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Pseudo-rotations and Steenrod squares revisited

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In this note we prove that if a closed monotone symplectic manifold admits a Hamiltonian pseudo-rotation, which may be degenerate, then the quantum Steenrod square of the cohomology class Poincaré dual to the point must be deformed. This result gives restrictions on the existence of pseudo-rotations, implying a form of uniruledness by pseudo-holomorphic spheres, and generalizes a recent result of the author. The new component in the proof consists in an elementary calculation with capped periodic orbits.

1. Setup

In this paper, (M,ω) denotes a closed monotone symplectic manifold of dimension $2n$, with the symplectic form rescaled so that $[\omega] = 2c_1(TM)$ on the image $H_2^S(M;\mathbb{Z})$ of the Hurewicz map $\pi_2(M) \to H_2(M;\mathbb{Z})$. For a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$, we denote by $\mathrm{Fix}(\phi)$ the set of its *contractible* fixed points, and by $x^{(k)}$ for $x \in \mathrm{Fix}(\phi)$ its image under the inclusion $\mathrm{Fix}(\phi) \subset \mathrm{Fix}(\phi^k)$. Contractible means that the homotopy class $\alpha(x,\phi)$ of the path $\alpha(x,H) = \{\phi_H^t(x)\}$ for a Hamiltonian $H \in \mathcal{H} \subset C^\infty([0,1]\times M,\mathbb{R})$ generating ϕ as the time-one map $\phi_H^1 = \phi$ of its Hamiltonian flow, is trivial[1]. Here $\mathcal{H} = \cap_{t\in[0,1]} \ker(I_t)$, $I_t(H) = \int_M H(t,-)\,\omega^n$. Further, recall that the minimal Chern number $N = N_M$ of (M,ω) is the index

$$[\mathbb{Z} : \mathrm{image}(c_1(TM) : \pi_2(M) \to \mathbb{Z})].$$

To an isolated fixed point x of ϕ one associates (cf. [7]) a local cohomology group $HF_{\mathrm{loc}}(\phi,x)$, which is naturally $\mathbb{Z}/(2)$-graded. If we choose a capping $\overline{x}$ of $\alpha(x,H)$, we obtain a $\mathbb{Z}$-graded version, $HF^*_{\mathrm{loc}}(H,\overline{x}) = HF^*_{\mathrm{loc}}(\widetilde{\phi},\overline{x})$. It depends only on the class $\widetilde{\phi}$ of $\{\phi_H^t\}_{t\in[0,1]}$ in the universal cover $\widetilde{\mathrm{Ham}}(M,\omega)$, and the capped orbit $\overline{x}$.

[1] This class does not depend on the choice of Hamiltonian by a classical argument in Floer theory.

We say that $\phi \in \mathrm{Ham}(M,\omega)$ is an $\mathbb{F}_2$ *Hamiltonian pseudo-rotation* if:

(i) It is perfect, that is for all iterations $k \geq 1$ of ϕ, $\mathrm{Fix}(\phi^k) = \mathrm{Fix}(\phi)$. In other words, ϕ admits no simple periodic orbits of order $k > 1$.

(ii) for all iterations $k \geq 1$,

$$N(\phi^k, \mathbb{F}_2) = \sum_{x \in \mathrm{Fix}(\phi)} \dim_{\mathbb{F}_2} HF^*_{\mathrm{loc}}(\phi^k, x^{(k)}) = \dim_{\mathbb{F}_2} H^*(M; \mathbb{F}_2).$$

