Pure and Applied Mathematics Quarterly
Volume 3, Number 3
(*Special Issue: In honor of
Leon Simon, Part 2 of 2*)
881—896, 2007

Minimal Graphs in $\mathbf{H}^2 \times \mathbf{R}$ and Their Projections

Michael Wolf

There has been a recent resurgence of interest in the subject of minimal surfaces in product three-manifolds, for example $\mathbf{H}^2 \times \mathbf{R}$ [Ros02], [MR04], [MIR05], [NR02], [Dan06], [FM05], [HET05]. The purpose of this note is to describe a construction and some examples originating in the harmonic maps literature whose consequences for minimal surfaces seem to have escaped notice.

We organize this note as follows. First we recall how harmonic maps to $\mathbf{H}^2$ (or a Riemannian complex) lead, via associated maps to real trees, to minimal surfaces in $\mathbf{H}^2 \times \mathbf{R}$. This occupies the first three sections and leads to a proposition describing minimal graphs in terms of harmonic maps to trees and their folds. In the next section, we apply the construction to some examples from the harmonic maps literature to create some minimal graphs whose shapes have not yet been explored much. We describe rapidly oscillating graphs, including one derived from a harmonic map to the Cayley graph of a free group. In terms of asymptotic behavior, we describe a non-trivial graph whose boundary values are constant except at a single point, and graphs which accumulate only over a Cantor set on S^1_∞. We relate Jenkins-Serrin constructions to trees of finite total valence in section five, and conclude in section six with a non-orientable example (over the product of a surface with $\mathbf{RP}^1$) corresponding to a harmonic map to a tree which does not fold.

The author is grateful to Bill Meeks and Matthias Weber for several stimulating and encouraging conversations.

Received October 13, 2006.

1. Harmonic Maps and their suspensions

A map $w : \mathcal{R} \to N$ between a Riemann surface $\mathcal{R}$ (with conformal structure given in a local coordinate z) and a Riemannian manifold N is harmonic if satisfies the Euler-Lagrange equation, say $\tau(w) = 0$; when N is a Riemannian surface equipped with a conformal metric $\rho(w)|dw|^2$, this equation has the form

$$w_{z\bar{z}} + (\log \rho)_w w_z w_{\bar{z}} = 0.$$

Some of the geometry of the map $w : \mathcal{R} \to N$ is captured by the Hopf differential $\Phi(z)dz^2$, a quadratic differential on the Riemann surface $\mathcal{R}$, defined by $\Phi = (w^* ds_N)^{(2,0)}$. Indeed a fundamental result of Sampson [Sam78] implies that, in some sense, the Hopf differential captures all of the geometry when N is a Riemannian surface of specified constant curvature.

Proposition. [Sam78] *Let N_1 and N_2 be Riemannian surfaces of constant curvature $K_{N_1} = K_{N_2} = K_0$. Suppose also that for a pair of harmonic maps $w_i : \mathcal{R} \to N_i$ from a closed Riemann surface $\mathcal{R}$, the Hopf differentials $\Phi_i = \Phi(w_i)$ agree. Then the pullback metrics $w_i^* ds_i$ agree (even pointwise).*

An open surfaces version of this is due to Wan [Wan92].

Proposition. [Wan92] *let Φ be a holomorphic quadratic differential on the hyperbolic disk $(\mathbf{D}^2, ds^2_{\mathbf{H}})$ with $||\Phi|| = |\Phi|(ds_{\mathbf{H}^2})^{-2}$ bounded. Then there is a unique harmonic map $w(\Phi) : \mathbf{D}^2 \to \mathbf{D}^2$ which is a quasiconformal surjective homeomorphism. Two such quasiconformal harmonic homeomorphisms, whose Hopf differentials (which necessarily have bounded norms) agree, differ by composition by an isometry.*

A version of the uniqueness portion of this was reproved in [HET05] (Theorem 4).

To connect the basic harmonic maps theory to the topic of minimal surfaces in $\mathbf{H}^2 \times \mathbf{R}$, we describe the construction of a *minimal suspension* from [Wol98]. To begin, recall that a holomorphic quadratic differential $\Phi = \varphi(z)dz^2$ determines two (singular) measured foliations $(\mathcal{F}_{\text{hor}}, \mu_{\text{hor}})$ and $(\mathcal{F}_{\text{vert}}, \mu_{\text{vert}})$, the horizontal and vertical measured foliations, respectively. The construction is the following: in a neighborhood where $\Phi \neq 0$, set

$$\zeta(z) = \int^z \pm\sqrt{\Phi} = \int^z \pm\sqrt{\varphi(z)}dz$$

so that $\Phi = d\zeta^2$, and put $(\mathcal{F}_{\text{hor}}, \mu_{\text{hor}}) = (\{\operatorname{Im}\zeta = \text{const}\}, |d\operatorname{Im}\zeta|)$ and $(\mathcal{F}_{\text{vert}}, \mu_{\text{vert}}) = (\{\operatorname{Re}\zeta = \text{const}\}, |d\operatorname{Re}\zeta|)$. The foliations extend to $\Phi^{-1}(0)$ where they have k-pronged singularities where $k = \operatorname{ord}\Phi + 2 \geq 3$ and they have the property that if $F : \mathcal{R} \to \mathcal{R}$ is a self-map of $\mathcal{R}$ that preserves the transversality of arcs to $\mathcal{F}$, then $F^*\mu = \mu$, by Cauchy's theorem.

