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On the stability of the anomaly flow

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We prove that the parabolic flow of conformally balanced metrics introduced in [13] is stable around Calabi-Yau metrics. The result shows that the flow can converge on a Kähler manifold even if the initial metric is not conformally Kähler.

1. Introduction

Anomaly flow is a geometric flow of Hermitian metrics studied in [3–5, 8–14]. The flow was originally considered in [10] on complex threefolds to study the Strominger system [15] and involves a real parameter α'. The flow was subsequently generalized to any complex dimension $n \geq 3$ for $\alpha' = 0$ in [13]. The latter evolves an initial Hermitian metric ω_0 on a compact complex manifolds M of complex dimension $n \geq 3$ with $c_1(M) = 0$ by

$$(1) \qquad \tfrac{\partial}{\partial t}(|\Omega|_\omega \omega^{n-1}) = i\partial\bar{\partial}\omega^{n-2}\,, \quad \omega(0) = \omega_0\,,$$

where Ω is a fixed complex volume form and $|\cdot|_\omega$ is the pointwise norm with respect to ω. By "complex volume form" we just mean a nowhere vanishing $(n,0)$-form, indeed for the purpose of the present paper we do not need to assume Ω to be holomorphic.

The well-posedness of the flow is proved in [13, Theorem 1] under the assumption on ω_0 to be conformally balanced (in such a case the components of $\omega(t)$ satisfy a parabolic system [13, Theorem 4]). Moreover, when ω_0 is conformally balanced, (1) is conformally equivalent to the Hermitian curvature flow introduced by Ustinovskiy in [16] (see [4]).

The flow (1) can only converge when M is Kähler. The research of the present paper is motivated by the following theorem about the long time existence and convergence of the flow when $|\Omega|_{\omega_0}^{1/(n-1)}\omega_0$ is a Kähler metric:

Theorem 1.1 (Phong, Picard and Zhang [13, Theorem 2]). *Let (M,χ) be a compact Kähler manifold with vanishing first Chern class and*

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let Ω be a complex volume form on M with constant norm with respect to χ. Let ω_0 be a Hermitian metric on M such that

$$|\Omega|_{\omega_0}\omega_0^{n-1} = \chi^{n-1}\,,$$

then (1) starting from ω_0 has a long-time solution which converges in C^∞–topology to the unique Ricci-flat Kähler metric $\omega_\infty \in [\chi] \in H^{1,1}(M)$.

The theorem gives an alternative proof of the Calabi-Yau theorem [17]. In [13] it is raised the problem of studying the convergence of the flow in Kähler manifolds when $|\Omega|_{\omega_0}^{1/(n-1)}\omega_0$ is not Kähler, for instance when $|\Omega|_{\omega_0}\omega_0^{n-1}$ is just closed and $[|\Omega|_{\omega_0}\omega_0^{n-1}] = [\chi^{n-1}]$, with χ Kähler. Here we prove that (1) is stable around Calabi-Yau metrics χ with $|\Omega|_\chi$ constant. In particular we have the convergence of (1), when $|\Omega|_{\omega_0}\omega_0^{n-1}$ is just close to the $(n-1)$-th power of a Kähler metric.

Theorem 1.2. *Let (M,χ) be a compact Kähler manifold with vanishing first Chern class and let Ω be a complex volume form on M with constant norm with respect to χ. For every $\epsilon > 0$ there exists $\delta > 0$ such that if ω_0 is a Hermitian metric on M satisfying*

$$\| \, |\Omega|_{\omega_0}\omega_0^{n-1} - \chi^{n-1}\|_{C^\infty} < \delta\,, \tag{2}$$

then flow (1) has a long-time solution $\omega(t) \in C^\infty(M\times[0,\infty),\Lambda^{1,1}_+)$ such that

$$\| \, |\Omega|_{\omega(t)}\omega(t)^{n-1} - \chi^{n-1}\|_{C^\infty} < \epsilon\,,\ \textit{for every } t\in[0,\infty)$$

and $|\Omega|_{\omega(t)}\omega(t)^{n-1}$ converges in C^∞–topology to a positive $(n-1,n-1)$-form ω_∞^{n-1} with ω_∞ astheno-Kähler.

If further ω_0 satisifies the conformally balanced condition $d(|\Omega|_{\omega_0}\omega_0^{n-1}) = 0$, then ω_∞ is Kähler Ricci flat.

