Pure and Applied Mathematics Quarterly
Volume 1, Number 1, 222—226, 2005

Excentric compactifications

Steven Zucker

The term *excentric* was coined by the author [6 : §1], [13 : §2]. It is accented on the first syllable, in contrast with the English word "eccentric", and conveys the following idea. For now, let W be a unipotent algebraic group. Then W/W (the trivial group) is the *reductive* quotient of W. When $U \subseteq W$ is a subgroup that is the center of something, then W/U is the (or an) *excentric* quotient of W.

We present the setting for these notes. Let D be a symmetric space of non-compact type, and Γ an arithmetically defined group of isometries of D; put informally, this means that some algebraic group $\mathcal{G}$ over $\mathbf{Q}$ has its real points giving the isometry group of D, and Γ is roughly $\mathcal{G}(\mathbf{Z})$. If Γ is not too big (i.e., is torsionfree, later neat), then $X = \Gamma\backslash D$ is a manifold. When D has an invariant complex structure, D is called *Hermitian*, as is X. The latter is called a locally symmetric variety, for X is a quasi-projective complex algebraic variety [2].

Typically, X is non-compact and one soon realizes that it is important to compactify it. There exist too many compactifications of X, so we select one or more to suit a given purpose. It is common enough to attach a Γ-equivariant boundary ∂D to D, and then take the quotient by Γ. Here are two such compactifications of X:

i) $\overline{X} = \Gamma\backslash\overline{D}$, the manifold-with-corners of Borel-Serre [3],

ii $X^{Sa} = \Gamma\backslash D^{Sa}$, a Satake compactification of X [9] (see also [11]). There are finitely many Satake compactifications. When X is Hermitian, one of these is topologically the Baily-Borel compactification X^*, a projective variety over C that is generally quite singular.

When X is Hermitian, there are also the smooth toroidal compactifications X^{tor} of Mumford et al. [1], constructed so that ∂X^{tor} is a divisor with normal crossings.

Received September 12, 2004.

A morphism $Y_1 \to Y_2$ of compactifications of X is the unique extension of the identity mapping of X, if it exists. For instance, for the three types of compactification above of a locally symmetric variety, there are morphisms

$$\begin{array}{ccc} & X^{\text{tor}} & \\ & \downarrow & \\ \overline{X} \to & X^*. & \end{array} \qquad (*)$$

We see that X^* is a common quotient of X and X^{tor}. In general, there is no morphism in either direction between $\overline{X}$ and X^{tor} .

One might take as a criterion for a good compactification that a (locally) homogeneous vector bundle $E \to X$ should extend to the compactification. Extending to $\overline{X}$ is trivial, as $\overline{X}$ is homotopy equivalent to X. It is wiser to take a quotient $\overline{X}^{\text{red}}$ of $\overline{X}$, the reductive Borel-Serre compactification, which is defined as follows. The open faces of $\overline{D}$ are of the form

$$e(R) \simeq D_R \times W_R,$$

with W_R the unipotent radical of R (real points). To get the open faces of $\overline{D}^{\text{red}}$, one collapses W_R to a point, yielding $e(R)^{\text{red}} \simeq D_R$. This is seen to define the reductive quotient $\overline{X}^{\text{red}}$ of $\overline{X}$, a stratified compactification of X. The bundle extension $\overline{E}^{\text{red}} \to \overline{X}^{\text{red}}$can be carried out by performing the Borel-Serre construction on the total space of E to produce $\overline{E} \to \overline{X}$, and then taking reductive quotients.

As for the extension of E to X^{tor}, this was done by Mumford [8], but we can alternatively take here the toroidal construction on the total space of E.

How different are $\overline{X}^{\text{red}}$ and X^{tor} ? There are two canonical notions (for compactifications of the same space): the greatest common quotient (GCQ) and the least common modification (LCM) [6]. These satisfy universal mapping properties:

$$\begin{array}{ccccc} Y_1 \to & \text{GCQ}(Y_1, Y_2) & \leftarrow Y_2 \\ \searrow & \Downarrow & \swarrow \\ & Q & \end{array} \qquad \begin{array}{ccccc} & M & \\ \swarrow & \Downarrow & \searrow \\ Y_1 \to & \text{LCM}(Y_1, Y_2) & \leftarrow Y_2 \end{array}$$

whenever Y_1 and $Y_2$are compactifications of X. It is easy to see that $\text{LCM}(Y_1, Y_2)$ is just the closure of the diagonal $\Delta_X \subset Y_1 \times Y_2$.

