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Perfecting the Nearly Perfect

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Abstract: We introduce a natural variant of the notion of nearly perfect complex. We show that this variant gives rise to canonical perfect complexes and prove several useful properties of this construction (including additivity of the associated Euler characteristics on suitable exact triangles). We then apply this approach to complexes arising from the étale cohomology of $\mathbb{G}_m$ on arithmetic surfaces and discuss links to Lichtenbaum's theory of Weil-étale cohomology.

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1. Introduction

In [8] Chinburg, Kolster, Pappas and Snaith introduced a rather ingenious generalization of the Euler characteristic of a perfect complex of modules for the group ring of a finite group. They defined a notion of 'nearly perfect complex' (or 'npc' for short) and associated to each such npc a canonical Euler characteristic. They then described applications of their approach to complexes arising from the cohomology of the multiplicative group $\mathbb{G}_m$ on arithmetic surfaces.

In [7] Köck, Snaith and the present author used an explicit mapping cone construction to both reinterpret and refine the Euler characteristic of [8].

In this article we further refine the approach of [7]. To this end, in §2 we introduce a natural variant of the notion of nearly perfect complex that is much more amenable to working in derived categories. We show that to each npc one can associate a 'derived nearly perfect complex' (or 'dnpc') and that in many cases of arithmetic interest this association is canonical. We then construct a functor from the category of dnpc's to a suitable derived category of perfect complexes

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and we show that, for any given dnpc, the Euler characteristic of the associated perfect complex is equal to the Euler characteristic (in the sense of [8]) of the underlying npc. This very natural approach has several technical advantages over the methods used in both [8] and [7]. In particular, it allows us to show that the Euler characteristic introduced by Chinburg et al is additive on suitable exact triangles and hence resolves a problem explicitly posed by Snaith in [14, Question 2.3.21].

In §3 we present some arithmetic applications of the above approach. We first use the methods of §2 to show that *canonical* perfect complexes underly all of the arithmetic results that are obtained by Chinburg et al in [8]. We then discuss links between our approach and the recent efforts of Lichtenbaum to define a cohomology theory ('Weil-étale cohomology') that is much better suited than étale cohomology for the purposes of formulating special value conjectures. In particular, by combining the methods developed in §2 with computations of Flach and the present author from [4] we are able to construct canonical perfect complexes which, for the purposes of formulating both special value conjectures and arithmetic duality theorems, provide adequate substitutes for the 'Weil-étale cohomology with compact support of $\mathbb{Z}$' and 'Weil-étale cohomology of $\mathbb{G}_m$' on schemes of the form Spec $\mathcal{O}$ with $\mathcal{O}$ the ring of algebraic integers in a number field. This is of interest since, whilst there is still no working definition of the Weil-étale topology in this setting, the canonical perfect complexes constructed here provide an adequate testing ground for the relevant aspects of Lichtenbaum's philosophy of leading term conjectures and also make clear links between this philosophy and the equivariant Tamagawa number conjecture that was formulated by Flach and the present author in [5] (see §3.3 for further details in this regard).

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2. Derived nearly perfect complexes

2.1. **Notation.** We fix a finite group G and always use the phrase 'G-module' to mean 'left G-module'. We write $\mathfrak{D}$ for the derived category of the abelian category of G-modules. We also write $\mathfrak{D}^{\mathrm{ct}}$, resp. $\mathfrak{D}^{\mathrm{perf}}$, for the full triangulated subcategory of $\mathfrak{D}$ consisting of those complexes which are isomorphic to a bounded complex of cohomologically trivial G-modules, resp. to a bounded complex of finitely generated projective G-modules. If X is an object of $\mathfrak{D}$ and i is an integer, then we let $H^i(X)$ denote the cohomology of X in degree i.

For any G-module A we write $A_{\rm div}$ for the submodule of all divisible elements of A and set $A_{\rm codiv} := A/A_{\rm div}$. We also let $A_{\rm tor}$ denote the $\mathbb{Z}$-torsion submodule of A and set $A_{\rm tf} := A/A_{\rm tor}$. We write A^*, $A^\wedge$ and $A^\vee$ for the groups $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$, $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ respectively (each endowed with the natural contragredient (left) G-action). We write κ_A and π_A for the homomorphisms $A^* \to A^\wedge$ and $A^\wedge \to A^\vee$ that are induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$ and projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ respectively. We often use the fact that if A is $\mathbb{Z}$-torsion-free, then one has an exact sequence

$$0 \to A^* \xrightarrow{\kappa_A} A^\wedge \xrightarrow{\pi_A} A^\vee \to 0.$$

2.2. **The definitions.** The following definition of 'nearly perfect complex' is due to Chinburg, Kolster, Pappas and Snaith [8, §2].

Definition 2.1. *A* nearly perfect complex (or 'npc') of G-modules *is a triple* $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \{\tau_i\}_{i\in\mathbb{Z}})$ *in which*

- $C^\bullet$ *is a bounded complex of cohomologically trivial G-modules for which* $H^i(C^\bullet)_{\rm codiv}$ *is finitely generated in each degree* i*,*
- *for each integer* i*,* L_i *is a torsion-free finitely generated G-module,*
- *for each integer* i*,* τ_i *is an isomorphism of G-modules* $L_i^\vee \xrightarrow{\sim} H^i(C^\bullet)_{\rm div}$.

We shall say that an npc of G-modules C as above is strict *if in addition one has*

- *for each integer* i*, either* $H^i(C^\bullet)_{\rm div} = 0$ *or* $H^{i-1}(C^\bullet)_{\rm codiv}$ *is finite.*

The notion of 'strictness' does not occur in [8]. However, we will see that all of the arithmetic examples considered in loc. cit. are in fact strict (cf. §3.1).

The key observation which we wish to make in this section is that data such as the above is much better studied in derived categories. To this end, we therefore introduce the following variant of the notion of npc.

Definition 2.2. *A* derived nearly perfect complex (or 'dnpc') of G-modules *is a triple* $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ *in which*

- $C^\bullet$ *is an object of* $\mathfrak{D}^{\rm ct}$ *for which* $H^i(C^\bullet)_{\rm codiv}$ *is finitely generated in each degree* i*,*
- *for each integer* i*,* L_i *is a torsion-free finitely generated G-module,*
- $\tau \in \mathrm{Hom}_{\mathfrak{D}}(\bigoplus_{i\in\mathbb{Z}} L_i^\wedge[-i], C^\bullet)$ *and in each degree* i *one has* $H^i(\tau) = \tau_i \circ \pi_{L_i}$ *where* τ_i *is an isomorphism of G-modules* $L_i^\vee \xrightarrow{\sim} H^i(C^\bullet)_{\rm div}$.

For any such C we set $C^{\rm ud} := \bigoplus_{i\in\mathbb{Z}} L_i^\wedge[-i]$.

It is clear that each dnpc of G-modules $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ gives rise to an npc of G-modules $(\tilde{C}^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \{\tau_i\}_{i\in\mathbb{Z}})$ where $\tilde{C}^\bullet$ is any bounded complex of cohomologically trivial G-modules which is isomorphic in $\mathfrak{D}$ to $C^\bullet$. In any such case we shall say that the dnpc and npc are 'associated'. The following result shows conversely that every npc of G-modules arises in this way and further that to each strict npc one can associate a *canonical* dnpc.

Lemma 2.3. *If $C = (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \{\tau_i\}_{i\in\mathbb{Z}})$ is an npc of G-modules, then for any object D of $\mathfrak{D}$ the natural 'passage to cohomology' homomorphism*

$$\mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{ud}}, D) \to \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_G(L_i^\wedge, H^i(D)_{\mathrm{div}}) \tag{1}$$

is surjective. Further, this map is bijective if either $H^i(D)$ is uniquely divisible in each degree i or both $D = C^\bullet$ and C is strict. In particular, if C is strict, then it is associated to a canonical dnpc of G-modules C^{der}.

Proof. For each integer i we set $X^i := H^i(D)$. Then it suffices to prove that for each i the natural map $h : \mathrm{Hom}_{\mathfrak{D}}(L_i^\wedge[-i], D) \to \mathrm{Hom}_G(L_i^\wedge, X^i_{\mathrm{div}})$ is surjective, resp. bijective if either X^{i-1} is uniquely divisible or both $D = C^\bullet$ and X^{i-1}_{codiv} is finite.

