5. Let $\phi \in \widehat{R^\circ}$. We define the *complexity* of ϕ as the set

$$\mathrm{comp}(\phi) = \{i \in \{1, \ldots, n\} \mid \phi(z_i) \in \langle q_i \rangle\}.$$

Remark 3.3.6. Any two elements of $\widehat{R^\circ}$ in the same $\mathbb{Z}^n$-orbit have the same complexity.

Recall that, after Proposition 1.1.9, any submodule of P_ϕ is a weight submodule. The next proposition shows that weights with empty complexity give rise to simple modules.

Proposition 3.3.7. *Suppose that $\bar{q}$ is generic. If $\phi \in \widehat{R^\circ}$, the following are equivalent:*

1) *the complexity of ϕ is empty,*

2) *P_ϕ is a simple object both of $\mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{Mod}$ and of $\mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{wMod}$.*

Proof. By Proposition 1.1.9, it is enough to consider the simplicity of P_ϕ as an object of $\mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{wMod}$.

Suppose M is a non-zero subobject of P_ϕ in the category $\mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{wMod}$. By Proposition 3.3.1, it must contain one of the v_k's with $k \in \mathbb{Z}^n$. This in turn implies, using Lemma 3.3.4, that if the complexity of ϕ is empty, then $v_0 \in M$ and thus $M = P_\phi$.

Conversely, suppose the complexity of ϕ is not empty and let i be an integer belonging to $\mathrm{comp}(\phi)$. Thus, $\phi(z_i) = q_i^\ell$ for some $\ell \in \mathbb{Z}$. Suppose first that $\ell < 0$. It follows from Lemma 3.3.4 that $x_i.v_k = 0$ whenever $k_i = \ell$. As

a consequence, $\oplus_{k, k_i \leq \ell} \Bbbk v_k$ is a strict $\mathcal{B}_n^{\bar{q}, \Lambda}$-submodule of P_ϕ. Suppose now that $\ell \geq 0$. In this case, it follows from Lemma 3.3.4 that $y_i . v_k = 0$ whenever $k_i = \ell + 1$. Consequently, $\oplus_{k, k_i \geq \ell + 1} \Bbbk v_k$ is a strict $\mathcal{B}_n^{\bar{q}, \Lambda}$-submodule of P_ϕ. Hence, in all cases, P_ϕ is not simple. □

In the next statement, we analyse the link between P_ϕ and $P_{\phi \circ \sigma_{e_\ell}}$ for an integer ℓ such that $1 \leq \ell \leq n$. Recall that e_i stands for the i-th vector of the canonical basis of $\mathbb{Z}^n$.

Proposition 3.3.8. *Suppose $q_i \neq 1$ for all $1 \leq i \leq n$. Fix an integer ℓ such that $1 \leq \ell \leq n$ and $\phi \in \widehat{R^\circ}$.*

1) *If $(\lambda_k)_{k \in \mathbb{Z}^n} \in \Bbbk^{(\mathbb{Z}^n)}$ satisfies the following conditions:*
   a) $\lambda_k = \lambda_{k+e_\ell}$, *if $k_\ell \neq -1$;*
   b) $\lambda_k = \dfrac{\phi(z_\ell) - 1}{q_\ell - 1} \lambda_{k+e_\ell}$, *if $k_\ell = -1$;*
   c) $\lambda_k = \lambda_{k+e_i}$, *if $i < \ell$;*
   d) $\lambda_k = \lambda_{i\ell}^{-1} \lambda_{k+e_i}$, *if $i > \ell$,*
   then, the $\Bbbk$-linear map

$$\begin{array}{ccc} P_{e_\ell . \phi} & \longrightarrow & P_\phi \\ w_k & \mapsto & \lambda_k v_{k+e_\ell}, \end{array}$$

   where $(w_k)_{k \in \mathbb{Z}^n}$ and $(v_k)_{k \in \mathbb{Z}^n}$ are respectively the canonical bases of $P_{e_\ell . \phi}$ and P_ϕ — as defined at the beginning of this subsection — is a morphism in $\mathcal{B}_n^{\bar{q}, \Lambda} - \mathsf{wMod}$.

2) *Assume in addition that $\phi(z_\ell) \neq 1$. Then, $P_{e_\ell . \phi}$ is isomorphic to P_ϕ in $\mathcal{B}_n^{\bar{q}, \Lambda} - \mathsf{wMod}$.*

Proof. The first item is a lengthy straightforward verification. The second item follows directly from the first one. □

4. Examples: the case where all the entries of Λ equal 1

As it will be shown in Section 5, it turns out that the study of simple weight modules of the extension $(R^\circ, \mathcal{B}_n^{\bar{q}, \Lambda})$ may be reduced to the case where the skew-symmetric matrix Λ has all its entries equal to 1. For this reason we concentrate, in the present section, to this special case. It is easier to handle since under this hypothesis, $\mathcal{B}_n^{\bar{q}, \Lambda}$ is a tensor product of n copies of $\mathcal{B}_1^{\bar{q}, \Lambda}$.

Hence, we first investigate the case where $n = 1$. We give a complete and explicit description of the projective weight modules P_ϕ attached to

each character ϕ of R°. From this we deduce a complete description of the simple weight modules and classify them. The second subsection comes back to the case of an arbitrary integer n. It is shown that simple weight modules in this case may be explicitly described using simple weight modules arising in the case $n = 1$. This is illustrated in the third subsection where the case $n = 2$ is studied in full details using very intuitive graphs to describe the modules P_ϕ.

In this section, we suppose that $\bar{q}$ is generic. We also assume that all the entries of the skew-symmetric matrix Λ equal 1. Under this last hypothesis, there is an isomorphism of $\Bbbk$-algebras

$$\mathcal{B}_n^{\bar{q},\Lambda} \quad \longrightarrow \quad \mathcal{B}_1^{q_1} \otimes \cdots \otimes \mathcal{B}_1^{q_n}$$

that sends each x_i to the elementary tensor $1 \otimes \cdots \otimes x_1 \otimes \cdots \otimes 1$, and each y_i to $1 \otimes \cdots \otimes y_1 \otimes \cdots \otimes 1$, where x_1 and y_1 are, respectively, in the i-th position, where, for $q \in \Bbbk^*$, we put $\mathcal{B}_1^q = \mathcal{B}_1^{\bar{q},\Lambda}$ with $\bar{q} = (q)$ and $\Lambda = (1)$. This isomorphism clearly induces in turn the isomorphism

$$\Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \quad \longrightarrow \quad \Bbbk[z_1^{\pm 1}] \otimes \cdots \otimes \Bbbk[z_1^{\pm 1}].$$

We are now ready to apply the results of Proposition 1.1.10 and Subsection 1.5.

From now on, we denote by $\mathbf{1}$ the unique element of $\Bbbk[\widehat{z_1^{\pm 1}, \ldots, z_n^{\pm 1}}]$ that sends $z_1, \ldots, z_n$ to 1.

