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Explicit Plancherel Theorems for $\mathcal{H}(q_1, q_2)$ and $\mathbb{SL}_2(F)$

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To J.-P. Serre, on the occasion of his 80th birthday.

Abstract: For each pair $q_1 \geq q_2 \geq 1$ of real numbers, the authors define a complex algebra (with identity) $\mathcal{H} = \mathcal{H}(q_1, q_2)$, an associated involutive Banach algebra $A = A(q_1, q_2)$ and its associated enveloping C^*-algebra $C^*(A)$, and a quotient C^*-algebra $C^*_r(A)$. Deformation arguments are used to obtain an explicit Plancherel formula for $C^*_r(A)$; the unitary dual of $C^*_r(A)$ is explicitly described, as is the unitary dual of $C^*(A)$. These results, together with the theory of types, are used to obtain the Plancherel measure for the group $\mathbb{SL}_2(F)$, where F is a complete non archimedean local field with arbitrary residual characteristic p. This includes an explicit description of the reduced dual. The methods, but not the results, are independent of p.
Key words and phrases: Hecke algebra, C^*-algebra, Plancherel measure, type, reduced dual.

0. Introduction

Let $q_1 \geq q_2 \geq 1$ be two real numbers. In this paper we consider an affine Hecke algebra $\mathcal{H} = \mathcal{H}(q_1, q_2)$ in two parameters: thus it is the complex algebra with two generators $s_i, i = 1, 2$ subject only to the relations $s_i^2 = c_i s_i + 1$ where $c_i = \sqrt{q_i} - \sqrt{q_i}^{-1}$. In §1 we quickly review the classification of the irreducible modules of $\mathcal{H}$. The algebra $\mathcal{H}$ can be equipped with an involution and a norm;

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in §2 we complete $\mathcal{H}$ with respect to this norm to obtain an involutive Banach algebra $A = A(q_1, q_2)$. The main result in §2 is the explicit determination of the topological structure of the full unitary dual $\hat{A}$ of the enveloping C^*-algebra $C^*(A)$ associated to A. (See Proposition 2.10.)

The algebra $\mathcal{H}$ is also equipped with a scalar product which provides it with the structure of a Hilbert algebra (§3.5). In §3 we determine an explicit Plancherel formula for the reduced C^*-algebra, $C^*_r(A)$ (a quotient of $C^*(A)$) which results from this structure (Theorem 3.14). To write down such a formula means to describe explicitly a certain measure and its support $\hat{A}_r$; we describe $\hat{A}_r$ as a closed subset of $\hat{A}$. This is accomplished in Proposition 3.15. Modulo some standard facts on C^*-algebras, sections 1 – 3 are self contained. Finally, in §4 we apply the formula(s) in §3 to obtain the Plancherel measure for the group $\mathbb{SL}_2(F)$ where F is a complete non archimedean local field with arbitrary residue characteristic p, and describe its reduced dual. For this we avail ourselves of the theory of types for $\mathbb{SL}_2(F)$ (see below). The main result here is Theorem 4.5.

Since others (see [Mat]) have previously derived explicit Plancherel formulas for affine Hecke algebras in two parameters, and yet others (see [Op]) have obtained formulae for far more general algebras, and since the Plancherel formula for $\mathbb{SL}_2(F)$ was written down some time ago in case $p \neq 2$, we feel bound to point out what we think is new in this account. As far as we know, previous versions (notably [Mat], [Op]) of a Plancherel formula for an affine Hecke algebra have used some form of residue calculus; this method can be traced back to the theory of Eisenstein series. Here we use a deformation argument, completely avoiding residue calculus. Namely we obtain a formula when $q_1 = q_2 = 1$ and deduce the general formula from that. Indeed the rational function that traditionally appears in this formula has a very clear and conceptual explanation from this point of view. It is a pleasant exercise to see that the formula we obtain in Theorem 3.14 is the same as that obtained in [Mat], page 47. We also note that Proposition 2.10 implies that the full unitary dual $\hat{A}$ as a topological space is independent of q_1, q_2.

Our approach to the Plancherel formula for $\mathbb{SL}_2(F)$ is very different from previous methods, and provides a conceptual route to that formula. The method, but not the answer, is independent of the residue characteristic of F; moreover it minimises the role of analysis: this appears primarily in the universal derivation of the Plancherel formula for the two parameter Hecke algebra in §3, and there we largely draw from the theory of C^*-algebras. Briefly, the underlying ideas are as follows.

If G is the group of F-points of a connected reductive group defined over F then Bernstein showed that the category $\mathcal{R}(G)$ of smooth representations of G decomposes into a product of full subcategories:

$$\mathcal{R}(G) = \prod_{\mathfrak{s} \in \mathcal{B}(G)} \mathcal{R}^{\mathfrak{s}}(G). \tag{0.1}$$

Here $\mathcal{B}(G)$ is the set of G-inertial equivalence classes [BK] of cuspidal pairs (L, σ) where L is the group of F-points of an F-Levi subgroup (of an F-parabolic) and σ is an irreducible supercuspidal representation of L.

Now suppose that μ is a Haar measure on G. If (K, λ) is a pair consisting of a compact open subgroup K of G and a smooth irreducible representation (λ, W) of K, write $(\check{\lambda}, \check{W})$ for the contragredient representation of (λ, W), and let $\mathcal{H}(G, \lambda)$ be the convolution algebra with respect to μ of compactly supported $\mathrm{End}_{\mathbb{C}}(\check{W})$-valued functions f on G such that

$$f(k_1 g k_2) = \check{\lambda}(k_1) f(g) \check{\lambda}(k_2)$$

for all $g \in G, k_1, k_2 \in K$. If (K, λ) is an $\mathfrak{s}$-type (see [BK]) there is an equivalence of categories

$$\mathbf{M}_\lambda : \mathcal{R}^{\mathfrak{s}}(G) \to \mathcal{H}(G, \lambda)\text{-Mod},$$

given on objects by $\mathcal{V} \mapsto \mathcal{V}_\lambda = \mathrm{Hom}_K(W, \mathcal{V})$.

Let $\hat{G}_r$ denote the reduced dual of G with Plancherel measure $\hat{\mu}$ corresponding to μ. For $\mathfrak{s}$ as above let $\hat{G}_r(\mathfrak{s})$ denote the set of equivalence classes of irreducible representations (π, H) of $\hat{G}_r$ such that the space of smooth vectors $(\pi_\infty, H_\infty) \in \mathcal{R}^{\mathfrak{s}}(G)$. According to [BHK] there is a disjoint union

$$\hat{G}_r = \cup_{\mathfrak{s} \in \mathcal{B}(G)} \hat{G}_r(\mathfrak{s}), \tag{0.2}$$

induced from the decomposition (0.1), and each $\hat{G}_r(\mathfrak{s})$ is an open subset of $\hat{G}_r$. If (K, λ) is an $\mathfrak{s}$-type we can define $\hat{G}_r(\lambda)$ analogously, and in fact $\hat{G}_r(\lambda) = \hat{G}_r(\mathfrak{s})$. The algebra $\mathcal{H}(G, \lambda)$ is in a natural way a normalised Hilbert algebra (see [BHK]), so that one can construct a C^*-algebra $C^*_r(G, \lambda)$ in which $\mathcal{H}(G, \lambda)$ naturally embeds, with dual space $C^*_r(G, \lambda)\hat{}$, and positive Borel measure $\hat{\mu}_{\mathcal{H}(G,\lambda)}$.

Theorem 4.3 of [BHK] says that there is a homeomorphism

$$\hat{m}_\lambda : \hat{G}_r(\lambda) \to C_r^*(G,\lambda)\check{}, \tag{0.3}$$

which is induced from $\mathbf{M}_\lambda$, and that if S is a measurable subset of $\hat{G}_r(\lambda)$ then

$$\hat{\mu}_\lambda(S) = \frac{\dim W}{\mu(K)} \hat{\mu}_{\mathcal{H}(G,\lambda)}(\hat{m}_\lambda(S)). \tag{0.4}$$

For the group $\mathbb{SL}_2(F)$, the second author has provided in [K] a type (J,λ) for each $\mathfrak{s}$, and he has described the algebra $\mathcal{H}(G,\lambda)$ explicitly. Using this, the principles above, and the results in §3 we describe each $\hat{G}_r(\lambda)$ and $\hat{\mu}_\lambda$ in §4 of this paper. We do **not** provide the dimension of λ in case $\mathfrak{s}$ corresponds to a supercuspidal representation of G: if the residue characteristic of $F \neq 2$ this is well known, and if the residue characteristic of F is 2 it can be computed in principle starting from the results in [KP]. We remark that the final version of the Plancherel formula in Theorem 4.5 is written down for that Haar measure μ on G in which $\mu(\mathbb{SL}_2(\mathfrak{o})) = 1$; here $\mathfrak{o}$ denotes the ring of integers in F. We leave it to the interested reader to verify that the formula here differs by a constant from those in in [GGPS], [SSh], which depend on another normalisation of Haar measure. We note that the descriptions of the sets $\hat{A}_r$ in §3 provide us with descriptions of the sets $\hat{G}_r(\lambda)$.

The results in §3 could also be used in a similar way to derive the Plancherel formula for the group $\mathbb{PGL}_2(F)$.

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1. The algebra $\mathcal{H}$ and its representations

1.1 We fix two real numbers $q_1 \geq q_2 \geq 1$ and we set $\gamma_i = q_i^{\frac{1}{2}},\ c_i = \gamma_i - \gamma_i^{-1},\ i = 1,2$. We let $\mathcal{H} = \mathcal{H}(q_1,q_2)$ be the complex algebra with identity $\mathbf{1}$ and two generators $s_i,\ i=1,2$ subject only to the relations

$$s_i^2 = c_i s_i + 1,\ i = 1,2. \tag{1.1.1}$$

We note that s_i is invertible: $s_i^{-1} = s_i - c_i,\ i = 1, 2$.

Viewed as a complex vector space the algebra $\mathcal{H}$ has a basis consisting of elements of the form $w = \prod_{i=1}^{k} u_i,\ u_i \in \{s_1, s_2\}$ where, for $1 \le i \le k-1$, $u_i \ne u_{i+1}$. (We allow for the case $k = 0$ as well; in that case, we set $w = \mathbf{1}$.) We refer to these elements as *words* and denote the set of words by $\mathcal{W}$; for each such word w we define its *length*, $l(w)$, by $l(w) = k$.

We set $d = s_1 s_2,\ \bar{d} = s_2 s_1$ and set $\mathcal{D} = \mathbb{C}[d, d^{-1}]$. We write $\mathcal{Z}$ for the center of $\mathcal{H}$. The elementary lemma below will be very useful for calculations.

