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# $p$-adic Banach Modules of Arithmetical Modular Forms and Triple Products of Coleman's Families

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*To dear Jean-Pierre Serre for his eightieth birthday with admiration*

**Abstract:** For a prime number $p \geq 5$, we consider three classical cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3)$$

of weights $k_1, k_2, k_3$, of conductors $N_1, N_2, N_3$, and of nebentypus characters $\psi_j \bmod N_j$.
According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each $f_j$ $(j = 1, 2, 3)$ (under suitable assumptions on $p$ and on $f_j$)

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j}) q^n\}$$

into a $p$-adic analytic family of cusp eigenforms $f_{j,k_j}$ of weights $k_j$ in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain $p$-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.
The conductor of the members of the families divides the prime-to-$p$ part of the level $N = \mathrm{LCM}(N_1, N_2, N_3)$.
The purpose of this paper is to describe a four variable $p$-adic $L$ function attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k) q^n \right\}$$

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of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha^{(1)}_{p,j}(k_j)) \geq 0$ where $\alpha^{(1)}_{p,j} = \alpha^{(1)}_{p,j}(k_j)$ is an eigenvalue (which depends on $k_j$) of Atkin's operator $U = U_p$ acting on Fourier expansions by $U(\sum_{n\geq 0}^{\infty} a_n q^n) = \sum_{n\geq 0}^{\infty} a_{np} q^n$.
Let us consider the product of three eigenvalues:

$$\lambda = \lambda(k_1, k_2, k_3) = \alpha^{(1)}_{p,1}(k_1)\alpha^{(1)}_{p,2}(k_2)\alpha^{(1)}_{p,3}(k_3)$$

and assume that the slope of this product

$$\sigma = v_p(\lambda(k_1, k_2, k_3)) = \sigma(k_1, k_2, k_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets $(k_1, k_2, k_3)$ in an appropriate $p$-adic neighbourhood of the fixed triplet of weights $(k_1, k_2, k_3)$. The each value $\sigma_j$ is fixed.
We consider the $p$-adic weight space $X$ containing all $(k_j, \psi_j)$. Our $p$-adic $L$-functions are Mellin transforms of certain measures with values in $\mathcal{A}$, where $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denotes an affinoid algebra associated with an affinoid space $\mathcal{B}$ as in [CoPB], where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integers $k_j$ and fixed Dirichlet characters $\psi_j \bmod N$).
We construct such a measure from higher twists of classical Siegel-Eisenstein series, which produce distributions with values in certain Banach $\mathcal{A}$-modules $\mathcal{M} = \mathcal{M}(N; \mathcal{A})$ of triple modular forms with coefficients in the algebra $\mathcal{A}$.

*Acknowledgement.* A part of these results was exposed in a talk at the French-German Seminar in Lille and the author is grateful to V.Gritsenko for the invitation.

It is a great pleasure for me to thank S.Boecherer, C.-G.Schmidt and R.Schulze-Pillot for valuable discussions.

Details of computations and proofs will appear elsewhere (a joint article with S.Boecherer in preparation for a special volume in the Contemporary Math. series of the AMS).

## Contents

## 1. Introduction

**Why study $L$-values attached to modular forms?** A popular proceedure in Number Theory is the following:

| Construct a generating function $f=\sum_{n=0}^\infty a_nq^n \in \mathbb{C}[[q]]$ of an arithmetical function $n\mapsto a_n$, for example $a_n=p(n)$ | $\rightsquigarrow$ | Compute $f$ via modular forms, for example $\displaystyle\sum_{n=0}^\infty p(n)q^n = (\Delta/q)^{-1/24}$ | $\rightsquigarrow$ | A number (solution) |
|---|---|---|---|---|
| Example 1 [Chand70]: (Hardy-Ramanujan) $p(n)=\dfrac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} +O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3)$, $\lambda_n=\sqrt{n-1/24}$, | | $\uparrow$ Good bases, finite dimensions, many relations and identities | | $\uparrow$ Values of $L$-functions, periods, congruences, ... |

Other examples: Birch and Swinnerton-Dyer conjecture, ... $L$-values attached to modular forms

*Our data: three primitive cusp eigenforms.*

$$f_j(z)=\sum_{n=1}^\infty a_{n,j}q^n\in \mathcal{S}_{k_j}(N_j,\psi_j),\ (j=1,2,3) \tag{1.1}$$

of weights $k_1,k_2,k_3$, of conductors $N_1,N_2,N_3$, and of nebentypus characters $\psi_j \bmod N_j$, $N=\mathrm{LCM}(N_1,N_2,N_3)$.

Let $p$ be a prime, $p\nmid N$. We view $f_j\in\overline{\mathbb{Q}}[\![q]\!]\overset{i_p}{\hookrightarrow}\mathbb{C}_p[\![q]\!]$ via a fixed embedding $\overline{\mathbb{Q}}\overset{i_p}{\hookrightarrow}\mathbb{C}_p$, $\mathbb{C}_p=\widehat{\overline{\mathbb{Q}}}_p$ is Tate's field.

Let $\chi$ denote a variable Dirichlet character mod $Np^v, v\geq 0$.

We view $k_j$ as a variable weight in the weight space $X=X_{Np^v}=\mathrm{Hom}_{cont}(Y,\mathbb{C}_p^*)$, $Y=(\mathbb{Z}/N\mathbb{Z})^*\times\mathbb{Z}_p^*\ni(y_0,y_p)$.

The space $X$ is a $p$-adic analytic space first used in Serre's [Se73] *"Formes modulaires et fonctions zêta p-adiques"*. Denote by $(k,\chi)\in X$ the homomorphism $(y_0,y_p)\mapsto\chi(y_0)\chi(y_p \bmod p^v)y_p^k$. We write simply $k_j$ for the couple $(k_j,\psi_j)\in X$.

*The purpose of this paper is to describe a four variable p-adic L function attached to Garrett's triple product of three Coleman's families.*

$$k_j\mapsto\left\{f_{j,k_j}=\sum_{n=1}^\infty a_{n,j}(k_j)q^n\right\}$$

of cusp eigenforms of three constant slopes $\sigma_j=\mathrm{ord}_p(\alpha^{(1)}_{p,j}(k_j))\geq 0$ where $\alpha^{(1)}_{p,j}(k_j)$, $\alpha^{(2)}_{p,j}(k_j)$ are the Satake parameters given as inverse roots of the Hecke $p$–polynomial $1-a_{p,j}X-\psi_j(p)p^{k_j-1}X^2=(1-\alpha^{(1)}_{p,j}(p)X)(1-\alpha^{(2)}_{p,j}(p)X)$. Each family of the associated primitive cusp eigenforms have conductors $C_{k_j}$ which divide $N$ if $k_j > 2\sigma_j + 2$ (see [CoM]).

We assume that $\mathrm{ord}_p(\alpha^{(1)}_{p,j}(k_j)) \le \mathrm{ord}_p(\alpha^{(2)}_{p,j}(k_j))$ and $\alpha^{(1)}_{p,j}(k_j) \ne \alpha^{(2)}_{p,j}(k_j)$

REMARK *of the referee: In the elliptic modular case, an equality is expected to*

*never happen if the weight > 1 (and this fact is proven when the weight $k$ is 2 and 3 by Coleman- Edixhoven and Ulmer under different conditions). However it happens in the Hilbert modular case, even when $k = 2$. Anyway once we assume that $\alpha^{(1)}_{p,j} \ne \alpha^{(2)}_{p,j}$ for the initial weight, this holds in an open neighborhood of the weight.*

The present paper extends a previous result: (see [PaTV]) where a two variable $p$-adic $L$-function was constructed interpolating on all $k$ a function $(k,s) \mapsto L^*(f_k, s, \chi)$ $(s = 1, \cdots, k-1)$ for such a family.

We use the theory of $p$-adic integration with values in spaces of nearly holomorphic modular forms (in the sense of Shimura, see [ShiAr]).

*A family of slope $\sigma > 0$ of cusp eigenforms $f_k$ of weight $k \ge 2$:*

| | |
|---|---|
| $k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k) q^n$<br>$\in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!]$<br>A model example<br>of a $p$-adic family<br>(not cusp and $\sigma = 0$):<br>Eisenstein series<br>$a_n(k) = \sum_{d\mid n} d^{k-1}, f_k = E_k$ | 1) the Fourier coefficients $a_n(k)$ of $f_k$<br>and one of the Satake<br>$p$-parameters $\alpha(k) := \alpha^{(1)}_p(k)$<br>are given by certain $p$-adic analytic<br>functions $k \mapsto a_n(k)$ for $(n,p) = 1$<br><br>2) the slope is constant and positive:<br>$\mathrm{ord}(\alpha(k)) = \sigma > 0$ |

The existence of families of slope $\sigma > 0$ was established in [CoPB]

| | |
|---|---|
| R.Coleman gave an example with<br>$p = 7,\ f = \Delta,\ k = 12$<br>$a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1.$ | A program in PARI for computing<br>such families is discussed in [CST98]<br>(see also the Web-page of W.Stein,<br>`http://modular.fas.harvard.edu/` ) |

*It was established by Coleman that:*

• The operator $U$ acts as a completely continuous operator on each $\mathcal{A}$-submodule $\mathcal{M}^\dagger(Np^v;\mathcal{A}) \subset \mathcal{A}[\![q]\!]$ (i.e. $U$ is a limit of finite-dimensional operators) $\Longrightarrow$ there exists the Fredholm determinant $P_U(T) = \det(Id - T\cdot U) \in \mathcal{A}[\![T]\!]$

• there is a version of the Riesz theory: for any inverse root $\alpha \in \mathcal{A}^*$ of $P_U(T)$ there exists an eigenfunction $g$, $Ug = \alpha g$ such that $ev_k(g) \in \mathbb{C}_p[\![q]\!]$ are classical cusp eigenforms for all $k$ in a neigbourhood $\mathcal{B} \subset X$ (see in [CoPB])

We refer to [CoPB], part B and [PaTV], section 1, for generalities on rigid analytic $p$-adic families of modular forms.

