Pure and Applied Mathematics Quarterly
Volume 3, Number 3
(*Special Issue: In honor of
Leon Simon, Part 2 of 2*)
647—688, 2007

Nodal Sets of Harmonic Functions

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Dedicated to Professor Leon Simon on the occasion of his sixtieth birthday

In the present survey paper, we shall discuss the relation between the growth of harmonic functions and the growth of nodal sets of those functions. The growth of harmonic functions is measured by their frequency. For any harmonic function u in the unit ball $B_1 \subset \mathbb{R}^n$, the frequency is defined as

$$N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}. \tag{0.1}$$

If u is a homogenous harmonic polynomial, its frequency is exactly its degree. In general, the frequency controls the growth of the harmonic functions. In the present paper, we shall discuss how the frequency controls the size of nodal sets. F.-H. Lin [26] made an important contribution by establishing the relation between the frequency and the size of nodal sets.

As we know, plane harmonic functions are simply the real parts of holomorphic functions in the complex plane. This simple identification is not enjoyed by harmonic functions in higher dimensional spaces. However, harmonic functions in $\mathbb{R}^n$, as analytic functions with interior estimates on derivatives, can be extended as holomorphic functions in $\mathbb{C}^n$. It turns out that such an extension is extremely important in the discussion of nodal sets of harmonic functions. This is because the nodal sets of (general analytic) functions in $\mathbb{R}^n$ are not stable in the sense

Received September 6, 2005.
The author is supported in part by an NSF grant.

that a simple perturbation may change the structure of nodal sets. In particular, the dimension of nodal sets may change by perturbations. However, this never happens for holomorphic functions in $\mathbb{C}^n$. In order to discuss nodal sets of harmonic functions, we shall first discuss complex nodal sets of the holomorphic extensions. It is not surprising that complex analysis plays an important role in our study. For example, we shall use repeatedly Rouché Theorem in $\mathbb{C}$ and in $\mathbb{C}^2$. It asserts that if an equidimensional holomorphic map has isolated zeroes then its holomorphic perturbation enjoys the same property and the number of isolated zeroes is preserved. Another property we shall use is the behavior of polynomials away from their zeroes. For a suitably normalized polynomial, a positive lower bound can be established for the modulus of the polynomial outside some balls around its zeroes.

The foundation of our discussion is a monotonicity formula for harmonic functions. Corollaries of such a monotonicity include the doubling condition of L^2-integrals and finite vanishing order. In fact, an integral quantity of harmonic functions in the unit ball controls the vanishing order of harmonic functions inside the ball.

This paper is based on a series of lectures given by the author in the Special Lecture Series in Peking University, in the summer of 2005. He would like to thank Gang Tian for the invitation and Peking University for its hospitality.

1. A Monotonicity Formula

In the present section, we shall discuss a monotonicity formula for harmonic functions. Important corollaries include doubling conditions and the control of vanishing order by the frequency. The entire section is taken from [12], [13] and ¿ [26] with few modifications..

Throughout this section, we always assume that u is a harmonic function in $B_1 \subset \mathbb{R}^n$, i.e.,

$$\triangle u = 0. \tag{1.1}$$

Define for any $r \in (0,1)$

$$D(r) = \int_{B_r} |\nabla u|^2, \quad H(r) = \int_{\partial B_r} u^2,$$

and

$$N(r) = \frac{rD(r)}{H(r)}. \tag{1.2}$$

We first note that $D(r)$ can be written as a surface integral. By (1.1), we have $\triangle u^2 = 2|\nabla u|^2$. With Green's formula, we may rewrite $D(r)$ as

$$D(r) = \frac{1}{2}\int_{B_r} \triangle u^2 = \int_{\partial B_r} u u_n. \tag{1.3}$$

As an example, we first calculate $N(r)$ for homogeneous harmonic polynomials.

Example 1.1. If u is a homogeneous harmonic polynomial of degree k, then $N(r)$ is a constant and $N(r) = k$.

Now we prove the following basic result.

Theorem 1.2. *$N(r)$ is a nondecreasing function of $r \in (0,1)$.*

Proof. First, we have

$$N'(r) = N(r)\left\{\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\right\}. \tag{1.4}$$

We need to calculate $D'(r)$ and $H'(r)$. A simple differentiation yields

$$D'(r) = \int_{\partial B_r} |\nabla u|^2 = \frac{1}{r}\int_{\partial B_r} < x|\nabla u|^2, \frac{x}{r} > .$$

By applying Green's formula for each $i = 1, \cdots, n$ and using $\triangle u = 0$, we have

$$\int_{\partial B_r} x_i |\nabla u|^2 \cdot \frac{x_i}{r} = \int_{B_r} \partial_i (x_i |\nabla u|^2) = \int_{B_r} |\nabla u|^2 + 2\sum_j \int_{B_r} x_i u_j u_{ij}$$

$$= \int_{B_r} |\nabla u|^2 - 2\sum_j \int_{B_r} \partial_j (x_i u_j) u_i + 2\sum_j \int_{\partial B_r} x_i u_j u_i \nu_j$$

$$= \int_{B_r} |\nabla u|^2 - 2\int_{B_r} u_i^2 + 2r\int_{\partial B_r} (\nu_i u_i) u_n.$$

Summing over i, we obtain

$$D'(r) = \frac{n-2}{r} D(r) + 2\int_{\partial B_r} u_n^2, \tag{1.5}$$

or with (1.3)

$$\frac{D'(r)}{D(r)} = \frac{n-2}{r} + \frac{2\int_{\partial B_r} u_n^2}{\int_{\partial B_r} u u_n}. \tag{1.6}$$

Next, we write $H(r)$ as

$$H(r) = \int_{|x|=r} u^2(x)dS_x = r^{n-1}\int_{|y|=1} u^2(ry)dS_y.$$

This implies

$$H'(r) = \frac{n-1}{r}H(r) + 2\int_{\partial B_r} uu_n,$$

or

$$\frac{H'(r)}{H(r)} = \frac{n-1}{r} + \frac{2\int_{\partial B_r} uu_n}{\int_{\partial B_r} u^2}. \tag{1.7}$$

By substituting (1.6) and (1.7) into (1.4), we get

$$N'(r) = 2N(r)\left\{\frac{\int_{\partial B_r} u_n^2}{\int_{\partial B_r} uu_n} - \frac{\int_{\partial B_r} uu_n}{\int_{\partial B_r} u^2}\right\},$$

which is nonnegative by Cauchy inequality. □

Now we discuss some corollaries of Theorem 1.2.

Corollary 1.3. *The limit of $N(r)$ as $r \to 0+$ exists and equals to the vanishing order of u at 0.*

Proof. The existence of the limit of $N(r)$ as $r \to 0+$ follows easily from the monotonicity of $N(r)$, by Theorem 1.2. To calculate the value of this limit, we write $u = P + R$, where P is a nonzero homogeneous polynomial of degree k and R is the remainder term. Both P and R are harmonic. Then we have by Example 1.1

$$\lim_{r\to 0} N(r) = \frac{\int_{B_1} |\nabla P|^2}{\int_{\partial B_1} P^2} = k.$$

This finishes the proof. □

Corollary 1.4. *There hold for any $r < 1$*

$$\frac{d}{dr}\log\frac{H(r)}{r^{n-1}} = 2\frac{N(r)}{r}, \tag{1.8}$$

and for any $0 < r_1 < r_2 < 1$

$$\frac{H(r_2)}{r_2^{n-1}} = \frac{H(r_1)}{r_1^{n-1}}\ \exp\{2\int_{r_1}^{r_2}\frac{N(r)}{r}dr\}, \tag{1.9}$$

and

$$\frac{H(r_2)}{r_2^{n-1}} \le \left(\frac{r_2}{r_1}\right)^{2N(r_2)}\frac{H(r_1)}{r_1^{n-1}}. \tag{1.10}$$

Proof. We may write (1.7) as

$$\frac{H'(r)}{H(r)} - \frac{n-1}{r} = 2\frac{D(r)}{H(r)} = 2\frac{N(r)}{r},$$

or

$$\frac{d}{dr}\log H(r) - \frac{d}{dr}\log r^{n-1} = 2\frac{N(r)}{r}.$$

This implies (1.8). A simple integration of (1.8) yields (1.9). To get (1.10), we simply note by Theorem 1.2

$$\exp\{2\int_{r_1}^{r_2} \frac{N(r)}{r} dr\} \le \exp\{2N(r_2)\int_{r_1}^{r_2} \frac{1}{r} dr\} = \left(\frac{r_2}{r_1}\right)^{2N(r_2)}.$$

This finishes the proof. □

The identity (1.8) plays an important role in subsequent discussions. An integration of (1.8) relates two surface integrals of u^2 through the function $N(r)$, as shown in (1.9).

The following result is the so called *doubling condition*.

Corollary 1.5. *There hold for any $R \in (0, 1/2)$*

$$\fint_{\partial B_{2R}} u^2 \le 2^{2N(1)} \fint_{\partial B_R} u^2, \tag{1.11}$$

$$\fint_{B_{2R}} u^2 \le 2^{-1} 2^{2N(1)} \fint_{B_R} u^2. \tag{1.12}$$

Proof. By Theorem 1.2 and taking $r_1 = R$ and $r_2 = 2R$ in (1.10), we obtain

$$\frac{H(2R)}{(2R)^{n-1}} \le 2^{2N(1)} \frac{H(R)}{R^{n-1}},$$

which is (1.11). To get (1.12), we simply integrate (1.11) from 0 to R. □

Next, we provide another proof of Theorem 1.2 and Corollary 1.4. We assume u is given by

$$u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} a_k r^k \varphi_k,$$

where u_k is a homogeneous harmonic polynomial of degree k in $\mathbb{R}^n$ and φ_k is a spherical harmonics of degree k on $\mathbb{S}^{n-1}$. We may assume $\{\varphi_k\}$ is orthonormal in $L^2(\mathbb{S}^{n-1})$, i.e, $\int_{\mathbb{S}^{n-1}} \varphi_k\varphi_l = \delta_{kl}$. Since each u_k is harmonic, we have by Green's formula

$$0 = \int_{B_r} u_k \triangle u_k = \int_{\partial B_r} u_k \partial_n u_k - \int_{B_r} |\nabla u_k|^2.$$

Note $u_k\partial_n u_k = kr^{2k-1}\varphi_k^2$. This implies

$$\int_{B_r} |\nabla u_k|^2 = k\int_{\partial B_r} r^{2k-1}\varphi_k^2 = kr^{2k-1}r^{n-1}.$$

Therefore, we obtain

$$D(r) = r^{n-1}\sum_{k=0}^{\infty} k|a_k|^2 r^{2k-1}, \quad H(r) = r^{n-1}\sum_{k=0}^{\infty} |a_k|^2 r^{2k},$$

and

$$N(r) = \frac{rD(r)}{H(r)} = \frac{\sum_{k=0}^{\infty} k|a_k|^2 r^{2k}}{\sum_{k=0}^{\infty} |a_k|^2 r^{2k}}.$$

Then, we have

$$N'(r) = 2\frac{\left(\sum_{k=0}^{\infty} k^2|a_k|^2 r^{2k}\right)\left(\sum_{k=0}^{\infty} |a_k|^2 r^{2k}\right) - \left(\sum_{k=0}^{\infty} k|a_k|^2 r^{2k}\right)^2}{r\left(\sum_{k=0}^{\infty} |a_k|^2 r^{2k}\right)^2} \geq 0,$$

by Cauchy inequality. Hence, we conclude $N(r)$ is increasing. Next, we note

$$\frac{H(r)}{r^{n-1}} = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

Then we have

$$\frac{d}{dr}\left(\log\frac{H(r)}{r^{n-1}}\right) = \frac{\sum_{k=0}^{\infty} 2k|a_k|^2 r^{2k-1}}{\sum_{k=0}^{\infty} |a_k|^2 r^{2k}} = \frac{2D(r)}{H(r)},$$

or

$$\frac{d}{dr}\left(\log\frac{H(r)}{r^{n-1}}\right) = 2\frac{N(r)}{r}.$$

This is (1.8).

