Math. Res. Lett.
Volume 28, Number 3, 863–887, 2021

Moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface

Anoop Singh

Let S be a finite subset of a compact connected Riemann surface X of genus $g \geq 2$. Let $\mathcal{M}_{lc}(n, d)$ denote the moduli space of pairs (E, D), where E is a holomorphic vector bundle over X and D is a logarithmic connection on E singular over S, with fixed residues in the centre of $\mathfrak{gl}(n, \mathbb{C})$, where n and d are mutually corprime. Let L denote a fixed line bundle with a logarithmic connection D_L singular over S. Let $\mathcal{M}'_{lc}(n, d)$ and $\mathcal{M}_{lc}(n, L)$ be the moduli spaces parametrising all pairs (E, D) such that underlying vector bundle E is stable and $(\bigwedge^n E, \tilde{D}) \cong (L, D_L)$ respectively. Let $\mathcal{M}'_{lc}(n, L) \subset \mathcal{M}_{lc}(n, L)$ be the Zariski open dense subset such that the underlying vector bundle is stable. We show that there is a natural compactification of $\mathcal{M}'_{lc}(n, d)$ and $\mathcal{M}'_{lc}(n, L)$ and compute their Picard groups. We also show that $\mathcal{M}'_{lc}(n, L)$ and hence $\mathcal{M}_{lc}(n, L)$ do not have any non-constant algebraic functions but they admit non-constant holomorphic functions. We also study the Picard group and algebraic functions on the moduli space of logarithmic connections singular over S, with arbitrary residues.

1. Introduction

Let X be a compact Riemann surface of genus$(X) = g \geq 2$. Fix a finite subset $S = \{x_1, \ldots, x_m\}$ of X such that $x_i \neq x_j$ for all $i \neq j$. Let E be a holomorphic vector bundle over X of rank $n \geq 1$ and degree d, where n and d are mutually coprime. For each $j = 1, \ldots, m$, fix $\lambda_j \in \mathbb{Q}$ such that $n\lambda_j \in \mathbb{Z}$. We consider pairs of the form (E, D), where D is a logarithmic connection in E singular over S with fixed residues $Res(D, x_j) = \lambda_j \mathbf{1}_{E(x_j)}$. For the construction of the moduli space of logarithmic connections, see [10], [18].

We put the condition on $\lambda_j \in \mathbb{Q}$, which is, $n\lambda_j \in \mathbb{Z}$, and on the residues $\lambda_j \mathbf{1}_{E(x_j)}$, which is,

$$d + n\sum_{j=1}^{m} \lambda_j = 0,$$

to ensure the following

1) The moduli space $\mathcal{M}_{lc}(n, d) \neq \emptyset$ [see [4], Proposition 1.2].
2) Under the monodromy representation $\rho : \pi_1(X_0, x_0) \to \mathrm{SL}(n, \mathbb{C})$, the image of the path homotopy class of the loop γ_j around $x_j \in S$ based at x_0, is a diagonal matrix with entries $\exp(-2\pi\sqrt{-1}\lambda_j)$ [see Section (3)].

In [5], the moduli space of rank n logarithmic connections singular exactly over one point has been considered and several properties, like algebraic functions, compactification and computation of Picard group have been studied. Also, the moduli space of rank one logarithmic connections singular over finitely many points with fixed residues has been considered in [15], [16], and it is proved that it has a natural symplectic structure and there are no non-constant algebraic functions on it.

In the present article, our aim is to study the algebraic functions and Picard group for the moduli space of rank n logarithmic connections singular over S with fixed residues and hence, we end up generalising several results in [5], and [17].

Let $\mathcal{U}(n, d)$ denote the moduli space of all stable vector bundles of rank n and degree d over X and $\mathcal{M}_{lc}(n, d)$ denote of the moduli space logarithmic connection (E, D) singular over S with fixed residues $Res(D, x_j) = \lambda_j \mathbf{1}_{E(x_j)}$ for all $j = 1, \ldots, m$, and $\mathcal{M}'_{lc}(n, d) \subset \mathcal{M}_{lc}(n, d)$ be the moduli space of logarithmic connection whose underlying vector bundle is stable. We show that there is a natural compactification of the moduli space $\mathcal{M}'_{lc}(n, d)$. More precisely, we prove the following (see Subsection 4.2 for the proof.)

Theorem 1.1. *There exists an algebraic vector bundle $\pi : \Xi \to \mathcal{U}(n, d)$ such that $\mathcal{M}'_{lc}(n, d)$ is embedded in $\mathbf{P}(\Xi)$ with $\mathbf{P}(\Xi) \setminus \mathcal{M}'_{lc}(n, d)$ as the hyperplane at infinity.*

Let $\mathcal{U}(n, d)$ denote the moduli space of all stable vector bundles over X. Then $\mathcal{U}(n, d)$ is an irreducible smooth complex projective variety of dimension $n^2(g - 1) + 1$ [see [13]].

Now, we have the natural homomorphism

$$p : \mathcal{M}'_{lc}(n, d) \to \mathcal{U}(n, d) \tag{1.1}$$

sending (E, D) to E. The morphism p induces a homomorphism

$$p^* : \mathrm{Pic}(\mathcal{U}(n,d)) \to \mathrm{Pic}(\mathcal{M}'_{lc}(n,d)) \tag{1.2}$$

of Picard groups, that sends an algebraic line bundle ξ over $\mathcal{U}(n,d)$ to an algebraic line bundle $p^*\xi$ over $\mathcal{M}'_{lc}(n,d)$. Here, $\mathrm{Pic}(Y)$ consists of algebraic line bundles over Y, where Y is an algebraic variety over $\mathbb{C}$. we show the following (see Subsection 4.2 for the proof.)

Theorem 1.2. *The homomorphism $p^* : \mathrm{Pic}(\mathcal{U}(n,d)) \to \mathrm{Pic}(\mathcal{M}'_{lc}(n,d))$ is an isomorphism of groups.*

Now, fix a holomorphic line bundle L over X of degree d, and fix a logarithmic connection D_L on L singular over S with residues $Res(D_L, x_j) = n\lambda_j$ for all $j = 1, \ldots, m$. Let $\mathcal{M}_{lc}(n,L) \subset \mathcal{M}_{lc}(n,d)$ be the moduli space parametrising isomorphism class of pairs (E, D) such that $(\bigwedge^n E, \tilde{D}) \cong (L, D_L)$, where $\tilde{D}$ is the logarithmic connection on $\bigwedge^n E$ induced by D. Let $\mathcal{M}'_{lc}(n,L) = \mathcal{M}_{lc}(n,L) \cap \mathcal{M}'_{lc}(n,d)$ and $\mathcal{U}_L(n,d) \subset \mathcal{U}(n,d)$ be the moduli space of stable vector bundles with $\bigwedge^n E \cong L$. Similarly, we have a natural morphism

$$p_0 : \mathcal{M}'_{lc}(n,L) \to \mathcal{U}_L(n,d) \tag{1.3}$$

of varieties, that induces a homomorphism of Picard groups, and we have

Proposition 1.3. *The homomorphism $p_0^* : Pic(\mathcal{U}_L(n,d)) \to Pic(\mathcal{M}'_{lc}(n,L))$ defined by $\xi \mapsto p_0^*\xi$ is an isomorphism of groups.*

From [[13], Proposition 3.4], we have $\mathrm{Pic}(\mathcal{U}_L(n,d)) \cong \mathbb{Z}$. Let Θ be the ample generator of the group $\mathrm{Pic}(\mathcal{U}_L(n,d))$. Then we have the Atiyah exact sequence (see [1]) associated to the line bundle Θ over $\mathcal{U}_L(n,d)$,

$$0 \to \mathcal{O}_{\mathcal{U}_L(n,d)} \xrightarrow{\imath} \mathrm{At}(\Theta) \xrightarrow{\sigma} T\mathcal{U}_L(n,d) \to 0, \tag{1.4}$$

where $\mathrm{At}(\Theta)$ is called **Atiyah algebra** of the holomorphic line bundle Θ. Let $\mathcal{C}(\Theta) \subset \mathrm{At}(\Theta)^*$ be the fibre bundle over $\mathcal{U}_L(n,d)$ such that for every $U \subset \mathcal{U}_L(n,d)$ a holomorphic section of $\mathcal{C}(\Theta)|_U$ gives a holomorphic splitting of (1.4). Then the two $\Omega^1_{\mathcal{U}_L(n,d)}$-*torsors* $\mathcal{C}(\Theta)$ and $\mathcal{M}'_{lc}(n,L)$ on $\mathcal{U}_L(n,d)$ are isomorphic. Finally, we show that there is no non-constant algebraic function on $\mathcal{M}'_{lc}(n,L)$, by showing the following (see Section (5) for the proof)

Theorem 1.4. *Assume that genus*$(X) \geq 3$*. Then*

$$\mathrm{H}^0(\mathcal{C}(\Theta), \mathcal{O}_{\mathcal{C}(\Theta)}) = \mathbb{C}. \tag{1.5}$$

Since $\mathcal{M}'_{lc}(n, L) \subset \mathcal{M}_{lc}(n, L)$ is an open dense subset, $\mathcal{M}_{lc}(n, L)$ does not have any non-constant algebraic function.

