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Uniqueness of Multiple-spike Solutions via the Method of Moving Planes

Chang-Shou Lin and Juncheng Wei

Dedicated to Professor L. Simon on the occasion of his sixtieth birthday

Abstract: We study the uniqueness of multiple-spike solutions for some singularly perturbed Neumann problems in a ball. We completely classify all two-peaked solutions and, except in some degenerate situations, also all three-peaked solutions. Our main idea is using the method of moving planes to show that in the case of two peaks both of them must be located on a line containing the origin and for three peaks all of them must lie in a two-dimensional hyperplane containing the origin. Then we compute the degree of these solutions (restricted in certain symmetry class) and show their uniqueness.

1. Introduction

We consider the following singularly perturbed semilinear elliptic problem

$$\begin{cases} \epsilon^2 \Delta u - bu + f(u) = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega \ \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in R^N, ν stands for the unit outer normal to $\partial\Omega$, $b > 0$ is a positive constant and $f(t)$ is a $C^{1+\sigma}(R) \cap C^2_{loc}(0, +\infty)$ function such that $f(0) = f^{'}(0) = 0$. Typical examples of the function $-bu + f(u)$ are

$$-au + f(u) = -u + u_+^p \text{ with } u_+ = \max(0, u), b = 1, \tag{1.2}$$

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$$-bu + f(u) = u(u-a)(1-u) \text{ with } 0 < a < \frac{1}{2}, b = 1+a, \tag{1.3}$$

where

$$1 < p < (\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2} \text{ when } N \geq 3; = +\infty \text{ when } N = 1, 2).$$

Equation (1.1) with (1.2) or (1.3) arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation ([18], [39], [51]) or of parabolic equations in chemotaxis, population dynamics and phase transitions ([5], [6],[32], [37]).

Associated with (1.1) is the energy functional J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega (\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{b}{2}u^2 - F(u))dx \qquad \text{for } u \in H^1(\Omega), \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$.

It is known that any solution u of (1.1) is a critical point of J_ϵ and vice versa. In this paper, we restrict ourselves to families of solutions $\{u_\epsilon\}_{0<\epsilon<\epsilon_0}$ of (1.1) with **finite** energy, i.e.

$$\epsilon^{-N} J_\epsilon[u_\epsilon] < +\infty \qquad \text{for } 0 < \epsilon < \epsilon_0. \tag{1.5}$$

It can be proved that for ϵ sufficiently small, any family of solutions of (1.1) satisfying (1.5) can have at most a finite number of local maximum points (see [34], [48]). Let the local maximum points be $\{P_1^\epsilon, ..., P_K^\epsilon\} \subset \bar{\Omega}$. Then one can show that for ϵ sufficiently small, we have

(S1) $P_i^\epsilon \neq P_j^\epsilon$ if $i \neq j$; there exists $\{P_1^0, ..., P_K^0\} \in \bar{\Omega}$ such that $P_j^\epsilon \to P_j^0 \in \bar{\Omega}$ as $\epsilon \to 0$, and u_ϵ attains a strict local maximum at $x = P_j^\epsilon$, for $j = 1, ..., K$,

(S2) $c_0^{-1} \geq u_\epsilon(P_j^\epsilon) \geq c_0 > 0$ for some constant c_0 independent of ϵ and

(S3) $u_\epsilon(x) \to 0$ as $\epsilon \to 0$ locally uniformly in $\bar{\Omega} \backslash \{P_1^0, ..., P_K^0\}$.

Solutions satisfying (S1)-(S3) and (1.5) are called K spike solutions.

In the pioneering papers [31], [32], [34] and [35], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ϵ sufficiently small the least-energy solution has only one local maximum point P^ϵ with $P^\epsilon \in \partial\Omega$. Moreover, $H(P^\epsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P.

Since then many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [4], [5], [6], [7], [11], [12], [13], [14], [15], [16],[21], [22], [23], [24], [25], [27], [28], [34], [35], [36], [37], [38], [44], [45], [46], [52], [53], and the references therein. (Recent surveys can be found in [39], [54].) It turns out that for the Neumann problem, there are arbitrarily many multiple interior spike solutions. In particular, it was proved in [22] (see also [6], [13]) that given any bounded domain Ω and a positive integer K, there exists an ϵ_K such that for $0 < \epsilon < \epsilon_K$ problem (1.1) has at least one $K-$interior spike solution with spikes located at $\{P_1^\epsilon, ..., P_K^\epsilon\}$. Moreover, as $\epsilon \to 0$,

$$\varphi_K(P_1^\epsilon, ..., P_K^\epsilon) \to \max_{(P_1,...,P_K)\in\Omega^K} \varphi_K(P_1, ..., P_K) \tag{1.6}$$

where

$$\varphi_K(P_1, ..., P_K) = \min_{i,j,k,i\neq j}\left(\frac{|P_i - P_j|}{2}, d(P_k, \partial\Omega)\right). \tag{1.7}$$

(1.6) shows that if $(P_1^\epsilon, ..., P_K^\epsilon) \to (P_1^0, ..., P_K^0)$ as $\epsilon \to 0$, then $(P_1^0, ..., P_K^0)$ attain the **sphere packing** positions in Ω:

$$\varphi_K(P_1^0, ..., P_K^0) = \max_{(P_1,...,P_K)\in\Omega^K} \varphi_K(P_1, ..., P_K). \tag{1.8}$$

(Multiple mixed-boundary-interior spike solutions are also obtained in [23].)

In this paper, we consider mainly the case where the domain Ω is the unit ball $B = \{x \in R^N | |x| < 1\}$. The results of [6], [13], [14], [22], [23] and [36] show that there can be an arbitrary number of multiple boundary spikes or multiple interior spikes. In general, the limiting positions of the boundary spikes or interior spikes have certain symmetries. As an example, let us consider the interior three spikes constructed in [22] (see Figure 1). The three locations P_1^0, P_2^0, P_3^0 form a perfect triangle which is symmetric under rotation by $\frac{2\pi}{3}$. Moreover, if $N \geq 3$, one can show that P_1^0, P_2^0, P_3^0 and the origin must lie in a two dimensional hyperplane. Naturally, one may ask: does the solution also have this **partial symmetry**?

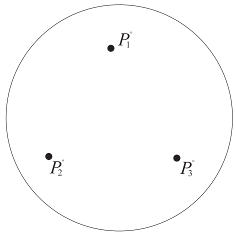

Figure 1

The first result on the partial symmetry of spike solutions of (1.1) is due to Lin and Takagi [30]. (Independently, Grossi [20] obtained the symmetry of the single interior spike solution.) In [30], Lin and Takagi showed that single boundary spike solutions must be axially symmetric, single interior spike solutions must be radially symmetric, and the two boundary spikes $P_1^\epsilon \in \partial\Omega$, $P_2^\epsilon \in \partial\Omega$ must satisfy $P_1^\epsilon = -P_2^\epsilon$. By using this information, they showed the uniqueness of the single boundary spike solution and of the two boundary spike solution. We remark that the uniqueness of the single boundary and single interior spike solutions in general domains is studied in [7], [51], [49].

In this paper, we study the partial symmetry for two spikes and three spikes (interior or boundary or mixed). The method of moving planes (MMP) gives us part of the partial symmetry but not the full result. To obtain the full partial symmetry, we have to show the uniqueness of the solutions. To this end, we compute the degree of the solutions.

To illustrate our idea, let us take a look again at the interior three spike solutions (see Figure 1). Suppose we have three interior spikes $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ with $P_j^\epsilon \to P_j^0, j = 1, 2, 3$ as $\epsilon \to 0$. To show the full partial symmetry, we have to show that $P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} P_1^\epsilon, j = 2, 3$. Note that this is an equation with $3N$ variables and many symmetries (in other words, degeneracies). To overcome these difficulties, we proceed in two steps:

Step 1. We use MMP to show that $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ and the origin must lie in a two-dimensional hyperplane and that u_ϵ is axially symmetric with respect to the hyperplane. This reduces our problem to R^2 with six scalar variables. Now the rotational invariance eliminates one more variable.

Step 2. We now show that $P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} P_1^0$ when $P_j^\epsilon \in R^2, j = 2, 3$. To achieve this, we have to compute the degree of u_ϵ restricted to the symmetry class obtained in Step 1. We use the Liapunov-Schmidt reduction method and asymptotic analysis to show that u_ϵ is nondegenerate and that the degree at u_ϵ is exactly $(-1)^5$. This proves the uniqueness.

MMP is a powerful method in showing symmetry for Dirichlet problems. For Neumann problems, it has been used recently to show partial symmetry for blow-up and concentration problems ([10], [29], [30]). On the other hand, the method of Liapunov-Schmidt reduction has been used in singularly perturbed problems to obtain existence and multiplicity of solutions ([1], [2], [3], [4], [5], [6], [7], [11], [13], [14], [17], [22], [23], [25], [27], [40], [41], [52], [53]). As far as we know, the results of this paper are the first in combining **both** methods and proving the partial symmetry for three spike solutions. In fact, we are able to completely classify **all** two spike solutions and, except for some degenerate cases, also **all** three spike solutions.

2. Main Results: partial symmetry and uniqueness of two and three spikes

We now state the main theorems of this paper. We always assume that $\Omega = B$ and that $P_j^\epsilon \in \bar{\Omega}, j = 1..., K$ are the K spikes (boundary or interior). Without loss of generality, we may assume that $b = 1$ in (1.1).

First we state the conditions on the function $f(t)$:

(f1) $f \in C^{1+\sigma}(R) \cap C^2_{loc}(0, +\infty)$ with $0 < \sigma \le 1, f(0) = 0, f'(0) = 0$ and $f(t) = 0$ for $t \le 0$.

(f2) There exists two positive constants α and β such that $0 < \alpha < \beta, (t - \alpha)(-t + f(t)) > 0$ for $0 < t < \alpha$ or $\alpha < t < \beta$, and $-\frac{1}{2}\beta^2 + F(\beta) > 0$ where $F(t) = \int_0^t f(s)ds$.

(f3) The problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0, w > 0 \text{ in } R^N, \\ w(0) = \max_{y \in R^N} w(y), \lim_{|y| \to +\infty} w(y) = 0, \end{cases} \tag{2.1}$$

has a unique solution w, which is nondegenerate, i.e.

$$\text{Kernel}(\Delta - 1 + f'(w)) = \text{span}\left\{\frac{\partial w}{\partial y_1}, ..., \frac{\partial w}{\partial y_N}\right\}. \tag{2.2}$$

By the well-known result of Gidas, Ni and Nirenberg [19], w is radially symmetric: $w(y) = w(|y|)$ and strictly decreasing: $w'(r) < 0$ for $r > 0, r = |y|$. Moreover, we have the following asymptotic behavior of w:

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \quad w'(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \tag{2.3}$$

for r large, where $A_N > 0$ is a generic constant.

The uniqueness of w is proved in [26] for the case $f(u) = u^p$. For a general nonlinearity, see [9]. For $f(u)$ defined by (1.3), the uniqueness of the entire solution was proved by Peletier and Serrin [43].

In what follows we always assume that $f(t)$ satisfies (f1), (f2) and (f3).

Our first theorem concerns the case $K = 2$.

Theorem 2.1. *Let $K = 2$. Then for ϵ sufficiently small, P_1^ϵ, the origin, P_2^ϵ must lie on a line with 0 between P_1^ϵ and P_2^ϵ. Without loss of generality, we may assume that P_1^ϵ and P_2^ϵ lie on the x_1-axis. Then u_ϵ is axially symmetric with respect to $x_j, j = 2, ..., N$.*

By Theorem 2.1 and asymptotic analysis (Lemma 5.4 of Section 5 and Lemma 7.1 of Section 7), up to a rotation, there are exactly three possibilities for the limiting positions of the two peaks. These locations are listed below (see Figure 2).

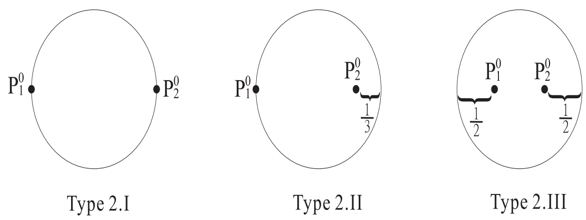

Figure 2

Our second theorem classifies all two-peaked solutions.

Theorem 2.2. *Let $K = 2$. Then for ϵ sufficiently small, up to a rotation, there are exactly three two-peaked solutions: $u_\epsilon^1, u_\epsilon^2, u_\epsilon^3$. The limiting locations of the three two-peaks are as above (Figure 2).*

Moreover, $u_\epsilon^1, u_\epsilon^3$ are symmetric with respect to $x_j, j = 1, ..., N$, u_ϵ^2 is symmetric with respect to $x_j, j = 2, ..., N$.

We note that the existence of type 2.I solutions was proved in [36] and the existence of type 2.III solutions was proved in [6], [13] and [22]. We remark that the existence and uniqueness of type 2.II solution is new. Combining Theorem 2.1 and Theorem 2.2, we have classified all two-peaked solutions.

Next we consider the three-peak case, which is more complicated.

Our third theorem shows that the three peaks and the origin must lie in a two-dimensional hyperplane.

Theorem 2.3. *Let $K = 3$ and let $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ be the three local maximum points of a three-peaked solution u_ϵ of (1.1). Then for ϵ sufficiently small, $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ and the origin must lie in a two-dimensional hyperplane. Without loss of generality, we may assume that the hyperplane is $\Gamma := \{(x_1, ..., x_N) | x_3 = ... = x_N = 0\}$. Then u_ϵ is axially symmetric with respect to $x_j, j = 3, ..., N$. Moreover, the origin must be in the interior of the triangle formed by $P_1^\epsilon P_2^\epsilon P_3^\epsilon$.*

Our last theorem concerns the uniqueness of three-peaked solutions. By Theorem 2.3 and some asymptotic analysis (Lemma 5.4 of Section 5 as well as Lemma 7.1 and Lemma 7.2 of Section 7), up to a rotation, there are exactly seven possibilities for the limiting positions of the three peaks. The locations are listed below (see Figure 3).

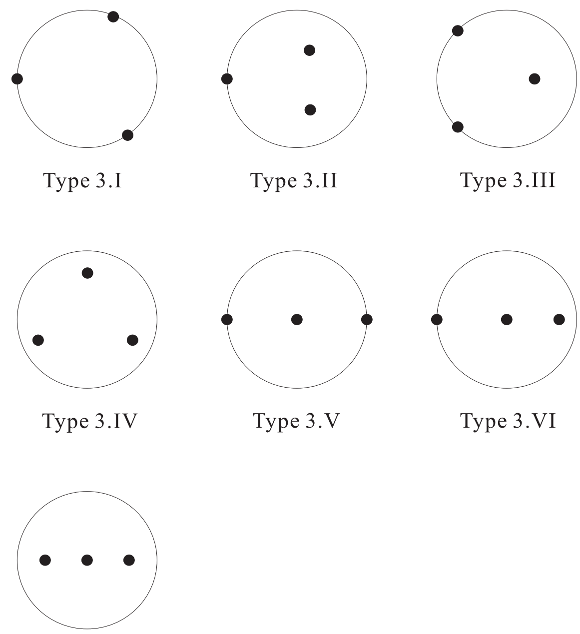

Figure 3

We have

Theorem 2.4. *Let $K = 3$. Then for ϵ sufficiently small, there are at least seven types of three-peaked solutions. Among them, the first four types are unique. Each of these four types of solution is symmetric in $x_j, j = 3, ..., N$ and inherits the partial symmetry of the locations, e.g., type 3.II, type 3.III solutions are symmetric in x_2, type 3.I and type 3.IV are symmetric in $x_j, j = 1, ..., N$ and rotationally invariant by $\frac{2\pi}{3}$.*

The other three types of solutions are symmetric with respect to $x_j, j = 3, ..., N$.