For any $p \in B_1$ and any $r \in (0, 1-|p|)$, we define

$$N(p,r) = \frac{r\int_{B_r(p)} |\nabla u|^2}{\int_{\partial B_r(p)} u^2}. \tag{1.13}$$

The quantity $N(0,1)$ is called the frequency.

Theorem 1.6. *For any $R \in (0,1)$, there exists a constant $N_0 = N_0(R) \ll 1$ such that the following holds. If $N(0,1) \leq N_0$, then u does not vanish in B_R. If $N(0,1) \geq N_0$, then there holds*

$$N(p, \frac{1}{2}(1-R)) \leq CN(0,1), \quad \textit{for any } p \in B_R,$$

where C is a positive constant depending only on n and R. In particular, the vanishing order of u at any point in B_R never exceeds $c(n,R)N(0,1)$.

Proof. The monotonicity of $N(0,r)$ implies that the vanishing order of u at 0 never exceeds $N(0,1)$. In the following, we shall prove for $R = 1/4$.

As in the proof of Corollary 1.5, we have for any $R \in (0, 1/2]$ and $\lambda \in (1,2]$

$$⨍_{B_{\lambda R}} u^2 \leq \lambda^{-1} \lambda^{2N(0,1)} ⨍_{B_R} u^2.$$

Note $B_{3/4}(p) \subset B_1$ and $B_{1/4} \subset B_{1/2}(p)$ for any $p \in B_{1/4}$. Hence, we have for any $p \in B_{1/4}$

$$⨍_{B_{\frac{3}{4}}(p)} u^2 \leq c(n) 4^{2N(0,1)} ⨍_{B_{\frac{1}{2}}(p)} u^2.$$

Now, we claim

$$⨍_{\partial B_{\frac{5}{8}}(p)} u^2 \leq c(n) 4^{2N(0,1)} ⨍_{\partial B_{\frac{1}{2}}(p)} u^2. \tag{1.14}$$

In fact, we have by (1.8)

$$\frac{d}{dr} \log ⨍_{\partial B_r(p)} u^2 = \frac{2N(p,r)}{r}. \tag{1.15}$$

Hence, the function

$$r \mapsto ⨍_{\partial B_r(p)} u^2$$

is increasing with respect to r. Then, we have

$$\int_{B_{\frac{3}{4}}(p)} u^2 \geq \int_{B_{\frac{3}{4}}(p) \setminus B_{\frac{5}{8}}(p)} u^2 \geq c(n) ⨍_{\partial B_{\frac{5}{8}}(p)} u^2,$$

and

$$\int_{B_{\frac{1}{2}}(p)} u^2 \leq c(n) ⨍_{\partial B_{\frac{1}{2}}(p)} u^2.$$

This finishes the proof of (1.14). Integrating (1.15), we obtain

$$\log \fint_{\partial B_{\frac{5}{8}}(p)} u^2 - \log \fint_{\partial B_{\frac{1}{2}}(p)} u^2 = \int_{\frac{1}{2}}^{\frac{5}{8}} \frac{2N(p,r)}{r} dr \geq 2N(p, \frac{1}{2}) \left(\log \frac{5}{8} - \log \frac{1}{2} \right).$$

This implies with (1.14)

$$c(n)N(p, \frac{1}{2}) \leq \log \left(c(n) 4^{2N(0,1)} \right),$$

or

$$N(p, \frac{1}{2}) \leq c(n)N(0,1) + c(n).$$

By the monotonicity of $N(p,r)$, we obtain for any $p \in B_{1/4}$ and any $r \leq 1/2$

$$N(p,r) \leq c(n)N(0,1) + c(n).$$

Therefore, the vanishing order of u at p never exceeds $c(n)N(0,1) + c(n)$.

To finish the proof, we claim $u(p) \neq 0$ for any $p \in B_{1/4}$ if $N(0,1) \leq \varepsilon(n) \ll 1$. To prove the claim, we assume by the normalization $\int_{\partial B_1} u^2 = 1$, which implies $\int_{B_1} |\nabla u|^2 \leq \varepsilon(n)$. Interior estimates yield

$$\sup_{B_{\frac{1}{2}}} |\nabla u| \leq c_1(n) \sqrt{\varepsilon(n)}.$$

By (1.11), we have

$$1 = \int_{\partial B_1} u^2 \leq 2^{n-1} 2^{2N(0,1)} \int_{\partial B_{\frac{1}{2}}} u^2 \leq c(n) \int_{\partial B_{\frac{1}{2}}} u^2.$$

Hence there exists a $p_0 \in \partial B_{1/2}$ such that $|u(p_0)| \geq c_2(n)$. Then, we have for any $p \in B_{1/2}$

$$|u(p)| \geq c_2(n) - c_1(n) \sqrt{\varepsilon(n)} > 0,$$

if $\varepsilon(n)$ is small. This completes the proof. □

Most of the results in the present section can be generalized to analytic functions in $\mathbb{C}$. Suppose $f = f(z)$ is an analytic function in the unit ball in $\mathbb{C}$ given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then we have

$$f'(z) = \sum_{k=0}^{\infty} n a_k z^k.$$

Similarly, we may define $H(r), D(r)$ and $N(r)$ by

$$H(r) = \int_{|z|=r} |f|^2 d\sigma = 2\pi \sum_{k=0}^{\infty} |a_k|^2 r^{2k+1},$$

$$D(r) = \int_{|z|\le r} |f'(z)| dz = \sum_{k=0}^{\infty} k^2 |a_k|^2 \int_{|z|\le r} |z|^{2k-2} dz = \pi \sum_{k=0}^{\infty} k |a_k|^2 r^{2k},$$

and

$$N(r) = \frac{rD(r)}{H(r)} = \frac{\sum_{k=0}^{\infty} k|a_k|^2 r^{2k}}{2\sum_{k=0}^{\infty} |a_k|^2 r^{2k}}.$$

Then, we may conclude as before that $N'(r) \ge 0$.

2. The Measure Estimate of Nodal Sets

In the present section, we shall estimate the measure of nodal sets of harmonic functions in terms of the frequency. The main result, Theorem 2.1, was proved in Lin in [26]. We will follow the proof in [26] closely, with the exception of the proof of Lemma 2.4. The presentation in this section is consistent with the rest of the paper.

We first examine an example. Consider in $\mathbb{R}^2$ the homogeneous harmonic polynomial u_d given by

$$u_d(x, y) = \mathrm{Re}(z^d) = r^d \cos(d\theta).$$

The nodal set of u_d consists of d straight lines intersecting at the origin. Hence, we have

$$\mathcal{H}^1(u_d^{-1}(0) \cap B_1) = 2d.$$

Note the measure of the nodal set depends on the degree linearly.

In general, we have the following result.

Theorem 2.1. *Suppose u is a harmonic function in B_1. Then there holds*

$$\mathcal{H}^{n-1}\{x \in B_{\frac{1}{2}}; u(x) = 0\} \le c(n)N, \tag{2.1}$$

where N is the frequency of u in B_1 as in (0.1).

We begin with a result of H. Cartan, which provides an estimate from below for the modulus of a polynomial in $\mathbb{C}$ away from its zeroes. For a proof, refer to Theorem 10, P19 in [25].

Lemma 2.2. *For any given number $H > 0$ and complex numbers $a_1, \cdots, a_d$, there is a collection of at most d circles in $\mathbb{C}$, with the sum of the radii equal to $2H$, such that for each z lying outside these circles there holds*

$$|z - a_1| \cdot |z - a_2| \cdots |z - a_d| > \left(\frac{H}{e}\right)^d.$$

Corollary 2.3. *Suppose $p(z) = \sum_{k=0}^{d} c_k z^k$ is a polynomial in $\mathbb{C}$ with $\sum_{k=0}^{d} |c_k|^2 \geq 1$. Then for any $H \in (0, 1)$, there is a collection of at most d circles in $\mathbb{C}$, with the sum of the radii $\leq 2H$, such that for any z with $|z| \leq 1$ lying outside these circles there holds*

$$|p(z)| > \left(\frac{H}{10}\right)^d.$$

Proof. We note

$$\frac{1}{2\pi} \int_{\partial D_1} |p|^2 = \sum_{k=0}^{d} |c_k|^2 \geq 1.$$

Since p has d zeroes in $\mathbb{C}$, we may assume

$$p(z) = c(z - a_1) \cdots (z - a_d).$$

Then for some $z_0 \in \partial D_1$, we have

$$1 \leq |p(z_0)| = |c| \cdot |z_0 - a_1| \cdots |z_0 - a_d| \leq |c|(1 + |a_1|) \cdots (1 + |a_d|),$$

which implies

$$|p(z)| \geq \frac{|z - a_1|}{1 + |a_1|} \cdots \frac{|z - a_d|}{1 + |a_d|}.$$

Note we only consider z with $|z| \leq 1$. We assume for some integer $d' \leq d$, $|a_k| \leq 2$ for $k \leq d'$ and $|a_k| > 2$ for $d' < k \leq d$. Then for $d' < k \leq d$, we have

$$\frac{|z - a_k|}{1 + |a_k|} \geq \frac{|a_k| - 1}{|a_k| + 1} > \frac{1}{3} \quad \text{for any } z \text{ with } |z| \leq 1.$$

Obviously, $1 + |a_k| \leq 3$ for $k \leq d'$. Hence we obtain

$$|p(z)| \geq \left(\frac{1}{3}\right)^{d - d'} |z - a_1| \cdots |z - a_{d'}|.$$

A simple application of Lemma 2.2 yields the desired result. □

Now we prove an important result in this section.

Lemma 2.4. *Let $f(z)$ be a nonzero analytic function in $B_1 = \{z \in \mathbb{C}; |z| \le 1\}$. Then there holds*

$$\#\{f^{-1}(0) \cap B_r\} \le 2N,$$

where $r \in (0,1)$ is universal and N is defined as

$$N = \frac{\int_{B_1} |f'|^2}{\int_{\partial B_1} |f|^2}.$$

Lemma 2.4 was first proved by Lin by using Taylor expansion in [26]. Here we shall prove by Rouché Theorem. The proof was adapted from [18].

Proof. Set $g(z) = f(z/M)$ for some $M > 0$ to be determined. Then we have

$$N = \frac{M \int_{B_M} |g'|^2}{\int_{\partial B_M} |g|^2}.$$

We claim for some sufficiently large M,

$$\#\{z \in B_{\frac{1}{2}}; g(z) = 0\} \le 2N,$$

where M is universal.