In the last Section (6), we consider the moduli spaces $\mathcal{N}_{lc}(n, d)$ and $\mathcal{N}'_{lc}(n, d)$ of logarithmic connections with arbitrary residues and show the similar results as Theorem 1.1 and Theorem 1.2, see Theorem 6.1 and Theorem 6.2. Consider the moduli spaces $\mathcal{N}_{lc}(n, L)$ and $\mathcal{N}'_{lc}(n, L)$ of logarithmic connections singular over S with arbitrary residues in the centre of $\mathfrak{gl}(n, \mathbb{C})$, and let

$$V = \left\{ (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m \;\middle|\; n\alpha_j \in \mathbb{Z} \;\text{ and }\; d + n\sum_{j=1}^{m} \alpha_j = 0 \right\}$$

Define a map

$$\Phi : \mathcal{N}'_{lc}(n, L) \to V \tag{1.6}$$

by $(E, D) \mapsto (\mathrm{tr}(Res(D, x_1))/n, \dots, \mathrm{tr}(Res(D, x_m))/n)$. Then we show the following [see Section (6) for the proof].

Theorem 1.5. *Every algebraic function on* $\mathcal{N}'_{lc}(n, L)$ *factors through the surjective map* $\Phi : \mathcal{N}'_{lc}(n, L) \to V$ *as defined in* (1.6).

We also show the similar result for $\mathcal{N}_{lc}(n, L)$, see Theorem 6.5.

2. Preliminaries

We denote by $S = x_1 + \cdots + x_m$ the reduced effective divisor on X associated with the finite set S. Let $\Omega^1_X(\log S)$ denote the sheaf of logarithmic differential 1-forms along S, see [14]. The notion of logarithmic connection was introduced by P. Deligne in [8]. We recall the definition of logarithmic connection in a holomorphic vector bundle E over X singular over S and residue of the logarithmic connection on the points of S.

Let E be a holomorphic vector bundle on X of rank $n \geq 1$. We will denote the fibre of E over any point $x \in X$ by $E(x)$.

A logarithmic connection on E singular over S is a $\mathbb{C}$-linear map

$$D : E \to E \otimes \Omega^1_X(\log S) = E \otimes \Omega^1_X \otimes \mathcal{O}_X(S) \tag{2.1}$$

which satisfies the Leibniz identity

$$D(fs) = fD(s) + df \otimes s, \tag{2.2}$$

where f is a local section of $\mathcal{O}_X$ and s is a local section of E.

Let D be a logarithmic connection in E singular over S. For any $x_\beta \in S$, the fiber $\Omega^1_X \otimes \mathcal{O}_X(S)(x_\beta)$ is canonically identified with $\mathbb{C}$ by sending a meromorphic form to its residue at x_β.

Let $v \in E(x_\beta)$ be any vector in the fiber of E over x_β. Let U be an open set around x_β and $s : U \to E$ be a holomorphic section of E over U such that $s(x_\beta) = v$. Consider the following composition

$$\begin{aligned} \Gamma(U, E) &\to \Gamma(U, E \otimes \Omega^1_X \otimes \mathcal{O}_X(S)) \\ &\to E \otimes \Omega^1_X \otimes \mathcal{O}_X(S)(x_\beta) = E(x_\beta), \end{aligned} \tag{2.3}$$

where the equality is given because of the identification $\Omega^1_X \otimes \mathcal{O}_X(S)(x_\beta) = \mathbb{C}$.

Let t be a local coordinate at x_β on U such that $t(x_\beta) = 0$, that is, the coordinate system (U, t) is centered at x_β and suppose that $\sigma \in \Gamma(U, E)$ such that $\sigma(x_\beta) = 0$. Then $\sigma = t\sigma'$ for some $\sigma' \in \Gamma(U, E)$. Now,

$$\begin{aligned} D(\sigma) = D(t\sigma') &= tD(\sigma') + dt \otimes \sigma' \\ &= tD(\sigma') + t\left(\frac{dt}{t} \otimes \sigma'\right), \end{aligned}$$

and $D(\sigma)(x_\beta) = 0$. Thus, we have a well defined endomorphism, denoted by

$$Res(D, x_\beta) \in \mathrm{End}(E)(x_\beta) = \mathrm{End}(E(x_\beta)) \tag{2.4}$$

that sends v to $D(s)(x_\beta)$. This endomorphism $Res(D, x_\beta)$ is called the **residue** of the logarithmic connection D at the point $x_\beta \in S$ (see [8] for the details).

If D is a logarithmic connection in E singular over S and $\theta \in \mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$, then $D + \theta$ is also a logarithmic connection in E, singular over S. Also, we have

$$Res(D, x_\beta) = Res(D + \theta, x_\beta),$$

for every $x_\beta \in S$.

Conversely, if D and D' are two logarithmic connections on E singular over S with

$$Res(D, x_\beta) = Res(D', x_\beta), \tag{2.5}$$

then $D' = D + \theta$, where $\theta \in \mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$.

Thus, the space of all logarithmic connections D' on a given holomorphic vector bundle E singular over S, and satisfying (2.5) with D fixed, is an affine space for $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$.

For each $i = 1, \dots, m$, fix $\lambda_i \in \mathbb{Q}$ such that $n\lambda_i \in \mathbb{Z}$, where n is the rank of the vector bundle E. By a pair (E, D) over X, we mean that

1) E is a holomorphic vector bundle of degree d and rank n over X.
2) D is a logarithmic connection in E singular over S with residues $Res(D, x_i) = \lambda_i \mathbf{1}_{E(x_i)}$ for all $i = 1, \dots, m$.

Then from [12], Theorem 3, we have

$$d + n \sum_{j=1}^{m} \lambda_i = 0 \tag{2.6}$$

Lemma 2.1. *Let (E, D) be a logarithmic connection on X. Suppose that F is a holomorphic subbundle of E such that the restriction $D' = D|_F$ of D to F is a logarithmic connection in F singular over S. Then $Res(D', x_j) = \lambda_j \mathbf{1}_{F(x_j)}$ for all $j = 1, \dots, m$.*

Proof. Follows from the definition of residues. □

A logarithmic connection D in a holomorphic vector bundle E is called **irreducible** if for any holomorphic subbundle F of E with $D(F) \subset \Omega^1_X(\log S) \otimes F$, then either $F = E$ or $F = 0$.

Proposition 2.2. *Let (E, D) be a logarithmic connection on X. Suppose that n and d are mutually coprime. Then D is irreducible.*

Proof. Let $0 \neq F$ be a holomorphic subbundle of E of rank r invariant under D, that is, $D(F) \subset F \otimes \Omega^1_X(\log S)$. Set $D' = D|_F$. Then from Lemma 2.1,

$Res(D', x_i) = \lambda_i \mathbf{1}_{F(x_i)}$, and from [12] Theorem 3, we have

$$\text{degree}(F) + r\sum_{i=1}^{m} \lambda_i = 0. \tag{2.7}$$

From (2.6) and (2.7), we get that $\mu(F) = \mu(E)$. Since F is a subbundle of E, if rank of F is less than rank of E, we get that $n|d$, which is a contradiction. Thus $F = E$. □

3. The moduli space of logarithmic connections with fixed residues

We say two pairs (E, D) and (E', D') of rank n and degree d are isomorphic if there exists an isomorphism $\Phi : E \to E'$ such that the following diagram

$$\begin{array}{ccc} E & \longrightarrow & E \otimes \Omega^1_X(\log S) \\ \downarrow \Phi & & \downarrow \Phi\otimes \mathbf{1}_{\Omega^1_X(\log S)} \\ E' & \longrightarrow & E' \otimes \Omega^1_X(\log S) \end{array} \tag{3.1}$$

commutes.

Let $\mathcal{M}_{lc}(n, d)$ denote the moduli space which parametrizes the isomorphic class of pairs (E, D). Then $\mathcal{M}_{lc}(n, d)$ is a separated quasi-projective scheme over $\mathbb{C}$ [see [10], Theorem 3.5].

Henceforth, we will assume following conditions

1) d and n are mutually coprime.
2) for each $i = 1, \dots, m$, $\lambda_i \in \mathbb{Q}$ such that $n\lambda_i \in \mathbb{Z}$.
3) $d, n, \lambda_1, ..., \lambda_m$ satisfies following relation

$$d + n\sum_{i=1}^{m} \lambda_i = 0. \tag{3.2}$$

Under the above condtions, from the Proposition 2.2, every logarithmic connection (E, D) in $\mathcal{M}_{lc}(n, d)$ is irreducible. Since the singular points of $\mathcal{M}_{lc}(n, d)$ corresponds to reducible logarithmic connections (see second paragraph on p.n. 790 of [5]), the moduli space $\mathcal{M}_{lc}(n, d)$ is smooth.