Lemma. *We have*

$$d^{-1} = \bar{d} - c_2 s_1 - c_1 s_2 + c_1 c_2,$$

and furthermore

$$s_1 d - d^{-1} s_1 = c_1 d + c_2, \quad s_1 d^{-1} - d s_1 = -c_1 d - c_2; \tag{1.1.2}$$

$$s_2 d - d^{-1} s_2 = c_2 d + c_1, \quad s_2 d^{-1} - d s_2 = -c_2 d - c_1. \tag{1.1.3}$$

Proof. This is a direct calculation.

1.2 Corollary.

(i) *$\mathcal{H}$ is free of rank two as a left $\mathcal{D}$-module. A basis is given by $\{\mathbf{1}, s_1\}$.*

(ii) *Set $z = d + d^{-1}$. Then $z = \bar{d} + \bar{d}^{-1}$, and $z \in \mathcal{Z}$. Moreover,*

$$\mathcal{Z} \cong \mathbb{C}[z],$$

and $\mathcal{D}$ is free of rank two as a left $\mathcal{Z}$-module with basis $\{\mathbf{1}, d\}$.

(iii) *Set $t = c_1 d + c_2 - (d - d^{-1}) s_1$. Then*

$$td = d^{-1} t$$

and

$$t^2 = -f(z) = -(z^2 - c_1 c_2 z - (c_1^2 + c_2^2 + 4)).$$

(iv) *We have $f(x) = 0$ for $x = \gamma_1 \gamma_2 + \frac{1}{\gamma_1 \gamma_2},\ x = -(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1})$.*

Proof. The last two assertions follow easily from Lemma 1.1. As for (i), note that Lemma 1.1 implies $\mathcal{D} + \mathcal{D}s_1$ is a subalgebra of $\mathcal{H}$ which contains s_1. Moreover,

$$s_2 = c_2 + s_2^{-1} = c_2 + s_2^{-1} s_1^{-1} s_1 = c_2 + d^{-1} s_1 \in \mathcal{D} + \mathcal{D}s_1,$$

hence $\mathcal{D} + \mathcal{D}s_1 = \mathcal{H}$. Since s_1 is not a zero-divisor, it will therefore suffice to show that

$$\mathcal{H} = \mathcal{D} \oplus \mathcal{D}s_1 \tag{1.2.1}$$

Suppose that $a \in \mathcal{D} \cap \mathcal{D}s_1$. Then $d^r a \in \mathbb{C}[d] \cap \mathbb{C}[d]s_1$ for large enough r. This latter intersection is $\{0\}$ since the words in $\mathbb{C}[d]$ all have even length while the words in $\mathbb{C}[d]s_1$ all have odd length. Since d is invertible it follows that $a = 0$ and so $\mathcal{D} \cap \mathcal{D}s_1 = 0$. Thus $\mathcal{H} = \mathcal{D} \oplus \mathcal{D}s_1$.

The first assertion in (ii) is a direct computation. For the other assertions, first note that (1.1.2) and (1.1.3) imply that $\mathbb{C}[z] \subseteq \mathcal{Z}$. Next, observe that $\mathcal{D}$ is an integral domain. Then (1.2.1) implies that $\mathcal{Z} \subseteq \mathcal{D}$. Indeed if $a + bs_1 \in \mathcal{Z}$ so that $d(a + bs_1) = (a + bs_1)d$, a short computation (use (1.1.1) and (1.2.1)) shows that $b(d - d^{-1}) = 0$, hence $b = 0$. Since $d^{\pm 2} = zd^{\pm 1} - 1$ and $d^{-1} = z - d$, we have $\mathcal{D} = \mathbb{C}[z] + \mathbb{C}[z]d$. Suppose then that $z_1 + z_2 d \in \mathcal{Z}$ where $z_i \in \mathbb{C}[z]$. Thus $s_1(z_1 + z_2 d) = (z_1 + z_2 d)s_1$, and this implies that $z_2(ds_1 - s_1 d) = 0$. By (i) $\mathcal{H} = \mathcal{D} \oplus \mathcal{D}s_1$, thus if $z_2 \neq 0$ it cannot be a zero divisor. Since $ds_1 \neq s_1 d$ it follows that $z_2 = 0$.

1.3 Given any complex unital algebra A we write A-Mod for the category of all unital left A-modules. Given a left $\mathcal{D}$-module N, we set $\mathrm{ind}N = \mathrm{ind}_{\mathcal{D}}^{\mathcal{H}} N = \mathrm{Hom}_{\mathcal{D}}(\mathcal{H}, N)$. Then $\mathrm{ind}N$ is a left $\mathcal{H}$-module via $a\phi(x) = \phi(xa)$, $\phi \in \mathrm{ind}N$, $a, x \in \mathcal{H}$ and we obtain in this way a functor

$$\mathrm{ind}_{\mathcal{D}}^{\mathcal{H}} : \mathcal{D}\text{-Mod} \to \mathcal{H}\text{-Mod}.$$

On the other hand, we have the usual functor of restriction

$$\mathrm{res} = \mathrm{res}_{\mathcal{D}}^{\mathcal{H}} : \mathcal{H}\text{-Mod} \to \mathcal{D}\text{-Mod}$$

It will be useful to us that $(\mathrm{res}, \mathrm{ind})$ is an *adjoint pair*. This means that, given a left $\mathcal{D}$-module N and a left $\mathcal{H}$-module M there is an isomorphism of complex vector spaces

$$T(M, N) : \mathrm{Hom}_{\mathcal{D}}(\mathrm{res}M, N) \cong \mathrm{Hom}_{\mathcal{H}}(M, \mathrm{ind}N)$$

and that, further, the collection of maps $T(M, N)$ is natural in M and N. In our case, the maps $T(M, N)$ are defined as follows. Given $f \in \mathrm{Hom}_{\mathcal{D}}(\mathrm{res}M, N)$ we set $T(M, N)f(m)(x) = f(xm),\ m \in M,\ x \in \mathcal{H}$.

1.4 We set $\mathrm{X} = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{D}, \mathbb{C})$. Then we may identify X with $\mathbb{C}^\times$ via $\chi \to \chi(d),\ \chi \in \mathrm{X}$. Given $\chi \in \mathrm{X}$ we write $\mathbb{C}_\chi$ for $\mathbb{C}$ viewed as a left $\mathcal{D}$-module via $b \cdot x = \chi(b)x,\ b \in \mathcal{D}, x \in \mathbb{C}$. Then the set of modules $\mathbb{C}_\chi$ is, up to equivalence, a complete set of irreducible left $\mathcal{D}$-modules.

Set $\mathbf{M} = \mathrm{ind}\mathcal{D}$ where $\mathcal{D}$ is viewed as a left $\mathcal{D}$-module via left multiplication. Then $\mathbf{M}$ is a left $\mathcal{H}$-module as above and is also a right $\mathcal{D}$-module, the action $f \to f \cdot b$ of $\mathcal{D}$ on $\mathbf{M}$ being given by

$$f \cdot b(x) = f(x)b,\ f \in \mathbf{M},\ b \in \mathcal{D},\ x \in \mathcal{H}.$$

One checks that these structures are compatible; that is, that $\mathbf{M}$ is an $(\mathcal{H}, \mathcal{D})$-bimodule.

Corollary 1.2 implies that $\mathbf{M}$ is free as a right $\mathcal{D}$-module; for example, one has the basis $\{\Phi_0, \Phi_1\}$ where

$$\Phi_0(\mathbf{1}) = 1,\ \Phi_0(s_1) = 0;$$
$$\Phi_1(\mathbf{1}) = 0,\ \Phi_1(s_1) = 1.$$

Using this basis to identify $\mathbf{M}$ with $\mathcal{D} \oplus \mathcal{D}$ one obtains a matrix representation $\sigma : \mathcal{H} \to \mathrm{M}_2(\mathcal{D})$ which is given on the generators s_i by

$$\sigma(s_1) = \begin{bmatrix} 0 & 1 \\ 1 & c_1 \end{bmatrix}, \quad \sigma(s_2) = \begin{bmatrix} c_2 & d^{-1} \\ d & 0 \end{bmatrix}.$$

It follows that $\sigma(d) = \begin{bmatrix} d & 0 \\ c_1 d + c_2\, d^{-1} \end{bmatrix}$.

If $f = f(d, d^{-1}) \in \mathcal{D}$ we define f^- by $f^-(d, d^{-1}) = f(d^{-1}, d)$. The map $f \to f^-$ is an algebra automorphism. We now define $\mathcal{D}^-$ to be the left $\mathcal{D}$-module whose underlying set is just $\mathcal{D}$ but where the module structure is given by

$$x \cdot y = x^- y,\ x \in \mathcal{D},\ y \in \mathcal{D}^-$$

and set $\mathbf{M}^- = \mathrm{ind}\mathcal{D}^-$. Then $\mathbf{M}^-$ is an $(\mathcal{H},\mathcal{D})$-bimodule where the right module structure is obtained as for $\mathbf{M}$ above, viewing $\mathcal{D}^-$ as a right $\mathcal{D}$-module under *ordinary* multiplication. The following simple result, whose proof is immediate, will be very useful in what follows.

1.5 Proposition.

(i) *For* $F \in \mathbf{M}$, *define the* $\mathcal{D}$*-valued function* $J(F)$ *on* $\mathcal{H}$ *by* $J(F)(h) = F(th)$, $h \in \mathcal{H}$. *Then* $J(F) \in \mathbf{M}^-$ *and* $J : \mathbf{M} \to \mathbf{M}^-$ *is an injective map of* $(\mathcal{H},\mathcal{D})$ *bi-modules.*

(ii) *For* $F \in \mathbf{M}^-$ *define the* $\mathcal{D}$*-valued function* $J^-(F)$ *on* $\mathcal{H}$ *by* $J^-(F)(h) = F(th)$, $h \in \mathcal{H}$. *Then* $J^-(F) \in \mathbf{M}$ *and* $J^- : \mathbf{M}^- \to \mathbf{M}$ *is an injective map of* $(\mathcal{H},\mathcal{D})$ *bi-modules.*

(iii) *We have* $J^- \circ J(F) = -f(z)F$, $F \in \mathbf{M}$.

1.6 We set $\tilde{\mathrm{X}} = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H},\mathbb{C})$. Then $\tilde{\mathrm{X}}$ has four elements, these elements being obtained by sending s_i to either γ_i or $-\gamma_i^{-1}$, $i = 1, 2$. Given $\rho \in \tilde{\mathrm{X}}$, we write $\mathbb{C}_\rho$ for the one-dimensional left $\mathcal{H}$-module corresponding to ρ.

For $\chi \in \mathrm{X}$, set $\mathbf{M}_\chi = \mathrm{ind}\mathbb{C}_\chi$. Then the map $F \to \chi \circ F$ is a surjective map of left $\mathcal{H}$-modules of M onto M_χ. Similarly, we obtain a two-dimensional complex matrix represention, σ_χ of $\mathcal{H}$ by applying χ to the entries of the matrix representation σ.

Proposition.