Remark 1. Observe that a perfect Hamiltonian diffeomorphism necessarily has no symplectically degenerate maxima (see [8]). Furthermore, in the case where ϕ is strongly non-degenerate, that is all the points in $\mathrm{Fix}(\phi^k)$ are non-degenerate, for all $k \geq 1$, then $HF_{\mathrm{loc}}(\phi, x) \cong \mathbb{K}$, and all iterations are *admissible*, that is $\lambda^k \neq 1$ for all eigenvalues $\lambda \neq 1$ of $D(\phi)_x$. By the Smith inequality in local Floer homology [2, 15], conditions (i) and (ii) imply for iterations $k = 2^m$, the stronger statement that for all $x \in \mathrm{Fix}(\phi)$, $\dim_{\mathbb{F}_2} HF^{\mathrm{loc}}(\phi^k, x^{(k)}) = \dim_{\mathbb{F}_2} HF^{\mathrm{loc}}(\phi, x)$. Moreover, [13, Theorem A] indicates that when a Hamiltonian diffeomorphism has a finite number of periodic points, then a condition like (ii) should be satisfied[2]. Showing this in general would bridge the gap between the initial Chance-McDuff conjecture (see for example [8]) and Theorem A. Dynamics of Hamiltonian pseudo-rotations in higher dimensions were recently studied by Ginzburg and Gürel [6]. We refer thereto for a further discussion of this interesting notion initially arising from [1], surveying results more closely related to the subject of this paper in Section 2.

2. Results

We call a symplectic manifold strongly uniruled if there exists a non-trivial 3-point genus-0 Gromov-Witten invariant $\langle [pt], a, b \rangle_\beta$, for $\beta \in H_2(M, \mathbb{Z}) \setminus \{0\}$. By [10, Lemma 2.1], if (M, ω) is not strongly uniruled, then the quantum square $\mu * \mu = 0$ for the degree $2n$ cohomology class μ Poincaré dual to the point class. A generally different stronger notion than $\mu * \mu = 0$, is that the quantum Steenrod square $\mathcal{QS}(\mu)$, defined in [16], of the volume class μ

[2]Note that semisimplicity of quantum cohomology from [13] implies that the quantum square, and hence the quantum Steenrod square, of the class $\mu = PD([pt])$ below is deformed.

satisfies

$$\mathcal{QS}(\mu) = h^{2n}\mu, \tag{1}$$

where h is a formal variable of degree 1. Note that $\mathcal{QS}(\mu)$ is equal to the classical Steenrod square $Sq(\mu) = h^{2n}\mu$ plus quantum corrections coming from certain pseudo-holomorphic curves. When (1) does not hold, we say that M is $\mathbb{Z}/(2)$-*Steenrod uniruled*. The main result of this note is the following.

Theorem A. *Let (M, ω) be a closed monotone symplectic manifold admitting an $\mathbb{F}_2$ Hamiltonian pseudo-rotation ϕ. Then (M, ω) is $\mathbb{Z}/(2)$-Steenrod uniruled.*

Remark 2. We observe following [14] that when (M, ω) is $\mathbb{Z}/(2)$-Steenrod uniruled, then it is geometrically uniruled: there exists a J-holomorphic curve through each point of M for each ω-compatible almost complex structure J. Hence, Theorem A provides a geometric obstruction to the existence of pseudo-rotations: for example a one-point blowup of the standard T^{2n}, $n > 1$, admits none. Other arguments in [14] rule out the existence of pseudo-rotations for $N > n + 1$. The existence of pseudo-rotations was ruled out in [8] (and references therein), in a strong way, by proving the Conley conjecture, for manifolds such that $\omega(A) \cdot c_1(A) \leq 0$ for all spherical homology classes $A \in H_2^S(M; \mathbb{Z})$.

Remark 3. Theorem A is the strongest result of its kind currently available. It was proven by the author in [14] under the additional assumption that (M, ω) satisfies the Poincaré duality property $c([M], \overline{H}) = -c([pt], H)$, $H \in \mathcal{H}$ for Hamiltonian spectral invariants (which holds in particular whenever the minimal Chern number of (M, ω) satisfies $N > n$). Under the additional assumption that ϕ is strongly non-degenerate, that is $\ker(D(\phi^k)_x - \mathrm{id}) = 0$, for all $k \in \mathbb{Z}_{>0}, x \in \mathrm{Fix}(\phi)$, Theorem A was also proved by Çineli, Ginzburg, and Gürel in [3], using different additional arguments extending [14]. While the specific short argument in this paper strongly relies on monotonicity, it is also the maximal level of generality admissible to the current technology: for example [3] does not apply per se in the possibly degenerate setting of Theorem A. However, both results have potential for future generalizations in different directions. A different result relating the existence of pseudo-rotations to pseudo-holomorphic spheres was proven in [4].