The similar technique as in [19], Theorem 11.1, can be used to show that $\mathcal{M}_{lc}(n, d)$ is an irreducible variety. Thus, altogether $\mathcal{M}_{lc}(n, d)$ is an irreducible smooth quasi-projective variety over $\mathbb{C}$.

From [[11] Theorem 2.8(A)], ${\mathcal{M}'}_{lc}(n,d)$ is a Zariski open subset of $\mathcal{M}_{lc}(n,d)$. Since $\mathcal{M}_{lc}(n,d)$ irreducible, $\mathcal{M}'_{lc}(n,d)$ is dense.

Consider $\mathcal{M}_{lc}(n,L)$ as defined above. Then it is a closed subvariety of $\mathcal{M}_{lc}(n,d)$. Moreover, ${\mathcal{M}'}_{lc}(n,L) = \mathcal{M}_{lc}(n,L) \cap \mathcal{M}'_{lc}(n,d)$ is a Zariski open dense subset of $\mathcal{M}_{lc}(n,L)$.

In particular, if we take $L_0 = \otimes_{i=1}^m \mathcal{O}_X(-n\lambda_i x_i)$ and D_{L_0} the logarithmic connection defined by the de Rham differential, then D_{L_0} is singular over S with residues $Res(D_{L_0}, x_i) = n\lambda_i$ for all $i = 1, \dots, m$. For this pair (L_0, D_{L_0}) we denote the moduli spaces $\mathcal{M}_{lc}(n,L)$ and ${\mathcal{M}'}_{lc}(n,L)$ by $\mathcal{M}_{lc}(n,L_0)$ and ${\mathcal{M}'}_{lc}(n,L_0)$ respectively.

Let $X_0 = X \setminus S$ and $x_0 \in X_0$. Let U_j be a simply connected open set in $X_0 \cup \{x_j\}$ containing x_0 and x_j. Then $\pi_1(U_j \setminus \{x_j\}, x_0) \cong \mathbb{Z}$, where 1 corresponds to the anticlockwise loop around x_j. We have a natural group homomorphism

$$h_j : \pi_1(U_j \setminus \{x_j\}, x_0) \to \pi_1(X_0, x_0).$$

for all $j = 1, \dots, m$. Suppose that $h_j(1) = \gamma_j$ for all $j = 1, \dots, m$. Then $\pi_1(X_0, x_0)$ admits a presentation with $2g + m$ generators $a_1, b_1, \dots, a_g, b_g, \gamma_1, \dots, \gamma_m$ with relation $\Pi_{i=1}^g [a_i, b_i] \Pi_{j=1}^m \gamma_j = 1$.

Let $(E, D) \in \mathcal{M}_{lc}(n, L_0)$. Then D determines a holomorphic (flat) connection on the holomorphic vector bundle $E|_{X_0}$ restricted to X_0. Since $Res(D, x_j) = \lambda_j \mathbf{1}_{E(x_j)}$, for $j = 1, \dots, m$, the image of γ_j under the monodromy representation is the $n \times n$ diagonal matrix with $\exp(-2\pi\sqrt{-1}\lambda_j)$ (see [8], p. 79, Proposition 3.11). Let

$$\mathcal{R}_g \subset \mathrm{Hom}(\pi_1(X_0, x_0), \mathrm{SL}(n, \mathbb{C}))$$

denote the space of those representations

$$\rho : \pi_1(X_0, x_0) \to \mathrm{SL}(n, \mathbb{C})$$

such that

$$\rho(\gamma_j) = \exp(-2\pi\sqrt{-1}\lambda_j)\mathrm{I}_{n\times n}$$

for all $j = 1, \dots, m$, where $\mathrm{I}_{n\times n}$ denotes the $n \times n$ identity matrix. Since the logarithmic connection D is irreducible, any representation in $\mathcal{R}_g$ is irreducible.

Consider the action of $\mathrm{SL}(n,\mathbb{C})$ on $\mathcal{R}_g$ by conjugation, that is, for any $T \in \mathrm{SL}(n,\mathbb{C})$ and $\rho \in \mathcal{R}_g$ the action is defined by

$$\rho.T = T^{-1}\rho T.$$

Let

$$\mathcal{B}_g = \mathcal{R}_g/\mathrm{SL}(n,\mathbb{C})$$

be the quotient space for the conjugation action. The algebraic structure of $\mathcal{R}_g$ induces an algebraic structure on $\mathcal{B}_g$. In literature, $\mathcal{B}_g$ is known as **Betti moduli space** (for instance see [18], [19]) and it is an irreducible smooth quasi-projective variety over $\mathbb{C}$. Thus, we have a holomorphic map

$$\Phi : \mathcal{M}_{lc}(n, L_0) \to \mathcal{B}_g \tag{3.3}$$

sending (E, D) to the equivalence class of its monodromy representaion under the conjugation action of $\mathrm{SL}(n,\mathbb{C})$.

For the inverse map of Φ, let $\rho \in \mathcal{B}_g$. Let (E_ρ, ∇_ρ) be the flat holomorphic vector bundle over X_0 associated to ρ. Then E_ρ over X_0 extends to a holomorphic vector bundle $\overline{E_\rho}$ over X, and the connection ∇_ρ on E_ρ extends to a connection $\overline{\nabla_\rho}$ such that $(\overline{E_\rho}, \overline{\nabla_\rho}) \in \mathcal{M}_{lc}(n, L_0)$[See [3], p. 159, Theorem 4.4]. Thus, Φ is a biholomorphism.

4. The Picard group of moduli space of logarithmic connections

Let

$$p : \mathcal{M}'_{lc}(n, d) \to \mathcal{U}(n, d)$$

be the forgetful map which forgets its logarithmic structure as defined in (1.1).

Let $E \in \mathcal{U}(n, d)$. Then E is indecomposable. Since d, n satisfy equation (3.2), from [4], Proposition 1.2, E admits a logarithmic connection D singular over S, with residues $Res(D, x_j) = \lambda_j \mathbf{1}_{E(x_j)}$ for all $j = 1, \ldots, m$.

Thus, the pair (E, D) is in the moduli space $\mathcal{M}'_{lc}(n, d)$, and hence p is surjective.

4.1. Torsors

We recall the definition of torsors and will show that the map

$$p : \mathcal{M}'_{lc}(n, d) \to \mathcal{U}(n, d)$$

is a $\Omega^1_{\mathcal{U}(n,d)}$-*torsor* on $\mathcal{U}(n,d)$, where $\Omega^1_{\mathcal{U}(n,d)}$ denotes the holomorphic cotangent bundle over $\mathcal{U}(n,d)$.

Let M be a connected complex manifold. Let $\pi : \mathcal{V} \to M$, be a holomorphic vector bundle.

A $\mathcal{V}$-*torsor* on M is a holomorphic fiber bundle $p : Z \to M$, and holomorphic map from the fiber product

$$\varphi : Z \times_M \mathcal{V} \to Z$$

such that

1) $p \circ \varphi = p \circ p_Z$, where p_Z is the natural projection of $Z \times_M \mathcal{V}$ to Z,
2) the map $Z \times_M \mathcal{V} \to Z \times_M Z$ defined by $p_Z \times \varphi$ is an isomorphism,
3) $\varphi(\varphi(z, v), w) = \varphi(z, v + w)$.

Proposition 4.1. *The isomorphic classes of* $\mathcal{V}$*-torsors over* M *are parametrized by* $\mathrm{H}^1(M, \mathcal{V})$.

Proposition 4.2. *Let* $p : \mathcal{M}'_{lc}(n,d) \to \mathcal{U}(n,d)$ *be the map as defined in* (1.1). *Then* $\mathcal{M}'_{lc}(n,d)$ *is a* $\Omega^1_{\mathcal{U}(n,d)}$-torsor *on* $\mathcal{U}(n,d)$.