To prove this we use the fact that for each object Y of $\mathfrak{D}$ there is a spectral sequence

$$E_2^{p,q} = \prod_{i\in\mathbb{Z}} \mathrm{Ext}^p_G(H^i(Y), H^{q+i}(D)) \Rightarrow H^{p+q}(R\,\mathrm{Hom}(Y, D)) \tag{2}$$

(cf. [16, III, 4.6.10]). Indeed, by combining this spectral sequence in the case $Y = L_i^\wedge[-i]$ together with the canonical isomorphism $H^0(R\,\mathrm{Hom}(Y, D)) \cong \mathrm{Hom}_{\mathfrak{D}}(Y, D)$ and the fact that $\mathrm{Ext}^p_G(L_i^\wedge, -) = 0$ for each $p > 1$ (since the G-module $L_i^\wedge$ is uniquely divisible and hence cohomologically trivial and so has a projective resolution of length one (cf. [1, Chap. VI, Th. (8.12)])), we obtain a short exact sequence

$$0 \to \mathrm{Ext}^1_G(L_i^\wedge, X^{i-1}) \to \mathrm{Hom}_{\mathfrak{D}}(L_i^\wedge[-i], D) \xrightarrow{h} \mathrm{Hom}_G(L_i^\wedge, X^i_{\mathrm{div}}) \to 0. \tag{3}$$

This implies that h is surjective, resp. bijective if $\mathrm{Ext}^1_G(L_i^\wedge, X^{i-1}) = 0$. Now if X^{i-1} is uniquely divisible, then it is a $\mathbb{Q}[G]$-module. As such, X^{i-1} is injective and hence, since $\mathbb{Z}[G] \to \mathbb{Q}[G]$ is flat, X^{i-1} is an injective $\mathbb{Z}[G]$-module and so $\mathrm{Ext}^1_G(L_i^\wedge, X^{i-1}) = 0$, as required. Hence it suffices to show that if $D = C^\bullet$ and X^{i-1}_{codiv} is finite, then $\mathrm{Ext}^1_G(L_i^\wedge, X^{i-1}) = 0$.

But if X^{i-1}_{codiv} is finite, then $\mathrm{Ext}^1_G(L_i^\wedge, X^{i-1}_{\mathrm{codiv}})$ vanishes because it is both uniquely divisible (since $L_i^\wedge$ is) and of finite exponent (since X^{i-1}_{codiv} is). The long exact sequence of $\mathrm{Ext}^\bullet_G(L_i^\wedge, -)$ groups of the tautological exact sequence $0 \to X^{i-1}_{\mathrm{div}} \to X^{i-1} \to X^{i-1}_{\mathrm{codiv}} \to 0$ thus implies that it suffices to prove $\mathrm{Ext}^1_G(L_i^\wedge, X^{i-1}_{\mathrm{div}}) = 0$. We recall next that X^{i-1}_{div} is isomorphic to $L^\vee_{i-1}$ (via τ_{i-1}). Since $\mathrm{Ext}^2_G(L_i^\wedge, -) = 0$ and also $\mathrm{Ext}^1_G(-, L^\wedge_{i-1}) = 0$ (because $L^\wedge_{i-1}$ is an injective $\mathbb{Z}[G]$-module), the vanishing of $\mathrm{Ext}^1_G(L_i^\wedge, X_{\mathrm{div}}^{i-1})$ is thus implied by the long exact sequence of $\mathrm{Ext}^\bullet_G(L_i^\wedge, -)$ groups of the exact sequence $0 \to L_{i-1}^* \to L_{i-1}^\wedge \to L_{i-1}^\vee \to 0$.

We have now proved that if the npc $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \{\tau_i\}_{i\in\mathbb{Z}})$ is strict and $D = C^\bullet$, then (1) is bijective. We thus obtain an associated dnpc $C^{\mathrm{der}} := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ by letting τ denote the (unique) element of $\mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{ud}}, C^\bullet)$ which corresponds via (1) to the tuple $(\tau_i \circ \pi_{L_i})_{i\in\mathbb{Z}}$. □

2.3. **The associated perfect complex.** In this section we define the category of dnpc's of G-modules and then construct a functor from this category to the derived category $\mathfrak{D}^{\mathrm{perf}}$ of perfect complexes of G-modules.

Definition 2.4. *The category $\mathfrak{C}^{\mathrm{dnpc}}$ of dnpc's of G-modules is the category with objects the collection of dnpc's of G-modules and with morphisms defined as follows: a morphism in $\mathfrak{C}^{\mathrm{dnpc}}$ from $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ to $\tilde{C} := (\tilde{C}^\bullet, \{\tilde{L}_i\}_{i\in\mathbb{Z}}, \tilde{\tau})$ is a pair $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 \in \mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{ud}}, \tilde{C}^{\mathrm{ud}})$ and $\alpha_2 \in \mathrm{Hom}_{\mathfrak{D}}(C^\bullet, \tilde{C}^\bullet)$ and $\alpha_2 \circ \tau = \tilde{\tau} \circ \alpha_1$ in $\mathfrak{D}$.*

Given another morphism $\beta : C' \to C$ in $\mathfrak{C}^{\mathrm{dnpc}}$, the composite $\alpha \circ \beta$ of α and β is the morphism $C' \to \tilde{C}$ in $\mathfrak{C}^{\mathrm{dnpc}}$ which is comprised of the pair $(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2)$, where both composites are formed in $\mathfrak{D}$. The identity morphism id_C of an object C of $\mathfrak{C}^{\mathrm{dnpc}}$ is the pair $(\mathrm{id}_{C^{\mathrm{ud}}}, \mathrm{id}_{C^\bullet})$.

Proposition 2.5. *Let $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ be a dnpc of G-modules and C^{perf} any complex which lies in an exact triangle in $\mathfrak{D}$ of the form*

$$C^{\mathrm{ud}} \xrightarrow{\tau} C^\bullet \xrightarrow{\tau'} C^{\mathrm{perf}} \xrightarrow{\tau''} C^{\mathrm{ud}}[1]. \tag{4}$$

(a) *Then C^{perf} belongs to $\mathfrak{D}^{\mathrm{perf}}$ and, for each integer i, there is natural exact sequence*

$$0 \to H^i(C^\bullet)_{\mathrm{codiv}} \to H^i(C^{\mathrm{perf}}) \to L_{i+1}^* \to 0. \tag{5}$$

(b) *If $\alpha : C \to \tilde{C}$ is any morphism in $\mathfrak{C}^{\mathrm{dnpc}}$ (as in Definition 2.4) and $\tilde{C}^{\mathrm{perf}}$ is any complex which lies in an exact triangle in $\mathfrak{D}$ of the form*

$$\tilde{C}^{\mathrm{ud}} \xrightarrow{\tilde{\tau}} \tilde{C}^\bullet \xrightarrow{\tilde{\tau}'} \tilde{C}^{\mathrm{perf}} \xrightarrow{\tilde{\tau}''} \tilde{C}^{\mathrm{ud}}[1],$$

then there exists a unique morphism $\alpha^{\mathrm{perf}} : C^{\mathrm{perf}} \to \tilde{C}^{\mathrm{perf}}$ in $\mathfrak{D}^{\mathrm{perf}}$ which makes the following diagram commute in $\mathfrak{D}$

$$\begin{tikzcd} C^{\mathrm{ud}} \arrow[r, "\tau"] \arrow[d, "\alpha_1"'] & C^\bullet \arrow[r, "\tau'"] \arrow[d, "\alpha_2"'] & C^{\mathrm{perf}} \arrow[r, "\tau''"] \arrow[d, "\alpha^{\mathrm{perf}}"'] & C^{\mathrm{ud}}[1] \arrow[d, "\alpha_1[1]"'] \\ \tilde{C}^{\mathrm{ud}} \arrow[r, "\tilde{\tau}"] & \tilde{C}^\bullet \arrow[r, "\tilde{\tau}'"] & \tilde{C}^{\mathrm{perf}} \arrow[r, "\tilde{\tau}''"] & \tilde{C}^{\mathrm{ud}}[1]. \end{tikzcd} \tag{6}$$

(c) *The association $C \mapsto C^{\mathrm{perf}}, \alpha \mapsto \alpha^{\mathrm{perf}}$ gives a functor, unique up to natural equivalence, from $\mathfrak{C}^{\mathrm{dnpc}}$ to $\mathfrak{D}^{\mathrm{perf}}$.*

Proof. Recall first that any G-module that is uniquely divisible is automatically cohomologically trivial. Hence both C^{ud} and $C^\bullet$ belong to $\mathfrak{D}^{\mathrm{ct}}$ and so the exact triangle (4) implies that C^{perf} also belongs to $\mathfrak{D}^{\mathrm{ct}}$. By a standard argument (cf. [7, Lem. 1.1]) we deduce that C^{perf} belongs to $\mathfrak{D}^{\mathrm{perf}}$ if its cohomology is finitely generated in all degrees. Thus, since both $H^i(C^\bullet)_{\mathrm{codiv}}$ and L^*_{i+1} are (by assumption) finitely generated it is enough to prove the existence of the exact sequences (5). But (5) follows directly upon combining the long exact cohomology sequence of (4) with the fact that $\mathrm{cok}(H^i(\tau))$ is canonically isomorphic to $H^i(C^\bullet)_{\mathrm{codiv}}$ and $\ker(H^{i+1}(\tau))$ is equal to $\ker(\pi_{L_{i+1}})$ and hence isomorphic (via $\kappa_{L_{i+1}}$) to L^*_{i+1}. This proves claim (a).