We prove Theorem 1.2 as follows:

after the change of variable $|\Omega|_\omega\omega^{n-1} = \tilde\omega^{n-1}$, equation (1) rewrites as

$$\frac{\partial}{\partial t}\tilde\omega^{n-1} = i\partial\bar\partial(|\Omega|_{\tilde\omega}^{-2}\tilde\omega^{n-2})\,, \tag{3}$$

(see Lemma 3.1).

If we set $E(\tilde\omega^{n-1}) = i\partial\bar\partial(|\Omega|_{\tilde\omega}^{-2}\tilde\omega^{n-2})$, then the linearization $DE(\alpha^{n-1})$ of E at any positive $(n-1,n-1)$-real form α^{n-1} satisfies

$$DE(\alpha^{n-1})(\psi) = -\frac{1}{n-1}|\Omega|_\alpha \Box\psi + \text{l.o.t.}$$

for any *closed* $\psi \in C^\infty(M, \Lambda_{\mathbb{R}}^{n-1,n-1})$ (see Lemma 3.3) and the flow (3) fits in Hamilton's framework [6, Section 5 and 6] with integrability condition $L = d$ (this is analogous to the argument used in [13]). That in particular implies the well-posedness of the flow (1) also when ω_0 is not conformally balanced.

Moreover if χ is Kähler, then

$$DE(\chi^{n-1})(\psi) = -\frac{1}{n-1}|\Omega|_\omega \Box\psi$$

for any closed $\psi \in C^\infty(M, \Lambda_{\mathbb{R}}^{n-1,n-1})$. This allows us to apply a general result about the stability of second order geometric flows with an integrability condition. We state this theorem in section 2 and we prove it in the last section.

Notation. Given a vector bundle F on a manifold M, we denote by $C^\infty(M,F)$ the space of smooth sections of F. If further $I \subset \mathbb{R}$ is an interval we denote by $C^\infty(M \times I, F)$ the space of smooth time depending sections of F. When we write $\|f\|_{C^\infty} < \delta$, we mean that $\|f\|_{C^k} < \delta$ for every $k \in \mathbb{N}$.

2. A stability result for second order geometric flows with an integrability condition

In [6] Hamilton proved the following general result about the short-time existence of second order geometric flows on compact manifolds.

Let M be an oriented compact manifold, F a vector bundle over M, U an open subbundle of F and

$$E\colon C^\infty(M,U) \to C^\infty(M,F)$$

a second order differential operator. Consider the geometric flow

$$\frac{\partial f}{\partial t} = E(f)\,, \quad f(0) = f_0\,, \tag{4}$$

where f_0 belongs to $C^\infty(M,U)$. For $f \in C^\infty(M,U)$, we denote by $DE(f)$: $C^\infty(M,F) \to C^\infty(M,F)$ the linearization of E at f and by $\sigma DE(f)$ the principal symbol of $DE(f)$. Following Hamilton's paper we assume that there exists a first order linear differential operator

$$L\colon C^\infty(M,F)\to C^\infty(M,G)\,,$$

with values in another vector bundle G over M, such that

1. $L(E(f))=0$ for all $f\in C^\infty(M,U)$;
2. for every $f\in C^\infty(M,U)$ and for every $(x,\xi)\in T^*M$ all the eigenvalues of $\sigma DE(f)(x,\xi)$ restricted to $\ker\sigma L(x,\xi)$ have strictly positive real part .

Because of the following result L is called an *integrability condition for E.*

Theorem 2.1 (Hamilton [6, Theorem 5.1]). *Under the above assumptions the geometric flow* (4) *has a unique short-time solution.*

Remark 2.2. Theorem 5.1 in [6] is in fact more general since the integrability condition L is allowed to smoothly depend on $f\in C^\infty(M,U)$. This generality is needed to prove the short time existence of the Ricci flow.

Using Theorem 2.1 we will be able to prove the following stability theorem for geometric flows with an integrability condition L.