## 2. Generalities on triple products

The triple product with a Dirichlet character $\chi$ is defined as the following complex $L$-function ( an Euler product of degree eight):

$$L(f_1\otimes f_2\otimes f_3, s, \chi) = \prod_{p\nmid N} L((f_1\otimes f_2\otimes f_3)_p, \chi(p)p^{-s}), \tag{2.2}$$

(2.3) where $L((f_1\otimes f_2\otimes f_3)_p, X)^{-1} =$

$$\det\left(1_8 - X\begin{pmatrix}\alpha^{(1)}_{p,1} & 0\\ 0 & \alpha^{(2)}_{p,1}\end{pmatrix}\otimes\begin{pmatrix}\alpha^{(1)}_{p,2} & 0\\ 0 & \alpha^{(2)}_{p,2}\end{pmatrix}\otimes\begin{pmatrix}\alpha^{(1)}_{p,3} & 0\\ 0 & \alpha^{(2)}_{p,3}\end{pmatrix}\right)$$

$$= \prod_\eta (1 - \alpha^{(\eta(1))}_{p,1}\alpha^{(\eta(2))}_{p,2}\alpha^{(\eta(3))}_{p,3}X)$$

$$= (1-\alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(1)}_{p,3}X)(1-\alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(2)}_{p,3}X)\cdot\ldots\cdot(1-\alpha^{(2)}_{p,1}\alpha^{(2)}_{p,2}\alpha^{(2)}_{p,3}X),$$

product taken over all 8 maps $\eta : \{1,2,3\} \to \{1,2\}$.

*The Satake parameters and Hecke $p$–polynomials of forms $f_j$:* Here the Satake parameters $\alpha^{(1)}_{p,j}$, $\alpha^{(2)}_{p,j}$ are given as inverse roots of the Hecke $p$–polynomials

$$1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1-\alpha^{(1)}_{p,j}(p)X)(1-\alpha^{(2)}_{p,j}(p)X).$$

We always assume that the weights are "balanced":

$$k_1 \ge k_2 \ge k_3 \ge 2, \text{ and } k_1 \le k_2 + k_3 - 2 \tag{2.4}$$

The non-balanced case (i.e. $k_1 > k_2 + k_3 - 2$) was treated in [HaTi], where $p$-adic measures for the square roots of special values of triple product $L$-functions, were constructed in the ordinary case. This construction can be probably extended to Coleman's case by techniques similar to the present paper.

*Critical values and functional equation.* We use the corresponding normalized $L$ function (see [De79], [Co], [Co-PeRi]), which has the form:

(2.5) $$\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k_3+1)\Gamma_{\mathbb{C}}(s-k_2+1)\Gamma_{\mathbb{C}}(s-k_1+1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

The Gamma-factor determines the critical values $s = k_1, \cdots, k_2+k_3-2$ of $\Lambda(s)$, which we explicitely evaluate (like in the classical formula $\zeta(2) = \frac{\pi^2}{6}$). A functional equation of $\Lambda(s)$ has the form:

$$s \mapsto k_1 + k_2 + k_3 - 2 - s.$$

. According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each $f_j$ ($j = 1, 2, 3$) (under suitable assumptions on $p$ and on $f_j$) into a $p$-adic analytic family

$$\mathbf{f}_j : k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j})q^n\}$$

of cusp eigenforms $f_{j,k_j}$ of weights $k_j$ in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain $p$-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.

## 3. Statement of the problem

Given three $p$-adic analytic families $\mathbf{f}_j$ of slope $\sigma_j \geq 0$, to construct a four-variable $p$-adic $L$-function attached to Garrett's triple product of these families. We show that this function interpolates the special values

$$(s, k_1, k_2, k_2) \longmapsto \Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, s, \chi)$$

at critical points $s = k_1, \cdots, k_2+k_3-2$ for balanced weights $k_1 \leq k_2+k_3-2$; we prove that these values are algebraic numbers after dividing out certain "periods".

However the construction uses directly modular forms, and not the $L$-values in question, and a comparison of special values of two functions is done after the construction.

Consider the product of the Satake parmeters

$$\lambda_p = \alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(1)}_{p,3} = \lambda_p(k_1, k_2, k_3)$$

We assume that $\mathrm{ord}_p\alpha^{(1)}_{p,j} \leq \mathrm{ord}_p\alpha^{(2)}_{p,j}$, and that the slope $\sigma = \mathrm{ord}_p(\lambda_p(k_1, k_2, k_3))$ is constant and positive for all triplets $(k_1, k_2, k_3)$ in a $p$-adic neighbourhood $\mathcal{B} \subset X^3$ of the fixed triplet of weights $(k_1, k_2, k_3)$.

*Our method includes:* • a version of Garrett's integral representation for the triple $L$-functions of the form: for $r = 0, \cdots, k_2 + k_3 - k_1 - 2$,
$\Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - r, \chi) =$

$$\int\int\int_{(\Gamma_0(N^2p^{2v})\backslash\mathbb{H})^3} \overline{\tilde{f}_{1,k_1}(z_1)\tilde{f}_{2,k_2}(z_2)\tilde{f}_{3,k_3}(z_3)}\mathcal{E}(z_1, z_2, z_3; -r, \chi) \prod_j (\frac{dx_j dy_j}{y_j^2})$$

where $\tilde{f}_{j,k_j} =: f^0_{j,k_j}$ is an eigenfunction of $U_p^*$ in $\mathcal{M}_{k_j}(Np, \psi_j)$, $f_{j,k_j,0}$ is the corresponding eigenfunction of $U_p$, and

$$\mathcal{E}(z_1, z_2, z_3; -r, \chi) \in \mathcal{M}_T(N^2p^{2v}) = \mathcal{M}_{k_1,r^*}(N^2p^{2v}, \psi_1) \otimes \mathcal{M}_{k_2,r^*}(N^2p^{2v}, \psi_2) \otimes \mathcal{M}_{k_3,r^*}(N^2p^{2v}, \psi_3)$$

is a (classical) nearly holomorphic triple modular form of triple weight $(k_1, k_2, k_3)$ of some type $r^* \geq 0$ (see [ShiAr]), and of fixed triple Nebentupus character $(\psi_1, \psi_2, \psi_3)$, obtained from a nearly holomorphic Siegel-Eisenstein series $F_{\chi,r} = G^\star(z, -r; k, (Np^v)^2, \boldsymbol{\psi})$, of degree 3, of weight $k = k_2 + k_3 - k_1$, and the Nebentypus character $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$ ([Sh83]).

We obtain $\mathcal{E}(z_1, z_2, z_3; -r, \chi)$ from a Siegel-Eisenstein series by applying to $F_{\chi,r}$ Boecherer's higher twist (see (11.22)) and Ibukiyama's differential operator (see (11.23)).

These operations act explicitely on the Fourier expansions. Then one uses:

• The theory of $p$-adic integration with values in Serre's type $\mathcal{A}$-modules $\mathcal{M}_T(\mathcal{A})$ of triple arithmetical nearly holomorphic modular forms over $p$-adic Banach algebras $\mathcal{A}$. We shall use the notation

$$\mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A}(\mathcal{B}_1)) \hat{\otimes} \mathcal{M}(\mathcal{A}(\mathcal{B}_2)) \hat{\otimes} \mathcal{M}(\mathcal{A}(\mathcal{B}_3))$$

for certain $\mathcal{A}$-modules of $p$-adic families of triple modular forms.

Explicit Fourier coefficients $a_{\chi,r}(R, \mathcal{T}) \in \overline{\mathbb{Q}}[R, T]$ of $\mathcal{E}(-r, \chi)$ are given by special polynomials of matricies $\mathcal{T} = (t_{ij})$, $R = (R_{ij})$ and of $\chi(\beta)\beta^r$ (with $\beta \in \mathbb{Z}_p^* \cap \mathbb{Q}$) i.e. the coefficients of $a_{\chi,r}$ by some elementry $p$-adic measures $\int_Y \chi y^r \mathrm{d}\mu_{\mathcal{T}} \in \mathcal{A}$. Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain $p$-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$ in the product of three copies of the weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$.

These measures on the group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$ produce the coefficients of $a_{\chi,r}$ of $\mathcal{E}(-r, \chi)$ of $\mathcal{M}_T(\mathcal{A})$ for all $p$-adic weights $x \in X$, given by $\int_Y x(y)\mathrm{d}\mu_{\mathcal{T}} \in \mathcal{A}$ (*an interpolation from* $x = \chi y_p^r$ *to all* $x \in X$).

• *The* spectral theory of triple Atkin's operator $U = U_{p,T}$. allows to evaluate the integral using at each weight $(k_1, k_2, k_3)$ the equality $\langle \mathbf{f}^0, \mathcal{E}(-r, \chi)\rangle = \langle \mathbf{f}^0, \pi_\lambda(\mathcal{E}(-r, \chi))\rangle$

with the projection $\pi_\lambda$ of $\mathcal{M}_T(\mathcal{A})$ to the $\lambda$-part $\mathcal{M}_T(\mathcal{A})^\lambda$, defined by:

$$\operatorname{Ker}\pi_\lambda := \bigcap_{n\geq 1} \operatorname{Im}(U_T - \lambda I)^n, \quad \operatorname{Im}\pi_\lambda := \bigcup_{n\geq 1} \operatorname{Ker}(U_T - \lambda I)^n.$$

Note that

$$\mathbf{f}^0 = f^0_{k_1,1} \otimes f^0_{k_2,2} \otimes f^0_{k_3,3}, \quad f^0_{k_j,j} = f^\rho_{k_j,j,0}|_{k_j} \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix} (\text{for } j = 1, 2, 3)$$

is a classical form, defined at triple weghts $(k_1, k_2, k_3)$ as the Weil-involution image of $\mathbf{f}_0 = f_{k_1,1,0} \otimes f_{k_2,2,0} \otimes f_{k_3,3,0}$ as in [PaTV], at p.555. Note that the functions $f^0$ are eigenfunctions of the adjoint triple operator $U_p^*$. We consider also the sequences of triple modular forms:

$$\mathbf{f}_0 = \{f_{k_1,1,0} \otimes f_{k_2,2,0} \otimes f_{k_3,3,0}\}_{(k_1,k_2,k_3)}, \ \mathbf{f}^0 = \{f^0_{k_1,1} \otimes f^0_{k_2,2} \otimes f^0_{k_3,3}\}_{(k_1,k_2,k_3)}.$$

The functions $\mathbf{f}_0$ form a $p$-adic family. The evaluation at triple weight $k = (k_1, k_2, k_3)$ of the $p$-adic coordinate with respect to $p$-adic family $\mathbf{f}_0$ is expressed through the triple Petersson scalar product with $\mathbf{f}^0$, which is algebraically orthogonal to $\mathbf{f}_0$ in the sense of Hida [Hi90] for classical weights (and in [PaTV] for Coleman's families).