(i) *Set* $\Gamma = \{\gamma_1\gamma_2,\ -\gamma_1\gamma_2^{-1},\ -\gamma_1^{-1}\gamma_2,\ (\gamma_1\gamma_2)^{-1}\}$. *Then* $\mathbf{M}_\chi$ *is irreducible if and only if* $\chi(d) \notin \Gamma$.

(ii) *Let* $\chi,\ \chi' \in X$ *and suppose that* $\chi(d) \notin \Gamma, \chi'(d) \notin \Gamma$. *Then the modules* $\mathbf{M}_\chi, \mathbf{M}_{\chi'}$ *are isomorphic if and only if either* $\chi' = \chi$ *or* $\chi' = \chi^{-1}$.

(iii) *Every irreducible left* $\mathcal{H}$*-module is isomorphic either to some* $\mathbb{C}_\rho$ *or to some* $\mathbf{M}_\chi$.

Proof. Assertion (i) follows easily from the fact that $(\mathrm{res}, \mathrm{ind})$ is an adjoint pair. As for (ii), the fact that $\mathbf{M}_\chi, \mathbf{M}_{\chi'}$ are inequivalent unless $\chi' = \chi,\ \chi^{-1}$ follows from the fact that the representations $\sigma_\chi, \sigma_{\chi'}$ have different traces. Now suppose that $\chi' = \chi^{-1}$. Then just as above we may define a homomorphism $J_\chi : \mathbf{M}_\chi \to \mathbf{M}_{\chi'}$ by setting $J_\chi(F)(x) = F(tx)$, $F \in \mathbf{M}_\chi$, $x \in \mathcal{H}$ and we may define $J_\chi^- : \mathbf{M}_{\chi'} \to \mathbf{M}_\chi$ similarly. Clearly, we have that $J_\chi^- \circ J_\chi$ is just multiplication by

$\chi(-f(z))$. Corollary 1.2 (iv) implies that J_χ is an isomorphism unless $\chi(d+d^{-1}) \in \{\gamma_1\gamma_2 + \frac{1}{\gamma_1\gamma_2},\ -(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1})\}$. But these are precisely the characters χ for which $\chi(d) \in \Gamma$. Since we are assuming that $\mathbf{M}_\chi$ is irreducible, we are done.

To prove (iii) let N be a non-zero irreducible left $\mathcal{H}$-module; by Corollary 1.2 it is finitely generated as a left $\mathcal{D}$-module, hence it has an irreducible $\mathcal{D}$-quotient. Assertion (iii) now follows from this and the fact that $(\mathrm{res}, \mathrm{ind})$ is an adjoint pair.

2. The algebra $C^*(A)$ and its dual

2.1 We define an involution $x \to x^*$ on $\mathcal{H}$ characterised by the following properties:

(i) ${s_i}^* = s_i,\ i = 1, 2;$

(ii) $x \to x^*$ is multiplication reversing and conjugate linear.

We denote by ρ_0 the element of $\tilde{\mathrm{X}}$ defined by $\rho_0(s_i) = \gamma_i,\ i = 1, 2$. Then we may define a norm $||\ ||$ on $\mathcal{H}$ by setting

$$||\sum_{w\in W} a_w w|| = \sum_{w\in W} |a_w|\rho_0(w).$$

Lemma. *We have*

(i)
$$||x^*|| = ||x||,\ x \in \mathcal{H}.$$

(ii)
$$||xy|| \leq ||x||||y||,\ x, y \in \mathcal{H}.$$

Proof. Only the second assertion needs proof and, here, it is enough to check that $||x||||y|| \leq ||xy||$ when $x, y \in W$. However, it is clear from (1.1.1) that if $x, y \in W$ then $xy = \sum b_w w$ with $b_w \geq 0$. It follows that $||xy|| = \sum b_w \rho(w) = \rho(xy) = \rho(x)\rho(y) = ||x||||y||$ and we are done.

2.2 We now may complete $\mathcal{H}$ with respect to $||\ ||$ to obtain an involutive Banach algebra. We denote this algebra by $A = A(q_1, q_2)$.

Proposition. *Let H be a Hilbert space, let $\mathcal{B}(H)$ be the algebra of bounded operators on H and write $||\ ||'$ for the operator norm on $\mathcal{B}(H)$. Let (π, H) be a* *representation of $\mathcal{H}$; that is π is an algebra homomorphism of $\mathcal{H}$ into* $\mathrm{End}_{\mathbb{C}}(H)$. *Then $\pi(\mathcal{H}) \subset \mathcal{B}(H)$. In fact, we have*

$$||\pi(x)||' \leq ||x||, \ x \in \mathcal{H}.$$

Proof. Consider first the case where $x = s_i, \ i = i, 2$, let $v \in H$ and let V be the subspace of H spanned by $\{v, \pi(s_i)v\}$. Then V has finite dimension and is invariant under $\pi(s_i)$. Writing V as a sum of its $\pi(s_i)$ eigenspaces and noting that $\rho_0(s_i)$ is the larger of the two eigenvalues of s_i, we see that $||\pi(s_i)v||_H \leq \rho_0(s_i)||v||_H$ where $|| \ ||_H$ is the Hilbert space norm on H. Since v is arbitrary, we have shown that $||\pi(s_i)||' \leq \rho_0(s_i)$.

Now, if $w \in W$ then by definition, $w = \prod u_i$ for elements $u_i \in \{s_1, s_2\}$. It follows that $||\pi(w)||' \leq \prod ||\pi(u_i)||' \leq \prod \rho_0(u_i) = \rho_0(w)$. Finally, if $x = \sum a_w w$ is an arbitrary element of $\mathcal{H}$ then we have

$$||\pi(x)||' \leq \sum |a_w| ||\pi(w)||' \leq \sum |a_w| \rho_0(w) = ||x||$$

as was to be shown.

2.3 Corollary. *Any representation (π, H) as above extends to a continuous representation of A.*

2.4 By a *unitary* representation of $\mathcal{H}$ we mean a representation (π, H) where H is a Hilbert space with scalar product $< \ | \ >_H$ and such that $< \pi(x)v|w >_H = < v|\pi(x^*)w >_H, \ x \in \mathcal{H}, \ v, w \in H$. We say that (π, H) is *topologically irreducible* if H has no proper non-zero closed $\pi(\mathcal{H})$-invariant subspaces.

Lemma.

(i) *Let (π, H) be a unitary representation of $\mathcal{H}$. Then π extends to a unitary representation of A.*
(ii) *Let (π, H) be a topologically irreducible unitary representation of $\mathcal{H}$. Then (π, H) is algebraically irreducible; that is, either $\pi \cong \sigma_\chi, \ \chi \in X$, or $\pi \in \tilde{X}$.*

Proof. The first assertion follows immediately from Corollary 2.3. As for the second, note that Proposition 2.2 implies that $\pi(\mathcal{H}) \subset \mathcal{B}(H)$. Proposition 2.3.1 of [D] then implies that $\mathcal{Z}$ acts on H by scalars. Now let $v \in H$ be non zero; then v is a topologically cyclic vector for π. Since $\mathcal{H}$ is free of finite rank as a

$\mathcal{Z}$-module, the remark above implies that $\pi(\mathcal{H})v$ is a finite dimensional subspace of H. But any finite dimensional subspace of H is closed (see e.g. [R]§4.15, page 82). Thus $\pi(\mathcal{H})v = H$.

Warning Remark. From now on, if A is an involutive Banach algebra, and H a Hilbert space, we shall say that $\pi : A \to \mathcal{B}(H)$ is a *representation* of A on H if π is a morphism of involutive Banach algebras. This is the definition used in [D]2.2.1; thus the unitary representation of A in the statement of Lemma 2.4 (i) above, is a representation. Any representation of an involutive Banach algebra is automatically continuous by [D]1.3.7.

2.5 For the corollary below we recall some elementary facts about C^*-algebras. If H is a Hilbert space we write $\mathcal{B}(H)$ for the algebra of bounded linear operators of H; it is an C^*-algebra under the operator norm and adjoint operation. If C is a C^*-algebra, a representation of C is a pair (π, H) where H is a Hilbert space, and π is a morphism $\pi : C \to \mathcal{B}(H)$ of involutive algebras; we remark that π is automatically continuous by [D]1.3.7.

Next, recall that given an involutive Banach algebra A with an approximate identity there is always a C^*-algebra $C^*(A)$ and a morphism of involutive algebras $\tau_A : A \to C^*(A)$ with the following properties:

(i) if π is a representation of A there is a unique representation ρ of $C^*(A)$ such that $\pi = \rho \circ \tau_A$, and $\rho(C^*(A))$ is the C^*-algebra generated by $\pi(A)$;

(ii) there is a bijection $\pi \to \rho_\pi$ from representations of A to representations of $C^*(A)$ which preserves nondegeneracy and irreducibility.

For this see [D]2.7; again the map τ_A is automatically continuous by [D]1.3.7. We call the algebra $C^*(A)$ the *enveloping C^*-algebra* of A.

Finally, recall that a C^*-algebra C is called *liminal* (or CCR) if for each irreducible C^*-algebra representation (π, H) of C the ring of operators $\pi(C)$ lies in the two sided ideal of compact operators in H.

The proof of the Corollary below follows immediately from Lemma 2.4, the remarks above, and the fact that $\mathcal{H}$ is dense in A.

Corollary. *Let $A = A(q_1, q_2)$ be as above.*

(i) *Denote by $C^*(A)$ the enveloping C^*-algebra of A, write $C^*(A)\hat{}$ for the set of equivalence classes of irreducible C^*-algebra representations of $C^*(A)$*

and write $\hat{\mathcal{H}}$ for the set of equivalence classes of irreducible unitary representations of $\mathcal{H}$. Then restriction induces a bijection of $C^*(A)\hat{}$ onto $\hat{\mathcal{H}}$.

(ii) *The algebra $C^*(A)$ is liminal.*

2.6 We endow the set $\hat{A}$ with the *Jacobson topology* (c.f. [D] chapter 3):

Recall that a two sided ideal of $C^*(A)$ is *primitive* if it is the kernel of a topologically irreducible representation of $C^*(A)$, and that any closed two sided ideal of $C^*(A)$ is the intersection of the primitive ideals which contain it ([D]2.9.7). Let Prim $(C^*(A))$ denote the set of primitive ideals of $C^*(A)$. If I is a closed two sided ideal of $C^*(A)$ define $V(I) \subset$ Prim $(C^*(A))$ by $V(I) = \{J | J \supset I\}$. One then obtains a topology on Prim $(C^*(A))$ in which the closed sets are precisely the subsets $V(I)$ ([D]3.1.1). Since $C^*(A)$ is (post)liminal ([D]3.1.6, 4.3.7), the map $[[\pi]] \mapsto \ker(\pi)$ is a bijection $\hat{A} \to \mathrm{Prim}(C^*(A))$, and we endow $\hat{A}$ with the topology that makes this map a homeomorphism.