Theorem 2.3. *Assume that E and L are as above. Let $\bar f\in C^\infty(M,U)$ be such that $E(\bar f)=0$. Let $\bar h$ be a fixed metric along the fibers of F. Assume*

(i) $DE(\bar f)\colon\ker L\to\ker L$ *is symmetric and negative semidefinite with respect to $\bar h$;*

(ii) $E(f)$ *is L^2-orthogonal to* $\ker DE(\bar f)$ *for every $f\in C^\infty(M,U)$;*

(iii) $DE(\bar f):\ker L\to\ker L$ *extends to an elliptic operator* $\Phi:C^\infty(M,F)\to C^\infty(M,F)$.

Then for every $\epsilon>0$ there exists $\delta>0$ such that if $f_0\in C^\infty(M,U)$ satisfies

$$\|f_0-\bar f\|_{C^\infty}<\delta,$$

then (4) *has a long-time solution $f\in C^\infty(M\times[0,\infty),U)$ such that*

$$\|f(t)-\bar f\|_{C^\infty}<\epsilon,\ \textit{for every}\ t\in[0,\infty).$$

Moreover, $f(t)$ converges to $f_\infty \in C^\infty(M,U)$ in C^∞–topology which satisfies $E(f_\infty)=0$.

3. Proof of Theorem 1.2

Let (M,ω_0) be a compact n-dimensional Hermitian manifold with vanishing first Chern class and let Ω be a fixed complex volume form.

Lemma 3.1. *Let $\omega(t)$ be a solution to the geometric flow* (1) *on M; then $\tilde\omega=|\Omega|_\omega^{1/(n-1)}\omega$ satisfies*

$$\frac{\partial}{\partial t}\tilde\omega^{n-1}=i\partial\bar\partial(|\Omega|_{\tilde\omega}^{-2}\tilde\omega^{n-2})\,. \tag{5}$$

Proof. Since

$$\omega=|\Omega|_\omega^{-1/(n-1)}\tilde\omega\,,$$

we have

$$\frac{\partial}{\partial t}(\tilde\omega^{n-1})=i\partial\bar\partial(|\Omega|_\omega^{-(n-2)/(n-1)}\tilde\omega^{n-2})\,.$$

Now in general for any conformal factor $f\in C^\infty(M,\mathbb{R}_+)$ one has

$$|\Omega|_{f\tilde\omega}=f^{-n/2}|\Omega|_{\tilde\omega}$$

and thus

$$|\Omega|_\omega=(|\Omega|_\omega^{-1/(n-1)})^{-n/2}|\Omega|_{\tilde\omega}=|\Omega|_\omega^{n/(2n-2)}|\Omega|_{\tilde\omega}$$

from which we deduce

$$|\Omega|_\omega=|\Omega|_{\tilde\omega}^{(2n-2)/(n-2)}\,,$$

and the claim follows. □

Now we focus on the geometric flow (5) and we show that it fits in the set-up of Theorem 2.3. The flow is governed by the operator

$$E\colon C^\infty(M,\Lambda_+^{n-1,n-1})\to C^\infty(M,\Lambda_{\mathbb{R}}^{n-1,n-1})$$

defined by

$$E(\omega^{n-1})=i\partial\bar\partial(|\Omega|_\omega^{-2}\,\omega^{n-2}) \tag{6}$$

where $\Lambda_+^{n-1,n-1}$ is the bundle of positive real $(n-1,n-1)$-forms on M and $\Lambda_{\mathbb{R}}^{n-1,n-1}$ is the bundle of real $(n-1,n-1)$-forms.

In order to study the linearization of E, we describe the principal part of the operator $\square$ in terms of the components of $(n-1,n-1)$-real forms on M.

Let ω be any Hermitian metric on M and $\psi\in C^\infty(M,\Lambda^{n-1,n-1}_{\mathbb{R}})$. Then ψ writes in a unique way as

$$\psi=\frac{1}{(n-1)!}h_0\omega^{n-1}-\frac{1}{(n-2)!}h_2\wedge\omega^{n-2}\,, \tag{7}$$

where h_0 is a smooth function and $h_2\in C^\infty(M,\Lambda^{1,1}_{\mathbb{R}})$ satisfies

$$h_2\wedge\omega^{n-1}=0\,.$$

Since

$$*h_2=-\frac{1}{(n-2)!}h_2\wedge\omega^{n-2}\,,$$

the form ψ can be alternatively written as

$$\psi=*(h_0\omega+h_2)\,.$$

Lemma 3.2. *If ψ is closed, then*

$$\square\,\psi=\partial\partial^*\psi=-\frac{2i}{(n-2)!}\partial\bar\partial h_0\wedge\omega^{n-2}+\frac{i}{(n-3)!}\partial\bar\partial h_2\wedge\omega^{n-3}+\text{l.o.t.}$$

where "l.o.t." *stands for "lower order terms" in ψ. Moreover if ω is Kähler we have*

$$\square\,\psi=\partial\partial^*\psi=-\frac{2i}{(n-2)!}\partial\bar\partial h_0\wedge\omega^{n-2}+\frac{i}{(n-3)!}\partial\bar\partial h_2\wedge\omega^{n-3}\,.$$