The existence of a type 3.I solution was proved in [36] and the existence of a type 3.IV solution was proved in [6], [13] and [22]. The existence of the other five types of solutions is new.

The uniqueness of the other three types 3.V, 3.VI, 3.VII remains open. The main problem is that we can not show that the solutions are symmetric with respect to the x_2-axis. Once this is shown, it can be proven that they are unique.

It is natural to ask what happens when $K \geq 4$. We pose the following conjecture at the end:

Conjecture: *If the limiting problem (1.8) has a certain partial symmetry, then for ϵ sufficiently small, the solutions obtained in* [22] *inherit that partial symmetry.*

In this paper, we shall study the uniqueness of type 3.IV solutions of three-peaked solutions in detail, since it is the most complicated and it has the largest number of degrees of freedom. The proof of the uniqueness of the other types will be given in the last section.

The structure of the paper is as follows:

In Section 3, we shall give the proof of Theorems 2.1 and 2.3 by applying the well-known method of moving planes (MMP) to Neumann problems. This is the MMP part of the paper.

From Section 4 to Section 6, we prove the uniqueness of type 3.IV solutions in the class of partial symmetric functions given by Theorem 2.3. This is the Liapunov-Schmidt reduction part of the paper.

In Section 4, we present some preliminaries on the reduction from the infinite dimensional space $H^1(\Omega)$ to a finite dimensional problem on the space of spikes. In Section 5, we classify all types of limiting positions of two or three spikes. In Section 6, we show the uniqueness of the type 3.IV solution by computing its Morse index and degree (restricted to certain symmetric class).

Finally in Section 7, we show how similar ideas can be adopted to prove the uniqueness of other types of solutions. Several technical estimates are contained in Appendices A, B, C and D.

Throughout the paper, we use C to denote various constants independent of ϵ small. It is always assumed that $\epsilon > 0$ is small and $\delta > 0$ is a fixed but small constant.

3. Method of Moving Planes and the Proofs of Theorems 2.1 and 2.3

In this section, we apply the well-known method of moving planes to (1.1). We follow the proofs given in Section 3 of [30], where it is shown that for two boundary spikes $P_1^\epsilon, P_2^\epsilon$ it holds that $P_1^\epsilon = -P_2^\epsilon$, provided that ϵ is sufficiently small.

To describe the local structure of spike-layer solutions, we need to introduce a diffeomorphism $z = \Phi_\epsilon^j(x)$ which is defined in a neighborhood of each concentration point P_j^ϵ. If $P_j^\epsilon \in \Omega$, then $\Phi_\epsilon^j(x) = x$ and $\Omega_{\epsilon,R}^j = B_{\epsilon R}(P_j^\epsilon) = \{x||x - P_j^\epsilon| \leq \epsilon R\}$. If $P_j^\epsilon \in \partial\Omega$, then $z = \Phi_\epsilon^j(x)$ maps the boundary portion of $\partial\Omega$ at P_j^ϵ to $(0, ..., 0, 1)$. For details we refer to the reader to [34] and [30]. We also assume that Φ_ϵ^j maps the interior of Ω to the lower half-space $R_-^N = \{z|z_N < 0\}$. Let $B_R^- := \{z \in R_-^N||z| < R\}$. Set $\Omega_{\epsilon,R}^j = (\Phi_\epsilon^j)^{-1}(B_{\epsilon R}^-)$.

We state a general result on the asymptotic behavior of $K-$spikes.

Proposition 3.1. *Let $\{u_\epsilon\}$ be a family of solutions to (1.1) with K spikes $P_j^\epsilon \in \bar{\Omega}, j = 1, ..., K$. Suppose that*

$$\lim_{\epsilon \to 0} \frac{d(P_j^\epsilon, \partial\Omega)}{\epsilon} < +\infty, \quad j = 1, ..., l$$

and

$$\lim_{\epsilon \to 0} \frac{d(P_j^\epsilon, \partial\Omega)}{\epsilon} = +\infty, \quad j = l + 1, ..., K.$$

Then $P_j^\epsilon \in \partial\Omega, j = 1, ..., l$ provided ϵ sufficiently small. Moreover, for any $\delta > 0$, there exists $R = R(\delta)$ and $\epsilon_1 = \epsilon(\delta)$ such that the following statements hold if $0 < \epsilon < \epsilon_1$

$$(i) \qquad \|u_\epsilon(x) - w(\Phi_j^\epsilon(x)/\epsilon)\|_{C^2(\bar{\Omega}_{\delta,R}^j)} \leq \delta,$$

for $j = 1, ..., K$;

$$(ii) \qquad u_\epsilon(x) \leq C\delta e^{-\mu dist(x,\{P_1^\epsilon,...,P_K^\epsilon\})} \text{ for } x \in \bar{\Omega} \setminus \cup_{j=1}^K \Omega_{\epsilon,R}^j$$

where C and μ are positive constants independent of ϵ.

The proof of Proposition 3.1 is similar to that of Theorem 2.1 of [34]. We omit the details here.

The main results in this section say that for two peaks, they must both lie on a line containing the origin. For three peaks, they must lie in a two-dimensional hyperplane containing the origin. Moreover, the corresponding solutions must be symmetric with respect to the line or the hyperplane. Since the proof of the two-peaked case is similar to that of [30], we focus our attention on three spikes. Without loss of generality, we assume that the three spikes $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ lie in a two-dimensional hyperplane $\{(x_1, ..., x_N)|x_3 = t_{\epsilon,3}, ..., x_N = t_{\epsilon,N}\}$. We need to show that $t_{\epsilon,j} = 0, j = 3, ..., N$ and u_ϵ is symmetric with respect to $x_j, j = 3, ..., N$, provided that ϵ is sufficiently small. Without loss of generality, we may assume that $t_\epsilon := t_{\epsilon,N} > 0$. Assume also that $(P_{1,1}^\epsilon)^2 + (P_{1,2}^\epsilon)^2 = \max_{j=1,2,3}((P_{j,1}^\epsilon)^2 + (P_{j,2}^\epsilon)^2)$ and that $P_{1,2}^\epsilon = 0, P_{1,1}^\epsilon > 0$. Let $\theta_\epsilon = \arctan(\frac{t_{\epsilon,N}}{P_{1,1}^\epsilon})$. Let $\bar{P}_j^\epsilon = (P_{j,1}^\epsilon, 0, ..., 0)$.

Set $e_\theta = (\sin\theta, 0, ..., 0, -\cos\theta)$ and Π_{N-1}^θ be the $(N-1)-$dimensional hyperplane perpendicular to the vector e_θ and x^θ denotes the reflection of x with respect to Π_{N-1}^θ. Set

$$w_\epsilon^\theta(x) = u_\epsilon(x) - u_\epsilon(x^\theta) \text{ for } x \in \Sigma_\theta$$

where Σ_θ is the connected component of $\Omega\backslash\Pi_{N-1}^\theta$ containing P_1^ϵ. Obviously w_ϵ^θ satisfies

$$\begin{cases} \epsilon^2\Delta w_\epsilon^\theta + c_\epsilon^\theta(x)w_\epsilon^\theta = 0 \text{ in } \Sigma_\theta, \\ \frac{\partial w_\epsilon^\theta}{\partial\nu}(x) = 0 \text{ on } \partial\Sigma_\theta\backslash\Pi_{N-1}^\theta, \\ w_\epsilon^\theta(x) = 0 \text{ on } \Sigma_\theta \cap \Pi_{N-1}^\theta, \end{cases} \tag{3.1}$$

where

$$c_\epsilon^\theta(x) = -1 + \frac{f(u_\epsilon(x)) - f(u_\epsilon(x^\theta))}{u_\epsilon(x) - u_\epsilon(x^\theta)}. \tag{3.2}$$

We prove our claim in a series of three steps.

Step 1: We first prove that

$$w_\epsilon^0(x) > 0 \text{ for } x \in \Sigma_0 = \{x \in \Omega | x_N > 0\}. \tag{3.3}$$

Note that since $P_j^\epsilon, j = 1, 2, 3$ are the local maximum points of u_ϵ (and one of them must be a global maximum point), we see that $w_\epsilon^0(P_j^\epsilon) > 0$ for some j. For contradiction, we assume that the set

$$E_\epsilon := \{x \in \Sigma_0 | w_\epsilon^0(x) < 0\}$$

is non-empty. We break the argument into three cases. (The following argument is for a subsequence of $\epsilon_i \to 0$. For simplicity, we use the same notation ϵ to denote ϵ_i.)

Case 1: $\frac{t_\epsilon}{\epsilon} \to +\infty$ as $\epsilon \to 0$.

In this case, we see that for arbitrarily large $R > 0$, we have $E_\epsilon \subset (\cup_{j=1}^3 B_{\epsilon R}(P_j^\epsilon))^c$. Hence $u_\epsilon \le \delta$ for $x \in E_\epsilon$ and ϵ small. Moreover $w_\epsilon^0(P_j^\epsilon) > 0$. Now that

$$c_\epsilon^\theta(x) \le -\frac{1}{2}$$

for $x \in E_\epsilon$. By (3.1), the minimum value of u_ϵ, if it is negative, must be obtained on the boundary of Σ_0. Since $\frac{\partial w_\epsilon^\theta}{\partial \nu} = 0$ on $\partial\Omega \cap \Sigma_0$ and $w_\epsilon^\theta = 0$ on $\partial \Sigma_0$, by the Maximum Principle, $w_\epsilon^0 > 0$ and E_ϵ is empty.

Case 2: $C^{-1} \le \frac{t_\epsilon}{\epsilon} \le C$ for some $C > 1$ independent of ϵ.

Let $x_\epsilon \in \bar{E}_\epsilon$ be such that

$$w_\epsilon^0(x_\epsilon) = \inf_{E_\epsilon} w_\epsilon^0(x) < 0. \tag{3.4}$$

Assume for the moment that

$$\text{limsup}_{\epsilon \to 0} \left\{ \frac{\min_{j=1,2,3} |x_\epsilon - P_j^\epsilon|}{\epsilon} \right\} \to +\infty.$$

Then by Proposition 3.1, $u_\epsilon(x_\epsilon) \to 0$ and $c_\epsilon(x_\epsilon) < -\frac{1}{2} < 0$ and $0 \le \epsilon^2 \Delta w_\epsilon^0(x_\epsilon) = -c_\epsilon^0(x_\epsilon) w_\epsilon^0(x_\epsilon) < 0$, a contradiction. Therefore we conclude that

$$|x_\epsilon - P_j^\epsilon| \le R\epsilon$$

for some $j = 1, 2, 3$. Without loss of generality, we may assume that $|x_\epsilon - P_1^\epsilon| \le R\epsilon$. Let

$$x = \bar{P}_1^\epsilon + \epsilon y, v_\epsilon(y) = u_\epsilon(x), x_\epsilon = \bar{P}_1^\epsilon + \epsilon y_\epsilon.$$

Set $y_\epsilon = (y_\epsilon^{'}, y_{\epsilon,N})$. Then $y_{\epsilon,N} \ge 0$ and let us assume that $y_{\epsilon,N} \to y_{0,N} \ge 0, y_\epsilon^{'} \to y^{'}$. We claim that $y_{0,N} > 0$. In fact, by our assumption, $\frac{P_{1,N}^\epsilon}{\epsilon} \to s > 0$. Then $v_\epsilon(y) \to w(y)$ in $C_{loc}^2(R^N)$. (If $P_1^\epsilon \in \partial\Omega$, then we need to extend v_ϵ be reflection. We omit the details.) If $y_{0,N} = 0$, then $\frac{\partial w_\epsilon^0(\epsilon y + P_1^\epsilon)}{\partial y_N} \to 2\frac{\partial w}{\partial y_N}(y^{'}, s) < 0$ which contradicts to the fact that

$\nabla w_\epsilon^0(x_\epsilon) = 0$. So $y_{0,N} > 0$. In this case, $\frac{\partial v_\epsilon}{\partial y_N} \to \frac{\partial w}{\partial y_N} > 0$ if $y_N > 0$. Hence $w_\epsilon^0(x_\epsilon) > 0$ for ϵ small. A contradiction again.

Case 3: $\frac{t_\epsilon}{\epsilon} = 0$.

In this case, if we write $P_i^\epsilon = P_i^0 + \epsilon(z'_{i,\epsilon}, \zeta_{i,\epsilon})$ with $z'_{i,\epsilon} \in R^{N-1}, \zeta_{i,\epsilon} > 0$, then $z'_{i,\epsilon} \to 0$ and $\zeta_{i,\epsilon} \to 0$ as $\epsilon \to 0$. Set $N_\epsilon := \sup_{x \in B_+} |w_\epsilon^0(x)|$ and let $\bar{x}_\epsilon \in \bar{B}_+$ be such that $|w_\epsilon^0(\tilde{x}_\epsilon)| = N_\epsilon$. It is easy to see that $\min_{j=1,\ldots,N}(|\tilde{x}_\epsilon - P_j^0|) \leq C\epsilon$. Without loss of generality, we may assume that $|\tilde{x}_\epsilon - P_1^0| \leq C\epsilon$.

Consider the following scaling

$$\tilde{w}_\epsilon^0(y) := \frac{1}{N_\epsilon} w_\epsilon^0(P_1^0 + \epsilon y). \tag{3.5}$$

Let us assume that $\tilde{y}_\epsilon := \frac{\tilde{x}_\epsilon - P_1^0}{\epsilon} \to \tilde{y}$, where $\tilde{y} = (\tilde{y}', \tilde{\eta})$ satisfying $\tilde{\eta} \geq 0$.

Similar to the proof of case 3 of Section 3 of [30], we see that $\tilde{w}_\epsilon^0(y) \to \tilde{v}(y)$ in C^2_{loc}, where

$$\tilde{v}(y) = c\frac{\partial w}{\partial y_N}(y). \tag{3.6}$$

Since P_1^ϵ is a local maximum point of u_ϵ, we obtain also that

$$c < 0. \tag{3.7}$$

One can also verify that for ϵ sufficiently small

$$C_0^{-1} \leq \frac{\zeta_\epsilon}{N_\epsilon} \leq C_0 \tag{3.8}$$

We now obtain a contradiction with (3.7). In fact, since $|\tilde{x}_\epsilon - P_1^0| \leq C\epsilon$, we see that $\tilde{w}_\epsilon(\tilde{y}_\epsilon) < 0$ but $\tilde{w}_\epsilon(\tilde{y}'_\epsilon, 0) = 0$. By the mean value theorem, there exists a $\xi_\epsilon > 0$ such that

$$\frac{\partial \tilde{w}_\epsilon^0}{\partial y_N}(\tilde{y}'_\epsilon, \xi_\epsilon) < 0 \tag{3.9}$$

which implies that as $\epsilon \to 0$, we have

$$0 \geq \lim_{\epsilon \to 0} \frac{\partial \tilde{w}_\epsilon^0}{\partial y_N}(\tilde{y}'_\epsilon, \xi_\epsilon) = \frac{\partial v}{\partial y_N}(y_*, 0) = c\frac{\partial^2 w}{\partial y_N^2}(y'_*, 0) = c\frac{w'(|y'_*|)}{|y'_*|} > 0$$

which is a contradiction.