We set

$$g(z) = \sum_{m=0}^{\infty} a_m z^m.$$

We may assume, without loss of generality, that

$$\frac{1}{2\pi} \int_{\partial B_1} |g|^2 = \sum_{m=0}^{\infty} |a_m|^2 = 1. \tag{2.2}$$

In the following, we set $N_* = [N]$, the integral part of N. Obviously, we have

$$N_* \le N \le N_* + 1.$$

By (1.10) for analytic functions, we get

$$\frac{1}{2\pi M} \int_{\partial B_M} |g|^2 \le \frac{M^{2N}}{2\pi} \int_{\partial B_1} |g|^2 = M^{2N},$$

which implies

$$\sum_{m=0}^{\infty} |a_m|^2 M^{2m} \le M^{2N}.$$

By $N \leq N_* + 1$, we have obviously

$$\sum_{m=0}^{\infty} |a_m|^2 M^{2m} \leq M^{2(N_*+1)}.$$

Therefore, we obtain

$$|a_m| \leq M^{N_*-m+1} \quad \text{for any } m \geq 0. \tag{2.3}$$

Note there holds for some universal constant $c > 0$

$$\frac{1}{2\pi}\int_{\partial B_1} |\sum_{m\geq 2N_*+1} a_m z^m|^2 = \sum_{m\geq 2N_*+1} |a_m|^2 \leq \frac{c}{M^{2N_*}}.$$

We then get by interior estimates

$$\sup_{B_{\frac{3}{4}}} |\sum_{m\geq 2N_*+1} a_m z^m| \leq \frac{c}{M^{N_*}}. \tag{2.4}$$

We choose M large, independent of N_*, such that

$$\sum_{m=2N_*+1}^{\infty} |a_m|^2 \leq \frac{1}{2}. \tag{2.5}$$

Set

$$P_*(z) = \sum_{m=0}^{2N_*} a_m z^m, \quad R_*(z) = \sum_{m=2N_*+1}^{\infty} a_m z^m. \tag{2.6}$$

Then $g = P_* + R_*$. Obviously, we have by (2.2) and (2.5)

$$\sum_{m=0}^{2N_*} |a_m|^2 \geq \frac{1}{2}.$$

Then by Corollary 2.3 with $d = 2N_*$ and possibly a different normalization constant, we conclude

$$\inf_{\partial B_r} |P_*| > \varepsilon^{2N_*},$$

for some $r \in (1/2, 3/4)$ and a universal constant $\varepsilon \in (0,1)$. By choosing M large enough, independent of N_*, we conclude by (2.4)

$$\sup_{B_{3/4}} |R_*| < \varepsilon^{2N_*}.$$

This implies

$$|g(z) - P_*(z)| < |P_*(z)| \quad \text{for any } |z| = r.$$

Then Rouché Theorem implies

$$\#\{z \in B_r; g(z) = 0\} \leq 2N_*,$$

and in particular

$$\#\{z \in B_{\frac{1}{2}}; g(z) = 0\} \leq 2N_*.$$

This finishes the proof, since $N_* \leq N$. □

We now state a different version of Lemma 2.4.

Lemma 2.5. *Suppose $f : B_1 \subset \mathbb{C} \to \mathbb{C}$ is analytic with*

$$|f(0)| = 1 \quad \text{and} \quad \sup_{B_1} |f| \leq 2^N,$$

for some positive constant N. Then there holds

$$\#\{z \in B_r; f(z) = 0\} \leq N,$$

where r is universal.

The proof of Lemma 2.5 is similar to that of Lemma 2.4, and is omitted.

To prove Theorem 2.1, we need the complexification of harmonic functions. Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^n$. Then there is an $R \in (0,1)$ such that $u(x)$ extends to a holomorphic function $\tilde{u}(z)$ on

$$\Omega = \{z = x + iy \in \mathbb{C}^n; x \in B_{\frac{3}{4}}, y \in B_R\}.$$

Moreover, there holds for some universal constant $c > 0$

$$\sup_{\Omega} |\tilde{u}| \leq c\|u\|_{L^2(\partial B_1)}. \tag{2.7}$$

To see this, we simply consider the Taylor expansion of $u = u(x)$ at any point $p \in B_{3/4}$ and replace $x-p \in \mathbb{R}^n$ by $z-p \in \mathbb{C}^n$. With the estimate of the derivatives of harmonic functions, the new complex series converges for $|z-p| < R$, with R independent of u. In the following, R will be fixed such that the above extension property and (2.7) hold. Hence, the constant c is also fixed, independent of u.

Now we are ready to prove Theorem 2.1. A key lemma in the proof is Lemma 2.5, where the universal constant r is small. In the proof of Theorem 2.1, we shall assume Lemma 2.5 holds for $r = 4/5$. The proof below can be modified easily when Lemma 2.5 holds only for small r.

Proof of Theorem 2.1. We assume without loss of generality that $\int_{\partial B_1} u^2 = 1$. By Corollary 1.4, Corollary 1.5 and Theorem 1.6, we have

$$\fint_{B_{\frac{1}{16}}(p)} u^2 \geq 4^{-cN} \quad \text{for any } p \in B_{\frac{1}{4}},$$

where c is a universal constant. In particular, there is a point $x_p \in B_{1/16}(p)$ such that $|u(x_p)| \geq 2^{-cN}$. Now we choose $p_1, \cdots, p_n \in \partial B_{1/4}(0)$, with p_j on the x_j-axis, $j = 1, \cdots, n$. Let $x_{p_j} \in B_{1/16}(p_j)$ be such that

$$|u(x_{p_j})| \geq 2^{-cN} \quad \text{for any } j = 1, \cdots, n.$$

For each j and $w \in \mathbb{S}^{n-1}$, we consider

$$f_j(w;t) = u(x_{p_j} + tw) \quad \text{for } t \in (-\frac{5}{8}, \frac{5}{8}).$$

It is obvious $f_j(w;t)$ is an analytic function of $t \in (-5/8, 5/8)$. Moreover, $f_j(w;t)$ extends to an analytic function $f_j(w;z)$ for $z = t + iy$, $|t| < 5/8$ and $|y| < y_0$. Then we have

$$|f_j(w;0)| \geq 2^{-cN},$$

and

$$|f_j(w;t+iy)| \leq C,$$

for some universal constant C. Applying Lemma 2.5, we obtain

$$\#\{t; u(x_{p_j} + tw) = 0, \ |t| < \frac{1}{2}\} \leq C(n)N,$$

and in particular

$$N_j(w) \equiv \#\{t; u(x_{p_j} + tw) = 0, \ x_{p_j} + tw \in B_{\frac{1}{16}}\} \leq C(n)N.$$

By the integral geometry estimate ([11], 3.2.22), we have

$$\mathcal{H}^{n-1}\{x \in B_{\frac{1}{16}}; u(x) = 0\} \leq c(n) \sum_{j=1}^{n} \int_{\mathbb{S}^{n-1}} N_j(w)dw \leq C(n)N.$$

Now Theorem 2.1 follows simply by a suitable finite covering of $B_{1/2}$ by balls of radius $1/16$. This completes the proof. □

3. Measure Estimates of Singular Sets in $\mathbb{R}^2$

Let u be a harmonic function in $B_1 \subset \mathbb{R}^n$. For any $p \in B_1$ with $u(p) = 0$ and $Du(p) \neq 0$, it is easy to see that $u^{-1}(0)$ is an $(n-1)$-dimensional analytic submanifold in a neighborhood of p. Now we set

$$\mathcal{S}(u) = \{p \in B_1; u(p) = 0, Du(p) = 0\}.$$

We first prove a result, which is due to Caffarelli and Friedman [3].

Lemma 3.1. *Let u be a nontrivial harmonic function in $B_1 \subset \mathbb{R}^n$. Then $\mathcal{S}(u)$ is contained in a countable union of $(n-2)$-dimensional analytic manifolds.*

Proof. For any $d \geq 2$, we set

$$\mathcal{S}_d(u) = \{p \in B_1;\ \partial^\nu u(p) = 0 \text{ for any } |\nu| < d, \\ \partial^{\nu_0} u(p) \neq 0 \text{ for some } |\nu_0| = d\}.$$

Then we have

$$\mathcal{S}(u) = \bigcup_{d\geq 2} \mathcal{S}_d(u).$$

This is a finite union by the finite vanishing order due to analyticity. We shall prove that $\mathcal{S}_d(u)$ is $(n-2)$-dimensional for each fixed $d \geq 2$.

For any $p \in \mathcal{S}_d(u)$, there exists a $|\beta| = d-2$ such that $\partial^2 v(p) \neq 0$ for $v = \partial^\beta u$. Obviously, v is a harmonic function in B_1. First, the Hessian matrix $(\partial^2 v(p))$ has a nonzero eigenvalue. Next, we may diagonalize $(\partial^2 v(p)) = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Then we have $\lambda_1 + \cdots + \lambda_n = 0$. By assuming $\lambda_1 \neq 0$, we have another nonzero eigenvalue and hence we may assume $\lambda_2 \neq 0$. Note

$$\partial\partial_1 v(p) = (\lambda_1, 0 \cdots, 0), \quad \partial\partial_2 v(p) = (0, \lambda_2, 0, \cdots, 0).$$

By applying the implicit function theorem to $\partial_1 v$ and $\partial_2 v$, we conclude that $\{\partial_1 v = 0, \partial_2 v = 0\}$ is an $(n-2)$-dimensional analytic manifold in a neighborhood of p. Obviously, this manifold contains $\mathcal{S}_d(u)$ in a neighborhood of p. This finishes the proof. □

Lemma 3.1 illustrates that $\mathcal{S}(u)$ is indeed the singular part of $\{u = 0\}$. We usually call $\mathcal{S}(u)$ the singular set of u.

The following conjecture concerns the size of singular sets in terms of the frequency. It was proposed by Lin in [26].

Conjecture 3.2. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^n$. Then there holds*

$$\mathcal{H}^{n-2}\{x \in B_{\frac{1}{2}}; u(x) = |Du|(x) = 0\} \leq cN^2,$$

where $c = c(n)$ is a general constant and N is the frequency of u in B_1 as in (0.1).

At this time, it is known to be true only for the case $n = 2$. The question for the general dimension remains open.

The rest of the section only concerns the case $n = 2$ and is adapted entirely from [18]. In fact, a better result is available for $n = 2$.

Theorem 3.3. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^2$. Then there holds*

$$\#\left(\mathcal{S}(u) \cap B_{\frac{1}{2}}\right) \leq cN,$$

where $c > 0$ is some universal constant and N is the frequency of u in B_1 as in (0.1).

Here we have the linear growth for the estimate on the singular set. Compare with the quadratic growth in Conjecture 3.2.

To prove Theorem 3.3, we shall identify $\mathbb{C} = \mathbb{R}^2$ and prove the following result.