Proof. Since ψ is closed we have

$$\square\,\psi=\partial\partial^*\psi=-\partial*\bar\partial*\psi=-\partial*(\bar\partial h_0\wedge\omega)-\partial*\bar\partial h_2+\text{l.o.t.}$$

On the other hand, from the closure of ψ, we deduce

$$\bar\partial h_2\wedge\omega^{n-2}=\frac{1}{n-1}\bar\partial h_0\wedge\omega^{n-1}+\text{l.o.t}\,.$$

Now we use the well-known splitting of $(1,2)$-forms as

$$\gamma=\gamma_++\gamma_-$$

where γ_+ is of the form $\alpha\wedge\omega$ with α a $(0,1)$-form and γ_- is such that $\gamma_-\wedge\omega^{n-2}=0$.

The fact we use is that $*\gamma_+ = \frac{i}{(n-2)!}\gamma_+ \wedge \omega^{n-3}$ and $*\gamma_- = -\frac{i}{(n-3)!}\gamma_- \wedge \omega^{n-3}$. Thus in our case, taking into account that $(\bar\partial h_2)_+ = \frac{1}{n-1}\bar\partial h_0 \wedge \omega + \text{l.o.t.}$, we have

$$\begin{aligned} *\bar\partial h_2 &= \frac{i}{(n-1)!}\bar\partial h_0 \wedge \omega^{n-2} - \frac{i}{(n-3)!}\left(\bar\partial h_2 - \frac{1}{n-1}\bar\partial h_0 \wedge \omega\right) \wedge \omega^{n-3} + \text{l.o.t.} \\ &= \frac{i}{(n-2)!}\bar\partial h_0 \wedge \omega^{n-2} - \frac{i}{(n-3)!}\bar\partial h_2 \wedge \omega^{n-3} + \text{l.o.t.} \end{aligned}$$

Therefore

$$\Box\,\psi = -\frac{2i}{(n-2)!}\partial\bar\partial h_0 \wedge \omega^{n-2} + \frac{i}{(n-3)!}\partial\bar\partial h_2 \wedge \omega^{n-3} + \text{l.o.t.},$$

where these lower order terms vanish if ω is closed since they all come from $d\omega$. □

Lemma 3.3. *Let ω be a Hermitian metric on M and let $\psi \in C^\infty(M, \Lambda_{\mathbb{R}}^{n-1,n-1})$ be closed; then*

$$DE(\omega^{n-1})(\psi) = -\frac{1}{n-1}|\Omega|_\omega \Box\psi + \text{l.o.t.} \tag{8}$$

Moreover if we assume that ω is Kähler then we have

$$DE(\omega^{n-1})(\psi) = -\frac{1}{n-1}|\Omega|_\omega \Box\psi\,.$$

Proof. Let $\omega(t)$, $t \in (-\epsilon, \epsilon)$ be a smooth curve of Hermitian metrics with $\omega(0) = \omega$ on M and we assume that

$$\psi := \tfrac{\partial}{\partial t}_{|t=0}\omega^{n-1}(t)$$

is closed. In order to simplify the notation we set

$$r(t) = |\Omega|^{-2}_{\omega(t)}\,, \quad r = r(0)\,, \quad \dot r = \tfrac{\partial}{\partial t}_{|t=0} r(t)\,, \quad \dot\omega = \tfrac{\partial}{\partial t}_{|t=0}\omega(t)\,.$$

We directly compute

$$\begin{aligned} \tfrac{\partial}{\partial t}_{|t=0} E(\omega(t)^{n-1}) &= i\partial\bar\partial(\dot r\,\omega^{n-2} + (n-2) r\,\dot\omega \wedge \omega^{n-3}) \\ &= i\partial\bar\partial \dot r \wedge \omega^{n-2} + (n-2) i r\,\partial\bar\partial\dot\omega \wedge \omega^{n-3} + \text{l.o.t.} \end{aligned}$$

