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Commutative subalgebras of $\mathcal{U}(\mathfrak{q})$ of maximal transcendence degree

Oksana Yakimova

We prove that the enveloping algebra $\mathcal{U}(\mathfrak{q})$ of a finite-dimensional Lie algebra $\mathfrak{q}$ contains a commutative subalgebra of the maximal possible transcendence degree $(\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$.

Introduction

Let $\mathfrak{q}$ be a finite-dimensional Lie algebra defined over a field $\mathbb{K}$ of characteristic zero. Then the universal enveloping algebra $\mathcal{U}(\mathfrak{q})$ is a filtered, associative, non-commutative (in general) algebra and one may ask a natural question:

(Q1) *how large can a commutative subalgebra of* $\mathcal{U}(\mathfrak{q})$ *be?*

The symmetric algebra $\mathcal{S}(\mathfrak{q})$ is the associated graded algebra of $\mathcal{U}(\mathfrak{q})$ and it carries the induced Poisson structure. If $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$ is a commutative algebra, then $\operatorname{gr}(\mathcal{A}) \subset \mathcal{S}(\mathfrak{q})$ is a Poisson-commutative subalgebra, i.e., the Poisson bracket vanishes on it. Basic properties of the coadjoint representation imply that in this situation,

$$\operatorname{tr.deg}\operatorname{gr}(\mathcal{A}) \leqslant (\operatorname{ind}\mathfrak{q} + \dim\mathfrak{q})/2 =: \boldsymbol{b}(\mathfrak{q}).$$

For a commutative algebra, the transcendence degree coincides with the Gelfand–Kirillov dimension and using a result of Borho and Kraft [BK, Satz 5.7], we obtain $\operatorname{tr.deg}\mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{q})$. This leads to a more precise formulation of the first question,

(Q2) *is there a commutative subalgebra* $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$ *such that* $\operatorname{tr.deg}\mathcal{A} = \boldsymbol{b}(\mathfrak{q})$*?*

Our main result, Theorem 1, asserts that the answer to (Q2) is positive.

For a nilpotent Lie algebra $\mathfrak{n}$, the existence of a commutative algebra $\mathcal{A} \subset \mathcal{U}(\mathfrak{n})$ with $\operatorname{tr.deg}\mathcal{A} = \boldsymbol{b}(\mathfrak{n})$ is shown in [GK, Lemme 9]. That algebra $\mathcal{A}$ plays a rôle in the proof of the Gelfand–Kirillov conjecture.

In case of a reductive Lie algebra $\mathfrak{g} = \operatorname{Lie} G$, we have $\operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$ and $\boldsymbol{b}(\mathfrak{g})$ is equal to the dimension of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Take $\gamma \in \mathfrak{g}^*$ such that $\dim \mathfrak{g}_\gamma = \operatorname{rk}\mathfrak{g}$. Let $\bar{\mathcal{A}}_\gamma \subset \mathcal{S}(\mathfrak{g})$ be the *Mishchenko–Fomenko subalgebra* associated with γ, see Definition 1. Then $\{\bar{\mathcal{A}}_\gamma, \bar{\mathcal{A}}_\gamma\} = 0$ [MF] and $\operatorname{tr.deg}\bar{\mathcal{A}}_\gamma = \boldsymbol{b}(\mathfrak{g})$ [PY]. The task of lifting $\bar{\mathcal{A}}_\gamma$ to $\mathcal{U}(\mathfrak{g})$ is known as *Vinberg's quantisation problem*. In full generality it is solved by L. Rybnikov [R06]. The solution produces a commutative subalgebra $\mathcal{A}_\gamma \subset \mathcal{U}(\mathfrak{g})$ such that $\operatorname{gr}(\mathcal{A}_\gamma) = \bar{\mathcal{A}}_\gamma$. Thus, $\operatorname{tr.deg}\mathcal{A}_\gamma = \boldsymbol{b}(\mathfrak{g})$ and this provides the positive answer to (Q2) in the reductive case.

The existence of a Poisson-commutative subalgebra $\bar{\mathcal{A}} \subset \mathcal{S}(\mathfrak{q})$ with $\operatorname{tr.deg}\bar{\mathcal{A}} = \boldsymbol{b}(\mathfrak{q})$ was conjectured by Mishchenko and Fomenko [MF′]. Their conjecture is proved by Sadetov [Sa]. In the proof he used a reduction to the semisimple case. The steps of that reduction are clarified in [VY]. There are two nice isomorphisms of certain invariants, see Sections 1.1 and 1.2, which are in the background of [Sa] and are proven in [VY]. Using these isomorphisms, we perform the same reduction on the level of $\mathcal{U}(\mathfrak{q})$.

Question (Q1) has two immediate generalisations. One can consider commutative subalgebras either of quotients of $\mathcal{U}(\mathfrak{q})$ or of some natural subrings. Both these instances turn out to be intricate. We will address quotients of $\mathcal{U}(\mathfrak{q})$ in a forthcoming paper. Some observations on commutative subalgebras of the invariant ring $\mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$, where $\mathfrak{l} \subset \mathfrak{q}$ is a Lie subalgebra, are presented in Section 4.

Throughout the paper $\mathfrak{g}$ stands for a reductive Lie algebra.

1. Basic facts on Lie and Poisson structures

The symmetric algebra $\mathcal{S}(\mathfrak{q})$ is the algebra of regular functions $\mathbb{K}[\mathfrak{q}^*]$ on $\mathfrak{q}^*$. For $\gamma \in \mathfrak{q}^*$, let $\hat{\gamma}$ be the corresponding skew-symmetric form on $\mathfrak{q}$ given by $\hat{\gamma}(\xi, \eta) = \gamma\big([\xi, \eta]\big)$. Note that the kernel of $\hat{\gamma}$ is equal to the stabiliser

$$\mathfrak{q}_\gamma = \{\xi \in \mathfrak{q} \mid \operatorname{ad}^*(\xi)\gamma = 0\}. \tag{1.1}$$

Let dF denote the differential of $F \in \mathcal{S}(\mathfrak{q})$ and $d_\gamma F$ denote the differential of F at $\gamma \in \mathfrak{q}^*$. Then $d_\gamma F \in \mathfrak{q}$. A well-known property of the Lie–Poisson bracket on $\mathcal{S}(\mathfrak{q})$ is that

$$\{F_1, F_2\}(\gamma) = \hat{\gamma}(d_\gamma F_1, d_\gamma F_2) \quad \text{for all} \quad F_1, F_2 \in \mathcal{S}(\mathfrak{q}).$$

The *index* of $\mathfrak{q}$, as defined by Dixmier, is the number

$$\text{(1.2)}\qquad \operatorname{ind}\mathfrak{q} = \min_{\gamma\in\mathfrak{q}^*}\dim\mathfrak{q}_\gamma = \dim\mathfrak{q} - \max_{\gamma\in\mathfrak{q}^*}\operatorname{rk}\hat{\gamma} = \dim\mathfrak{q} - \max_{\gamma\in\mathfrak{q}^*}\dim(\mathfrak{q}\gamma),$$

where $\mathfrak{q}\gamma = \{\operatorname{ad}^*(\xi)\gamma \mid \xi\in\mathfrak{q}\}$. The set of *regular* elements of $\mathfrak{q}^*$ is

$$\mathfrak{q}^*_{\mathsf{reg}} = \{\eta\in\mathfrak{q}^* \mid \dim\mathfrak{q}_\eta = \operatorname{ind}\mathfrak{q}\}.$$

Set $\mathfrak{q}^*_{\mathsf{sing}} = \mathfrak{q}^*\setminus\mathfrak{q}^*_{\mathsf{reg}}$.

Suppose that $\mathfrak{q} = \operatorname{Lie}Q$ is an algebraic Lie algebra and Q is a connected affine algebraic group defined over $\mathbb{K}$. Then $\dim(\mathfrak{q}x) = \dim(Qx)$ for each x in $\mathfrak{q}^*$. By Rosenlicht's theorem, see e.g. [VP, Sect. 2.3], we have $\operatorname{tr.deg}\mathbb{K}(\mathfrak{q}^*)^Q = \operatorname{ind}\mathfrak{q}$.

Return to an arbitrary $\mathfrak{q}$. For any subalgebra $A\subset\mathcal{S}(\mathfrak{q})$ and any $x\in\mathfrak{q}^*$ set

$$d_xA = \langle d_xF \mid F\in A\rangle_{\mathbb{K}} \subset T^*_x\mathfrak{q}^*.$$

Then

$$\operatorname{tr.deg}A = \max_{x\in\mathfrak{q}^*}\dim d_xA.$$

If A is Poisson-commutative, then $\hat{x}(d_xA, d_xA) = 0$ for each $x\in\mathfrak{q}^*$ and thereby

$$\text{(1.3)}\qquad \operatorname{tr.deg}A \leqslant \frac{\dim\mathfrak{q} - \operatorname{ind}\mathfrak{q}}{2} + \operatorname{ind}\mathfrak{q} = \boldsymbol{b}(\mathfrak{q})$$

as mentioned in the Introduction.