After passing from the holomorphic quadratic differential to a measured foliation, the next step is to pass to an almost combinatorial object — a real tree. These trees are in some sense simpler than (singular) measured foliations in that they are one-dimensional Euclidean complexes, but they are also quite singular in that they are frequently not locally compact. The passage from a measured foliation $(\widetilde{\mathcal{F}}, \mu)$ on a surface S to a real tree (T, d) is straightforward: one lifts $(\mathcal{F}, \mu)$ on S to $(\widetilde{\mathcal{F}}, \tilde{\mu})$ on the universal cover $\widetilde{S}$ and then sets T to be the leaf space of $\widetilde{\mathcal{F}}$. Defining the map $p : \widetilde{S} \to T$ to be the projection along the leaves, we define a metric on T by $d = p_*\mu$. See [FW01] for details.

The minimal suspension (see [Wol98]) of a harmonic map $w : \mathcal{R} \to N$ where N is a Riemannian manifold (or more generally, a Riemannian complex with enough structure so that stationary points of the energy functional may be defined) is defined as follows. Begin with the Hopf differential $\Phi(w)$ of the harmonic map w and construct its horizontal foliation $(\mathcal{F}_{\text{hor}}(\Phi), \mu_{\text{hor}}(\Phi))$: these leaves integrate the directions of maximal stretch of the differential dw of w. Lift $(\mathcal{F}_{\text{hor}}, \mu_{\text{hor}})$ to a measured foliation $(\widetilde{\mathcal{F}}_{\text{hor}}, \mu_{\text{hor}})$ on the universal cover $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ and then, as above, project along the leaves of $\widetilde{\mathcal{F}}_{\text{hor}}(\Phi)$ to obtain a real tree (T_Φ, d_Φ). This map $p : \widetilde{\mathcal{R}} \to (T_\Phi, d_\Phi)$ is also harmonic (see [Wol96]).

Notation. In our examples, the holomorphic quadratic differential Φ will usually be fixed throughout the discussion, and so we will abbreviate the notation of the tree from (T_Φ, d_Φ) to (T, d).

We claim that the product map $f = (\tilde{w}, p) : \widetilde{\mathcal{R}} \to N \times (T, 2d)$ is then both conformal and harmonic, hence minimal. Certainly as the product of harmonic maps, the map f is harmonic. To se that it is also conformal, note that away from the zeroes of Φ, the map f is locally a map to $N \times \mathbf{R}$; a classical (and straightforward) calculation implies that f is conformal there. On the other hand, at the zeroes of Φ, the map $\tilde{w} : \widetilde{\mathcal{R}} \to N$ is conformal, and as f_* takes tangent vectors to $\widetilde{\mathcal{R}}$ at that point to horizontal tangent vectors to $N \times T$, we see that the conformality of $\tilde{w}$ extends to give the conformality of f at those points.

Informally, the basic reason for the conformality is that by projecting along the directions of maximal stretch, the projection does no stretching in that direction, and so all of its stretching is in the direction of minimal stretch. The product map, because of its additional stretching of dp in the minimal stretch direction of dw actually stretches everywhere equal amounts and is thus conformal.

The factor of 2 in the metric space $(T, 2d)$ in some sense reflects the choice of the Hopf differential to be $(w^*ds_N)^{2,0}$ instead of perhaps $4(w^*ds_N)^{2,0}$: this of course would reflect the factor of $\frac{1}{2}$ in $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

This construction has been discovered a number of times for the case of the Riemann surface $\mathcal{R}$ being a torus: see for example [MIR05].

2. Vertex Folds

The final operation comes from the study of real trees (see [MO93] and [Sko96]), and is known as a vertex fold of a tree. A vertex in a tree is a point whose complement has more than two components (for example the origin in the tree T_n of coordinate axes in $\mathbf{R}^n$). A vertex fold is possible if two of those components are isometric by a map which fixes the vertex. (For example, a vertex fold at the origin in the tree T_n just described might map all of the positive axes to one ray and all of the negative axes to another ray.)

In this note, we will be concerned with foldings of trees which result in a map $i : T \to \mathbf{R}$ with the following property: for any isometric embedding $e : I \to T$ of a compact arc I into T, there are a finite number of preimages $v_1, \dots, v_k \in I$ of vertices in T so that $i \circ e : I \to \mathbf{R}$ is an isometry on the complement of $v_1, \dots, v_k$. Such a map $i : T \to \mathbf{R}$ will be called a *complete vertex folding* of the tree T, or when there is no ambiguity, a *vertex folding* of T.

Remarks. (i) Suppose that the tree $T = T_\Phi$ associated to the Hopf differential Φ admits a complete vertex folding (or even a folding of a single vertex). This implies that at the very least that all of the zeroes of Φ are even. Thus so are the zeroes of $e^{i\theta}\Phi$ for any $\theta \in (0, 2\pi]$. As a consequence, the tree $T_{e^{i\theta}\Phi}$ (associated to the holomorphic quadratic differential $e^{i\theta}\Phi$) admits vertex folds on small intervals through its vertices.