We decompose ψ according to (7). Then [1, Lemma 2.5] implies

$$\dot\omega = \frac{h_0}{(n-1)(n-1)!}\omega - \frac{1}{(n-1)!}h_2$$

and using

$$\dot r = \frac{nh_0}{(n-1)(n-1)!}r$$

we obtain

$$\begin{aligned}\tfrac{d}{dt}_{|t=0}E(\omega(t)^{n-1}) = &\frac{n}{(n-1)(n-1)!}ir\,\partial\bar\partial h_0\wedge\omega^{n-2}\\ &+\frac{n-2}{(n-1)(n-1)!}ir\,\partial\bar\partial h_0\wedge\omega^{n-2}\\ &-\frac{n-2}{(n-1)!}ir\,\partial\bar\partial h_2\wedge\omega^{n-3}+\text{l.o.t.}\,,\end{aligned}$$

i.e.

$$\tfrac{d}{dt}_{|t=0}E(\omega(t)^{n-1}) = \frac{2}{(n-1)!}ir\,\partial\bar\partial h_0\wedge\omega^{n-2} -\frac{n-2}{(n-1)!}ir\,\partial\bar\partial h_2\wedge\omega^{n-3}+\text{l.o.t.} \tag{9}$$

and Lemma 3.2 implies (8). Since the lower order terms in (9) vanish if ω is closed, the claim follows. □

Proof of Theorem 1.2. Let (M,χ) be a compact Kähler manifold of complex dimension n. Let Ω be a complex volume form on M with constant norm with respect to χ and let $E\colon C^\infty(M,\Lambda^{n-1,n-1}_+)\to C^\infty(M,\Lambda^{n-1,n-1}_{\mathbb{R}})$ be as in (6). Then we have

$$E(\chi^{n-1})=0\,.$$

Now Lemma 3.3 and Hodge theory imply that all the assumptions of Theorem 2.3 are satisfied when we consider

$$\begin{aligned}&F=\Lambda^{n-1,n-1}_{\mathbb{R}}\,,\quad U=\Lambda^{n-1,n-1}_+\,,\\ &L=d\colon C^\infty(M,\Lambda^{n-1,n-1}_{\mathbb{R}})\to C^\infty(M,\Lambda^{2n-1}_{\mathbb{C}})\,.\end{aligned}$$

Hence for every $\epsilon>0$ there exists $\delta>0$ such that if $\tilde\omega_0$ is a Hermitian metric on M satisfying

$$\|\tilde\omega_0^{n-1}-\chi^{n-1}\|_{C^\infty}<\delta\,, \tag{10}$$

then there exists a smooth family of Hermitian metrics $\tilde\omega(t)$, $t\in[0,\infty)$, such that

$$\|\tilde\omega(t)^{n-1}-\chi\|_{C^\infty}<\epsilon\,,\quad \tfrac{\partial}{\partial t}\tilde\omega=i\partial\bar\partial(|\Omega|^{-2}_{\tilde\omega}\tilde\omega^{n-2})\,,\quad \tilde\omega(0)=\tilde\omega_0 \tag{11}$$

and $\tilde{\omega}(t)$ converges in C^∞–topology to a Hermitian metric $\tilde{\omega}_\infty$ such that

$$\partial\bar{\partial}(|\Omega|^{-2}_{\tilde{\omega}_\infty}\,\tilde{\omega}^{n-2}_\infty) = 0\,. \tag{12}$$

Now let ω_0 be a Hermitian metric on M satisfying (2); then $\tilde{\omega}_0 := |\Omega|^{1/(n-1)}_{\omega_0}\omega_0$ satisfies (10). Thus there exists $\tilde{\omega} \in C^\infty(M \times [0,\infty), \Lambda^{1,1}_{\mathbb{R}})$ satisfying (11) and converging in C^∞-topology to a $\tilde{\omega}_\infty$ for which (12) holds. Therefore $\omega = |\Omega|^{-2/(n-2)}_{\tilde{\omega}}\tilde{\omega}$ is a solution to the anomaly flow (1) satisfying

$$\||\Omega|_{\omega(t)}\omega(t)^{n-1} - \chi\|_{C^\infty} < \epsilon\,,$$

and $|\Omega|_{\omega(t)}\omega(t)^{n-1}$ converges in C^∞-topology to

$$\omega^{n-1}_\infty = |\Omega|^{-2(n-1)/(n-2)}_{\tilde{\omega}_\infty}\tilde{\omega}^{n-1}_\infty\,.$$

By a change of variable we obtain that ω_∞ is astheno-Kähler, i.e.

$$\partial\bar{\partial}(\omega^{n-2}_\infty) = 0\,,$$

and the first part of the claim follows.