Proof. Let $E \in \mathcal{U}(n,d)$. Then $p^{-1}(E) \subset \mathcal{M}'_{lc}(n,d)$ is an affine space for $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$ and the fiber of the cotangent bundle $\pi : \Omega^1_{\mathcal{U}(n,d)} \to \mathcal{U}(n,d)$ at E is isomorphic to $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$, that is, $\Omega^1_{\mathcal{U}(n,d),E} \cong \mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$. There is a natural action of $\Omega^1_{\mathcal{U}(n,d),E}$ on $p^{-1}(E)$, that is,

$$\Omega^1_{\mathcal{U}(n,d),E} \times p^{-1}(E) \to p^{-1}(E)$$

sending (ω, D) to $\omega + D$. This action on the fibre is faithful and transitive. This action will induce a holomorphic map on the fibre product

$$\varphi : \Omega^1_{\mathcal{U}(n,d)} \times_{\mathcal{U}(n,d)} \mathcal{M}'_{lc}(n,d) \to \mathcal{M}'_{lc}(n,d), \tag{4.1}$$

which satisfies the above conditions in the definition of the torsor. □

4.2. The Picard group of moduli space of logarithmic connection

Remark 4.3. *Note that*

$$p : \mathcal{M}'_{lc}(n,d) \to \mathcal{U}(n,d)$$

as defined in (1.1) *is a fibre bundle (not a vector bundle) with fibre* $p^{-1}(E)$ *which is an affine space modelled over* $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$*. Moreover, from Proposition 4.2,* $\mathcal{M}'_{lc}(n,d)$ *is a* $\Omega^1_{\mathcal{U}(n,d)}$-torsor *on* $\mathcal{U}(n,d)$*. We know that the dual of an affine space (modelled over a vector space over* $\mathbb{C}$*) is a vector space over* $\mathbb{C}$*, in the same spirit, the dual of a torsor is a vector bundle. We use this fact to construct an algebraic vector bundle over* $\mathcal{U}(n,d)$*. For another construction of this algebraic vector bundle over* $\mathcal{U}(n,d)$*, see the Remark* 3.2 *and the third paragraph on the p.n.*792 *in [5].*

Proof of Theorem 1.1. For any $E \in \mathcal{U}(n,d)$, the fiber $p^{-1}(E)$ is an affine space modelled on $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$. The dual

$$p^{-1}(E)^{\vee} = \{\varphi : p^{-1}(E) \to \mathbb{C} \mid \varphi \text{ is an affine linear map}\}$$

is a vector space over $\mathbb{C}$.

Let

$$\pi : \Xi \to \mathcal{U}(n,d)$$

be the algebraic vector bundle such that for every Zariski open subset U of $\mathcal{U}(n,d)$, a section of Ξ over U is an algebraic function $f : p^{-1}(U) \to \mathbb{C}$ whose restriction to each fiber $p^{-1}(E)$, is an element of $p^{-1}(E)^{\vee}$. Thus, a fiber $\Xi(E) = \pi^{-1}(E)$ of Ξ at $E \in \mathcal{U}(n,d)$ is $p^{-1}(E)^{\vee}$. Let $(E,D) \in \mathcal{M}'_{lc}(n,d)$, and define a map

$$\Phi_{(E,D)} : p^{-1}(E)^{\vee} \to \mathbb{C},$$

by

$$\Phi_{(E,D)}(\varphi) = \varphi[(E,D)],$$

which is nothing but the evaluation map. Now, the kernel $\mathrm{Ker}(\Phi_{(E,D)})$ defines a hyperplane in $p^{-1}(E)^{\vee}$ denoted by $H_{(E,D)}$.

Let $\mathbf{P}(\Xi)$ be a projective bundle defined by hyperplanes in the fiber $p^{-1}(E)^{\vee}$, that is, we have

$$\tilde{\pi} : \mathbf{P}(\Xi) \to \mathcal{U}(n,d)$$

induced from π. Define a map

$$\iota : \mathcal{M}'_{lc}(n,d) \to \mathbf{P}(\Xi)$$

by sending (E,D) to the equivalence class of $H_{(E,D)}$, which is clearly an open embedding. Set $Y = \mathbf{P}(\Xi) \setminus \mathcal{M}'_{lc}(n,d)$. Then $\tilde{\pi}^{-1}(E) \cap Y$ is a projective hyperplane in $\tilde{\pi}^{-1}(E)$ for every $E \in \mathcal{U}(n,d)$, and hence Y is a hyperplane at infinity. This completes the proof. □

The techniques used in the proof of the following theorem are adopted from the proof of the Theorem 3.1 in [5].

Proof of Theorem 1.2. First we show that p^* in (1.2) is injective. Let $\xi \to \mathcal{U}(n,d)$ be an algebraic line bundle such that $p^*\xi$ is a trivial line bundle over $\mathcal{M}'_{lc}(n,d)$. Giving a trivialization of $p^*\xi$ is equivalent to giving a nowhere vanishing section of $p^*\xi$ over $\mathcal{M}'_{lc}(n,d)$. Fix $s \in \mathrm{H}^0(\mathcal{M}'_{lc}(n,d), p^*\xi)$ a nowhere vanishing section. Take any point $E \in \mathcal{U}(n,d)$. Then,

$$s|_{p^{-1}(E)} : p^{-1}(E) \to \xi(E)$$

is a nowhere vanishing map. Notice that $p^{-1}(E) \cong \mathbb{C}^N$ and $\xi(E) \cong \mathbb{C}$, where $N = n^2(g-1)+1$. Now, any nowhere vanishing algebraic function on an affine space $\mathbb{C}^N$ is a constant function, that is, $s|_{p^{-1}(E)}$ is a constant function and hence corresponds to a non-zero vector $\alpha_E \in \xi(E)$. Since s is constant on each fiber of p, the trivialization s of $p^*\xi$ descends to a trivialization of the line bundle ξ over $\mathcal{U}(n,d)$, and hence giving a nowhere vanishing section of ξ over $\mathcal{U}(n,d)$. Thus, ξ is a trivial line bundle over $\mathcal{U}(n,d)$. The surjectivity of p^* follows from the Theorem 1.1 and the fact that $\mathrm{Pic}(\mathbf{P}(\Xi)) \cong \tilde{\pi}^*\mathrm{Pic}(\mathcal{U}(n,d)) \oplus \mathbb{Z}\mathcal{O}_{\mathbf{P}(\Xi)}(1)$. □

5. Algebraic functions on the moduli space

Let $\Omega^1_{\mathcal{U}_L(n,d)}$ denote the holomorphic cotangent bundle on $\mathcal{U}_L(n,d)$. Then, we have following proposition.

Proposition 5.1. *Let $p_0 : \mathcal{M}'_{lc}(n,L) \to \mathcal{U}_L(n,d)$ be the map as defined in (1.3). Then $\mathcal{M}'_{lc}(n,L)$ is a $\Omega^1_{\mathcal{U}_L(n,d)}$-torsor on $\mathcal{U}_L(n,d)$.*

Proof. First note that for any $E \in \mathcal{U}_L(n,d)$, the holomorphic cotangent space
$\Omega^1_{\mathcal{U}_L(n,d),E}$ at E is isomorphic to $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{ad}(E))$, where $\mathrm{ad}(E) \subset \mathrm{End}(E)$ is the subbundle consists of endomorphism of E whose trace is zero. Also, $p_0^{-1}(E)$ is an affine space modelled over $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{ad}(E))$. Thus, there is a natural action of $\Omega^1_{\mathcal{U}_L(n,d),E}$ on $p_0^{-1}(E)$, that is,

$$\Omega^1_{\mathcal{U}_L(n,d),E} \times p_0^{-1}(E) \to p_0^{-1}(E)$$

sending (ω, D) to $\omega + D$, which is faithful and transitive. □

Proposition 5.2. *There exists an algebraic vector bundle* $\pi : \Xi' \to \mathcal{U}_L(n,d)$ *such that* $\mathcal{M}'_{lc}(n,L)$ *is embedded in* $\mathbf{P}(\Xi')$ *with* $\mathbf{P}(\Xi') \setminus \mathcal{M}'_{lc}(n,L)$ *as the hyperplane at infinity.*

Proof. See the proof of the Theorem 1.1. □

Proposition 5.3. *The homomorphism* $p_0^* : Pic(\mathcal{U}_L(n,d)) \to Pic(\mathcal{M}'_{lc}(n,L))$ *defined by* $\xi \mapsto p_0^*\xi$ *is an isomorphism of groups.*

Proof. See the proof of the Theorem 1.2. □

Now, from [13], Proposition 3.4, (ii), we have $Pic(\mathcal{U}_L(n,d)) \cong \mathbb{Z}$. Thus, in view of Proposition 5.3, we have

$$Pic(\mathcal{M}'_{lc}(n,L)) \cong \mathbb{Z}. \tag{5.1}$$

Let Θ be the ample generator of the group $Pic(\mathcal{U}_L(n,d))$. We have the *symbol exact sequence* for the holomorphic line bundle Θ given as follows,

$$\begin{aligned} 0 \to \mathcal{E}nd_{\mathcal{O}_{\mathcal{U}_L(n,d)}}(\Theta) &\xrightarrow{\imath} \mathcal{D}iff^1(\Theta,\Theta) \\ &\xrightarrow{\sigma} T\mathcal{U}_L(n,d) \otimes \mathcal{E}nd_{\mathcal{O}_{\mathcal{U}_L(n,d)}}(\Theta) \to 0, \end{aligned} \tag{5.2}$$

where $\mathcal{D}iff^1(\Theta,\Theta)$ denotes the sheaf of first order holomorphic differential operator from Θ to itself, and $T\mathcal{U}_L(n,d)$ is the holomorphic tangent bundle over $\mathcal{U}_L(n,d)$. Since Θ is a holomorphic line bundle, the *symbol exact sequence* (5.2) becomes (1.4) because in that case $\mathrm{At}(\Theta) = \mathcal{D}iff^1(\Theta,\Theta)$, for more details see [1], [6].