We remark that since $C^*(A)$ is liminal, the space $\hat{A}$ is T_1 ([D] 4.2.3): points in $\hat{A}$ are closed, or again, given two distinct points $x, y \in \hat{A}$ there is always an open neighborhood of x which does not contain y.

Proposition. *Let $\hat{A}_j$, $j = 1, 2$ be the subset of $\hat{A}$ consisting of equivalence classes of representations of dimension j. Then $\hat{A} = \hat{A}_1 \cup \hat{A}_2$ and $\hat{A}_2$ is an open subset of $\hat{A}$.*

Proof. This is an immediate consequence of Theorem 3.6.3 of [D]. Alternatively, we know from §1.4 and §2.4 that $\hat{A}_1$ has four elements, hence by the T_1-property, it is a closed subset of $\hat{A}$.

2.7 Our goal in the rest of this section is to describe the topological space $\hat{A}$ explicitly. We need some preliminary lemmas.

First, we identify X in §1.4 with $\mathbb{C}^\times$. Then the subset X_U of X consisting of characters χ for which $|\chi(d)| = 1$, is identified with the unit circle S^1; we endow it with the topology it inherits as a subset of $\mathbb{C}^\times$ via the map $\chi \to \chi(d)$. Let X'_U be the subset of characters $\chi \in X_U$ for which $\chi(d) \notin \Gamma$. Then, clearly, X'_U is open in X_U. Finally, let Y be the subset of X'_U consisting of characters χ for which $\mathrm{Im}\chi(d) \geq 0$. We identify Y with a subset of the unit circle.

Lemma. *Let $\mathbb{C}^2$ be the space of two-dimensional complex column vectors viewed as a Hilbert space with respect to the usual scalar (dot) product. Then the representations $\sigma_\chi,\ \chi \in X_U$ are unitary.*

Proof. We need to check that $(\sigma_\chi(s_i))^t = \overline{\sigma_\chi(s_i)},\ i = 1, 2$. But this follows immediately from 1.4 and the fact that $\chi(d)^{-1} = \overline{\chi(d)}$.

2.8 Lemma. *Let $\chi \in X$. The representation σ_χ is unitarizable if and only if either*

(i) $|\chi(d)| = 1$

or

(ii) $\chi(d)$ *is real and* $f(\chi(z)) < 0$.

Proof. By Corollary 1.2 (ii) we have that $z^* = z$. Thus we must have that $\chi(d) + \chi(d^{-1})$ is real whenever σ_χ is unitarizable. It follows easily that either $|\chi(d)| = 1$ or $\chi(d)$ is real. In light of Lemma 2.7, we may now assume that $\chi(d)$ is real. In that case, we are looking for a two by two complex matrix $\mathcal{A}_\chi$ with the property that $\mathcal{A}_\chi^t = \bar{\mathcal{A}}_\chi$ such that $\det\mathcal{A}_\chi > 0$ and $(\sigma_\chi(s_i))^t\mathcal{A}_\chi = \mathcal{A}_\chi\sigma_\chi(s_i),\ i = 1, 2$. A direct calculation shows that, up to a real scalar, we must have

$$\mathcal{A}_\chi = \begin{bmatrix} c_1\chi(d) + c_2 & \chi(d^{-1}) - \chi(d) \\ \chi(d^{-1}) - \chi(d) & c_1\chi(d^{-1}) + c_2 \end{bmatrix}. \tag{2.8.1}$$

Our result now follows from the fact that $\det\mathcal{A}_\chi = -f(\chi(z))$.

2.9 We note that all σ_χ satisfying 2.8(ii) above are irreducible. There may however, be at most two points in 2.8(i) above where reducibility occurs. Removing such points and using Proposition 1.6 and Corollary 2.5, we see that the map $\chi(d) \mapsto \sigma_\chi$ induces a bijection of sets

$$f : \Xi' \to \hat{A}_2,$$

where

$$\Xi' = (-\frac{\gamma_1}{\gamma_2}, -1] \cup Y \cup [1, \gamma_1\gamma_2).$$

Our first goal in describing $\hat{A}$ is to show that with the standard topology on Ξ', f is a homeomorphism. We begin by identifying the space of each irreducible unitarizable representation σ_χ in 2.8 with the Hilbert space $\mathbb{C}^2$ in a continuous way, and describing σ_χ under this identification. With this in mind, and with $\mathcal{A}_\chi$ as in (2.8.1), we write

$$\nu_\chi = \begin{cases} c_2 + c_1, & \text{if } \chi(d) \in (1, \gamma_1\gamma_2) \\ 1, & \text{if } \chi(d) \in Y \\ c_2 - c_1, & \text{if } \chi(d) \in (-\frac{\gamma_1}{\gamma_2}, -1), \end{cases}$$

and set

$$\mathcal{B}_\chi = \begin{cases} \frac{1}{\nu_\chi}\mathcal{A}_\chi, & \text{if } \chi(d) \in (-\frac{\gamma_1}{\gamma_2}, -1) \cup (1, \gamma_1\gamma_2) \\ I, & \text{if } \chi(d) \in Y. \end{cases}$$

Next, let

$$\tilde{\mu}_\chi^{(1)} = \begin{cases} -\gamma_1^{-1}\chi(d) + \gamma_1\chi(d)^{-1} + c_2, & \text{if } \chi(d) \in (-\frac{\gamma_1}{\gamma_2}, -1) \cup (1, \gamma_1\gamma_2) \\ 1, & \text{if } \chi(d) \in Y. \end{cases}$$

and

$$\tilde{\mu}_\chi^{(2)} = \begin{cases} \gamma_1\chi(d) - \gamma_1^{-1}\chi(d)^{-1} + c_2, & \text{if } \chi(d) \in (-\frac{\gamma_1}{\gamma_2}, -1) \cup (1, \gamma_1\gamma_2) \\ 1, & \text{if } \chi(d) \in Y. \end{cases}$$

We write $\mu_\chi^{(i)} = \frac{\tilde{\mu}_\chi^{(i)}}{\nu_\chi}$ for $i \in \{1, 2\}$.

Let $\lambda = \sqrt{1 + \gamma_1^2}$. With respect to the right $\mathcal{D}$-basis Φ_0, Φ_1 of §1.4 we define $\Psi_1' = \lambda^{-1}\begin{bmatrix} 1 \\ \gamma_1 \end{bmatrix}$ and $\Psi_2' = \lambda^{-1}\begin{bmatrix} -\gamma_1 \\ 1 \end{bmatrix}$. The vectors Ψ_1', Ψ_2' form a right $\mathcal{D}$-basis for $\mathbf{M}$ and for any specialisation to $\mathbf{M}_\chi$ this is an orthonormal basis for the usual dot product. The vectors Ψ_1', Ψ_2' are eigenvectors for the matrix $\mathcal{B}_\chi$ with respective (positive) eigenvalues $\mu_\chi^{(1)}, \mu_\chi^{(2)}$.

Henceforth we let R_2 denote the set of all topologically irreducible two-dimensional matrix representations of $C^*(A)$; that is, R_2 is the set of all topologically irreducible C^*-algebra representations of $C^*(A)$ with Hilbert space $\mathbb{C}^2$ (c.f. Lemma 2.7).

We then have the following useful lemma.

Lemma.

With notation as above:

(i) The map $\chi(d) \to \mathcal{B}_\chi$ is continuous on matrix coefficients, for $\chi(d) \in \Xi'$.

(ii) For $i \in \{1, 2\}$ define $\Psi_i(\chi) = \sqrt{\mu_\chi^{(i)}}^{-1} \Psi'_i$, for $\chi(d) \in \Xi'$. Then $\Psi_i(\chi)$ belongs to the eigenspace corresponding to the eigenvalue $\mu_\chi^{(i)}$. The vectors $\Psi_1(\chi), \Psi_2(\chi)$ provide an orthonormal basis for the form $\langle v|w\rangle_\chi = v^* \mathcal{B}_\chi w$.

(iii) Set $\mu_\chi = \sqrt{\frac{\mu_\chi^{(1)}}{\mu_\chi^{(2)}}}$, for $\chi(d) \in \Xi'$. Then with respect to the basis $\{\Psi_1(\chi), \Psi_2(\chi)\}$ the action of $\mathcal{H}$ on $\mathbf{M}_\chi$ is given by

$$[\sigma_\chi](s_1) = \begin{bmatrix} \gamma_1 & 0 \\ 0 & -\gamma_1^{-1} \end{bmatrix}$$

and

(2.9.1)
$$[\sigma_\chi](s_2) = \lambda^{-2} \begin{bmatrix} c_2 + \gamma_1(\chi(d) + \chi(d)^{-1}) & (\chi(d)^{-1} - \chi(d)\gamma_1^2 - \gamma_1 c_2)\mu_\chi \\ (\chi(d) - \gamma_1^2\chi(d)^{-1} - \gamma_1 c_2)\mu_\chi^{-1} & \gamma_1^2 c_2 - \gamma_1(\chi(d)^{-1} + \chi(d)) \end{bmatrix}.$$

The map $\chi(d) \mapsto [\sigma_\chi]$ induces an injection of sets $g : \Xi' \to R_2$.

(iv) Set $b_\chi = -\gamma_1 \sqrt{\tilde{\mu}_\chi^{(1)} \tilde{\mu}_\chi^{(2)}}$

$$= -\gamma_1 \sqrt{((\gamma_1\chi(d)^{-1} - \gamma_1^{-1}\chi(d)) + c_2)((\gamma_1\chi(d) - \gamma_1^{-1}\chi(d)^{-1}) + c_2)}.$$

Then

$$(2.9.2) \qquad [\sigma_\chi](s_2) = \lambda^{-2} \begin{bmatrix} c_2 + \gamma_1(\chi(d) + \chi(d)^{-1}) & b_\chi \\ b_\chi & \gamma_1^2 c_2 - \gamma_1(\chi(d)^{-1} + \chi(d)) \end{bmatrix}.$$

The matrices in (2.9.1) and (2.9.2) are hermitian; moreover the matrix (2.9.2) is defined when $\chi(d) \in \{-\frac{\gamma_1}{\gamma_2}, \gamma_1 \cdot \gamma_2\}$, whereas the matrix (2.9.1) is not.

Proof. Straightforward.

2.10 For the explicit description of the topological space $\hat{A}$ below, we employ the following construction/definition:

Definition. Let Ξ be a locally compact Hausdorff topological space. Let x_0 be a point of Ξ, and write Ξ' for the subspace $\Xi\backslash\{x_0\}$. Let x_0^1, x_0^2 be two copies of x_0. We define a new space $\tilde{\Xi}$ as follows.

(1) The set $\tilde{\Xi} = \Xi' \cup \{x_0^1, x_0^2\}$.
(2) The topology on $\tilde{\Xi}$ is the weakest such that
(a) any neighborhood base for $x \in \Xi'$ will be a neighborhood base for $x \in \tilde{\Xi}$, and
(b) If U is an open neighborhood of x_0 then, for $i \in \{1, 2\}$, any set of the form $(U\backslash\{x_0\}) \cup \{x_0^i\}$ is an open neighborhood of x_0^i.