3. Proof

First, we recall a few preliminaries. The notion of mean-index, introduced in symplectic topology in [12] is described as follows. For a Hamiltonian $H \in \mathcal{H}$ generating $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M, \omega)$ and capped periodic orbit $\overline{x}$ of H, we set

$$\Delta(H, \overline{x}) = \Delta(\widetilde{\phi}_H, \overline{x}) = \lim_{k \to \infty} \frac{1}{k} CZ(\widetilde{\phi}^k, \overline{x}^{(k)}),$$

where $\overline{x}^{(k)}$ is $\overline{x}$ iterated k times, which is indeed a capped periodic orbit of a Hamiltonian generating $\widetilde{\phi}^k$. The limit exists, since the Conley-Zehnder index comes from a quasi-morphism $\widetilde{Sp}(2n, \mathbb{R}) \to \mathbb{R}$ (see [5]). The mean-index satisfies the following properties that we use below:

(i) *homogeneity:* $\Delta(\widetilde{\phi}^k, \overline{x}^{(k)}) = k \cdot \Delta(\widetilde{\phi}, \overline{x})$, for all $k \in \mathbb{Z}_{>0}$.

(ii) *recapping:* $\Delta(\widetilde{\phi}, \overline{x} \# A) = \Delta(\widetilde{\phi}, \overline{x}) - 2 \langle c_1(TM), A \rangle$, $A \in H_2^S(M; \mathbb{Z})$.

(iii) *distance to index:* $CZ(\widetilde{\phi}, \overline{x}) \in [\Delta(\widetilde{\phi}, \overline{x}) - n, \Delta(\widetilde{\phi}, \overline{x}) + n]$,

(iv) *support of local Floer cohomology:* $HF^r_{\mathrm{loc}}(\widetilde{\phi}, \overline{x}) = 0$, unless $r \in [\Delta(\widetilde{\phi}, \overline{x}) - n, \Delta(\widetilde{\phi}, \overline{x}) + n]$.

(v) *symplectically degenerate maxima:* by definition, $\overline{x}$ is not a SDM of ϕ, if $HF^r_{\mathrm{loc}}(\widetilde{\phi}, \overline{x}) = 0$, unless $r \in [\Delta(\widetilde{\phi}, \overline{x}) - n, \Delta(\widetilde{\phi}, \overline{x}) + n)$.

Remark 4. Observe that, by the symmetry of the Conley-Zehnder index and a simple duality argument, if the reversal $\overline{x}^{(-1)}$ of $\overline{x}$ is not an SDM of ϕ^{-1}, then $HF^r_{\mathrm{loc}}(\widetilde{\phi}, \overline{x}) = 0$, unless $r \in (\Delta(\widetilde{\phi}, \overline{x}) - n, \Delta(\widetilde{\phi}, \overline{x}) + n]$.