About the second part of the claim, we use that if ω_0 satisfies the conformally balanced condition $d(|\Omega|_{\omega_0}\omega_0^{n-1}) = 0$, then $d(|\Omega|_{\omega(t)}\omega(t)^{n-1}) = 0$ for every t and hence ω_∞ satisfies

$$d(|\Omega|_{\omega_\infty}\omega^{n-1}_\infty) = 0\,, \quad \partial\bar{\partial}(\omega^{n-2}_\infty) = 0\,.$$

In view of [13, Lemma 1] (see also [7]), ω_∞ is Kähler-Einstein. □

4. Proof of Theorem 2.3

In this section we prove the general result about the stability of geometric flows in Hamilton's set-up described in section 2.

Proof. We adapt the proof of the main theorem in [2].
Fix a metric connection ∇ on $(F, \bar{h})$ and a volume form on M. The space $C^\infty(M, F)$ has the natural structure of tame Fréchet space given by the Sobolev norms $\|.\|_{H^n}$ induced by $\bar{h}$, ∇ and the volume form of M.

On $C^\infty(M \times [a,b], F)$ we consider the grading

$$\|f\|_{n,[a,b]} = \sum_{2j \le n} \int_a^b \|\partial_t^j f(t)\|_{H^{n-2j}}\, dt.$$

Hamilton in [6, sections 5 and 6] proved that, with respect to this grading, for any $T > 0$ the map

$$\mathcal{F}\colon C^\infty(M \times [0,T], U) \to C^\infty(M \times [0,T], F) \times C^\infty(M, F)$$

$$\mathcal{F}(f) = (\partial f/\partial t - E(f), f(0))$$

satisfies the assumptions of the Nash-Moser theorem, i.e. $\mathcal{F}$ is smooth tame, $D\mathcal{F}(f)$ is bijective for every $f \in C^\infty(M \times [0,T], U)$ and the family of the inverses

$$C^\infty(M \times [0,T], U) \times C^\infty(M \times [0,T], F) \times C^\infty(M, F) \to C^\infty(M \times [0,T], F)$$

$$(f, (g,k)) \mapsto D\mathcal{F}(f)^{-1}(g,k)$$

is a smooth tame map. Hence the Nash-Moser theorem can be applied and $\mathcal{F}$ is locally invertible with smooth tame inverse. As a direct consequence, arguing as in Proposition 5.3 in [2] we have the following

Claim 1. *For all $\epsilon, T > 0$, there exists $\delta' > 0$ such that if $f_0 \in C^\infty(M, U)$ satisfies*

$$\|f_0 - \bar{f}\|_{C^\infty} < \delta',$$

then (4) has a solution $f \in C^\infty(M \times [0,T], U)$ and

$$\|f - \bar{f}\|_{n,[0,T]} < \epsilon$$

for all $n \in \mathbb{N}$.

Then we show the following

Claim 2. *For δ' small enough the L^2-norm of $E(f)$ with respect to $\bar{h}$ has an exponential decay.*

Fix a small time $\tau > 0$ arbitrary. By Claim 1 we have that there exists $\delta' > 0$ such that if $\|f_0 - \bar{f}\|_{C^\infty} < \delta'$, then problem (4) has a solution

$f \in C^\infty(M \times [0, T+2\tau], U)$ with $\|f - \bar f\|_{l,[0,T+2\tau]}$ bounded for every l. Now since

$$\frac{\partial}{\partial t} E(f) = DE(f)(E(f)),$$

we have

$$\frac{d}{dt}\|E(f)\|^2_{L^2} = 2\langle \frac{\partial}{\partial t} E(f), E(f)\rangle_{L^2} = 2\langle DE(f)(E(f)), E(f)\rangle_{L^2}.$$