In particular, $\tilde{\Xi}$ is non Hausdorff, but it is T_1.

We shall say that $\tilde{\Xi}$ *is the space obtained from* Ξ *by replacing* x_0 *with a double point.* One can vary this construction by taking a finite set of points $\{x_1, \ldots, x_n\}$ in Ξ in place of one point x_0.

The algebra $\mathcal{H}$ has four irreducible one dimensional representations, corresponding to the possibilities $s_i \mapsto \gamma_i$ or $s_i \mapsto -\gamma_i^{-1}$ (§1.6). In what follows we shall write $\rho_{\{x,y\}}$ for the one dimensional representation $\rho(s_1) = x, \rho(s_2) = y$. Each of these representations is unitarizable via the usual scalar product on $\mathbb{C}$, and we shall use the same symbol to denote the corresponding representation for $C^*(A)$.

Proposition. *Let*

$$\Xi = [-\frac{\gamma_1}{\gamma_2}, -1] \cup Y \cup [1, \gamma_1\gamma_2],$$

and let

$$\Xi' = (-\frac{\gamma_1}{\gamma_2}, -1] \cup Y \cup [1, \gamma_1\gamma_2).$$

(i) Via the identification $\mathbb{C}^\times \to X$ *of §1.4 the map* $\chi(d) \to \sigma_\chi$ *induces a homeomorphism* $f : \Xi' \to \hat{A}_2$.

(ii) The space $\hat{A}$ *is homeomorphic to the space obtained from* Ξ *by replacing* $-\frac{\gamma_1}{\gamma_2}$ *and* $\gamma_1\gamma_2$ *with double points. Under this identification* $\rho_{\{\gamma_1,\gamma_2\}}$ *and* $\rho_{\{-\gamma_1^{-1},-\gamma_2^{-1}\}}$ *correspond to the double point at* $\gamma_1\gamma_2$*; and* $\rho_{\{-\gamma_1^{-1},\gamma_2\}}$ *and* $\rho_{\{\gamma_1,-\gamma_2^{-1}\}}$ *correspond to the double point at* $-\frac{\gamma_1}{\gamma_2}$.

In particular the Hausdorff space $\hat{A}_2$ *is open and dense in* $\hat{A}$.

Remarks. (i) Of course the subintervals in the definition of Ξ' may be empty.

(ii) In particular, part (ii) of the Proposition illustrates Theorem 4.4.5 of [D].

Proof. To prove (i), let $h : R_2 \to \hat{A}_2$ be the map which sends a representation to its equivalence class. Then the map $f : \Xi' \to A_2$ factors: $f = h \circ g$, where $g : \Xi' \to R_2$ is the injection of sets defined in Lemma 2.9(iii).

Furthermore the map $h : R_2 \to \hat{A}_2$ is continuous and open, by Theorem 3.5.8 of [D]. Here, R_2 is endowed with the topology of weak pointwise convergence over $C^*(A)$: a typical basic open neighborhood $V_{(v,w;a_1,\dots,a_n;\epsilon)}(\pi_0)$ for π_0 is provided by choosing $v, w \in \mathbb{C}^2$, a finite set $a_1, \dots, a_n \in C^*(A)$, $\epsilon > 0$ and then defining

$$V_{(v,w;a_1,\dots,a_n;\epsilon)}(\pi_0) = \cap_{i=1}^{n} \{\pi \,|\, |(\pi(a_i)v, w) - (\pi_0(a_i)(v), w)| < \epsilon\},$$

where $(\ ,\ )$ denotes the usual dot product for $\mathbb{C}^2$.

On the other hand, the injection g is a homeomorphism onto its image in R_2. For, on each segment defining Ξ' the matrix coefficients above are continuous maps hence, by definition of the weak pointwise convergent topology, so is the map $\chi \mapsto [\sigma_\chi]$. Moreover, this map is open. Indeed the formula (2.9.1) says that $\chi(d) \mapsto [\sigma_\chi](s_2)_{11}$ is an open map; this implies that the image under $\chi(d) \mapsto [\sigma_\chi]$ of any open segment containing $\chi_0(d)$ will contain a suitable open set $V_{(e_1,e_1;s_2;\epsilon)}(\pi_{\chi_0}) \cap \text{image}\ (g)$. Here e_1, e_2 denote the standard basis elements for $\mathbb{C}^2$.

Part (i) now follows, since $f : \Xi' \to \hat{A}_2$ is bijective.

In proving (ii), we identify Ξ' with $\hat{A}_2$, via (i). To avoid excessive notation, we let $\rho^+ = \rho_{\{\gamma_1,\gamma_2\}}$, $\rho^- = \rho_{\{-\gamma_1^{-1},-\gamma_2^{-1}\}}$ in the statement of (ii), and similarly $\rho_+ = \rho_{\{\gamma_1,-\gamma_2^{-1}\}}$, $\rho_- = \rho_{\{-\gamma_1^{-1},\gamma_2\}}$. We write $\rho^\pm$ for an element of $\{\rho^+, \rho^-\}$ and we write $\rho_\pm$ for an element of $\{\rho_+, \rho_-\}$.

We shall assume that $\gamma_1 > 1$: the proof that follows is readily adapted if $\gamma_1 = 1$.

Since $\hat{A}_2$ is open in $\hat{A}$ (Proposition 2.6), it is enough to show the following:

(A) Any set of the form $(\delta, \gamma_1 \cdot \gamma_2) \cup \{\rho^\pm\}$ where $\delta \in (1, \gamma_1 \cdot \gamma_2)$, is open in $\hat{A}$; any set of the form $(-\frac{\gamma_1}{\gamma_2}, \lambda) \cup \{\rho_\pm\}$ where $\lambda \in (-\frac{\gamma_1}{\gamma_2}\,, -1)$, is open in $\hat{A}$,

and

(B) Any open set in $\hat{A}$ containing $\rho^{\pm}$ must contain a set $(\delta, \gamma_1 \cdot \gamma_2) \cup \{\rho^{\pm}\}$ where $\delta \in (1, \gamma_1 \cdot \gamma_2)$; any open set in $\hat{A}$ containing $\rho_{\pm}$ must contain a set $(-\frac{\gamma_1}{\gamma_2}, \lambda) \cup \{\rho_{\pm}\}$ where $\lambda \in (-\frac{\gamma_1}{\gamma_2}, -1)$.

Proof of (A): We shall prove the first assertion: any set of the form $(\delta, \gamma_1 \cdot \gamma_2) \cup \{\rho^+\}$, or of the form $(\delta, \gamma_1 \cdot \gamma_2) \cup \{\rho^-\}$ is open in $\hat{A}$.

For this it suffices to prove the following:

(A1) Let S be any open segment of Ξ' which is the complement of an interval $[\delta, \gamma_1 \cdot \gamma_2)$, where $1 < \delta < \gamma_1 \cdot \gamma_2$. Then the closure of S in $\hat{A}$ is S together with δ and the two double points ρ_+, ρ_- corresponding to $-\frac{\gamma_1}{\gamma_2}$.

Indeed, (A1) implies that any set of the form $(\delta, \gamma_1 \cdot \gamma_2) \cup \{\rho^+, \rho^-\}$ is open in $\hat{A}$. Since points are closed in $\hat{A}$, this in turn implies the first assertion of (A).

To prove (A1), observe that the closure of S in $\hat{A}$ is the set $V(I)$ where $I = \cap J$, and where J runs through all the ker σ_χ for $\chi(d) \in S$. But $\sigma_\chi(a) = 0$ if and only if its associated matrix $[\sigma_\chi](a) = 0$. The descriptions (2.9.1) and (2.9.2) imply that for any $a \in C^*(A)$, $[\sigma_\chi](a)_{ij} \to 0$ if $i \neq j$ as $\chi(d) \to -\frac{\gamma_1}{\gamma_2}$, while $[\sigma_\chi](a)_{11} \to \rho_+(a)$ and $[\sigma_\chi(a)]_{22} \to \rho_-(a)$. This implies immediately that if $a \in I$, then $a \in \ker(\rho_+) \cap \ker(\rho_-)$, and (A1) follows.

Proof of (B): Any open set containing ρ^+ contains a set of the form $(\delta, \gamma_1 \cdot \gamma_2) \cup \{\rho^+\}$, and similarly for ρ^-.

Let O be an open set containing ρ^+. Then, under the identification $\hat{A} \to$ Prim $(C^*(A))$, O is the complement of a closed set $V(I)$: $O = \{\pi | \ker(\pi) \not\supset I\}$. In particular there is $a \in I$ and $\rho^+(a) \neq 0$. But $\lim_{\chi(d) \to \gamma_1 \cdot \gamma_2} (\pi_\chi(a))_{11} = \rho^+(a) \neq 0$ implies that $(\pi_\chi(a))_{11} \neq 0$ if $\chi(d)$ is sufficiently close to $\gamma_1 \cdot \gamma_2$. In other words there is a number $\delta : \gamma_1 \cdot \gamma_2 > \delta > 1$ such that if $\chi(d) \in (\delta, \gamma_1 \cdot \gamma_2)$ then $(\pi_\chi(a))_{11} \neq 0$. But, via our identifications, this means that $O \cap \hat{A}_2$ contains the segment $(\delta, \gamma_1 \cdot \gamma_2)$.

The second assertion in (B) is proved in a similar way.

3. Some harmonic analysis

3.1 We define a functional $\Lambda : \mathcal{H} \to \mathbb{C}$ by setting $\Lambda(\mathbf{1}) = 1$, $\Lambda(w) = 0$, $w \in W$, $w \neq 1$. For $x, y \in \mathcal{H}$ we set $< x|y >= \Lambda(xy^*)$.

Lemma. *For any two words $w, u \in \mathcal{H}$ we have $\Lambda(wu^*) = \delta_{w,u}$.*

Proof. We proceed by induction on $l(w)$, the case $l(w) = 0$ being trivial. We also may suppose that $u \neq \mathbf{1}$. Suppose without loss of generality that $w = w's_1$ for some word w'. If $u = u's_2$ for some word u' then $w \neq u$ and wu^* is a word so that $\Lambda(wu^*) = 0$. If $u = u's_1$ for some word u' then $wu^* = c_1 w's_1u'^* + w'u'^*$. We have that $\Lambda(w's_1u'^*) = 0$ while $\Lambda(w'u'^*) = \delta_{w',u'} = \delta_{w,u}$ by induction. This gives us what we want.

3.2 Proposition.

(i) $< \ | \ >$ is a scalar product.
(ii) $< x|y >=< y^*|x^* >, \ x, y \in \mathcal{H}$.
(iii) $< xy|z >=< y|x^*z >, \ x, y, z \in \mathcal{H}$.