From the definition of morphisms in $\mathfrak{C}^{\mathrm{dnpc}}$ we know that the first square of (6) commutes and so the existence of a morphism α^{perf} in $\mathfrak{D}^{\mathrm{perf}}$ which completes the diagram to give a morphism of exact triangles follows directly from the axioms of a triangulated category. To establish uniqueness of α^{perf} we set $D := \tilde{C}^{\mathrm{perf}}$ and consider the exact sequence

$$\mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{ud}}[1], D) \xrightarrow{\iota} \mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{perf}}, D) \xrightarrow{\iota'} \mathrm{Hom}_{\mathfrak{D}}(C^\bullet, D)$$

induced by applying $\mathrm{Hom}_{\mathfrak{D}}(-, D)$ to the upper row of (6). It suffices to prove that ι' is injective, or equivalently that $\iota = 0$. But since both C^{perf} and D belong to $\mathfrak{D}^{\mathrm{perf}}$ the spectral sequence (2) implies that $\mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{perf}}, D)$ is finitely generated and so we need only observe that $\mathrm{Hom}_{\mathfrak{D}}(C^{\mathrm{ud}}[1], D)$ is uniquely divisible. Indeed, this follows easily from the fact that, for any natural number n, the 'multiplication by n' morphism $C^{\mathrm{ud}}[1] \xrightarrow{\times n} C^{\mathrm{ud}}[1]$ is a quasi-isomorphism. This proves claim (b).

To prove claim (c) we need to be more precise about C^{perf} and α^{perf}. Thus, for each dnpc C, we now *fix* an exact triangle $\Delta(C)$ of the form (4). We write C^{perf}_Δ for the complex C^{perf} which occurs in $\Delta(C)$ and $\alpha^{\mathrm{perf}}_\Delta : C^{\mathrm{perf}}_\Delta \to \tilde{C}^{\mathrm{perf}}_\Delta$ for the morphism in $\mathfrak{D}^{\mathrm{perf}}$ obtained from a morphism $\alpha : C \to \tilde{C}$ in $\mathfrak{C}^{\mathrm{dnpc}}$ by making the construction of claim (b) with the upper, resp. lower, row of (6) equal to $\Delta(C)$, resp. $\Delta(\tilde{C})$. Then the uniqueness assertion of claim (b) allows one to show that for any pair of morphisms in $\mathfrak{C}^{\mathrm{dnpc}}$ one has $(\alpha \circ \beta)^{\mathrm{perf}}_\Delta = \alpha^{\mathrm{perf}}_\Delta \circ \beta^{\mathrm{perf}}_\Delta$ (where the latter composite is taken in $\mathfrak{D}^{\mathrm{perf}}$) and also that $(\mathrm{id}_C)^{\mathrm{perf}}_\Delta = \mathrm{id}_{C^{\mathrm{perf}}}$ for any object C of $\mathfrak{C}^{\mathrm{dnpc}}$. Thus we obtain a functor $\Pi_\Delta : \mathfrak{C}^{\mathrm{dnpc}} \to \mathfrak{D}^{\mathrm{perf}}$ by setting $\Pi_\Delta(C) = C^{\mathrm{perf}}_\Delta$ and $\Pi_\Delta(\alpha) = \alpha^{\mathrm{perf}}_\Delta$.

If now we make the constructions of the previous paragraph with respect to a different choice $\Delta'(C)$ of triangle (4) for each object C of $\mathfrak{C}^{\mathrm{dnpc}}$, then we obtain an analogous functor $\Pi_{\Delta'} : \mathfrak{C}^{\mathrm{dnpc}} \to \mathfrak{D}^{\mathrm{perf}}$. To complete the proof of claim (c) we shall construct a natural equivalence $\tau : \Pi_\Delta \to \Pi_{\Delta'}$. Given any object C of $\mathfrak{C}^{\mathrm{dnpc}}$ we let $\tau_C : C^{\mathrm{perf}}_\Delta \to C^{\mathrm{perf}}_{\Delta'}$ denote the isomorphism in $\mathfrak{D}^{\mathrm{perf}}$ obtained from the construction of claim (b) with $C = \tilde{C}$, $\alpha = \mathrm{id}_C$ and with the upper, resp. lower, row of (6) equal to $\Delta(C)$, resp. $\Delta'(C)$. To prove that $\tau : \Pi_\Delta \to \Pi_{\Delta'}$ is a natural equivalence we must show that $\tau_{\tilde{C}} \circ \alpha_{\Delta}^{\mathrm{perf}} = \alpha_{\Delta'}^{\mathrm{perf}} \circ \tau_C$ in $\mathfrak{D}^{\mathrm{perf}}$ for each morphism $\alpha : C \to \tilde{C}$ in $\mathfrak{C}^{\mathrm{dnpc}}$. But this follows from the uniqueness assertion of claim (b) and the fact that both $\tau_{\tilde{C}} \circ \alpha_{\Delta}^{\mathrm{perf}}$ and $\alpha_{\Delta'}^{\mathrm{perf}} \circ \tau_C$ complete the following diagram (where the upper row is $\Delta(C)$ and the lower is $\Delta'(\tilde{C})$) to give a morphism of exact triangles

$$\begin{array}{ccccccc} C^{\mathrm{ud}} & \xrightarrow{\tau} & C^{\bullet} & \longrightarrow & C_{\Delta}^{\mathrm{perf}} & \longrightarrow & C^{\mathrm{ud}}[1] \\ {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\alpha_2}\big\downarrow & & & & {\scriptstyle\alpha_1[1]}\big\downarrow \\ \tilde{C}^{\mathrm{ud}} & \xrightarrow{\tilde{\tau}} & \tilde{C}^{\bullet} & \longrightarrow & \tilde{C}_{\Delta'}^{\mathrm{perf}} & \longrightarrow & \tilde{C}^{\mathrm{ud}}[1]. \end{array}$$

□

2.4. **Euler characteristics.** We write $K_0(\mathbb{Z}[G])$ for the Grothendieck group of the category of finitely generated projective G-modules. We recall that to each isomorphism class of objects P of $\mathfrak{D}^{\mathrm{perf}}$ one can associate a canonical 'projective Euler characteristic' $\chi^{\mathrm{perf}}(P)$ in $K_0(\mathbb{Z}[G])$ (this construction is certainly well known, but for details see [7, Lem. 1.2]).

Definition 2.6. *The* Euler characteristic $\chi(C)$ *of a dnpc of G-modules C is the element $\chi^{\mathrm{perf}}(C^{\mathrm{perf}})$ of $K_0(\mathbb{Z}[G])$.*

Before stating the next result we recall that in [8, Th. 2.3] Chinburg, Kolster, Pappas and Snaith use a recursive construction to associate to each npc of G-modules $\tilde{C}$ a canonical 'Euler characteristic' element in $K_0(\mathbb{Z}[G])$ which we denote by $\chi^{\mathrm{CKPS}}(\tilde{C})$.