We prove that $U$ is a completely continuos $\mathcal{A}$-linear operator on a certain Coleman's submodule $\mathcal{M}(\mathcal{A})^\dagger$ of Serre's type module $\mathcal{M}(\mathcal{A})$. Then the projection $\pi_\lambda$ exists (on this submodule) due to general results of Serre and Coleman, see [CoPB], [SePB].

We show that there exists an element $\tilde{\mathcal{E}}(-r,\chi) \in \mathcal{M}(\mathcal{A})^\dagger$ such that at each weight $(k_1, k_2, k_3)$ the equality holds: $\left\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\right\rangle = \left\langle \mathbf{f}^0, \pi_\lambda(\tilde{\mathcal{E}}(-r,\chi))\right\rangle$, and the product can be expressed through certain coefficients the series $\tilde{\mathcal{E}}(-r,\chi)$ which are the same as those of $\mathcal{E}(-r,\chi)$.

• Key point: modular admissible measures. Let us write for simplicity: $\mathcal{E}(-r,\chi)$ for $\tilde{\mathcal{E}}(-r,\chi)$

$\mathcal{M}_T(\mathcal{A})$ instead of $\mathcal{M}_T(\mathcal{A})^\dagger$ (Coleman's submodule)

One defines admissible $p$-adic measures $\tilde{\Phi}^\lambda$ with values in Banach $\mathcal{A}$-modules $\mathcal{M}_T^\lambda(\mathcal{A})$ which are locally free of finite rank, using the test functions: $\int_Y \chi y_p^r \tilde{\Phi}^\lambda = \pi_\lambda(\mathcal{E}(-r,\chi))$.

Consider the evaluation maps $ev_{\mathsf{s}} : \mathcal{A} \to \mathbb{C}_p$ for any $p$-adic triple weights $\mathsf{s} = (s_1, s_2, s_3) \in \mathcal{B}$.

• Passage from values in modular forms to scalar values: apply an algebraic $\mathcal{A}$-linear form $\mathcal{M}_T^\lambda(\mathcal{A}) \xrightarrow{\ell_T} \mathcal{A}$ to the constructed measure $\tilde{\Phi}^\lambda$ (in modular forms), and the evaluation maps $\mathcal{A} \xrightarrow{ev_{\mathbf{s}}} \mathbb{C}_p$ for any $p$-adic triple weights $\mathbf{s} \in X^3$.

The linear form $\ell_T$ is an algebraic version of the Petersson product (a geometric meaning of $\ell_T$: the first coordinate in an (orthogonal) $\mathcal{A}$-basis of eigenfunctions of all Hecke operators $T_q$ for $q \nmid Np$, with the first basis element $\mathbf{f}_0 \in \mathcal{M}^\lambda(\mathcal{A})$).

*Using the evaluation map and the Mellin transform.* We obtain the measure $\mu = \ell_T(\tilde{\Phi}^\lambda)$ with values in $\mathcal{A}$ on the profinite group $Y$.

• Construct an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the $p$-adic Mellin transform $\mathcal{L}_\mu(x) = \int_Y x(y)\, d\mu(y) \in \mathcal{A}$, $x \in X$.

• Solution: the function in question $\mathcal{L}_\mu(x, \mathbf{s})$ is given by evaluation of $\mathcal{L}_\mu(x)$ at $\mathbf{s} = (s_1, s_2, s_3) \in \mathcal{B}$: this is a $p$-adic analytic function in four variables

$$(x, \mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$$

$$\mathcal{L}_\mu(x, \mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_\mu(x)) \quad (x \in X,\ \mathbf{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_\mu(x) \in \mathcal{A}).$$

*Final step: comparison between $\mathbb{C}$ and $\mathbb{C}_p$.* • We check an equality relating the values of the constructed analytic function $\mathcal{L}_\mu(x, \mathsf{s})$ at the arithmetical characters $x = y_p^r \chi \in X$, and at triple weights $\mathsf{s} = (k_1, k_2, k_3) \in \mathcal{B}$, with the normalized critical special values

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \ \ (r = 0, \cdots, k_2 + k_3 - k_1 - 2),$$

for certain Dirichlet characters $\chi$ mod $Np^v, v \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ (the Tate field of $p$-adic numbers). The normalisation of $L^*$ includes at the same time Gauss sums, Petersson scalar products, powers of $\pi$, the product $\lambda_p(k_1, k_2, k_3)$, and a certain finite Euler product.

We refer to Theorem 0.3 at p.556 of [PaTV], where the explicit form of such finite Euler product is given in the case of the $L$-function of one Coleman's family.

## 4. Arithmetical nearly holomorphic modular forms

*Arithmetical nearly holomorphic modular forms (the elliptic case).* Let $\mathcal{A}$ be a commutative ring (a subring of $\mathbb{C}$ or $\mathbb{C}_p$)

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr] are certain formal series

$$g = \sum_{n=0}^{\infty} a(n; R) q^n \in \mathcal{A}[\![q]\!][R], \text{ with the property}$$

that for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}$ of a given weight $k$ and Dirichlet character $\psi$. The coefficients $a(n;R)$ are polynomials in $\mathcal{A}[R]$. If $\deg_R a(n;R) \le r$ for all $n$, we call $g$ nearly holomorphic of type $r$ (it is annihilated by $(\frac{\partial}{\partial \bar z})^{r+1}$, see [ShiAr]).

. We use the notation $\mathcal{M}_{k,r}(N,\psi,\mathcal{A})$ or $\tilde{\mathcal{M}}(N,\psi,\mathcal{A})$ for $\mathcal{A}$-modules of such forms (In our constructions the weight $k$ varies).

A known example (see the introduction to [ShiAr]) is given by the series

$$\begin{aligned}
&-12R + E_2 := -12R + 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n \\
&= \frac{3}{\pi^2}\lim_{s\to 0} y^s \sum_{m_1,m_2\in\mathbb{Z}}{}' (m_1+m_2z)^{-2}|m_1+m_2z|^{-2s}, (R = (4\pi y)^{-1})
\end{aligned}$$

where $\sigma_1(n) = \sum_{d|n} d$.

The action of the Shimura differential operator

$$\delta_k : \mathcal{M}_{k,r}(N,\psi,\mathcal{A}) \to \mathcal{M}_{k+2,r+1}(N,\psi,\mathcal{A}),$$

is given over $\mathbb{C}$ by $\delta_k(f) = (\dfrac{1}{2\pi i}\dfrac{\partial}{\partial z} - \dfrac{k}{4\pi y})f$.

This operator is a correction of the Ramanujan operator

$$\theta(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} n a_n q^n = \frac{1}{2\pi i}\frac{\partial}{\partial z}(\sum_{n=0}^{\infty} a_n q^n) = q\frac{\partial}{\partial q}(\sum_{n=0}^{\infty} a_n q^n),$$

which does not preserve the modularity. For example $\theta\Delta = E_2\Delta$, where $E_2$ is a quasimodular form (in the sense of Kaneko and Zagier, see [Ka-Za]).

Notice that $\delta_k f = (\theta - kR)f$, and that Serre's operator $f \mapsto \theta f - \frac{k}{12}E_2 f$ takes $\mathcal{M}_k$ to $\mathcal{M}_{k+2}$.

Note that that the arithmetical twist operator

$$\theta_\chi(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} \chi(n) a_n q^n$$

is a natural analog of the Ramanujan operator.

*Triple arithmetical modular forms.* Let $\mathcal{A}$ be a commutative ring. The tensor product over $\mathcal{A}$

$$\mathcal{M}_{\mathbf{k},r,T}(N,\psi,\mathcal{A}) := \mathcal{M}_{k_1,r}(N,\psi_1,\mathcal{A}) \otimes \mathcal{M}_{k_2,r}(N,\psi_2,\mathcal{A}) \otimes \mathcal{M}_{k_3,r}(N,\psi_3,\mathcal{A})$$

consists of triple arithmetical modular forms as certain formal series of the form

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3)q_1^{n_1}q_2^{n_2}q_3^{n_3}$$
$$\in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H},\ R_j = (4\pi y_j)^{-1},$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight $(k_1,k_2,k_3)$ and character $(\psi_1,\psi_2,\psi_3)$, $j = 1,2,3$. The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$. Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.

## 5. Siegel-Eisenstein series

*Siegel modular groups.* Let $J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}$. The symplectic group

$$\mathrm{Sp}_m(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2m}(\mathbb{R}) | {}^t g \cdot J_{2m} g = J_{2m} \right\},$$

acts on the Siegel upper half plane

$$\mathbb{H}_m = \left\{ z = {}^t z \in M_m(\mathbb{C}) | \mathrm{Im}\, z > 0 \right\}$$

by $g(z) = (az+b)(cz+d)^{-1}$, where we use the bloc notation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{R})$. We use the congruence subgroup $\Gamma_0^m(N) = \{\gamma \in \mathrm{Sp}_m(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\} \subset \mathrm{Sp}_m(\mathbb{Z})$.