Proof. For (i) we must show that $< \ | \ >$ is sesquilinear, hermitian and positive-definite. This, as well as (ii), and (iii), is a routine computation using Lemma 3.1.

3.3 Corollary. *The set of words is orthonormal for $< \ | \ >$.*

3.4 Proposition.

(i) *For each element $x \in \mathcal{H}$, the map $y \to xy$ of $\mathcal{H}$ to $\mathcal{H}$ is continuous with respect to the topology induced by $< \ | \ >$.*
(ii) *The set of $xy, \ x, y \in \mathcal{H}$ is dense in $\mathcal{H}$.*

Proof. Assertion (ii) is trivial since $\mathcal{H}$ is unital, while (i) follows from Proposition 2.2 since $< \ | \ >$ is positive definite by Lemma 3.1.

3.5 Recall ([D]A 54) that a *Hilbert algebra* is an involutive algebra B equipped with a scalar product $< \ | \ >$ which provides B with the structure of a Hausdorff pre-Hilbert space, satisfying properties (ii) and (iii) of Proposition 3.2, and (i) and (ii) of Proposition 3.4 with respect to $< \ | \ >$.

Corollary. *$\mathcal{H}$ is a Hilbert algebra with respect to $< \ | \ >$.*

3.6 Let $\mathbf{H}$ be the Hilbert space completion of $\mathcal{H}$ with respect to $< \ | \ >$, so that the action of left multiplication of $\mathcal{H}$ on itself extends to give a unitary representation of $\mathcal{H}$ on $\mathbf{H}$ ([D]A 54). This extends to a representation of A by Lemma 2.4(i), and property 2.5(i) then implies there is a unique morphism of $C^*(A)$ into $\mathcal{B}(\mathbf{H})$. We denote the image of $C^*(A)$ in $\mathcal{B}(\mathbf{H})$ by $C^*_r(A)$: this is a C^*-algebra by property 2.5(i) above. The action of $\mathcal{H}$ on $\mathbf{H}$ is faithful, hence by property (i) of 2.5 above, $\mathcal{H}$ embeds in $C^*_r(A)$. We write $\hat{A}_r$ for the dual of $C^*_r(A)$. Since $C^*_r(A)$ is a quotient of $C^*(A)$ it is liminal, and moreover we may identify $\hat{A}_r$ with a closed subset of $\hat{A}$ by Proposition 3.2.1 of [D]. In particular $\hat{A}_r$ is a T_1 space as well.

We can now state what may be referred to as the Plancherel theorem in this context. The modifications of results in Dixmier that are needed to justify this formula may be found in [BHK]3.2:

Proposition. *There is a positive Borel measure $\hat{\mu} = \hat{\mu}_{\mathcal{H}}$ on $\hat{A}$ which is unique with the following property*

$$\Lambda(x) = \int_{\hat{A}} tr\pi(x)d\hat{\mu}(\pi),\ x \in \mathcal{H}.$$

Further, the support of this measure is $\hat{A}_r$.

Remarks. (i) The Borel structure on $\hat{A}$ is that given by the Jacobson topology of §2.6. Since A is liminal, Proposition 4.6.1 in [D] implies this structure is equivalent to the Mackey Borel structure of [D]3.8.2. By *Borel measure* we simply mean a positive measure on $\hat{A}$, as defined in [D] B 30, for example.

(ii) We remind the reader that if m is a Borel measure on a topological space X, the *support* of m is the smallest closed set F such that $m(X\backslash F) = 0$.

(iii) For the convenience of the reader we shall elaborate the final statement of the Proposition. The algebras $C^*(A)$ (resp. $C^*_r(A)$) are liminal, so $\hat{A}$ (resp. $\hat{A}_r$) is homeomorphic to Prim $(C^*(A))$ (resp. Prim $(C^*_r(A))$. (See §2.6 for the definitions.) Let I denote the kernel of the representation $\pi : C^*(A) \to \mathcal{B}(\mathbf{H})$ in §3.6: it is a closed two-sided ideal. By definition the *support* of π consists of those classes of irreducible representations of $C^*(A)$ whose kernels contain I ([D] 3.4.6). This set corresponds precisely to $\hat{A}_r$ by [D] 2.11.5, and it is a closed set by definition of the Jacobson topology. But from Theorem 8.6.8 of [D], this is precisely the support (as defined in **Remark** (ii)) of our measure $\hat{\mu}$.

3.7 Corollary. *Let ν be the restriction of $\hat{\mu}$ to $\hat{A}_2$. Then there are non-negative numbers $\kappa_\rho,\ \rho \in \hat{A}_1$ such that*

$$\Lambda(x) = \int_{\hat{A}_2} tr\pi(x)d\nu(\pi) + \sum_{\rho \in \hat{A}_1} \kappa_\rho \rho(x),\ x \in \mathcal{H}. \tag{3.7.1}$$

Remark. From §1.4 and §2.4 we know that $\hat{A}_1$ consists of four points.

3.8 Our goal for the remainder of this section is to elucidate Proposition 3.6 and Corollary 3.7; in particular we want to describe the measures $\hat{\mu}$, ν and the topological space $\hat{A}_r$ explicitly.

We shall begin by defining another functional Λ' on $\mathcal{H}$ by

$$\Lambda'(x) = \Lambda(t^2 x) = \Lambda(-f(z)x).$$

We then have the following rather surprising result, which will be crucial in the proof of Propositions 3.10 and 3.12 below.

Lemma. *We have*

$$\Lambda'(\mathbf{1}) = 2;\ \Lambda'(d^{\pm 2}) = -1;\ \Lambda'(d^n) = 0 \ \textit{otherwise.}$$

In particular, the restriction of Λ' to $\mathcal{D}$ is independent of q_1, q_2.

Proof. Since $\Lambda(xy) = \Lambda(yx),\ x, y \in \mathcal{H}$ by Proposition 3.2(ii), we have $\Lambda(t^2 d^n) = \Lambda(td^{-n}t) = \Lambda(t^2 d^{-n})$. Thus we need only compute $\Lambda'(d^n)$ for $n \geq 0$.

Now $d^{-1} = s_2^{-1} s_1^{-1} = (s_2 - c_2)(s_1 - c_1)$ so that

$$\Lambda(d^{-1}) = c_1 c_2$$

and a similar calculation shows that

$$\Lambda(d^{-2}) = c_1^2 + c_2^2 + c_1^2 c_2^2.$$

Further, from Corollary 1.2 we have that

$$t^2 = -d^{-2} - d^2 + c_1 c_2 d^{-1} + c_1 c_2 d + c_1^2 + c_2^2 + 2.$$

Our result now follows by a direct calculation, keeping in mind that $\Lambda(\mathbf{1}) = 1$ while $\Lambda(d^n) = 0,\ n \geq 1$.

3.9 Lemma. *Let $\hat{A}_U$ (resp. $\hat{A}_R$) denote the subset of $\hat{A}_2$ consisting of equivalence classes of representations σ_χ for which $|\chi(d)| = 1$ (resp.$\chi(d)$ is real but $\chi(d) \neq \pm 1$). Then*

(i) $\hat{A}_2 = \hat{A}_U \cup \hat{A}_R$;

(ii) $\hat{A}_R$ *is homeomorphic to*

$$\begin{cases} (-\frac{\gamma_1}{\gamma_2}, -1) \cup (1, \gamma_1\gamma_2), & \text{for } \gamma_1 > \gamma_2 \geq 1 \\ (1, \gamma_1\gamma_2), & \text{for } \gamma_1 = \gamma_2 > 1 \\ \emptyset, & \text{for } \gamma_1 = \gamma_2 = 1. \end{cases}$$

In particular, $\hat{A}_R$ *is open in* $\hat{A}_2$*, and* $\hat{A}_U$ *is closed in* $\hat{A}_2$*.*

Proof. This follows from Lemma 2.8 and Proposition 2.10.

3.10 Now we can determine $\hat{A}_r \cap \hat{A}_2$.

Proposition. *The measure* ν *of Corollary 3.7 is supported on* $\hat{A}_U$*.*

Proof. By Corollary 1.2 (iv) we have that $\rho(-f(z)x) = 0,\ \rho \in \tilde{\mathrm{X}},\ x \in \mathcal{H}$. It follows from (3.7.1) that

$$\Lambda'(x) = \int_{\hat{A}_2} \mathrm{tr}\pi(-f(z)x) d\nu(\pi) = \int_{\hat{A}_2} \omega_\pi(-f(z))\mathrm{tr}\pi(x) d\nu(\pi),\ x \in \mathcal{H}$$

where ω_π is the central character of π. Then we have

$$\Lambda'(x) = \int_{\hat{A}_U} \omega_\pi(-f(z)\mathrm{tr}\pi(x) d\nu(\pi) + \int_{\hat{A}_R} \omega_\pi(-f(z)\mathrm{tr}\pi(x) d\nu(\pi).$$

From Lemma 3.8 we have that $|\Lambda'(d^n)| \leq 2,\ n \in \mathbb{Z}$. We also have that

$$|\int_{\hat{A}_U} \omega_\pi(-f(z)\mathrm{tr}\pi(d^n) d\nu(\pi)| \leq \int_{\hat{A}_U} |\omega_\pi(-f(z)\mathrm{tr}\pi(d^n)| d\nu(\pi) \leq 2M \int_{\hat{A}_U} d\nu(\pi)$$

where M is the maximum value of the quadratic polynomial $|\omega_\pi(-f(z))|$ on the interval $[-2, 2]$. Since

$$2\int_{\hat{A}_U} d\nu(\pi) = \int_{\hat{A}_U} \mathrm{tr}\pi(\mathbf{1}) d\nu(\pi) \leq \int_{\hat{A}} \mathrm{tr}\pi(\mathbf{1}) d\mu(\pi) = \Lambda(\mathbf{1}) = 1,$$

we see that

$$|\int_{\hat{A}_U} \omega_\pi(-f(z)\mathrm{tr}\pi(d^n) d\nu(\pi)| \leq M,\ n \in \mathbb{Z}.$$

On the other hand, $\omega_\pi(-f(z)) > 0$ for $\pi \in \hat{A}_R$ and $K = \max\{|\chi(d)|, |\chi(d^{-1}|\} > 1$ for $\sigma_\chi \in \hat{A}_R$. But $|\mathrm{tr}\sigma_\chi(d^{2n})| \geq K^{2n}$ from which it follows easily that

$$\int_{\hat{A}_R} \omega_\pi(-f(z)\mathrm{tr}\pi(d^{2n}) d\nu(\pi)$$

is unbounded if $\nu(\hat{A}_R) \neq 0$. Thus $\nu(\hat{A}_R) = 0$ which was to be shown.