Now, while we cannot in general expect trees to fold, in the case of a simply connected domain, there exists a tremendous simplification: local folds present the only obstruction, as (roughly) local square roots extend to global roots. More precisely, if $p = i \circ \pi : \Omega \to T \to \mathbf{R}$ is the induced harmonic projection to the tree, then we may write $p(z) = \operatorname{Re} \int^z \sqrt{\Phi}$, with $\Phi = 4(\partial p)^2$.

Moreover, since Φ admits a square root, then so does $e^{i\theta}\Phi$, and thus there is a well-defined projection $p_\theta = i_\theta \circ \pi_\theta : \Omega \to T_\theta \to \mathbf{R}$ which factors through the tree T_θ associated to $e^{i\theta}\Phi$. That tree T_θ is in general quite different from the tree T associated to Φ metrically and even combinatorially.

(ii) It is important to note that in the presence of non-trivial topology, local folds (or the existence of local square roots of quadratic differentials) will not necessarily imply the existence of global folds (or global square roots). See [MS93] for existence of quadratic differentials in surfaces of positive genus all of whose zeroes are of even order but which are not squares of abelian differentials. We comment on a minimal surface implication of this in Section 6.

3. Minimal graphs through minimal suspensions

3.1. **Minimal suspensions.** Our first main structural observation is that, given a minimal suspension of a harmonic map, one can often, though not always, obtain a minimal graph in $N \times \mathbf{R}$ by vertex folding that minimal suspension. Moreover, all minimal graphs are obtained this way. Before stating a result in this direction, we give a very simple example.

Example 0. The harmonic map $h : \mathbf{C} \to \mathbf{R}$ defined by $h = \operatorname{Re} z^2 = x^2 - y^2$ has Hopf differential $\Phi = z^2$. The horizontal foliation for Φ is the collection of hyperbolas which are asymptotic to the lines $\{y = \pm x\}$ (with those lines being the singular leaves). The (dual) real tree to this singular measured foliation is the set consisting of the pair of coordinate axes with distance defined by $d((0,0),(x,0)) = \int_0^x \sqrt{\Phi} = \frac{1}{2}x^2$ or $d((0,0),(0,y)) = \frac{1}{2}y^2$.

Now, there is a harmonic map $w : \mathbf{C} \to \mathbf{E}^2$ with Hopf differential $\Phi(w) = -z^2 dz^2$, for example $w = z - \frac{1}{3}\bar{z}^3$. Because the Hopf differentials of h and w are negatives of each other, the map $f = (w, h) : \Omega \subset \mathbf{C} \to \mathbf{E}^3$ is a minimal graph, for Ω a small domain containing the origin. $\square$

Note that this example can be approached from several perspectives: one can construct the minimal surface beginning either with the harmonic map $w : \Omega \to$

$\mathbf{E}^2$ between surfaces, the harmonic "height" function $h : \Omega \to \mathbf{R}$, or even the harmonic map $\pi = p \circ h : \Omega \to (T, d)$ from the domain to the real tree. We will explore each of these perspectives below.

Finally, observe that the harmonic function $h : \Omega \to \mathbf{R}$ factors as $i \circ \pi : \Omega \to \mathbf{R}$, where $\pi : \Omega \to (T, d)$ is the harmonic map from Ω to the tree and $i : T \to \mathbf{R}$ is the vertex-folding map of the pair of coordinate axes to the reals $\mathbf{R}$.

3.2. **Minimal suspensions and minimal graphs.** We conclude this section by summarizing our construction and proving a converse.

Proposition. (i) *A harmonic map $w : \Omega \to N$ between a simply connected Riemann surface and a Riemannian manifold N gives rise to a minimal graph $f = (w, \pi) : \Omega \to N \times T$, where T is a real tree.*

(ii) *If in the above, the tree T admits a complete vertex folding $i : T \to \mathbf{R}$, then the composition $i \circ f : \Omega \to N \times \mathbf{R}$ is a minimal graph over N.*

(iii) *Conversely, a minimal graph $F : \Omega \to N \times \mathbf{R}$ over N factors as $i \circ (w, \pi) : \Omega \to N \times T \to N \times \mathbf{R}$.*

Proof. We are left to prove the third statement. Certainly the projection $\pi : F(\Omega) \to N$ is harmonic, so we need only explain why the resulting tree T admits a vertex fold to the reals $\mathbf{R}$. But observe that the map $F(\Omega) \to \mathbf{R}$ is harmonic with Hopf differential the negative of that of π: in particular, the tree T is obtained as the leaf space of the *minimal* (rather than the maximal, in our basic construction) stretch foliation. But this minimal stretch foliation integrates $\ker d\pi$, or, in terms of the coordinate X^3 for the second factor in $N \times \mathbf{R}$, integrates $\ker dX^3$. Certainly, the vertices of T are then the images of the locations of horizontal tangent spaces, and thus the local folds of the tree dual to the level sets of $F(\Omega)$ are a map from $T \to \{X^3\}$. The X^3 heights then well-define a global assignment of X^3-values to T, and so the local vertex fold isometries then extend to a global isometry $T \to \mathbf{R}$, as required. $\square$

Remarks. (i) From the proposition, we see that the class of harmonic homeomorphisms into $\mathbf{H}^2$ differs from the class of minimal graphs over $\mathbf{H}^2$ on whether the associated tree folds to $\mathbf{R}^2$ or not.