Consequently, E_ϵ must be empty for ϵ sufficiently small.

This finishes Step 1.

Step 2. Let

$$\theta_0 = \sup \{\bar{\theta} | w_\epsilon^\theta > 0 \text{ for } x \ \in \Sigma_\theta \text{ and } 0 \leq \theta \leq \bar{\theta}\}.$$

By the same argument as in Step 1, we see that

$$\theta_0 \geq \theta_\epsilon.$$

Note that by the definition of θ_0, $w_\epsilon^{\theta_0} \geq 0$ in Σ_{θ_0}, and (ii) if $w_\epsilon^{\theta_0}(x) > 0$ for some $x \in \Sigma_{\theta_0}$ then $w_\epsilon^{\theta_0} > 0$ in Σ_{θ_0} by the maximum principle. Hence we see that $w_\epsilon^{\theta_0} \equiv 0$ on Σ_{θ_0}. Since P_1^ϵ is a local maximum point, we see that $\theta_0 \leq \theta_\epsilon$. Hence $\theta_0 \equiv \theta_\epsilon$ and $w_\epsilon^{\theta_\epsilon}(x) \equiv 0$ for $x \in \Sigma_{\theta_\epsilon}$. Since u_ϵ has exactly three local maximum points, this implies that $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ must lie in the hyperplane $\partial\Sigma_{\theta_\epsilon}$. This shows that $P_j^\epsilon, j = 1, 2, 3$ and the origin lie in a two-dimensional hyperplane.

Step 3: By Step 2, $P_j^\epsilon, j = 1, 2, 3$ and the origin lie in a two-dimensional hyperplane. Without loss of generality, we may assume that the hyperplane is $\Gamma = \{x_3 = ... = x_N = 0\}$. Now we show that

$$w_\epsilon^0(x) \equiv 0 \text{ on } \Sigma_0.$$

Suppose that there is a sequence $\epsilon_j \to 0$ such that

$$N_i = \sup_{x \in \Sigma_0} |w_i^0(x)| > 0,$$

where w_i^0 stands for $w_{\epsilon_i}^0$. Choose an $x_i \in \bar{\Sigma}_0$ so that $|w_i^0(x_i)| = N_i$. As in Case 2 above, we can show that

$$\text{limsup}_{i \to +\infty} \min_{j=1,2,3} (\frac{|x_i - P_j^\epsilon|}{\epsilon}) < +\infty.$$

Without loss of generality, we may assume that $\frac{|x_i - P_1^\epsilon|}{\epsilon} \leq C$. As before, let

$$v_i(y) = \frac{1}{N_i} w_i^0(P_1^\epsilon + \epsilon y).$$

Then, along a subsequence, $\{v_i\}$ converges to v in C_{loc}^2, and similar as before

$$v(y) = c \frac{\partial w}{\partial y_N}(y)$$

for some constant $c \neq 0$. Since $u_\epsilon(P_1^\epsilon) = 0$, we have $\frac{\partial v_i}{\partial y_N}(0) = 0$, so that $\frac{\partial v}{\partial y_N}(0) = 0$ which implies that $c = 0$ since $\frac{\partial^2 w}{\partial y_N^2}(0) \neq 0$. A contradiction. Consequently, $w_\epsilon^0(x) \equiv 0$ on Σ_0, i.e., $u_\epsilon(x^{'}, x_N) = u_\epsilon(x^{'}, x_N)$. Similarly, we have u_ϵ is symmetric in $x_j, j = 3, ..., N$.

□

4. Preliminaries I: Reduction to Finite-Dimensional Problem

From this section until Section 6, we shall prove the uniqueness of type 3.IV solutions. (Existence is given in [6], [13] and [22].) Our main idea is to show that type 3.IV solutions are nondegenerate (in some symmetry class) and to compute the Morse index of such solutions. We remark that the uniqueness and Morse index of boundary spikes have been studied in [6] and [49]. The analysis here is more complicated due to the fact that we are dealing with exponentially small orders.

We first introduce a general framework. This framework is a combination of the Liapunov-Schmidt reduction method and the variational principle. The Liapunov-Schmidt reduction method has been introduced and used in a lot of papers. See [1], [4], [5], [6], [7], [17], [22], [23], [25], [40], [41], [52], [53] and the references therein. A combination of the Liapunov-Schmidt reduction method and the variational principle was used in [5], [11], [13], [14], [22] and [23]. We shall follow the procedure in [22] which consists of the following steps:

Step 1. Choose good approximate functions.

Recall that $\Omega = B$. Let w be the unique solution of (2.1). We fix a point $P \in \bar{\Omega}$ and introduce the following functions as good approximate functions – the *"projection"* of w in $H^1(\Omega)$. This projection was first introduced in [48] and later studied in [47]. The idea of projecting a function has been used in other problems. See [5], [8], [38], [42], [52], [53] and the references therein.

We define $w_{\epsilon,P}$ to be the unique solution of

$$\begin{cases} \epsilon^2\Delta w_{\epsilon,P} - w_{\epsilon,P} + f(w(\frac{x-P}{\epsilon})) = 0 \text{ in } \Omega, \\ w_{\epsilon,P} > 0 \text{ in } \Omega, \ \frac{\partial w_{\epsilon,P}}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Set

$$\bar{w}_{\epsilon,P} = w(\frac{x-P}{\epsilon}), w_{\epsilon,P} = \bar{w}_{\epsilon,P}(x) + \varphi_{\epsilon,P}(x). \tag{4.2}$$

Then $\varphi_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon^2\Delta \varphi_{\epsilon,P} - \varphi_{\epsilon,P} = 0 \text{ in } \Omega, \\ \frac{\partial \varphi_{\epsilon,P}}{\partial \nu} = -\frac{\partial}{\partial \nu}\bar{w}_{\epsilon,P} \text{ on } \partial\Omega. \end{cases} \tag{4.3}$$

To study the properties of $\varphi_{\epsilon,P}$, we need to introduce the so-called distance function: let $P \in \Omega$, we define

$$d_P := d(P, \partial\Omega) = 1 - |P|. \tag{4.4}$$

For $P \neq 0$, it is easy to compute that

$$\nabla_P d_P = -\frac{P}{|P|}, \tag{4.5}$$

$$\frac{\partial^2}{\partial P_i \partial P_j} d_P = -\frac{1}{|P|}(\delta_{ij} - \frac{P_i P_j}{|P|^2}) \tag{4.6}$$

where $P = (P_1, ..., P_N)$. (Note that d_P is not differentiable at $P = 0$.)

We state the following useful lemma about the properties of $\varphi_{\epsilon,P}$ and the computations of some integrals. The proof of it is technical and thus delayed to Appendix A.

Lemma 4.1. *Let $\Omega = B$ and $P \in \Omega$.*

(1) For ϵ sufficiently small, we have

$$\varphi_{\epsilon,P}(x) = (1 + o(1))w(\frac{x-P}{\epsilon}), \ \textit{for} \ \ x \in \partial\Omega, \tag{4.7}$$

and

$$-\epsilon \log \varphi_{\epsilon,P}(P) \to 2d_P, \quad \textit{as } \epsilon \to 0. \tag{4.8}$$

(2) If we further assume that $|P| \geq d_0$ for some $d_0 > 0$, then we have

$$\varphi_{\epsilon,P}(P + \epsilon y) = \varphi_{\epsilon,P}(P)(1 + o(1))e^{-<\nabla d_P, y>}, \ \textit{for } P + \epsilon y \in \bar{\Omega}, \tag{4.9}$$

(4.10) $$\varphi_{\epsilon,P}(P)=(c_N+o(1))(d_P(1-d_P))^{-\frac{N-1}{2}}\epsilon^{N-1}e^{-2d_P/\epsilon},$$

where $c_N>0$ is a generic constant (depending on N only), and

$$\int_\Omega f^{'}(\bar{w}_{\epsilon,P})\frac{\partial \bar{w}_{\epsilon,P}}{\partial P_i}\varphi_{\epsilon,P}(x)dx$$

(4.11) $$=(-\gamma_1+o(1))\epsilon^{N-1}\varphi_{\epsilon,P}(P)(\nabla d_P)_i+O(e^{-(2+\sigma)d_P/\epsilon})$$

where $(\nabla d_P)_i$ denotes the $i-th$ component of ∇d_P (which is $-P_i/|P|$ in our case) and

(4.12) $$\gamma_1=\int_{R^N}f(w)e^{-y_1}dy>0,\sigma=\min(p-1,1).$$

(3) For ϵ sufficiently small and $P_1,P_2\in\Omega$, $\frac{|P_1-P_2|}{\epsilon}\to+\infty$, we have

$$\int_\Omega f^{'}(\bar{w}_{\epsilon,P_1})\bar{w}_{\epsilon,P_2}\frac{\partial \bar{w}_{\epsilon,P_1}}{\partial P_{1,i}}$$

(4.13) $$=\epsilon^{N-1}(-\gamma_1+o(1))w(\frac{|P_1-P_2|}{\epsilon})(\nabla_{P_1}(|P_1-P_2|))_i+O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})$$

where γ_1 is given by (4.12).

Step 2. Finite-dimensional reduction.

We now describe the so-called Liapunov-Schmidt finite dimension reduction procedure. Most of the material is from Sections 3,4 and 5 in [22]. See also Sections 4, 5 and 6 in [23].

We first introduce some notations.

We observe that solving (1.1) is equivalent to finding a zero of the following nonlinear equation:

(4.14) $$S_\epsilon[u]:=\Delta u-u+f(u_+)=0,u\in H^2_\nu(\Omega_\epsilon),$$

where

$$\Omega_\epsilon=\{y|\epsilon y\in\Omega\},$$

(4.15) $$H^2_\nu(\Omega_\epsilon):=\{u\in H^2(\Omega_\epsilon)|\frac{\partial u}{\partial\nu}=0 \text{ on } \partial\Omega_\epsilon\}.$$

For any $u,v\in H^1(\Omega)$, we define the inner product and the norm as follows:

$$<u,v>_\epsilon=\epsilon^{-N}\int_\Omega(\epsilon^2\nabla u\cdot\nabla v+u\cdot v),\quad \|u\|_\epsilon=<u,v>_\epsilon^{\frac{1}{2}}.$$

We consider $K-$interior spikes. The case of boundary spikes or mixed boundary-interior spikes will be discussed in Section 7. Fix $\mathbf{P}=(P_1,...,P_K)\in\mathbf{\Omega}^K$. Let $\varphi_K(\mathbf{P})=\varphi_K(P_1,...,P_K)$ be defined at (1.7). We assume that

$$\mathbf{P}\in\Lambda_\delta=\{\mathbf{P}\in\Omega^K|\varphi_K(\mathbf{P})\geq 2\delta\} \tag{4.16}$$

where δ is a small but fixed positive constant.

To simplify notations, we use the following simplified symbols:

$$\partial_{j,i}:=\frac{\partial}{\partial P_{j,i}}, w_{\epsilon,\mathbf{P}}=\sum_{j=1}^K w_{\epsilon,P_j}.$$

We remark that the variable of $w_{\epsilon,\mathbf{P}}$ is in Ω. Sometimes, we also consider $w_{\epsilon,\mathbf{P}}(\epsilon y)$ for $y\in\Omega_\epsilon$. We denote $w_{\epsilon,\mathbf{P}}(\epsilon y)$ as $w_{\epsilon,\mathbf{P}}$.

Now we define the approximate kernel and cokernel respectively as follows:

$$\mathcal{K}_{\epsilon,\mathbf{P}}:=\text{span }\{\partial_{j,i}w_{\epsilon,\mathbf{P}}|j=1,...,K,i=1,...,N\}\subset H^2_\nu(\Omega_\epsilon), \tag{4.17}$$

$$\mathcal{C}_{\epsilon,\mathbf{P}}:=\text{span }\{\partial_{j,i}w_{\epsilon,\mathbf{P}}|j=1,...,K,i=1,...,N\}\subset L^2(\Omega_\epsilon). \tag{4.18}$$

(Note that $\partial_{j,i}w_{\epsilon,\mathbf{P}}\in H^2_\nu(\Omega_\epsilon)$ as one can differentiate equation (4.1).)

We also need the following spaces

$$\mathcal{K}^\perp_{\epsilon,\mathbf{P}}=\ \{u\in H^2_\nu(\Omega_\epsilon)|\int_{\Omega_\epsilon}u\partial_{j,i}w_{\epsilon,\mathbf{P}}=0,j=1,...,K,i=1,...,N\}, \tag{4.19}$$

$$\mathcal{C}^\perp_{\epsilon,\mathbf{P}}=\ \{u\in L^2(\Omega_\epsilon)|\int_{\Omega_\epsilon}u\partial_{j,i}w_{\epsilon,\mathbf{P}}=0,j=1,...,K,i=1,...,N\}. \tag{4.20}$$

Set

$$\tilde{L}_{\epsilon,\mathbf{P}}(\phi)=\Delta\phi-\phi+f'(w_{\epsilon,\mathbf{P}})\phi,\quad \mathcal{L}_{\epsilon,\mathbf{P}}=\pi^\perp_{\epsilon,\mathbf{P}}\circ\tilde{L}_{\epsilon,\mathbf{P}}, \tag{4.21}$$

for $\phi\in H^2_\nu(\Omega_\epsilon)$, where $\pi^\perp_{\epsilon,\mathbf{P}}$ is the projection from $L^2(\Omega_\epsilon)$ into $\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$.

We recall the following result in [47] (see Propositions 3.1 and 3.2 in [47]).

Lemma 4.2. *For $\epsilon<<1$, $\mathcal{L}_{\epsilon,\mathbf{P}}:\mathcal{K}^\perp_{\epsilon,\mathbf{P}}\to\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$ is one-to-one and onto. Moreover, the inverse of $\mathcal{L}_{\epsilon,\mathbf{P}}$ exists and bounded (independent of $\epsilon>0$).*

Next, we have

Lemma 4.3. *For ϵ sufficiently small, $\mathbf{P} \in \Lambda_\delta$, there exists a unique $v_{\epsilon,\mathbf{P}} \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$ such that*

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) \in \mathcal{C}_{\epsilon,\mathbf{P}}. \tag{4.22}$$

Moreover, $v_{\epsilon,\mathbf{P}}$ is C^2 in $\mathbf{P}$ and

$$\|v_{\epsilon,\mathbf{P}}\|_\epsilon \le Ce^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon} \tag{4.23}$$

$$\|\partial_{j,i} v_{\epsilon,\mathbf{P}}\|_\epsilon \le C\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon} \tag{4.24}$$

where $\sigma = \min(1, p-1)$.

Proof: The proof of this Lemma is similar to that of Lemma 2.4 of [51]. For the sake of completeness, we include it in Appendix B. □

Step 3. Solve the finite dimensional problem.