Furthermore, keeping in mind the duality between Floer homology and Floer cohomology [9], for a quantum cohomology class $\mu \in QH^*(M; \Lambda_{\mathbb{F}_2}) \setminus \{0\}$ and a Hamiltonian $H \in \mathcal{H}$ with isolated contractible fixed points $\mathrm{Fix}(\phi^1_H)$, we recall that the Hamiltonian spectral invariant $c(\mu, H)$ of μ is *carried* by a capped 1-periodic orbit $\overline{x}$ of H, if the following holds: in the cohomology version $C(H)$ of the filtered canonical complex from [14, Theorem D] on (a completion of) $\oplus HF^*_{\mathrm{loc}}(H, \overline{z})$, $\overline{z}$ running over cappings of $\alpha(H, z)$, $z \in \mathrm{Fix}(\phi^1_H)$, a chain-level representative of filtration level $c(\mu, H)$ of the image $PSS_H(\mu)$ of μ under the PSS isomorphism [11] from the quantum cohomology $QH^*(M, \Lambda_{\mathbb{F}_2})$ to $HF^*(H) = H^*(C(H))$ contains a non-zero contribution from the summand $HF^*_{\mathrm{loc}}(H, \overline{x})$ and $\mathcal{A}_H(\overline{x}) = c(\mu, H)$. We refer the reader to [7, 8] for a different detailed description of this notion, recording only the following two facts:

(i) *spectrality:* for (M,ω) rational, in particular monotone, for each non-zero class $\mu \in QH^*(M;\Lambda_{\mathbb{F}_2})$, and $H \in \mathcal{H}$ with $\#\mathrm{Fix}(\phi^1_H) < \infty$, $c(\mu, H)$ is carried by at least one capped 1-periodic orbit $\overline{x}$ of H.

(ii) *contribution to local Floer cohomology:* if μ is homogeneous of degree k, and $\overline{x}$ carries $c(\mu,H)$, then $HF^k_{\mathrm{loc}}(H,\overline{x}) \neq 0$.

The proof of our main result relies on the following observations. First, as mentioned in Remark 1, no fixed point of ϕ or ϕ^{-1} is a symplectically degenerate maximum (see [8]). In particular, if the capping $\overline{x}$ of a contractible fixed point $x \in \mathrm{Fix}(\phi)$ carries a cohomology class μ of Conley-Zehnder index n in $HF^n(\widetilde{\phi}) \cong QH^{2n}(M,\Lambda_{\mathbb{F}_2})$ for a lift $\widetilde{\phi}$ of ϕ to the universal cover $\widetilde{\mathrm{Ham}}(M,\omega)$ of $\mathrm{Ham}(M,\omega)$, then its mean-index $\Delta = \Delta(\widetilde{\phi},\overline{x})$ satisfies $n < \Delta + n$. Hence

$$\Delta(\widetilde{\phi},\overline{x}) > 0. \tag{2}$$

Second, the following result was proven in [14] specifically in the setting of a *pseudo-rotation* assuming that (1) holds. Here $\mu \in QH^{2n}(M,\Lambda_{\mathbb{F}_2})$ denotes the cohomology class Poincaré dual to the point.

Theorem B. *Let ψ be an $\mathbb{F}_2$ pseudo-rotation of (M,ω) that is not $\mathbb{Z}/(2)$-Steenrod uniruled. Then*

$$c(\mu,\widetilde{\psi}^2) \geq 2 \cdot c(\mu,\widetilde{\psi}) \tag{3}$$

for each $\widetilde{\psi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ covering ψ.

We proceed to the proof of the main result, which follows a calculation from [7].

Proof of Theorem A. Choose $H \in \mathcal{H}$, such that the path $\{\phi^t_H\}_{t\in[0,1]}$ represents the class $\widetilde{\phi}$ lifting ϕ. By the perfectness property (i) of a pseudo-rotation, the spectrality property (i) of $c(\mu, H^{(k)}), k \in \mathbb{Z}_{>0}$, and the pigeonhole principle, there exists a fixed point $x \in \mathrm{Fix}(\phi)$, and an increasing sequence k_i such that $c(\mu, H^{(r_i)})$ for $r_i = 2^{k_i}$ is carried by a capping y_i of the 1-periodic orbit of the r_i-iterated Hamiltonian $H^{(r_i)}$ corresponding to $x^{(r_i)}$. By taking a power of ϕ, we can assume that $r_1 = 1$, and set $y = y_1$. Write y_i as a recapped iteration of y, for an auxiliary homology class $A_i \in H^S_2(M;\mathbb{Z})$,

$$y_i = y^{(r_i)} \# A_i. \tag{4}$$

We claim that for r_i large, $\omega(A_i) \leq 0$, and $c_1(A_i) > 0$ contradicting monotonicity. Indeed, write $\mathcal{A}_i$ for the action functional of $H^{(r_i)}$, and $\mathcal{A} := \mathcal{A}_1$. If