4.1. The case $n = 1$

The observations in the Introduction show that one can build simple weight modules of $\mathcal{B}_n^{\bar{q},\Lambda}$ from those appearing in the case $n = 1$. So we start by studying this case. In this subsection, we suppose $n = 1$.

Remark 4.1.1. Let $\phi \in \widehat{R^\circ}$. It is clear that the following statements are equivalent:

1) $\mathrm{comp}(\phi) = \{1\}$;

2) $\mathbf{1} \in \mathbb{Z}.\phi$;

3) $\mathbb{Z}.\phi = \mathbb{Z}.\mathbf{1}$.

Fix now an element $\phi \in \widehat{R^\circ}$. As a consequence of Proposition 3.3.1 and Corollary 3.2.3, the set of weights of P_ϕ is $\mathbb{Z}.\phi$, with $\mathbb{Z}.\phi$ being equipotent

to $\mathbb{Z}$ and each weight space of P_ϕ is one dimensional over $\Bbbk$.

According to Proposition 3.3.7, P_ϕ is simple if and only if the complexity of ϕ is empty, that is if and only if ϕ is in the orbit of $\mathbf{1}$ under the action of $\mathbb{Z}$ (cf. Remark 4.1.1). We thus analyse these two cases.

In order to visualize the action under consideration, we represent it by a graph as follows. The vectors of the canonical basis of P_ϕ are displayed horizontally along a line and constitute the set of vertices of the graph. By Lemma 3.3.4, x_1 (resp. y_1) acts on a given v_α sending it to a scalar multiple of $v_{\alpha+1}$ (resp. $v_{\alpha-1}$). In case the corresponding scalar is non-zero, we draw an arrow, labeled by the corresponding generator, from v_α to $v_{\alpha+1}$ (resp. $v_{\alpha-1}$); in case it is zero, we do not draw any arrow. A careful observation allows us to "see" in this graph the unique maximal submodule and corresponding simple quotient.

Example 4.1.2. Suppose $\phi \notin \mathbb{Z}.\mathbf{1}$ or, equivalently, $\mathrm{comp}(\phi) = \emptyset$.

1) By Proposition 3.3.7, P_ϕ is a simple $\mathcal{B}_1^{\bar{q},\Lambda}$-weight module. Its corresponding scheme is as follows:
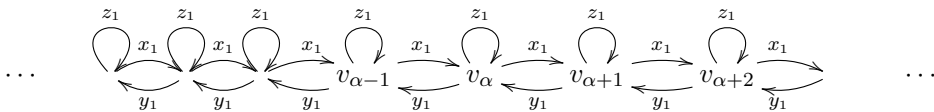

    where we have added in the scheme the action of z_1 for the sake of completeness.

2) By Proposition 3.3.8, for all $\Psi \in \mathbb{Z}.\phi$, P_ϕ and P_Ψ are isomorphic in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$.

Recall from Remark 3.3.3 the submodule N_ϕ of P_ϕ.

Example 4.1.3. Suppose $\phi \in \mathbb{Z}.\mathbf{1}$.

1) First case: $\phi \in (-\mathbb{N}).\mathbf{1}$. In this case there exists a unique non negative integer α such that $\phi = (-\alpha)\mathbf{1}$, or equivalently such that $\phi(z_1) = q_1^\alpha$.
   a) The action of x_1, y_1, z_1 can be pictured as follows:
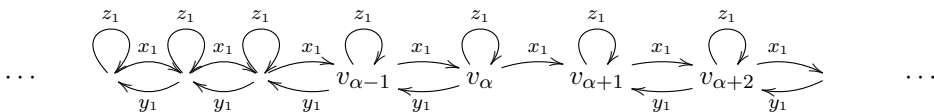
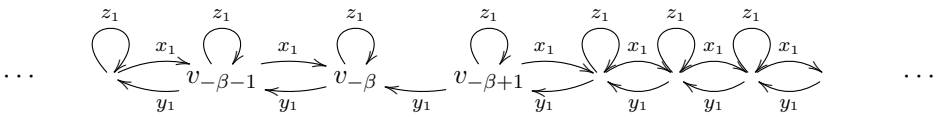b) Applying Proposition 1.1.9, one gets at once that

$$N_\phi = \bigoplus_{k \geq \alpha+1} \Bbbk.v_k.$$

In particular the set of weights of N_ϕ is $\mathbb{N}^*.\mathbf{1}$.

c) It follows that the set of weights of S_ϕ is $(-\mathbb{N}).\mathbf{1}$.

2) Second case: $\phi \in \mathbb{N}^*.\mathbf{1}$. In this case there exists a unique positive integer β such that $\phi = \beta.\mathbf{1}$, or equivalently such that $\phi(z_1) = q_1^{-\beta}$.

a) The action of x_1, y_1, z_1 can be pictured as follows:

b) Arguing as in the first case, $N_\phi = \bigoplus_{k \leq -\beta} \Bbbk.v_k$. Hence, the set of weights of N_ϕ is $(-\mathbb{N}).\mathbf{1}$.

c) As a consequence, the set of weights of S_ϕ is $\mathbb{N}^*.\mathbf{1}$.

3) By Proposition 3.3.8, we get that:

a) for all $\rho, \psi \in (-\mathbb{N}).\mathbf{1}$, P_ρ and P_ψ are isomorphic in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$;

b) for all $\rho, \psi \in \mathbb{N}^*.\mathbf{1}$, P_ρ and P_ψ are isomorphic in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$.

c) Now, consider $\rho \in (-\mathbb{N}).\mathbf{1}$ and $\psi \in \mathbb{N}^*.\mathbf{1}$. From the analysis of the first two cases, it follows that the sets of weights of N_ρ and N_ψ are $\mathbb{N}^*.\mathbf{1}$ and $(-\mathbb{N}).\mathbf{1}$, respectively. Since we are in the generic setting, these sets are not equal. Hence, N_ρ and N_ψ are not isomorphic, implying that P_ρ and P_ψ are not isomorphic, neither. Using again both previous cases, the sets of weights of S_ρ and S_ψ are $(-\mathbb{N}).\mathbf{1}$ and $\mathbb{N}^*.\mathbf{1}$, respectively. Thus, S_ρ and S_ψ are not isomorphic.

The next two theorems provide a description in terms of orbits of the isomorphism classes of projective and simple weight modules, respectively.

Theorem 4.1.4. *Let $\phi, \psi \in \widehat{R^\circ}$.*

1) If the modules P_ϕ and P_ψ are isomorphic, either in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$ or in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{Mod}$, then $\mathbb{Z}.\phi = \mathbb{Z}.\psi$.

2) If $\mathbb{Z}.\phi = \mathbb{Z}.\psi \neq \mathbb{Z}.\mathbf{1}$, then P_ϕ and P_ψ are isomorphic, both in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$ and in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{Mod}$.

3) If $\mathbb{Z}.\phi = \mathbb{Z}.\psi = \mathbb{Z}.\mathbf{1}$, then P_ϕ and P_ψ are isomorphic if and only if both ϕ and ψ belong to $(-\mathbb{N}).\mathbf{1}$ or they both belong to $\mathbb{N}^*.\mathbf{1}$.