Proposition 2.7. *If C is a dnpc of G-modules and $\tilde{C}$ is any npc of G-modules that is associated to C (in the sense described in §2.2), then one has $\chi(C) = \chi^{\mathrm{CKPS}}(\tilde{C})$ in $K_0(\mathbb{Z}[G])$.*

Proof. As usual, we write $C = (C^{\bullet}, \{L_i\}_{i\in\mathbb{Z}}, \tau)$. We first recall that in [7, p. 253] a complex of G-modules $(Q\oplus R)^{\bullet}$ is constructed together with a quasi-isomorphism of complexes of G-modules $\mu : (Q\oplus R)^{\bullet} \to C^{\mathrm{ud}}$ and also a morphism of complexes of G-modules $\theta : (Q \oplus R)^{\bullet} \to C^{\bullet}$ with the property that $H^i(\theta) \circ H^i(\mu)^{-1}$ is equal to the composite $L_i^{\wedge} \xrightarrow{\tau_i \circ \pi_{L_i}} H^i(C^{\bullet})_{\mathrm{div}} \subseteq H^i(C^{\bullet})$ in each degree i. Now $(Q \oplus R)^{\bullet}$ is by construction a bounded complex of projective G-modules and so $\mathrm{Hom}_{\mathfrak{D}}((Q \oplus R)^{\bullet}, C^{\bullet})$ is equal to the group of homotopy classes of morphisms of complexes of G-modules from $(Q \oplus R)^{\bullet}$ to $C^{\bullet}$. The explicit construction of loc. cit. thus allows us to choose θ so that $\theta = \tau \circ \mu$ in $\mathrm{Hom}_{\mathfrak{D}}((Q \oplus R)^{\bullet}, C^{\bullet})$. The mapping cone $\mathrm{Cone}_C^{\bullet}$ of θ that is used in loc. cit. is then isomorphic in $\mathfrak{D}$ to any complex C^{perf} which lies in an exact triangle of the form (4). We may therefore deduce that $\chi(C) = \chi^{\mathrm{perf}}(C^{\mathrm{perf}}) = \chi^{\mathrm{perf}}(\mathrm{Cone}_C^{\bullet}) = \chi^{\mathrm{CKPS}}(\tilde{C})$, where the first equality is by the definition of $\chi(C)$ and the last follows directly from [7, Th. 1.3]. $\square$

It is well known that any exact triangle $P_1 \to P_2 \to P_3 \to P_1[1]$ in $\mathfrak{D}^{\mathrm{perf}}$ gives rise to an equality $\chi^{\mathrm{perf}}(P_2) = \chi^{\mathrm{perf}}(P_1) + \chi^{\mathrm{perf}}(P_3)$ in $K_0(\mathbb{Z}[G])$. In the next result we prove a natural analogue of this additivity property for dnpc's and thereby answer a question posed by Snaith in [14, Question 2.3.21].

Definition 2.8. *The* 1-shift *of a dnpc* $C := (C^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ *of* G*-modules is the dnpc of* G*-modules defined by*

$$C[1] := (C^\bullet[1], \{L_{i+1}\}_{i\in\mathbb{Z}}, \tau[1])$$

where $C^\bullet[1]$ *and* $\tau[1]$ *denote the usual* 1*-shifts of the complex* $C^\bullet$ *and of the morphism* τ *respectively. We then define* an exact triangle of dnpc's of G-modules *to be a triple of the form* $(\alpha_1, \alpha_2, \alpha_3)$ *where* $\alpha_1 : C_1 \to C_2$*,* $\alpha_2 : C_2 \to C_3$ *and* $\alpha_3 : C_3 \to C_1[1]$ *are morphisms in* $\mathfrak{C}^{\mathrm{dnpc}}$ *and the sequence of morphisms*

$$C_1^\bullet \xrightarrow{\alpha_{1,2}} C_2^\bullet \xrightarrow{\alpha_{2,2}} C_3^\bullet \xrightarrow{\alpha_{3,2}} C_1^\bullet[1]$$

is an exact triangle in $\mathfrak{D}$*.*

Theorem 2.9. *For any exact triangle of dnpc's of* G*-modules as above one has an equality* $\chi(C_2) = \chi(C_1) + \chi(C_3)$ *in* $K_0(\mathbb{Z}[G])$*.*

Proof. For each $j \in \{1,2,3\}$ we set $C_j := (C_j^\bullet, \{L_{j,i}\}_{i\in\mathbb{Z}}, \tau_j)$, and we consider the following diagram in $\mathfrak{D}$

$$\begin{array}{ccccccc}
C_1^{\mathrm{ud}} & \xrightarrow{\alpha_{1,1}} & C_2^{\mathrm{ud}} & \xrightarrow{\alpha_{2,1}} & C_3^{\mathrm{ud}} & \xrightarrow{\alpha_{3,1}} & C_1^{\mathrm{ud}}[1] \\
{\scriptstyle\tau_1}\downarrow & & {\scriptstyle\tau_2}\downarrow & & {\scriptstyle\tau_3}\downarrow & & {\scriptstyle\tau_1[1]}\downarrow \\
C_1^\bullet & \xrightarrow{\alpha_{1,2}} & C_2^\bullet & \xrightarrow{\alpha_{2,2}} & C_3^\bullet & \xrightarrow{\alpha_{3,2}} & C_1^\bullet[1] \\
{\scriptstyle\tau_1'}\downarrow & & {\scriptstyle\tau_2'}\downarrow & & {\scriptstyle\tau_3'}\downarrow & & {\scriptstyle\tau_1[1]'}\downarrow \\
C_1^{\mathrm{perf}} & & C_2^{\mathrm{perf}} & & C_3^{\mathrm{perf}} & & C_1^{\mathrm{perf}}[1] \\
{\scriptstyle\tau_1''}\downarrow & & {\scriptstyle\tau_2''}\downarrow & & {\scriptstyle\tau_3''}\downarrow & & {\scriptstyle\tau_1[1]''}\downarrow \\
C_1^{\mathrm{ud}}[1] & & C_2^{\mathrm{ud}}[1] & & C_3^{\mathrm{ud}}[1] & & C_1^{\mathrm{ud}}[2].
\end{array} \tag{7}$$

The second row and each column of this diagram is an exact triangle (by assumption and by the definition (4) of each complex C_j^{perf} respectively) and all squares commute (by virtue of the definition of a morphism of dnpc's). Hence, if the upper row of the diagram is also an exact triangle, then the Octahedral axiom implies the existence of an exact triangle in $\mathfrak{D}^{\mathrm{perf}}$ of the form $C_1^{\mathrm{perf}} \to C_2^{\mathrm{perf}} \to C_3^{\mathrm{perf}} \to C_1^{\mathrm{perf}}[1]$. But, given any such triangle, the claimed equality would follow directly from (the definition of each term $\chi(C_j) := \chi^{\mathrm{perf}}(C_j^{\mathrm{perf}})$ and) the well-known additivity of projective Euler characteristics on exact triangles in $\mathfrak{D}^{\mathrm{perf}}$.

Lemma 2.10. *The upper row of (7) is an exact triangle in $\mathfrak{D}$ if the complex*

$$(8) \quad \cdots \to H^i(C_1^\bullet)_{\mathrm{div}} \xrightarrow{H^i(\alpha_{1,2})} H^i(C_2^\bullet)_{\mathrm{div}} \xrightarrow{H^i(\alpha_{2,2})} H^i(C_3^\bullet)_{\mathrm{div}} \xrightarrow{H^i(\alpha_{3,2})} H^{i+1}(C_1^\bullet)_{\mathrm{div}} \to \cdots$$

(where, to be specific, we regard $H^i(C_2^\bullet)_{\mathrm{div}}$ as placed in degree 0) has finite cohomology groups in all degrees.

Proof. After recalling the explicit definition of each complex C_j^{ud} it is easily shown that the upper row of (7) is an exact triangle in $\mathfrak{D}$ precisely when the following sequence is exact

$$(9) \qquad \cdots \to L^\wedge_{1,i} \xrightarrow{H^i(\alpha_{1,1})} L^\wedge_{2,i} \xrightarrow{H^i(\alpha_{2,1})} L^\wedge_{3,i} \xrightarrow{H^i(\alpha_{3,1})} L^\wedge_{1,i+1} \to \cdots .$$

Assuming the cohomology of (8) to be finite in degree 0, we now prove that (9) is exact at the term $L^\wedge_{2,i}$. We leave the reader to verify that an entirely similar argument allows one to deduce that the exactness of (9) at any other term follows from the assumption that the corresponding cohomology group of (8) is finite.

We consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & L^*_{1,i} \arrow[r, "\delta'_1"] \arrow[d, "\kappa_1"'] & L^*_{2,i} \arrow[r, "\delta'_2"] \arrow[d, "\kappa_2"'] & L^*_{3,i} \arrow[r, "\delta'_3"] \arrow[d, "\kappa_3"'] & {} \\
{} \arrow[r] & L^\wedge_{1,i} \arrow[r, "\delta_1"] \arrow[d, "H^i(\tau_1)"'] & L^\wedge_{2,i} \arrow[r, "\delta_2"] \arrow[d, "H^i(\tau_2)"'] & L^\wedge_{3,i} \arrow[r, "\delta_3"] \arrow[d, "H^i(\tau_3)"'] & {} \\
{} \arrow[r] & H^i(C_1^\bullet)_{\mathrm{div}} \arrow[r, "\overline{\delta}_1"] \arrow[d] & H^i(C_2^\bullet)_{\mathrm{div}} \arrow[r, "\overline{\delta}_2"] \arrow[d] & H^i(C_3^\bullet)_{\mathrm{div}} \arrow[r, "\overline{\delta}_3"] \arrow[d] & {} \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