*A Siegel modular form.* $f \in \mathcal{M}_k(\Gamma_0^m(N),\chi)$ of degree $m > 1$, weight $k$ and a Dirichlet chracter $\chi \bmod N$ is a holomorphic function $f\colon \mathbb{H}_m \to \mathbb{C}$ such that for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(N)$ one has

$$f(\gamma(z)) = \chi(\det\, d) \det(cz+d)^k f(z).$$

The Fourier expansion of $f$ uses the symbol

$$q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}z)) = \prod_{i=1}^m q_{ii}^{\mathcal{T}_{ii}} \prod_{i<j} q_{ij}^{2\mathcal{T}_{ij}} \in \mathbb{C}[q_{11},\cdots,q_{mm},q_{ij},q_{ij}^{-1}]_{1\le i<j\le m}$$

for each Fourier coefficient, where $q_{ij} = \exp(2\pi(\sqrt{-1}z_{i,j}))$, and $\mathcal{T}$ in the semi-group $B_m = \{\mathcal{T} = {}^t\mathcal{T} \ge 0 | \mathcal{T} \text{ half-integral}\}$. We may consider $f(z) = \sum_{\mathcal{T}\in B_m} a(\mathcal{T})q^{\mathcal{T}} \in \mathbb{C}[\![q^{B_m}]\!]$ as a formal $q$-expansion in $\mathbb{C}[\![q^{B_m}]\!]$ (the subring of $\mathbb{C}[\![q_{11},...,q_{mm}]\!][q_{ij},q_{ij}^{-1}]$) generated by all $q^{\mathcal{T}}$).

*Siegel-Eisenstein series.*

EXAMPLE 5.1 ( [Nag2], p.408).

$$\begin{aligned}E_4^{(2)}(z) =&1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^{-2} + 13440q_{12}^{-1}\\ &+ 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \dots\\ E_6^{(2)}(z) =&1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^{-2} + 44352q_{12}^{-1}\\ &+ 166320 + 44352q_{12} - 504q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \dots .\end{aligned}$$

**Arithmetical nearly holomorphic Siegel modular forms.**

*Arithmetical Siegel modular forms.* Consider a commutative ring $\mathcal{A}$, the formal variables $q = (q_{i,j})_{i,j=1,\dots,m}$, $R = (R_{i,j})_{i,j=1,\dots,m}$, and the ring of formal Fourier series

$$\mathcal{A}[\![q^{B_m}]\!][R_{i,j}] = \left\{ f = \sum_{\mathcal{T}\in B_m} a(\mathcal{T}, R)q^{\mathcal{T}} \;\middle|\; a(\mathcal{T}, R) \in \mathcal{A}[R_{i,j}] \right\} \tag{5.6}$$

(over the complex numbers this notation corresponds to $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}z))$, $R = (4\pi \mathrm{Im}(z))^{-1}$).

The formal Fourier expansion of a nearly holomorphic Siegel modular form $f$ with coefficients in $\mathcal{A}$ is a certain element of $\mathcal{A}[\![q^{B_m}]\!][R_{i,j}]$. We call $f$ arithmetical in the sense of Shimura [ShiAr], if $\mathcal{A} = \overline{\mathbb{Q}}$.

## 5.1. Algebraic differential operators of Maass and Shimura.

*Maass differential operator.* Let us consider the Maass differential operator (see [Maa]) $\Delta_m$ of degree $m$, acting on complex $\mathcal{C}^\infty$-functions on $\mathbb{H}_m$ by:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \qquad \tilde{\partial}_{ij} = 2^{-1}(1 + \delta_{ij})\partial/\partial_{ij}, \tag{5.7}$$

its algebraic version is the Ramanujan operator of degree $m$:

$$\Theta_m := \det(\frac{1}{2\pi i}\tilde{\partial}_{ij}) = \det(\theta_{ij}) = \frac{1}{(2\pi i)^m}\Delta_m, \tag{5.8}$$

where $\Theta_m(q^{\mathcal{T}}) = \det(\mathcal{T})q^{\mathcal{T}}$.

*Shimura differential operator.* The Shimura differential operator (see [Shi76, ShiAr]):

$$\delta_k f(z) = \det(R)^{k+1-\varkappa}\Theta_m \left[\det(R)^{\varkappa-1-k} f\right], \text{ where } R = (4\pi y)^{-1},$$

acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

(5.9) $\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \widetilde{\mathcal{M}}_k^m(N,\psi;\overline{\mathbb{Q}}) \to \widetilde{\mathcal{M}}_{k+2rm}^m(N,\psi;\overline{\mathbb{Q}})$,

where

$$\delta_k f(z) = \left(\frac{-1}{4\pi}\right)^m \det(y)^{-1} \det(z-\bar{z})^{\varkappa-k} \Delta_m \left[\det(z-\bar{z})^{k-\varkappa+1} f(z)\right].$$

*Universal polynomials* $Q(R,\mathcal{T};k,r)$. Let $f = \sum_{\mathcal{T}\in B_m} c(\mathcal{T}) q^{\mathcal{T}} \in \mathcal{M}_k^m(N,\psi)$ be a formal holomorphic Fourier expansion. One shows that $\delta_k^{(r)} f$ is given by

$$\delta_k^{(r)} f = \sum_{\mathcal{T}\in B_m} Q(R,\mathcal{T};k,r) c(\mathcal{T}) q^{\mathcal{T}}.$$

Here we use a universal polynomial (5.10) which can be defined for all $k \in \mathbb{C}$, and it expresses the action of the Shimura operator on the exponential (of degree $m$):

$$\delta_k^{(r)}(q^{\mathcal{T}}) = Q(R,\mathcal{T};k,r) q^{\mathcal{T}}.$$

If $m = 1$, $r$ arbitrary (see [Shi76]), $\delta_k^{(r)} = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{\Gamma(k+r)}{\Gamma(k+j)} R^{r-j} \theta^j$,

$Q(R,n;k,r) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{\Gamma(k+r)}{\Gamma(k+j)} R^{r-j} n^j$.

If $r = 1$, $m$ arbitrary, one has (see [Maa]):

$$\delta_k f(z) = \sum_{\mathcal{T}\in B_m} c(\mathcal{T}) \sum_{l=0}^{m} (-1)^{m-l} c_{m-l}(k+1-\varkappa) \mathrm{tr}\left({}^t\rho_{m-l}(R) \cdot \rho_l^\star(\mathcal{T})\right) q^{\mathcal{T}}$$

where $R = (4\pi y)^{-1} = (R_{i,j}) \in \mathrm{M}_m(\mathbb{R})$, $c_m(\alpha) = \dfrac{\Gamma_m(\alpha+\kappa)}{\Gamma_m(\alpha+\kappa-1)}$, $\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s-(j/2))$.

Here we use the natural representation $\rho_r : \mathrm{GL}_m(\mathbb{C}) \longrightarrow \mathrm{GL}(\wedge^r \mathbb{C}^m)$ $(0 \le r \le m)$ of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r\mathbb{C}^m$. Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r} \times \binom{m}{r}$ composed of the subdeterminants of $z$ of degree $r$. Put $\rho_r^\star(z) = \det(z)\rho_{m-r}({}^t z)^{-1}$.

Then the representations $\rho_r$ and $\rho_r^\star$ turn out to be polynomial representations.

. In general (see [CourPa], Theorem 3.14) one has:

$$
\begin{aligned}
(5.10)\qquad & Q(R,\mathcal{T}) = Q(R,\mathcal{T};k,r)\\
&= \sum_{t=0}^{r}\binom{r}{t}\det(\mathcal{T})^{r-t}\sum_{|L|\le mt-t} R_L(\kappa-k-r)Q_L(R,\mathcal{T}),\\
& Q_L(R,\mathcal{T}) = \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\mathcal{T})\right)\cdot\ldots\cdot\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\mathcal{T})\right)).
\end{aligned}
$$

In (5.10), $L$ goes over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, and $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ in (5.10) are polynomials in $\beta$ of degree $(mt - |L|)$ (used with $\beta = \kappa - k - r$).

Note the differentiation rule of degree $m$ (see [Sh83], p.466): $\Delta(fg) = \sum_{r=0}^{m}\mathrm{tr}\left({}^t\rho_r(\tilde\partial/\partial z)f\cdot\rho^\star_{m-r}(\tilde\partial/\partial z)g\right)$. As in (5.8), we write here $\tilde\partial/\partial z = (\tilde\partial/\partial z_{ij})$ for the matrix with entries $\tilde\partial_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij}$.

.

Example 5.2 (Siegel-Eisenstein series of odd degree and higher level).

$$
\begin{aligned}
(5.11)\qquad & G^*(z,s;k,\boldsymbol{\psi},N)\\
&= \det(y)^s\sum_{c,d}\boldsymbol{\psi}(\det c)\det(cz+d)^{-k}|\det(cz+d)|^{-2s}\ \cdot\\
&\cdot\tilde\Gamma(k,s)L_N(k+2s,\boldsymbol{\psi})\left(\prod_{i=1}^{[m/2]}L_N(2k+4s-2i,\boldsymbol{\psi}^2)\right),\ \textit{where}
\end{aligned}
$$

$(c,d)$ *runs over all "non-associated coprime symmetric pairs" with* $\det(c)$ *coprime to* $N$, $\kappa = (m+1)/2$, *and for* $m$ *odd the* $\Gamma$*-factor has the form:* $\tilde\Gamma(k,s) = i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma_m(k+s)$.

We use this series with $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, $k = k_2 + k_3 - k_1 \ge 2$, $m = 3$, $\kappa = \dfrac{m+1}{2} = 2$, $[m/2] = 1$.

.

Theorem 5.3 (Siegel, Shimura [Sh83], P. Feit [Fei86]). *Let* $m$ *be an* odd integer *such that* $2k > m$, *and* $N > 1$ *be an integer, then:*

*For an integer* $s$ *such that* $s = -r$, $0 \le r \le k - \kappa$, *there is the following Fourier expansion*

$$
(5.12)\qquad G^\star(z,-r) = G^\star(z,-r;k,\boldsymbol{\psi},N) = \sum_{A_m\ni\mathcal{T}\ge 0} a(\mathcal{T},R)q^{\mathcal{T}},
$$

*where for $s > (m+2-2k)/4$ in (5.12) the only non-zero terms occur for positive definite $\mathcal{T} > 0$,*

$$a(\mathcal{T}, R) = M(\mathcal{T}, \boldsymbol{\psi}, k-2r) \cdot \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k-2r, r), \tag{5.13}$$

$$M(\mathcal{T}, k-2r, \boldsymbol{\psi}) = \prod_{\ell | \det(2\mathcal{T})} M_\ell(\mathcal{T}, \boldsymbol{\psi}(\ell)\ell^{-k+2r}) \tag{5.14}$$

*polynomials $Q(R, \mathcal{T}; k-2r, r)$ are given by (5.10), and for all $\mathcal{T} > 0$, $\mathcal{T} \in A_m$, is a finite Euler product, in which $M_\ell(\mathcal{T}, x) \in \mathbb{Z}[x]$.* □

## 6. Statement of the Main Result

*Main Theorem (on p-adic analytic function in four variables).* The following Main Theorem corresponds to Theorem 0.3 at p.556 of [PaTV] in the situation of the $L$-function attached to one Coleman's family.