(ii) Of course the associate surfaces to such a graph corresponding to Φ arise as the surfaces corresponding to $\epsilon^{i\theta}\Phi$.

3.3. **Minimal suspensions and branched minimal multigraphs.** If the tree $T = T_\Phi$ associated to the Hopf differential Φ of the harmonic map has vertices of odd-ordered valence, then the minimal suspension cannot fold even locally. Nevertheless, if the underlying harmonic map is a homeomorphism, the construction described above extends to construct a branched minimal multigraph over the image of the map, with branch points over the images of the zeroes of Φ of odd valence. In particular, here the point is that the projection map $p : \Omega \to \mathbf{R}$ from the domain Ω to the reals has the form $p(z) = \mathrm{Re} \int^z \sqrt{\Phi}$, so near a zero of odd order, the two branches of $\sqrt{\Phi}$ define a connected branched multigraph over Ω; by way of contrast, near a zero of even order, each branch (of the two possible branches) defines a smooth graph over Ω, so the two-fold covering is disconnected. With this in mind, let $w : \Omega \to N^2$ be a harmonic homemorphism from a simply connected domain $\Omega \subset \mathbf{C}$ to a two-manifold N. Then define $\hat{p} = \mathrm{Re} \int^z \pm\sqrt{\Phi}$ to be either a connected two-sheeted multigraph over Ω, or in case Φ has only even order zeroes, choose one of the two branches of $\hat{p}$. Then $f = (w, \hat{p})$ defines either a minimal graph over N^2 or a two-sheeted minimal branched surface with branch points over the images of the odd order zeroes of the Hopf differential Φ of w. Our discussion then of the minimal suspension then extends to show

Theorem. Let $\tilde{w} : \Omega \to N^2$ be a harmonic homeomorphism between a simply connected domain $\Omega \subset \mathbf{C}$ to a 2-manifold N. Then there is a branched minimal immersion $\tilde{w}^* : \Omega \to N^2 \times \mathbf{R}$ which is a two-valued minimal multigraph over $\tilde{w}(\Omega)$, branched over the images of the odd order zeroes of the Hopf differential of $\tilde{w}$ and horizontal there, where this multigraph is obtained through the construction above. Conversely, all two-sheeted minimal multigraphs where the images of the branch points are horizontal are so obtained.

Proof. Again, only the converse is left to discuss. If the two-sheeted multigraph admits a decomposition into a union of two smooth graphs, then we are in the situation of the previous proposition. Otherwise, we look locally near a branch point, and consider a sheet of the multigraph over a neighborhood of the branch point, i.e. a discontinuous graph over the neighborhood which is smooth away from a ray emanating from the branch point. The projection of this graph to the domain $\tilde{w}(\Omega)$ is harmonic, and because the branch point has a multiplicity two horizontal tangent plane, we see that the Hopf differential of the projection map has a zero at the branch point. Morevoer, because the suspension of that projection will agree with the original surface on the closure of the discontinuous graph, it will extend to a branched multigraph in the same way as the original multigraph. The result follows from the uniqueness of the analytic continuations of all of the local multigraphs. □

4. Examples derived from minimal suspensions of harmonic maps to $\mathbf{H}^2$

In this section, we suspend some well-known examples of harmonic maps to $\mathbf{H}^2$ to obtain minimal graphs over $\mathbf{H}^2$.

In [NR02], the question of the existence of minimal graphs in $\mathbf{H}^2 \times \mathbf{R}$ over $\mathbf{H}^2$ is raised and answered by Jenkins-Serrin and Plateau constructions. In view of the construction from [Wol98] above, other examples are abundant.

Example 1a. Let $\mathcal{R}$ be a closed Riemann surface and let $\alpha \neq 0$ be a holomorphic 1-form on $\mathcal{R}$. Then α^2 is a holomorphic quadratic differential on $\mathcal{R}$ and so [Wol89] asserts that there is a unique harmonic diffeomorphism $w : \mathcal{R} \to (M, \rho)$ in the homotopy class of a diffeomorphism $\mathcal{R} \to M$ between $\mathcal{R}$ and a surface M equipped with a (not conformally equivalent) hyperbolic metric ρ. Upon lifting to the universal cover, we obtain a harmonic map $\tilde{w} : \mathbf{D}^2 \to \mathbf{H}^2$ with Hopf differential $\tilde{\alpha}^2$, the lift of α^2. The tree associated to a squared differential clearly folds to $\mathbf{R}$, and so we obtain a minimal graph over $\mathbf{H}^2$.