Theorem 3.4. *There exists a universal constant $M > 1$ such that for a harmonic function u in $B_M \subset \mathbb{R}^2$, with $u(0) = 0$, satisfying*

$$\frac{M \int_{B_M} |\nabla u|^2}{\int_{\partial B_M} u^2} \leq N,$$

there holds

$$\#\{x \in B_{\frac{1}{2}}; u_{x_1}(x) = u_{x_2}(x) = 0\} \leq 2N.$$

Proof. Let (r, θ) denote polar coordinates in $\mathbb{R}^2$ and we write u in the following form

$$u(r, \theta) = \sum_{m=1}^{\infty} a_m r^m \cos(m\theta + \theta_m),$$

where $\theta_m \in [0, 2\pi)$. We may assume, without loss of generality, that

$$\frac{1}{\pi} \int_{\partial B_1} u^2 = \sum_{m=1}^{\infty} a_m^2 = 1. \tag{3.1}$$

In the following, we set

$$N_* = \inf\{n \in \mathbb{Z}_+;\ n \ge N\}.$$

Obviously, we have

$$N_* - 1 \le N \le N_*.$$

By (1.10), we get

$$\frac{1}{\pi M}\int_{\partial B_M} u^2 \le \frac{M^{2N(0,M)}}{\pi}\int_{\partial B_1} u^2 = M^{2N(0,M)},$$

which implies

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \le M^{2N(0,M)}.$$

By $N(0,M) \le N \le N_*$, we have obviously

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \le M^{2N_*}.$$

Therefore, we obtain

$$|a_m| \le M^{N_*-m}, \quad \text{for any } m \ge 1. \tag{3.2}$$

We first choose M large, independent of N_*, such that

$$\sum_{m=2N_*}^{\infty} |a_m|^2 \le \frac{1}{2}. \tag{3.3}$$

Now we identify $\mathbb{C} = \mathbb{R}^2$ and write $z = x_1 + ix_2$. Setting $f(z) = u_{x_1} - iu_{x_2}$, we note f is holomorphic and $f^{-1}(0) = |Du|^{-1}(0)$. A straightforward calculation yields

$$f(z) = \sum_{m=1}^{\infty} m a_m e^{i\theta_m} z^{m-1}.$$

Set

$$P(z) = \sum_{m=1}^{2N_*-1} m a_m e^{i\theta_m} z^{m-1}, \quad R(z) = \sum_{m=2N_*}^{\infty} m a_m e^{i\theta_m} z^{m-1}. \tag{3.4}$$

Then we get $f = P + R$. Note P is a polynomial in $\mathbb{C}$ of degree $2N_* - 2$ and its coefficients satisfy

$$\sum_{m=1}^{2N_*-1} |m a_m|^2 \ge \frac{1}{2}.$$

By Corollary 2.3, there exists an $r \in (1/2, 1)$ and a universal $\varepsilon > 0$ such that

$$|P(z)| > \varepsilon^{2N_*-2} \quad \text{for any } |z| = r.$$

Moreover, by choosing a universal M large enough, independent of N_*, we have by (3.3) for any $|z| < 1$

$$|R(z)| \leq \sum_{m \geq 2N_*} |ma_m| \leq \sum_{m \geq 2N_*} \frac{m}{M^{m-N_*}} \leq \frac{c}{M^{N_*}} < \varepsilon^{2N_*-2}.$$

This implies

$$|f(z) - P(z)| < |P(z)| \quad \text{for any } |z| = r.$$

By Rouché Theorem, we have

$$\#\{f^{-1}(0) \cap B_r\} \leq 2N_* - 2,$$

or

$$\#\{f^{-1}(0) \cap B_{\frac{1}{2}}\} \leq 2N_* - 2.$$

This finishes the proof, since $N_* - 1 \leq N$. □

Theorem 3.3 follows from Theorem 3.4 and Theorem 1.6 easily. The proof of Theorem 3.3 makes an essential use of the identification $\mathbb{R}^2 = \mathbb{C}$. In the following, we study the singular set from another point of view. Instead of identifying $\mathbb{R}^2$ as $\mathbb{C}$, we put $\mathbb{R}^2$ into $\mathbb{C}^2$ and then consider the complexification of harmonic functions.

Suppose u is a harmonic function defined in $B_1 \subset \mathbb{R}^2$. As discussed in the previous section, u extends to a holomorphic function $\tilde{u}(z)$ in $D_R \subset \mathbb{C}^2$ for some universal $R \in (0, 1)$. Moreover, there holds for some universal constant $c > 0$

$$\sup_{D_R} |\tilde{u}| \leq c\|u\|_{L^2(\partial B_1)}. \tag{3.5}$$

In the following, we always denote by $\tilde{u}$ the complexification of u. We shall also use $B_r(x)$ and $D_r(z)$ to denote open balls of radius r centered at x and z in $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively. When the center is the origin, we shall simply write B_r and D_r. The singular set $\tilde{u}$ is defined as

$$\mathcal{S}(\tilde{u}) = \{z \in D_R; \tilde{u}(z) = \tilde{u}_{z_1}(z) = \tilde{u}_{z_2}(z) = 0\}.$$

Then we have the following result concerning complex singular sets.

Theorem 3.5. *Let u be a (real) harmonic function in $B_1 \subset \mathbb{R}^2$. Then for some universal constants $R_0 \in (0,1)$ and $c > 0$ there holds*

$$\#\left(\mathcal{S}(\tilde{u}) \cap D_{R_0}\right) \leq cN^2,$$

where N is the frequency of u in B_1 as in (0.1).

A significant aspect of Theorem 3.5 is that a property of the complexified $\tilde{u}$ is determined by its restriction on the real space $u = \tilde{u}|_{\mathbb{R}^2}$. Here we make an important remark about the complexification $\tilde{u}$. Since u is a harmonic function, the holomorphic function $\tilde{u}$ satisfies $\partial_{z_1 z_1}\tilde{u} + \partial_{z_2 z_2}\tilde{u} = 0$. Theorem 3.5 asserts that the singular set of $\tilde{u}$ is isolated and that the number of singular points can be estimated in terms of the frequency of the (real) function u. This result does not hold for general holomorphic functions v satisfying

$$\partial_{z_1 z_1} v + \partial_{z_2 z_2} v = 0. \tag{3.6}$$

The following example is taken from [24].

Example 3.6. Let $v(z) = (z_1 - iz_2)^2$. Obviously v satisfies (3.6). However, the singular set of v is not even isolated.

Hence in order to have an isolated singular set for a holomorphic function $v = v(z_1, z_2)$ satisfying (3.6), all the coefficients in the Taylor expansion of v have to be real.

To prove Theorem 3.5, we shall first provide a simple but crucial calculation for harmonic polynomials in $\mathbb{R}^2$ and their complexification $\tilde{u}$ in $\mathbb{C}^2$. In the polar coordinate system (r, θ) in $\mathbb{R}^2$, the homogeneous polynomial $P_d(x) = r^d \cos d\theta$ is harmonic. If we consider a linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a^2 + b^2 = 1,\ a, b \in \mathbb{R}, \tag{3.7}$$

then $P_d(T\cdot)$ is also harmonic. In fact, any homogeneous harmonic polynomial of degree d can be written in this way. Note T in (3.7) is simply a rotation in $\mathbb{R}^2$.

Now we consider the gradient of homogeneous harmonic polynomials. We identify $\mathbb{R}^2 = \mathbb{C}$ and use the complex coordinate $z = x_1 + ix_2$. Consider the homogeneous polynomial

$$\bar{z}^d = (x_1 - ix_2)^d = r^d \cos d\theta - ir^d \sin d\theta.$$

We use its real part and complex part to construct a homogeneous polynomial map $Q_d : \mathbb{R}^2 \to \mathbb{R}^2$ as follows

$$Q_d(x) = Q_d(x_1, x_2) = \begin{pmatrix} r^d \cos d\theta \\ -r^d \sin d\theta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z^d + \bar{z}^d \\ i(z^d - \bar{z}^d) \end{pmatrix},$$

or

$$Q_d(x) = \frac{1}{2} \begin{pmatrix} (x_1 + ix_2)^d + (x_1 - ix_2)^d \\ i((x_1 + ix_2)^d - (x_1 - ix_2)^d) \end{pmatrix}. \tag{3.8}$$

Each component is a homogeneous harmonic polynomial. In fact Q_d is the gradient of some homogeneous harmonic polynomial of degree $d+1$.

As before, we consider a linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ given in (3.7). If Q_d is a homogeneous polynomial map given in (3.8), then $T^t Q_d(T\cdot)$ is also a homogeneous polynomial map given by the gradient of some homogeneous harmonic polynomial of degree $d+1$. In fact, the converse is also true. A homogeneous polynomial map of degree d can be expressed as $T^t Q_d(T\cdot)$ for some linear transform T in (3.7) if it is the gradient of some homogeneous harmonic polynomial of degree $d+1$.

Now we extend the map $Q_d : \mathbb{C}^2 \to \mathbb{C}^2$ simply by replacing $x = (x_1, x_2)$ by $z = (z_1, z_2)$,

$$Q_d(z) = Q_d(z_1, z_2) = \frac{1}{2} \begin{pmatrix} (z_1 + iz_2)^d + (z_1 - iz_2)^d \\ i((z_1 + iz_2)^d - (z_1 - iz_2)^d) \end{pmatrix}. \tag{3.9}$$

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transform given in (3.7). Set

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 - bz_2 \\ bz_1 + az_2 \end{pmatrix}.$$

Then we have

$$z_1' + iz_2' = (a + ib)(z_1 + iz_2), \quad z_1' - iz_2' = (a - ib)(z_1 - iz_2).$$

Set $\alpha = a + ib$. We obtain

$$Q_d(Tz) = \frac{1}{2} \begin{pmatrix} \alpha^d (z_1 + iz_2)^d + \bar{\alpha}^d (z_1 - iz_2)^d \\ i\big(\alpha^d (z_1 + iz_2)^d - \bar{\alpha}^d (z_1 - iz_2)^d\big) \end{pmatrix},$$

and

$$T^t Q_d(Tz) = \frac{1}{2} \begin{pmatrix} \alpha^{d+1} (z_1 + iz_2)^d + \bar{\alpha}^{d+1} (z_1 - iz_2)^d \\ i\big(\alpha^{d+1} (z_1 + iz_2)^d - \bar{\alpha}^{d+1} (z_1 - iz_2)^d\big) \end{pmatrix}. \tag{3.10}$$

We conclude easily

$$|T^t Q_d(Tz)|^2 = \frac{1}{2}\big(|z_1 + iz_2|^{2d} + |z_1 - iz_2|^{2d}\big).$$

A direct calculation shows

$$|z_1 \pm iz_2|^2 = |z_1|^2 + |z_2|^2 \pm 2(y_1x_2 - x_1y_2).$$

Then we obtain

$$\begin{aligned}|T^t Q_d(Tz)|^2 =& \frac{1}{2}\Big(\big(|z_1|^2 + |z_2|^2 + 2(y_1x_2 - x_1y_2)\big)^d \\ &+ \big(|z_1|^2 + |z_2|^2 - 2(y_1x_2 - x_1y_2)\big)^d\Big).\end{aligned}$$

Notice that only the even powers of $y_1x_2 - x_1y_2$ appear in the right side. Hence we get

$$|T^t Q_d(Tz)| \geq |z|^d. \tag{3.11}$$

Next we shall generalize (3.11) to nonhomogeneous harmonic polynomial maps.

Lemma 3.7. *Suppose $Q : \mathbb{C}^2 \to \mathbb{C}^2$ is given by*

$$Q(z) = \sum_{k=0}^{d} c_k T_k^t Q_k(T_k z), \tag{3.12}$$

where for $k = 0, 1, \cdots, d$, $Q_k : \mathbb{C}^2 \to \mathbb{C}^2$ is the homogeneous harmonic polynomial map given by (3.9), $T_k : \mathbb{C}^2 \to \mathbb{C}^2$ is a linear transform given by

$$T_k = \begin{pmatrix} a_k & -b_k \\ b_k & a_k \end{pmatrix}, \quad \text{with } a_k^2 + b_k^2 = 1,\ a_k, b_k \in \mathbb{R},$$

and c_k is a complex number such that $\sum_{k=0}^{d} |c_k|^2 \geq 1$. Then there exists an $r \in (1/2, 1)$ such that

$$|Q(z)| > \varepsilon^d \quad \text{for any } z \in \partial D_r,$$

for some universal constant $\varepsilon \in (0, 1)$.