Dualising the exact sequence (1.4), we get following exact sequence,

$$0 \to \Omega^1_{\mathcal{U}_L(n,d)} \xrightarrow{\sigma^*} \mathrm{At}(\Theta)^* \xrightarrow{\imath^*} \mathcal{O}_{\mathcal{U}_L(n,d)} \to 0 \tag{5.3}$$

Consider $\mathcal{O}_{\mathcal{U}_L(n,d)}$ as trivial line bundle $\mathcal{U}_L(n,d) \times \mathbb{C}$. Let

$$s : \mathcal{U}_L(n,d) \to \mathcal{U}_L(n,d) \times \mathbb{C}$$

be a holomorphic map defined by $E \mapsto (E,1)$. Then s is a holomorphic section of the trivial line bundle $\mathcal{U}_L(n,d) \times \mathbb{C}$.

Let $S = \mathrm{Im}(s) \subset \mathcal{U}_L(n,d) \times \mathbb{C}$ be the image of s. Then $S \to \mathcal{U}_L(n,d)$ is a fibre bundle. Consider the inverse image ${\imath^*}^{-1}S \subset \mathrm{At}(\Theta)^*$, and denote it by $\mathcal{C}(\Theta)$. Then for every open subset $U \subset \mathcal{U}_L(n,d)$, a holomorphic section of $\mathcal{C}(\Theta)|_U$ over U gives a holomorphic splitting of (1.4). For instance, suppose

$\gamma : U \to \mathcal{C}(\Theta)|_U$ is a holomorphic section. Then γ will be a holomorphic section of $\mathrm{At}(\Theta)^*|_U$ over U, because $\mathcal{C}(\Theta) = \imath^{*-1}S \subset \mathrm{At}(\Theta)^*$. Since $\gamma \circ \imath = \imath^*(\gamma) = \mathbf{1}_U$, so we get a holomorphic splitting γ of (1.4). Thus, $\Theta|_U$ admits a holomorphic connection. Conversely, given any holomorphic splitting of (1.4) over an open subset $U \subset \mathcal{U}_L(n,d)$, we get a holomorphic section of $\mathcal{C}(\Theta)|_U$.

Let

$$\psi : \mathcal{C}(\Theta) \to \mathcal{U}_L(n,d) \tag{5.4}$$

be the canonical projection. Then using the short exact sequence (5.3), $\mathcal{C}(\Theta)$ is a $\Omega^1_{\mathcal{U}_L(n,d)}$-*torsor* on $\mathcal{U}_L(n,d)$

Proposition 5.4. *There is an isomorphism of algebraic varieties*

$$f : \mathcal{C}(\Theta) \to \mathcal{M}'_{lc}(n,L) \tag{5.5}$$

such that $p_0 \circ f = \psi$, where p_0 and ψ are defined in (1.3) *and* (5.4) *respectively.*

Proof. From the Proposition 4.1, isomorphism class of $\Omega^1_{\mathcal{U}_L(n,d)}$-torsors over $\mathcal{U}_L(n,d)$ is given by a cohomology class in $\mathrm{H}^1(\mathcal{U}_L(n,d), \Omega^1_{\mathcal{U}_L(n,d)})$.

Let $\alpha, \beta \in \mathrm{H}^1(\mathcal{U}_L(n,d), \Omega^1_{\mathcal{U}_L(n,d)})$ be the cohomology class corresponding to $\mathcal{C}(\Theta)$ and $\mathcal{M}'_{lc}(n,L)$ respectively. Since the $\dim_{\mathbb{C}}(\mathrm{H}^1(\mathcal{U}_L(n,d), \Omega^1_{\mathcal{U}_L(n,d)})) = 1$, there exists $c \in \mathbb{C}$ such that $\beta = c\,\alpha$. Thus, $\mathcal{C}(\Theta)$ and $\mathcal{M}'_{lc}(n,L)$ are isomorphic as a fibre bundle over $\mathcal{U}_L(n,d)$. Now, to complete the proof, it is sufficient to show that $\alpha \neq 0$ and $\beta \neq 0$. Θ being an ample line bundle, its first Chern class $c_1(\Theta) \neq 0$ and $\alpha = c_1(\Theta)$. From [7], Theorem 2.11, we conclude that $\beta \neq 0$. □

Let $\alpha_j \in \mathbb{Q}$, for $j = 1, \ldots, m$, such that $n\alpha_j \in \mathbb{Z}$ and $d + n\sum_{j=1}^m \alpha_j = 0$. Fix a holomorphic line bundle L of degree d, and fix a logarithmic connection D'_L on L singular over S with residues $Res(D'_L, x_j) = n\alpha_j$ for $j = 1, \ldots, m$.

Let $\mathcal{V}_{lc}(n,L)$ denote the moduli space parametrising all pairs (E, D) such that

1) E is a holomorphic vector bundle of rank n over X with $\bigwedge^n E \cong L$.
2) D is a logarithmic connection on E singular over S with $Res(D, x_i) = \alpha_i \mathbf{1}_{E(x_i)}$ for every $i = 1, \ldots, m$.
3) the logarithmic connection on $\bigwedge^n E$ induced by D coincides with the given logarithmic connection D'_L on L.

Let $\mathcal{V}'_{lc}(n, L)$ denote the subset of $\mathcal{V}_{lc}(n, L)$ whose underlying vector bundle is stable.

From Proposition 5.4, we have

Corollary 5.5. *There is an isomorphism between* $\mathcal{M}'_{lc}(n, L)$ *and* $\mathcal{V}'_{lc}(n, L)$*.*

Proof. From above Proposition 5.4 both the varieties are isomorphic to $\mathcal{C}(\Theta)$. □

Corollary 5.6. $\mathcal{M}_{lc}(n, L)$ *and* $\mathcal{V}_{lc}(n, L)$ *are birationally equivalent.*

Proof. Since $\mathcal{M}_{lc}(n, L)$ and $\mathcal{V}_{lc}(n, L)$ being irreducible quasi-projective varieties over $\mathbb{C}$, and $\mathcal{M}'_{lc}(n, L)$ and $\mathcal{V}'_{lc}(n, L)$ are dense open subset of $\mathcal{M}_{lc}(n, L)$ and $\mathcal{V}_{lc}(n, L)$, respectively. From Corollary 5.5, we are done. □

We will show that $\mathcal{M}'_{lc}(n, L)$ does not admit any algebraic function. In view of Proposition 5.4, it is enough to show that $\mathcal{C}(\Theta)$ does not have any non constant algebraic function. The proof of the Theorem 1.4 is very similar to the proof of the Theorem 4.3 in [5].

Proof of Theorem 1.4. Let $\mathrm{At}(\Theta)$ be the Atiyah bundle over $\mathcal{U}_L(n, d)$ associated to ample line bundle Θ as described in (1.4), and $\mathbf{P}(\mathrm{At}(\Theta))$ be the projectivization of $\mathrm{At}(\Theta)$, that is, $\mathbf{P}(\mathrm{At}(\Theta))$ parametrises hyperplanes in $\mathrm{At}(\Theta)$. Let $\mathbf{P}(T\mathcal{U}_L)$ be the projectivization of the tangent bundle $T\mathcal{U}_L(n, d)$. Notice that $\mathbf{P}(T\mathcal{U}_L)$ is a subvariety of $\mathbf{P}(\mathrm{At}(\Theta))$, and $\mathbf{P}(T\mathcal{U}_L)$ is the zero locus of the of a section of the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathrm{At}(\Theta))}(1)$. Now, observe that $\mathcal{C}(\Theta) = \mathbf{P}(\mathrm{At}(\Theta)) \setminus \mathbf{P}(T\mathcal{U}_L)$. Then we have

$$\begin{aligned} \mathrm{H}^0(\mathcal{C}(\Theta), \mathcal{O}_{\mathcal{C}(\Theta)}) &= \varinjlim_k \mathrm{H}^0(\mathbf{P}\mathrm{At}(\Theta), \mathcal{O}_{\mathbf{P}\mathrm{At}(\Theta)}(k)) \\ &= \varinjlim_k \mathrm{H}^0(\mathcal{U}_L(n, d), \mathcal{S}^k \mathrm{At}(\Theta)) \end{aligned} \tag{5.6}$$

where $\mathcal{S}^k \mathrm{At}(\Theta)$ denotes the k-th symmetric powers of $\mathrm{At}(\Theta)$. Consider the symbol operator

$$\sigma : \mathrm{At}(\Theta) \to T\mathcal{U}_L(n, d) \tag{5.7}$$

given in (1.4). This induces a morphism

$$\mathcal{S}^k(\sigma) : \mathcal{S}^k \mathrm{At}(\Theta) \to \mathcal{S}^k T\mathcal{U}_L(n, d) \tag{5.8}$$

of k-th symmetric powers. Now, because of the following composition

$$\mathcal{S}^{k-1}\mathrm{At}(\Theta) = \mathcal{O}_{\mathcal{U}_L(n,d)} \otimes \mathcal{S}^{k-1}\mathrm{At}(\Theta) \hookrightarrow \mathrm{At}(\Theta) \otimes \mathcal{S}^{k-1}\mathrm{At}(\Theta) \to \mathcal{S}^{k}\mathrm{At}(\Theta),$$

we have

$$\mathcal{S}^{k-1}\mathrm{At}(\Theta) \subset \mathcal{S}^{k}\mathrm{At}(\Theta) \quad \text{for all } k \geq 1. \tag{5.9}$$