For any subalgebra $\mathfrak{l}\subset\mathfrak{q}$, let $\mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ denote the *Poisson centraliser* of $\mathfrak{l}$, i.e.,

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{l}} = \{F\in\mathcal{S}(\mathfrak{q}) \mid \{\xi, F\} = 0 \ \text{ for all } \ \xi\in\mathfrak{l}\}.$$

The algebra of symmetric invariants $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is the *Poisson centre* of $\mathcal{S}(\mathfrak{q})$. The canonical symmetrisation map $\operatorname{symm}\colon\mathcal{S}(\mathfrak{q})\to\mathcal{U}(\mathfrak{q})$ is an isomorphism of $\mathfrak{q}$-modules. Hence we have an isomorphism of vector spaces $\mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ and $\mathcal{U}(\mathfrak{q})^{\mathfrak{l}} = \{u\in\mathcal{U}(\mathfrak{q}) \mid [u,\mathfrak{l}] = 0\}$ for each $\mathfrak{l}$.

According to [MY, Prop. 1.1],

$$\text{(1.4)}\qquad \operatorname{tr.deg}A \leqslant \boldsymbol{b}(\mathfrak{q}) - \boldsymbol{b}(\mathfrak{l}) + \operatorname{ind}\mathfrak{l}$$

for a Poisson-commutative subalgebra $A\subset\mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$.

Definition 1. For $\gamma \in \mathfrak{q}^*$, let $\bar{\mathcal{A}}_\gamma \subset \mathcal{S}(\mathfrak{q})$ be the corresponding *Mishchenko–Fomenko subalgebra*, which is generated by all *γ-shifts* $\partial_\gamma^k H$ with $k \geqslant 0$ of all elements $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$.

Note that $\partial_\gamma^k H$ is a constant for $k = \deg H$. We have $\{\bar{\mathcal{A}}_\gamma, \bar{\mathcal{A}}_\gamma\} = 0$ [MF].

1.1. Abelian ideals and their invariants

Let $\mathfrak{h} \lhd \mathfrak{q}$ be an Abelian ideal consisting of ad-nilpotent elements. Then $\mathfrak{h} = \operatorname{Lie} H$, where H is a unipotent algebraic group acting on $\mathfrak{q}^*$ regularly. Since $\mathcal{U}(\mathfrak{h})$ is commutative, we have $\mathcal{U}(\mathfrak{h}) = \mathcal{S}(\mathfrak{h})$. Set $\mathbb{F} = \mathbb{K}(\mathfrak{h}^*)$. Then $\mathfrak{h} \subset \mathbb{F}$. Let $\mathfrak{h} \otimes_{\mathfrak{h}} \mathbb{F}$ be a one-dimensional vector space over $\mathbb{F}$ spanned by $\delta = w \otimes \dfrac{1}{w}$ with a non-zero $w \in \mathfrak{h}$. Here $v \otimes 1 = v\delta$ for each $v \in \mathfrak{h}$.

Remark. Notation $\mathfrak{h} \otimes_{\mathfrak{h}} \mathbb{F}$ is borrowed from [VY]. It should be understood in the following way. Let us regard $\mathfrak{h}$ as $\mathfrak{h}{\cdot}1$. Then $\mathfrak{h} \otimes_{\mathfrak{h}} \mathbb{F}$ is an $\mathbb{F}$-vector space spanned by $1 \otimes 1$ with the property $v \otimes 1 = 1 \otimes v$ for each $v \in \mathfrak{h}$.

The tensor product $\mathfrak{q} \otimes_{\mathfrak{h}} \mathbb{F}$, taken over $\mathfrak{h}$, is an $\mathbb{F}$-vector space of dimension $\dim \mathfrak{q} - \dim \mathfrak{h} + 1$ and as such it can be identified with $(\mathfrak{q}/\mathfrak{h}) \otimes_{\mathbb{K}} \mathbb{F} \oplus \mathbb{F}\delta$. The group H acts on $\mathbb{F}$ trivially and we have an $\mathbb{F}$-linear action of H on $\mathfrak{q} \otimes_{\mathfrak{h}} \mathbb{F}$. Set $\hat{\mathfrak{q}} = (\mathfrak{q} \otimes_{\mathfrak{h}} \mathbb{F})^H$. The elements of $\hat{\mathfrak{q}}$ are linear combinations of elements of $\mathfrak{q}$ with coefficients from $\mathbb{F}$. Therefore $\hat{\mathfrak{q}}$ is a subset of the localised enveloping algebra $\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}$. Note that $[\xi, w^{-1}] = -w^{-2}[\xi, w] \in \mathbb{F}\delta$ for any $\xi \in \mathfrak{q}$ and a non-zero $w \in \mathfrak{h}$. Hence

$$[\hat{\mathfrak{q}}, \hat{\mathfrak{q}}] \subset \mathfrak{q} \otimes_{\mathfrak{h}} \mathbb{F} \subset \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}.$$

Clearly, the commutator of two H-invariant elements is again an H-invariant. Thus,

$$[\hat{\mathfrak{q}}, \hat{\mathfrak{q}}] \subset \hat{\mathfrak{q}} \tag{1.5}$$

and $\hat{\mathfrak{q}}$ is a finite-dimensional Lie algebra over $\mathbb{F}$. Furthermore, $\delta \in \hat{\mathfrak{q}}$. In view of the fact that $[\hat{\mathfrak{q}}, \mathfrak{h}] = 0$, one can write the Lie bracket of $\hat{\mathfrak{q}}$ in down to earth terms:

$$\left[\sum_{j=1}^N c_j\xi_j, \sum_{j=1}^N b_j\eta_j\right] = \sum_{i,j} c_jb_i[\xi_j, \eta_i] \quad \text{for} \quad \sum_{j=1}^N c_j\xi_j, \sum_{j=1}^N b_j\eta_j \in \hat{\mathfrak{q}}$$

with $c_j, b_j \in \mathbb{F}$, $\xi_j, \eta_j \in \mathfrak{q}$. Working over $\mathbb{F}$, we let $\mathcal{U}(\hat{\mathfrak{q}})$ stand for the enveloping algebra of $\hat{\mathfrak{q}}$. Then clearly $\mathbb{F} \subset \mathcal{U}(\hat{\mathfrak{q}})$. At the sam time, $\delta \in \mathcal{U}(\hat{\mathfrak{q}})$ and $\delta \notin \mathbb{F} \subset \mathcal{U}(\hat{\mathfrak{q}})$. Let $\mathcal{U}_\delta(\hat{\mathfrak{q}})$ be the subalgebra of $\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}$ generated by $\hat{\mathfrak{q}}$. Then $\mathcal{U}_\delta(\hat{\mathfrak{q}}) \cong \mathcal{U}(\hat{\mathfrak{q}})/(\delta - 1)$ as an associative $\mathbb{F}$-algebra.

Example 2. Suppose that $\mathfrak{q} = \mathfrak{s} \ltimes \mathfrak{h}$, where $\mathfrak{s}$ is a subalgebra of $\mathfrak{q}$. Then

$$\hat{\mathfrak{q}} = \{\xi \in \mathfrak{s} \otimes_{\mathbb{K}} \mathbb{F} \mid [v, \xi] = 0 \ \forall v \in \mathfrak{h}\} \oplus \mathbb{F}\delta, \tag{1.6}$$

see [Y17, Lemma 2.1]. We note that there is an unfortunate misprint in Remark 2.6 in [Y17] and that the Lie bracket on $\hat{\mathfrak{q}}$, which is defined by the inclusion $\hat{\mathfrak{q}} \subset \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}$, is the same as the one extended from $\mathfrak{s}$.

Let $\{\xi_1, \ldots, \xi_m\}$ be a basis for a complement of $\mathfrak{h}$ in $\mathfrak{q}$ and $\{\eta_1, \ldots, \eta_r\}$ be a basis of $\mathfrak{h}$. Then

$$\hat{\mathfrak{q}} = \mathbb{F}\delta \oplus \left\{ \sum_{i=1}^{m} c_i \xi_i \mid c_i \in \mathbb{F}, \sum_{i=1}^{m} c_i [\xi_i, \eta_j] = 0 \ \forall j, 1 \leqslant j \leqslant r \right\}, \tag{1.7}$$

where each $[\xi_i, \eta_j] \in \mathfrak{h}$ is regarded as an element of $\mathbb{F}$. The rank of the $m \times r$-matrix $(\boldsymbol{m}_{ij})$ with $\boldsymbol{m}_{ij} = [\xi_i, \eta_j]$ is equal to the dimension of $\mathfrak{q}\alpha \subset \mathfrak{h}^*$ for a generic $\alpha \in \mathfrak{h}^*$. Hence

$$\begin{aligned} \dim_{\mathbb{F}} \hat{\mathfrak{q}} &= \dim \mathfrak{q} - \dim \mathfrak{h} - \max_{\alpha \in \mathfrak{h}^*} \dim(\mathfrak{q}\alpha) + 1 \\ &= \min_{\alpha \in \mathfrak{h}^*} \dim \mathfrak{q}_\alpha - \dim \mathfrak{h} + 1. \end{aligned} \tag{1.8}$$

In these terms, $\sum_{i=1}^{m} c_i \xi_i \in \hat{\mathfrak{q}}$ if and only if $\sum_{i=1}^{m} c_i(\alpha) \xi_i \in \mathfrak{q}_\alpha$ whenever all c_i are defined for $\alpha \in \mathfrak{h}^*$.