Proof. Set $\alpha_k = a_k + ib_k$, for $k = 0, 1, \cdots, d$. We claim

$$|Q(z)|^2 = \frac{1}{2}\left(|\sum_{k=0}^{d} c_k \alpha_k^{k+1}(z_1 + iz_2)^k|^2 + |\sum_{k=0}^{d} c_k \bar{\alpha}_k^{k+1}(z_1 - iz_2)^k|^2\right). \tag{3.13}$$

To prove this, we set for simplicity

$$w_1 = z_1 + iz_2, \quad w_2 = z_1 - iz_2.$$

Then we obtain by (3.10)

$$Q(z)=\sum_{k=0}^{d} c_kT_k^tQ_k(T_kz)=\frac{1}{2}\sum_{k=0}^{d} c_k\begin{pmatrix}\alpha_k^{k+1}w_1^k+\bar\alpha_k^{k+1}w_2^k\\ i(\alpha_k^{k+1}w_1^k-\bar\alpha_k^{k+1}w_2^k)\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}\sum_{k=0}^d c_k\alpha_k^{k+1}w_1^k+\sum_{k=0}^d c_k\bar\alpha_k^{k+1}w_2^k\\ i(\sum_{k=0}^d c_k\alpha_k^{k+1}w_1^k-\sum_{k=0}^d c_k\bar\alpha_k^{k+1}w_2^k)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}I+II\\ I-II\end{pmatrix}.$$

This implies

$$|Q(z)|^2=\frac{1}{4}\big(|I+II|^2+|I-II|^2\big)=\frac{1}{2}\big(|I|^2+|II|^2\big)$$
$$=\frac{1}{2}\left(|\sum_{k=0}^{d} c_k\alpha_k^{k+1}w_1^k|^2+|\sum_{k=0}^{d} c_k\bar\alpha_k^{k+1}w_2^k|^2\right).$$

This finishes the proof of (3.13).

We now apply Corollary 2.3 to polynomials

$$\sum_{k=0}^{d} c_k\alpha_k^{k+1}w^k,\quad \text{and}\quad \sum_{k=0}^{d} c_k\bar\alpha_k^{k+1}w^k.$$

For any $H\in(0,1)$, there is a collection of discs $\{D_{r_k}(p_k)\}$ and $\{D_{s_l}(q_l)\}$ in $\mathbb{C}$, with $\sum r_k\le 2H$ and $\sum s_l\le 2H$, such that for each $z=(z_1,z_2)\in D_1$ with

$$z_1+iz_2\notin\cup D_{r_k}(p_k),\quad\text{or}\quad z_1-iz_2\notin\cup D_{s_l}(q_l),$$

there holds

$$|Q(z)|>\left(\frac{H}{10}\right)^d.$$

Now we consider the set

$$\mathcal{B}_{r,s}(p,q)=\{(z_1,z_2)\in\mathbb{C}^2; z_1+iz_2\in D_r(p), z_1-iz_2\in D_s(q)\}.$$

Consider the linear transform in $\mathbb{C}^2$ from (z_1,z_2) to (w_1,w_2)

$$w_1=\frac{1}{\sqrt2}(z_1+iz_2),\quad w_2=\frac{1}{\sqrt2}(z_1-iz_2).$$

In the new coordinate system, $\mathcal{B}_{r,s}(p,q)$ is a polydisc

$$D_{r/\sqrt2}(\frac{p}{\sqrt2})\times D_{s/\sqrt2}(\frac{q}{\sqrt2})\subset\mathbb{C}\times\mathbb{C}=\mathbb{C}^2.$$

Hence by setting $r=r_k$ and $s=s_l$, there is a collection of polydiscs $\{D_{r_k/\sqrt2}(p_k/\sqrt2)$ $\times D_{s_l/\sqrt2}(q_l/\sqrt2)\}$ such that if $w=(w_1,w_2)$ is not in these polydiscs then $|Q(z)|>$

$(H/10)^d$ for the corresponding $z = (z_1, z_2)$. By choosing $H > 0$ small enough, we may find an $r \in (1/2, 1)$ such that

$$\partial D_r \cap \left(\cup \{ D_{r_k/\sqrt{2}}(\frac{p_k}{\sqrt{2}}) \times D_{s_l/\sqrt{2}}(\frac{q_l}{\sqrt{2}}) \} \right) = \emptyset.$$

Therefore we obtain

$$|Q(z)| > \left(\frac{H}{10} \right)^d, \quad \text{for any } z \in \partial D_r.$$

This finishes the proof. □

The next result is the 2-dimensional version of the Rouché Theorem. For a general form and a proof, refer to [27].

Lemma 3.8. *Suppose $f, g : D_1 \subset \mathbb{C}^2 \to \mathbb{C}^2$ are holomorphic in D_1 and C^1 up to the boundary ∂D_1. If*

$$|f(z_1, z_2) - g(z_1, z_2)| < |g(z_1, z_2)| \quad \textit{for any } (z_1, z_2) \in \partial D_1,$$

then $f^{-1}(0)$ and $g^{-1}(0)$ are isolated in D_1 and the number of points in $f^{-1}(0)$ is the same as that in $g^{-1}(0)$, counting the multiplicity.

Corollary 3.9. *Suppose $f : D_1 \subset \mathbb{C}^2 \to \mathbb{C}^2$ is holomorphic in D_1 and continuous up to the boundary ∂D_1 and that Q is given in Lemma 3.7. If for the universal $\varepsilon > 0$ in Lemma 3.7, there holds*

$$|f(z_1, z_2) - Q(z_1, z_2)| < \varepsilon^d, \quad \textit{for any } (z_1, z_2) \in D_1 \setminus D_{\frac{1}{2}},$$

then

$$\#\{f^{-1}(0) \cap D_{\frac{1}{2}}\} \le d^2.$$

Proof. By Lemma 3.7, there exists an $r \in (1/2, 1)$ such that

$$|f(z_1, z_2) - Q(z_1, z_2)| < |Q(z_1, z_2)|, \quad \text{for any } (z_1, z_2) \in \partial D_r.$$

Bezout formula ([1], Corollary 1, P200) implies

$$\#\{Q^{-1}(0)\} \le d^2.$$

Here the multiplicity is counted. Hence by Lemma 3.8, we obtain

$$\#\{f^{-1}(0) \cap D_r\} \le d^2.$$

This finishes the proof. □

Remark 3.10. Suppose P is a harmonic polynomial of degree $d+1$, with $P(0) = 0$. We may write

$$P = \sum_{m=1}^{d+1} a_m \Phi_m,$$

where Φ_m is a homogeneous harmonic polynomial of degree m with $\int_{\mathbb{S}^1} \Phi_m^2 = 1$, for any $1 \leq m \leq d+1$. Obviously, $\{\Phi_m|_{\mathbb{S}^1}\}$ is orthogonal in $L^2(\mathbb{S}^1)$. Now we assume $\int_{\mathbb{S}^1} P^2 \geq 1$. This implies $\sum_{m=1}^{d+1} a_m^2 \geq 1$. Then it is easy to see that DP, considered as a map from $\mathbb{C}^2$ to $\mathbb{C}^2$, can be written as in (3.12), with $\sum_{k=0}^{d} |c_k|^2 \geq 1/2$.

Now we begin to prove Theorem 3.5. We shall prove the following result. The constant N in Theorem 3.11 means different from that in (0.1).

Theorem 3.11. *There are two universal constants $M > 1$ and $r \in (0,1)$ such that for a harmonic function u in $B_M \subset \mathbb{R}^2$, with $u(0) = 0$, satisfying*

$$\frac{M \int_{B_M} |\nabla u|^2}{\int_{\partial B_M} u^2} \leq N,$$

there holds

$$\#\{z \in D_r; \tilde{u}_{z_1}(z) = \tilde{u}_{z_2}(z) = 0\} \leq 4N^2.$$

The proof of Theorem 3.11 is similar to that of Theorem 3.4.

Proof. For simplicity, we shall use the same notation to denote harmonic functions and their complexifications. Let (r, θ) denote polar coordinates in $\mathbb{R}^2$ and we write u in the following form

$$u(r,\theta) = \sum_{m=1}^{\infty} a_m \Phi_m(r,\theta) \quad \text{and} \quad \Phi_m(r,\theta) = r^m \varphi_m(\theta),$$

where $\varphi_m(\theta)$ satisfies

$$\int_{\mathbb{S}^1} \varphi_m^2(\theta) d\theta = 1 \quad \text{and} \quad \varphi_m''(\theta) + m^2 \varphi_m(\theta) = 0.$$

Moreover, we may assume, without loss of generality, that

$$\int_{\partial B_1} u^2 = \sum_{m=1}^{\infty} a_m^2 = 1. \tag{3.14}$$

In the following, we set

$$N_* = \inf\{n \in \mathbb{Z}_+;\ n \geq N\}.$$

Obviously, we have

$$N_* - 1 \leq N \leq N_*.$$

By (1.10), we get

$$\frac{1}{M}\int_{\partial B_M} u^2 \leq M^{2N(0,M)} \int_{\partial B_1} u^2 = M^{2N(0,M)},$$

which implies

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \leq M^{2N(0,M)}.$$

By $N(0,M) \leq N \leq N_*$, we have obviously

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \leq M^{2N_*}.$$

Therefore, we obtain

$$|a_m| \leq M^{N_*-m} \quad \text{for any } m \geq 1. \tag{3.15}$$

Since $\{\varphi_m\}$ is orthonormal in $L^2(\mathbb{S}^1)$, there holds for some universal constant $c > 0$

$$\int_{\partial B_1} |\sum_{m\geq 2N_*} a_m \Phi_m|^2 = \sum_{m\geq 2N_*} |a_m|^2 \leq \frac{c}{M^{2N_*}}.$$

We first choose M large, independent of N_*, such that

$$\sum_{m=2N_*}^{\infty} |a_m|^2 \leq \frac{1}{2}. \tag{3.16}$$

By (3.5), we get for some universal $R \in (0,1)$,

$$\sup_{D_R} |\sum_{m\geq 2N_*} a_m \Phi_m| \leq \frac{c}{M^{N_*}}.$$

Interior estimates for holomorphic functions imply

$$\sup_{D_{\frac{R}{2}}} |D(\sum_{m\geq 2N_*} a_m \Phi_m)| \leq \frac{c}{RM^{N_*}}. \tag{3.17}$$

Set

$$P_* = \sum_{m=1}^{2N_*-1} a_m \Phi_m, \quad R_* = \sum_{m=2N_*}^{\infty} a_m \Phi_m. \tag{3.18}$$

Then $u = P_* + R_*$. Obviously, we have by (3.14) and (3.16)

$$\sum_{m=1}^{2N_*-1} |a_m|^2 \geq \frac{1}{2}.$$

Then DP_* satisfies the assumptions in Lemma 3.7, with $d = 2N_* - 2$ and possibly a different normalization constant. See the Remark 3.10. By choosing M large enough, independent of N_*, we conclude by (3.17)

$$\sup_{D_{\frac{R}{2}}} |DR_*| < \varepsilon^{2N_*-2},$$

where ε is the universal constant as in Corollary 3.9, or Lemma 3.7. This implies

$$|Du(z) - DP_*(z)| < \varepsilon^{2N_*-2} \quad \text{for any } z \in D_{\frac{R}{2}}.$$

By applying Corollary 3.9 to Du and DP_* in $D_{R/2}$, we conclude

$$\#\{|Du|^{-1}(0) \cap D_{R/4}\} \leq (2N_* - 2)^2.$$

This finishes the proof, since $N_* - 1 \leq N$. □

Now we may prove Theorem 3.5.

Proof of Theorem 3.5. Recall N is defined in (0.1).

First, we consider the case that N is small. Let $N_0 = N_0(1/4)$ be the constant in Theorem 1.6. If $N \leq N_0$, then u is never zero in $B_{1/4}$ by Theorem 1.6. Harnack inequality and interior estimates for harmonic functions and holomorphic functions imply that $\tilde{u}$ has no zeroes in D_{R_1}, for some universal $R_1 < 1$. Therefore we have $\mathcal{S}(\tilde{u}) \cap D_{R_1} = \emptyset$.