Thus, we get a short exact sequence of vector bundles over $\mathcal{U}_L(n,d)$,

$$0 \to \mathcal{S}^{k-1}\mathrm{At}(\Theta) \to \mathcal{S}^{k}\mathrm{At}(\Theta) \xrightarrow{\mathcal{S}^k(\sigma)} \mathcal{S}^k T\mathcal{U}_L(n,d) \to 0. \tag{5.10}$$

In other words, we get a filtration

$$0 \subset \mathcal{S}^{0}\mathrm{At}(\Theta) \subset \mathcal{S}^{1}\mathrm{At}(\Theta) \subset \cdots \subset \mathcal{S}^{k-1}\mathrm{At}(\Theta) \subset \mathcal{S}^{k}\mathrm{At}(\Theta) \subset \cdots \tag{5.11}$$

such that

$$\mathcal{S}^{k}\mathrm{At}(\Theta)/\mathcal{S}^{k-1}\mathrm{At}(\Theta) \cong \mathcal{S}^k T\mathcal{U}_L(n,d) \quad \text{for all } k \geq 1. \tag{5.12}$$

Above filtration in (5.11) gives following increasing chain of $\mathbb{C}$-vector spaces

$$\mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{O}_{\mathcal{U}_L(n,d)}) \subset \mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^1\mathrm{At}(\Theta)) \subset \cdots \tag{5.13}$$

To prove (4.2), it is enough to show that

$$\mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^{k-1}\mathrm{At}(\Theta)) \cong \mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^{k}\mathrm{At}(\Theta)) \quad \text{for all } k \geq 1. \tag{5.14}$$

Since,

$$\frac{\mathcal{S}^{k}\mathrm{At}(\Theta)}{\mathcal{S}^{k-2}\mathrm{At}(\Theta)} \cong \frac{\mathcal{S}^k T\mathcal{U}_L(n,d)}{\mathcal{S}^{k-1} T\mathcal{U}_L(n,d)},$$

we have following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}^{k-1}\mathrm{At}(\Theta) & \longrightarrow & \mathcal{S}^{k}\mathrm{At}(\Theta) & \xrightarrow{\mathcal{S}^k(\sigma)} & \mathcal{S}^k T\mathcal{U}_L(n,d) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{S}^{k-1} T\mathcal{U}_L(n,d) & \longrightarrow & \frac{\mathcal{S}^{k}\mathrm{At}(\Theta)}{\mathcal{S}^{k-2}\mathrm{At}(\Theta)} & \longrightarrow & \mathcal{S}^k T\mathcal{U}_L(n,d) & \longrightarrow & 0 \end{array} \tag{5.15}$$

which gives rise to a following commutative diagram of long exact sequences (5.16)

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^k T\mathcal{U}_L(n,d)) & \xrightarrow{\delta'_k} & \mathrm{H}^1(\mathcal{U}_L(n,d), \mathcal{S}^{k-1}\mathrm{At}(\Theta)) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^k T\mathcal{U}_L(n,d)) & \xrightarrow{\delta_k} & \mathrm{H}^1(\mathcal{U}_L(n,d), \mathcal{S}^{k-1} T\mathcal{U}_L(n,d)) & \longrightarrow \cdots \end{array}$$

To show (5.14), it is enough to prove that the boundary operator δ'_k is injective for all $k \geq 1$, which is equivalent to showing that the boundary operator

$$\delta_k : \mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^k T\mathcal{U}_L(n,d)) \to \mathrm{H}^1(\mathcal{U}_L(n,d), \mathcal{S}^{k-1} T\mathcal{U}_L(n,d)) \tag{5.17}$$

is injective for every $k \geq 1$.

Now, we will describe δ_k using the first Chern class $c_1(\Theta) \in \mathrm{H}^1(\mathcal{U}_L(n,d), T^*\mathcal{U}_L(n,d))$ of the ample line bundle Θ over $\mathcal{U}_L(n,d)$. The cup product with $kc_1(\Theta)$ gives rise to a homomorphism

$$\begin{aligned} \mu : \mathrm{H}^0(\mathcal{U}_L(n,d), &\mathcal{S}^k T\mathcal{U}_L(n,d)) \\ &\to \mathrm{H}^1(\mathcal{U}_L(n,d), \mathcal{S}^k T\mathcal{U}_L(n,d) \otimes T^*\mathcal{U}_L(n,d)) \end{aligned} \tag{5.18}$$

Also, we have a canonical homomorphism of vector bundles

$$\beta : \mathcal{S}^k T\mathcal{U}_L(n,d) \otimes T^*\mathcal{U}_L(n,d) \to \mathcal{S}^{k-1} T\mathcal{U}_L(n,d)$$

which induces a morphism of $\mathbb{C}$-vector spaces

$$\begin{aligned} \beta^* : \mathrm{H}^1(\mathcal{U}_L(n,d), &\mathcal{S}^k T\mathcal{U}_L(n,d) \otimes T^*\mathcal{U}_L(n,d)) \\ &\to \mathrm{H}^1(\mathcal{U}_L(n,d), \mathcal{S}^{k-1} T\mathcal{U}_L(n,d)). \end{aligned} \tag{5.19}$$

So, we get a morphism

$$\begin{aligned} \tilde{\mu} = \beta^* \circ \mu : \mathrm{H}^0(\mathcal{U}_L(n,d), &\mathcal{S}^k T\mathcal{U}_L(n,d)) \\ &\to \mathrm{H}^1(\mathcal{U}_L(n,d), \mathcal{S}^{k-1} T\mathcal{U}_L(n,d)). \end{aligned} \tag{5.20}$$

Then $\tilde{\mu} = \delta_k$. It is sufficient to show that $\tilde{\mu}$ is injective.

Moreover, we have natural projection

$$\eta : T^*\mathcal{U}_L(n,d) \to \mathcal{U}_L(n,d) \tag{5.21}$$

and

$$\eta_*\eta^*\mathcal{O}_{\mathcal{U}_L(n,d)} = \oplus_{k\geq 0}\mathcal{S}^k T\mathcal{U}_L(n,d). \tag{5.22}$$

Thus, we have

$$\begin{aligned} &\mathrm{H}^j(T^*\mathcal{U}_L(n,d), \mathcal{O}_{T^*\mathcal{U}_L(n,d)}) \\ &\qquad = \oplus_{k\geq 0}\mathrm{H}^j(\mathcal{U}_L(n,d), \mathcal{S}^k T\mathcal{U}_L(n,d)) \quad \text{for all } j \geq 0. \end{aligned} \tag{5.23}$$

Now, we use Hitchin fibration to compute $\mathrm{H}^j(T^*\mathcal{U}_L(n,d), \mathcal{O}_{T^*\mathcal{U}_L(n,d)})$. Let

$$h : T^*\mathcal{U}_L(n,d) \to B_n = \oplus_{i=2}^n \mathrm{H}^0(X, K_X^i) \tag{5.24}$$

be the Hitchin map defined by sending a pair (E, ϕ) to $\sum_{i=1}^n trace(\phi^i)$. Notice that the base of the Hithcin map h in (5.24) is a vector space over $\mathbb{C}$ of dimension $n^2(g-1)+1$.

Let $b \in B_n$. Then $h^{-1}(b) = A \setminus F$, where A is some abelian variety and F is a subvariety of A with $\mathrm{codim}(F, A) \geq 3$ (for more details see [2], [9]), and we will be using this fact showing that $\tilde{\mu}$ is injective.

Let $g : T^*\mathcal{U}_L(n,d) \to \mathbb{C}$ be an algebaric function. Then its restriction $g|_{h^{-1}(b)} : h^{-1}(b) \to \mathbb{C}$ to $h^{-1}(b)$ for every $b \in B_n$ is an algebraic function. Since codim(A,F) ≥ 3, $g|_h^{-1}(b)$ extended to a unique algebraic function $\tilde{g} : A \to \mathbb{C}$. A being an abelian variety, $\tilde{g}$ is a constant function. Thus, on each fibre $h^{-1}(b)$, g is constant, and hence gives an algebraic function on B_n.