Main Theorem 6.1. 1) *The function $\mathcal{L}_{\boldsymbol{f}} : (s, k_1, k_2, k_3) \mapsto \frac{\langle \boldsymbol{f}^0, \mathcal{E}(-r,\chi) \rangle}{\langle \boldsymbol{f}^0, \boldsymbol{f}_0 \rangle}$ extends to a p-adic analytic function on four variables $(\chi \cdot y_p^r, k_1, k_2, k_3) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$;*

2) *Comparison of complex and p-adic values: for all $(k_1, k_2, k_3)$ in an affinoid neighborhood $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$, satisfying $k_1 \le k_2 + k_3 - 2$: the values at $s = k_2 + k_3 - 2 - r$ coincide with the normalized critical special values*

$$\begin{aligned} &L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \\ &(r = 0, \cdots, k_2 + k_3 - k_1 - 2), \end{aligned} \tag{6.15}$$

*for Dirichlet characters $\chi \bmod Np^v, v \ge 1$, such that all three corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductors:*

$$\begin{aligned} &\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2 \bar{\psi}_3 \chi, \\ &\chi_3 \bmod Np^v = \psi_1 \bar{\psi}_3 \chi, \boldsymbol{\psi} = \chi^2 \psi_1 \psi_2 \overline{\psi}_3. \end{aligned} \tag{6.16}$$

*The normalisation of $L^*$ in (6.15) is the same as in Theorem C below.*

3) *Dependence on $x \in X$: let $H = [2\mathrm{ord}_p(\lambda)] + 1$. For any fixed $(k_1, k_2, k_3) \in \mathcal{B}$ and $x = \chi \cdot y_p^r$ the function*

$$x \longmapsto \frac{\langle \boldsymbol{f}^0, \mathcal{E}(-r, \chi) \rangle}{\langle \boldsymbol{f}^0, \boldsymbol{f}_0 \rangle}$$

*extends to a p-adic analytic function of type $o(\log^H(\cdot))$ of the variable $x \in X$.*

REMARK. The function $\mathcal{L}_{\mathbf{f}}$ depends on the variables $(s, k_1, k_2, k_3)$ in a different way: it is a mixture of the $p$-adic Mellin transform (in $s$), and of a rigid analytic function (in $k_1, k_2, k_3$).

*Outline of the proof.* The proof follows the lines given in Sections 5-7 in [PaTV] (the case of the $L$-function of one Coleman's family).

1) • (compare with Section 5 in [PaTV]). At each classical weight $(k_1, k_2, k_3)$ let us use the equality

$$\left\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\right\rangle = \left\langle \mathbf{f}^0, \pi_\lambda(\mathcal{E}(-r,\chi))\right\rangle$$

which is deduced from the definition of the projector $\pi_\lambda$: $\operatorname{Ker}\pi_\lambda := \bigcap_{n\geq 1} \operatorname{Im}(U_T - \lambda I)^n$, $\operatorname{Im}\pi_\lambda := \bigcup_{n\geq 1}\operatorname{Ker}(U_T-\lambda I)^n$.

Notice that the coefficients of $\mathcal{E}(-r,\chi) \in \mathcal{M}(\mathcal{A})$ depend $p$-adic analytically on $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, where $\mathcal{A} = \mathcal{A}(\mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3)$ is the $p$-adic Banach algebra of rigid-analytic functions on $\mathcal{B}$.

Interpolation to all $p$-adic weights: • At each classical weight $(k_1, k_2, k_3)$ the scalar product $\left\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\right\rangle$ is given by the first coordinate of $\pi_\lambda(\mathcal{E}(-r,\chi))$ with respect to an orthogonal basis of $\mathcal{M}^\lambda(\mathcal{A})$ containing $\mathbf{f}_0$ with respect to Hida's algebraic Petersson product $\langle g, h\rangle_a := \left\langle g^\rho | \begin{pmatrix} 0 & -1 \\ Np & 0\end{pmatrix}, h\right\rangle$, see [Hi90].

Let us extend the linear form $\ell(h) = \frac{\langle \mathbf{f}^0, h\rangle}{\langle \mathbf{f}^0, \mathbf{f}_0\rangle}$ (defined firs only for classical weights), to Coleman's type submodule of overconvergent families $h \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}^\lambda(\mathcal{A})$ as the first coordinate of $h$ with respect to some $\mathcal{A}$-basis of eigenfunctions of all (triple) Hecke operators $T_q$ for $q \nmid Np$, having the first basis vector $\mathbf{f}_0 \in \mathcal{M}^\lambda(\mathcal{A})^\dagger$.

The linear form $\ell$ can be characterized as a normalized eigenfunction of the adjoint Atkin's operator, acting on the dual $\mathcal{A}$-module of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell(\mathbf{f}_0) = 1$.

In order to extend $\ell$ to $h = \mathcal{E}(-r,\chi)$, we need to choose a certain representative of $\mathcal{E}(-r,\chi)$ in the $\mathcal{A}$-submodule $\mathcal{M}^\lambda(\mathcal{A})^\dagger$ , which is locally free of finite rank.

*A representative of* $\mathcal{E}(-r,\chi)$ *in the (locally free of finite rank* $\mathcal{A}$*-submodule)* $\mathcal{M}^\lambda(\mathcal{A})^\dagger$. (compare with Section 6 in [PaTV] in the case of the $L$-function of one Coleman's family). Choose a (local) basis $\ell^1, \cdots, \ell^n$ given by some *triple Fourier coefficients* of the dual (locally free of finite rank) $\mathcal{A}$-module $\mathcal{M}^\lambda(\mathcal{A})^{\dagger *}$.

Then define

$$\ell = \beta_1\ell^1 + \cdots + \beta_n\ell^n,$$

where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and $\ell_i$ denotes the dual basis of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell^j(\ell_i) = \delta_{ij}$.

At each $p$-adic weight $(k_1, k_2, k_3) \in \mathcal{B}$ let us define

$$\ell(\mathcal{E}(-r,\chi)) := \beta_1 \ell^1(\mathcal{E}(-r,\chi)) + \cdots + \beta_n \ell^n(\mathcal{E}(-r,\chi)) \text{ (belongs to } \mathcal{A}),$$

where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and $\ell^i(\mathcal{E}(-r,\chi)) \in \mathcal{A}$ are certain Fourier coefficients of the seies $\mathcal{E}(-r,\chi)$.

*Conclusion.* There exists an element $\tilde{\mathcal{E}}(-r,\chi) \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}(\mathcal{A})^\dagger$ such that

$$\ell(\mathcal{E}(-r,\chi)) = \ell(\tilde{\mathcal{E}}(-r,\chi))$$

(at each triple weight $(k_1, k_2, k_3)$). In fact, let us define

$$\tilde{\mathcal{E}}(-r,\chi) := \ell^1(\mathcal{E}(-r,\chi))\ell_1 + \cdots + \ell^n(\mathcal{E}(-r,\chi))\ell_n$$

$\Rightarrow \ell(\tilde{\mathcal{E}}(-r,\chi)) = \ell(\ell_1)\ell^1(\mathcal{E}(-r,\chi)) + \cdots + \ell(\ell_n)\ell^n(\mathcal{E}(-r,\chi))$
$= \beta_1 \ell^1(\mathcal{E}(-r,\chi)) + \cdots + \beta_n \ell^n(\mathcal{E}(-r,\chi))$
$= \ell(\mathcal{E}(-r,\chi))$ (at each weight $(k_1, k_2, k_3)$).

Thus, the dependence of $\ell(\mathcal{E}(-r,\chi)) \in \mathcal{A}$ on $(k_1, k_2, k_3) \in X^3$ is $p$-adic analytic.

In order to prove the remaining statements 2), 3), the dependence on $x = \chi \cdot y_p^r$ is studied in the next section.

## 7. Distributions and admissible measures

*Distributions and measures with values in Banach modules.* We refer to Section 4 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman's family.

| | | |
|---|---|---|
| | $\mathcal{A}$ | (a $p$-adic Banach algebra) |
| | $V$ | (an $\mathcal{A}$-module) |
| **Notation.** | $\mathcal{C}(Y,\mathcal{A})$ | (the $\mathcal{A}$-Banach algebra |
| | $\cup$ | of continuous functions on $Y$ ) |
| | $\mathcal{C}^{loc-const}(Y,\mathcal{A})$ | (the $\mathcal{A}$-algebra |
| | | of locally constant functions on $Y$ ) |

Definition 7.1 (Distributions and measures). *a) A* distribution $\mathcal{D}$ *on* $Y$ *with values in* $V$ *is an* $\mathcal{A}$*-linear form*

$$\mathcal{D} : \mathcal{C}^{loc-const}(Y,\mathcal{A}) \to V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$

*b) A* measure $\mu$ *on* $Y$ *with values in* $V$ *is a continuous* $\mathcal{A}$*-linear form*

$$\mu : \mathcal{C}(Y,\mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function $\varphi$, and any bounded distribution $\mu$, using the Riemann sums.

**Admissible measures of Amice-Vélu.**

*Admissible measures.* Let $h$ be a positive integer. A more delicate notion of an $h$-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

DEFINITION 7.2.