Proof. The proof of the first item is clear since $\mathbb{Z}.\phi$ and $\mathbb{Z}.\psi$ are the sets of weights of P_ϕ and P_ψ, respectively. The other statements have already been proved. $\qquad\square$

Theorem 4.1.5. *Let $\phi, \psi \in \widehat{R^\circ}$.*

1) *If S_ϕ and S_ψ are isomorphic, either in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$ or in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{Mod}$, then $\mathbb{Z}.\phi = \mathbb{Z}.\psi$.*

2) *If $\mathbb{Z}.\phi = \mathbb{Z}.\psi \neq \mathbb{Z}.\mathbf{1}$, then S_ϕ and S_ψ are isomorphic.*

3) *If $\mathbb{Z}.\phi = \mathbb{Z}.\psi = \mathbb{Z}.\mathbf{1}$, then S_ϕ and S_ψ are isomorphic if and only if both ϕ and ψ belong to $(-\mathbb{N}).\mathbf{1}$ or both of them belong to $\mathbb{N}^*.\mathbf{1}$.*

Proof. Again the proof of the first item is clear since, by the previous results, the sets of weights of S_ϕ and S_ψ are, respectively, subsets of $\mathbb{Z}.\phi$ and $\mathbb{Z}.\psi$. All the rest has already been proved. $\qquad\square$

Corollary 4.1.6. *Let $\phi, \psi \in \widehat{R^\circ}$. The following are equivalent:*

1) *S_ϕ and S_ψ are isomorphic either in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{wMod}$ or in $\mathcal{B}_1^{\bar{q},\Lambda} - \mathsf{Mod}$;*

2) *S_ϕ and S_ψ have the same set of weights.*

Proof. Clearly, the first statement implies the second one. Now, let $w(S_\phi)$ and $w(S_\psi)$ be the sets of weights of S_ϕ and S_ψ, respectively, and suppose $w(S_\phi) = w(S_\psi)$. There are different possible cases:
First case: $\phi \notin \mathbb{Z}\mathbf{1}$. Example 4.1.2 shows that $S_\phi = P_\phi$ and $\mathbb{Z}\phi = w(S_\phi) = w(S_\psi)$. But $\psi \in w(S_\psi)$, hence $\mathbb{Z}\phi = \mathbb{Z}\psi$ and Theorem 4.1.5 proves that S_ϕ and S_ψ are isomorphic.
Second case: $\phi \in (-\mathbb{N})\mathbf{1}$. By Example 4.1.3, $(-\mathbb{N})\mathbf{1} = w(S_\phi) = w(S_\psi)$. Hence, $\psi \in (-\mathbb{N})\mathbf{1}$ and $\mathbb{Z}\phi = \mathbb{Z}\mathbf{1}$. Finally, Theorem 4.1.5 proves that S_ϕ and S_ψ are isomorphic.
Third case: $\phi \in \mathbb{N}^*\mathbf{1}$. The same argument as in the second case applies to prove that S_ϕ and S_ψ are isomorphic. $\qquad\square$

4.2. Simple weight modules for arbitrary n

Recall from the introduction of this section that $\mathcal{B}_n^{\bar{q},\Lambda}$ is identified to the n-fold tensor product of algebras $\mathcal{B}_1^{q_1} \otimes \cdots \otimes \mathcal{B}_1^{q_n}$ and that, likewise, R° identifies to the tensor product of n copies of $\Bbbk[z_1^{\pm 1}]$. In the rest of this section, we will freely use these identifications.

Notice further that we are in position to apply Proposition 1.1.10.

Let $\phi \in \widehat{R^\circ}$. By Proposition 1.1.10, under the above identification, there is a unique n-tuple $(\phi_1, \ldots, \phi_n)$ of elements of $\widehat{\Bbbk[z_1^{\pm 1}]}$ such that $\phi = \phi_1 \cdots \phi_n$.

We are interested in two weight $\mathcal{B}_n^{\bar{q}, \Lambda}$-modules, namely S_ϕ and $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$.

Lemma 4.2.1. *Let $(\phi_1, \ldots, \phi_n)$ be a n-tuple of elements of $\widehat{\Bbbk[z_1^{\pm 1}]}$. The $\mathcal{B}_n^{\bar{q}, \Lambda}$-module $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ is simple.*

Proof. Since $\bar{q}$ is generic, according to Proposition 3.3.1, for any i, $1 \leq i \leq n$, the weight spaces of the $\mathcal{B}_1^{q_i}$-module P_{ϕ_i} are at most one dimensional $\Bbbk$-subspaces; hence, the same holds for the quotient S_{ϕ_i}. Hence, the statement follows from Lemma 1.5.4. $\qquad\square$

Theorem 4.2.2. *Suppose Λ is the skew-symmetric $n \times n$-matrix with all entries equal to 1 and suppose $\bar{q}$ is generic.*

1) *For all n-tuple $(\phi_1, \ldots, \phi_n)$ of elements of $\widehat{\Bbbk[z_1^{\pm 1}]}$, the $\mathcal{B}_n^{\bar{q}, \Lambda}$-module $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ is a simple weight module and any simple weight $\mathcal{B}_n^{\bar{q}, \Lambda}$-module is of that form.*

2) *Let $(\phi_1, \ldots, \phi_n)$ and $(\psi_1, \ldots, \psi_n)$ be elements of $\left(\widehat{\Bbbk[z_1^{\pm 1}]} \right)^n$. The $\mathcal{B}_n^{\bar{q}, \Lambda}$-modules $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ and $S_{\psi_1} \otimes \cdots \otimes S_{\psi_n}$ are isomorphic as weight modules if and only if, for all $1 \leq i \leq n$, S_{ϕ_i} and S_{ψ_i} are isomorphic in $\mathcal{B}_1^{q_i} - \mathsf{wMod}$.*

Proof. The proof of the first item follows from Lemma 4.2.1, Proposition 1.4.3, and Corollary 1.5.6. We now prove the second statement. The *if* part of the equivalence is evident. Suppose now that $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ and $S_{\psi_1} \otimes \cdots \otimes S_{\psi_n}$ are isomorphic. Given i such that $1 \leq i \leq n$, let $w(S_{\phi_i})$ and $w(S_{\psi_i})$ be the sets of weights of S_{ϕ_i} and S_{ψ_i}, respectively. It is clear that the set of weights of $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ is the image under the bijective map $\left(\widehat{\Bbbk[z_1^{\pm 1}]} \right)^n \longrightarrow \widehat{R^\circ}, (\chi_1, \ldots, \chi_n) \mapsto \chi_1 \cdots \chi_n$ — as defined in the second item of Proposition 1.1.10 — of the subset $w(S_{\phi_1}) \times \cdots \times w(S_{\phi_1})$. Of course, the same holds for $S_{\psi_1} \otimes \cdots \otimes S_{\psi_n}$. Hence, these images must be equal since $S_{\phi_1} \otimes \cdots \otimes S_{\phi_n}$ and $S_{\psi_1} \otimes \cdots \otimes S_{\psi_n}$ are isomorphic. But that latter map is bijective. So, for $1 \leq i \leq n$, $w(S_{\phi_i}) = w(S_{\psi_i})$. Corollary 4.1.6 yields the desired conclusion. $\qquad\square$

4.3. The case $n = 2$

In this short section we concentrate on the case $n = 2$. Our aim is to illustrate how the case of an arbitrary n can be understood on the basis of the case $n = 1$. We do this by using graphs attached to the modules under consideration, following the approach of Section 4.1. The action of x_1 is displayed horizontaly from left to right, the action of y_1 is displayed horizontally from right to left, the action of x_2 is displayed vertically from bottom to top and the action of y_2 is displayed vertically from top to bottom.