3.11 Set $\hat{A}_{ri} = \hat{A}_r \cap \hat{A}_i$, $i = 1, 2$. Then, with notation of Proposition 2.10, the map $\chi \to \sigma_\chi$ is a homeomorphism of Y onto $\hat{A}_{r2}$; this follows from Proposition 3.10, Lemma 3.9, and Proposition 2.10. We let ν' be the Borel measure on Y that corresponds to ν under this homeomorphism. We then have the following

Corollary. *We have*

$$\Lambda'(x) = \int_Y tr\sigma_\chi(x)\chi(-f(z))d\nu'(\chi), \ x \in \mathcal{H}. \tag{3.11.1}$$

3.12 We may now state one of our major results.

Proposition. *Fix Haar measure μ_X on X so that $\mu_X(X_U) = 1$ and fix measure ν_0 on $\hat{A}_2$ so that ν_0 is supported on the set of representations σ_χ, $\chi \in Y$ and so that the map $\chi \to \sigma_\chi$, $\chi \in Y$ is measure preserving. Then ν is absolutely continuous with respect to ν_0 and we have*

$$d\nu(\sigma_\chi) = \frac{\chi(z^2 - 4)}{\chi(f(z))} d\nu_0(\sigma_\chi).$$

Proof. Consider for the moment the case that $q_1 = q_2 = 1$. In that case, we have that

$$\Lambda(d^n) = \delta_{n,0}.$$

However, it is standard that the measure μ_{X} is the unique Borel measure on X_U with the property that

$$\delta_{n,0} = \int_{X_U} \chi(d^n) d\mu_{\mathrm{X}}(\chi).$$

We conclude easily that

$$\Lambda(d^n) = \int_{\mathrm{Y}} \mathrm{tr}\sigma_\chi(d^n) d\mu_X(\chi)$$

so that

$$\Lambda'(d^n) = \int_Y \mathrm{tr}\sigma_\chi(d^n)\chi(-f(z)) d\mu_X(\chi) \tag{3.12.1}$$

in case $q_1 = q_2 = 1$.

Now let q_1, q_2 be arbitrary. Lemma 3.8 implies that the functionals on $\mathcal{D}$ defined by the right hand sides of the expressions (3.11.1), and (3.12.1) are equal. Since $f(z) = z^2 - 4$ in case $q_1 = q_2 = 1$ we conclude that that

$$d\nu'(\chi) = \frac{\chi(z^2-4)}{\chi(f(z))} d\mu_X(\chi).$$

Our proposition follows immediately.

3.13 We must now compute the constants $\kappa(\rho)$ which appear in Corollary 3.7. Since each of the characters ρ is a component of $\hat{A}_r$ it follows that either $\kappa_\rho = 0$ or else the Hilbert space completion $\mathbf{H}$ of $\mathcal{H}$ has a non-zero $\mathcal{H}$ subspace isomorphic to $\mathbb{C}_\rho$. A standard argument now shows that either $\kappa(\rho) = 0$ or else $|\rho(d)| < 1$.

Now if $q_1 \neq q_2$ there are two characters $\rho_i,\ i = 1,2$ which satisfy this condition. These are determined by

$$\rho_1(s_i) = -\gamma_i^{-1},\ i = 1,2;$$

$$\rho_2(s_1) = -\gamma_1^{-1}, \quad \rho_2(s_2) = \gamma_2.$$

In case $q_1 = q_2$ then ρ_1 above is the unique character satisfying $|\rho(d)| < 1$.

Proposition. *The constants* $\kappa(\rho_1), \kappa(\rho_2)$ *are given by*

$$\kappa(\rho_1) = \frac{1}{2}\left(\frac{q_1-1}{q_1+1} + \frac{q_2-1}{q_2+1}\right) \qquad \kappa(\rho_2) = \frac{1}{2}\left(\frac{q_1-1}{q_1+1} - \frac{q_2-1}{q_2+1}\right).$$

Proof. We have (see 1.4) that $\mathrm{tr}\sigma_\chi(c_i - 2s_i) = 0,\ i = 1,2$. It follows that

$$c_i = \Lambda(c_i - 2s_i) = (\gamma_i + \gamma_i^{-1})\kappa_1 + (-1)^{i+1}(\gamma_i + \gamma_i^{-1})\kappa_2,\ i = 1,2.$$

Our result now follows easily.

3.14 We have now proved the following:

Theorem. *With all notation as above, we have*

$$\Lambda(x) = \int_{\hat{A}_{r2}} tr\sigma_\chi(x)\frac{\chi(z^2-4)}{\chi(f(z))} d\nu_0(\sigma_\chi) + \kappa(\rho_1)\rho_1(x) + \kappa(\rho_2)\rho_2(x),\ x \in \mathcal{H}.$$

3.15 We have the following description of $\hat{A}_r$, which follows from Proposition 2.10 and the remarks following the statement of Proposition 3.6. We employ the notation of Proposition 2.10; moreover, we denote the closure of a subset B in a topological space A by $\mathrm{Cl}_A(B)$.

Proposition.

(i) The map $\chi \to \sigma_\chi$ induces a homeomorphism from the space $\hat{A}_{r2}$ onto Y.

(ii) (a) If $\gamma_1 > \gamma_2 \geq 1$ then $\hat{A}_r$ is just $\hat{A}_{r2}$ together with two isolated points, corresponding to the irreducible representations $\rho_{\{-\gamma_1^{-1}, -\gamma_2^{-1}\}}, \rho_{\{-\gamma_1^{-1}, \gamma_2\}}$.

(b) If $\gamma_1 = \gamma_2 > 1$ then $Cl(\hat{A}_{r2})$ in $\hat{A}_r$ is homeomorphic to the space obtained from $Cl_{\mathbb{R}^2}(Y)$ by replacing -1 with a double point. These two points correspond to the one dimensional representations $\rho_{\{1,-1\}}, \rho_{\{-1,1\}}$. The space $\hat{A}_r$ consists of $Cl(\hat{A}_{r2})$ together with an isolated point corresponding to the representation $\rho_{\{-\gamma_1^{-1}, -\gamma_2^{-1}\}}$.

(c) If $\gamma_1 = \gamma_2 = 1$ then $\hat{A}_{r2}$ is dense in $\hat{A}_r$. The space $\hat{A}_r$ is homeomorphic to the space obtained from $Cl_{\mathbb{R}^2}(Y)$ by replacing each of -1, 1 with a double point. The two points in place of -1 correspond to the one dimensional representations $\rho_{\{1,-1\}}, \rho_{\{-1,1\}}$. The two points in place of 1 correspond to the one dimensional representations $\rho_{\{1,1\}}, \rho_{\{-1,-1\}}$.

Proof. For part (i) see the remarks preceding Corollary 3.11. As for part (ii), the sets described in (ii)(a) – (c) are the supports of the measure in each case; this follows from the preceding sections, and the description of $\hat{A}$ in 2.10.

4. An application

4.1 In this section we illustrate the preceding theory by deducing the Plancherel formula for the group $\mathbb{SL}_2(F)$ where F is a local non-archimedean field; this will include a description of the reduced dual. For this we follow the strategy outlined in the introduction. In [K] the second author has explicitly described a complete list of types for $G = \mathbb{SL}_2(F)$ other than those inertial pairs where $L = \mathbb{SL}_2(F) = G$; in other words for each $\mathfrak{s}$ he describes a pair (K, λ). For each pair (K, λ) he also gives an explicit description of the algebra $\mathcal{H}(G, \lambda)$. We shall use these descriptions together with (0.1) – (0.4), to describe the Plancherel measure $\hat{\mu}$ for $\mathbb{SL}_2(F)$ explicitly.

To describe the decomposition (0.1) for $\mathbb{SL}_2(F)$, let $\mathfrak{o}$ denote the ring of integers in F, $\mathfrak{p}$ its maximal ideal, and $\mathbb{F}_q = \mathfrak{o}/\mathfrak{p}$ its residue field with q elements, where $q = p^n$ for some rational prime p. Fix a generator $\varpi \in \mathfrak{p}$; we implicitly employ the additive valuation η on F with the property that $\eta(\varpi) = 1$. Let L, N, denote the diagonal, and upper unipotent subgroups of $\mathbb{SL}_2(F)$ respectively, and let $B = LN$ denote the group of upper triangular matrices in $\mathbb{SL}_2(F)$.

In this framework each subcategory $\mathcal{R}^{\mathfrak{s}}(G)$ in (0.1) is one of the following listed below.

(1) Supercuspidal elements. For each irreducible supercuspidal representation $(\sigma, \mathcal{V})$ of G we write $\mathfrak{s}(\sigma)$ for its inertial equivalence class. Then $\mathcal{R}^{\mathfrak{s}(\sigma)}(G)$ is the full subcategory of $\mathcal{R}(G)$ whose objects are isomorphic to sums of copies of σ.

(2) Induced elements. The group L can be identified with $F^{\times}$; it has a unique maximal compact subgroup L^0 which is isomorphic with $\mathfrak{o}^{\times}$. For each quasicharacter $\psi : \mathfrak{o}^{\times} \to \mathbb{C}^{\times}$ we let $\mathcal{R}^{\psi}(L)$ be the full subcategory of $\mathcal{R}(L)$ whose objects $(\pi, \mathcal{V})$ satisfy $\pi(x)v = \psi(x)v$ for all $x \in L^0, v \in \mathcal{V}$. We set $\mathfrak{s}(\psi) = \{\psi, \psi^{-1}\}$ and let $\mathcal{R}^{\mathfrak{s}(\psi)}(G)$ be the full subcategory of $\mathcal{R}(G)$ whose objects are subrepresentations of representations of G of the form $\mathrm{Ind}_B^G(\tau)$ (normalised induction), with τ an object in $\mathcal{R}^{\psi}(L)$ or $\mathcal{R}^{\psi^{-1}}(L)$.

4.2 Next for each $\mathfrak{s}$ in §4.1 we describe the associated type (K, λ) and its algebra $\mathcal{H}(G, \lambda)$.

(1) Supercuspidal elements. It is a fact (see [KP], [KS]) that for each $\mathfrak{s}(\sigma)$ in 4.1(1) above there is a pair (K, λ) where K is a compact open subgroup of G and λ is an irreducible smooth representation of K with the property that

$$\sigma = \text{c-Ind}_K^G(\lambda)$$

(compact induction). The pair (K, λ) is an $\mathfrak{s}(\sigma)$-type (see [BK]§5). The algebra $\mathcal{H}(G, \lambda)$ is just the trivial $\mathbb{C}$ -algebra, and under the equivalence of categories (guaranteed by the existence of $\mathfrak{s}$-types) $\mathcal{R}^{\mathfrak{s}(\sigma)}(G) \to \mathcal{H}(G, \lambda)$-Mod which sends the space $\mathcal{V}$ of σ to its space of λ-invariants $\mathcal{V}_\lambda$, the representation σ corresponds to the trivial $\mathcal{H}(G, \lambda)$ module.