Let $\mathcal{U}(\mathfrak{q}) = \bigcup_{d \geqslant 0} \mathcal{U}_d(\mathfrak{q})$ be the standard filtration on $\mathcal{U}(\mathfrak{q})$. Set $\mathcal{W}_d = \mathcal{U}_d(\mathfrak{q})\mathcal{U}(\mathfrak{h})$. Then clearly $\mathcal{U}(\mathfrak{q}) = \bigcup_{d \geqslant 0} \mathcal{W}_d(\mathfrak{q})$. Assume that $\mathcal{W}_{-1} = 0$. Since $\mathfrak{h}$ is an Abelian ideal, we have a non-canonical isomorphism of commutative alebras

$$\mathrm{gr}_{\mathcal{W}} \mathcal{U}(\mathfrak{q}) = \bigoplus_{d \geqslant 0} \mathcal{W}_d / \mathcal{W}_{d-1} \cong \mathcal{S}(\mathfrak{q}).$$

The new filtration extends to $\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}$ and on $\mathcal{U}_\delta(\hat{\mathfrak{q}}) \subset \mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}$ it coincides with the standard filtration inherited by the quotient $\mathcal{U}(\hat{\mathfrak{q}})/(\delta - 1)$ from $\mathcal{U}(\hat{\mathfrak{q}})$.

The algebra $\hat{\mathfrak{q}}$ coincides with the algebra $\widetilde{\mathfrak{q}} = \widetilde{\mathfrak{q}}(I_0)$, defined in [VY, Sect. 4], in the particular case $I_0 = \{0\}$. Therefore $\hat{\mathfrak{q}}$ is the quotient of the Lie algebra of all rational maps

$$\xi\colon \mathfrak{h}^* \to \mathfrak{q} \ \text{ such that } \ \xi(\alpha) \in \mathfrak{q}_\alpha \ \text{ whenever } \ \xi(\alpha) \ \text{ is defined}$$

by $\hat{\mathfrak{h}} := \{\xi \in \mathfrak{h} \otimes_{\mathbb{K}} \mathbb{F} \mid \alpha(\xi(\alpha)) = 0 \text{ for each } \alpha \in \mathfrak{h}^* \text{ such that } \xi(\alpha) \text{ is defined}\}$.

Lemma 3. (i) $(\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F})^H = \mathcal{U}_\delta(\hat{\mathfrak{q}})$. (ii) $\boldsymbol{b}(\hat{\mathfrak{q}}) = \boldsymbol{b}(\mathfrak{q}) - \dim \mathfrak{h} + 1$.

Proof. (i) Set $\mathcal{F} = (\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F})^H$. Note that $\mathcal{U}_\delta(\hat{\mathfrak{q}}) \subset \mathcal{F}$ by the construction. Since the action of H on $\mathfrak{h}^*$ is trivial, we have also $\mathcal{F} = \mathcal{U}(\mathfrak{q})^H \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F}$. Because $\hat{\mathfrak{q}}$ is a Lie algebra over $\mathbb{F}$, it suffices to show that $\mathcal{U}(\mathfrak{q})^H \subset \mathcal{U}_\delta(\hat{\mathfrak{q}})$. Employing the filtration $\mathcal{U}(\mathfrak{q}) = \bigcup_{d\geqslant 0} \mathcal{W}_d(\mathfrak{q})$ and the symmetrisation map one readily reduces the claim to the level of $\mathcal{S}(\mathfrak{q})^H$.

The assertion that $\mathcal{S}(\mathfrak{q})^H \otimes_{\mathcal{S}(\mathfrak{h})} \mathbb{F} \cong \mathcal{S}(\hat{\mathfrak{q}})/(\delta - 1)$ is contained implicitly in [VY, Lemma 21]. For the sake of completeness, we briefly recall the argument. Set $\boldsymbol{F} = \mathcal{S}(\mathfrak{q})^H \otimes_{\mathcal{S}(\mathfrak{h})} \mathbb{F}$.

Now we consider $\hat{\mathfrak{q}}$ as subset of $\boldsymbol{F}$ identifying δ with 1. Both, $\boldsymbol{F}$ and the subalgebra $\mathcal{S}_\delta(\hat{\mathfrak{q}}) \subset \boldsymbol{F}$ generated by $\hat{\mathfrak{q}}$, are vector spaces over $\mathbb{K}(\mathfrak{h}^*)$. Thus, it suffices to verify the equality $\boldsymbol{F} = \mathcal{S}_\delta(\hat{\mathfrak{q}})$ at generic $\alpha \in \mathfrak{h}^*$.

Let $Y_\alpha \subset \mathfrak{q}^*$ be the preimage of a point $\alpha \in \mathfrak{h}^*$ under the canonical restriction $\mathfrak{q}^* \to \mathfrak{h}^*$. Since $\mathfrak{h}$ is commutative, H acts on Y_α. By [VY, Lemma 20], $Y_\alpha/H = \operatorname{Spec}(\mathbb{K}[Y_\alpha]^H)$ and the restriction map $\pi_\alpha\colon Y_\alpha \to (\mathfrak{q}_\alpha)^*$ defines an isomorphism $Y_\alpha/H \cong (\mathfrak{q}_\alpha/\mathfrak{h})^* \times \{\alpha\}$.

Let $\boldsymbol{F}_\alpha \subset \boldsymbol{F}$ be the subset of elements that are defined at α. Then for any $\alpha \in \mathfrak{h}^*$, we have a map

$$\epsilon_\alpha\colon \boldsymbol{F}_\alpha \to \mathbb{K}[Y_\alpha]^H \cong \mathbb{K}[(\mathfrak{q}_\alpha/\mathfrak{h})^* \times \{\alpha\}] \cong \mathcal{S}(\mathfrak{q}_\alpha/\mathfrak{h}).$$

Eq. (1.8) and the discussion after it imply that $\mathfrak{q}_\alpha/\mathfrak{h}$ embeds into $\epsilon_\alpha(\hat{\mathfrak{q}} \cap \boldsymbol{F}_\alpha)$ for a generic point α. Hence, if α is generic, then $\epsilon_\alpha(\mathcal{S}_\delta(\hat{\mathfrak{q}}) \cap \boldsymbol{F}_\alpha) \cong \mathcal{S}(\mathfrak{q}_\alpha/\mathfrak{h})$, cf. the proof of [VY, Lemma 21]. Therefore, $\epsilon_\alpha(\boldsymbol{F}_\alpha) = \epsilon_\alpha(\mathcal{S}_\delta(\hat{\mathfrak{q}}) \cap \boldsymbol{F}_\alpha)$ and we can conclude that $\boldsymbol{F} = \mathcal{S}_\delta(\hat{\mathfrak{q}})$.

(ii) This part is proven in [VY, Sect. 5]. Let $\alpha \in \mathfrak{h}^*$ and $\gamma \in Y_\alpha$ be generic. Set $k = \dim(H\gamma)$. Since the form $\alpha([\ ,\ ])$ defines a non-degenerate pairing between $\mathfrak{q}/\mathfrak{q}_\alpha$ and $\mathfrak{h}/\mathfrak{h}_\gamma$, we have also $k = \dim(\mathfrak{q}\alpha)$. Note that

$$\dim_{\mathbb{F}} \hat{\mathfrak{q}} = \dim \mathfrak{q} - k - \dim \mathfrak{h} + 1$$

by Eq. (1.8).

The numerical characteristics of $\hat{\mathfrak{q}}$, like index, can be computed locally, at α, so to say. In particular, $\dim_{\mathbb{F}} \hat{\mathfrak{q}} - \operatorname{ind} \hat{\mathfrak{q}} = \operatorname{rk}(\hat{\gamma}|_{\mathfrak{q}_\alpha \times \mathfrak{q}_\alpha})$.