Set $\mathcal{B} = \mathrm{d}(\mathrm{H}^0(B_n, \mathcal{O}_{B_n})) \subset \mathrm{H}^0(B_n, \Omega^1_{B_n})$ the space of all exact algebraic 1-form. Define a map

$$\theta : \mathrm{H}^0(T^*\mathcal{U}_L(n,d), \mathcal{O}_{T^*\mathcal{U}_L(n,d)}) \to \mathcal{B} \tag{5.25}$$

by $t \mapsto dg$, where g is the function which is defined by descent of t. Then θ is an isomorphism.

From (5.23) and (5.25), we have

$$\theta : \oplus_{k\geq 0}\mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^k T\mathcal{U}_L(n,d)) \to \mathcal{B} \tag{5.26}$$

which is an isomorphism.

Let $T_h = T_{T^*\mathcal{U}_L(n,d)/B_n} = \mathcal{K}er(dh)$ be the relative tangent sheaf on $T^*\mathcal{U}_L(n,d)$, where $dh : T(T^*\mathcal{U}_L(n,d)) \to h^*TB_n$ morphism of bundles.

Note that $\mathrm{H}^0(B_n, \Omega^1_{B_n}) \subset \mathrm{H}^0(T^*\mathcal{U}_L(n,d), T_h)$, and hence from (5.26), we have an injective homomorphism

$$\nu : \mathcal{B} = \oplus_{k\geq 0}\theta(\mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^kT\mathcal{U}_L(n,d))) \to \mathrm{H}^0(T^*\mathcal{U}_L(n,d), T_h). \tag{5.27}$$

Consider the morphism

$$\mathrm{H}^0(T^*\mathcal{U}_L(n,d), T_h) \to \mathrm{H}^1(T^*\mathcal{U}_L(n,d), T_h \otimes T^*T^*\mathcal{U}_L(n,d))$$

defined by taking cup product with the first Chern class

$$c_1(\eta^*\Theta) \in \mathrm{H}^1(T^*\mathcal{U}_L(n,d), T^*T^*\mathcal{U}_L(n,d)).$$

Using the pairing $T_h \otimes T^*T^*\mathcal{U}_L(n,d) \to \mathcal{O}_{T^*\mathcal{U}_L(n,d)}$, we get a homomorphism

$$\psi : \mathrm{H}^0(T^*\mathcal{U}_L(n,d), T_h) \to \mathrm{H}^1(T^*\mathcal{U}_L(n,d), \mathcal{O}_{T^*\mathcal{U}_L(n,d)}) \tag{5.28}$$

Since $c_1(\eta^*\Theta) = \eta^*(c_1\Theta)$, we have

$$k\psi \circ \nu \circ \theta(\omega_k) = \tilde{\mu}(\omega_k), \tag{5.29}$$

for all $\omega_k \in \mathrm{H}^0(\mathcal{U}_L(n,d), \mathcal{S}^kT\mathcal{U}_L(n,d)))$. Since ν and θ are injective homomorphisms, it is enough to show that $\psi|_{\nu(\mathcal{B})}$ is injective homomorphism. Let $\omega \in \mathcal{B} \setminus \{0\}$ be a non-zero exact 1-form. Choose $b \in B_n$ such that $\omega(b) \neq 0$. As previously discussed $h^{-1}(b) = A \setminus F$, where A is an abelian variety and F is a subvariety of A such that $\mathrm{codim}(F, A) \geq 3$. Now, $\psi(\nu(\omega)) \in \mathrm{H}^1(T^*\mathcal{U}_L(n,d), \mathcal{O}_{T^*\mathcal{U}_L(n,d)})$ and we have restriction map

$$\mathrm{H}^1(T^*\mathcal{U}_L(n,d), \mathcal{O}_{T^*\mathcal{U}_L(n,d)}) \to \mathrm{H}^1(h^{-1}(b), \mathcal{O}_{h^{-1}(b)}).$$

Since $\omega(b) \neq 0$, $\psi(\nu(\omega)) \in \mathrm{H}^1(h^{-1}(b), \mathcal{O}_{h^{-1}(b)})$. Because of the following isomorphisms

$$\mathrm{H}^1(h^{-1}(b), \mathcal{O}_{h^{-1}(b)}) \cong \mathrm{H}^1(A, \mathcal{O}_A) \cong \mathrm{H}^0(A, TA),$$

it follows that $\psi(\nu(\omega)) \neq 0$. This completes the proof. □

Since $\mathcal{M}'_{lc}(n,L)$ is a open dense subset of $\mathcal{M}_{lc}(n,L)$, we have following

Corollary 5.7. $\mathrm{H}^0(\mathcal{M}_{lc}(n,L), \mathcal{O}_{\mathcal{M}_{lc}(n,L)}) = \mathbb{C}$.

Now, for the pair (L_0, D_{L_0}) where $L_0 = \otimes_{i=1}^m \mathcal{O}_X(-n\lambda_i x_i)$ and D_{L_0} the logarithmic connection defined by the de Rham differential as described in Section (3), consider the moduli space $\mathcal{M}_{lc}(n, L_0)$. We show that the moduli space $\mathcal{M}_{lc}(n, L_0)$ admits non-constant holomorphic functions. Consider the Betti moduli space $\mathcal{B}_g$ described in Section (3), which is an affine variety.

Let $\gamma_j \in \pi_1(X_0, x_0)$. Define a function $f_{jk} : \mathcal{B}_g \to \mathbb{C}$ by $\rho \mapsto \text{trace}(\rho(\gamma_j)^k)$ for $k \in \mathbb{N}$. Then f_{jk} are non-constant algebraic functions on $\mathcal{B}_g$ for $j = 1, \ldots, m$ and $k \in \mathbb{N}$. Thus $\mathcal{M}_{lc}(n, L_0)$ is not isomorphic to B_g as algebraic varieties.

Since $\mathcal{M}_{lc}(n, L_0)$ is biholomorphic to $\mathcal{B}_g$, $f_{jk} \circ \Phi : \mathcal{M}_{lc}(n, L_0) \to \mathbb{C}$ are non-constant holomorphic functions for all $j = 1, \ldots, m$ and $k \in \mathbb{N}$.

6. The Moduli space of logarithmic connection with arbitrary residues

Let X be a compact Riemann surface of genus$(g) \geq 3$ and $S = \{x_1, \ldots, x_m\}$ be a subset of distinct points of X as in Section (2). By a pair (E, D) over X, we mean that

1) E is a holomorphic vector bundle over X of degree d and rank n.
2) n and d are mutually coprime.
3) D is a logarithmic connection in E singular over S.

We call such a pair (E, D) logarithmic connection on X singular over S.

Now, given such a pair (E, D), from [12], Theorem 3, we have

$$d + \sum_{j=1}^{m} \text{tr}(Res(D, x_j)) = 0, \tag{6.1}$$

where $Res(D, x_j) \in \text{End}(E(x_j))$, for all $j = 1, \ldots, m$.

Let $\mathcal{N}_{lc}(n, d)$ be the moduli space which parametrises isomorphism class of pairs (E, D). Then $\mathcal{N}_{lc}(n, d)$ is a separated quasi-projective scheme over $\mathbb{C}$ (see [10]). Let $\mathcal{N}'_{lc}(n, d)$ be a subset of $\mathcal{N}_{lc}(n, d)$, whose underlying vector bundle is stable. Let (E, D) and (E, D') be two points in $\mathcal{N}'_{lc}(n, d)$. Then

$$D - D' \in \text{H}^0(X, \text{End}(E) \otimes \Omega^1_X(\log S)). \tag{6.2}$$

Next, for $\theta \in \text{H}^0(X, \text{End}(E) \otimes \Omega^1_X(\log S))$, we have $(E, D + \theta) \in \mathcal{N}'_{lc}(n, d)$. Notice the difference between the affine spaces when residue is fixed and otherwise. Thus, the space of all logarithmic connections D on a given stable vector bundle E singular over S, is an affine space modelled over $\mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X(\log S))$. Let

$$q : \mathcal{N}'_{lc}(n,d) \to \mathcal{U}(n,d) \tag{6.3}$$

be the natural projection defined by sending (E, D) to E. Given $E \in \mathcal{U}(n,d)$. Choose a set of complex numbers $\alpha_1, \ldots, \alpha_m$ which satisfies the following equation

$$d + n\sum_{j=1}^{m} \alpha_j = 0. \tag{6.4}$$

Since E is stable, from [4], Proposition 1.2, E admits a logarithmic connection D singular over S. Thus, q is a surjective map, and dimension of each fibre $q^{-1}(E)$ is $n^2(g-1+m)$. We have following result very similar to the Theorem 1.1.