Fix any $\mathbf{P} \in \Lambda_{2\delta}$. Let $v_{\epsilon,\mathbf{P}}$ be the unique solution of (4.22) given by Lemma 4.3. Now we define

$$M_\epsilon(\mathbf{P}) = M_\epsilon(P_1, ..., P_K) := \epsilon^{-N} J_\epsilon[w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}] \tag{4.25}$$

$$M_\epsilon(\mathbf{P}) : \Lambda_{2\delta} \to R,$$

where J_ϵ is the energy functional introduced in (1.4) of Section 1.

By Lemma 4.3, $M_\epsilon(\mathbf{P}) \in C^2(\Lambda_{2\delta})$. Then we have the following reduction theorem.

Lemma 4.4. *(Proposition 4.1 of [22]) $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}, \mathbf{P}^\epsilon \in \Lambda_{2\delta}$ is a critical point of J_ϵ if and only if P^ϵ is a critical point of $M_\epsilon(\mathbf{P})$.*

Therefore, to prove the existence and uniqueness of solutions of (1.1), we just need to concentrate on the study of critical points of $M_\epsilon(\mathbf{P})$, which is a finite-dimensional problem. We shall compute $\nabla M_\epsilon(\mathbf{P})$ and $\nabla^2 M_\epsilon(\mathbf{P})$ in the next two sections.

5. Preliminaries II: computations of $\nabla M_\epsilon(\mathbf{P})$ and $\nabla^2 M_\epsilon(\mathbf{P})$

In this section, we first obtain a general formula for the locations of K interior spikes $(P_1^\epsilon, ..., P_K^\epsilon)$ and classify all types of limiting locations. Then we compute the (first and second order) derivatives of $M_\epsilon(\mathbf{P})$.

The following theorem shows that there will be no spike collapsing to the boundary or with each other.

Lemma 5.1. *Let $(P_1^\epsilon, ..., P_K^\epsilon)$ be the K local maximum points of a $K-$peaked solution u_ϵ of (1.1). Suppose that $K \leq 3$. Let $P_j^\epsilon \to P_j^0$ for $j = 1, ..., K$. Then we have*

$$P_i^0 \neq P_j^0, \text{ for } i \neq j, \tag{5.1}$$

and if $P_i^0 \in \partial\Omega$, then $P_i^\epsilon \in \partial\Omega$ for ϵ sufficiently small.

Lemma 5.1 eliminates the collision of spikes or collision of spikes with the boundary. Note that if the mean curvature of the domain is not constant, one can construct multiple spikes concentrating at one local minimum point of the mean curvature (see [24]). The proof of Lemma 5.1 is technical and thus we delay the proof of it to the appendix C.

By Lemma 5.1, if $P_j^\epsilon, j = 1, 2, 3$ are three interior spikes, then $\varphi_K(P_1^\epsilon, ..., P_K^\epsilon) \geq \delta_0$ for some $\delta_0 > 0$. Now we choose $\delta = \frac{\delta_0}{4}$. By Lemma 4.4, $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$ is a solution with three interior spikes if and only if $\mathbf{P}^\epsilon$ is a critical point of M_ϵ, since $\mathbf{P}^\epsilon \in \Lambda_{2\delta}$.

The asymptotic expansion of $M_\epsilon(\mathbf{P})$ in Λ_δ is given in Proposition 4.1 of [22].

Lemma 5.2. *(Proposition 4.1 of* [22]*.) For ϵ sufficiently small and $\mathbf{P} \in \Lambda_\delta$, we have*

$$M_\epsilon(\mathbf{P}) = KI(w) - \frac{1}{2}(\gamma_1 + o(1))(\sum_{i=1}^K \varphi_{\epsilon,P_i}(P_i)) - (\gamma_1 + o(1))\sum_{k\neq l} w(|P_k - P_l|/\epsilon) \tag{5.2}$$

where

$$I(w) = \frac{1}{2}\int_{R^N} |\nabla w|^2 + \frac{1}{2}\int_{R^N} w^2 - \int_{R^N} F(w) \tag{5.3}$$

and γ_1 is given by (4.12).

We now show that the asymptotic expansion in (5.2) holds true in C^2 sense. Set

$$\tilde{M}_\epsilon(\mathbf{P}) := -\frac{\gamma_1}{2}\sum_{j=1}^{K}\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon) - \gamma_1\sum_{k\neq l} w(|P_k - P_l|/\epsilon). \tag{5.4}$$

By (4.10) of Lemma 4.1 and (2.3), we see that if $|P_j^\epsilon| \geq \frac{1}{10}, j = 1, ..., K$, then we have

$$\tilde{M}_\epsilon(\mathbf{P}) := -\frac{c_N(\gamma_1 + o(1))}{2}\epsilon^{\frac{N-1}{2}}\sum_{j=1}^{N} c(P_j^\epsilon)e^{-2d_{P_j^\epsilon}/\epsilon}$$
$$-A_N(\gamma_1 + o(1))\epsilon^{\frac{N-1}{2}}\sum_{k\neq l}(|P_k - P_l|)^{-\frac{N-1}{2}}e^{-|P_k-P_l|/\epsilon}, \tag{5.5}$$

where c_N is given in (4.10) of Lemma 4.1, $A_N > 0$ is given by (2.3), and

$$c(P) = (d_P(1 - d_P))^{-\frac{N-1}{2}}. \tag{5.6}$$

The following lemma is our key estimate.

Lemma 5.3. *Suppose that $\mathbf{P}^\epsilon \in \Lambda_\delta$ and ϵ is sufficiently small.*

(1) If $|P_j^\epsilon| \geq d_0$ for some j and $d_0 > 0$, then we have

$$\partial_{j,i}M_\epsilon(\mathbf{P}) = \partial_{j,i}\tilde{M}_\epsilon(\mathbf{P}) + O(\tilde{M}_\epsilon(\mathbf{P})), i = 1, ..., N. \tag{5.7}$$

(2). If $|P_j^\epsilon| \leq \frac{1}{10}$ for some j, then we have

$$\partial_{j,i}M_\epsilon(\mathbf{P}) = -(\gamma_1 + o(1))\sum_{k\neq j}\partial_{j,i}(w(|P_k - P_j|/\epsilon)) + O(\tilde{M}_\epsilon(\mathbf{P})), i = 1, ..., N. \tag{5.8}$$

(3) Suppose that $\mathbf{P}^\epsilon$ is a critical point of $M_\epsilon(\mathbf{P})$ such that $|P_j^\epsilon| \geq d_0, j = 1, ..., N$ for some $d_0 > 0$. Then we have

(5.9)

$$\partial_{l,m}\partial_{j,i}M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} = \partial_{l,m}\partial_{j,i}\tilde{M}_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} + O(\epsilon^{-1}\tilde{M}_\epsilon(\mathbf{P}^\epsilon)), l = 1, ..., K, m = 1, ..., N.$$

More precisely, we have

$$\partial_{l,m}\partial_{j,i}M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$= \epsilon^{N-2}(\gamma_1 + o(1))w(|P_j^\epsilon - P_l^\epsilon|/\epsilon)e_{jl,m}^\epsilon e_{jl,i}^\epsilon(1 - \delta_{jl})$$
$$-\epsilon^{N-2}(\gamma_1 + o(1))\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)e_{j,i}^\epsilon e_{l,m}^\epsilon\delta_{jl}$$
$$-\epsilon^{N-2}(\gamma_1 + o(1))\sum_{k\neq j} w(|P_j^\epsilon - P_k^\epsilon|/\epsilon)e_{jk,i}^\epsilon e_{jk,m}^\epsilon\delta_{jl}, \tag{5.10}$$

where

$$e_j^\epsilon = \frac{P_j^\epsilon}{|P_j^\epsilon|}, e_{jk}^\epsilon = \frac{P_j^\epsilon - P_k^\epsilon}{|P_j^\epsilon - P_k^\epsilon|}, j \neq k, \tag{5.11}$$

and $e_{j,i}^\epsilon$ *and* $e_{jk,i}^\epsilon$ *denote the* $i-th$ *component of the vectors* e_j^ϵ *and* e_{jk}^ϵ*, respectively.*

Remark: The reason that we have to consider cases when P_j^ϵ is close to the origin is that the function d_P is not differentiable at the origin. However if we know the rate of P_j^ϵ approaching the origin, we may still be able to compute the derivatives (see [51]). This is a delicate issue that needs further investigation. For the purpose of this paper, Lemma 5.3 is good enough.

The proof of Lemma 5.3 is very technical and we will present it in Appendix D.

Let $\mathbf{P}^\epsilon = (P_1^\epsilon, ..., P_K^\epsilon)$ be a critical point of $M_\epsilon(\mathbf{P})$. Namely, we have

$$\partial_{j,i} M_\epsilon(\mathbf{P})|_{\mathbf{P}=\mathbf{P}^\epsilon} = 0, j = 1, ..., K, i = 1, ..., N. \tag{5.12}$$

(5.10) shows that if $\mathbf{P}^\epsilon$ is a critical point of M_ϵ and $|P_j^\epsilon| \geq \frac{1}{10}, j = 1, ..., K$, then we have

$$\begin{aligned} &(\gamma_1 + o(1)) c(P_j^\epsilon) e^{-2d_{P_j^\epsilon}/\epsilon} (\nabla_{P_j^\epsilon} d_{P_j^\epsilon})_i \\ &+(\gamma_1 + o(1)) \sum_{l \neq j} (|P_l^\epsilon - P_j^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_l^\epsilon - P_j^\epsilon|/\epsilon} (\nabla_{P_j^\epsilon} |P_j^\epsilon - P_l^\epsilon|)_i \\ &+O(\epsilon(\sum_{j=1}^K e^{-2d_{P_j^\epsilon}/\epsilon} + \sum_{l \neq k} e^{-|P_l^\epsilon - P_k^\epsilon|/\epsilon})) = 0, j = 1, ..., K, i = 1, ..., N. \end{aligned} \tag{5.13}$$

(5.8) and (5.13) enable us to classify all types of locations for the limiting positions of interior two or interior three peaks. Moreover, it also gives us the estimate on the speed of $\mathbf{P}^\epsilon$ approaching these limiting positions (Lemma 6.1 in the next section).

Lemma 5.4. *Let* $K = 2$ *or* 3 *and* $\tilde{P}_j^\epsilon \in \Omega, j = 1, ..., K$ *be the* K *local maximum points of* u_ϵ*. Suppose that* $\tilde{P}_j^\epsilon \to P_j^0, j = 1, ..., K$*, then up to a rotation,* $\mathbf{P}^0 = (P_1^0, ..., P_K^0)$ *must belong to type 2.III (for* $K = 2$*, Figure 2) and either type 3.IV or type 3.VII (for* $K = 3$*, Figure 3).*

Proof: Let u_ϵ be a solution of (1.1) with two interior spikes or three interior spikes. By Lemma 4.4, we have $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$, where $\mathbf{P}^\epsilon \in \Lambda_\delta$. Let $\tilde{P}_j^\epsilon, j = 1, 2, ..., K$ be the K local maximum points of u_ϵ. Then, up to a rotation, we have $\tilde{P}_j^\epsilon - P_j^\epsilon = o(1), j = 1, ..., K$. By Lemma 5.1, $P_j^0 \neq P_i^0$ for $i \neq j$.

Let us consider $K = 2$ first. From (5.8), it is easy to see that both $|P_1^\epsilon|$ and $|P_2^\epsilon|$ must be larger than $\frac{1}{10}$ (as otherwise, (5.8) is not balanced). Thus we may assume that $|P_1^\epsilon| \geq \frac{1}{10}, |P_2^\epsilon| \geq \frac{1}{10}$.

Since both $\tilde{P}_1^\epsilon$ and $\tilde{P}_2^\epsilon$ are in the interior, (5.13) implies that we must have

$$|P_1^\epsilon - P_2^\epsilon| = 2d_{P_1^\epsilon} + o(1) = 2d_{P_2^\epsilon} + o(1)$$

and that P_1^0 and P_2^0 must be anti-pole, i.e. $P_1^0 = -P_2^0$. This shows that the limiting positions must be type 2.III.

For $K = 3$, we proceed similarly. By Lemma 4.1, $P_j^\epsilon \to P_j^0, j = 1, 2, 3$, where $P_j^0 \in \Omega, P_k^0 \neq P_l^0$ if $k \neq l$. We consider two cases.

Case 1. Suppose one of the points $P_j^0, j = 1, 2, 3$ is the origin, say $P_2^0 = 0$. Then we consider (5.13) at $j = 1$ and conclude that $2d_{P_1^0} = |P_1^0 - P_2^0|$. Similarly we have $2d_{P_2^0} = |P_1^0 - P_2^0|$. From equation (5.8) at $j = 2$, we see that

$$P_1^0 + P_2^0 = 0$$

which shows that (P_1^0, P_2^0, P_3^0) is type 3.VII.

Case 2. Suppose $P_j^0 \neq 0, j = 1, 2, 3$. Then $|P_j^\epsilon| \geq d_0$ for $j = 1, 2, 3$ and some $d_0 > 0$. We use (5.13) at $j = 1, 2, 3$ to conclude that (P_1^0, P_2^0, P_3^0) must be type 3.IV (see the proof of Lemma 6.1 in the next section).

□

6. UNIQUENESS OF THE TYPE 3.IV SOLUTIONS

In this section, we prove the uniqueness of the type 3.IV solution for ϵ sufficiently small. Let P_1^0, P_2^0, P_3^0 be the limiting positions as shown in Figure 3 and let u_ϵ be a three-peaked solution whose local maximum points are $\tilde{P}_j^\epsilon, j = 1, 2, 3$. By MMP

(Section 3), the solution u_ϵ is symmetric in $x_j, j = 3, ..., N$. Let

$$H^2_{\nu,s}(\Omega_\epsilon) = \{u \in H^2_\nu(\Omega_\epsilon) | u \text{ is symmetric with respect to } x_j, j = 3, ..., N\}. \tag{6.1}$$

Consider the following minimization problem

$$\min_{P_j \in \hat{\Lambda}_{2\delta}} \left\| u_\epsilon - \sum_{j=1}^3 w_{\epsilon,P_j} \right\|_{L^2(\Omega_\epsilon)} \tag{6.2}$$

where

$$\hat{\Lambda}_{2\delta} = \{(P_1, P_2, P_3) \mid |P_j - P_j^0| \le 2\delta, P_{j,i} = 0, j = 1, 2, 3, i \ge 3\}. \tag{6.3}$$

It is easy to see that (6.2) can be attained and thus we have

$$u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + \phi_\epsilon \tag{6.4}$$

where $\mathbf{P}^\epsilon \in \hat{\Lambda}_\delta, \phi_\epsilon \in H^2_{\nu,s}(\Omega)$. Moreover, $\phi_\epsilon \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}^\epsilon}$. Since

$$S[w_{\epsilon,\mathbf{P}^\epsilon} + \phi_\epsilon] = 0 \in \mathcal{C}_{\epsilon,\mathbf{P}^\epsilon}, \phi_\epsilon \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}^\epsilon},$$

by Lemma 4.3, we see that

$$\phi_\epsilon = v_{\epsilon,\mathbf{P}^\epsilon} \tag{6.5}$$

where $v_{\epsilon,\mathbf{P}^\epsilon}$ is defined by Lemma 4.3. (Note that P_j^ϵ may not be a local maximum point of u_ϵ. But it is easy to show that up to a permutation, $P_j^\epsilon = \tilde{P}_j^\epsilon + o(1), j = 1, 2, 3.$)

For $\mathbf{P} \in \hat{\Lambda}_\delta$, we may define $\hat{P}_j = (P_{j,1}, P_{j,2})$, $\hat{\mathbf{P}} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ and

$$\hat{M}_\epsilon(\hat{\mathbf{P}}) = M_\epsilon(\mathbf{P}). \tag{6.6}$$

Similar to Lemma 4.4, we have that $\hat{\mathbf{P}}^\epsilon$ is a critical point of $\hat{M}_\epsilon(\hat{\mathbf{P}})$ if and only if $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$ is a critical point of J_ϵ.