Write $\mathfrak{q} = \mathfrak{m} \oplus (\mathfrak{r} \oplus \mathfrak{h})$, where $\mathfrak{r} \oplus \mathfrak{h} = \mathfrak{q}_\alpha$. Here $\hat{\gamma}(\mathfrak{q}_\alpha, \mathfrak{h}) = 0$ and $\hat{\gamma}$ is non-degenerate on $\mathfrak{m} \times (\mathfrak{h}/\mathfrak{h}_\gamma)$. The block structure of $\hat{\gamma}$ shows that

$$\operatorname{rk} \hat{\gamma} = 2k + \operatorname{rk}(\hat{\gamma}|_{\mathfrak{r} \times \mathfrak{r}}) = 2k + \operatorname{rk}(\hat{\gamma}|_{\mathfrak{q}_\alpha \times \mathfrak{q}_\alpha}).$$

Hence $\dim \mathfrak{q} - \operatorname{ind} \mathfrak{q} = \operatorname{rk} \hat{\gamma} = 2k + \dim_{\mathbb{F}} \hat{\mathfrak{q}} - \operatorname{ind} \hat{\mathfrak{q}}$ and

$$\boldsymbol{b}(\hat{\mathfrak{q}}) = \dim_{\mathbb{F}} \hat{\mathfrak{q}} - \frac{1}{2}(\dim_{\mathbb{F}} \hat{\mathfrak{q}} - \operatorname{ind} \hat{\mathfrak{q}}) = \frac{1}{2}(\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q}) - \dim \mathfrak{h} + 1.$$

This completes the proof. □

We will need another auxiliary statement. Suppose that $z \in \mathfrak{q}$ is a non-zero central element. Let $\mathcal{A} \in \mathcal{U}(\mathfrak{q})$ be a commutative subalgebra. For $c \in \mathbb{K}$, let $\mathcal{A}(c)$ be the image of $\mathcal{A}$ in $\mathcal{U}(\mathfrak{q})/(z-c)$. Then the following assertion is true.

Lemma 4. *There is a non-zero $c \in \mathbb{K}$ such that* $\operatorname{tr.deg} \mathcal{A}(c) \geqslant \operatorname{tr.deg} \mathcal{A} - 1$.

Proof. We consider $\mathcal{A}$ as a subalgebra of $\mathcal{U}(\mathfrak{q})_z = \mathcal{U}(\mathfrak{q}) \otimes_{\mathbb{K}[z]} \mathbb{K}(z)$. On $\mathcal{U}(\mathfrak{q})_z$, there is an increasing filtration by the finite-dimensional $\mathbb{K}(z)$-vector spaces $\mathcal{U}_d(\mathfrak{q})_z = \langle \mathcal{U}_d(\mathfrak{q}) \rangle_{\mathbb{K}(z)}$. The associated graded algebra $\operatorname{gr}_z(\mathcal{U}(\mathfrak{q})_z)$ is isomorphic to $\mathcal{S}(\mathfrak{q})_z = \mathcal{S}(\mathfrak{q}) \otimes_{\mathbb{K}[z]} \mathbb{K}(z)$. Let $\bar{\mathcal{A}} \subset \mathcal{S}(\mathfrak{q})_z$ be the graded image of $\mathcal{A}$. According to [BK, Satz 5.7], $\operatorname{tr.deg}_{\mathbb{K}(z)} \bar{\mathcal{A}} = \operatorname{tr.deg}_{\mathbb{K}(z)} \mathcal{A}$. Note that actually $\bar{\mathcal{A}} \subset \mathcal{S}(\mathfrak{q})$.

The quotient $\mathcal{U}(\mathfrak{q})/(z-c)$ inherits the standard filtration from $\mathcal{U}(\mathfrak{q})$. Moreover, Diagram 1 is commutative.

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{q}) & \longrightarrow & \mathcal{U}(\mathfrak{q})/(z-c) \\ \big\downarrow{\scriptstyle \operatorname{gr}_z} & & \big\downarrow{\scriptstyle \operatorname{gr}} \\ \mathcal{S}(\mathfrak{q}) & \longrightarrow & \mathcal{S}(\mathfrak{q})/(z-c) \end{array}$$

Figure 1: Filtration of the localised algebra and quotients.

Let $\bar{\mathcal{A}}(c)$ be the image of $\bar{\mathcal{A}}$ in $\mathcal{S}(\mathfrak{q})/(z-c)$. One of the basic facts in algebraic geometry states that there is a non-zero $c \in \mathbb{K}$ such that $\operatorname{tr.deg} \bar{\mathcal{A}}(c) = \operatorname{tr.deg}_{\mathbb{K}(z)} \bar{\mathcal{A}}$. Hence, for this c, we have

$$\operatorname{tr.deg} \mathcal{A}(c) \geqslant \operatorname{tr.deg} \bar{\mathcal{A}}(c) = \operatorname{tr.deg}_{\mathbb{K}(z)} \bar{\mathcal{A}} = \operatorname{tr.deg}_{\mathbb{K}(z)} \mathcal{A} \geqslant \operatorname{tr.deg} \mathcal{A} - 1$$

as desired. □

1.2. Invariants of a Heisenberg algebra

Recall that a $(2n{+}1)$-dimensional Heisenberg Lie algebra over $\mathbb{K}$ is a Lie algebra $\mathfrak{h}$ with a basis $\{x_1,\ldots,x_n,y_1,\ldots,y_n,z\}$ such that $n\geqslant 1$, $[x_i,x_j]=[y_i,y_j]=0$ for all i,j, $[\mathfrak{h},z]=0$, and $[x_i,y_j]=\delta_{ij}z$. Suppose that $\mathfrak{q}=\mathfrak{l}\ltimes\mathfrak{h}$, where $\mathfrak{l}$ is a subalgebra and $[\mathfrak{q},z]=0$. Assume further that the subspace $\mathfrak{v}=\langle x_j,y_j\mid 1\leqslant j\leqslant n\rangle_{\mathbb{K}}$ is $\mathfrak{l}$-stable. According to [VY, Lemma 18] and its corollary,

$$(\mathcal{S}(\mathfrak{q})[z^{-1}])^{\mathfrak{h}}\cong\mathcal{S}(\mathfrak{l})\otimes_{\mathbb{K}}\mathbb{K}[z,z^{-1}]. \tag{1.9}$$

This isomorphism can be made very explicit. For $\xi\in\mathfrak{l}$, set

$$\hat{\xi}=\xi+\frac{1}{2z}\sum_{i=1}^{n}([\xi,x_i]y_i-[\xi,y_i]x_i)\in\mathcal{U}(\mathfrak{q})[z^{-1}].$$

The following statement is elementary in nature and is certainly known. Similar ideas have been used in [PPY, Sect. 4.8].

Lemma 5. *We have $[v,\hat{\xi}]=0$ for all $\xi\in\mathfrak{l}$ and all $v\in\mathfrak{h}$.*

Proof. It is enough to show that $[x_j,\hat{\xi}]=[y_j,\hat{\xi}]=0$ for all $j\in\{1,\ldots,n\}$. We have

$$\begin{aligned}
[x_j,\hat{\xi}]&=[x_j,\xi]+\frac{1}{2z}\left(\left(\sum_{i=1}^{n}[x_j,[\xi,x_i]]y_i\right)+[\xi,x_j]z-\sum_{i=1}^{n}[x_j,[\xi,y_i]]x_i\right)\\
&=[x_j,\xi]+\frac{1}{2}[\xi,x_j]+\frac{1}{2z}\left(\sum_{i=1}^{n}[[x_j,\xi],x_i]y_i-\sum_{i=1}^{n}[[x_j,\xi],y_i]x_i\right)\\
&=[x_j,\xi]+[\xi,x_j]=0;\\
[y_j,\hat{\xi}]&=[y_j,\xi]+\frac{1}{2z}\left(\left(\sum_{i=1}^{n}[y_j,[\xi,x_i]]y_i\right)+[\xi,y_j]z-\sum_{i=1}^{n}[y_j,[\xi,y_i]]x_i\right)\\
&=[y_j,\xi]+\frac{1}{2}[\xi,y_j]+\frac{1}{2z}\left(\sum_{i=1}^{n}[[y_j,\xi],x_i]y_i-\sum_{i=1}^{n}[[y_j,\xi],y_i]x_i\right)\\
&=[y_j,\xi]+[\xi,y_j]=0.
\end{aligned}$$

This completes the proof. □

It follows from [VY, Lemma 18] that $(\mathcal{S}(\mathfrak{q})[z^{-1}])^{\mathfrak{h}}$ is generated by the symbols $\mathrm{gr}(z\hat{\xi})$ of the elements $z\hat{\xi}$ with $\xi\in\mathfrak{l}$ and by z,z^{-1}. The same lemma states that (1.9) is a natural isomorphism of Poisson algebras. For $\xi, \eta \in \mathfrak{l}$ and $\zeta = [\xi, \eta]$, we have therefore

$$\{\mathrm{gr}(z\hat{\xi}), \mathrm{gr}(z\hat{\eta})\} = z\mathrm{gr}(z\hat{\zeta}), \tag{1.10}$$

where the Poisson bracket is taken in $\mathcal{S}(\mathfrak{q})$. Observe next that the commutator $[\hat{\xi}, \hat{\eta}]$ belongs to $(\mathcal{U}(\mathfrak{q})[z^{-1}])^{\mathfrak{h}}$.