Next, we consider $N \geq N_0$. By Theorem 1.6 there holds for any $p \in B_{1/4}$

$$\frac{\int_{B_{\frac{1}{4}}(p)} |\nabla u|^2}{4 \int_{\partial B_{\frac{1}{4}}(p)} u^2} \leq CN,$$

for some positive constant C independent of u. For any $p \in B_{1/4}$, with $u(p) = 0$, by the scaled version of Theorem 3.11, we have

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2}(p)\} \leq cN^2,$$

for some positive constants $R_2 < 1$ and c, independent of u and p. To finish the proof, we consider two cases. If u is never zero in $B_{R_2/2}$, then $\tilde{u}$ is never zero in

$D_{2R_1R_2}$, as in the first part of the proof. This implies that $\mathcal{S}(\tilde{u}) \cap D_{2R_1R_2} = \emptyset$. If $u(p) = 0$ for some $p \in B_{R_2/2}$, then we have

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2}(p)\} \le cN^2,$$

which implies

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{\frac{R_2}{2}}\} \le cN^2.$$

This finishes the proof by taking $R_0 = \min\{R_1, 2R_1R_2, R_2/2\}$. □

To finish this section, we provide an example to show that the number of complex singular points is indeed in the quadratic order of the frequency. Hence the estimate in Theorem 3.5 is optimal.

Example 3.12. For any integer $d \ge 2$ and any small $\varepsilon > 0$, consider the harmonic polynomial u in the polar coordinate

$$u(x) = \varepsilon r \cos\theta - \frac{1}{d+1} r^{d+1} \cos(d+1)\theta.$$

Then it is easy to see that

$$Du(x) = \begin{pmatrix} \varepsilon - r^d \cos d\theta \\ r^d \sin d\theta \end{pmatrix}.$$

By (3.9), we have

$$D\tilde{u}(z) = \begin{pmatrix} \varepsilon - \frac{1}{2}\big((z_1 + iz_2)^d + (z_1 - iz_2)^d\big) \\ -\frac{i}{2}\big((z_1 + iz_2)^d - (z_1 - iz_2)^d\big) \end{pmatrix}.$$

A simple calculation shows that $D\tilde{u}(z) = 0$ has d^2 solutions close to the origin. Obviously, the frequency of u is in the order of d.

4. Measure Estimates of Singular Sets in $\mathbb{R}^n$

In the present section, we shall discuss singular sets of harmonic functions in multi-dimensional spaces.

We first examine an example.

Example 4.1. Consider the harmonic polynomial in $\mathbb{R}^3$

$$u(x_1, x_2, x_3) = x_1^2 x_3 + x_2^2 x_3 - \frac{2}{3} x_3^2 - \varepsilon x_3.$$

A simple calculation shows that

$$Du(x_1, x_2, x_3) = (2x_1x_3, x_2x_3, x_1^2 + x_2^2 - 2x_3^2 - \varepsilon).$$

Then we have

$$\mathcal{S}(u) = \begin{cases} \emptyset & \text{for } \varepsilon < 0, \\ \{(0,0,0)\} & \text{for } \varepsilon = 0, \\ \{(x_1, x_2, 0); x_1^1 + x_2^2 = \varepsilon\} & \text{for } \varepsilon > 0. \end{cases}$$

The dimension of the singular set $\mathcal{S}(u)$ changes according to the sign of ε.

Example 4.1 illustrates that a serious problem arises if we study singular sets only in real spaces. This suggests that we shall study the singular set for the holomorphic extension of harmonic functions in the complex space.

We shall first prove a structure result for singular sets of harmonic functions. The proof is adapted from [15].

Lemma 4.2. *Let u be a nontrivial harmonic function in $B_1 \subset \mathbb{R}^n$. Then there holds*

$$\mathcal{S}(u) = \mathcal{S}^*(u) \bigcup \mathcal{S}_*(u),$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and the Hausdorff dimension of $\mathcal{S}_*(u)$ is at most $n-3$. Moreover, for any $p \in \mathcal{S}^*(u)$ the leading polynomial of u at p is a polynomial of two variables after some rotation of coordinates.

Proof. The proof consists of several steps. For each fixed $d \geq 2$, we consider

$$\mathcal{S}_d(u) = \{p \in \mathcal{S}(u);\ \partial^\nu u(p) = 0 \text{ for any } |\nu| < d,$$
$$\partial^{\nu_0} u(p) \neq 0 \text{ for some } |\nu_0| = d\}.$$

Step 1. We first study local behaviors at each point. For each point $y \in B_{1/2} \cap \mathcal{S}_d(u)$, set for any $r \in \big(0, \frac{1}{2}(1 - |y|)\big)$,

$$u_{y,r}(x) = \frac{u(y + rx)}{\big(⨍_{\partial B_r(y)} |u|^2\big)^{\frac{1}{2}}} \quad \text{for any } x \in B_2. \tag{4.1}$$

Then we have

$$u_{y,r} \to P \quad \text{in } L^2(B_2) \quad \text{as } r \to 0, \tag{4.2}$$

where $P = P_y$ is a non-zero homogeneous harmonic polynomial of d-degree. Moreover, $\|P\|_{L^2(\partial B_1)} = 1$. Note P is the normalized leading polynomial of u at y.

Since P is a non-zero homogeneous polynomial of d-degree, we have

$$\mathcal{S}_d(P) = \{x; \partial^\nu P(x) = 0 \text{ for any } |\nu| \le d-1\}.$$

Obviously $0 \in \mathcal{S}_d(P)$ by the homogeneity of P. It is easy to see $\mathcal{S}_d(P)$ is a linear subspace and

$$P(x) = P(x+z) \quad \text{for any } x \in \mathbb{R}^n \text{ and } z \in \mathcal{S}_d(P). \tag{4.3}$$

Next, we observe that $\dim\mathcal{S}_d(P) \le n-2$ for $d \ge 2$. In fact, (4.3) implies P is a function of n-$\dim\mathcal{S}_d(P)$ variables. If $\dim\mathcal{S}_d(P) = n-1$, P would be a d-degree monomial of one variable harmonic function. Hence $d < 2$, which is a contradiction.

Step 2. We define for each $j = 0, 1, 2, \dots, n-2$,

$$\mathcal{S}^j_d(u) = \{y \in \mathcal{S}_d(u);\ \dim\mathcal{S}_d(P_y) = j\}.$$

We claim that $\mathcal{S}^j_d(u)$ is on a countable union of j-dimensional C^1 graphs. In fact, we shall prove that for any $y \in \mathcal{S}^j_d(u)$ there exists an $r = r(y)$ such that $\mathcal{S}^j_d(u) \cap B_r(y)$ is contained in a (single piece of) j-dimensional C^1 graph.

To show this, we let ℓ_y be the j-dimensional linear subspace $\mathcal{S}_d(P_y)$ for any $y \in \mathcal{S}^j_d(u)$. For any $\{y_k\} \subset \mathcal{S}^j_d(u)$ with $y_k \to y$, we first prove

$$\text{Angle } < \overline{yy_k},\ \ell_y > \to 0. \tag{4.4}$$

To prove (4.4), we may assume $y = 0$ and $p_k = \dfrac{y_k}{|y_k|} \to \xi \in \mathbb{S}^{n-1}$. Note $p_k \in \mathcal{S}_d(u_{0,|y_k|})$ for

$$u_{0,|y_k|}(x) = \frac{u(|y_k|x)}{\left(⨍_{\partial B_{|y_k|}} u^2\right)^{\frac{1}{2}}}.$$

See (4.1) for notations. Obviously, $u_{0,|y_k|}$ is a harmonic function. It is easy to see that P_y vanishes at ξ with an order at least d, i.e.,

$$D^\nu P_y(\xi) = 0 \quad \text{for any } |\nu| \le d-1. \tag{4.5}$$

In fact, (4.5) holds since $u_{0,|y_k|} \to P_y$ in $C^d(\bar{B}_1)$ and $p_k \to \xi$ as $k \to \infty$, and $D^\nu u_{0,|y_k|}(p_k) = 0$ for any $|\nu| \le d-1$. Since P_y is a homogeneous polynomial of d-degree, then we have $\xi \in \ell_y$. This implies (4.4).

By (4.4), we obtain that for any $y \in \mathcal{S}_d^j(u)$ and small $\varepsilon > 0$ there exists an $r = r(y, \varepsilon)$ such that

$$\mathcal{S}_d^j(u) \cap B_r(y) \subset B_r(y) \cap C_\varepsilon(\ell_y), \tag{4.6}$$

where

$$C_\varepsilon(\ell_y) = \{z \in \mathbb{R}^n; \operatorname{dist}(z, \ell_y) \le \varepsilon |z|\}.$$

Let P_k and P be leading polynomials of u at y_k and $y = 0$, respectively. Then we have

$$P_k \to P \quad \text{uniformly in } C^d(B_1). \tag{4.7}$$

This implies

$$\ell_{y_k} \to \ell_y \quad \text{as } k \to \infty,$$

as subspaces in $\mathbb{R}^n$. By an argument similar as proving (4.4), we may prove that the constant r in (4.6) can be chosen uniformly for any point $z \in \mathcal{L}_d^j(u)$ in a neighborhood of y. In other words, for any $y \in \mathcal{S}_d^j(u)$ and any small $\varepsilon > 0$ there exists an $r = r(\varepsilon, y)$ such that

$$\mathcal{S}_d^j(u) \cap B_r(z) \subset B_r(z) \cap C_\varepsilon(\ell_z) \quad \text{for any } z \in \mathcal{S}_d^j(u) \cap B_r(y).$$

For $\varepsilon > 0$ small enough, this clearly implies that $\mathcal{S}_d^j(u) \cap B_r(y)$ is contained in a j-dimensional Lipschitz graph. By (4.4) this graph is C^1.

Step 3. Now we set

$$\mathcal{S}^*(u) = \bigcup_{d \ge 2} \mathcal{S}_d^{n-2}(u) \quad \text{and} \quad \mathcal{S}_*(u) = \bigcup_{j=0}^{n-3} \bigcup_{d \ge 2} \mathcal{S}_d^j(u).$$

This finishes the proof. □

Remark 4.3. In fact, we can prove that $\mathcal{S}^*(u)$ is on a countable union of $(n-2)$-dimensional $C^{1,\beta}$ manifolds, for some $0 < \beta < 1$.

The main result in the present section is the following theorem, which was proved in [21].

Theorem 4.4. *Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^n$. Then there holds*

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \cap B_{\frac{1}{2}}) \le C(N),$$

where $C(N)$ is a positive constant depending only on n and N, and N is the frequency of u in B_1 as in (0.1).

The key result is the following lemma for functions in $\mathbb{R}^n$.

Lemma 4.5. *Let P be a homogeneous harmonic polynomial of degree $d \geq 2$ and of two variables in $\mathbb{R}^n$. Then there exist positive constants ε and r, depending on P, such that for any harmonic function u in B_1 with*

$$|u - P|_{L^\infty(B_1)} < \varepsilon,$$

there holds

$$\mathcal{H}^{n-2}(|Du|^{-1}\{0\} \cap B_r) \leq c(n)(d-1)^2 r^{n-2}.$$

Proof. First, there exists a universal constant $R \in (0, 1)$ such that $u(x)$ can be extended into a holomorphic function $u(z)$ in D_R with

$$|u - P|_{C^1(D_R)} \leq c(n)|u - P|_{L^\infty(B_1)} \leq c\varepsilon. \tag{4.8}$$

In the following, we assume $\int_{\partial B_1} P^2 = 1$.