To avoid clumsy notation, we drop the hat now. Thus our problem is reduced to a six-dimensional problem. Moreover, by rotation, we may fix $P_{11} = 0$. Let $\mathbf{p} = (P_{12}, P_{21}, P_{22}, P_{31}, P_{32})$. Then if $\mathbf{P}^\epsilon$ is a critical point, the corresponding $\mathbf{p}^\epsilon$ is also a critical point of $M_\epsilon(\mathbf{P})$. Thus all we need to prove is the uniqueness of the critical point of $M_\epsilon(\mathbf{P})$ for $\mathbf{P}$ in the set

$$\omega = \{(P_{12}, P_{21}, P_{22}, P_{31}, P_{32}) | \mathbf{P} \in \hat{\Lambda}_\delta, P_{11} = 0\},$$

which is a five-dimensional problem.

We begin with the following lemma which computes the speed of $\mathbf{P}^\epsilon$ approaching $\mathbf{P}^0$. This kind of estimate is needed for the proof of uniqueness of spikes. See [7] and [49].

Lemma 6.1. *Let $P_j^\epsilon, j = 1, 2, 3$ be as above. Then there exists a unique vector $\vec{a} \in R^2$ such that*

$$P_j^\epsilon = P_1^0 + \epsilon e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} \vec{a} + o(\epsilon), j = 1, 2, 3 \tag{6.7}$$

where $\vec{a} = (0, a_2)$ for some fixed number a_2.

Proof: Our main tool is equation (5.13).

Adding all the three equations in (5.13), we obtain that

$$\sum_{j=1}^{3} e^{-|P_j^\epsilon|/\epsilon} \frac{P_j^\epsilon}{|P_j^\epsilon|} + o(\sum_{j=1}^{3} e^{2|P_j^\epsilon|/\epsilon}) = 0. \tag{6.8}$$

Since

$$\frac{P_1^\epsilon}{|P_1^\epsilon|} = \vec{e}_2 = (0, 1)$$

and

$$\frac{P_j^\epsilon}{|P_j^\epsilon|} = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} \vec{e}_2 + o(1), \quad j = 2, 3,$$

we deduce from (5.13) that

$$|P_j^\epsilon| = |P_1^\epsilon| + o(\epsilon), d_{P_j^\epsilon} = d_{P_1^\epsilon} + o(\epsilon), \quad j = 2, 3. \tag{6.9}$$

Next we examine equation (5.13) at $j = 1$. We have

$$\frac{P_1^0}{|P_1^0|} + a_1^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon} \frac{P_2^0 - P_1^0}{|P_2^0 - P_1^0|} + a_1^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon} \frac{P_3^0 - P_1^0}{|P_3^0 - P_1^0|} = o(1)$$

where a_1^0 is a generic constant. Since the following decomposition is unique,

$$\frac{P_1^0}{|P_1^0|} = -\frac{P_2^0 - P_1^0}{|P_2^0 - P_1^0|} - \frac{P_3^0 - P_1^0}{|P_3^0 - P_1^0|}$$

we see that

$$a_1^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon} = 1 + o(1), a_1^0 e^{(2d_{P_1^\epsilon} - |P_3^\epsilon - P_1^\epsilon|)/\epsilon} = 1 + o(1),$$

which implies that

$$|P_2^\epsilon - P_1^\epsilon| = 2d_{P_1^\epsilon} + \epsilon a_0 + o(\epsilon), |P_3^\epsilon - P_1^\epsilon| = 2d_{P_1^\epsilon} + \epsilon a_0 + o(\epsilon) \tag{6.10}$$

where a_0 is a generic constant. Similarly, we have

$$|P_i^\epsilon - P_j^\epsilon| = 2d_{P_i^\epsilon} + \epsilon a_0 + o(\epsilon), \quad i \neq j, \quad i,j = 1,2,3. \tag{6.11}$$

From (6.9) and (6.11), we see that

$$P_j^\epsilon = e^{\sqrt{-1}\frac{2(j-1)\pi}{3}} P_1^\epsilon + o(\epsilon), \quad j = 2,3. \tag{6.12}$$

Substituting (6.12) into (6.10), we see that

$$P_1^\epsilon = P_1^0 + \epsilon \vec{a} + o(\epsilon) \tag{6.13}$$

for some unique $\vec{a}$. (Note that $\vec{a} = a_2 \vec{e}_2$ for some a_2.) By (6.12), (6.7) holds.

□

By Lemma 6.1, any critical point $\mathbf{P}^\epsilon$ of $M_\epsilon(\mathbf{P})$ in $B_\delta(\mathbf{P}^0)$ must satisfy $\mathbf{P}^\epsilon = \mathbf{P}^0 + \epsilon \mathbf{a} + o(\epsilon)$ for some fixed $\mathbf{a}$. Let $\mathbf{Q}^\epsilon = \mathbf{P}^0 + \epsilon \mathbf{a}$.

Our next lemma shows that every critical point $\mathbf{P}^\epsilon$ must be nondegenerate.

Lemma 6.2. *Let $\mathbf{P}^\epsilon \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ be a critical point of $M_\epsilon(\mathbf{P})$. Then for ϵ sufficiently small, we have*

$$\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i} \geq C\epsilon^{\frac{3N-5}{2}} e^{-2\varphi_K(\mathbf{P}^\epsilon)/\epsilon} |\eta|^2 \tag{6.14}$$

where C is independent of $\epsilon, \eta = (\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22}, \eta_{31}, \eta_{32}) \in R^6, \eta_{11} = 0$ and $|\eta|^2 = \sum_{i,j} \eta_{ij}^2$.

Proof: We have by Lemma 5.3 (2),

$$\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i}$$

$$= (\gamma_1 + o(1))\epsilon^{N-2} \sum_j \varphi_{\epsilon, P_j^\epsilon}(P_j^\epsilon) \sum_l < e_l^\epsilon, \eta_l >^2 \tag{6.15}$$

$$+(\gamma_1 + o(1))\epsilon^{N-2} w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)(1 + o(1)) \sum_{j \neq l} < e_{jl}^\epsilon, \eta_j - \eta_l >^2 .$$

Since $\mathbf{P}^\epsilon \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$, $|P_i^\epsilon - P_j^\epsilon| = |P_1^\epsilon - P_2^\epsilon| + o(\epsilon)$ for $i \neq j$ and $d_{P_j^\epsilon} = d_{P_1^\epsilon} + o(\epsilon)$. Hence

$$\varphi_{\epsilon, P_j^\epsilon}(P_j^\epsilon) \sim w(|P_1^\epsilon - P_2^\epsilon|), \quad j = 1,2,3.$$

(6.15) shows that

$$\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i}$$

$$\geq C\epsilon^{N-2} e^{-2\varphi_K(\mathbf{P}^\epsilon)/\epsilon} \Big(\sum_l < e_l^\epsilon, \eta_l >^2 + \sum_{j\neq l} < e_{jl}^\epsilon, \eta_j - \eta_l >^2\Big) \tag{6.16}$$

for some $C > 0$ independent of ϵ.

We now show that

$$\sum_{j,l,m,i} \partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \eta_{l,m}\eta_{j,i} \geq C\epsilon^{N-2} e^{-2\varphi_K(\mathbf{P}^\epsilon)/\epsilon} |\eta|^2$$

where C is independent of ϵ and $|\eta|^2 = \sum_{i,j} \eta_{ij}^2$.

To this end, it is enough to show that

$$\sum_l < e_l^\epsilon, \eta_l >^2 + \sum_{j\neq l} < e_{jl}^\epsilon, \eta_j - \eta_l >^2 \geq C|\eta|^2 \tag{6.17}$$

if $\eta_{11} = 0$. In fact, the left hand side of (6.17) is equal to 0 if and only if

$$< e_l^\epsilon, \eta_l >= 0, < e_{jl}^\epsilon, \eta_l - \eta_j >= 0.$$

For $j = 1, < e_1^\epsilon, \eta_1 >= 0$, then $\eta_{12} = 0$. Hence $\eta_1 = 0$, which implies that $< e_{12}^\epsilon, \eta_2 >=< e_2^\epsilon, \eta_2 >= 0$. Hence $\eta_2 = 0$. Similarly, we have $\eta_3 = 0$. Thus (6.17) holds true.

□

(6.14) shows that the matrix $(\partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon})$ is negatively definite if we restrict to the space $\{\eta_{11} = 0\}$. Thus the Morse index is 5.

Finally we have

Lemma 6.3. *For $\delta > 0$ small, there exists a unique critical point of $M_\epsilon(\mathbf{P})$ over $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$.*

Proof:

First, by restricting to the symmetric class of functions, we can adopt the arguments of [22] to show there exists a critical point $\mathbf{P}^\epsilon$ of $M_\epsilon(\mathbf{P})$. By Lemma 6.1, $\mathbf{P}^\epsilon = P_0 + \epsilon\mathbf{a} + o(\epsilon)$ and any other critical point of $M_\epsilon(\mathbf{P})$ is in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$.

We now show that $\mathbf{P}^\epsilon$ is unique.

By Lemma 6.2, there are only finite number of critical points of $M_\epsilon(\mathbf{P})$ in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ (since each critical point is nondegenerate). Let k_ϵ be the number of critical points. At each critical point, we have by Lemma 6.2,

$$deg(\nabla M_\epsilon, B_{\delta_i\epsilon}(\mathbf{Q}_i^\epsilon), 0) = (-1)^5 = -1$$

where $\delta_i > 0$ are small constants so that $B_{\delta_i\epsilon}(\mathbf{Q}_i^\epsilon)$ contains only one critical point (i.e. $\mathbf{Q}_i^\epsilon$) of $M_\epsilon(\mathbf{P})$.

Hence by the additivity of the degree we have

$$deg(\nabla M_\epsilon, B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = k_\epsilon(-1)^5. \tag{6.18}$$

On the other hand, it is easy to see that $\tilde{M}_\epsilon(\mathbf{P})$ has only one critical point in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ (because of the nondegeneracy of $(\nabla^2\tilde{M}_\epsilon(\mathbf{P}))$). For $\mathbf{P} \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$, we have

$$e^{-2d_{P_i}/\epsilon} = (1+O(\delta))e^{-2d_{Q_i^\epsilon}}, w(|P_i - P_j|/\epsilon) = (1+O(\delta))w(|Q_i^\epsilon - Q_j^\epsilon|/\epsilon),$$

$$M_\epsilon(\mathbf{P}) = (1+O(\delta))M_\epsilon(\mathbf{Q}^\epsilon).$$

By (1) of Lemma 5.3, we have $\nabla M_\epsilon(\mathbf{P}) = \nabla\tilde{M}_\epsilon(\mathbf{P}) + O(\tilde{M}_\epsilon(\mathbf{P}))$. Note that $\nabla M_\epsilon(\mathbf{P}) \neq 0$ and $\nabla\tilde{M}_\epsilon(\mathbf{P}) \neq 0$ on $\partial B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$. By a continuity argument, we obtain that

$$deg(\nabla M_\epsilon, B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = deg(\nabla\tilde{M}_\epsilon(\mathbf{P}), B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = -1. \tag{6.19}$$

Comparing (6.18) and (6.19), we deduce that $k_\epsilon = 1$.

□

Lemma (6.3) shows that type 3.IV solution is unique, up to a rotation, provided that ϵ is sufficiently small.

7. Existence and uniqueness of other types of solutions

In the previous sections, we have proved the uniqueness of type 3.IV solutions. It is easy to see that same techniques (with much simpler computations) show that the type 2.III solutions are unique. It remains to deal with boundary spikes and mixed-interior-boundary spikes.

To be able to deal with the boundary peak case, we use different approximate functions. (Here the fact that $\Omega = B$ plays an important role.)

We begin with Theorem 1.1 of [30]: for every $P \in \partial\Omega$, there exists a unique boundary spike solution $u_{\epsilon,P}$ which concentrates at P. Moreover $u_{\epsilon,P}$ is axially symmetric with respect to the straight line joining 0 and P. So for $P \in \partial\Omega$, we may choose our approximate function as follows

$$w_{\epsilon,P} = u_{\epsilon,P}, \text{ for } P \in \partial\Omega. \tag{7.1}$$

For $P \in \partial\Omega$, we denote the $i-$th tangential derivative at P_j as $\partial_{j,i} = \frac{\partial}{\partial \tau_{P_j,i}}, i = 1, ..., N-1$.

In this case, we see that

$$S_\epsilon(w_{\epsilon,P}) = 0, \text{ if } P \in \partial\Omega. \tag{7.2}$$

Let $\mathbf{P} = (P_1, ..., P_K)$ be such that $P_j \in \partial\Omega, i = 1, ..., K_1, P_j \in \Omega, j = K_1 + 1, ..., K$. We define a new function (which was introduced first in [23])

$$\tilde{\varphi}_K(\mathbf{P}) = \min_{i=1,...,K_1, j=1,...,K, i\neq j} (\frac{1}{2}|P_i - P_j|, \varphi_{K-K_1}(P_{K_1+1}, ..., P_K)). \tag{7.3}$$

We can also define $w_{\epsilon,\mathbf{P}}, \mathcal{K}_{\epsilon,\mathbf{P}}, \mathcal{C}_{\epsilon,\mathbf{P}}, \mathcal{K}^\perp_{\epsilon,\mathbf{P}}, \mathcal{C}^\perp_{\epsilon,\mathbf{P}}, \mathcal{L}_{\epsilon,\mathbf{P}}, v_{\epsilon,\mathbf{P}}, M_\epsilon(\mathbf{P})$, etc.

Then similar to Section 4, we will obtain the following

$$S_\epsilon(w_{\epsilon,\mathbf{P}}) = O(e^{-2\tilde{\varphi}_K(\mathbf{P})/\epsilon}) \tag{7.4}$$

$$\|v_{\epsilon,\mathbf{P}}\|_\epsilon \leq Ce^{-(1+\sigma)\tilde{\varphi}_K(\mathbf{P})/\epsilon} \tag{7.5}$$

$$\|\partial_{j,i} v_{\epsilon,\mathbf{P}}\|_\epsilon \leq C\epsilon^{-2} e^{-(1+\sigma)\tilde{\varphi}_K(\mathbf{P})/\epsilon} \tag{7.6}$$

where $\sigma = \min(1, p-1)$.