Lemma 6. *In the above notation, we have* $[\hat{\xi}, \hat{\eta}] = \hat{\zeta}$.

Proof. Write $\hat{u} = u + T(u)$ for $u \in \mathfrak{l}$. Then $T(u) \in \mathcal{U}(\mathfrak{h})[z^{-1}]$ and hence $[\hat{v}, T(u)] = 0$ for all $v \in \mathfrak{l}$. Now $[\hat{\xi}, \hat{\eta}] = \zeta + [\xi, T(\eta)]$. Next we consider the elements

$$\overline{T}(u) = \frac{1}{2}\sum_{i=1}^{n}([u, x_i]y_i - [u, y_i]x_i)) \in \mathcal{S}^2(\mathfrak{v}).$$

By the construction, $\mathrm{gr}(z\hat{u}) = zu + \overline{T}(u)$. Since $[\mathfrak{l}, \mathfrak{h}] \subset \mathfrak{h}$, each $\{\xi, \mathrm{gr}(z\hat{u})\}$ is again an $\mathfrak{h}$-invariant. In particular,

$$\{\xi, \mathrm{gr}(z\hat{\eta})\} - \mathrm{gr}(z\hat{\zeta}) = \{\xi, \overline{T}(\eta)\} - \overline{T}(\zeta) \in \mathcal{S}^2(\mathfrak{v})^{\mathfrak{h}}.$$

Because $\mathcal{S}^2(\mathfrak{v})^{\mathfrak{h}} = 0$, this difference is zero.

Observe that

$$[[u, x_i], y_i] + [x_i, [u, y_i]] = [u, z] = 0$$

for each i and hence $[[u, x_i], y_i] - [[u, y_i], x_i] = 0$. This implies the equality $zT(u) = \mathrm{symm}(\overline{T}(u))$. Thereby $[\xi, zT(\eta)]$ equals the symmetrisation of $\{\xi, \overline{T}(\eta)\} = \overline{T}(\zeta)$. Thus $[\xi, T(\eta)] = T(\zeta)$ and we are done. □

Corollary 7. *We have* $(\mathcal{U}(\mathfrak{q})[z^{-1}])^{\mathfrak{h}} \cong \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{K}} \mathbb{K}[z, z^{-1}]$. □

The isomorphism (1.9) implies that $\mathrm{ind}\, \mathfrak{q} = \mathrm{ind}\, \mathfrak{l} + 1$. Hence

$$\boldsymbol{b}(\mathfrak{q}) = \boldsymbol{b}(\mathfrak{l}) + n + 1. \tag{1.11}$$

2. On algebraic extensions

Let $A = \bigcup_{n \geqslant 0} A_n$ be an increasingly filtered associative algebra such that $\dim A_n < \infty$ for each $n \geqslant 0$. Assume that $A_{-m} = 0$ for all $m \geqslant 1$. Suppose that the associated graded algebra $\overline{A} = \mathrm{gr}\, A$ is a commutative domain and a finitely generated $\mathbb{K}$-algebra. For each $a \in A_n \setminus A_{n-1}$, set $\bar{a} =$

$\mathrm{gr}(a) = a + A_{n-1}$. For a subspace $V \subset A$, let $\overline{V} = \mathrm{gr}(V)$ be the subspace of $\overline{A}$ spanned by $\mathrm{gr}(v)$ with $v \in V$.

Let B be a subalgebra of A. Then [BK, Satz 5.7] asserts that the Gelfand–Kirillov dimensions of B and $\overline{B}$ are equal. This result implies that for commutative subalgebras $B \subset C \subset A$, we have

$$C \text{ is algebraic over } B \iff \mathrm{gr}(C) \text{ is algebraic over } \mathrm{gr}(B). \tag{2.1}$$

Lemma 8 (cf. [R03, Lemma 1]). *Keep the above assumptions on A and let $B \subset C$ be commutative subalgebras of A such that C is algebraic over B. If $[x, B] = 0$ for some $x \in A$, then also $[x, C] = 0$.*

Proof. Since $\overline{A}$ is commutative, we have $[A_n, A_m] \subset A_{n+m-1}$ for all $m, n \geqslant 0$. Assume that there is $x \in A_m \setminus A_{m-1}$ such that $[x, B] = 0$ and $[x, C] \neq 0$. Let $k \geqslant 1$ be the minimal number such that

$$[x, C \cap A_n] \subset A_{m+n-k} \text{ for all } n \geqslant 0, \tag{2.2}$$

but $[x, C \cap A_n] \not\subset A_{m+n-k-1}$ for some n. For $\bar{u} \in \overline{A}_\ell$ with $u \in C$, set

$$\{\bar{x}, \bar{u}\}_k = [x, u] + A_{m+\ell-k-1}.$$

If $\bar{u} = \mathrm{gr}(u')$ with $u' \in C$, then $u - u' \in (C \cap A_{\ell-1})$ and because of (2.2), we have $[x, u - u'] \in A_{m+\ell-k-1}$. Thereby $\{\bar{x}, y\}_k$ is a well-defined element of $A_{m+\ell-k}/A_{m+\ell-k-1}$ for each $y \in (\overline{C} \cap \bar{A}_\ell)$. The linear map $\{\bar{x}, \ \}\colon \mathrm{gr}(C) \to \overline{A}$ satisfies the Leibniz rule by the construction and $\{\bar{x}, \overline{B}\} = 0$.

There is $u \in (C \cap A_n)$ such that $\{\bar{x}, \bar{u}\}_k \neq 0$. Since u is algebraic over B, the symbol $\bar{u}$ is algebraic over $\overline{B}$. Let

$$\boldsymbol{Q}(\bar{u}) = \bar{b}_N \bar{u}^N + \cdots + \bar{b}_1 \bar{u} + \bar{b}_0 = 0$$

with $b_j \in B$ be a non-trivial equation on $\bar{u}$ of the smallest possible degree. Since $\bar{u}$ is a homogeneous element of $\overline{A}$, we can assume that all summands $\bar{b}_j \bar{u}^j$ have one and the same degree in $\overline{A}$.

Consider the symbol of $[x, \widetilde{\boldsymbol{Q}}(u)]$, where $\widetilde{\boldsymbol{Q}}(X) = \sum b_j X^j$. This symbol is equal to the product

$$\{\bar{x}, \bar{u}\}_k (N \bar{b}_N \bar{u}^{N-1} + \cdots + 2\bar{b}_2 \bar{u} + \bar{b}_1) = \{\bar{x}, \boldsymbol{Q}(\bar{u})\}_k = 0.$$

Because $\overline{A}$ is a domain, we have obtained an equation on $\bar{u}$ of smaller than N degree. This contradiction proves that $[x, C] = 0$ whenever $[x, B] = 0$. $\square$

Corollary 9. *Let $B \subset A$ be as in Lemma 8. Then the algebraic closure of B in the centraliser $Z_A(B) \subset A$ is a commutative subalgebra.* □

Remark. A well-known fact is that the algebraic closure of a Poisson-commutative subalgebra is again Poisson-commutative. Lemma 8, which is inspired by [R03, Lemma 1], can be regarded as a non-commutative generalisation of this statement.

Our main example of A is $\mathcal{U}(\mathfrak{q})$. Here $\operatorname{gr} A = \mathcal{S}(\mathfrak{q})$ is a finitely generated commutative algebra, which is a domain.

3. The inductive argument

Let $\mathfrak{n} \lhd \mathfrak{q}$ be the nilpotent radical of $\mathfrak{q}$. Note that $\mathfrak{n}$ is an algebraic Lie algebra.

Lemma 10 ([D, Lemma 4.6.2], cf. [VY, Lemma 17]). *Suppose that each commutative non-zero characteristic ideal of $\mathfrak{n}$ is one-dimensional and $\mathfrak{n} \neq 0$. Then either $\mathfrak{n} = \mathbb{K}$ or $\mathfrak{n}$ is a Heisenberg Lie algebra.* □

Remark. It is a borderline issue, whether to consider $\mathbb{K}$ as a Heisenberg Lie algebra. In [VY, Lemma 17], the convention is that $\mathbb{K}$ is included into the class of Heisenberg algebras. Note that the results of [VY, Section 4] are valid for $\mathbb{K}$ as well by a trivial reason.