In this diagram we set $\kappa_a := \kappa_{L_{a,i}}$ and $\delta_a := H^i(\alpha_{a,1})$ for each $a \in \{1,2,3\}$ and write $\overline{\delta}_a$, resp. δ'_a, for the map induced by $H^i(\alpha_{a,2})$, resp. the restriction of δ_a (this makes sense because each column of the above diagram is exact). Now since $L^\wedge_{1,i}$ is uniquely divisible and $L^*_{3,i}$ is finitely generated one has $\mathrm{Hom}_G(L^\wedge_{1,i}, L^*_{3,i}) = 0$. Hence, since $\mathrm{im}(\overline{\delta}_1) \subseteq \ker(\overline{\delta}_2)$, the commutativity of the diagram implies

$\mathrm{im}(\delta_1) \subseteq \ker(\delta_2)$. We must show that $\mathrm{im}(\delta_1) = \ker(\delta_2)$. But, if N denotes the cardinality of the (assumed to be finite) group $\ker(\overline{\delta}_2)/\mathrm{im}(\overline{\delta}_1)$, then for any $x \in \ker(\delta_2)$ one has $Nx \in \mathrm{im}(\delta_1) + \mathrm{im}(\kappa_2)$ and hence $x \in \mathrm{im}(\delta_1) + \frac{1}{N}\mathrm{im}(\kappa_2)$. This induces an injection of $\ker(\delta_2)/\mathrm{im}(\delta_1)$ into the finitely generated module $\frac{1}{N}\mathrm{im}(\kappa_2)/(\frac{1}{N}\mathrm{im}(\kappa_2) \cap \mathrm{im}(\delta_1))$. But $\ker(\delta_2)/\mathrm{im}(\delta_1)$ is a subquotient of $L^{\wedge}_{2,i}$ and so is uniquely divisible. Being both finitely generated and uniquely divisible, it follows that $\ker(\delta_2)/\mathrm{im}(\delta_1) = 0$, as required. □

To complete the proof of Theorem 2.9 it now only remains to prove that the complex (8) has finite cohomology groups. We will show that the cohomology of (8) is finite in degree 0, leaving the reader to verify by an entirely similar argument that the cohomology is finite in any other degree.

To do this we use the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i(C_1^\bullet)_{\mathrm{div}} & \longrightarrow & H^i(C_1^\bullet) & \longrightarrow & H^i(C_1^\bullet)_{\mathrm{codiv}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow{\scriptstyle H^i(\alpha_{1,2})} & & \Big\downarrow{\scriptstyle \epsilon'} & & \\
0 & \longrightarrow & H^i(C_2^\bullet)_{\mathrm{div}} & \longrightarrow & H^i(C_2^\bullet) & \longrightarrow & H^i(C_2^\bullet)_{\mathrm{codiv}} & \longrightarrow & 0
\end{array}$$

where the rows are tautological exact sequences and ϵ and ϵ' are defined so as to make the diagram commute. Applying the Snake lemma to this diagram gives an exact sequence $\ker(\epsilon') \to \mathrm{cok}(\epsilon) \xrightarrow{\beta} \mathrm{cok}(H^i(\alpha_{1,2})) \to \mathrm{cok}(\epsilon')$. Now $\ker(\epsilon')$ is finitely generated (because $H^i(C_1^\bullet)_{\mathrm{codiv}}$ is) and $\mathrm{cok}(\epsilon)$ is $\mathbb{Z}$-torsion (because $H^i(C_2^\bullet)_{\mathrm{div}} \cong L^{\vee}_{2,i}$ is) and so $\ker(\beta)$ is finite. But $H^i(\alpha_{2,2})$ maps $\mathrm{cok}(H^i(\alpha_{1,2}))$ injectively into $H^i(C_3^\bullet)$ and so, writing $\iota : \mathrm{cok}(\epsilon) \to H^i(C_3^\bullet)_{\mathrm{div}}$ for the map that is induced by $H^i(\alpha_{2,2})$, the commutative diagram

$$\begin{array}{ccc}
\mathrm{cok}(\epsilon) & \xrightarrow{\beta} & \mathrm{cok}(H^i(\alpha_{1,2})) \\
\Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle H^i(\alpha_{2,2})} \\
H^i(C_3^\bullet)_{\mathrm{div}} & \xrightarrow{\subseteq} & H^i(C_3^\bullet)
\end{array}$$

implies that $\ker(\iota)$ is finite. This shows that the cohomology of (8) is finite in degree 0, as required. This then completes the proof of Theorem 2.9. □

3. Arithmetic applications

3.1. **Arithmetic surfaces.** We first apply the approach developed in §2 to show that *canonical* perfect complexes underly all of the arithmetic constructions that are made by Chinburg et al in [8]. To do this we let X be a regular two dimensional scheme which is proper over $\mathrm{Spec}\,\mathbb{Z}$, geometrically connected and for which the Brauer group $\mathrm{Br}(X)$ is finite (which is conjectured to be true in all cases). For convenience we say that we are in Case 'L', resp. in Case 'S', if the characteristic of the function field of X is non-zero, resp. is zero (in which case the constructions of [8] depend upon results of Lichtenbaum, resp. Saito). We set $\Lambda := \mathbb{Z}$ in Case 'L' and $\Lambda := \mathbb{Z}[\frac{1}{2}]$ in Case 'S'. We let G be a finite group which acts on X in such a way that the inertia group in G of each point of X is trivial. We write $\mathfrak{D}(\Lambda(G))$ for the derived category of the abelian category of $\Lambda[G]$-modules and define subcategories $\mathfrak{D}(\Lambda(G))^{\mathrm{ct}}$ and $\mathfrak{D}(\Lambda(G))^{\mathrm{perf}}$ of $\mathfrak{D}(\Lambda(G))$ just as in §2.1. We also write $K_0(\Lambda[G])$ for the Grothendieck group of the category of finitely generated projective $\Lambda[G]$-modules.

Proposition 3.1. *To each pair (X, G) as above one can associate a canonical object C_X^{perf} of $\mathfrak{D}(\Lambda(G))^{\mathrm{perf}}$. The projective Euler characteristic of C_X^{perf} in $K_0(\Lambda(G))$ is equal to the element $\chi^G(X, \Lambda \otimes_{\mathbb{Z}} \mathbb{G}_m)$ that is defined by Chinburg et al in* [8, Cor. 3.4].

Proof. We define the concept of (strict) nearly perfect complexes and derived nearly perfect complexes of $\Lambda[G]$-modules by adapting Definitions 2.1 and 2.2 in the obvious way (in Case 'S').

We first recall the construction of an npc of $\Lambda(G)$-modules made in [8, Cor. 3.4]. To do this we set $\mathbb{G}'_m := \Lambda \otimes_{\mathbb{Z}} \mathbb{G}_m$. Then $R\Gamma(X, \mathbb{G}'_m)$ belongs to $\mathfrak{D}(\Lambda(G))^{\mathrm{ct}}$ [8, Prop. 3.2] and so we may choose a bounded complex of cohomologically trivial $\Lambda(G)$-modules $C_X^\bullet$ which is isomorphic in $\mathfrak{D}(\Lambda(G))$ to $R\Gamma(X, \mathbb{G}'_m)$. Since $\mathrm{Br}(X)$ is assumed to be finite, by results of [10, §3.4, §4] in Case 'L', resp. [13] in Case 'S', one knows that $H^i(C_X^\bullet)$ is finite if $i \notin \{1, 3\}$ in Case 'L', resp. if $i \notin \{0, 1, 3, 4\}$ in Case 'S'; that the Λ-modules $H^0(C_X^\bullet)$ and $H^1(C_X^\bullet)$ are finitely generated; and that for $i \in \{0, 1\}$, there is a natural isomorphism of $\Lambda[G]$-modules $\lambda_i : H^{4-i}(C_X^\bullet) \cong \mathrm{Hom}_\Lambda(H^i(C_X^\bullet), \mathbb{Q}/\Lambda)$. For $i \in \{3, 4\}$, we set $L_i := H^{4-i}(C_X^\bullet)_{\mathrm{tf}}$ and write $\tau_i : \mathrm{Hom}_\Lambda(L_i, \mathbb{Q}/\Lambda) \to H^i(C_X^\bullet)_{\mathrm{div}}$ for the isomorphism that is induced by λ_i. Setting $L_i := 0$ and $\tau_i := 0$ for $i \notin \{3, 4\}$ one obtains an npc of $\Lambda[G]$-modules by defining $C_X := (C_X^\bullet, \{L_i\}_{i\in\mathbb{Z}}, \{\tau_i\}_{i\in\mathbb{Z}})$.