Now, the generic 1-form has leaf structure which is rather wild; in particular, we can find examples of such differentials where the leaves are dense on $\mathcal{R}$, or minimal in the (dynamical) sense that almost every leaf is dense. This has the effect that the level set of a height (a leaf of the maximal stretch foliation, or where $\operatorname{Im} \alpha = 0$) comes within distance ϵ of branch points of the foliation not just infinitely often, but also within a uniformly bounded time intervals. (This last statement follows if we construct a train track supporting the foliation with all of the branches of the track split sufficiently so as to admit a transverse measure of less than ϵ. (See [PH92] for more details.)) Now, near such branch points, there are distinct components of the level set (represented by leaves whose images on the tree are identified by the vertex fold) whose paths on the Riemann surface have very different dynamics. As a result, the boundary behavior of the minimal graph is highly oscillatory: because each level set on the surface comes within ϵ of a different component of the same level set after at most a uniformly bounded distance, the number of such components of a level set grows exponentially with the length of a single component. Thus, on a circle of hyperbolic radius r, the number of points on which a particular height is taken grows exponentially with r.

Example 1b. We may also begin from a different direction, taking the tree as the basic object instead of the harmonic map as above. For example, to engineer the oscillatory behavior described above, let us take the tree T to be the Cayley graph of the free group $\mathbf{Z} * \mathbf{Z} = \langle a, b\rangle$ on two generators using the generators $\{a, a^{-1}, b, b^{-1}\}$ as the elements at distance one from the identity. Of course, the tree T has the basic "quadruple antenna" shape to it, with each vertex having valence 4 and all vertices at distance one from their four closest neighbors.

Our goal is to construct a harmonic map from the disk D^2 to the tree T. To this end, observe that $\mathbf{Z} * \mathbf{Z}$ is the fundamental group of a one-holed torus $T^2 - \{pt\}$. Indeed, if we tile the hyperbolic disk with ideal squares (with one of the squares having ideal vertices at $\{\pm 1, \pm i\}$), then the incidence graph of the tiling is (up to a scaling) an isometric image of T, and the action of $\mathbf{Z} * \mathbf{Z}$ on this graph is as the covering group of the disk over the one-holed torus. It is this enough to find a harmonic map from the (hyperbolic) once-punctured torus to the quotient of the incidence graph by the group, i.e. the bouquet of two circles, or more particularly, the union $\mu \cup \lambda$ of a meridian μ and longitude λ on the torus (which share a single point).

To define such a map, in each ideal square of the ideal square tiling connect an interior point to the four ideal vertices by rays. (In the tile described above with ideal vertices $\{\pm 1, \pm i\}$, these rays would be the ray from the origin along the coordinate axes.) The complementary regions of these rays may then be foliated by arcs connecting the punctures. The leaf space then descends to be the given meridian and longitude of the torus. Let $w_0 : T^2 - \{pt\} \to \mu \cup \lambda$ be the map which sends each leaf on $T^2 - \{pt\}$ to its representative on $\mu \cup \lambda$. This map w_0 can be taken to have finite energy, and thus it is easy to see (see e.g. [Wol95]) that there is a harmonic representative w in this homotopy class of maps; moreover, this harmonic map $w : T^2 - \{pt\} \to \mu \cup \lambda$ lifts to a harmonic map $\tilde{w} : D^2 \to T$ whose Hopf differential has maximal stretch (measured) foliation dual to the tree $(T, 2d)$.

Now, we constructed the tree T to admit vertex folds inducing a map $i : T \to \mathbf{R}$; because i is an isometry, the vertices are mapped onto the integers, and the open edges (components of the complement of the vertices) are mapped onto the intervals between the integers. It is an elementary consequence of the folding method that the preimage of any integer occurs with a frequency which is exponential in the distance along the tree. Consequently, as the tree isometrically embeds in the hyperbolic disk, the minimal suspension $\tilde{w}$ takes on a height h on a disk of hyperbolic radius r with a frequency that is exponential with r.

Example 1c. As we noted in section 1, the thesis of T.Y.-H. Wan, [Wan92] provides a generalization of the result in [Wol89], i.e. that for φ a derivative of a Bloch function (see [Pom92] for details: g is a Bloch function if $|g'(z)|(1-r^2)^2 < \infty$ on $\mathbf{D}^2$), there is a unique harmonic map $w(\varphi) : \mathbf{D}^2 \to \mathbf{H}^2$ with Hopf differential φdz^2. This gives an infinite dimensional Banach (quotient) space of examples by following the procedures above. Here we restrict to the subspace of Bloch functions whose zeroes are of odd order.

Example 2. In [Wol91] we found an example of a harmonic map from a hyperbolic cylinder to itself which was not quasi-conformal but was the identity on the boundary of the cylinder. Li-Tam [LT93] later independently found the same harmonic map, in the more natural setting of a harmonic map from the upper-half-plane model of $\mathbf{H}^2$ to itself which takes on the values of the identity on $\mathbf{R}$.