*a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{P}^h(Y, \mathcal{A})$ denote the $\mathcal{A}$-module of* locally polynomial functions *of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}^\times$; in particular,*

$$\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$$

*(the $\mathcal{A}$-submodule of* locally constant functions*). Let also denote $\mathcal{C}^{loc-an}(Y, \mathcal{A})$ the $\mathcal{A}$-module of* locally analytic functions*, so that*

$$\mathcal{P}^1(Y, \mathcal{A}) \subset \mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A}) \subset \mathcal{C}(Y, \mathcal{A}).$$

*b) Let $V$ be a normed $\mathcal{A}$-module with the norm $|\cdot|_{p,V}$. For a given positive integer $h$ an $h$-admissible measure on $Y$ with values in $V$ is an $\mathcal{A}$-module homomorphism*

$$\tilde{\Phi} : \mathcal{P}^h(Y, \mathcal{A}) \to V$$

*such that for fixed $a \in Y$ and for $v \to \infty$ the following* growth condition *is satisfied:*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)}) \tag{7.17}$$
$$\text{for all} \quad h' = 0, 1, \ldots, h-1, a_p := y_p(a)$$

The condition (7.17) allows to integrate the locally-analytic functions on $Y$ along $\tilde{\Phi}$ using Taylor's expansions! This means: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y, \mathcal{A}) \to V$.

7.1. $U_p$**–Operator and the method of canonical projection.** We refer to Section 5 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman's family.

*Using the canonical projection* $\pi_\lambda$. We construct our $H$-admissible measure $\widetilde{\Phi}^\lambda$ : $\mathcal{P}^H(Y,\mathcal{A}) \to \mathcal{M}(\mathcal{A})$ out of a sequence of distributions $\Phi_r : \mathcal{P}^1(Y,\mathcal{A}) \to \mathcal{M}(\mathcal{A})$ defined on local monomials $y_p^r$ of each degree $r$ by the rule

$$\int_Y \chi y_p^r \, d\widetilde{\Phi}^\lambda = \pi_\lambda(\tilde{\mathcal{E}}(-r,\chi)), \text{ where } \tilde{\mathcal{E}}(-r,\chi) \in M = \mathcal{M}(\mathcal{A}).$$

Here $\tilde{\mathcal{E}}(-r,\chi)$ takes values in an $\mathcal{A}$-module

$$M = \mathcal{M}(\mathcal{A}) \subset \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3]$$

of nearly holomorphic (overconvergent) triple modular forms over $\mathcal{A}$ (for $0 \le r \le H-1$, $H = [2\mathrm{ord}_p\lambda_p]+1$), and the formal series $\tilde{\mathcal{E}}(-r,\chi)$ was constructed in the proof of 1) of Main Theorem.

*Definition of the canonical projection* $\pi_\lambda$. Here $\mathcal{A}$ is an $\mathbb{C}_p$-algebra, and $\lambda \in \mathcal{A}^\times$ is a fixed non-zero eigenvalue of triple Atkin's operator $U_T = U_{T,p}$, acting on $\mathcal{M}(\mathcal{A})$,

$$\pi_\lambda : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$$

is the canonical projection operator onto the maximal $\mathcal{A}$-submodule $\mathcal{M}(\mathcal{A})^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent (we call $\mathcal{M}(\mathcal{A})^\lambda$ the $\lambda$-characteristic submodule of $\mathcal{M}(\mathcal{A})$).
The projector $\pi_\lambda$ is defined by its kernel:

$$\mathrm{Ker}\,\pi_\lambda := \bigcap_{n\ge1} \mathrm{Im}\,(U_T-\lambda I)^n, \quad \mathrm{Im}\,\pi_\lambda := \bigcup_{n\ge1} \mathrm{Ker}\,(U_T-\lambda I)^n.$$

## 8. Triple modular forms

Triple modular forms are certain formal series

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}$$
$$\in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H}, \ R_j = (4\pi y_j)^{-1},$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight $(k_1,k_2,k_3)$ and character $(\psi_1,\psi_2,\psi_3)$, $j=1,2,3$. The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$, and the triple Atkin's operator is given by

$$U_T(g) = \sum_{n_1,n_2,n_3=0}^{\infty} a(pn_1,pn_2,pn_3;pR_1,pR_2,pR_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}.$$

**Eigenfunctions of $U_p$ and of $U_p^*$.**

*Functions $f_{j,0}$ and $f_j^0$.* Recall that for any primitive cusp eigenform $f_j = \sum_{n=1}^{\infty} a_n(f)q^n$, there is an eigenfunction $f_{j,0} = \sum_{n=1}^{\infty} a_n(f_{j,0})q^n \in \overline{\mathbb{Q}}[\![q]\!]$ of $U = U_p$ with the eigenvalue $\alpha = \alpha_{p,j}^{(1)} \in \overline{\mathbb{Q}}$ ($U(f_0) = \alpha f_0$) given by

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j | V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j | \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \tag{8.18}$$

$$\sum_{n=1}^{\infty} a_n(f_{j,0}) n^{-s} = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} a_n(f_j) n^{-s} (1 - \alpha_{p,j}^{(1)} p^{-s})^{-1}.$$

Moreover, there is an eigenfunction $f_j^0$ of $U_p^*$ given by

$$f_j^0 = f_{j,0}^{\rho}\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, \text{ where } f_{j,0}^{\rho} = \sum_{n=1}^{\infty} \overline{a(n, f_0)} q^n. \tag{8.19}$$

Therefore, $U_T(f_{1,0} \otimes f_{2,0} \otimes f_{3,0}) = \lambda(f_{1,0} \otimes f_{2,0} \otimes f_{3,0})$.

## 9. Critical values of the $L$ function $L(f_1 \otimes f_2 \otimes f_3, s, \chi)$

(compare with Section 7 in [PaTV]).

*Choice of Dirichlet characters.* For an arbitrary Dirichlet character $\chi \bmod Np^v$ consider the following Dirichlet characters:

$$\chi_1 \bmod Np^v = \chi, \ \chi_2 \bmod Np^v = \psi_2 \bar{\psi}_3 \chi, \tag{9.20}$$
$$\chi_3 \bmod Np^v = \psi_1 \bar{\psi}_3 \chi, \boldsymbol{\psi} = \chi^2 \psi_1 \psi_2 \overline{\psi}_3;$$

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_{0,1}, \chi_{0,2}, \chi_{0,3}$ are $Np$-completes (i.e. have the same prime divisors as resp. those of $Np$).

Theorem A (algebraic properties of the triple product). *Assume that $k_2 + k_3 - k_1 \geq 2$, then for all pairs $(\chi, r)$ such that the corresonding Dirichlet characters $\chi_j$ have $Np$-complete conductors, and $0 \leq r \leq k_2 + k_3 - k_1 - 2$, we have that*

$$\frac{\Lambda(f_1^{\rho} \otimes f_2^{\rho} \otimes f_3^{\rho}, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)}{\langle f_1^{\rho} \otimes f_2^{\rho} \otimes f_3^{\rho}, f_1^{\rho} \otimes f_2^{\rho} \otimes f_3^{\rho} \rangle_T} \in \overline{\mathbb{Q}}$$

*where*

$$\langle f_1^{\rho} \otimes f_2^{\rho} \otimes f_3^{\rho}, f_1^{\rho} \otimes f_2^{\rho} \otimes f_3^{\rho} \rangle_T := \langle f_1^{\rho}, f_1^{\rho} \rangle_N \langle f_2^{\rho}, f_2^{\rho} \rangle_N \langle f_3^{\rho}, f_3^{\rho} \rangle_N$$
$$= \langle f_1, f_1 \rangle_N \langle f_2, f_2 \rangle_N \langle f_3, f_3 \rangle_N.$$

## 10. Theorems B-D

*Recall: the $p$-adic weight space and the Mellin transform.* (for generalities we also refer to the introduction of [PaTV] in the case of the $L$-function of one Coleman's family). The $p$-adic weight space is the group $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^\times)$ of (continuouos) $p$-adic characters of the commutative profinite group $Y = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^*$

The group $X$ is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A $p$-adic $L$-function $L_{(p)} : X \to \mathbb{C}_p$ is a certain meromorphic function on $X$. Such a function usually come from a $p$-adic measure $\mu$ on $Y$ (*bounded* or *admissible* in the sense of Amice-Vélu, see [Am-V]). The $p$-adic Mellin transform of $\mu$ is given for all $x \in X$ by

$$L_{(p)}(x) = \int_{Y_{N,p}} x(y)\mathrm{d}\mu(y), L_{(p)} : X \to \mathbb{C}_p$$

*Theorem B (on admissible measures attached to the triple product:fixed balanced weights case). Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde{\mu}^\lambda_{f_1\otimes f_2\otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function $\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p$, given for all $x \in X_{N,p}$ by the integral $\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y)\mathrm{d}\tilde{\mu}^\lambda_{f_1\otimes f_2\otimes f_3}(y)$, and having the following properties:*

(i) *for all pairs $(r, \chi)$ such that $\chi \in X^{\mathrm{tors}}_{N,p}$, and all three corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductor ($j = 1, 2, 3$), and $r \in \mathbb{Z}$ is an integer with $0 \le r \le k_2 + k_3 - k_1 - 2$, the following equality holds:*

$$\mathcal{D}_{(p)}(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = i_p\Big(\frac{(\psi_1\psi_2)(2)C_\chi^{4(k_2+k_3-2-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda_p^{2v}}$$

$$\frac{\Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2 + k_3 - 2 - r, \psi_1\psi_2\chi)}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{T,Np}}\Big)$$

*where $v = \mathrm{ord}_p(C_\chi)$, $G(\chi)$ denotes the Gauß sum of a primitive Dirichlet character $\chi_0$ attached to $\chi$ (modulo the conductor of $\chi_0$),*

(ii) *if $\mathrm{ord}_p\lambda_p = 0$ then the holomorphic function in (i) is a* bounded $\mathbb{C}_p$-analytic function*;*

(iii) *in the general case (but assuming that $\lambda_p \neq 0$) the holomorphic function in (i) belongs to the* type $o(\log(x_p^H))$ with $H = [2\mathrm{ord}_p\lambda_p]+1$ *and it can be represented as the Mellin transform of the $H$-admissible $\mathbb{C}_p$-valued measure $\tilde{\mu}^\lambda_{f_1\otimes f_2\otimes f_3}$ (in the sense of Amice-Vélu) on $Y$*

(iv) *Let $k = k_2 + k_3 - k_1 \ge 2$. If $H \le k - 2$ then the function $\mathcal{D}_{(p)}$ is uniquely* *determined by the above conditions (i).* Let us describe now $p$-adic measures attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k)q^n\}(j=1,2,3). \tag{10.21}$$

Consider the product of three eigenvalues:

$$\lambda = \lambda_p(k_1,k_2,k_3) = \alpha_{p,1}^{(1)}(k_1)\alpha_{p,2}^{(1)}(k_2)\alpha_{p,3}^{(1)}(k_3)$$

and assume that the slope of this product

$$\sigma = \mathrm{ord}_p(\lambda(k_1,k_2,k_3)) = \sigma(k_1,k_2,k_3) = \sigma_1+\sigma_2+\sigma_3$$

is constant and positive for all triplets $(k_1,k_2,k_3)$ in an appropriate $p$-adic neighbourhood of the fixed triplet of weights $(k_1,k_2,k_3)$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denote an affinoid algebra $\mathcal{A}$ associated with an analytic space $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, a neighbourhood around $(k_1,k_2,k_3) \in X^3$ (with a given $k$ and $\psi$ mod $N$).