In our case, the following was known in the twentieth century:

Proposition 1. *i*)$\text{GCQ}(\overline{X}^{\text{red}}, X^{\text{tor}}) = X^*$ [7];

ii)LCM($\overline{X}^{\text{red}}, X^{\text{tor}}) \to X^{\text{tor}}$ is a homotopy equivalence [4].

Let h be the composite mapping $X^{\text{tor}} \to \text{LCM}(\overline{X}^{\text{red}}, X^{\text{tor}}) \to \overline{X}^{\text{red}}$ defined by any homotopy inverse to (ii) in Proposition 1.

Conjecture 1 [4]. $E^{\text{tor}} \simeq h^* \overline{E}^{\text{red}}$.

I like to interpret this assertion as saying that $\overline{X}^{\text{red}}$is more fundamental than the algebraic varieties X^{tor} for homogeneous vector bundles in the Hermitian case.

Next, the space X has a canonical quasi-isometry class of Riemannian metrics g_{inv}, induced by the invariant metrics on D. In the Hermitian case, each toroidal compactification X^{tor} imparts a Poincaré metric g_P to X. The Chern forms for an invariant connection on the homogeneous vector bundle E are L^∞in both metrics. Both classes of metrics have finite volume, and we have from [12]

$$H^\bullet_{(\infty),g_{inv}}(X) \to H^\bullet_{(p),g_{inv}}(X) \to H^\bullet(\overline{X}^{\text{red}}) \ \ (1 << p < \infty),$$

$$H^\bullet_{(\infty),g_P}(X) \to H^\bullet_{(p),g_P}(X) \to H^\bullet(X^{\text{tor}}) \ \ (1 < p < \infty).$$

(The second line is different from the treatment in [8].) Furthermore, under the isomorphisms in the above, the Chern forms of an invariant connection map to the Chern classes of $\overline{E}^{\text{red}}$ and E^{tor} respectively.

Now is the time to bring in the excentric compactifications of X. Let $e(R)$ be, as before, the R-stratum of $\overline{X}$ for the **Q**-parabolic subgroup R of $\mathcal{G}$, and let $Z(R)$ denote the R-stratum of X^{tor}. Both have an action of U_P , the center of W_P , when R is *subordinate* to P; that means that P is the "smallest" maximal parabolic subgroup containing R, and we have $U_P \subseteq W_R$. In the toroidal case, the tori that occur are of the form $T_P = \Gamma(U_P)\backslash U_P(\mathrm{C})$. We take the quotients at the respective boundary strata,

$$D_R \times W_R \simeq e(R) \to e(R)^{exc} =: e(R)/U_P \simeq D_R \times (W_R/U_P),$$

(recall the opening paragraph) and $Z(R) \to Z(R)/U_P$, obtaining the excentric compactifications $\overline{X}^{exc}$(with morphisms $\overline{X} \to \overline{X}^{\text{exc}} \to \overline{X}^{\text{red}}$) and $X^{\text{tor,exc}}$(a quotient ofX^{tor}). The two excentric quotients are still different in general, but less so than $\overline{X}^{\text{red}}$ and X^{tor}. For instance, one can see rather easily that the corresponding strata of $\overline{X}^{\text{exc}}$ and $X^{\text{tor,exc}}$ are homotopy equivalent.

There are bundle extensions $\overline{E}^{\text{exc}} \to \overline{X}^{\text{exc}}$(the pullback of $\overline{E}^{\text{red}}$) and $E^{\text{tor,exc}} \to X^{\text{tor,exc}}$ (which pulls back to E^{tor}). We have the following analogue of Prop. 1 and Conj. 1:

Proposition 2. i) In the canonical diagram

$$\begin{array}{ccc} \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) & \xrightarrow{\beta} & X^{\mathrm{tor,exc}} \\ \downarrow \alpha & & \\ \overline{X}^{\mathrm{exc}} & & \end{array}$$

both projections α and β are homotopy equivalences.

ii)Let $k\ :X^{\mathrm{tor,exc}} \to \overline{X}^{\mathrm{exc}}$ be the mapping defined by composing α with a homotopy inverse to β in i). Then $k^*\overline{E}^{\mathrm{exc}} \simeq E^{\mathrm{tor,exc}}$.