Example 4.3.1. Let $\phi \in \Bbbk[\widehat{z_1^{\pm 1}, z_2^{\pm 1}}]$ and suppose $\mathrm{comp}(\phi) = \emptyset$.

In this case, the action of x_1, y_1, x_2, y_2 can be pictured as follows:
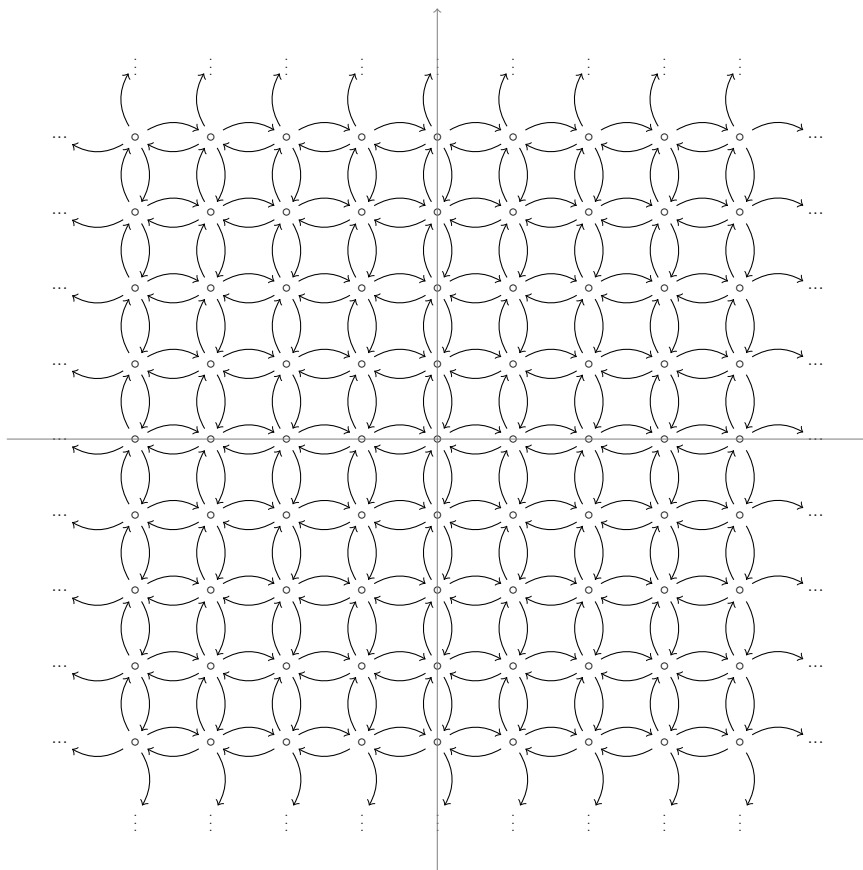
It is clear that in this graph, there is a path going from any vertex to the vertex v_0. It follows that any non-trivial submodule of P_ϕ must contain v_0 and hence must coincide with P_ϕ. So, P_ϕ is simple.

Example 4.3.2. Let $\phi \in \Bbbk[\widehat{z_1^{\pm 1}, z_2^{\pm 1}}]$ and suppose $\mathrm{comp}(\phi) = \{1\}$. In this case, there exists an integer α_1 such that $\phi(z_1) = q_1^{\alpha_1}$.

Let us suppose that $\alpha_1 \in \mathbb{N}$. The action of x_1, y_1, x_2, y_2 can be pictured as follows:
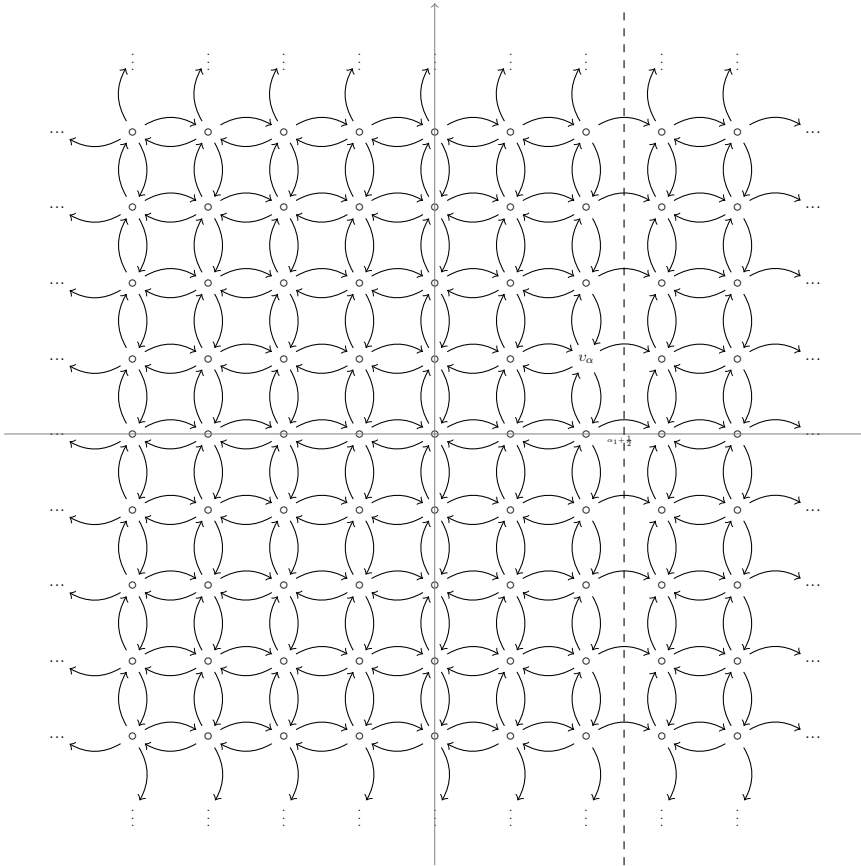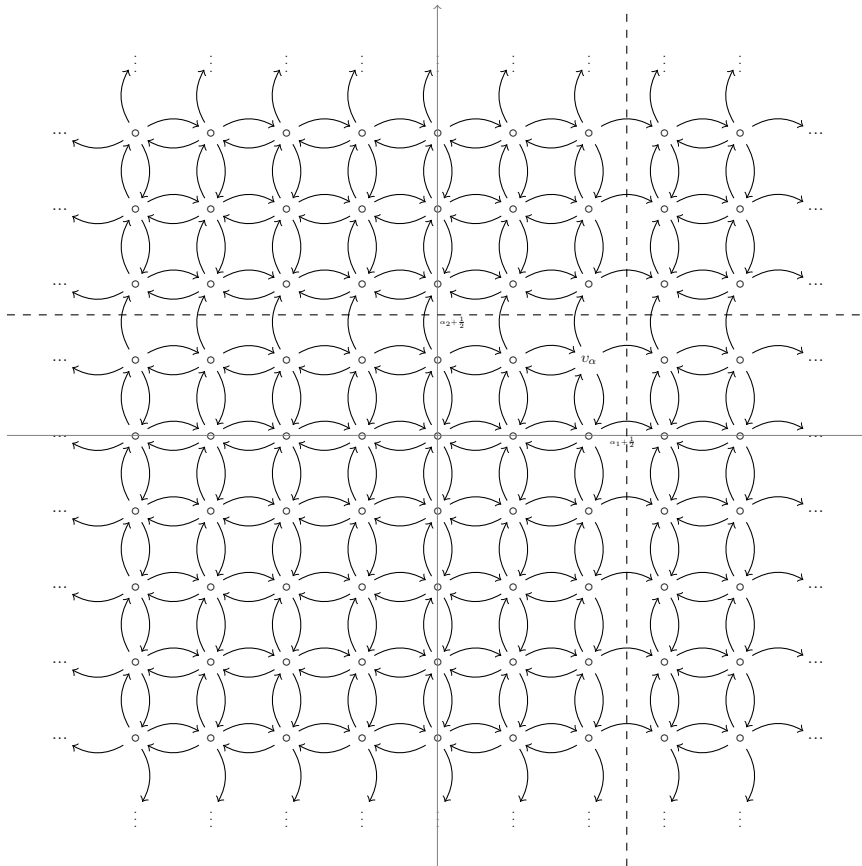