(M,ω) is not $\mathbb{Z}/(2)$-Steenrod uniruled, then by (4) and Theorem B,

$$r_i\mathcal{A}(y) - \omega(A_i) = \mathcal{A}_i(y_i) = c(\mu, H^{(r_i)}) \geq r_i c(\mu, H) = r_i\mathcal{A}(y).$$

Hence

$$\omega(A_i) \leq 0.$$

However, we know that y_i carries $c(\mu, H^{(r_i)})$, hence $\Delta(H^{(r_i)}, y_i) \in (0, 2n)$ and also $\Delta(H,y) \in (0,2n)$. Hence $r_i\Delta(H,y) > 2n$ for r_i large enough, and

$$2n > \Delta(H^{(r_i)}, y_i) = r_i\Delta(H,y) - 2c_1(A_i).$$

Therefore

$$c_1(A_i) > 0. \qquad \square$$

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References

[1] D. V. Anosov and A. B. Katok, *New examples in smooth ergodic theory. Ergodic diffeomorphisms*, Trudy Moskov. Mat. Obšč. **23** (1970), 3–36.

[2] E. Cineli and V. L. Ginzburg, *On the iterated Hamiltonian Floer homology*, `arXiv:1902.06369`, (2019).

[3] E. Cineli, V. L. Ginzburg, and B. Z. Gürel, *From pseudo-rotations to holomorphic curves via quantum Steenrod squares*, `arXiv:1909.11967`, (2019).

[4] E. Cineli, V. L. Ginzburg, and B. Z. Gürel, *Pseudo-rotations and holomorphic curves*, `arXiv:1905.07567`, (2019).

[5] M. Entov and L. Polterovich, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), no. 3, 773–826.

[6] V. L. Ginzburg and B. Gürel, *Hamiltonian pseudo-rotations of projective spaces*, Invent. Math. **214** (2018), no. 3, 1081–1130.

[7] V. L. Ginzburg and B. Z. Gürel, *Conley conjecture for negative monotone symplectic manifolds*, Int. Math. Res. Not. IMRN (2012), no. 8, 1748–1767.

[8] V. L. Ginzburg and B. Z. Gürel, *Conley conjecture revisited*, Int. Math. Res. Not. IMRN (2019), no. 3, 761–798.

[9] R. Leclercq and F. Zapolsky, *Spectral invariants for monotone Lagrangians*, J. Topol. Anal. **10** (2018), no. 3, 627–700.

[10] D. McDuff, *Hamiltonian S^1-manifolds are uniruled*, Duke Math. J. **146** (2009), no. 3, 449–507.

[11] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, in: Contact and Symplectic Geometry (Cambridge, 1994), Vol. 8 of Publ. Newton Inst., 171–200, Cambridge Univ. Press, Cambridge (1996).

[12] D. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. **45** (1992), no. 10, 1303–1360.

[13] E. Shelukhin, *On the Hofer-Zehnder conjecture*, `arXiv:1905.04769`, (2019).

[14] E. Shelukhin, *Pseudo-rotations and Steenrod squares*, `arXiv:1905.05108`, (2019).

[15] E. Shelukhin and J. Zhao, *The $\mathbb{Z}/p\mathbb{Z}$-equivariant product-isomorphism in fixed point Floer cohomology*, `arXiv:1905.03666`, (2019).

[16] N. Wilkins, *A construction of the quantum Steenrod squares and their algebraic relations*, `arXiv:1810.02738`, (2018).

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