Corollary. *Conjecture* 1 *is true.*

The corollary is an immediate consequence of (ii) in Prop. 2. We give some indication of the proof of Prop. 2 [13] in the following outline:

1. The proof of the assertion in (i) about β goes, more or less, like the argument in [4] (for (ii) in Prop. 1 above). We show that β has contractible fibers.

2. From (*), we get

$$\begin{array}{ccc} & & X^{\mathrm{tor,exc}} \\ & & \downarrow \\ \overline{X}^{\mathrm{exc}} & \to & X^*. \end{array}$$

The problem of determining the fibers of β fibers over X^*. This brings in partial compactifications of homogeneous cones, and then the duality noted in [5 : §2.3].

3. The means for deducing the assertion in (i) about α goes under the name LCM-basechange. This is a rather simple notion. Suppose that $Y_1 \longrightarrow Y_2$ is a morphism of compactifications of a space X, and that Y_3 is a third compactification of X. It is easy to see that one has an inclusion

$$\mathrm{LCM}(Y_1, Y_3) \subseteq Y_1 \times Y_2\ \mathrm{LCM}(Y_2, Y_3).$$

We say that LCM-*basechange holds* in the given situation if the inclusion is an equality. In that case, the projections $\mathrm{LCM}(Y_1, Y_3) \longrightarrow Y_3$ and $\mathrm{LCM}(Y_2, Y_3) \longrightarrow Y_3$ have the same fiber.

4. Statement (ii) is verified directly.

References

[1] Ash, A., Mumford, D., Rapoport, M., Tai, Y.-S., Smooth Compactification of Locally Symmetric Varieties, Math. Sci. Press, Brookline, MA, 1975.

[2] Baily, W., Borel, A., Compactification of arithmetic quotients of bounded symmetric domains. Ann. of Math. 84 (1966), 442-528.

[3] Borel, A., Serre, J.-P., Corners and arithmetic groups. Comm. Math. Helv. 4 (1973), 436-491.

[4] Goresky, M., Tai, Y.-S., Toroidal and reductive Borel-Serre compactifications of locally symmetric spaces. Amer. J. Math. 121 (1999), 1095-1151.

[5] Harris, M., Zucker, S., Boundary cohomology of Shimura varieties, I: Coherent cohomology on toroidal compactifications. Ann. Sci. ENS 27 (1994), 249-344.

[6] Harris, M., Zucker, S., Boundary cohomology of Shimura varieties, II: Hodge theory at the boundary. Invent. Math. 116 (1994), 243-307.

[7] Ji, L., The greatest common quotient of Borel-Serre and the toroidal compactifications, Geom. funct. anal. 8 (1994), 978C1015.

[8] Mumford, D., Hirzebruchs proportionality theorem in the non-compact case. Invent. Math. 42 (1977), 239-272.

[9] Satake, I., On compactifications of the quotient spaces for arithmetically defined discontinuous groups. Ann. of Math. 72 (1960), 555-580.

[10] Zucker, S., L_2 cohomology and intersection homology of locally symmetric varieties, II. Compositio Math. 59 (1986), 339-398.

[11] Zucker, S., Satake compactifications. Commentarii Math. Helvetica 58 (1983), 312-343. (Erratum on p. 337 of L^p-cohomology and Satake compactifications, J. Noguchi, T. Ohsawa (eds.), Prospects in Complex Geometry: Proceedings, Katata/Kyoto 1989. Springer LNM 1468 (1991), 317-339.)

[12] Zucker, S., On the reductive Borel-Serre compactification: L^p-cohomology of arithmetic groups (for large p). Amer. J. Math. 123 (2001), 951-984.

[13] Zucker, S., On the reductive Borel-Serre compactification, II: Excentric quotients and least common modifications (to appear).

Department of Mathematics Johns Hopkins University Baltimore
E-mail: zucker@math.jhu.edu