The graph above is clearly divided into two parts, delimited by the vertical line intersecting the x-axis at $x = \alpha_1 + 1/2$. From any vertex located at the left hand side of this line, there is a path to v_0. Hence, any submodule containing one such vertex must coincide with P_ϕ. In addition, the $\Bbbk$-vector space with basis the vertices located to the right of this line is a proper submodule of P_ϕ. It follows that this subspace must be maximum among non-trivial submodules. That is, we have:

$$N_\phi = \bigoplus_{\{k \mid k_1 > \alpha_1\}} \Bbbk.v_k.$$

We let the interested reader deal with the case where $\alpha_1 < 0$. Clearly, the case where $\mathrm{comp}(\phi) = \{2\}$ is similar, interchanging horizontal and vertical directions.

Example 4.3.3. Let $\phi \in \mathbb{k}[\widehat{z_1^{\pm 1}, z_2^{\pm 1}}]$ and suppose $\mathrm{comp}(\phi) = \{1, 2\}$. In this case, there exist integers α_1 and α_2 such that $\phi(z_1) = q_1^{\alpha_1}$, and $\phi(z_2) = q_2^{\alpha_2}$.

Let us suppose that $\alpha_1, \alpha_2 \in \mathbb{N}$. Then, the action of x_1, y_1, x_2, y_2 can be pictured as follows, where $\alpha = (\alpha_1, \alpha_2)$:

In this case, the graph is again divided into two parts, delimited by the vertical line intersecting the x-axis at $x = \alpha_1 + 1/2$ and the horizontal line intersecting the y-axis at $y = \alpha_2 + 1/2$.

From any vertex located at the left hand side of the vertical line and below the horizontal one, there is a path to v_0. Hence, any submodule containing one such vertex must coincide with P_ϕ. In addition, the $\mathbb{k}$-vector

space with basis the vertices located either to the right of the vertical line or above the horizontal one is a submodule of P_ϕ. It follows that this subspace must be maximum among non-trivial submodules. That is, we have:

$$N_\phi = \bigoplus_{\{k|k_1 > \alpha_1 \, or \, k_2 > \alpha_2\}} \Bbbk.v_k.$$

5. Quantum Weyl algebras and Zhang twists

The aim of this section is to show how weight modules of the extension $(R^\circ, \mathcal{B}_n^{\bar{q},\Lambda})$ may be described using their analogues of the special case where Λ has all its entries equal to 1. The method that we use to reduce the general case to this special case is based on the notion of Zhang twist as introduced in [Z].

We first recall this notion, then show that $\mathcal{B}_n^{\bar{q},\Lambda}$ is indeed a Zhang twist of $\mathcal{B}_n^{\bar{q},(1)}$ and finally establish that all simple weight modules over $\mathcal{B}_n^{\bar{q},\Lambda}$ are in fact obtained as Zhang twists of simple weight modules over $\mathcal{B}_n^{\bar{q},(1)}$.

For all this section, we fix $\bar{q} = (q_1, \ldots, q_n) \in (\Bbbk^*)^n$ and a skew-symmetric $n \times n$ matrix $\Lambda = (\lambda_{ij})$. It is clear from the definition of $\mathcal{A}_n^{\bar{q},\Lambda}$ that it is $\mathbb{Z}^n$-graded by

$$\deg(x_i) = e_i \qquad \text{and} \qquad \deg(y_i) = -e_i.$$

From Proposition 2.2.6, we get that the homogeneous part of degree 0 with respect to this grading is reduced to

$$R = \Bbbk[z_1, \ldots, z_n];$$

the subalgebra of $\mathcal{A}_n^{\bar{q},\Lambda}$ generated by z_i, $1 \leq i \leq n$, the latter being algebraically independent.

5.1. Some $\Bbbk$-algebra automorphisms of $\mathcal{A}_n^{\bar{q},(1)}$

Denote by (1) the $n \times n$ skew-symmetric matrix with all entries equal to 1. It is straightforward to check that, for $1 \leq i \leq n$, there are $\Bbbk$-algebra

automorphisms τ_i of $\mathcal{A}_n^{\bar{q},(1)}$ such that

$$
\begin{array}{rccl}
\tau_i & : & \mathcal{A}_n^{\bar{q},(1)} & \longrightarrow \quad \mathcal{A}_n^{\bar{q},(1)} \\[2mm]
& & x_j & \mapsto \quad \begin{cases} x_j & \text{whenever } 1 \le j \le i, \\ \lambda_{ij}^{-1} x_j & \text{whenever } i < j \le n. \end{cases} \\[4mm]
& & y_j & \mapsto \quad \begin{cases} y_j & \text{whenever } 1 \le j \le i, \\ \lambda_{ij} y_j & \text{whenever } i < j \le n. \end{cases}
\end{array}
$$

Clearly, the above automorphisms pairwise commute. Hence, there is a morphism of groups as follows

$$
\begin{array}{ccc}
\mathbb{Z}^n & \longrightarrow & \mathrm{Aut}_{\Bbbk}\left(\mathcal{A}_n^{\bar{q},(1)} \right) \\[2mm]
e_i & \mapsto & \tau_i
\end{array}
$$

It is also clear that the automorphisms τ_i are homogeneous of degree 0 with respect to the $\mathbb{Z}^n$-grading of $\mathcal{A}_n^{\bar{q},(1)}$ defined in this section.