Since both $H^2(C_X^\bullet)$ and $H^3(C_X^\bullet)_{\mathrm{codiv}} \cong \mathrm{Hom}_\Lambda(H^1(C_X^\bullet)_{\mathrm{tor}}, \mathbb{Q}/\Lambda)$ are finite, it is clear that the npc C_X is strict. Following Lemma 2.3 and Proposition 2.5 we may therefore set $C_X^{\mathrm{perf}} := C_X^{\mathrm{der,perf}}$. With this definition, the equality $\chi^{\mathrm{perf}}(C_X^{\mathrm{perf}}) = \chi^G(X, \mathbb{G}'_m)$ then follows directly from Proposition 2.7 and the definition of $\chi^G(X, \mathbb{G}'_m)$ that is given in [8, Cor. 3.4]. □

Remark 3.2. *(Refined Euler characteristics)* The existence of C_X^{perf} as in Proposition 3.1 also allows a more direct (and natural) approach to the main arithmetic construction of [7]. Indeed, the same argument as used to prove Proposition 2.7 shows that the element $\chi^{\mathrm{rel}}(C^*, \mu_{\mathbb{Q}})$ that is defined in [7, pp. 248-249] is simply equal to the refined Euler characteristic $\chi^{\mathrm{rel}}(C_X^{\mathrm{perf}}, \mu_{\mathbb{Q}})$ in the sense described in [2, §1.2] and [7, §3].

3.2. **Weil-étale Cohomology.** Let X be a scheme of finite type over a finite field. In this subsection we use the Weil-étale topology $\mathcal{W}$ on X as defined by Lichtenbaum in [11] and studied by Geisser in [9]. In particular, we recall that there exists a natural morphism of topoi γ from the Weil-étale topos $X_{\mathcal{W}}$ on X to the étale topos $X_{\text{ét}}$ on X.

We fix a finite group G and assume that there exists an étale morphism $\pi : X \to Y$ which is Galois with group G. For each subgroup H of G we write $X \to X^H$ for the corresponding H-cover and $\pi_H : X^H \to Y$ for the induced étale morphism.

Proposition 3.3. *Let $\mathcal{E}$ be any sheaf, or bounded below complex of sheaves, on $Y_{\text{ét}}$ such that the pullback $\mathcal{F} := \pi^*(\mathcal{E})$ satisfies all of the following conditions:*

(i) *$H^i(X_{\mathcal{W}}, \gamma^*(\mathcal{F}))$ is finitely generated for all integers i and vanishes for almost all i.*
(ii) *$H^i(X_{\text{ét}}, \mathcal{F})_{\text{div}} \subseteq H^i(X_{\text{ét}}, \mathcal{F})_{\text{tor}}$ for all i.*
(iii) *For each subgroup H of G, $H^i(X^H_{\text{ét}}, \pi^*_H(\mathcal{E}))$ vanishes for almost all i.*

Then there exists a canonical dnpc $C_{X,\mathcal{F}} := (C^\bullet_{X,\mathcal{F}}, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ of G-modules with the following two properties:

(a) *$C^\bullet_{X,\mathcal{F}} = R\Gamma(X_{\text{ét}}, \mathcal{F})$.*
(b) *$C^{\text{perf}}_{X,\mathcal{F}}$ is canonically isomorphic in $\mathfrak{D}^{\text{perf}}$ to $R\Gamma(X_{\mathcal{W}}, \gamma^*(\mathcal{F}))$.*

Proof. We note first that, under condition (iii), the argument of [8, Prop. 3.2] shows that $R\Gamma(X_{\text{ét}}, \mathcal{F})$ belongs to $\mathfrak{D}^{\text{ct}}$. Next, we recall from [9, Th. 3.3] that there exists a natural morphism $\theta : R\Gamma(X_{\text{ét}}, \mathcal{F}) \to R\Gamma(X_{\mathcal{W}}, \gamma^*(\mathcal{F}))$ in $\mathfrak{D}$ and that if D is any complex which lies in an exact triangle in $\mathfrak{D}$ of the form

$$(10) \qquad R\Gamma(X_{\text{ét}}, \mathcal{F})[-1] \xrightarrow{\theta[-1]} R\Gamma(X_{\mathcal{W}}, \gamma^*(\mathcal{F}))[-1] \xrightarrow{\psi} D \xrightarrow{\mu} R\Gamma(X_{\text{ét}}, \mathcal{F}),$$

then $H^i(D)$ is uniquely divisible in each degree i. Hence, if we set $H(D) := \bigoplus_{i\in\mathbb{Z}} H^i(D)[-i]$, then Lemma 2.3 implies there exists a unique morphism ι in $\operatorname{Hom}_{\mathfrak{D}}(H(D), D)$ such that $H^i(\iota)$ is equal to the identity map on $H^i(D)$ in each degree i.

In each degree i condition (i) combines with the exact cohomology sequence of (10) to give a short exact sequence $0 \to M_i \to H^i(D) \to H^i(X_{\text{ét}}, \mathcal{F})_{\text{div}} \to 0$ where M_i is the torsion-free finitely generated module $\operatorname{im}(H^i(\psi))$. From condition (ii) we deduce that this sequence induces an identification $M_i \otimes_{\mathbb{Z}} \mathbb{Q} = H^i(D)$ and hence also an isomorphism $M_i \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong H^i(X_{\text{ét}}, \mathcal{F})_{\text{div}}$.

We now set $C^\bullet_{X,\mathcal{F}} := R\Gamma(X_{\text{ét}}, \mathcal{F})$, $L_i := M_i^*$ for each $i \in \mathbb{Z}$ and $\tau := \mu \circ \iota$. Then $C_{X,\mathcal{F}} := (C^\bullet_{X,\mathcal{F}}, \{L_i\}_{i\in\mathbb{Z}}, \tau)$ is a dnpc of G-modules which has property (a).

Further, $C^{\mathrm{ud}}_{X,\mathcal{F}} = H(D)$ and there exists a commutative diagram in $\mathfrak{D}$ of the form

$$\begin{array}{ccccccc} D & \xrightarrow{\mu} & R\Gamma(X_{\text{ét}},\mathcal{F}) & \xrightarrow{\theta} & R\Gamma(X_{\mathcal{W}},\gamma^*(\mathcal{F})) & \xrightarrow{\psi[1]} & D[1] \\ {\scriptstyle\iota}\uparrow & & \| & & & & \uparrow{\scriptstyle\iota[1]} \\ H(D) & \xrightarrow{\tau} & R\Gamma(X_{\text{ét}},\mathcal{F}) & \longrightarrow & C^{\mathrm{perf}}_{X,\mathcal{F}} & \longrightarrow & H(D)[1] \end{array}$$

where the upper, resp. lower, row is the exact triangle obtained by rotating (10), resp. from the definition of $C^{\mathrm{perf}}_{X,\mathcal{F}}$. From the argument of Proposition 2.5(b) we may thus deduce the existence of a unique isomorphism in $\mathfrak{D}^{\mathrm{perf}}$ from $C^{\mathrm{perf}}_{X,\mathcal{F}}$ to $R\Gamma(X_{\mathcal{W}},\gamma^*(\mathcal{F}))$ which completes the above diagram to give a morphism of exact triangles. This shows that $C_{X,\mathcal{F}}$ has property (b). $\square$

Remark 3.4. *(Explicit examples)* If X is both smooth and proper and n is any non-negative integer, then Lichtenbaum has conjectured that condition (i) of Proposition 3.3 is valid with $\mathcal{E}$ equal to the 'motivic complex' $\mathbb{Z}(n)$ (for more details in this regard see [9, §6]). By now, condition (i) has been verified in each of the following special cases:

- X is a smooth projective variety, $\mathcal{E} = \mathbb{Z}$ [11, Th. 3.2].
- X is a smooth projective variety of dimension at most 2, $\mathcal{E} = j_!\mathbb{Z}$ where $j : Z \to X$ is an open dense embedding [11, Th. 3.3].
- X is a quasi-projective smooth curve, $\mathcal{E} = \mathbb{G}_m$ [11, Th. 7.1].

Condition (ii) (and (iii)) of Proposition 3.3 can be checked by computing explicitly étale cohomology groups. For example, it can be shown to be valid in each of the following cases:

- X is a smooth curve, $\mathcal{E} = j_!\mathbb{Z}$ with $j : Z \to X$ an open dense embedding or $\mathcal{E} = \mathbb{G}_m$.
- X is a smooth projective surface, $\mathcal{E} = \mathbb{G}_m$ (cf. the proof of Proposition 3.1).

In particular, Proposition 3.3 suggests the following question: if X is as in Proposition 3.1 (in Case 'L'), then what is the precise relationship between $R\Gamma(X_{\mathcal{W}},\gamma^*(\mathbb{G}_m))$ and C^{perf}_X?