Moreover, we have the following equations for the equilibrium positions: for $j = 1, ..., K_1$, we have

$$(\gamma_1 + o(1)) \sum_{l\neq j} (|P_l^\epsilon - P_j^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_l^\epsilon - P_j^\epsilon|/\epsilon} \partial_{j,i}(|P_j^\epsilon - P_l^\epsilon|) \tag{7.7}$$

$$+O(\epsilon(\sum_{j=1}^{K} e^{-2d_{P_j^\epsilon}/\epsilon}+\sum_{l\neq k} e^{-|P_l^\epsilon-P_k^\epsilon|/\epsilon}))=0, j=1,...,K_1, i=1,...,N-1,$$

(here $\partial_{j,i}$ means the $i-th$ tangential derivative at P_j^ϵ) and for $j=K_1,...,K$ (if $|P_j^\epsilon|\geq \delta_0$), we have

$$(\gamma_1+o(1))c_N c(P_j^\epsilon)e^{-2d_{P_j^\epsilon}/\epsilon}(\nabla_{P_j^\epsilon} d_{P_j^\epsilon})_i$$

$$+(\gamma_1+o(1))A_N\sum_{l\neq j}(|P_l^\epsilon-P_j^\epsilon|)^{-\frac{N-1}{2}}e^{-|P_l^\epsilon-P_j^\epsilon|/\epsilon}(\nabla_{P_j^\epsilon}|P_j^\epsilon-P_l^\epsilon|)_i \tag{7.8}$$

$$+O(\epsilon(\sum_{j=1}^{K} e^{-2d_{P_j^\epsilon}/\epsilon}+\sum_{l\neq k} e^{-|P_l^\epsilon-P_k^\epsilon|/\epsilon}))=0, j=K_1+1,...,K, i=1,...,N.$$

The next lemma is an extension of Lemma 5.4.

Lemma 7.1. *Let $K=2$ or 3 and $\tilde{P}_j^\epsilon\in\Omega, j=1,...,K$ be the K local maximum points of u_ϵ. Suppose that $\tilde{P}_j^\epsilon\to P_j^0, j=1,...,K$, then up to a rotation, $\mathbf{P}^0=(P_1^0,...,P_K^0)$ must be of of the three types (for $K=2$, Fig. 2) and of the seven types (for $K=3$, Fig. 3).*

Proof: As in the proof of Lemma 5.4, by Lemma 5.1, $P_j^0\neq P_i^0$ for $i\neq j$.

By Lemma 5.4, we only need to consider boundary spikes or mixed interior and boundary spikes.

Let us consider $K=2$ first. By Theorem 2.1, we may assume that $P_1^\epsilon, P_2^\epsilon$ lie on the x_1-axis. We may further assume that $P_j^\epsilon=(l_j^\epsilon,0,...,0), j=1,2, l_1^\epsilon<l_2^\epsilon$. If both P_1^ϵ and P_2^ϵ lie on the boundary, then necessarily by [30], $l_1^\epsilon=-1, l_2^\epsilon=1$, which is type 2.I. Suppose that $P_1^\epsilon\in\partial\Omega$, we may assume that $l_1^\epsilon=-1$. In this case, (7.8) implies that $|P_1^\epsilon-P_2^\epsilon|=2d_{P_2^\epsilon}+o(1)$ and therefore $l_1^\epsilon\to\frac{1}{3}$, which is type 2.II.

For $K=3$, by Theorem 2.3, we may assume that $P_1^\epsilon, P_2^\epsilon, P_3^\epsilon$ lie on a two-dimensional plane, say $(x_1,x_2)-$plane. That is we have $P_j^\epsilon=(P_{j,1}^\epsilon, P_{j,2}^\epsilon,0,...,0), j=1,2,3$.

Suppose first we have three boundary spikes: $P_j^\epsilon\in\partial\Omega, j=1,2,3$. We may assume that $P_1^\epsilon=(-1,0,...,0)$. We examine equation (7.7) for $j=1$ and $\partial_{1,i}=\frac{\partial}{\partial P_{1,2}^\epsilon}$. Then it is easy to see that we must have

(7.9)
$$e^{-|P_2^\epsilon - P_1^\epsilon|/\epsilon}(\frac{P_{2,2}^\epsilon - P_{1,2}^\epsilon}{|P_2^\epsilon - P_1^\epsilon|} + e^{-|P_3^\epsilon - P_1^\epsilon|/\epsilon}(\frac{P_{3,2}^\epsilon - P_{1,2}^\epsilon}{|P_3^\epsilon - P_1^\epsilon|} + o(e^{-|P_2^\epsilon - P_1^\epsilon|/\epsilon} + e^{-|P_3^\epsilon - P_1^\epsilon|/\epsilon}) = 0$$

which shows that

$$|P_2^\epsilon - P_1^\epsilon| = |P_3^\epsilon - P_1^\epsilon| + o(1)$$

Similarly, we will have

$$|P_1^\epsilon - P_2^\epsilon| = |P_3^\epsilon - P_2^\epsilon| + o(1),$$

$$|P_1^\epsilon - P_3^\epsilon| = |P_2^\epsilon - P_3^\epsilon| + o(1).$$

As $\epsilon \to 0$, it shows that P_1^0, P_2^0 and P_3^0 must form a perfect triangle, which is type 3.I.

The other cases are similar. We omit the details.

□

As for the existence, we have

Lemma 7.2. *For ϵ sufficiently small and $K = 2$, up to a rotation, there are at least three solutions. The limiting positions are shown in Fig. 2. Moreover, these solutions inherit the partial symmetries of their limiting positions.*

Similarly for ϵ sufficiently small and $K = 3$, up to a rotation, there are at least seven solutions. The limiting positions are shown in Fig. 3. Moreover, these solutions inherit the partial symmetries of their limiting positions.

Proof: The existence of all the above solutions follow a general procedure: we first restrict to a symmetric class and then use the idea in [22] and [23]. As an example, we consider the existence of type 2.II solutions. We fix $P_1 = (-1, 0, ..., 0)$ and let $P_2 = (l_2, 0, ..., 0), |l_2 - \frac{1}{3}| < \delta$. Then we solve the following problem

(7.10)
$$S_\epsilon(w_{\epsilon,P_1} + w_{\epsilon,P_2} + v) \in \mathcal{C}_{\epsilon,P_2},\ \in \mathcal{K}_{\epsilon,P_2} \cap H^2_{\nu,s}(\Omega_\epsilon),$$

where

$$H^2_{\nu,s}(\Omega_\epsilon) = \{u \in H^2_\nu(\Omega_\epsilon) | u \text{ is symmetric with respect to } x_j, j = 2, ..., N\}.$$

Problem (7.10) can be solved since the degeneracy at P_1 is eliminated by restricting to $H^2_{\nu,s}(\Omega_\epsilon)$.

Then we define a reduced energy functional

$$\tilde{M}_\epsilon(P_2) = J_\epsilon[w_{\epsilon,P_1} + w_{\epsilon,P_2} + v_\epsilon]$$

where v_ϵ is obtained in (7.10). It is easy to compute that asymptotically we have

$$\tilde{M}_\epsilon(P_2) = \epsilon^N[2I(w) - (c_1 + o(1))w(|P_1 - P_2|/\epsilon) - (\frac{\gamma_1}{2} + o(1))\varphi_{\epsilon,P_2}(P_2)] \tag{7.11}$$

where c_1 is some positive constant, $I(w) = \frac{1}{2}\int_{R^N}(|\nabla w|^2 + w^2) - \int_{R^N} F(w)$ is the energy of the ground state w.

Similar to [22], we now maximize the reduced energy

$$\max_{|l_2 - \frac{1}{3}| \leq \delta} \tilde{M}_\epsilon(P_2), P_2 = (l_2, 0, ..., 0). \tag{7.12}$$

Then by the energy expansion of (7.11), it is easy to see that the maximum is attained at some l_2^ϵ with $l_2^\epsilon \to \frac{1}{3}$ and the corresponding solution $u_\epsilon = w_{\epsilon,P_1} + w_{\epsilon,P_2^\epsilon}$ is a solution of type 2.II.

The proof of the existence of the other types of solutions is similar. We omit the details.

□

As for uniqueness, we consider case by case separately.

The uniqueness of the type 2.I solution is given in [30].

For type 2.II solutions, by Theorem 2.1, we may assume that $P_1^\epsilon = (-1, 0, ..., 0), P_2^\epsilon = (l^\epsilon, 0, ..., 0)$. By rotation, we may fix P_1^ϵ. Moreover, we may consider solutions which are axially symmetric with respect to the x_1-axis. This reduces the total degrees of freedom into one. Now it is easy to show that $M_\epsilon(\mathbf{P})$ has a nondegenerate local maximum at P_2^ϵ. This shows that the Morse index is 1 and uniqueness follows.

The uniqueness of type 2.III follows by the same proof as the uniqueness of type 3.IV.

Next we consider $K = 3$.

For type 3.I solutions, by Theorem 1.3, $P_j^\epsilon, j = 1, 2, 3$ and the origin must lie in a hyperplane. The problem becomes two-dimensional. Now we fix $P_1^\epsilon = (-1, 0, ..., 0)$. Then we have two degrees of freedom. It is easy to see that the Morse index is 2.

For type 3.II and type 3.III, the proofs are similar.

8. Appendix A: Proof of Lemma 4.1

In this appendix, we prove Lemma 4.1 of Section 4.

Recall that $\varphi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \varphi_{\epsilon,P}(x) - \varphi_{\epsilon,P}(x) = 0 \text{ in } \Omega, \\ \frac{\partial \varphi_{\epsilon,P}}{\partial \nu} = -\frac{\partial}{\partial \nu} w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{8.1}$$

On $\partial\Omega$, we have

$$\frac{\partial}{\partial \nu}\varphi_{\epsilon,P}(x) = (-w'(\frac{x-P}{\epsilon}))\frac{< x-P, \nu >}{|x-P|}\frac{1}{\epsilon}$$

$$= \epsilon^{-1}(1 + O(\epsilon))(\frac{\epsilon}{|x-P|})^{\frac{N-1}{2}} e^{-\frac{|x-P|}{\epsilon}} \frac{< x-P, \nu >}{|x-P|}. \tag{8.2}$$

We consider an auxiliary problem (considered in [38] and [50])

$$\begin{cases} \epsilon^2 \Delta \varphi^D_{\epsilon,P}(x) - \varphi^D_{\epsilon,P}(x) = 0 \text{ in } \Omega, \\ \varphi^D_{\epsilon,P} = w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{8.3}$$

Let $\varphi^D_{\epsilon,P} = e^{-\Psi_{\epsilon,P}(x)/\epsilon}$, where $\Psi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \Psi_{\epsilon,P}(x) - |\nabla \Psi_{\epsilon,P}(x)|^2 + 1 = 0 \text{ in } \Omega, \\ \Psi_{\epsilon,P} = -\epsilon \log w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{8.4}$$

By Lemma 3.6 of [38], we see that as $\epsilon \to 0$,

$$\Psi_{\epsilon,P}(P) \to 2d_P, \quad \text{as } \epsilon \to 0. \tag{8.5}$$

It is proved in [50] that

$$\frac{\partial \Psi_{\epsilon,P}(x)}{\partial \nu} = (-1 + O(\epsilon))\frac{\partial}{\partial \nu}|x-P| = (-1 + O(\epsilon))\frac{< x-P, \nu >}{|x-P|} \text{ on } \partial\Omega. \tag{8.6}$$

Thus comparing (8.1), (8.2) and (8.6), we obtain that

$$\varphi_{\epsilon,P}(x) = (1+O(\epsilon))e^{-\Psi_{\epsilon,P}(x)/\epsilon}, \ x \in \bar{\Omega}. \tag{8.7}$$

Hence $\varphi_{\epsilon,P}(x) = (1+O(\epsilon))w(\frac{x-P}{\epsilon})$ on $\partial\Omega$ and (4.7) of Lemma 4.1 is proved.

(4.8) follows from (8.5).

To compute the exact asymptotic expansion of $\varphi_{\epsilon,P}(P)$, we use the Green's function. Let $G_\epsilon(x,z)$ be the Green's function of

$$\begin{cases} \epsilon^2\Delta G_\epsilon(x,z) - G_\epsilon(x,z) + \delta(z-x) = 0 \ \text{ in } \Omega, \\ \frac{\partial G_\epsilon(x,z)}{\partial \nu_x} = 0 \text{ on } \partial\Omega. \end{cases} \tag{8.8}$$

Then we have

$$\varphi_{\epsilon,P}(x) = \int_{\partial\Omega} G_\epsilon(x,z)\frac{\partial \varphi_{\epsilon,P}}{\partial \nu}(z)dz. \tag{8.9}$$

We decompose

$$G_\epsilon(x,z) = K_\epsilon(|x-z|) + H_\epsilon(x,z)$$

where $K_\epsilon(r)$ is the fundamental solution of $\epsilon^2\Delta - 1$ in $R^N\backslash\{0\}$.

Then H_ϵ satisfies

$$\begin{cases} \epsilon^2\Delta H_\epsilon - H_\epsilon = 0 \ \text{ in } \Omega, \\ \frac{\partial H_\epsilon(x,z)}{\partial \nu} = -\frac{\partial K_\epsilon(|x-z|)}{\partial \nu} \text{ on } \partial\Omega. \end{cases} \tag{8.10}$$

Since on $\partial\Omega$, we have

$$\frac{\partial}{\partial \nu}H_\epsilon(x) = (-K'(\frac{x-P}{\epsilon}))\frac{< x-P, \nu >}{|x-P|}\frac{1}{\epsilon}$$

$$= \epsilon^{-1}(d_N + O(\epsilon))(\frac{\epsilon}{|x-P|})^{\frac{N-1}{2}} e^{-\frac{|x-P|}{\epsilon}}\frac{< x-P, \nu >}{|x-P|} \tag{8.11}$$

for some generic number $d_N > 0$.

As before, we have

$$H_\epsilon(x,z) = d_N\varphi_{\epsilon,x}(z) = d_N w(\frac{z-P}{\epsilon}) \text{ for } z \in \partial\Omega.$$

So we have

$$\varphi_{\epsilon,P}(P) = \int_{\partial\Omega}(2+O(\epsilon))K_\epsilon(|z-P|)\frac{\partial \varphi_{\epsilon,P}}{\partial \nu}dz$$

$$= (c_N + o(1))\epsilon^{-1}\int_{\partial\Omega}(\frac{\epsilon}{|z-P|})^{N-1}e^{-2|z-P|/\epsilon}\frac{< z-P, \nu >}{|z-P|}dz$$

$$= (c_N + o(1))\epsilon^{N-2} \int_{\partial\Omega} (\frac{1}{|z-P|})^{N-1} e^{-2|z-P|/\epsilon} \frac{< z-P, \nu >}{|z-P|} dz. \tag{8.12}$$

Let P be such that $|P| \geq d_0$ for some $d_0 > 0$. Then the integral in (8.12) is a typical Laplace integral and can computed by the classical Laplace method: namely, we let $z = \sqrt{\epsilon} y$ and then obtain

$$\varphi_{\epsilon,P}(P) = (c_N + o(1))(d_P(1-d_P))^{-\frac{N-1}{2}} \epsilon^{\frac{3N}{2}-\frac{5}{2}} e^{-2d_P/\epsilon}$$

for some positive constant $C_N > 0$. This proves (4.10) of Lemma 4.1.