Moreover a general result for families of symmetric operators on Hilbert spaces combined with the Sobolev Embedding theorem and elliptic regularity of $DE(\bar f)$ (see [2] Corollary 5.6) imply that given $a > 0$ we can choose δ' so small that we have

$$\langle DE(f)(E(f)), E(f)\rangle_{L^2} \leq (1-a)\langle DE(\bar f)(E(f)), E(f)\rangle_{L^2} + a\|E(f)\|^2_{L^2}$$

for every time in the interval $[0, T+\tau]$. Now let λ be half the smallest positive eigenvalue of $-DE(\bar f)_{|\ker L}$. Take $a = \frac{\lambda}{2\lambda+1}$ so that

$$\langle DE(f)(E(f)), E(f)\rangle_{L^2} \leq \frac{\lambda+1}{2\lambda+1}\langle DE(\bar f)(E(f)), E(f)\rangle_{L^2} + \frac{\lambda}{2\lambda+1}\|E(f)\|^2_{L^2}.$$

By assumption (i) and (ii) we have $\langle DE(\bar f)(E(f)), E(f)\rangle_{L^2} \leq -2\lambda\|E(f)\|^2_{L^2}$ thus

$$\langle DE(f)(E(f)), E(f)\rangle_{L^2} \leq -\lambda\|E(f)\|^2_{L^2}$$

which implies

$$\frac{d}{dt}\|E(f)\|^2_{L^2} \leq -2\lambda\|E(f)\|^2_{L^2}.$$

Using Gronwall's lemma we get

$$\|E(f(t))\|^2_{L^2} \leq e^{-2\lambda t}\|E(f_0)\|^2_{L^2}, \qquad \text{for all } t \in [0, T+\tau),$$

and the Claim 2 follows.

By integrating the last formula we get

$$\|E(f)\|^2_{0,[t,T+\tau]} = \int_t^{T+\tau} \|E(f(s))\|^2_{L^2}\, ds \le \|E(f_0)\|^2_{L^2} \frac{e^{-2\lambda t}}{2\lambda}\,. \tag{13}$$

Since using the parabolic Sobolev embedding theorem (see [2] Corollary 5.8) there exist m and $C > 0$ such that for every $t \in [0,T]$

$$\|E(f(t))\|_{H^n} \le C\|E(f)\|_{m,[t,T+\tau]}\,,$$

we will need the following estimate in order to prove H^n-exponential decay.

Claim 3. *Let $\tau_0 \in (0,T)$. For δ' small enough we have that for every $m \in \mathbb{N}$ there exists $C > 0$ such that*

$$\|E(f)\|_{m,[t,T+\tau]} \le C\|E(f)\|_{0,[t-\tau_0,T+\tau]}\,,$$

for every $t \in [\tau_0, T]$.

We prove by induction on m that for every $\tau_0 \in (0,T)$ we can choose δ' small enough such that there exists a positive C (depending on m, τ_0 and an upper bound on δ') such that for every $g \in C^\infty(M \times [0,T+\tau], F)$ solving

$$\frac{\partial}{\partial t} g = DE(f)(g)\,,$$

the following estimate holds

$$\|g\|_{m,[t,T+\tau]} \le C\|g\|_{0,[t-\tau_0,T+\tau]}\,, \tag{14}$$

for every $t \in [\tau_0, T]$. Then we deduce the claim by setting $g = E(f)$.
For $m = 0$ the estimate (14) is trivial. We assume the above statement true up to $m = N$. For a smooth family of linear second order differential operator P we set

$$|[P]|_N = \sum_{2j \le N} [\tfrac{\partial^j}{\partial t^j} P]_{N-2j}$$

where $[P]_N$ is the supremum of the norm of P and its space covariant derivatives up to degree N. By [2, Lemma 6.10] for δ' small enough there exists $C > 0$, depending on T and an upper bound on δ', such that for $t \in [0, T+\tau)$

and $g\in C^\infty(M\times[0,T+\tau],F)$ we have

$$\|g\|_{N+2,[t,T+\tau]}\leq C\left(\|\tfrac{\partial}{\partial t}g-DE(f)(g)\|_{N,[t,T+\tau]}+\|g(t)\|_{H^{N+1}}\right)\\ +C|[DE(f)]|_N\left(\|\tfrac{\partial}{\partial t}g-DE(f)(g)\|_{0,[t,T+\tau]}+\|g(t)\|_{H^1}\right)\,,$$

which implies

$$\|g\|_{N+2,[t,T+\tau]}\leq C(1+|[DE(f)]|_N)\\ \times\left(\|\tfrac{\partial}{\partial t}g-DE(f)(g)\|_{N,[t,T+\tau]}+\|g(t)\|_{H^{N+1}}\right)\,.$$