In the upper-half-plane model, the map has the form $(x, y) \mapsto (x, \psi(y))$ and so the Hopf differential is invariant under horizontal translations. (Here $\psi(y) = \frac{1}{\sqrt{c}} \sinh[\sqrt{c}(y-1) + \sinh^{-1} \sqrt{c}]$ for $c \geq 0$.) In particular, the Hopf differential has no zeroes, and so the associated tree has no vertices. Moreover because of the translation invariance of the map in the x-direction, the Hopf differential Φ (which we check to be non-trivial when $c \neq 0$) has maximal stretch foliation given by the vertical lines in the upper-half-plane. The transverse measure, being translation invariant in the x-direction, is proportional to Lebesgue measure on the x-axis.

a) Thus the suspended harmonic map is a minimal graph on $\mathbf{H}^2 \times \mathbf{R}$ with boundary values growing linearly over $\mathbf{R}$ and taking boundary values $\pm\infty$ at the point at infinity.

b) The hyperbolic metric pulls back to the Riemannian surface $(M, \sigma|dz|^2)$ via the formula

$$w^* ds^*_{\mathbf{H}} = \Phi dz^2 + \sigma(\mathcal{H} + \frac{|\Phi|^2}{\sigma^2 \mathcal{H}}) dz d\bar{z} + \overline{\Phi} d\bar{z}^2 \tag{1}$$

where $\mathcal{H}$ satisfies the equation

$$\Delta \log \mathcal{H} = \mathcal{H} - 2\frac{|\Phi|^2}{\sigma^2 \mathcal{H}} + 2K(\sigma) \tag{2}$$

where $K(\sigma) = -1$ everywhere. Thus, upon replacing Φ by $-\Phi$ in (1), and using that Φ appears in (2) only as a modulus, we see that such a modified version of (1) defines a pulled back hyperbolic metric on M. In addition, since the original map was a diffeomorphism, the identity map $\mathrm{id} : (M, \sigma) \to (M, -\Phi dz^2 + \sigma(\mathcal{H} + \frac{|\Phi|^2}{\sigma^2 \mathcal{H}}) dz d\bar{z} - \overline{\Phi} d\bar{z}^2)$ defines a harmonic diffeomorphism from the Riemann surface $\mathcal{R}$ underlying (M, σ) onto $\mathbf{H}^2$.

This time, however, the maximal stretch foliation are the horizontal lines, and since we can compute that $\int_0^{y_0} \sqrt{|\Phi|} dy$ is finite for all y_0, we find that the tree is a half-line, with an endpoint corresponding to the image of the real axis. The minimal surface then has constant boundary values over every point on S^1_∞ except for a single point: at that point, all positive boundary values are taken on say by taking the limits over the horocycles (which are the level sets here) in $\mathbf{H}^2$.

c) Interpolating between these alternatives are the harmonic maps corresponding to $e^{i\theta}\Phi$.

In the language of [HET05], the surfaces of examples (a) and (b) are conjugate, while those of example (c) are the associate surfaces for this example.

Example 3. There are two types of complete hyperbolic metrics on Riemann surfaces of genus $g > 0$ which have punctures. The first type of complete metric is of finite volume and has a metric near the puncture isometric to a neighborhood of the origin in the punctured plane equipped with the metric $ds^2 = (|z| \log |z|)^{-2} |dz|^2$. The second type of metric has infinite volume, and the homotopy class of the link of the puncture is represented by a simple closed geodesic of positive length: in Fermi coordinates, the metric is given by $ds^2 = dr^2 + \cosh^2 r dt^2$, where r measures distance from the geodesics. (Alternatively, the neighborhood is given by the metric in the hyperbolic plane as follows. Choose two complete geodesics Γ_1 and Γ_2 in $\mathbf{H}^2$ which have four distinct endpoints on the circle S^1_∞ at infinity. There is a shortest geodesic γ connecting the pair Γ_1 and

Γ_2, and if we identify Γ_1 and Γ_2 then the region bounded by the arcs $\Gamma_1 = \Gamma_2$ and γ defines a neighborhood isometric (up to choosing Γ_1 and Γ_2 at an appropriate distance apart) to the neighborhood of a puncture in the infinite volume case.

In [Wol91], we demonstrated the existence of a harmonic map which mapped a punctured surface with a finite volume hyperbolic metric homeomorphically into a surface topologically equivalent to itself, but equipped with a hyperbolic metric of infinite volume. The map is homotopically equivalent to a surjective homeomorphism, but had the feature of having image within the convex region bounded by the closed geodesics homotopic to the punctures. In such a map, the horocycle of distance d from a fixed point has image of distance $O(e^{-c\sqrt{d}})$ from the geodesic.

Thus the lift of such a map to the universal cover would map the hyperbolic plane onto a region bounded by a countable collection of geodesics which do not share endpoints on the circle of infinity. The accumulation set on S^1_∞ of this image set is then a Cantor set: this is because it is a closed set on S^1_∞ complementary to the orbit by the (countably infinite) surface group of an interval on S^1_∞.

Now, such a map exists for each meromorphic quadratic differential on a compact Riemann surface with second order poles at the punctures with non-trivial real residues there, so we can certainly choose such a map whose Hopf differential is a square of an abelian differential. Thus the minimal suspension of such a map folds and we get a minimal graph in $\mathbf{H}^2 \times \mathbf{R}$ which is of similar character to a Jenkins-Serrin surface: in particular, the graph is asymptotic to vertical planes over the (countably infinite) collection of infinite geodesics in $\mathbf{H}^2$ described above. It does, however, have an accumulation set over a Cantor set on S^1_∞.