An algebraic Lie algebra $\mathfrak{q}$ has an algebraic Levi decomposition $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$, where $\mathfrak{l}$ is reductive. In the non-algebraic case, $\mathfrak{q} = \mathfrak{s} \ltimes \mathfrak{r}$, where $\mathfrak{r}$ is the solvable radical of $\mathfrak{q}$ and $\mathfrak{s}$ is semisimple. As is well-known, $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n}$. Moreover, $\mathfrak{n} \neq 0$ in the non-algebraic case, because otherwise $\mathfrak{q}$ were reductive. The case of a non-algebraic $\mathfrak{q}$ is more involved and requires an additional lemma.

Lemma 11. *Let $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{z}$ be a Heisenberg Lie algebra, where $\mathfrak{z} = \mathbb{K}z$ is the centre of $\mathfrak{n}$ and $\dim \mathfrak{v} \geqslant 2$. Suppose that $\mathfrak{h}$ is an ideal of $\mathfrak{q}$. Set*

$$\widetilde{\mathfrak{l}} = \{\xi \in \mathfrak{q} \mid [\xi, \mathfrak{v}] \subset \mathfrak{v}\}.$$

Then $\mathfrak{q} = \widetilde{\mathfrak{l}} + \mathfrak{h}$ and $\widetilde{\mathfrak{l}} \cap \mathfrak{h} = \mathfrak{z}$.

Proof. The equality $\widetilde{\mathfrak{l}} \cap \mathfrak{h} = \mathfrak{z}$ follows from the structure of $\mathfrak{h}$. Take any $\xi \in \mathfrak{q}$. Then $\operatorname{ad}(\xi)$ defines a linear map from $\mathfrak{v}$ to $\mathfrak{h}/\mathfrak{v} \cong \mathfrak{z}$. Any such map can be presented as a commutator with some $\eta \in \mathfrak{v}$. Hence there is $v \in \mathfrak{v}$ such that $\operatorname{ad}(\xi) - \operatorname{ad}(v)$ preserves $\mathfrak{v}$. Here $\xi - v \in \widetilde{\mathfrak{l}}$ and we are done. □

The construction of Section 1.2 generalises easily to the non-algebraic setting leading to the following statement.

Corollary 12. *Keep the assumptions and notation of Lemma 11 and suppose additionally that* $[\mathfrak{q},\mathfrak{z}]=0$. *Then* $(\mathcal{U}(\mathfrak{q})[z^{-1}])^{\mathfrak{h}} \cong \mathcal{U}(\widetilde{\mathfrak{l}})[z^{-1}]$. □

One more observation is required before we can start the induction.

Lemma 13. *In the reductive case, quantum MF-subalgebras* $\mathcal{A}_\gamma$ *are defined over* $\mathbb{Q}$ *and hence over any field of characteristic zero. If* $\gamma\in\mathfrak{g}^*_{\mathsf{reg}}$, *then* $\mathrm{tr.deg}\,\mathcal{A}_\gamma=\boldsymbol{b}(\mathfrak{g})$.

Proof. Recall the construction from [R06]. Let G be a connected **complex** reductive algebraic group. Set $\mathfrak{g}=\mathrm{Lie}\,G$. The universal enveloping algebra $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ contains a certain commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$, which is known as the *Feigin–Frenkel centre*. Set $\mathfrak{l}=[\mathfrak{g},\mathfrak{g}]$, $r=\dim\mathfrak{l}$. According to [R08], $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the centraliser in $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ of the following quadratic element

$$\mathcal{H}[-1]=\sum_{a=1}^{r}x_at^{-1}x_at^{-1},$$

where $\{x_1,\ldots,x_r\}$ is any basis of $\mathfrak{l}$ that is orthonormal w.r.t. the Killing form.

For any $\gamma\in\mathfrak{g}^*$ and a non-zero $z\in\mathbb{C}$, the map

$$(3.1)\qquad \varrho_{\gamma,z}\colon \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])\to\mathcal{U}(\mathfrak{g}),\qquad xt^r\mapsto z^rx+\delta_{r,-1}\gamma(x),\quad x\in\mathfrak{g},$$

defines a G_γ-equivariant algebra homomorphism. The image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under $\varrho_{\gamma,z}$ is a commutative subalgebra $\mathcal{A}_\gamma$ of $\mathcal{U}(\mathfrak{g})$, which does not depend on z [R06]. Moreover, $\bar{\mathcal{A}}_\gamma\subset\mathrm{gr}(\mathcal{A}_\gamma)$ for each $\gamma\in\mathfrak{g}^*$ [R06]. If $\gamma\in\mathfrak{g}^*_{\mathsf{reg}}$, then $\bar{\mathcal{A}}_\gamma$ is a maximal w.r.t. inclusion Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ [PY] and hence $\mathrm{gr}(\mathcal{A}_\gamma)=\bar{\mathcal{A}}_\gamma$.

If $\mathfrak{l}$ is simple, then $H[-1]=\mathrm{gr}(\mathcal{H}[-1])$ spans $\mathcal{S}^2(\mathfrak{l}t^{-1})^{\mathfrak{g}}$. In general, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the centraliser of $\mathrm{symm}(\mathcal{S}^2(\mathfrak{g}t^{-1})^{\mathfrak{g}})$. The subspace $\mathcal{S}^2(\mathfrak{g}t^{-1})^{\mathfrak{g}}$ has a $\mathbb{Q}$-form and behaves well under field extensions. Its centraliser in $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ shares the same properties.

If $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$ and $\mathbb{K}\subset\mathbb{L}$, then

$$\mathcal{S}^2_{\mathbb{L}}(\mathfrak{g}(\mathbb{L})t^{-1})^{\mathfrak{g}(\mathbb{L})}=\mathcal{S}^2(\mathfrak{g}t^{-1})^{\mathfrak{g}}\otimes_{\mathbb{K}}\mathbb{L}$$

for $\mathfrak{g}(\mathbb{L})=\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{L}$. If $\gamma\in\mathfrak{g}^*$ and $\gamma(\mathbb{L})\in\mathfrak{g}(\mathbb{L})^*$ is its continuation, then $\bar{\mathcal{A}}_{\gamma(\mathbb{L})}=\bar{\mathcal{A}}_\gamma\otimes_{\mathbb{K}}\mathbb{L}$. Playing with extensions $\mathbb{Q}\subset\overline{\mathbb{Q}}\subset\mathbb{C}$ and $\mathbb{K}\subset\overline{\mathbb{K}}$, one shows that $\mathfrak{z}(\widehat{\mathfrak{g}})$ produces a quantum MF-subalgebra over any $\mathbb{K}$. In more details, since $\bar{\mathcal{A}}_\gamma \subset \mathrm{gr}(\mathcal{A}_\gamma)$ holds over $\mathbb{C}$, it holds over $\mathbb{Q}$ and $\overline{\mathbb{K}}$, thereby it holds over $\mathbb{K}$. By [PY], $\mathrm{tr.deg}\, \bar{\mathcal{A}}_\gamma = \boldsymbol{b}(\mathfrak{g})$ for $\gamma \in \mathfrak{g}^*_{\mathsf{reg}}$ over $\overline{\mathbb{K}}$. Hence also $\mathrm{tr.deg}\, \bar{\mathcal{A}}_\gamma = \boldsymbol{b}(\mathfrak{g})$ for $\gamma \in \mathfrak{g}^*_{\mathsf{reg}}$ over $\mathbb{K}$ and $\mathrm{tr.deg}\, \mathcal{A}_\gamma = \boldsymbol{b}(\mathfrak{g})$ over $\mathbb{K}$. □

Theorem 1. *For each finite-dimensional Lie algebra* $\mathfrak{q}$*, there is a commutative algebra* $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$ *with* $\mathrm{tr.deg}\, \mathcal{A} = \boldsymbol{b}(\mathfrak{q})$.

Proof. There is no harm in assuming that $\mathfrak{q}$ is indecomposable. The case of a simple (reductive) Lie algebra $\mathfrak{g}$ is settled by a result of Rybnikov [R06], here $\mathrm{tr.deg}\, \mathcal{A}_\gamma = \boldsymbol{b}(\mathfrak{g})$ for a quantum Mishchenko–Fomenko subalgebra $\mathcal{A}_\gamma$, see also Lemma 13 and the Introduction. Therefore suppose that $\mathfrak{n} \neq 0$. In this case we argue by induction on $\dim \mathfrak{q}$. The induction begins with $\dim \mathfrak{q} = 1$, where $\boldsymbol{b}(\mathfrak{q}) = 1$ and there is nothing to prove.