3.3. **Number field analogues.** There is as yet no working theory of Weil-étale cohomology for arithmetic schemes. Indeed, whilst Lichtenbaum has suggested in [12] a possible definition of the Weil-étale topology for $\mathrm{Spec}\,\mathcal{O}$ with $\mathcal{O}$ the ring of algebraic integers in any number field, Flach has recently shown that, with respect to the topology defined in loc. cit., the cohomology groups (with compact support) of the constant sheaf $\mathbb{Z}$ are not always finitely generated (thus contradicting the axioms of any conjectural Weil-étale topology). However, motivated by Proposition 3.3, we now observe that in this case the approach of Flach and the present author in [4] leads to a direct construction of canonical perfect complexes which are well suited to the formulation of special value conjectures.

To describe this construction we fix a finite Galois extension of number fields L/K of group G and a finite G-stable set S of places of L that contains all archimedean places and all which ramify in L/K. We write $\mathcal{O}_S$ for the ring of S-integers in L and, for each $w \in S$, we write L_w for the completion of L at w. For any sheaf $\mathcal{F}$ on $(\mathrm{Spec}\,\mathcal{O}_S)_{\text{ét}}$ we follow [4, (3)] in defining the 'cohomology with compact support' by setting

$$C_c^{\bullet}(\mathcal{F}) := \mathrm{cone}(R\Gamma((\mathrm{Spec}\,\mathcal{O}_S)_{\text{ét}}, \mathcal{F}) \to \bigoplus_{w \in S} R\Gamma((\mathrm{Spec}\,L_w)_{\text{ét}}, j_w^*(\mathcal{F})))[-1]$$

where $j_w : (\mathrm{Spec}\,L_w)_{\text{ét}} \to (\mathrm{Spec}\,\mathcal{O}_S)_{\text{ét}}$ is the natural morphism (and the mapping cone is defined by using Godement resolutions).

Then $C_c^{\bullet}(\mathbb{Z})$ belongs to $\mathfrak{D}^{ct}$ and the same computations as used in [4, pp. 1356-1357] show that $H^i(C_c^{\bullet}(\mathbb{Z})) = 0$ if $i \notin \{1,2,3\}$, that $H^1(C_c^{\bullet}(\mathbb{Z}))$ identifies with X_S^* where X_S is the kernel of the natural diagonal morphism $\bigoplus_S \mathbb{Z} \to \mathbb{Z}$, that $H^2(C_c^{\bullet}(\mathbb{Z}))$ is finite and that $H^3(C_c^{\bullet}(\mathbb{Z}))$ identifies with $(\mathcal{O}_S^{\times})^{\vee}$. Hence (3) induces an isomorphism $\mathrm{Hom}_{\mathfrak{D}}((\mathcal{O}_S^{\times})^{\wedge}[-3], C_c^{\bullet}(\mathbb{Z})) \cong \mathrm{Hom}_G((\mathcal{O}_S^{\times})^{\wedge}, (\mathcal{O}_S^{\times})^{\vee})$ and there exists a dnpc of G-modules

$$C_c(\mathbb{Z}) := (C_c^{\bullet}(\mathbb{Z}), \{L_i\}_{i \in \mathbb{Z}}, \tau)$$

which has $L_3 := (\mathcal{O}_S^{\times})_{\mathrm{tf}}$, $L_i := 0$ for $i \neq 3$ and τ equal to the element of $\mathrm{Hom}_{\mathfrak{D}}(L_3^{\wedge}[-3], C_c^{\bullet}(\mathbb{Z})) \cong \mathrm{Hom}_G((\mathcal{O}_S^{\times})^{\wedge}, (\mathcal{O}_S^{\times})^{\vee})$ which corresponds to the canonical projection $\pi_{\mathcal{O}_S^{\times}}$.

We recall that [4, Prop. 3.1] defines a canonical object Ψ_S of $\mathfrak{D}^{\mathrm{perf}}$ that is acyclic outside degrees 0 and 1. Indeed, whilst the definition of Ψ_S in [4] explicitly mentions the assumption $\mathrm{Pic}(\mathcal{O}_S) = 0$, it is straightforward to check that if $\mathrm{Pic}(\mathcal{O}_S) \neq 0$, then the methods of [4] define objects Ψ_S and $\tilde{\Psi}_S$ of $\mathfrak{D}$ which satisfy all properties described in [4, Prop. 3.1] (with the exception that in [4, (29)] the term X_S is replaced by a G-module which is a 1-extension of X_S by $\mathrm{Pic}(\mathcal{O}_S)$) and also lie in exact triangles of the form [4, (31), (32), (85)].

Proposition 3.5. *$C_c(\mathbb{Z})^{\mathrm{perf}}$ has each of the following properties.*

(a) *There exists a canonical morphism in $\mathfrak{D}$ from $C_c^{\bullet}(\mathbb{Z})$ to $C_c(\mathbb{Z})^{\mathrm{perf}}$.*
(b) *For each prime p and natural number n, there is a natural isomorphism in $\mathfrak{D}(\mathbb{Z}/p^n[G])^{\mathrm{perf}}$ between $C_c(\mathbb{Z})^{\mathrm{perf}} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/p^n$ and $C_c^{\bullet}(\mathbb{Z}/p^n)$.*
(c) *$H^i(C_c(\mathbb{Z})^{\mathrm{perf}}) = 0$ if $i \notin \{1,2,3\}$; there are canonical identifications $H^1(C_c(\mathbb{Z})^{\mathrm{perf}}) = X_S^*$ and $H^3(C_c(\mathbb{Z})^{\mathrm{perf}}) = ((\mathcal{O}_S^{\times})_{\mathrm{tor}})^{\vee}$ and a canonical exact sequence $0 \to \mathrm{Pic}(\mathcal{O}_S)^{\vee} \to H^2(C_c(\mathbb{Z})^{\mathrm{perf}}) \to (\mathcal{O}_S^{\times})^* \to 0$.*
(d) *The $\mathbb{R}$-linear dual of the Dirichlet regulator map induces an isomorphism $\lambda : H^1(C_c(\mathbb{Z})^{\mathrm{perf}}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^2(C_c(\mathbb{Z})^{\mathrm{perf}}) \otimes_{\mathbb{Z}} \mathbb{R}$.*

(e) *Set* $P_S := R\operatorname{Hom}_{\mathbb{Z}}(C_c(\mathbb{Z})^{\mathrm{perf}}, \mathbb{Z}[-2])$. *Then the Artin-Verdier duality theorem induces an isomorphism* $q : P_S \xrightarrow{\sim} \Psi_S$ *in* $\mathfrak{D}$, *where* Ψ_S *is the object of* $\mathfrak{D}^{\mathrm{perf}}$ *discussed above. Further, if we identify* $H^0(P_S)$ *and* $H^1(P_S)_{\mathrm{tf}}$ *with* $\mathcal{O}_S^\times$ *and* X_S *by means of the linear duals of the identifications in (c), and* $H^0(\Psi_S)$ *and* $H^1(\Psi_S)_{\mathrm{tf}}$ *with* $\mathcal{O}_S^\times$ *and* X_S *by means of* [4, Prop. 3.1], *then* q *induces the identity maps on both* $\mathcal{O}_S^\times$ *and* X_S.

(f) *Let* d_S^i *denote the differential of* P_S *in degree* i. *If* $\operatorname{Pic}(\mathcal{O}_S) = 0$, *then* $H^0(P_S)$ *and* $H^1(P_S)$ *identify with* $\mathcal{O}_S^\times$ *and* X_S *respectively and the class in* $\operatorname{Ext}^2_G(X_S, \mathcal{O}_S^\times)$ *of the tautological exact sequence*

$$0 \to \mathcal{O}_S^\times \to P_S^0/\operatorname{im}(d_S^{-1}) \xrightarrow{d_S^0} \ker(d_S^1) \to X_S \to 0$$

is, when computed with respect to a projective resolution of X_S, *equal to the 'fundamental class' defined by Tate in* [15].