It is straightforward to create examples (say by using surfaces with either one or two holes) where the graph has height always tending to $+\infty$ over the lifts of the boundary geodesics and examples where the graph has height tending to both $+\infty$ and $-\infty$, depending on which boundary geodesic is chosen.

5. Examples derived from harmonic maps of $\mathbf{C}$ to $\mathbf{H}^2$ and the Jenkins-Serrin construction

Au-Tam-Wan [ATT02] have made a beautiful study of the harmonic maps $w(\Phi) : \mathbf{C} \to \mathbf{H}^2$ with Hopf differential Φ whose growth is small. In particular, none of the harmonic maps are surjective diffeomorphisms onto hyperbolic space, but instead are diffeomorphisms onto a region bounded by hyperbolic geodesics. In this section, we discuss the minimal graphs related to some of their simpler examples (see also [Han96], [HTTW95] for important ingredients in this study.)

Example 5. Consider a polynomial in the plane all of whose zeroes are even order, or equivalently, which is a square. Han [Han96] shows that $w(\Phi)$ has image bounded by $\deg(\Phi)+2$ geodesics (i.e. the convex hull of $\deg(\Phi)+2$ distinct points on S^1_∞); moreover, Han-Tam-Treibergs-Wan [HTTW95] show that, when the minimal suspension is a complete surface, the image $w(\Phi)$ is such a convex hull only when Φ is a polynomial. (To be precise, their condition is almost this one: they require that, in the notation of (1), the metric $\mathcal{H}|dz|^2$ should be complete. The induced metric on the minimal suspension is $(\mathcal{H} + |\Phi|^2/\sigma^2\mathcal{H} + 2|\Phi|^2)|dz|^2$ which is complete if and only if $\mathcal{H}|dz|^2$ is complete, the equivalence following by elementary algebra.)

It is immediate that the tree T_Φ associated to Φ is a planar tree (with finite total valence and $\deg(\Phi) + 2$ ends): hence the ends admit a natural ordering. Moreover, the fold of the minimal suspension then takes the values $+\infty$ and $-\infty$ alternately on the vertical planes over the geodesics. As a consequence, these maps are examples of the Jenkins-Serrin minimal surfaces in $\mathbf{H}^2 \times \mathbf{R}$ found by Nelli-Rosenberg [NR02].

Indeed, the result of [HTTW95] proves the converse: a Jenkins-Serrin minimal surface in a domain bounded by the convex hull of an *even* number of vertical lines over S^1_∞ is the fold of a minimal suspension of a polynomial as its induced metric is complete and its projection is a harmonic map.

Thus the set of Jenkins-Serrin graphs with $2m + 2$ vertical planar asymptotes in $\mathbf{H}^2 \times \mathbf{R}$ is parametrized by the collection of degree m (complex) polynomials on $\mathbf{C}$ or alternatively by the collection of $2m + 2$-ended planar trees all of whose vertices have even valence, and a complex scaling constant. (Much of this has also been recently observed by Fernández-Mira [FM05].)

6. A (non-orientable) minimal suspension by a tree which folds locally but not globally

While this note was being prepared, some articles ([HET05] and [Dan06]) with a similar theme have appeared: these focus their attention not on trees which admits folds but on Hopf differentials whose zeroes are even order (or which have no zeroes). Indeed, in all of our previous examples, the folding of the relevant tree could be ignored in favor of the local analytic condition of the Hopf differential having zeroes of even order as the geometric condition of the tree admitting local folds. As noted above in Section 2, this is because on a simply connected domain, a local branch of the square root of the Hopf differential admits a well-defined global extension.

In this section, we discuss an example where no such global folding is possible, but for which a partial folding results in a non-orientable example.

Example 6a. We being by recalling the example of Masur-Smillie [MS93] of a genus two Riemann surface with a holomorphic quadratic differential Φ with even order zeroes which is not a square of a linear differential. We build the surface and the differential simultaneously as follows. Let C_0 be a cylinder of circumference 1, and let C_{-1} and C_{+1} denote cylinders each of circumference 1/2. Each of the two components $\partial_+ C_0$ and $\partial_- C_0$ of the boundary ∂C_0 of C_0 is a circle of circumference 1. Identify a single pair of antipodal points on each of those curves so that each component is now a bouquet, say ∂_+^* and ∂_-^*, of two circles of circumference $\frac{1}{2}$. Then glue C_{+1} to ∂_+^* and C_{-1} to ∂_-^* so that, away from the two preimages of the vertex, the boundary map is a local isometry and the arcs on C_{+1} and C_{-1} which leave the vertex of the bouquet orthogonally to ∂C_+ meets the same vertex on the other side of the cylinders C_+ and C_- (i.e. the identifications do not "twist" the cylinders C_{+1} and C_{-1}).

It is easy to check that the holomorphic quadratic differentials dz^2 on C_{-1}, C_0, and C_{+1} extend to give a holomorphic quadratic differential on the identification space outside of the vertices of the bouquets of circles; those vertices are removable singularities for the differential, and the foliation structure implies that the differential has second order zeroes at those points. Let $\mathcal{R}$ denote the Riemann surface underlying this example, and Φ the constructed quadratic differential.