• Suppose first that there is an Abelian Ideal $\mathfrak{h} \lhd \mathfrak{q}$ such that $\mathfrak{h} \subset \mathfrak{n}$ and $[\mathfrak{q}, \mathfrak{h}] \neq 0$ or $\dim \mathfrak{h} > 1$. Let H, $\mathbb{F}$, and $\hat{\mathfrak{q}}$ be the same as in Section 1.1. We have

$$\dim_{\mathbb{F}} \hat{\mathfrak{q}} \leqslant \dim_{\mathbb{F}}(\mathfrak{q} \otimes_{\mathfrak{h}} \mathbb{F}) = \dim_{\mathbb{K}} \mathfrak{q} - \dim \mathfrak{h} + 1.$$

Moreover, $\dim_{\mathbb{F}} \hat{\mathfrak{q}} < \dim_{\mathbb{F}}(\mathfrak{q} \otimes_{\mathfrak{h}} \mathbb{F})$ if $[\mathfrak{q}, \mathfrak{h}] \neq 0$. By the assumptions on $\mathfrak{h}$, $\dim_{\mathbb{F}} \hat{\mathfrak{q}} < \dim_{\mathbb{K}} \mathfrak{q}$. By the inductive hypothesis, $\mathcal{U}(\hat{\mathfrak{q}})$ contains a commutative subalgebra $\widetilde{\mathcal{A}}_1$ such that $\mathrm{tr.deg}_{\mathbb{F}} \widetilde{\mathcal{A}}_1 = \boldsymbol{b}(\hat{\mathfrak{q}})$. Without loss of generality, assume that $\mathcal{A}_1$ contains the central element $\delta \in \hat{\mathfrak{q}}$. By Lemma 4, there is a non-zero $c \in \mathbb{F}$ such that $\mathrm{tr.deg}_{\mathbb{F}} \mathcal{A}_1 = \boldsymbol{b}(\hat{\mathfrak{q}}) - 1$ for the image $\mathcal{A}_1 = \widetilde{\mathcal{A}}_1(c)$ of $\widetilde{\mathcal{A}}_1$ in $\mathcal{U}(\hat{\mathfrak{q}})/(\delta - c)$. Here $\mathcal{U}(\hat{\mathfrak{q}})/(\delta - c) \cong \mathcal{U}_\delta(\hat{\mathfrak{q}})$.

According to Lemma 3,

$$\boldsymbol{b}(\hat{\mathfrak{q}}) = \boldsymbol{b}(\mathfrak{q}) - \dim \mathfrak{h} + 1 \quad \text{and} \quad \mathcal{U}_\delta(\hat{\mathfrak{q}}) = (\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F})^H.$$

Now we consider $\mathcal{A}_1$ as an subalgebra of $(\mathcal{U}(\mathfrak{q}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{F})^H$. After multiplying the elements of $\mathcal{A}_1$ by suitable elements of $\mathbb{F}$, we may safely assume that $\mathcal{A}_1 \subset \mathcal{U}(\mathfrak{q})^H$. Let $\mathcal{A} = \mathsf{alg}\langle \mathcal{A}_1, \mathfrak{h} \rangle \subset \mathcal{U}(\mathfrak{q})$ be the algebra generated by $\mathcal{A}_1$ and $\mathfrak{h}$. Then $\mathcal{A}$ is commutative and $\mathrm{tr.deg}_{\mathbb{K}} \mathcal{A} = \mathrm{tr.deg}_{\mathbb{F}} \mathcal{A}_1 + \dim_{\mathbb{K}} \mathfrak{h} = \boldsymbol{b}(\mathfrak{q})$.

• Suppose now that $\mathfrak{n}$ contains no commutative characteristic ideals $\mathfrak{h}$ such that $\dim \mathfrak{h} > 1$ or $[\mathfrak{q}, \mathfrak{h}] \neq 0$. Then either $\dim \mathfrak{n} = 1$ or $\mathfrak{n}$ is a Heisenberg Lie algebra, see Lemma 10. Let $\mathfrak{z} \subset \mathfrak{n}$ be the centre of $\mathfrak{n}$. Since $\mathfrak{z}$ is an Abelian ideal of $\mathfrak{q}$, we have $[\mathfrak{q}, \mathfrak{z}] = 0$. We will treat the cases of an algebraic and a non-algebraic $\mathfrak{q}$ separately. For both of them, let $z \in \mathfrak{z}$ be a non-zero element.

• Consider first the algebraic case, where $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$. If $\mathfrak{n} = \mathfrak{z}$, then $\mathfrak{q}$ is a sum of two ideals. This contradicts our assumption on $\mathfrak{q}$. Thus, $\mathfrak{n}$ is a Heisenberg Lie algebra such that $\dim \mathfrak{n} \geqslant 3$ and $[\mathfrak{q}, \mathfrak{z}] = 0$. By Corollary 7,

$(\mathcal{U}(\mathfrak{q})[z^{-1}])^{\mathfrak{n}} \cong \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{K}} \mathbb{K}[z, z^{-1}]$. Since $\mathfrak{l}$ is reductive, there is a quantum Mishchenko–Fomenko subalgebra $\mathcal{A}_\gamma \subset \mathcal{U}(\mathfrak{l})$ with $\text{tr.deg}\,\mathcal{A}_\gamma = \boldsymbol{b}(\mathfrak{l})$. Let $\mathcal{A}_1$ be the image of this subalgebra in $(\mathcal{U}(\mathfrak{q})[z^{-1}])^{\mathfrak{n}}$. After multiplying the elements of $\mathcal{A}_1$ by suitable powers of z, we may safely assume that $\mathcal{A}_1 \subset \mathcal{U}(\mathfrak{q})^{\mathfrak{n}}$. Set $\mathcal{A} = \mathsf{alg}\langle \mathcal{A}_1, x_1, \ldots, x_n, z\rangle \subset \mathcal{U}(\mathfrak{q})$ in the notation of Section 1.2. Then $\mathcal{A}$ is commutative and $\text{tr.deg}\,\mathcal{A} = \boldsymbol{b}(\mathfrak{l}) + n + 1$. In view of (1.11), $\text{tr.deg}\,\mathcal{A} = \boldsymbol{b}(\mathfrak{q})$.

• Finally let $\mathfrak{q}$ be a non-algebraic Lie algebra. Then $\mathfrak{q} = \mathfrak{s} \ltimes \mathfrak{r}$. If $\mathfrak{z} = \mathfrak{n}$, then $[\mathfrak{r}, \mathfrak{n}] = [\mathfrak{r}, \mathfrak{z}] = 0$ and $\mathfrak{r}$ is the nilpotent radical of $\mathfrak{q}$. Since $[\mathfrak{q}, \mathfrak{z}] = 0$, we have also $\mathfrak{q} = \mathfrak{s} \oplus \mathfrak{z}$, which contradicts our assumption on $\mathfrak{q}$. Hence $\dim \mathfrak{n} \geqslant 3$.

According to Corollary 12, $(\mathcal{U}(\mathfrak{q})[z^{-1}])^{\mathfrak{n}} \cong \mathcal{U}(\widetilde{\mathfrak{l}})[z^{-1}]$. This isomorphism implies that $\text{ind}\,\widetilde{\mathfrak{l}} = \text{ind}\,\mathfrak{q}$. Note that $\dim(\widetilde{\mathfrak{l}}) = \dim \mathfrak{q} - \dim \mathfrak{n} + 1$. By the inductive hypothesis, $\mathcal{U}(\widetilde{\mathfrak{l}})$ contains a commutative subalgebra $\mathcal{C}$ such that $\text{tr.deg}\,\mathcal{C} = \boldsymbol{b}(\widetilde{\mathfrak{l}})$. It produces a commutative subalgebra $\mathcal{A}_1 \subset \mathcal{U}(\mathfrak{q})^{\mathfrak{n}}$ of the same transcendence degree. The rest of the argument does not differ from the algebraic case above. □

4. Commutative subalgebras in subrings of invariants

Let $\mathfrak{l} \subset \mathfrak{q}$ be a subalgebra. Then $\text{gr}(\mathcal{U}(\mathfrak{q})^{\mathfrak{l}}) = \mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$. Since $\mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$ is a domain, combining Eq. (1.4) with [BK, Satz 5.7], we obtain

$$\text{tr.deg}\,\mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{q}) - \boldsymbol{b}(\mathfrak{l}) + \text{ind}\,\mathfrak{l} =: \boldsymbol{b}^{\mathfrak{l}}(\mathfrak{q}), \tag{4.1}$$

for any commutative subalgebra $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$. Note that if $\mathfrak{l}$ is Abelian, then $\boldsymbol{b}(\mathfrak{l}) = \dim \mathfrak{l} = \text{ind}\,\mathfrak{l}$ and hence $\boldsymbol{b}^{\mathfrak{l}}(\mathfrak{q}) = \boldsymbol{b}(\mathfrak{q})$.

For $\mathfrak{l} = \mathfrak{q}$, we have $\boldsymbol{b}^{\mathfrak{l}}(\mathfrak{q}) = \text{ind}\,\mathfrak{q}$. This shows already that the upper bound cannot be achieved in all cases. There are Lie algebras such that $\mathcal{U}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{K}$ and $\text{ind}\,\mathfrak{q} \geqslant 1$. An easy example is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}_3$, where $\text{ind}\,\mathfrak{b} = 1$.