*Theorem C (on p-adic measures for families of triple products). Put* $H = [2\mathrm{ord}_p(\lambda)]+1$. *There exists a sequence of distributions* $\Phi_r$ *on* $Y$ *with values in* $\mathcal{M} = \mathcal{M}(\mathcal{A})$ *giving an* $H$*-admissible measure* $\tilde{\Phi}^\lambda$ *with values in* $\mathcal{M}^\lambda \subset \mathcal{M}$ *with the following properties:*

*There exists an* $\mathcal{A}$*-linear form* $\ell = \ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}$ *(given by (11.24), such that the composition*

$$\tilde{\mu} = \tilde{\mu}_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda} := \ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda}(\tilde{\Phi}^\lambda)$$

*is an* $H$*-admissible measure with values in* $\mathcal{A}$*, and for all* $(k_1,k_2,k_3)$ *in the affinoid neighborhood* $\mathcal{B} = \mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3$*, as above, satisfying* $k_1 \le k_2+k_3-2$ *we have that the evaluated integrals*

$$ev_{(k_1,k_2,k_3)}\left((\ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda})(\tilde{\Phi}^\lambda)(y_p^r\chi)\right)$$

*on the arithmetical chracters* $y_p^r\chi$ *coincide with the critical special values*

$$\Lambda^*(f_{1,k_1}\otimes f_{2,k_2}\otimes f_{3,k_3}, k_2+k_3-2-r,\chi)$$

*for* $r = 0,\cdots,k_2+k_3-k_1-2$*, and for all Dirichlet characters* $\chi$ mod $Np^v, v\ge 1$*, with all three corresonding Dirichlet characters* $\chi_j$ *given by (6.16), having* $Np$*-complete conductors. Here the normalisation of* $\Lambda^*$ *includes at the same time certain Gauss sums, Petersson scalar products, powers of* $\pi$ *and of* $\lambda(k_1,k_2,k_3)$*, and a certain finite Euler product.* The precise form of the Euler-like $p$-factor is given by a general motivic setting as in [Pa94], [Co], [Co-PeRi]; we also refer to Section 7 of [PaTV] in the case of the $L$-function of one Coleman's family. However, our modular construction of the admissible measures of Theorem C does not use these explicit formulae. Moreover, these measures are uniquely determined by general unicity properties by all but a finite number of values on characters $y_p^r\chi$.

*The p-adic Mellin transform and four variable p-adic analytic functions.* Any $h$-admissible measure $\tilde\mu$ on $Y$ with values in a $p$-adic Banach algebra $\mathcal{A}$ can be caracterized its *Mellin transform* $\mathcal{L}_{\tilde\mu}(x)$ $\mathcal{L}_{\tilde\mu}: X \to \mathcal{A}$, defined by $\mathcal{L}_{\tilde\mu}(x) = \int_Y x(y)d\tilde\mu(y)$, where $x \in X,\ \mathcal{L}_{\tilde\mu}(x) \in \mathcal{A}$,

Key property of $h$-admissible measures $\tilde\mu$: its Mellin transform $\mathcal{L}_{\tilde\mu}$ is analytic with values in $\mathcal{A}$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{B}) = \mathcal{A}_1\hat\otimes\mathcal{A}_2\hat\otimes\mathcal{A}_3 = \mathcal{A}(\mathcal{B}_1)\hat\otimes\mathcal{A}(\mathcal{B}_2)\hat\otimes\mathcal{A}(\mathcal{B}_3)$ denote again the Banach algebra $\mathcal{A}$ where $\mathcal{B}$ is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integer $k$ and Dirichlet character $\psi$ mod $N$).

*Theorem D (on p-adic analytic function in four variables). Put* $H = [2\mathrm{ord}_p(\lambda)] + 1$. *There exists a p-adic analytic function in four variables* $(x, \boldsymbol{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde\mu} : (x, \boldsymbol{s}) \longmapsto ev_{\boldsymbol{s}}(\mathcal{L}_{\tilde\mu(x)}) \quad (x \in X, \quad \mathcal{L}_{\tilde\mu}(x) \in \mathcal{A}).$$

*with values in* $\mathbb{C}_p$, *such that for all* $(k_1, k_2, k_3)$ *in the affinoid neighborhood as above* $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, *satisfying* $k_1 \le k_2 + k_3 - 2$, *we have that the special values* $\mathcal{L}_{\tilde\mu}(x, \boldsymbol{s})$ *at the arithmetical chracters* $x = y_p^r\chi$, *and* $\boldsymbol{s} = (k_1, k_2, k_3) \in \mathcal{B}$ *coincide with the normalized critical special values*

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \cdots, k_2 + k_3 - k_1 - 2),$$

*for Dirichlet characters* $\chi$ mod $Np^v, v \ge 1$, *such that all three corresonding Dirichlet characters* $\chi_j$ *given by (6.16), have* $Np$*-complete conductors where the same normalisation of* $L^*$ *as in Theorem C.*

*Moreover, for any fixed* $\boldsymbol{s} = (k_1, k_2, k_3) \in \mathcal{B}$ *the function*

$$x \longmapsto \mathcal{L}_{\tilde\mu}(x, \boldsymbol{s})$$

*is p-adic analytic of type* $o(\log^H(\cdot))$.

Indeed, we obtain the function in question $\mathcal{L}_\mu(x, \mathbf{s})$ by evaluation at

$$\mathbf{s} = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B}:$$

this is a $p$-adic analytic function in four variables $(x, \mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde\mu}(x, \mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_{\tilde\mu})(x) \quad (x \in X,\ \mathbf{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_{\tilde\mu}(x) \in \mathcal{A}).$$

This is a joint work in progress with S.Boecherer, we use:

1) the higher twists of the Siegel-Eisenstein series, introduced in [Boe-Schm],

2) Ibukiyama's differential operators (see [Ibu], [BSY]).

## 11. Scheme of the Proof

### 11.1. Boecherer's higher twist.

*Boecherer's Higher Twist.* 1) We define the higher twist of the series $F_{\chi,r} = \sum_{\mathcal{T}} a_{\chi,r}(R,\mathcal{T})q^{\mathcal{T}}$ by some Dirichlet characters $\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3$ as the following formal nearly holomorphic Fourier expansion:

$$F_{\chi,r} = \sum_{\mathcal{T}} \bar{\chi}_1(t_{12})\bar{\chi}_2(t_{13})\bar{\chi}_3(t_{23})a_{\chi,r}(R,\mathcal{T})q^{\mathcal{T}}. \tag{11.22}$$

The seies (11.22) is a Siegel modular form of some higher level, but it has additional symmetries with respect to symplectic embedding $\iota_3 : \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \to \mathrm{Sp}_3$: its triple Nebentypus character does not depend on $\chi$ mod $Np^v$, and is equal to $(\psi_1, \psi_2, \psi_3)$, if we choose Dirichlet characters as in (6.16):

$$\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2\bar{\psi}_3\chi,$$
$$\chi_3 \bmod Np^v = \psi_1\bar{\psi}_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3.$$

We use the Siegel-Eisenstein series $F_{\chi,r}$ which depends on the character $\chi$, but its precise nebentypus character is $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, and it is defined by $F_{\chi,r} = G^\star(z, -r; k, (Np^v)^2, \boldsymbol{\psi})$, where $z$ denotes a variable in the Siegel upper half space $\mathbb{H}_3$, and the normalized series $G^\star(z, s; k, (Np^v)^2, \boldsymbol{\psi})$ is given by (5.11).

This series depends on $s = -r$, and for the critical values at integral points $s \in \mathbb{Z}$ such that $2-k \le s \le 0$, it represents a (*nearly*) *holomorphic* Siegel modular form in the sense of Shimura [ShiAr]:

$$F_{\chi,r} = \sum_{\mathcal{T}} \det(\mathcal{T})^{k-2r-\kappa} Q(R,\mathcal{T}; k-2r, r) a_{\chi,r}(\mathcal{T})q^{\mathcal{T}}.$$

### 11.2. Ibukiyama's differential operator.

*Ibukiyama's differential operator.* 2) We use an algebraic version of Ibukiyama's differential operator, which generalizes the algebraic "pull-back": it transforms a nearly holomorphic Siegel modular form of weight $k$

to a nearly holomorphic triple modular form of weight $(k_1, k_2, k_3)$ $(k = k_2 + k_3 - k_1)$.

On a holomorphic Siegel modular form $F = \sum_{\mathcal{T}} a(\mathcal{T})q^{\mathcal{T}}$, this operator has the form

$$\mathcal{L}_k^{\lambda,\nu}(F) = \sum_{\mathcal{T}} \mathcal{P}(k_1, k_2, k_3, 0, \mathcal{T})a(\mathcal{T})q_1^{t_{11}}q_2^{t_{22}}q_3^{t_{33}}, \tag{11.23}$$

where $\lambda = k_1 - k_3 \geq \mu = k_1 - k_2 \geq 0$, and $\mathcal{P}(k_1, k_2, k_3; r; \mathcal{T})$ is certain Ibukiyama's polynomial, equal to $(t_{11}t_{22}t_{33})^{\lambda}(t_{12}t_{13}t_{23})^{\mu}$, if $r = 0$.