Consider the foliation of Φ arising from the foliations of the cylinders by straight line segments orthogonal to the boundary. By construction, this foliation is not orientable; to see this, consider the leaves near the leaf connecting the zeroes. Thus Φ is not the square of a linear differential, despite being such a square locally. Indeed, one sees that the leaf space is $\mathbf{P}^1$.

As Φ is a holomorphic quadratic differential on $\mathcal{R}$, there is a (non-conformal) harmonic map $w_\Phi : \mathcal{R} \to M$ from $\mathcal{R}$ to a hyperbolic surface M of genus two. Because the leaf space is a manifold and the minimal suspension construction is local, the minimal suspension of w_Φ produces a (conformal) minimal map $f : \mathcal{R} \to M \times \mathbf{RP}^1$ which is not orientable.

In this case, the lift to a minimal graph in $\mathbf{H}^2 \times \mathbf{R}$ is of course, orientable, but is equivariant with respect to an index two subgroup of $\pi_1\mathcal{R}$, but not equivariant with respect to $\pi_1\mathcal{R}$.

Example 6b. In the construction, we may twist the cylinder C_0 about an interior circle through a distance ℓ. If ℓ is rational, we obtain a minimal suspension in the manifold $M \times \mathbf{RP}^1$, but if ℓ is irrational, the leaf space is not Hausdorff, so there is no group acting on the minimal suspension of the lift $\tilde{w}_\Phi$ of w_Φ to allow the quotient surface to live in a manifold.

References

[ATT02] T. Au, L.-F. Tam, and T.Wan. Hopf differentials and the images of harmonic maps. *Comm. Anal. Geom.*, 10:515=573, 2002.

[Dan06] B. Daniel. The gauss map of minimal surfaces in the heisenberg group. preprint, 2006.

[FM05] I. Fern/'andez and P. Mira. Harmonic maps and mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$. preprint, 2005.

[FW01] Benson Farb and Michael Wolf. Harmonic splittings of surfaces. *Topology*, 40(6):1395–1414, 2001.

[Han96] Zheng-Chao Han. Remarks on the geometric behavior of harmonic maps between surfaces. In *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, pages 57–66. A K Peters, Wellesley, MA, 1996.

[HET05] L. Hauswirth, R. Sa Earp, and E. Toubiana. Associate and conjugate minimal immersions in $m \times r$. preprint, 2005.

[HTTW95] Z.-C. Han, L.-F. Tam, A. Treibergs, and T. Wan. Harmonic maps from the complex plane into surfaces wthi nonpositive curvature. *Comm. Anal. Geom.*, 3:85–114, 1995.

[LT93] P. Li and L.-F. Tam. Uniqueness and regularity of proper harmonic maps ii. *Indiana J. of Math.*, 42:591–635, 1993.

[MIR05] W. H. Meeks III and H. Rosenberg. The theory of minimal surfaces in $M \times \mathbb{R}$. *Comment. Math. Helv.*, 80:811–858, 2005.

[MO93] J. Morgan and J.P. Otal. Relative growth rates of closed geodesics on a surface under varying hyperbolic structures. *Comm. Math. Helv.*, 68:171–208, 1993.

[MR04] William H. Meeks, III and Harold Rosenberg. Stable minimal surfaces in $M \times \mathbb{R}$.*J. Differential Geom.,* 68(3):515C534, 2004.

[MS93] Howard Masur and John Smillie. Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms. *Comment. Math. Helv.*, 68(2):289–307, 1993.

[NR02] Barbara Nelli and Harold Rosenberg. Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *Bull. Braz. Math. Soc. (N.S.)*, 33(2):263–292, 2002.

[PH92] R. C. Penner and J. L. Harer. *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992.

[Pom92] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[Ros02] Harold Rosenberg. Minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$. *Illinois J. Math.*, 46(4):1177–1195, 2002.

[Sam78] J. H. Sampson. Some properties and applications of harmonic mappings. *Ann. Sci. École Norm. Sup. (4)*, 11(2):211–228, 1978.

[Sko96] R. Skora. Splittings of surfaces. *Journal A.M.S.*, 9:605–616, 1996.

[Wan92] T. Y.-H. Wan. Constant mean curvature surface, harmonic maps, and universal teichmuller space. *J. Differential Geom.*, 35:643–657, 1992.

[Wol89] M. Wolf. The teichmüller theory of harmonic maps. *J. Diff. Geometry*, 29:449–479, 1989.

[Wol91] M. Wolf. Infinite energy harmonic maps and degeneration of hyperbolic surfaces in moduli space. *J. Diff. Geometry*, 33:487–539, 1991.

[Wol95] M. Wolf. On the existence of jenkins-strebel differentials using harmonic maps from surfaces to graphs. *Ann. Acad. Sci. Fenn.*, 20:269–278, 1995.

[Wol96] M. Wolf. On realizing measured foliations via quadratic differentials of harmonic maps to $\mathbb{R}$-trees. *J. D'Analyse Math.*, 68:107–120, 1996.

[Wol98] M. Wolf. Measured foliations and harmonic maps of surfaces. *J. Diff. Geometry*, 49:437–467, 1998.

Michael Wolf
Department of Mathematics
Rice University Houston, TX 77005
E-mail: mwolf@math.rice.edu