Sections 1.1 and 1.2 combined with the proof of Theorem 1 show that there are two positive and very useful cases. Namely, if $\mathfrak{l}$ is either an Abelian ideal of $\mathfrak{q}$ consisting of ad-nilpotent elements or a normal Heisenberg subalgebra such that $[\mathfrak{l}, \mathfrak{l}]$ lies in the centre of $\mathfrak{q}$, then $\mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$ contains a commutative subalgebra $\mathcal{A}$ such that $\text{tr.deg}\,\mathcal{A} = \boldsymbol{b}^{\mathfrak{l}}(\mathfrak{q})$.

Therefore it stands to reason to look for appropriate classes of pairs $(\mathfrak{q}, \mathfrak{l})$. We will concentrate on the case, where $\mathfrak{q} = \mathfrak{g}$ is reductive. The study of $\mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$ is motivated by the application to the branching rules $\mathfrak{g} \downarrow \mathfrak{l}$.

Speculation. Take $\mathfrak{g} = \mathfrak{gl}_{2n}$. Then $\mathfrak{g}$ contains a commutative subalgebra $\mathfrak{l}$ of dimension n^2. For example, $\mathfrak{l} = \langle E_{ij} \mid i \leqslant n, j > n \rangle_{\mathbb{K}}$. Note that $\boldsymbol{b}(\mathfrak{g}) = 2n^2 + n$. To the best of my knowledge, no one ever looked at commutative subalgebras of $\mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$ or their Poisson-commutative counterparts. That could be an interesting class of commutative subalgebras of $\mathcal{U}(\mathfrak{g})$ of the maximal possible transcendence degree. If contrary to my expectations, $\mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$ does not contain a commutative subalgebra of the transcendence degree $2n^2 + n$, then any maximal commutative subalgebra of $\mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$ would provide an example of a maximal w.r.t. inclusion commutative subalgebra that does not have the maximal possible transcendence degree.

4.1. Centralisers

Consider the case $\mathfrak{l} = \mathfrak{q}_\gamma$ with $\gamma \in \mathfrak{q}^*$. Here $\bar{\mathcal{A}}_\gamma \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$. If a reasonable quantisation exists, it has to lie in $\mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$. This is indeed the case for the quantum MF-subalgebra $\mathcal{A}_\gamma \subset \mathcal{U}(\mathfrak{g})$ of a reductive Lie algebra $\mathfrak{g}$, cf. Eq. (3.1). Moreover, $\text{tr.deg}\, \mathcal{A}_\gamma = \boldsymbol{b}^{\mathfrak{l}}(\mathfrak{g})$, see [MY, Lemma 2.1 & Prop. 4.1]. If $\gamma \in \mathfrak{g}^*_{\mathsf{sing}}$ and γ is semisimple, then $\mathfrak{l} = \mathfrak{g}_\gamma$ is a proper Levi subalgebra of $\mathfrak{g}$. The importance of $\mathcal{A}_\gamma$ in the description of the branching rule $\mathfrak{g} \downarrow \mathfrak{l}$ is discovered in [HKRW].

4.2. Symmetric subalgebras

Suppose now that $\mathfrak{l} = \mathfrak{g}_0 = \mathfrak{g}^\sigma$, where σ is an involution of $\mathfrak{g}$. Poisson-commutative subalgebras $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{l}}$ such that $\text{tr.deg}\, \mathcal{Z} = \boldsymbol{b}^{\mathfrak{l}}(\mathfrak{g})$ are constructed in [PY′]. Unfortunately, no quantisation of those subalgebras is known in general.

Example 14. Take $\mathfrak{g} = \mathfrak{so}_{n+1}$, $\mathfrak{l} = \mathfrak{so}_n$. Then $\mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$ is commutative and is generated by the centres $\mathcal{Z}\mathcal{U}(\mathfrak{g})$, $\mathcal{Z}\mathcal{U}(\mathfrak{l})$. Furthermore, $\boldsymbol{b}^{\mathfrak{l}}(\mathfrak{g}) = \text{tr.deg}\, \mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$.

5. On the notion of maximality

Suppose that $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$ is a commutative subalgebra such that $\text{tr.deg}\, \mathcal{A} = \boldsymbol{b}(\mathfrak{q})$. It does not have to be *maximal w.r.t. inclusion*, but it is not far from it. Assume that $\mathcal{A} \subset \mathcal{C} \subset \mathcal{U}(\mathfrak{q})$ and $[\mathcal{C}, \mathcal{C}] = 0$, then each element of $\mathcal{C}$ is algebraic over $\mathcal{A}$ and $\mathcal{C} \subset Z_{\mathcal{U}(\mathfrak{q})}(\mathcal{A})$.

Proposition 15. *Let $\mathcal{A}$ be as above. Then $Z_{\mathcal{U}(\mathfrak{q})}(\mathcal{A})$ is a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{q})$. With obvious changes the statement holds for commutative subalgebras $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$.*

Proof. Set $\mathcal{C} = Z_{\mathcal{U}(\mathfrak{q})}(\mathcal{A})$. Since $\mathcal{C}$ is an algebraic extension of $\mathcal{A}$, it is commutative according to Corollary 9. If $[\mathcal{C}, x] = 0$ for some $x \in \mathcal{U}(\mathfrak{q})$, then also $[\mathcal{A}, x] = 0$ and $x \in Z_{\mathcal{U}(\mathfrak{q})}(\mathcal{A})$. Hence $\mathcal{C}$ is maximal. □

If such an $\mathcal{A}$ is algebraically closed in $\mathcal{U}(\mathfrak{q})$, then it is maximal. Also if $\mathrm{gr}(\mathcal{A})$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$, then $\mathcal{A}$ is maximal. Both properties hold for the quantum Mishchenko–Fomenko subalgebras $\mathcal{A}_\gamma \subset \mathcal{S}(\mathfrak{g})$ with $\gamma \in \mathfrak{g}^*_{\mathsf{reg}}$ [PY]. According to [MY], $\mathcal{A}_\gamma$ is a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_\gamma}$ for any $\gamma \in \mathfrak{g}^*$ if $\mathfrak{g}$ is of type A or C.

The inductive steps in the proof of Theorem 1 involve localisation. Therefore it is difficult to check, whether the constructed subalgebras are maximal or not.

Example 16. Consider an easy example of a semi-direct product $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{h}$ with a Heisenberg Lie algebra. Take $\mathfrak{l} = \mathfrak{sl}_2$ with a standard basis $\{e, h, f\}$ and $\mathfrak{h} = \langle x, y, z\rangle_{\mathbb{K}}$. Suppose that $[e, y] = x,\ [e, x] = 0,\ [f, x] = y,\ [f, y] = 0$. Then over $\mathbb{K}[z, z^{-1}]$ the $\mathfrak{h}$-invariants $\mathcal{U}(\mathfrak{q})^{\mathfrak{h}}[z^{-1}]$ are generated by

$$zh + xy,\ \ 2ez - x^2,\ \ 2fz + y^2.$$

Furthermore, $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is generated by z and

$$H_2 = z(h^2 + 4ef) + 2(hxy - fx^2 + ey^2).$$

Identify $\mathfrak{sl}_2 \cong \mathfrak{sl}_2^*$. Then we can take the quantum MF-subalgebra of $\mathcal{U}(\mathfrak{sl}_2)$ associated with either h or e. In both cases, we pass to $\mathcal{U}(\mathfrak{q})^{\mathfrak{h}}$ and add x and z as prescribed by the proof of Theorem 1.

The first algebra $\mathcal{A}$ is $\mathbb{K}[z, x, zh + xy, \mathrm{symm}(H_2)]$. Calculations in the centraliser $\mathcal{U}(\mathfrak{q})^{\mathbb{K}x}$ show that this one is maximal. The second algebra $\mathcal{A}$ is different:

$$\mathcal{A} = \mathbb{K}[z, x, 2ez - x^2, \mathrm{symm}(H_2)] \subset \mathbb{K}[z, x, e, \mathrm{symm}(H_2)].$$

It is not maximal.

In the Abelian reduction step, we obtain $\mathcal{A} = \mathsf{alg}\langle \mathcal{A}_1, \mathfrak{h}\rangle$. If $\mathcal{A} \subset \mathcal{C} \subset \mathcal{U}(\mathfrak{q})$ and $\mathcal{C}$ is commutative, then clearly $\mathcal{C} \subset \mathcal{U}(\mathfrak{q})^{\mathfrak{h}}$. However, some complications related to denominators may appear here as well.

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Institut für Mathematik, Friedrich-Schiller-Universität Jena
Jena, 07737, Germany
E-mail address: oksana.yakimova@uni-jena.de

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