. As a result we obtain a sequence of triple modular distributions $\Phi_r(\chi)$ with values in the tensor product $\mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A})\widehat{\otimes}_{\mathcal{A}}\mathcal{M}(\mathcal{A})\widehat{\otimes}_{\mathcal{A}}\mathcal{M}(\mathcal{A})$ of three Banach $\mathcal{A}$-modules of arithmetical nearly holomorphic modular forms (the normalizing factor $2^r$ is neeeded in order to prove certain congruences between $\Phi_r$). Note that $\mathcal{M}_T(\mathcal{A})$ is again a Banach $\mathcal{A}$-module on which $U_T$ acts as a completely continuous operator.

The important property of the triple modular forms $\Phi_r(\chi)$: the nebentypus character is fixed and is equal to $(\psi_1, \psi_2, \psi_3)$ (for all $(k_1, k_2, k_3)$ and $\chi$ in question).

Using this property we compute the canonical projection $\pi_\lambda(\Phi_r(\chi))$ of the triple modular form $\Phi_r(\chi)$ onto the $\lambda$-characteristic $\mathcal{A}$-submodule ${\mathcal{M}_T}^{\lambda}(\mathcal{A})$ of the triple Atkin's operator $U_{T,p}$:

$$\pi_\lambda : \mathcal{M}_T(\mathcal{A}) \to \mathcal{M}_T^\lambda(\mathcal{A}).$$

. We prove that the resulting sequence of modular distributions $\pi_\lambda(\Phi_r)$ on the profinite group $Y$ produces a certain $p$-adic admissible measure $\tilde{\Phi}^\lambda$ (in the sense of Amice-Vélu, [Am-V]) with values in a certain locally free $\mathcal{A}$-submodule of finite rank

$$\mathcal{M}_T^\lambda(\mathcal{A}) \subset \mathcal{M}_T(\mathcal{A}) \subset \bigcup_{v\geq 0} \mathcal{M}_T(Np^v, \psi_1, \psi_2, \psi_3; \mathcal{A})$$

of formal nearly holomorphic triple modular forms of all levels $Np^v$ and the fixed nebentypus characters $(\psi_1, \psi_2, \psi_3)$. We use congruences between triple modular forms $\Phi_r(\chi) \in \mathcal{M}_T(\mathcal{A})$ (they have explicit formal Fourier coefficients).

Then we use a general admissibility criterion saying that these congruences imply $H$-admissibility for their projections in $\mathcal{M}_T^\lambda(\mathcal{A})$, where $H = [2\mathrm{ord}_p(\lambda)] + 1$.

11.3. **Algebraic linear form.** We refer here to Section 6 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman's family.

. 3) From $\mathcal{M}_T^\lambda(\mathcal{A})$ to $\mathcal{A}$: we use a $\overline{\mathbb{Q}}$-valued linear forms of type

$$\mathcal{L} \ : h \longmapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, h\rangle}{\langle f_1^0, f_{1,0}\rangle\langle f_2^0, f_{2,0}\rangle\langle f_3^0, f_{3,0}\rangle}$$

where $f_j^0$ is the eigenfunction (8.18) of the conjugate Atkin's operator $U_p^*$, and $f_{j,0}$ is the eigenfunction (8.19) of $U_p$. The linear form $\mathcal{L}$ is defined on the finite dimensional $\overline{\mathbb{Q}}$-vector characteristic subspace

$$\begin{aligned}&h \in \mathcal{M}_{\mathbf{k}}(\overline{\mathbb{Q}})^{\lambda(\mathbf{k})} \subset \\ &\mathcal{M}_{k_1,r^*}(Np, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_2,r^*}(Np, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_3,r^*}(Np, \psi_3; \overline{\mathbb{Q}}).\end{aligned}$$

This map is then extended to an $\mathcal{A}$-linear map

$$\ell = \ell_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}; \tag{11.24}$$

on the locally free $\mathcal{A}$-module of finite rank $\mathcal{M}(\mathcal{A})^\lambda$.

. This map produces a sequence of $\mathcal{A}$-valued distributions $\mu_r^\lambda(\chi) \in \mathcal{A}$ in such a way that for all suitable weights $\mathbf{k} \in \mathcal{B}$ one has

$$ev_{\mathbf{k}}(\mu_r^\lambda(\chi)) = \mathcal{L}(ev_{\mathbf{k}}(\pi_\lambda(\Phi_r)(\chi))), \lambda \in \mathcal{A}^\times, \lambda(\mathbf{k}) \in \overline{\mathbb{Q}}^\times,$$

where $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{B}$, $ev_{\mathbf{k}} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{\mathbf{k}} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_{\mathbf{k}}(\mathbb{C}_p).$$

. More precisely, we consider three auxilliary families of modular forms

$$\begin{aligned}&\tilde{f}_{j,k_j}(z) = \\ &\sum_{n=1}^{\infty} \tilde{a}_{n,j,k_j} e(nz) \in S_{k_j}(\Gamma_0(N_j p^{\nu_j}), \psi_j),\ (1 \le j \le 3, \nu_j \ge 1),\end{aligned} \tag{11.25}$$

with the same eigenvalues as those of (10.21),

for all Hecke operators $T_q$, with $q$ prime to $Np$. In our construction we use as $\tilde{f}_{j,k_j}$ certain "easy transforms" of primitive cusp forms in (1.1). In particular, we choose as $\tilde{f}_j$ certain eigenfunctions $\tilde{f}_{j,k_j} = f_{j,k_j}^0$ of the adjoint Atkin's operator $U_p^*$, in this case we denote by $f_{j,k_j,0}$ the corresponding eigenfunctions of $U_p$.

The $\overline{\mathbb{Q}}$-linear form $\mathcal{L}$ produces a $\mathbb{C}_p$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ starting from the modular $p$-adic admissible measure $\tilde{\Phi}^\lambda$ of stage 3), where $\ell$ : $\mathcal{M}_T(\mathbb{C}_p) \to \mathbb{C}_p$ denotes a $\mathbb{C}_p$-linear form, interpolating $\mathcal{L}$.

11.4. **Evaluation of $p$-adic integrals.** We refer to Section 7 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman's family.

*L-values and p-adic integrals.* 4) We show that for any appropriate Dirichlet character $\chi$ mod $Np^v$ the integral

$$\mu_r^\lambda(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r(\chi))) \in \mathcal{A}$$

evaluated at $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, coincides (up to a normalisation) with the special $L$-value

$$L^*(f^\rho_{1,k_1} \otimes f^\rho_{2,k_2} \otimes f^\rho_{3,k_3}, k_2 + k_3 - 2 - r, \psi_1\psi_2\chi)$$

under the above assumptions on $\chi$ and $r$).

*A general integral representation of Garrett's type.* The basic idea how a Dirichlet character $\chi$ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $\mathcal{A}$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ established at stage 4), does not depend on this technical computation.

In order to control the denominators of the modular forms

$$\pi_\lambda(\tilde{\mathcal{E}}(-r, \chi)) \in \mathcal{M}^\lambda(\mathcal{A}) =: \Phi_r(\chi),$$

used in the construction (the admissibility condition) we use the following result.

## 12. Criterion of admissibility

THEOREM 12.1 (Criterion of admissibility ). *Let $\alpha \in \mathcal{A}^*$, $0 < |\alpha|_p < 1$ and suppose that there exists a positive integer $\varkappa$ such that the following conditions are satisfied:*

*1) for all $r = 0, 1, \cdots, h-1$ with $h = [\varkappa \mathrm{ord}_p \alpha] + 1$, and $v \geq 1$,*

$$\Phi_r(a + (Np^v)) \in \mathcal{M}(Np^{\varkappa v}) \qquad \text{(the level condition)} \tag{12.26}$$

*2) the following congruence for the coefficients holds: for all $t = 0, 1, \cdots, h-1$*

$$U^{\varkappa v} \sum_{r=0}^{t} \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \bmod p^{vt} \tag{12.27}$$

(the divisibility condition)

*Then the linear form given by* $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)} y_p^r) := \pi_\alpha(\Phi_r(a + (Np^v))$ *on local monomials (for all $r = 0, 1, \cdots, h-1$), is an $h$-admissible measure:* $\tilde{\Phi}^\alpha : \mathcal{P}^h(Y, \overline{\mathbb{Q}}) \to \mathcal{M}^\alpha \subset \mathcal{M}$

Proof uses the commutative diagram:

$$\begin{array}{ccccc}
\mathcal{M}(Np^{v+1},\psi;\mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^{\alpha}(Np^{v+1},\psi;\mathcal{A}) & & \\
U^v \big\downarrow & & \big\downarrow \wr\, U^v & & \\
\mathcal{M}(Np,\psi;\mathcal{A}) & \underset{\pi_{\alpha,0}}{\longrightarrow} & \mathcal{M}^{\alpha}(Np,\psi;\mathcal{A}) & = \mathcal{M}^{\alpha}(Np^{v+1},\psi;\mathcal{A}). &
\end{array}$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman's Theorem A.4.3 [CoPB].

On the right: $U$ acts on the locally free $\mathcal{A}$-module $\mathcal{M}^{\alpha}(Np^{v+1},\mathcal{A})$ via the matrix:

$$\begin{pmatrix} \alpha \cdots & \cdots & * \\ 0 \ \ \alpha & \cdots & * \\ 0 \ \ 0 & \ddots & \cdots \\ 0 \ \ 0 & \cdots & \alpha \end{pmatrix} \text{ where } \alpha \in \mathcal{A}^{\times}$$

$$\Longrightarrow U^v \text{ is an isomorphism over } \mathcal{A},$$

and one controls the denominators of the modular forms of all levels $v$ by the relation:

(12.28) $$\pi_{\alpha,v}(h) = U^{-v}\pi_{\alpha,0}(U^v h) =: \pi_{\alpha}(h)$$

The equality (12.28) can be used as the definition of $\pi_{\alpha}$ at any level $Np^v$.

The growth condition (see (7.17)) for $\pi_{\alpha}(\Phi_r)$ is deduced from the congruences (12.27) between modular forms, using the relation (12.28).

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