Up to shrink δ' we may assume

$$|[DE(f)]|_N\leq 1+|[DE(\bar f)]|_N$$

in order to rewrite the last estimate as

$$\|g\|_{N+2,[t,T+\tau]}\leq C(2+|[DE(\bar f)]|_N)\\ \times\left(\|\tfrac{\partial}{\partial t}g-DE(f)(g)\|_{N,[t,T+\tau]}+\|g(t)\|_{H^{N+1}}\right)\,. \tag{15}$$

Now assume that g satisfies

$$\frac{\partial}{\partial t}g=DE(f)(g)$$

and let $\chi\colon\mathbb{R}\to[0,1]$ be smooth and such that

$$\begin{cases}\chi(s)=0 & \text{for } s\leq t-\frac{\tau_0}{2}\\ \chi(s)=1 & \text{for } s\geq t\,.\end{cases}$$

Then $\tilde g=\chi g$ satsfies

$$\frac{\partial}{\partial t}\tilde g=DE(f)(\tilde g)+\dot\chi g$$

and from (15) we deduce

$$\|g\|_{N+2,[t,T+\tau]}\leq\|\tilde g\|_{N+2,[t-\frac{\tau_0}{2},T+\tau]}\\ \leq C(2+|[DE(\bar f)]|_N)\|\dot\chi g\|_{N,[t-\frac{\tau_0}{2},T+\tau]}\leq C'\|g\|_{N,[t-\frac{\tau_0}{2},T+\tau]}$$

and the claim follows using the inductive hypothesis.

Finally putting these together with (13) we have that for δ' small enough we have

$$\|E(f(t))\|_{H^n} \leq C\|E(f_0)\|_{L^2}\mathrm{e}^{-\lambda t} \tag{16}$$

for $t \in [\tau_0, T]$. The constant C may depend on T, τ_0 and an upperbound on δ', but not on t and f_0.

Now we choose $\delta \leq \delta'$ such that if $\|f_0 - \bar{f}\|_{C^\infty} \leq \delta$ then

$$C\|E(f_0)\|_{L^2}\frac{\mathrm{e}^{-\lambda\tau_0}}{\lambda}\sum_{j=0}^{\infty}\mathrm{e}^{-\lambda j(T-\tau_0)} + \|f(\tau_0) - \bar{f}\|_{H^n} \leq \delta' \,. \tag{17}$$

Using (16) and (17) and working as in [2] we have that for any $t \in [NT - (N-1)\tau_0, (N+1)T - N\tau_0]$ with $N \in \mathbb{N}$

$$\|f - \bar{f}\|_{H^n} \leq C\|E(f_0)\|_{L^2}\frac{\mathrm{e}^{-\lambda\tau_0}}{\lambda}\sum_{j=0}^{N}\mathrm{e}^{-\lambda j(T-\tau_0)} + \|f(\tau_0) - \bar{f}\|_{H^n} \leq \delta'.$$

This allows us to conclude that the solution f is defined in $M \times [0, \infty)$.

Now let $f_\infty = f_0 + \int_0^\infty E(f)ds \in C^\infty(M, F)$; since

$$\lim_{t\to\infty}\|f(t) - f_\infty\|_{H^n} \leq \lim_{t\to\infty} C\|E(f_0)\|_{L^2}\mathrm{e}^{-\lambda t} = 0\,, \text{ for } n \text{ large enough}$$

$f(t)$ converges to f_∞ in C^∞-topology. Possibly shrinking δ, we will have $f_\infty \in C^\infty(M, U)$. Using again [2, Proposition 5.7], we have that up to shrink δ,

$$\|f(t) - \bar{f}\|_{C^\infty} < \epsilon$$

for every $t \in [0, \infty)$.

Finally

$$E(f_\infty) = \lim_{t\to 0} E(f(t)) = 0$$

and the claim follows. □

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