From now on, given $g \in \mathbb{Z}^n$, we let τ_g denote the $\Bbbk$-algebra automorphism — $\mathbb{Z}^n$-homogeneous of degree 0 — that is the image of g under the above group morphism. More precisely, if $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$, we have that

$$
\tau_g = \tau_1^{g_1} \cdots \tau_n^{g_n}.
$$

5.2. Twisting of $\mathcal{A}_n^{\bar{q},(1)}$

This subsection relies on Zhang twists as defined in [Z]. However, we are interested in left modules rather than in right ones. So, the convenient context for us is that of [RZ]; we will follow it. We will thus consider left twists (see [RZ, Def. 1.2.1]).

Clearly, $\tau = (\tau_g)_{g \in \mathbb{Z}^n}$ is a normalised left twisting system of $\mathcal{A}_n^{\bar{q},(1)}$ in the sense of [RZ]. Thus, we may associate to τ and $\mathcal{A}_n^{\bar{q},(1)}$ the graded $\Bbbk$-algebra $^{\tau}\left(\mathcal{A}_n^{\bar{q},(1)} \right)$. As a graded $\Bbbk$-vector space, $^{\tau}\left(\mathcal{A}_n^{\bar{q},(1)} \right) = \mathcal{A}_n^{\bar{q},(1)}$, but the algebra structure is given by a new associative product, that we denote $*$, which is defined by:

$$
y * z = \tau_g(y) z
$$

whenever z is an homogeneous element of degree g of the vector space $\mathcal{A}_n^{\bar{q},(1)}$ and y is any element of $\mathcal{A}_n^{\bar{q},(1)}$. This new product has the same unit as the original one and $\left(^{\tau}\left(\mathcal{A}_n^{\bar{q},(1)} \right), * \right)$ is a $\mathbb{Z}^n$-graded $\Bbbk$-algebra with respect to the original grading.

At this stage, we have the following result.

Theorem 5.2.1. *There is a $\Bbbk$-algebra isomorphism $\mathcal{A}_n^{\bar{q},\Lambda} \longrightarrow {}^\tau \left(\mathcal{A}_n^{\bar{q},(1)} \right)$ such that $x_i \mapsto x_i$ and $y_i \mapsto y_i$.*

Proof. We first check that there is a $\Bbbk$-algebra homomorphism as desired. Let $1 \leq i < j \leq n$. We have that:

$$x_i * x_j - \lambda_{ij} x_j * x_i = \tau_j(x_i)x_j - \lambda_{ij}\tau_i(x_j)x_i$$
$$= x_i x_j - \lambda_{ij}\lambda_{ji} x_j x_i = x_i x_j - x_j x_i = 0;$$

$$y_i * y_j - \lambda_{ij} y_j * y_i = \tau_j^{-1}(y_i)y_j - \lambda_{ij}\tau_i^{-1}(y_j)y_i$$
$$= y_i y_j - \lambda_{ij}\lambda_{ji} y_j y_i = y_i y_j - y_j y_i = 0;$$

$$x_i * y_j - \lambda_{ji} y_j * x_i = \tau_j^{-1}(x_i)y_j - \lambda_{ji}\tau_i(y_j)x_i$$
$$= x_i y_j - \lambda_{ji}\lambda_{ij} y_j x_i = x_i y_j - y_j x_i = 0;$$

$$y_i * x_j - \lambda_{ji} x_j * y_i = \tau_j(y_i)x_j - \lambda_{ji}\tau_i^{-1}(x_j)y_i$$
$$= y_i x_j - \lambda_{ji}\lambda_{ij} x_j y_i = y_i x_j - x_j y_i = 0.$$

Now, given $1 \leq i \leq n$:

$$x_i * y_i - q_i y_i * x_i - 1 = \tau_i^{-1}(x_i)y_i - q_i \tau_i(y_i)x_i - 1 = x_i y_i - q_i y_i x_i - 1 = 0.$$

This establishes the existence of such a $\Bbbk$-algebra homomorphism. Note that this homomorphism sends the PBW basis of the $\Bbbk$-vector space $\mathcal{A}_n^{\bar{q},\Lambda}$ to the PBW basis of the $\Bbbk$-vector space $\mathcal{A}_n^{\bar{q},(1)}$. As a consequence, it is an isomorphism. $\square$

5.3. Twisting of $\mathcal{B}_n^{\bar{q},(1)}$.

In this subsection, we extend the results of Subsection 5.2 to the localisation $\mathcal{B}_n^{\bar{q},(1)}$ of $\mathcal{A}_n^{\bar{q},(1)}$. Let $1 \leq i \leq n$. In the notation of paragraph 5.1, it is clear that $\tau_i(z_j) = z_j$, for all $1 \leq j \leq n$. It follows that the automorphism τ_i extends to an automorphism — still denoted τ_i — of $\mathcal{B}_n^{\bar{q},(1)}$. Hence, we get a morphism of groups

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathrm{Aut}_\Bbbk \left(\mathcal{B}_n^{\bar{q},(1)} \right) \\ e_i & \mapsto & \tau_i. \end{array}$$

The same construction as in Subsection 5.2 leads to the following result.

Theorem 5.3.1. *There is a $\Bbbk$-algebra isomorphism $\mathcal{B}_n^{\bar{q},\Lambda} \longrightarrow {}^\tau\left(\mathcal{B}_n^{\bar{q},(1)}\right)$ such that $x_i \mapsto x_i$ and $y_i \mapsto y_i$.*

5.4. The $\mathbb{Z}^n$-grading of P_ϕ

Let $\phi \in \Bbbk[\widehat{z_1^{\pm 1}, \ldots, z_n^{\pm 1}}]$.

Recall from Subsection 3.3 that P_ϕ has a $\Bbbk$-basis $\{v_k,\ k \in \mathbb{Z}^n\}$. Consider the $\mathbb{Z}^n$-grading of the $\Bbbk$-vector space P_ϕ whose k-th component is $\Bbbk v_k$. We know, after Lemma 3.3.4, that this grading is also a grading of the $\mathcal{B}_n^{\bar{q},\Lambda}$-module P_ϕ. Hence, we can define the twist ${}^\tau(P_\phi)$ of the $\mathcal{B}_n^{\bar{q},(1)}$-module P_ϕ. See [RZ] for the definition of the twist of a module. It is worth mentioning at this point the following general fact: for any $\mathbb{Z}^n$-graded $\mathcal{B}_n^{\bar{q},(1)}$-module M, M and ${}^\tau M$ have the same lattice of $\mathbb{Z}^n$-graded submodules. Using Theorem 5.2.1, we get that ${}^\tau(P_\phi)$ is naturally endowed with a structure of $\mathcal{B}_n^{\bar{q},\Lambda}$-module, by restriction of scalars.