We first prove for $n = 2$. By the calculation in the previous section, (3.11) specifically, we have

$$|DP(z)| \geq c_0|z|^{d-1}.$$

By taking ε small in (4.8), we get

$$|Du(z) - DP(z)| < |PD(z)| \quad \text{for any } |z| = R.$$

Note Bezout's formula ([1], Corollary 1, P200) implies

$$\#\{|DP|^{-1}(0)\} = (d-1)^2, \quad \text{(including the multiplicity)}.$$

By Lemma 3.8, Rouché Theorem in $\mathbb{C}^2$, we have

$$\#\{|Du|^{-1}(0) \cap D_R\} \leq (d-1)^2.$$

This implies in particular

$$\#\{|Du|^{-1}(0) \cap B_R\} \leq (d-1)^2.$$

Next, we discuss the general dimension. We temporarily use $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n)$ to denote coordinates in $\mathbb{R}^n$ and use $\tilde{z}$ to denote the corresponding complex coordinates. In the following, we set $Du = (f, \tilde{f})$ and $DP = (g, 0)$. Here we treat f and g as maps from $\mathbb{C}^n$ to $\mathbb{C}^2$. Then we have

$$|g(\tilde{z})|^2 \geq c_0(|\tilde{z}_1|^2 + |\tilde{z}_2|^2)^{\frac{d-1}{2}},$$

and

$$|f - g|_{L^\infty(D_R)} \leq c\varepsilon.$$

Now we introduce a change of coordinate $\tilde{x} = Ox$ in $\mathbb{R}^n$ with an orthogonal matrix $O = (o_{ij})$ to be chosen. Then in $\mathbb{C}^n$, we have $\tilde{z} = Oz$. In the following, we shall evaluate f and g in z. For simplicity, we still write f and g, instead of $f \circ O$ and $g \circ O$. Then we have

$$|g(z)|^2 \geq c_0 \left(|\sum_{i=1}^n o_{1i} z_i|^2 + |\sum_{i=1}^n o_{2i} z_i|^2 \right)^{\frac{d-1}{2}}. \tag{4.9}$$

Note only the first two rows of the matrix O appear in (4.9).

For any $p \in \mathbb{R}^n$ and any $1 \leq i < j \leq n$, let $\mathbb{P}_{ij}(p)$ denote the 2-dimensional hyperplane

$$\{(p_1, \cdots, p_{i-1}, z_i, p_{i+1}, \cdots, p_{j-1}, z_j, p_{j+1}, \cdots, p_n)\}$$

and simply write $\mathbb{P}_{ij}(p) = \{(z_i, z_j)\}$ when there is no confusion. We also set $\mathbb{P}_{ij} = \mathbb{P}_{ij}(0)$.

Fix any $1 \leq i < j \leq n$. We consider f and g restricted on $\mathbb{P}_{ij}$. A straightforward calculation shows that

(4.10)

$$\begin{aligned}
&\left|g|_{\mathbb{P}_{ij}}\right|^2 \geq c_0 \left(|o_{1i} z_i + o_{1j} z_j|^2 + |o_{2i} z_i + o_{2j} z_j|^2 \right)^{\frac{d-1}{2}} \\
\geq & c_0 \left(\min\{\frac{1}{2}(o_{1i}^2 + o_{1j}^2 + o_{2i}^2 + o_{2j}^2), \frac{(o_{1i} o_{2j} - o_{1j} o_{2i})^2}{o_{1i}^2 + o_{1j}^2 + o_{2i}^2 + o_{2j}^2}\} \cdot (|z_i|^2 + |z_j|^2) \right)^{\frac{d-1}{2}}.
\end{aligned}$$

Therefore we require that in the orthogonal matrix O any 2×2 submatrices in the first two rows have nonzero determinants. If we write $g = (g_1, g_2)$, then each g_i is a product of $d-1$ homogeneous linear functions with real-valued coefficients. We may again apply Lemma 3.8, Rouché Theorem in $\mathbb{C}^2$, to get the following conclusion. There exists a constant δ_{ij} such that for any holomorphic function $v : D_R \subset \mathbb{C}^2 = \{(z_i, z_j)\} \to \mathbb{C}^2$ with

$$|v - g|_{\mathbb{P}_{ij}}| < \delta_{ij}, \quad \text{for any } (z_i, z_j) \in D_R, \tag{4.11}$$

there holds

$$\#(v^{-1}\{0\} \cap D_R^2) \leq (d-1)^2. \tag{4.12}$$

Here we use D_R^2 to denote the ball (centered at origin) with radius R in $\mathbb{C}^2$.

Take

$$\delta = \frac{1}{2} \min_{1 \le i < j \le n} \delta_{ij}.$$

For any $p \in \mathbb{R}^n$ and any $1 \le i < j \le n$, set $v_{ij,p} = f\big|\mathbb{P}_{ij}(p)$. By taking ε small in (4.8), we may find a small $r \in (0, R)$ such that for any $p \in B_r$ there holds

$$\big|v_{ij,p} - g|_{\mathbb{P}_{ij}}\big|_{L^\infty(D_R^2)} < 2\delta \le \delta_{ij}.$$

Then we have

$$\#(v_{ij,p}^{-1}\{0\} \cap D_R^2) \le (d-1)^2.$$

Obviously $|Du|^{-1}\{0\} \cap \mathbb{P}_{ij}(p) \subset v_{ij,p}^{-1}\{0\}$. If we set π_{ij} as the projection

$$\pi_{ij}(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \in \mathbb{R}^{n-2},$$

then we have shown, in particular, that for $q \in B_r^{n-2} \subset \mathbb{R}^{n-2}$ and any $1 \le i < j \le n$

$$\#(|Du|^{-1}\{0\} \cap \pi_{ij}^{-1}(q) \cap B_r) \le (d-1)^2.$$

Hence the integral geometric formula ([11], 3.2.22) implies

$$\begin{aligned}&\mathcal{H}^{n-2}(|Du|^{-1}\{0\} \cap B_r)\\ &\le \sum_{1 \le i < j \le n} \int_{B_r^{n-2}} \#(|Du|^{-1}\{0\} \cap \pi_{ij}^{-1}(q) \cap B_r) d\mathcal{H}^{n-2} q \le c(n)(d-1)^2 r^{n-2}.\end{aligned}$$

This finishes the proof. □

As the first step in proving Theorem 4.4, we shall show the following result.

Lemma 4.6. *Suppose u is a nonconstant harmonic function in $B_1$with $\|u\|_{L^2(B_1)}= 1$. Then there exist positive constants $C(u)$ and $\varepsilon(u)$, depending on n and u, and a finite collection of balls $\{B_{r_i}(x_i)\}$ with $r_i \le 1/8$ and $x_i \in \mathcal{S}(u)$ such that for any harmonic function v in B_1, with*

$$|u - v|_{L^\infty(B_1)} < \varepsilon(u),$$

there hold

$$\mathcal{H}^{n-2}\left(\mathcal{S}(v) \cap B_{1/2} \backslash \cup B_{r_i}(x_i)\right) \le C(u),$$

and

$$\sum r_i^{n-2} \le \frac{1}{2^{n-1}}.$$

Lemma 4.6 illustrates that the singular set $\mathcal{S}(v)$ of v is decomposed into two parts, a good part and a bad part. The good part has a measure estimate and the bad part is covered by small balls. The key point here is that the estimate for the good part and the covering for the bad part can be made uniform for harmonic functions v close to some u.

Proof. Let u be given in Lemma 4.6. By Lemma 4.2, we have

$$\mathcal{S}(u) = \mathcal{S}^*(u) \cup \mathcal{S}_*(u),$$

where $\mathcal{S}_*(u)$ has the Hausdorff dimension not exceeding $n-3$, $\mathcal{S}^*(u)$ is on a countable union of $(n-2)$-dimensional C^1 manifolds and for any $p \in \mathcal{S}^*(u)$ the leading polynomial of u at p is a homogeneous harmonic polynomial of 2 variables after an appropriate rotation. In particular, we have

$$\mathcal{H}^{n-2}\left(\mathcal{S}_*(u)\right) = 0.$$

Then there exist at most countably many balls $B_{r_i}(x_i)$ with $r_i \le 1/8$ and $x_i \in \mathcal{S}_*(u)$ such that

$$\mathcal{S}_*(u) \subset \bigcup_i B_{r_i}(x_i), \tag{4.13}$$

and

$$\sum r_i^{n-2} \le \frac{1}{2^{n-1}}. \tag{4.14}$$

We claim for any $y \in \mathcal{S}^*(u) \cap B_{3/4}$, there exist positive constants $R = R(y,u) < 1/8$, $r = r(y,u)$, $\eta = \eta(y,u)$ and $c = c(y,u)$, with $r < R$, such that if the function v satisfies

$$|u - v|_{L^\infty(B_R(y))} < \eta, \tag{4.15}$$

then

$$\mathcal{H}^{n-2}\left\{\mathcal{S}(v) \cap B_r(y)\right\} \le c r^{n-2}. \tag{4.16}$$

We will postpone the proof of (4.16).

It is obvious that the collection of $\{B_{r_i}(x_i)\}$ and $\{B_{r(y)}(y)\}$, $y \in \mathcal{S}^*(u)$, covers $\mathcal{S}(u)$. By the compactness of $\mathcal{S}(u)$, there exist $x_i \in \mathcal{S}_*(u)$, $i = 1, \cdots, k = k(u)$, and $y_j \in \mathcal{S}^*(u)$, $j = 1, \cdots, l = l(u)$, such that

$$\mathcal{S}(u) \cap B_{3/4} \subset \left(\bigcup_{i=1}^{k} B_{r_i}(x_i) \right) \bigcup \left(\bigcup_{j=1}^{l} B_{s_j}(y_j) \right), \tag{4.17}$$

with $r_i \le 1/8$, $i = 1, \cdots, k$, and $s_j \le 1/8$, $j = 1, \cdots, l$. Since $\mathcal{S}(u)$ is closed, there exists a positive constant $\rho = \rho(u)$ such that

$$\begin{aligned} &\left\{ x \in B_{3/4} \; ; \; dist(x, \mathcal{S}(u)) < \rho \right\} \\ &\subset \left(\bigcup_{i=1}^{k} B_{r_i}(x_i) \right) \bigcup \left(\bigcup_{j=1}^{l} B_{s_j}(y_j) \right). \end{aligned} \tag{4.18}$$

It is easy to see that for such a ρ there exists a positive constant $\delta = \delta(u)$ such that $|u - v|_{C^1(B_{3/4})} < \delta$ implies

$$\mathcal{S}(v) \cap B_{1/2} \subset \left\{ x \in B_{3/4}; \; dist(x, \mathcal{S}(u)) < \rho \right\}. \tag{4.19}$$

Denote

$$\mathcal{B}_u = \bigcup_{i=1}^{k} B_{r_i}(x_i), \quad \mathcal{G}_u = \bigcup_{j=1}^{l} B_{s_j}(y_j).$$

Now we take $\varepsilon(u) < \delta(u)$ small enough such that, for any harmonic function v in B_1, the condition

$$|u - v|_{L^\infty(B_1)} < \varepsilon(u)$$

implies for each $j = 1, \cdots, l = l(u)$,

$$|u - v|_{L^\infty(B_R(y_j))} < \eta(y_j, u).$$

Therefore there hold by (4.13), (4.14), (4.17)-(4.19),

$$\mathcal{S}(v) \cap B_{1/2} \subset (\mathcal{S}(v) \cap \mathcal{B}_u) \cup (\mathcal{S}(v) \cap \mathcal{G}_u),$$

$$\mathcal{H}^{n-2}(\mathcal{S}(v) \cap \mathcal{G}_u) \le c \sum_{j=1}^{l} s_j^{n-2} \equiv C(u),$$

and

$$\mathcal{B}_u = \bigcup_{i=1}^{k} B_{r_i}(x_i), \; r_i \le \frac{1}{8} \; \text{ and } \; \sum_{i=1}^{k} r_i^{n-2} \le \frac{1}{2^{n-1}}.$$