Next we prove (4.9) of Lemma 4.1. To this end, we note that for $x = P + \epsilon y$

$$\varphi_{\epsilon,P}(x) = \int_{\partial\Omega} G_\epsilon(x,z) \frac{\partial \varphi_{\epsilon,P}}{\partial \nu}(z) dz$$

$$= \epsilon^{-1}(c_N + o(1)) \int_{\partial\Omega} (\frac{\epsilon}{|z-x|})^{-\frac{N-1}{2}} (\frac{\epsilon}{|z-P|})^{-\frac{N-1}{2}} e^{-\frac{|z-x|+|z-P|}{\epsilon}} \frac{< z-P, \nu >}{|z-P|} dz$$

$$= \epsilon^{-1}(c_N + o(1)) \int_{\partial\Omega} (\frac{\epsilon}{|z-x|})^{-\frac{N-1}{2}} (\frac{\epsilon}{|z-P|})^{-\frac{N-1}{2}} e^{-\frac{2|z-P|}{\epsilon}} e^{-\frac{<z-P,y>}{|z-P|}} \frac{< z-P, \nu >}{|z-P|} dz$$

$$= (1 + o(1)) \varphi_{\epsilon,P}(P) e^{-<\nabla d_P, y>}$$

which proves (4.9) of Lemma 4.1.

Finally we prove (4.11) and (4.13) of Lemma 4.1.

For $P \in \Omega$, we define

$$\Omega_{\epsilon,P} := \{ y | \epsilon y + P \in \Omega \}. \tag{8.13}$$

If $P = 0$, we denote $\Omega_{\epsilon,P}$ as Ω_ϵ.

For $P \in \Omega$, we have

$$\int_\Omega f'(\bar{w}_{\epsilon,P}) \frac{\partial \bar{w}_{\epsilon,P}}{\partial P_i} \varphi_{\epsilon,P}(x) dx$$

$$= (-1 + o(1)) \varphi_{\epsilon,P}(P) \epsilon^{N-1} \int_{R^N} f'(w) \frac{\partial w}{\partial y_i} e^{-<\nabla d_P, y>} dy \quad \text{(by Lemma 4.1 (2))}$$

$$= (-\gamma_1 + o(1)) \epsilon^{N-1} \varphi_{\epsilon,P}(P) (\nabla d_P)_i + O(e^{-(2+\sigma)d_P/\epsilon}) \tag{8.14}$$

where γ_1 is given in (4.12). This prove (4.11).

For $P_1, P_2 \in \Omega$ with $|P_1 - P_2|/\epsilon \to +\infty$, we have

$$\int_\Omega f'(\bar{w}_{\epsilon,P_1}) \bar{w}_{\epsilon,P_2} \frac{\partial \bar{w}_{\epsilon,P_1}}{\partial P_{1,i}}$$

$$= (-1+o(1))\epsilon^{N-1}\int_{\Omega_{\epsilon,P_1}} f'(w(y))\frac{\partial w}{\partial y_i}w(y+\frac{P_1-P_2}{\epsilon})dy + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})$$

$$= \epsilon^{N-1}\int_{R^N} f(w)\frac{\partial}{\partial y_i}w(y+\frac{P_1-P_2}{\epsilon})dy + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})$$

$$= \epsilon^{N-1}(-\gamma_1+o(1))w(\frac{|P_1-P_2|}{\epsilon})(\nabla_{P_1}(|P_1-P_2|))_i + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon}) \tag{8.15}$$

which proves (4.13).

□

9. Appendix B: Proof of Lemma 4.3

In this appendix, we prove Lemma 4.3. This is similar to the proof of Lemma 2.4 of [51].

The existence of $v_{\epsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$ such that $S_\epsilon(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}) \in \mathcal{C}_{\epsilon,\mathbf{P}}$ follows from Section 3 in [22]. The C^2-smoothness of $v_{\epsilon,\mathbf{P}}$ in $\mathbf{P}$ follows from Lemma 3.5 in [22]. For estimate (4.23), please see Lemma 3.4 of [22]. It remains to estimate $\partial_{j,i}v_{\epsilon,\mathbf{P}}$ and prove (4.24). We decompose

$$\partial_{j,i}v_{\epsilon,\mathbf{P}} = \sum_{s=1,\ldots,K,t=1,\ldots,N} \alpha^{\epsilon}_{st,ji}\partial_{s,t}w_{\epsilon,\mathbf{P}} + v^{\perp}_{\epsilon,\mathbf{P}}, v^{\perp}_{\epsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}},$$

where $\alpha^{\epsilon}_{st,ji}$ are scalar constants.

We first note that by (4.23)

$$\int_{\Omega_\epsilon} \partial_{j,i}v_{\epsilon,\mathbf{P}}\partial_{l,m}w_{\epsilon,\mathbf{P}} = -\int_{\Omega_\epsilon} v_{\epsilon,\mathbf{P}}\partial_{j,i}\partial_{l,m}w_{\epsilon,\mathbf{P}} = O(\epsilon^{-2}e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon}).$$

Hence

$$\sum_{s,t}\alpha^{\epsilon}_{st,ji}\int_{\Omega_\epsilon}\partial_{l,m}w_{\epsilon,\mathbf{P}}\partial_{j,i}w_{\epsilon,\mathbf{P}} = O(\epsilon^{-2}e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon}).$$

Since

$$\int_{\Omega_\epsilon}\partial_{l,m}w_{\epsilon,\mathbf{P}}\partial_{s,t}w_{\epsilon,\mathbf{P}} = \epsilon^{-2}(\Gamma_0+o(1))\delta_{sl}\delta_{mt},$$

where $\Gamma_0 = \int_{R^N}(\frac{\partial w}{\partial y_1})^2 > 0$, we obtain $\alpha^{\epsilon}_{st,ji} = O(\epsilon^{-2}e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon})$.

Next we observe that

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) = \sum_{s,t} \beta^\epsilon_{s,t}(\mathbf{P}) \partial_{s,t} w_{\epsilon,\mathbf{P}}, \tag{9.1}$$

where $\beta^\epsilon_{s,t}(\mathbf{P}) \in C^1$ and $\beta^\epsilon_{s,t}(\mathbf{P}) = O(\epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon})$.

Differentiating the equation (9.1) by $\partial_{l,m}$, we have,

$$S^{'}_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(\partial_{l,m} w_{\epsilon,\mathbf{P}} + \partial_{l,m} v_{\epsilon,\mathbf{P}}) - \sum_{s,t} \beta^\epsilon_{s,t}(\mathbf{P}) \partial_{l,m} \partial_{s,t} w_{\epsilon,\mathbf{P}} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Substituting the decomposition of $\partial_{j,i} v_{\epsilon,\mathbf{P}}$ into the above equation, we obtain that

$$S^{'}_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(v^\perp_{\epsilon,\mathbf{P}}) + S^{'}_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(\partial_{l,m} w_{\epsilon,\mathbf{P}} + \sum_{s,t} \alpha^\epsilon_{st,ji} \partial_{s,t} w_{\epsilon,\mathbf{P}})$$

$$- \sum_{s,t} \beta^\epsilon_{s,t}(\mathbf{P}) \partial_{s,t} \partial_{l,m} w_{\epsilon,\mathbf{P}} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

It is easy to see that

$$\pi^\perp_{\epsilon,\mathbf{P}} \circ S^{'}_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) : \mathcal{K}^\perp_{\epsilon,\mathbf{P}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{P}}$$

is again invertible for ϵ sufficiently small. Hence (since $v^\perp_{\epsilon,\mathbf{P}} \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$)

$$\|v^\perp_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} \le C \|\pi^\perp_{\epsilon,\mathbf{P}} \circ S^{'}_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})(\partial_{l,m} w_{\epsilon,\mathbf{P}} + \sum_{s,t} \alpha^\epsilon_{st,ji} \partial_{s,t} w_{\epsilon,\mathbf{P}})|\|_{L^2(\Omega_\epsilon)}$$

$$+ C \sum_{s,t} |\beta^\epsilon_{s,t}(\mathbf{P})| \|\partial_{s,t} \partial_{l,m} w_{\epsilon,\mathbf{P}}\|_{L^2(\Omega_\epsilon)}$$

$$\le C \epsilon^{-2} e^{-(1+\sigma)\varphi_K(\mathbf{P})/\epsilon}.$$

(4.24) is thus proved.

□

10. Appendix C: Proof of Lemma 5.1

In this appendix, we prove Lemma 5.1.

We first exclude the case when multiple interior spikes collapse to the boundary. In fact this follows from the proof of Lemma 2.3 of [48]. We prove by contradiction.

Suppose now that $P_j^\epsilon \in \Omega, \frac{d(P_j^\epsilon,\partial\Omega)}{\epsilon} \to +\infty, P_j^\epsilon \to P_0 \in \partial\Omega, j = 1,...,n_1 \leq K$. Suppose that all the other spikes stay away from P_0. That is $|P_j^\epsilon - P_0| \geq \delta_0 > 0$ for $j = n_2 = 1,...,K$.

We recall the following Pohozaev identity: suppose that u satisfies $\epsilon^2\Delta u - u + f(u) = 0$ in a domain Ω, then for any $y \in R^N$, we have

$$\int_{\Omega_0} [(NF(u) - \frac{N-2}{2}uf(u)) - u^2]$$

$$= \int_{\partial\Omega_0} \Bigg[(\epsilon^2 (x-y, \nabla u)\frac{\partial u}{\partial \nu} - \epsilon^2 < x-y, \nu > \frac{|\nabla u|^2}{2})$$

$$+ < x-y, \nu > (\frac{u^2}{2} - F(u)) + \epsilon^2 \frac{N-2}{2} u \frac{\partial u}{\partial \nu} \Bigg].$$

Since y is arbitrary, we deduce that

$$\int_{\partial\Omega_0} \left[\nu(\epsilon^2 \frac{|\nabla u|^2}{2} + \frac{u^2}{2} - F(u)) - \epsilon^2 \nabla u \frac{\partial u}{\partial \nu} \right] = 0. \tag{10.1}$$

Now we choose $\Omega_0 = \{x \in \Omega | d(x, P_0) \leq \frac{1}{10}\}$ so that $|\nu(x) - \nu(P_0)| \leq \frac{1}{10}$ where $\nu(x)$ is the normal derivative at $x \in \ \partial\Omega \cap \partial\Omega_0$. Let

$$\rho_\epsilon = \min_{j,k,l=1,...,n_1, k\neq l} (d(P_j^\epsilon, \partial\Omega), |P_k^\epsilon - P_l^\epsilon|)/\epsilon.$$

Then similar to the proof of (2.8) and (2.12) of [48], we have

$$u_\epsilon \geq Ce^{-\rho_\epsilon} \quad \text{on } \partial\Omega_0 \cap \partial\Omega \tag{10.2}$$

and

$$u_\epsilon, |\nabla u_\epsilon| \leq Ce^{-\frac{\delta}{\epsilon}} \quad \text{on } \partial\Omega_0 \backslash \partial\Omega. \tag{10.3}$$

Substituting (10.2) and (10.3) into (10.1), we have

$$0 = \int_{\partial\Omega_0 \cap \partial\Omega} [\nu(\epsilon^2 \frac{|\nabla u_\epsilon|^2}{2} + \frac{u_\epsilon^2}{2} - F(u_\epsilon))] + O(e^{-\delta/\epsilon})$$

$$\geq C \int_{\partial\Omega_0 \cap \partial\Omega} u_\epsilon^2 + O(e^{-\delta/\epsilon}) \geq Ce^{-\rho_\epsilon} \tag{10.4}$$

which is a contradiction.

Second, we exclude the case when multiple interior spikes collapse in the interior. Namely, we have $P_j^\epsilon, j = 1, ..., n_2, 2 \le n_2 \le K$ where $P_j^\epsilon \to P_0 \in \Omega, j = 1, ..., n_2$. Suppose that all the other spikes stay away from P_0. That is $|P_j^\epsilon - P_0| \ge \delta_0 > 0, j = n_2 + 1, ..., K$. Similar to the proof of Lemma 3.1 of [46], we see that equation (7.7) remains true: so we have (all the boundary terms are higher order terms)

$$(10.5) \quad \sum_{k \neq j} w(|P_k^\epsilon - P_j^\epsilon|/\epsilon) \nabla_{P_j^\epsilon}(|P_k^\epsilon - P_j^\epsilon|) + o(\sum_{k \neq j} w(|P_k^\epsilon - P_j^\epsilon|/\epsilon)) = 0, j = 1, ..., n_2.$$

Since $2 \le n_2 \le K \le 3$, we have either $n_2 = 2$ or $n_2 = 3$.

If $n_2 = 2$, (10.5) becomes

$$w(|P_1^\epsilon - P_2^\epsilon|/\epsilon) \nabla_{P_1^\epsilon}(|P_1^\epsilon - P_2^\epsilon|)$$

$$+ o(w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)) = 0$$

which is impossible.

If $n_2 = 3$, we let $|P_1^\epsilon - P_2^\epsilon| = \min_{i \neq j, i,j=1,2,3} |P_i^\epsilon - P_j^\epsilon|$. We consider two cases. The first case is that $\lim_{\epsilon \to 0} \frac{w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)}{w(|P_3^\epsilon - P_2^\epsilon|/\epsilon)} < +\infty$. In this case, let us assume that $< P_3^\epsilon - P_1^\epsilon, P_2^\epsilon - P_1^\epsilon > \ge 0$. Then for equation (10.5) at $j = 1$, we have

$$w(|P_1^\epsilon - P_2^\epsilon|/\epsilon) + w(|P_3^\epsilon - P_1^\epsilon|/\epsilon) < \nabla_{P_1^\epsilon}(|P_3^\epsilon - P_1^\epsilon|), \nabla_{P_1^\epsilon}(|P_2^\epsilon - P_1^\epsilon|) > + o(w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)) = 0,$$

which is impossible. If $\lim_{\epsilon \to 0} \frac{w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)}{w(|P_3^\epsilon - P_1^\epsilon|/\epsilon)} < +\infty$, the proof is similar.

The second case is that $\lim_{\epsilon \to 0} \frac{w(|P_3^\epsilon - P_2^\epsilon|/\epsilon)}{w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)} = 0$. This reduces to the $n_2 = 2$ case.

Finally, we need to exclude the case when multiple interior spikes collapse to a boundary spike or multiple boundary spikes collapse. That is we have $|P_j^\epsilon - P_1^\epsilon| \to 0, j = 2, ..., n_3 \le K$, where $P_1^\epsilon \in \partial\Omega$. Without loss of generality, we may assume that $P_1^\epsilon = P_0 = (-1, ..., 0)$. If $K = 2$, by Theorem 2.1, P_2^ϵ and P_1^ϵ are on different sides. Thus, $|P_2^\epsilon - P_1^\epsilon| \ge 1$. So we may assume that $K = 3$. By Theorem 2.3, we have $|P_3^\epsilon - P_1^\epsilon| \ge 1$ and $P_2^\epsilon - P_1^\epsilon| \to 0$. There are two cases to be considered.

Case 1. $P_2^\epsilon \in \Omega, P_2^\epsilon \to P_0 = P_1^\epsilon$.