Theorem 6.1. *There exists an algebraic vector bundle $\tilde{\pi} : \tilde{\Xi} \to \mathcal{U}(n,d)$ such that $\mathcal{N}'_{lc}(n,d)$ is embedded in $\mathbf{P}(\tilde{\Xi})$ with $\mathbf{P}(\tilde{\Xi}) \setminus \mathcal{N}'_{lc}(n,d)$ as the hyperplane at infinity.*

Proof. Proof is very similar to the proof of Theorem 1.1. □

Next, the morphism q defined in (6.3) induces a homomorphism

$$q^* : \mathrm{Pic}(\mathcal{U}(n,d)) \to \mathrm{Pic}(\mathcal{N}'_{lc}(n,d)) \tag{6.5}$$

of Picard groups, that sends line bundle η over $\mathcal{U}(n,d)$ to a line bundle $q^*\eta$ over $\mathcal{N}'_{lc}(n,d)$ as described in Subsection (4.2). Again, we record a result similar to the Theorem 1.2.

Theorem 6.2. *The homomorphism $q^* : \mathrm{Pic}(\mathcal{U}(n,d)) \to \mathrm{Pic}(\mathcal{N}'_{lc}(n,d))$ is an isomorphism of groups.*

Proof. Proof is very similar to the proof of Theorem 1.2. □

Now, fix a pair (L, D_L), where L is a holomorphic vector bundle of degree d and D_L is a fixed logarithmic connections on L singular over S. Let $\mathcal{N}_{lc}(n, L)$ denote the moduli space parametrising all pairs (E, D) such that

1) E is a holomorphic vector bundle over X of rank n and degree d with $\bigwedge^n E \cong L$, and n and d are mutually coprime.

2) D is a logarithmic connection in E singular over S with $Res(D, x_j) \in Z(\mathfrak{gl}(n, \mathbb{C}))$, and $\mathrm{tr}(Res(D, x_j)) \in \mathbb{Z}$, where $Z(\mathfrak{gl}(n, \mathbb{C}))$ denotes the centre of $\mathfrak{gl}(n, \mathbb{C})$.
3) the logarithmic connection on $\bigwedge^n E$ induced by D coincides with the given logarithmic connection D_L on L.

Then, Lemma (2.1) holds for such a pair (E, D), and by Proposition 2.2, (E, D) is irreducible.

Let $\mathcal{N}'_{lc}(n, L)$ be the subset of $\mathcal{N}_{lc}(n, L)$ whose underlying vector bundle is stable. Let

$$q_0 : \mathcal{N}'_{lc}(n, L) \to \mathcal{U}_L(n, d) \tag{6.6}$$

be the natural projection sending (E, D) to E. Then, we have following results similar to Proposition 5.2 and Proposition 5.3.

Proposition 6.3. *There exists an algebraic vector bundle $\pi : \tilde{\Xi}' \to \mathcal{U}_L(n, d)$ such that $\mathcal{N}'_{lc}(n, L)$ is embedded in $\mathbf{P}(\tilde{\Xi}')$ with $\mathbf{P}(\tilde{\Xi}') \setminus \mathcal{N}'_{lc}(n, L)$ as the hyperplane at infinity.*

Proposition 6.4. *The homomorphism $q_0^* : \mathrm{Pic}(\mathcal{U}_L(n, d)) \to \mathrm{Pic}(\mathcal{N}'_{lc}(n, L))$ defined by $\xi \mapsto q_0^*\xi$ is an isomorphism of groups.*

Note that $q_0 : \mathcal{N}'_{lc}(n, L) \to \mathcal{U}_L(n, d)$ is not a $\Omega^1_{\mathcal{U}_L(n,d)}$-torsor, and therefore we cannot apply the same technique as in previous Section (5) to compute the algebraic functions on $\mathcal{N}'_{lc}(n, L)$.

Next, let

$$V = \left\{ (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m \;\middle|\; n\alpha_j \in \mathbb{Z} \;\text{ and }\; d + n\sum_{j=1}^{m} \alpha_j = 0 \right\}$$

Define a map

$$\Phi : \mathcal{N}'_{lc}(n, L) \to V \tag{6.7}$$

by $(E, D) \mapsto (\mathrm{tr}(Res(D, x_1))/n, \ldots, \mathrm{tr}(Res(D, x_m))/n)$.

Proof of Theorem 1.5. Let $(\alpha_1, \ldots, \alpha_m) \in V$. Then $\Phi^{-1}((\alpha_1, \ldots, \alpha_m))$ is the moduli space of logarithmic connections with fixed residues $\alpha_j \mathbf{1}_{E(x_j)}$, which is isomorphic to $\mathcal{M}'_{lc}(n, L)$ follows from Corollary 5.5. Let $g : \mathcal{N}'_{lc}(n, d) \to \mathbb{C}$ be an algebraic function. Then g restricted to each fibre of Φ

is an algebraic function on the moduli space isomorphic to $\mathcal{M}'_{lc}(n, L)$. Now, from Theorem 1.4, g is constant on each fibre and thus defining a function from $V \to \mathbb{C}$. This completes the proof. □

Similarly, we define a map

$$\Psi : \mathcal{N}_{lc}(n, L) \to V \tag{6.8}$$

by $(E, D) \mapsto (\mathrm{tr}(Res(D, x_1))/n, \ldots, \mathrm{tr}(Res(D, x_m))/n)$. We have following

Theorem 6.5. *Every algebraic function on $\mathcal{N}_{lc}(n, L)$ factors through the surjective map $\Psi : \mathcal{N}_{lc}(n, L) \to V$ as defined in* (6.8)*.*

Proof. Let $g : \mathcal{N}_{lc}(n, L) \to \mathbb{C}$ be an algebraic function. Then restriction of g to each fibre of Ψ is a constant function, follows from Corollary 5.7, and hence defining a function from $V \to \mathbb{C}$. □

Acknowledgements

The author would like to thank referees for their detailed and helpful comments. The author is deeply grateful to Prof. Indranil Biswas for suggesting the problem, and helpful discussions, and would like to thank his Ph.D. advisor Prof. N. Raghavendra for numerous discussions and his guidance.

References

[1] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207.

[2] A. Beauville, M. S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor*, Jour. reine angew. Math. **398** (1989), 169–179.

[3] A. Borel, P. P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics **2**, Academic Press, Inc., Boston, MA, (1987), xii+355 pp.

[4] I. Biswas, A. Dan, and A. Paul, *Criterion for logarithmic connections with prescribed residues*, Manuscripta Math. **155** (2018), 77–88.

[5] I. Biswas and N. Raghavendra, *Line bundles over a moduli space of logarithmic connections on a Riemann surface*, Geom. Funct. Anal. **15** (2005), 780–808.

[6] I. Biswas and N. Raghavendra, *The Atiyah- Weil criterion for holomorphic connections*, Indian J. pure appl. Math. **39** (2008), no. 1, 3–47.

[7] I. Biswas and N. Raghavendra, *Curvature of the determinant bundle and the Kähler form over the moduli of parabolic bundles for a family of pointed curves*, Asian Jour. Math. **2** (1998), 303–324.

[8] P. Deligne, Equations Différentielles á Points Singuliers Réguliers, Lecture Notes in Mathematics, Vol. 163, Springer, Berlin, (1970).

[9] N. J. Hitchin, *Stable bundles and integrable systems*, Duke Math. Jour. **54** (1987), 91–114.

[10] N. Nitsure, *Moduli of semistable logarithmic connections*, Jour. Amer. Math. Soc. **6** (1993), 597–609.

[11] M. Maruyama, *Openness of a family of torsion free sheaves*, Jour. Math. Kyoto Univ. **16** (1976), 627–637.

[12] M. Ohtsuki, *A residue formula for Chern classes associated with logarithmic connections*, Tokyo J. Math. **5** (1982), no. 1.

[13] S. Ramanan, *The moduli space of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69–84.

[14] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291.

[15] R. Sebastian, *Torelli theorems for moduli space of logarithmic connections and parabolic bundles*, Manuscripta Math. **136** (2011), 249–271.

[16] A. Singh, *Moduli space of rank one logarithmic connections over a compact Riemann surface*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 3, 297–301.

[17] A. Singh, *On the moduli space of λ-connections*, Proc. Amer. Math. Soc. **149** (2021), no. 2, 459–470.

[18] C. T. Simpson, *Moduli of representations of fundamental group of a smooth projective variety. I*,Inst. Hautes Études Sci. Publ. Math. **79** (1994), 47–129.

[19] C. T. Simpson, *Moduli of representations of fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. **80** (1994), 5–79.

Harish-Chandra Research Institute, HBNI
Chhatnag Road, Jhusi, Prayagraj 211 019, India
E-mail address: anoopsingh@hri.res.in

Received August 13, 2019
Accepted April 1, 2020