Now we prove (4.16) under the assumption (4.15). For any $y \in \mathcal{S}^*(u) \cap B_{3/4}$, there holds

$$u(x + y) = P(x) + \psi(x) \text{ for any } x \in B_{\frac{1}{4}},$$

where P is a nonzero d-degree homogeneous harmonic polynomial with $2 \le d \le N$ and $\psi(x)$ satisfies, by interior estimates, for any $|x| < 1/8$,

$$|\psi(x)| \le C|x|^{d+1}, \tag{4.20}$$

where C is a positive constant depending only on N and n. By an appropriate rotation P is a function of two variables. Hence we may assume P is defined in $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$. We abuse the notation by saying that P is defined in $\mathbb{R}^2$. Let ε_* and r_* be the constants given in Lemma 4.5 for P. By (4.20), we may take a positive constant $R = R(y,u) < 1/8$ such that

$$|\frac{1}{R^d}\psi|_{L^\infty(B_R)} < \frac{1}{2}\varepsilon_*.$$

Choose η small, depending on R and ε_*, such that (4.15) implies

$$|\frac{1}{R^d}(u-v)|_{L^\infty(B_R(y))} < \frac{1}{2}\varepsilon_*.$$

Then there holds

$$|\frac{1}{R^d}\left(v - P(\cdot - y)\right)|_{L^\infty(B_R(y))} < \varepsilon_*.$$

By considering the transformation $x \mapsto y + Rx$, we have

$$|\frac{1}{R^d}v(y+R\,\cdot) - P|_{L^\infty(B_1)} < \varepsilon_*.$$

Hence we may apply Lemma 4.5 to P. After transforming back to $B_R(y)$ we get for some $r \le Rr_*$

$$\mathcal{H}^{n-2}(|Dv|^{-1}\{0\} \cap B_r) \le c(n)(d-1)^2 r^{n-2}.$$

Therefore, we obtain (4.16). □

The proof of Theorem 4.4 is based on an iteration of Lemma 4.6. In order to do this, we need to introduce a class of compact harmonic functions. Consider a positive integer N. We denote by $\mathcal{H}_N$ the collection of all harmonic functions u in $B_1 \subset \mathbb{R}^n$

$$⨍_{B_{2r}(x_0)} u^2(x)\,dx \le 4^N ⨍_{B_r(x_0)} u^2(x)\,dx, \tag{4.21}$$

for all $x_0 \in B_{2/3}$ and $0 < 2r < \operatorname{dist}(x_0, \partial B_1)$. Obviously, $\mathcal{H}_N$ is invariant under dilations and translations. Specifically, if $u \in \mathcal{H}_N$, then we have $u_{x_0,r} = u(x_0 + r\cdot) \in \mathcal{H}_N$ for any $x_0 \in B_{2/3}$ and $0 < 2r < \operatorname{dist}(x_0, \partial B_1)$. The class $\mathcal{H}_N$ has the following important compactness property.

Lemma 4.7. *For any fixed positive integer N, the collection*

$$\{u \in \mathcal{H}_N; \int_{B_{1/2}} u^2(x)dx = 1\}$$

is compact under the local L^∞-metric.

Proof. The proof is straightforward. Suppose $u_k \in \mathcal{H}_N$ satisfies $\int_{B_{1/2}} u_k^2(x)\,dx = 1$. By (4.21) and some covering argument there holds for any $R \in (0,1)$

$$\|u_k\|_{L^2(B_R)} \le c(N,R), \quad k = 1, 2, \cdots .$$

Then there is a subsequence $u_{k'}$ such that $u_{k'}$ converges to a harmonic function u locally in $C^2(B_1)$. In (4.21) with u replaced with u_k, we may take the limit $k \to \infty$. Hence (4.21) holds for u and then $u \in \mathcal{H}_N$. It is obvious that $\int_{B_{1/2}} u^2(x)dx = 1$. □

Now we prove the following result.

Theorem 4.8. *Let N be a positive integer. Then there holds for any $u \in \mathcal{H}_N$*

$$\mathcal{H}^{n-2}\left\{\mathcal{S}(u) \cap B_{1/2}\right\} \le C,$$

where C is a positive constant depending on N and n.

Theorem 4.4 follows readily from Theorem 4.8. To prove Theorem 4.8, we need an improved version of Lemma 4.6.

Lemma 4.9. *Suppose N is a given positive integer. Then there exists a positive constant C, depending on N and n, such that for any $u \in \mathcal{H}_N$ there exists a finite collection of balls $\{B_{r_i}(x_i)\}$, with $r_i \le 1/4$ and $x_i \in \mathcal{S}(u)$, such that there hold*

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2} \setminus \cup B_{r_i}(x_i)\right) \le C,$$

and

$$\sum r_i^{n-2} \le \frac{1}{2}.$$

By comparing Lemma 4.6 and Lemma 4.9, we note that the constant C in Lemma 4.6 depends on the function u and that the constant C in Lemma 4.9 depends only on the class $\mathcal{H}_N$, independent of the specific functions in this class.

Proof. We set

$$\mathcal{H}_N^1 = \{u \in \mathcal{H}_N; \int_{B_{\frac{1}{2}}} u^2 = 1\}.$$

Take an arbitrary function $u_0 \in \mathcal{H}_N^1$. Consider any $u \in \mathcal{H}_N^1$ with $|u_0 - u|_{L^\infty(B_{7/8})} < \eta_0$. We take $\eta_0 = \eta_0(u_0)$ small such that

$$\eta_0 \le \varepsilon(u_0),$$

where $\varepsilon(u_0)$ is the constant given in Lemma 4.6. Then by Lemma 4.6, there exist a positive constant $C(u_0)$ and finitely many balls $\{B_{r_i}(x_i)\}$, with $x_i \in \mathcal{S}(u_0)$ and $r_i \le 1/8$, such that for any $u \in \mathcal{H}_N^1$, with $|u_0 - u|_{L^\infty(B_{7/8})} < \eta_0$, there hold

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{\frac{1}{2}} \setminus \cup_{i\ge1} B_{r_i}(x_i)\right) \le C(u_0),$$

and

$$\sum_{i\ge1} r_i^{n-2} \le \frac{1}{2^{n-1}}.$$

If $\mathcal{S}(u) \cap B_{r_i}(x_i) \ne \emptyset$, we may take $\tilde{x}_i \in \mathcal{S}(u) \cap B_{r_i}(x_i)$. Obviously $B_{r_i}(x_i) \subset B_{2r_i}(\tilde{x}_i)$. Therefore, for such a u by renaming radii and centers, we find a finite collection of balls $\{B_{r_i}(x_i)\}$, with $x_i \in \mathcal{S}(u)$ and $r_i \le 1/4$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2} \setminus \cup B_{r_i}(x_i)\right) \le C(u_0),$$

and

$$\sum_{i\ge1} r_i^{n-2} \le \frac{1}{2}.$$

By Lemma 4.7, $\mathcal{H}_N^1$ is compact under local L^∞-metric. Hence there exist $u_1, \cdots, u_p \in \mathcal{H}_N^1$ and $\eta_1 = \eta(u_1), \cdots, \eta_p = \eta(u_p)$ such that for any $u \in \mathcal{H}_N^1$ there exists a k with $1 \le k \le p$ with the property

$$|u - u_k|_{L^\infty(B_{7/8})} \le \eta_k.$$

Denote

$$C = \max\{C(u_1), \cdots, C(u_p)\}.$$

Such a constant C is finite and depends only on the class $\mathcal{H}_N$. This finishes the proof. □

Now we are ready to prove Theorem 4.8. The iteration scheme in the proof was first used by Hardt and Simon in [23].

Proof of Theorem 4.8. We use an iteration process to prove Theorem 4.8. To begin with, define

$$\phi_0 = \{B_{1/2}\}.$$

We claim that we may find $\phi_1, \phi_2, \cdots$, each of which consists of a collection of balls, such that for any $\ell \geq 1$

$$\text{rad}(B) \leq \frac{1}{2} \cdot \frac{1}{2^\ell} \quad \text{for any } B \in \phi_\ell,$$

$$\sum_{B \in \phi_\ell} [\text{rad}(B)]^{n-2} \leq \frac{1}{2^\ell},$$

and

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B \in \phi_{\ell-1}} B \sim \bigcup_{B \in \phi_\ell} B\right) \leq \frac{C}{2^\ell},$$

where C is the positive constant given in Lemma 4.9. Observe that

$$\mathcal{S}(u) \cap B_{\frac{1}{2}} \subset \bigcup_{\ell=1}^{\infty}\left(\mathcal{S}(u) \cap \big(\bigcup_{B \in \phi_{\ell-1}} B \sim \bigcup_{B \in \phi_\ell} B\big)\right)$$
$$\cup \bigcap_{\ell=0}^{\infty}\left(\mathcal{S}(u) \cap \bigcup_{j=\ell}^{\infty} \bigcup_{B \in \phi_j} B\right).$$

Hence we have

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2}\right) \leq C\left\{\sum_{\ell \geq 1} \frac{1}{2^{\ell-1}} + \inf_{\ell \geq 1} \sum_{j=\ell}^{\infty} \frac{1}{2^j}\right\} \leq 2C.$$

To prove the claim we construct $\{\phi_\ell\}$ by an induction. Note $\phi_0 = \{B_{1/2}\}$. Suppose $\phi_0, \phi_1, \ldots, \phi_{\ell-1}$ are already defined for some $\ell \geq 1$. To construct ϕ_ℓ, we take $B = B_r(y) \in \phi_{\ell-1}$, with $r \leq 1/2$. Consider the transformation $x \mapsto y + 2rx$. Then, $\tilde{u}(x) = u(y + 2rx)$ is a harmonic function in B_1. Obviously, $\tilde{u} \in \mathcal{H}_N$. Hence we may apply Lemma 4.9 to $\tilde{u}$ to obtain a collection of balls $\{B_{s_i}(z_i)\}$, with $s_i \leq 1/4$ and $z_i \in \mathcal{S}(\tilde{u})$, such that there hold

$$\mathcal{H}^{n-2}\left(\mathcal{S}(\tilde{u}) \cap B_{\frac{1}{2}} \backslash \cup B_{s_i}(z_i)\right) \leq C,$$

and

$$\sum s_i^{n-2} \leq \frac{1}{2}.$$

Now transform $B_{1/2}$ back to $B_r(y)$ by $x \mapsto (x-y)/2r$. We obtain that, for $B = B_r(y) \in \phi_{\ell-1}$, there exist finitely many balls $\{B_{r_i}(x_i)\}$ in $B_{2r}(y)$, with $r_i \le r/2$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u)\cap B_r(y)\backslash\bigcup_i B_{r_i}(x_i)\right)\le Cr^{n-2}\ ,$$

and

$$\sum_i r_i^{n-2}\le \frac{1}{2}r^{n-2}.$$

Then we set

$$\phi_\ell^B=\bigcup_i\{B_i(x_i)\},$$

and

$$\phi_\ell=\bigcup_{B\in\phi_{\ell-1}}\phi_\ell^B\ .$$

Hence we obtain

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u)\cap\bigcup_{B\in\phi_{\ell-1}}B\sim\bigcup_{B\in\phi_\ell}B\right)\le C\left(\sum_{B_{r_i}(x_i)\in\phi_{\ell-1}}r_i^{n-2}\right),$$

and by induction

$$r_i\le\frac{1}{2}\cdot\frac{1}{2^\ell},\qquad\sum_{B_{r_i}(x_i)\in\phi_\ell}r_i^{n-2}\le\frac{1}{2^\ell},$$

for each $\ell \ge 1$. This concludes the proof. □

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