Let $\tilde{\epsilon} = \frac{\epsilon}{\delta_\epsilon}$, where $\delta_\epsilon = |P_1^\epsilon - P_2^\epsilon|$.

Equations (7.7) and (7.8) remains true as long as $\tilde{\varphi}_K(P_1^\epsilon, ..., P_K^\epsilon)/\epsilon \to +\infty$. Now we look at equation (7.8) for $j = 2, i = 1$. We have

$$2(\gamma_1 + o(1))c(P_2^\epsilon)e^{-2d_{P_2^\epsilon}/\epsilon}(\nabla_{P_2^\epsilon} d_{P_2^\epsilon})_1$$

$$(\gamma_1 + o(1))(|P_1^\epsilon - P_2^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_1^\epsilon - P_2^\epsilon|/\epsilon}(\nabla_{P_2^\epsilon}|P_2^\epsilon - P_1^\epsilon|)_1 = 0. \tag{10.6}$$

Note that both $\nabla_{P_2^\epsilon} d_{P_2^\epsilon}$ and $\nabla_{P_2^\epsilon}(|P_2^\epsilon - P_1^\epsilon|)$ are pointing in the same direction. Equation (10.6) can not hold.

Case 2. $P_1^\epsilon \in \partial\Omega, P_2^\epsilon \in \partial\Omega, |P_2^\epsilon - P_1^\epsilon| \to 0$.

In this case, we apply (7.7) to $j = 1, \frac{\partial}{\partial \tau_{P_1^\epsilon}} = \frac{\partial}{\partial x_{1,2}^\epsilon}$ and we obtain

$$(\gamma_1 + o(1))(|P_1^\epsilon - P_2^\epsilon|)^{-\frac{N-1}{2}} e^{-|P_1^\epsilon - P_2^\epsilon|/\epsilon}(\nabla_{P_2^\epsilon}|P_2^\epsilon - P_1^\epsilon|)_2 = 0 \tag{10.7}$$

which is impossible.

□

11. Appendix D: Proof of Lemma 5.3

In this appendix, we prove Lemma 5.3.

Proof of (1) and (2) of Lemma 5.3: Observe that

$$\nabla_{j,i} M_\epsilon(\mathbf{P}) = < w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})$$

$$= < w_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}})$$

$$+ < v_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}})v_{\epsilon,\mathbf{P}}\partial_{j,i}(w_{\epsilon,\mathbf{P}})$$

$$+ < w_{\epsilon,\mathbf{P}}, \partial_{j,i}(v_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(v_{\epsilon,\mathbf{P}})$$

$$+ < v_{\epsilon,\mathbf{P}}, \partial_{j,i}(v_{\epsilon,\mathbf{P}}) >_\epsilon - \epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(v_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon})$$

$$= < w_{\epsilon,\mathbf{P}}, \partial_{j,i} w_{\epsilon,\mathbf{P}} >_\epsilon - \epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon})$$

$$= \epsilon^{-N} \int_\Omega [\sum_{l=1}^K f(\bar{w}_{\epsilon,P_l}) - f(\sum_{l=1}^K w_{\epsilon,P_l})](\partial_{j,i} w_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon}) \quad \text{(by Lemma 3.1)}$$

$$= \epsilon^{-N}\int_\Omega [\sum_{l=1}^K (f(\bar{w}_{\epsilon,P_l}) - f(w_{\epsilon,P_l})) - \sum_{l\neq j} f'(w_{\epsilon,P_j}) w_{\epsilon,P_l}]\partial_{j,i} w_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon})$$

$$= \epsilon^{-N}\int_\Omega [f(\bar{w}_{\epsilon,P_j}) - f(w_{\epsilon,P_j})]\partial_{j,i} w_{\epsilon,P_j} - \epsilon^{-N}\sum_{l\neq j}\int_\Omega f'(w_{\epsilon,P_j}) w_{\epsilon,P_l}\partial_{j,i} w_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon}).$$

(11.1)

$$= \int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j})(-\varphi_{\epsilon,P_j})\partial_{j,i}\bar{w}_{\epsilon,P_j} - \sum_{l\neq j}\int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j})\bar{w}_{\epsilon,P_l}\partial_{j,i}\bar{w}_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P})/\epsilon})$$

We discuss two cases.

Case 1. If $|P_j^\epsilon| \leq \frac{1}{10}$, then the first term in (11.1)

$$\int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j})(-\varphi_{\epsilon,P_j})\partial_{j,i}\bar{w}_{\epsilon,P_j} = O(e^{-2d_{P_j^\epsilon}}/\epsilon)$$

which is a higher order term, comparing with the second order term (since $|P_k^\epsilon - P_j^\epsilon| \leq \frac{11}{10} < 2d_{P_j^\epsilon}$). This proves (5.8) of Lemma 5.3.

Case 2. If $|P_j^\epsilon| \geq d_0$ for some $d_0 > 0$, then (11.1) equals

(11.2) $$\epsilon^{N-1}(\gamma_1 + o(1))\varphi_{\epsilon,P_j}(P_j)(\nabla d_{P_j})_i + \epsilon^{N-1}(\gamma_1 + o(1))\sum_{l\neq j} w(|P_j - P_l|/\epsilon)(\nabla|P_j - P_l|)_i$$

by (4.11) and (4.13) of Lemma 4.1.

By using Lemma 4.1, we see that (5.7) holds.

□

Proof of (3) of Lemma 5.3: Let $\mathbf{P}^\epsilon$ be a critical point of $M_\epsilon(\mathbf{P})$ in Λ_δ such that $|P_j^\epsilon| \geq d_0, j = 1, ..., K$ for some $d_0 > 0$. We now expand,

$$\partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$= < \partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}), \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$+ < w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}, \partial_{l,m}\partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$-\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}\partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$= < \partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}), \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

(since $\mathbf{P}^\epsilon$ is a critical point of $M_\epsilon(\mathbf{P})$)

$$=<\partial_{l,m}w_{\epsilon,\mathbf{P}},\partial_{j,i}w_{\epsilon,\mathbf{P}}>_\epsilon|_{\mathbf{P}=\mathbf{P}^\epsilon}-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$+<\partial_{l,m}w_{\epsilon,\mathbf{P}},\partial_{j,i}v_{\epsilon,\mathbf{P}}>_\epsilon|_{\mathbf{P}=\mathbf{P}^\epsilon}-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$+<\partial_{l,m}v_{\epsilon,\mathbf{P}},\partial_{j,i}w_{\epsilon,\mathbf{P}}>_\epsilon|_{\mathbf{P}=\mathbf{P}^\epsilon}-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$+<\partial_{l,m}v_{\epsilon,\mathbf{P}},\partial_{j,i}v_{\epsilon,\mathbf{P}}>_\epsilon|_{\mathbf{P}=\mathbf{P}^\epsilon}-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}\partial_{j,i}v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$=I_1+I_2+I_3+I_4$$

where $I_i, i=1,...,4$ are defined at the last equality.

We now estimate each term. Certainly the estimate of I_2 is the same as that of I_3. By Lemma 4.3,

$$I_4=O(\|\partial_{l,m}v_{\epsilon,\mathbf{P}^\epsilon}\|_\epsilon\|\partial_{j,i}v_{\epsilon,\mathbf{P}^\epsilon}\|_\epsilon)=O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}). \tag{11.3}$$

Next we consider I_2:

(11.4)

$$I_2=\epsilon^{-N}\int_\Omega[f(\bar{w}_{\epsilon,P_l^\epsilon})\partial_{l,m}\bar{w}_{\epsilon,P_l^\epsilon}-f(w_{\epsilon,\mathbf{P}^\epsilon}+v_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,P_l^\epsilon}]\partial_{j,i}w_{\epsilon,P_j^\epsilon}=O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}).$$

Similarly, we have

$$I_3=O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}). \tag{11.5}$$

Hence it remains to compute I_1 only. We divide it into two cases: $j\neq l$ and $j=l$.

When $j\neq l$, we have by Lemma 4.1

$$I_1=\epsilon^{-N}\int_\Omega[f'(\bar{w}_{\epsilon,P_l^\epsilon})\partial_{l.m}\bar{w}_{\epsilon,P_l^\epsilon}-f'(w_{\epsilon,\mathbf{P}^\epsilon})\partial_{l,m}w_{\epsilon,P_l^\epsilon}]\partial_{j,i}w_{\epsilon,P_j^\epsilon}$$

$$=\epsilon^{-N}\int_\Omega[f'(\bar{w}_{\epsilon,P_l^\epsilon})\partial_{l,m}\bar{w}_{\epsilon,P_l^\epsilon}-(f'(w_{\epsilon,P_l^\epsilon})+f'(w_{\epsilon,P_j^\epsilon}))\partial_{l,m}w_{\epsilon,P_l^\epsilon}]\partial_{j,i}\bar{w}_{\epsilon,P_j^\epsilon}+O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$=\epsilon^{-N}\int_\Omega[f'(\bar{w}_{\epsilon,P_l^\epsilon})\partial_{l,m}\bar{w}_{\epsilon,P_l^\epsilon}-f'(w_{\epsilon,P_l^\epsilon})\partial_{l,m}w_{\epsilon,P_l^\epsilon}]\partial_{j,i}\bar{w}_{\epsilon,P_j^\epsilon}$$

$$-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,P_j^\epsilon})\partial_{l,m}w_{\epsilon,P_l^\epsilon}\partial_{j,i}\bar{w}_{\epsilon,P_j^\epsilon}+O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$=-\epsilon^{-N}\int_\Omega f'(w_{\epsilon,P_j^\epsilon})\partial_{l,m}w_{\epsilon,P_l^\epsilon}\partial_{j,i}\bar{w}_{\epsilon,P_j^\epsilon}+O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} w(\frac{|P_j^\epsilon - P_l^\epsilon|}{\epsilon}) \int_{\Omega_{\epsilon,P_j^\epsilon}} f'(w) \frac{\partial w}{\partial y_i} e^{-<e_{jl}^\epsilon, y>} (e_{jl}^\epsilon)_m + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} w(|P_j^\epsilon - P_l^\epsilon|/\epsilon)(\gamma_1 + o(1))(e_{jl}^\epsilon)_m (e_{jl}^\epsilon)_i + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon}) \tag{11.6}$$

For $j = l$, we have

$$I_1 = \epsilon^{-N} \int_\Omega [f'(\bar{w}_{\epsilon,P_j^\epsilon}) \partial_{j,m} \bar{w}_{\epsilon,P_j^\epsilon} - f'(w_{\epsilon,\mathbf{P}^\epsilon}) \partial_{j,m} w_{\epsilon,P_j^\epsilon}] \partial_{j,i} w_{\epsilon,P_j^\epsilon}$$

$$= \epsilon^{-N} \int_\Omega [f'(\bar{w}_{\epsilon,P_j^\epsilon}) \partial_{j,m} \bar{w}_{\epsilon,P_j^\epsilon} - f'(w_{\epsilon,P_j^\epsilon}) \partial_{j,m} w_{\epsilon,P_j^\epsilon}] \partial_{j,i} w_{\epsilon,P_j^\epsilon}$$

$$-\epsilon^{-N} \sum_{k \neq j} \int_\Omega f''(\bar{w}_{\epsilon,P_j^\epsilon}) w_{\epsilon,P_k^\epsilon} \partial_{j,m} w_{\epsilon,P_j^\epsilon} \partial_{j,i} w_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= I_{1,1} - I_{1,2}.$$

For $I_{1,1}$, we have

$$I_{1,1} = \epsilon^{-N} \int_\Omega [\partial_{j,m} f(\bar{w}_{\epsilon,P_j^\epsilon}) - \partial_{j,m} f(w_{\epsilon,P_j^\epsilon})] \partial_{j,i} \bar{w}_{\epsilon,P_j^\epsilon}$$

$$= \epsilon^{-N} \int_\Omega [(-\frac{\partial}{\partial x_m} f(\bar{w}_{\epsilon,P_j^\epsilon}) - f(w_{\epsilon,P_j^\epsilon}))](-\frac{\partial}{\partial x_i} \bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-N} \int_\Omega (f(w_{\epsilon,P_j^\epsilon}) - f(\bar{w}_{\epsilon,P_j^\epsilon})(\frac{\partial^2}{\partial x_i \partial x_m} \bar{w}_{\epsilon,P_j^\epsilon}) + +O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} \int_{\Omega_{\epsilon,P_j^\epsilon}} f'(w)(y) \varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon + \epsilon y) \frac{\partial^2 w}{\partial y_i \partial y_m} + +O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} \varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon) \int_{R^N} f'(w) e^{-<\nabla_{P_j^\epsilon} d_{P_j^\epsilon}, y>} \frac{\partial^2 w}{\partial y_i \partial y_m} dy + +O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

(by (4.9) of Lemma 4.1)

$$= \epsilon^{-2} \varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon) \int_{R^N} (-f''(w) \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_m} e^{-<\nabla_{P_j^\epsilon} d_{P_j^\epsilon}, y>} dy + +O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} \varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon) e_{j,i}^\epsilon e_{j,m}^\epsilon (-\gamma_1 + o(1)). \tag{11.7}$$

For $I_{1,2}$, we have

$$I_{1,2} = \epsilon^{-N} \sum_{k \not{j}} \int_\Omega f''(\bar{w}_{\epsilon,P_j^\epsilon}) \bar{w}_{\epsilon,P_k^\epsilon} \partial_{j,m} \bar{w}_{\epsilon,P_j^\epsilon} \partial_{j,i} \bar{w}_{\epsilon,P_j^\epsilon} + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} \sum_{k \neq j} \int_{R^N} f''(w) \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_m} w(y + \frac{P_j^\epsilon - P_k^\epsilon}{\epsilon}) dy + O(e^{-(2+\sigma)\varphi_K(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} \sum_{k \neq j} w(\frac{|P_j^\epsilon - P_k^\epsilon|}{\epsilon}) e_{jk,i}^\epsilon e_{jk,m}^\epsilon (\gamma_1 + o(1)). \tag{11.8}$$

Combining all together, we have

$$\begin{aligned} &\partial_{l,m}\partial_{j,i} M_\epsilon(\mathbf{P})\big|_{\mathbf{P}=\mathbf{P}^\epsilon} \\ &= \epsilon^{-2}(\gamma_1 + o(1)) w(\frac{|P_j^\epsilon - P_k^\epsilon|}{\epsilon}) e_{jl,m}^\epsilon e_{jl,i}^\epsilon (1 - \delta_{jl}) \\ &\quad -\epsilon^{-2}(\gamma_1 + o(1)) \varphi_{\epsilon, P_j^\epsilon}(P_j^\epsilon) e_{j,i}^\epsilon e_{l,m}^\epsilon \delta_{jl} \\ &\quad -\epsilon^{-2}(\gamma_1 + o(1)) \sum_{k \neq j} w(\frac{|P_j^\epsilon - P_k^\epsilon|}{\epsilon}) e_{jk,i}^\epsilon e_{jk,m}^\epsilon \delta_{jl}, \end{aligned}$$

which is exactly (5.9).

□

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Chang-Shou Lin

Department of Mathematics, Chung Cheng University

Minghsiung, Chia Yi, Taiwan

E-mail: cslin@math.ccu.edu.tw

Juncheng Wei
Department of Mathematics
The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail: wei@math.cuhk.edu.hk