Proof. Claim (a) follows immediately from the definition of $C_c(\mathbb{Z})^{\mathrm{perf}}$ and claim (b) is a consequence of the exact triangle (4) with $C = C_c(\mathbb{Z})$, the exact sequence of sheaves $0 \to \mathbb{Z} \xrightarrow{\times p^n} \mathbb{Z} \to \mathbb{Z}/p^n \to 0$ on $(\operatorname{Spec}\mathcal{O}_S)_{\text{ét}}$ and the fact that $C^{\mathrm{ud}} \xrightarrow{\times p^n} C^{\mathrm{ud}}$ is a quasi-isomorphism. Claim (c) follows by combining the computations of [4, pp. 1356-1357] with the exact cohomology sequence of (4) with $C = C_c(\mathbb{Z})$ and claim (d) is then obvious. To prove claim (e) we consider the following diagram in $\mathfrak{D}$

$$\begin{array}{ccccccc}
X_S \otimes_{\mathbb{Z}} \mathbb{Q}[-2] & \xrightarrow{\mathrm{id}} & X_S \otimes_{\mathbb{Z}} \mathbb{Q}[-2] & \longrightarrow & 0 & \longrightarrow \\
{\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow & \\
\tilde{\Psi}_S & \xrightarrow{\gamma} & R\operatorname{Hom}_{\mathbb{Z}}(C_c^\bullet(\mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-3]) & \longrightarrow & \widehat{\mathcal{O}_S^\times}/\mathcal{O}_S^\times[0] & \longrightarrow \\
{\scriptstyle\beta}\downarrow & & \downarrow & & \downarrow{\scriptstyle\mathrm{id}} & \\
P_S & \longrightarrow & R\operatorname{Hom}_{\mathbb{Z}}(C_c^\bullet(\mathbb{Z}), \mathbb{Z}[-2]) & \longrightarrow & \widehat{\mathcal{O}_S^\times}/\mathcal{O}_S^\times[0] & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow &
\end{array} \tag{11}$$

The third row of this diagram is the exact triangle which results from applying $R\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}[-2])$ to (4) with $C = C_c(\mathbb{Z})$ and then using the fact that $R\operatorname{Hom}_{\mathbb{Z}}((\mathcal{O}_S^\times)^\wedge[-3], \mathbb{Z}[-2])$ identifies naturally with $\widehat{\mathcal{O}_S^\times}/\mathcal{O}_S^\times[0]$ where $\widehat{\mathcal{O}_S^\times}$ is the profinite completion of $\mathcal{O}_S^\times$; the second row of (11) is the exact triangle of [4, (31)] (which is constructed by using the Artin-Verdier duality theorem); the second column of (11) is induced by the exact triangle

$$R\operatorname{Hom}_{\mathbb{Z}}(C_c^\bullet(\mathbb{Z}), \mathbb{Z}[-3]) \to R\operatorname{Hom}_{\mathbb{Z}}(C_c^\bullet(\mathbb{Z}), \mathbb{Q}[-3]) \to \\ R\operatorname{Hom}_{\mathbb{Z}}(C_c^\bullet(\mathbb{Z}), \mathbb{Q}/\mathbb{Z}[-3]) \to R\operatorname{Hom}_{\mathbb{Z}}(C_c^\bullet(\mathbb{Z}), \mathbb{Z}[-2])$$

and the fact that, since $H^i(C_c^\bullet(\mathbb{Z}))$ is torsion for each $i \neq 1$, there are natural isomorphisms

$$R\operatorname{Hom}(C_c^\bullet(\mathbb{Z}), \mathbb{Q}[-3]) \cong R\operatorname{Hom}(H^1(C_c^\bullet(\mathbb{Z}))[-1], \mathbb{Q}[-3]) \\ \cong H^1(C_c^\bullet(\mathbb{Z}))^\wedge[-2] \cong X_S \otimes_\mathbb{Z} \mathbb{Q}[-2];$$

α is the morphism which makes the upper left hand square commute (this morphism exists because $\operatorname{Hom}_\mathfrak{D}(X_S \otimes_\mathbb{Z} \mathbb{Q}[-2], \widehat{\mathcal{O}_S^\times}/\mathcal{O}_S^\times[0]) = 0$ and is unique because both the natural 'passage to cohomology' homomorphism $\operatorname{Hom}_\mathfrak{D}(X_S \otimes_\mathbb{Z} \mathbb{Q}[-2], \tilde{\Psi}_S) \to \operatorname{Hom}_G(X_S \otimes_\mathbb{Z} \mathbb{Q}, H^2(\tilde{\Psi}_S))$ and the map $H^2(\gamma)$ are bijective); by using [4, Lem. 7], one checks that the lower right hand square of (11) commutes and so the existence of a morphism β which makes the lower left hand square commute and allows the left hand column to be completed to an exact triangle is a consequence of the Octahedral axiom. Now α is equal to the morphism which occurs in the exact triangle of [4, (32)] and so there exists an isomorphism $q : P_S \cong \Psi_S$ in $\mathfrak{D}$ which, taken together with the identity morphisms on $X_S \otimes_\mathbb{Z} \mathbb{Q}[-2]$ and $\tilde{\Psi}_S$, gives an isomorphism of exact triangles between [4, (32)] and the left hand column of (11). The fact that, with respect to the stated identifications, q induces the identity map on both $\mathcal{O}_S^\times$ and X_S can then be checked by a explicit diagram chase using (11). This completes the proof of claim (e) and claim (f) then follows directly from claim (e) and the proof of [4, Th. 3.2]. □

Remark 3.6. Proposition 3.5(f) is very reminiscent of Lichtenbaum's use of fundamental classes to define the 'Weil-étale topology' of [12]. Further, Proposition 3.5(c) shows that in each degree i with $0 \leq i \leq 3$ one has $H^i(C_c(\mathbb{Z})^{\mathrm{perf}}) = H^i(\overline{\operatorname{Spec}\mathcal{O}_S}, \phi_!\mathbb{Z})$ where the latter group is as defined and computed by Lichtenbaum in [12, Th. 6.3, Prop. 6.4].

Remark 3.7. Proposition 3.5(e) combines with [6, (29)] to show that the 'equivariant Tamagawa number conjecture' of [5, Conj. 4] is equivalent, when specialized to the pair $(h^0(\operatorname{Spec} L), \mathbb{Z}[G])$, to an equality

$$\hat{\delta}^1_{\mathbb{Z}[G],\mathbb{R}}(\zeta^*_{L/K,S}(0)) = \chi^{\mathrm{rel}}(C_c(\mathbb{Z})^{\mathrm{perf}}, -\lambda) \tag{12}$$

where $\hat{\delta}^1_{\mathbb{Z}[G],\mathbb{R}}$ is the canonical homomorphism from the multiplicative group of the centre $Z(\mathbb{R}[G])$ of $\mathbb{R}[G]$ to the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ that is defined in [5, Lem. 9], $\zeta^*_{L/K,S}(0)$ is the leading term at 0 of the S-truncated Zeta function of L/K (which is denoted by $L_S^*(\mathbb{Q}(0)_L)$ in [6]) and the right hand side of (12) is the refined Euler characteristic of the pair $(C_c(\mathbb{Z})^{\mathrm{perf}}, -\lambda)$ (cf. Remark 3.2). Now (12) is a precise analogue for number fields of the equivariant version of Lichtenbaum's conjecture [11, Conj. 8.1e)] that is studied in [3] (see, in particular, Conj. C(K/k), Rem. 1 and Rem. 3 in [3, §2.2]). Also, if $L = K$ and S is the set of archimedean places of L, then (12) is equivalent to the formula proved by Lichtenbaum in [12, Th. 8.1]. Thus, for the purposes of formulating special value conjectures, it seems reasonable to regard $C_c(\mathbb{Z})^{\mathrm{perf}}$ as an adequate substitute for the 'Weil-étale cohomology with compact support of $\mathbb{Z}$ on $\operatorname{Spec}\mathcal{O}_S$'.

Remark 3.8. Given the interpretation of $C_c(\mathbb{Z})^{\mathrm{perf}}$ in terms of the 'Weil-étale cohomology of $\mathbb{Z}$' that is suggested in Remark 3.7, Proposition 3.5(e) suggests an interpretation of Ψ_S as the 'Weil-étale cohomology of $\mathbb{G}_m$ on $\operatorname{Spec}\mathcal{O}_S$'. Indeed, with this interpretation, the isomorphism q in Proposition 3.5(e) is an analogue of the duality theorem in Weil-étale cohomology for curves over finite fields that is proved by Lichtenbaum in [11, Th. 6.5]. We note also that, as per the standard expectations for Weil-étale cohomology, there exists a natural morphism in $\mathfrak{D}$ from $R\Gamma((\operatorname{Spec}\mathcal{O}_S)_{\text{ét}}, \mathbb{G}_m)$ to Ψ_S (obtained as the composite of the morphisms $R\Gamma((\operatorname{Spec}\mathcal{O}_S)_{\text{ét}}, \mathbb{G}_m) \to \tilde{\Psi}_S$ and $\tilde{\Psi}_S \to \Psi_S$ which occur in [4, (85), resp. (32)]).

Remark 3.9. Motivated by the results of the last two subsections it seems reasonable to ask the following question: if X is as in Proposition 3.1 (in case 'S'), can one give a (conjectural) interpretation of C_X^{perf} in terms of Weil-étale cohomology, or use C_X^{perf} to give a conjectural formula for the leading term of the $Z(\mathbb{C}[G])$-valued Zeta function of X at $s = 1$?

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