Now, suppose that $\bar{q}$ is generic. By Prop. 3.3.1, the weight decomposition of P_ϕ is

$$P_\phi = \bigoplus_{\mathbb{k} \in \mathbb{Z}^n} \Bbbk.v_{\mathbb{k}}$$

and each line $\Bbbk.v_k$ is the weight space of P_ϕ of weight $\phi \circ \sigma_k$. That is: the weight decomposition of P_ϕ and its $\mathbb{Z}^n$-grading coincide. On the other hand, any submodule of P_ϕ is a weight submodule — cf. Proposition 1.1.9. It follows that any submodule or quotient of P_ϕ is $\mathbb{Z}^n$-homogeneous. As a consequence, the quotient S_ϕ is again $\mathbb{Z}^n$-graded and we can define its twist ${}^\tau(S_\phi)$. Using Theorem 5.2.1, we get that ${}^\tau(S_\phi)$ is naturally endowed with a structure of $\mathcal{B}_n^{\bar{q},\Lambda}$-module, by restriction of scalars.

5.5. An isomorphism of $\mathcal{B}_n^{\bar{q},\Lambda}$-modules

In this subsection we want to consider simultaneously $\mathcal{B}_n^{\bar{q},(1)}$-modules and $\mathcal{B}_n^{\bar{q},\Lambda}$-modules. Hence, to avoid any ambiguity, we introduce the following notation. For all $\phi \in \Bbbk[\widehat{z_1^{\pm 1}, \ldots, z_n^{\pm 1}}] = R^\circ$, let

$$P_\phi^{(1)} = \mathcal{B}_n^{\bar{q},(1)} \otimes_{R^\circ} \Bbbk_\phi \qquad \text{and} \qquad P_\phi^\Lambda = \mathcal{B}_n^{\bar{q},\Lambda} \otimes_{R^\circ} \Bbbk_\phi.$$

Likewise, we denote by $\{v_{\mathbb{k}}^{(1)},\ k \in \mathbb{Z}^n\}$ the $\Bbbk$-basis of $P_\phi^{(1)}$ introduced in Subsection 3.3 and by $(v_{\mathbb{k}}^\Lambda,\ k \in \mathbb{Z}^n)$ the corresponding $\Bbbk$-basis of P_ϕ^Λ. Let us

consider the $\Bbbk$-vector space morphism ι_ϕ such that

$$
\begin{array}{rcl}
\iota_\phi \;\; : \;\; {}^\tau\left(P_\phi^{(1)}\right) & \longrightarrow & P_\phi^\Lambda \\
v_\Bbbk^{(1)} & \mapsto & v_\Bbbk^\Lambda.
\end{array}
$$

A direct application of Lemma 3.3.4 gives the following result.

Theorem 5.5.1. *The map ι_ϕ is an isomorphism of $\mathcal{B}_n^{\bar{q},\Lambda}$-modules.*

Suppose now that $\bar{q}$ is generic. Analogously, we denote by S_ϕ^Λ the unique simple quotient of P_ϕ^Λ and by $S_\phi^{(1)}$ the unique simple quotient of $P_\phi^{(1)}$. Using the previous theorem, we have the following composition of $\mathcal{B}_n^{\bar{q},\Lambda}$-linear maps

$$
{}^\tau\left(P_\phi^{(1)}\right) \xrightarrow{\;\iota_\phi\;} P_\phi^\Lambda \longrightarrow S_\phi^\Lambda,
$$

where the second map is the canonical projection. This composition is surjective, hence its kernel is a maximal strict $\mathcal{B}_n^{\bar{q},\Lambda}$-submodule of ${}^\tau\left(P_\phi^{(1)}\right)$. But the lattice of submodules of the ${}^\tau\left(\mathcal{B}_n^{\bar{q},(1)}\right)$-module ${}^\tau\left(P_\phi^{(1)}\right)$ coincides with the lattice of submodules of the $\mathcal{B}_n^{\bar{q},(1)}$-module $P_\phi^{(1)}$. Hence, the kernel of the above map is $N_\phi^{(1)}$ — using an obvious notation. As a consequence, we get that S_ϕ^Λ is isomorphic to ${}^\tau\left(P_\phi^{(1)}\right)/N_\phi^{(1)}$ as a $\mathcal{B}_n^{\bar{q},\Lambda}$-module. That is, we have proved the following theorem.

Theorem 5.5.2. *Suppose that $\bar{q}$ is generic. There is an isomorphism of $\mathcal{B}_n^{\bar{q},\Lambda}$-modules between S_ϕ^Λ and ${}^\tau\left(S_\phi^{(1)}\right)$.*

We finish this section with an easy but useful observation.

Remark 5.5.3. Let M be a $\mathbb{Z}^n$-graded $\mathcal{B}_n^{\bar{q},(1)}$-module. Then, for all $\phi \in \widehat{\Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]}$, we have an equality

$$
({}^\tau M)(\phi) = M(\phi).
$$

This is obvious since the automorphisms of the twisting system τ all act trivially on $z_1, \ldots, z_n$.

It follows that the set of weights of ${}^\tau M$ coincides with the set of weights of M.

6. Classification of simple weight modules of $\mathcal{B}_n^{\bar{q},\Lambda}$ in the generic case

In this final section, we collect the results of the previous ones to give an explicit description and a classification of the simple weight modules of the extension $(R^\circ, \mathcal{B}_n^{\bar{q},\Lambda})$ when $\bar{q}$ is generic.

We also show that the natural representation of $\mathcal{B}_n^{\bar{q},\Lambda}$ by skew differential operators on the convenient quantum affine space is a simple weight module.

For all this section, we fix $\bar{q} = (q_1, \ldots, q_n) \in (\mathbb{k}^*)^n$ that we assume generic and a skew-symmetric matrix $\Lambda = (\lambda_{ij})$ with entries in $\mathbb{k}^*$. We let τ be the normalised left twisting system associated to Λ, as defined in Sections 5.2 and 5.3.

As proven in Section 5.3, there is an algebra isomorphism $\mathcal{B}_n^{\bar{q},\Lambda} \longrightarrow {}^\tau\left(\mathcal{B}_n^{\bar{q},(1)}\right)$ such that $x_i \mapsto x_i$ and $y_i \mapsto y_i$. Notice that, under the above isomorphism, the subalgebra of $\mathcal{B}_n^{\bar{q},\Lambda}$ generated by $z_1, \ldots, z_n$ and their inverses is in one-to-one correspondence with the subalgebra of ${}^\tau\left(\mathcal{B}_n^{\bar{q},(1)}\right)$ generated by $z_1, \ldots, z_n$ and their inverses. Further, the latter subalgebra is in one-to-one correspondence with the subalgebra of $\mathcal{B}_n^{\bar{q},(1)}$ generated by $z_1, \ldots, z_n$ by means of the identity map. Both subalgebras are Laurent polynomial algebras in $z_1, \ldots, z_n$. By abuse of notation, we all denote them by R°.

We keep the notation of Section 5.5.

Construction of simple weight modules. Let S be a simple weight module of $\mathcal{B}_n^{\bar{q},\Lambda}$.

By Proposition 1.4.3, there exists $\phi \in \widehat{R^\circ}$ such that

$$S \cong S_\phi^\Lambda.$$

On the other hand, we deduce from Section 5.5 that

$$S_\phi^\Lambda \cong {}^\tau\left(S_\phi^{(1)}\right)$$

and S_ϕ^Λ and $S_\phi^{(1)}$ have the same set of weights.

Further, by Theorem 4.2.2 (see also Corollary 1.5.6), if $(\phi_1, \ldots, \phi_n)$ is the unique n-tuple of elements of $\mathbb{k}[z_1^{\pm 1}]$ such that $\phi = \phi_1 \cdots \phi_n$, then

$$S_\phi^{(1)} \cong S_{\phi_1}^{(1)} \otimes \cdots \otimes S_{\phi_n}^{(1)}.$$

It is clear, in addition, that the set of weights of $S_{\phi_1}^{(1)} \otimes \cdots \otimes S_{\phi_n}^{(1)}$ is the image under the map of Proposition 1.1.10 (2) of $w\left(S_{\phi_1}^{(1)}\right) \times \cdots \times w\left(S_{\phi_n}^{(1)}\right)$, where, for $1 \le i \le n$, $w\left(S_{\phi_i}^{(1)}\right)$ denotes the set of weights of $S_{\phi_i}^{(1)}$. These last weight spaces are described in Examples 4.1.2 and 4.1.3 according to whether ϕ_i is or not in the $\mathbb{Z}$-orbit of $\mathbf{1}$.

Denote by $\mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{wSimple}$ the set of simple weight $\mathcal{B}_n^{\bar{q},\Lambda}$-modules — note that the previous discussion shows this collection is actually a set — and by $\sim$ the equivalence relation on this set defined by isomorphism. We deduce from the arguments above that there is a surjective map

$$\kappa_n \; : \; \widehat{R^\circ} \longrightarrow \mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{wSimple} \longrightarrow (\mathcal{B}_n^{\bar{q},\Lambda} - \mathsf{wSimple})/\sim$$

that sends any $\phi \in \widehat{R^\circ}$ to the isomorphism class of the $\mathcal{B}_n^{\bar{q},\Lambda}$-module $^\tau\left(S_\phi^{(1)}\right)$.

Isomorphisms between simple weight modules. Our final aim is to describe the fibers of the map κ_n.

Let $\phi, \psi \in \widehat{R^\circ}$. Since weight spaces and $\mathbb{Z}^n$-grading coincide for $S_\phi^{(1)}$ and $S_\psi^{(1)}$, the $\mathcal{B}_n^{\bar{q},(1)}$-modules $S_\phi^{(1)}$ and $S_\psi^{(1)}$ are isomorphic if and only if the $\mathcal{B}_n^{\bar{q},\Lambda}$-modules $^\tau\left(S_\phi^{(1)}\right)$ and $^\tau\left(S_\psi^{(1)}\right)$ are isomorphic. But, the former occur if and only if $S_\phi^{(1)}$ and $S_\psi^{(1)}$ have the same set of weights (see, in particular, Corollary 4.1.6 and Theorem 4.2.2). It follows from this that ϕ and ψ have the same image under κ_n if and only if the corresponding simple modules $S_\phi^{(1)}$ and $S_\psi^{(1)}$ have the same set of weights.

Example 6.1. The natural representation of $\mathcal{B}_n^{\bar{q},\Lambda}$. As already discussed, $\mathcal{B}_n^{\bar{q},\Lambda}$ may be seen as a ring of q-difference operators on quantum spaces. We now describe this representation and show it is a simple weight module. For this, we follow [R2, Sec. 4.1]. Let E_n^Λ be the $\Bbbk$-algebra generated by y_i, $1 \le i \le n$ subject to the relations $y_i y_j = \lambda_{ij} y_j y_i$, for all $1 \le i < j \le n$. This is a noetherian integral domain and we denote by F_n^Λ its skew-field of fractions. It is easy to check that, for $1 \le i \le n$, there is an automorphism of $\Bbbk$-algebra as follows

$$\begin{array}{rrcl} \xi_i & : & F_n^\Lambda & \longrightarrow & F_n^\Lambda \\ & & y_j & \mapsto & q_i^{\delta_{ij}} y_j \end{array}$$

and that this automorphism restrict to an automorphism of E_n^Λ. Further, for $1 \le i \le n$, we consider the following $\Bbbk$-vector space endomorphisms of F_n^Λ:

$$m_i \;:\; \begin{array}{ccc} F_n^\Lambda & \longrightarrow & F_n^\Lambda \\ f & \mapsto & y_i f \end{array} \quad \text{and} \quad \partial_i \;:\; \begin{array}{ccc} F_n^\Lambda & \longrightarrow & F_n^\Lambda \\ f & \mapsto & (q_i - 1) y_i^{-1} (\xi_i(f) - f) \end{array}$$

It is easy to see that, actually, the operators m_i and ∂_i stabilise E_n^Λ. It turns out that we have a $\Bbbk$-algebra morphism as follows:

$$\begin{array}{ccc} \mathcal{B}_n^{\bar{q},\Lambda} & \longrightarrow & \mathrm{End}_{\Bbbk}(F_n^\Lambda) \\ x_i & \mapsto & \partial_i \\ y_i & \mapsto & m_i \\ z_i & \mapsto & \xi_i \end{array}$$

(see [R2, Prop. 4.1.3]) which defines an action of $\mathcal{B}_n^{\bar{q},\Lambda}$ on F_n^Λ and, by restriction, an action of $\mathcal{B}_n^{\bar{q},\Lambda}$ on E_n^Λ.

For $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, let $y^k = y_1^{k_1} \cdots y_n^{k_n} \in E_n^\Lambda$. As it is well known, the family $\{y_k, k \in \mathbb{N}^n\}$ is a basis of the vector space E_n^Λ. Further, for $k \in \mathbb{N}^n$, it is easy to see that y^k is a weight vector of weight $-k\mathbf{1}$, in the notation of Section 3.2 and of the introduction to Section 4. It is also easy to verify that E_n^Λ is simple.

But, on the other hand, we know that the set of weights of $S_{\mathbf{1}}^\Lambda$ is $(-\mathbb{N})^n \mathbf{1}$. This shows that E_n^Λ and $S_{\mathbf{1}}^{(\Lambda)}$ are two simple weight modules of $\mathcal{B}_n^{\bar{q},\Lambda}$ with the same set of weights. Hence, they are isomorphic.

Acknowledgements

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