(2) Induced elements. Given $\mathfrak{s}(\psi)$ as in 4.1(2) define $\mathrm{sw}(\psi)$ by

$$\mathrm{sw}(\psi) = \begin{cases} 1, & \text{if } 1 + \mathfrak{p} \subset \ker \psi \\ n, & \text{where } 1 + \mathfrak{p}^n \subset \ker \psi, \text{ but } 1 + \mathfrak{p}^{n-1} \not\subset \ker \psi, \text{ otherwise.} \end{cases}$$

Now define

$$K = K_\lambda = \{ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in G | c_{11}, c_{22} \in \mathfrak{o}^{\times}, c_{12} \in \mathfrak{o}, c_{21} \in \mathfrak{p}^{\mathrm{sw}(\psi)} \},$$

and

$$\lambda = \lambda_\psi : K \to \mathbb{C}^{\times}$$
$$\lambda_\psi((c_{ij})) = \psi(c_{11}).$$

Then λ is a character and the pair $(K, \lambda) = (K_\lambda, \lambda_\psi)$ is an $\mathfrak{s}(\psi)$ type (see [K]§2.1).

To describe the algebra $\mathcal{H}(G, \lambda)$ in each case, we write $\mathbb{C}[\mathbb{Z}]$ for the group algebra on $\mathbb{Z}$: its elements are functions $f : \mathbb{Z} \to \mathbb{C}$ with finite support, and it has a basis $\{e_n\}_{n\in\mathbb{Z}}$ where $e_n(m) = \delta_{m,n}$. The algebra $\mathbb{C}[\mathbb{Z}]$ has a conjugate linear involution $f \mapsto f^*$ characterised by $e_n^* = e_{-n}$; the convolution product is given by $f * g(\ell) = \Sigma_{\mathbb{Z}} f(n)g(\ell - n)$; and the algebra identity is e_0. For $f, g \in \mathbb{C}[\mathbb{Z}]$ we define

$$\langle f|g\rangle = f * g^*(0).$$

Now let $\Pi = \begin{bmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{bmatrix}$ and define h_Π to be the function on G which is supported only on the double coset $K\Pi K$ and given by $h_\Pi(k_1\Pi k_2) = q^{-1}\psi(k_1^{-1}k_2)$ for $k_1, k_2 \in K$.

Recall ([BHK]§3.1) that a Hilbert algebra A with identity e, and inner product $< | >$ is *normalised* if $< e|e >= 1$.

We then have the following result ([K]§3.1, §3.3):

Proposition. *Normalise Haar measure μ on G so that $\mu(K_\lambda) = 1$. Then (i) if $\psi^2 \neq 1$ the element h_Π is invertible and there is an isomorphism of normalised Hilbert algebras*

$$\Phi_\psi : \mathcal{H}(G, \lambda_\psi) \to \mathbb{C}[\mathbb{Z}]$$

given by sending h_Π to e_1. (ii) Suppose $\psi^2 = 1$. Then there is an isomorphism of normalised Hilbert algebras

$$\Phi_\lambda : \mathcal{H}(G, \lambda_\psi) \to \begin{cases} \mathcal{H}(1,1), & \text{if } \psi \neq 1 \\ \mathcal{H}(q,q), & \text{if } \psi = 1. \end{cases}$$

4.3 From now on we put

$$\vartheta = \begin{cases} q, & \text{if } \psi = 1, \\ 1, & \text{if } \psi^2 = 1, \psi \neq 1, \end{cases}$$

and we set $\mathcal{H}(\vartheta) = \mathcal{H}(\vartheta, \vartheta)$. We also write

$$\Phi_\lambda^* : \mathcal{H}(G, \lambda)\text{-Mod} \to \begin{cases} \mathbb{C}[\mathbb{Z}]\text{-Mod}, & \text{if } \psi^2 \neq 1 \\ \mathcal{H}(\vartheta)\text{-Mod}, & \text{if } \psi^2 = 1 \end{cases}$$

for the equivalence induced by Φ_λ in Proposition 4.2.

Then combining (0.2) and Proposition 4.2(ii) we have equivalences of categories:

$$\Phi_\psi^* \circ \mathbf{M}_\psi : \mathcal{R}^{\mathfrak{s}(\psi)} \to \begin{cases} \mathbb{C}[\mathbb{Z}]\text{-Mod}, & \text{if } \psi^2 \neq 1, \\ \mathcal{H}(\vartheta)\text{-Mod}, & \text{if } \psi^2 = 1. \end{cases} \tag{4.3.1}$$

Let ℓ^1 denote the space of sequences $f : \mathbb{Z} \to \mathbb{C}$ for which $\Sigma_{\mathbb{Z}}|f(n)| < \infty$. It has an involution $f^*(n) = f(-n)$, and is an algebra via convolution; this equips it with the structure of an involutive Banach algebra. Indeed it is the L^1-algebra for the locally compact abelian group $\mathbb{Z}$. We shall write $A(\vartheta)$ for the algebra denoted by $A(\vartheta, \vartheta)$ in §2.2. Then (4.3.1) induces homeomorphisms (see (0.3))

$$\hat{m}_\lambda : \hat{G}_r(\lambda) \to \begin{cases} C_r^*(\ell^1)\hat{}, & \text{if } \psi^2 \neq 1, \\ C_r^*(A(\vartheta))\hat{}, & \text{if } \psi^2 = 1 \end{cases}$$

induced from $(\rho, \mathcal{V}) \mapsto \Phi_\lambda^*((\rho_\infty)_\lambda, (\mathcal{V}_\infty)_\lambda)$.

Definition. If $\Phi_\psi^* \circ \mathbf{M}_\psi(\rho, \mathcal{V}) = (\sigma, M)$ we say that (σ, M) *corresponds* to $(\pi, \mathcal{V})$.

4.4 These correspondences, and the resulting homeomorphisms, can be made explicit in terms of parameters as follows.

First, given ψ and a number $t \in \mathbb{C}$ we define a quasicharacter χ_t on $L = F^\times$ by

$$\chi_t|\mathfrak{o}^\times = \psi; \; \chi_t(\varpi) = q^{-t}.$$

This provides a one dimensional representation $(\chi_t, \mathbb{C}_t)$ of L. We write $(\rho_t, \mathcal{V}_t)$ for the representation of G in $\mathcal{R}^{\mathfrak{s}(\psi)}$ obtained from $(\chi_t, \mathbb{C}_t)$ via normalised induction from $B = LN$. On the other hand $\mathbb{C}_t$ has the structure of a left $\mathcal{D}$-module (notation of §1) via $d \cdot s = q^{-t}s, s \in \mathbb{C}$.

Assume first that $\psi^2 = 1$. Then from the constructions in §1 we obtain a left $\mathcal{H}(\vartheta)$-module (σ_t, M_t) where $M_t = \mathrm{Hom}_{\mathcal{D}}(\mathcal{H}(\vartheta), \mathbb{C}_t)$.

On the other hand if $\psi^2 \neq 1$ then $\mathbb{C}_t$ above has the structure of a left $\mathbb{C}[\mathbb{Z}]$-module via $e_1 \cdot s = q^{-t}s, s \in \mathbb{C}$.

Then

Proposition. *([K]§4.2) (i) Suppose that $\psi^2 \neq 1$. Then $\mathbb{C}_t$ corresponds to $(\rho_t, \mathcal{V}_t)$.*

(ii) Suppose that $\psi^2 = 1$. Then (σ_t, M_t) corresponds to $(\rho_t, \mathcal{V}_t)$.

4.5 Now we can proceed to the description of the Plancherel measures $\hat{\mu}_{\lambda_\psi}$ in (0.4) on each open set $\hat{G}_r(\lambda_\psi)$. In what follows we shall employ the following conventions.

First, we write S^1 for the unit circle: we identify it with the closed interval $[-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$ with the end points identified via the map $x \mapsto q^{-ix}$. We shall write

$\nu_0(t)$ for that measure on S^1 such that $d\,\nu_0(t) = \frac{\ln q}{2\pi} d\,t$ where $d\,t$ denotes Lebesgue measure on $[-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$.

Second, if $(\rho, \mathcal{V}) \in \mathcal{R}^{\mathfrak{s}(\psi)}$ is preunitary and irreducible, we write $[[\tilde{\rho}, \tilde{\mathcal{V}}]]$ for the resulting object in $\hat{G}_r(\lambda_\psi)$.

Third, if $t \in \mathbb{C}$ we define $L(1,t) = (1 - q^{-t})^{-1}$.

Finally, we employ the definition/construction in §2.10, and we remind the reader that K_λ always denotes the compact open subgroup of §4.2.

Theorem. *Normalize Haar measure μ on G so that $\mu(\mathbb{SL}_2(\mathfrak{o})) = 1$. For each $\lambda = \lambda_\psi$ we can describe the open and closed set $\hat{G}_r(\lambda)$ and the associated measure $\hat{\mu}_\lambda$ as follows.*

(1) Supercuspidal elements. *Each $\hat{G}_r(\lambda)$ is a singleton set S, and*

$$\hat{\mu}_\lambda(S) = \frac{dim\lambda}{\mu(K_\lambda)}.$$

(2) Induced elements. *(i) Suppose that $\psi^2 \neq 1$. Then $\hat{G}_r(\lambda_\psi)$ is homeomorphic with S^1. This homeomorphism is realised by $t \mapsto [[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]]$, and for $t \in [-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$, we have*

$$d\,\hat{\mu}_{\lambda_\psi}([[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]]) = \frac{(q+1)}{q^{1-sw(\psi)}} d\,\nu_0(t).$$

(ii) Suppose that $\psi^2 = 1$ but $\psi \neq 1$. Then $\hat{G}_r(\lambda_\psi)$ is homeomorphic to the space obtained from

$$\{t \in S^1 | -\frac{\pi}{\ln q} \leq t \leq 0\},$$

by replacing each of $t = 0, -\frac{\pi}{\ln q}$ with a double point. For $-\frac{\pi}{\ln q} < t < 0$ this homeomorphism is realised by $t \mapsto [[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]]$.

Let $Z_\psi = \{[[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]] \mid -\frac{\pi}{\ln q} < t < 0\}$; then

$$d\,\hat{\mu}_{\lambda_\psi}|Z_\psi([[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]]) = \frac{(q+1)}{q^{1-sw(\psi)}}\, d\,\nu_0(t).$$

The complement of Z_ψ in $\hat{G}_r(\lambda_\psi)$ is a set of measure zero.

(iii) Suppose that $\psi = 1$. Then $\hat{G}_r(\lambda_\psi)$ is homeomorphic to the space obtained from

$$\{t \in S^1 | -\frac{\pi}{\ln q} \leq t \leq 0\},$$

by replacing $t = -\frac{\pi}{\ln q}$ *with a double point, together with an isolated point* [[St]].

For $-\frac{\pi}{\ln q} < t \leq 0$ *this homeomorphism is realised by* $t \mapsto [[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]]$.

Let $Z_\psi = \{[[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]] \mid -\frac{\pi}{\ln q} < t \leq 0\}$. *For* $[[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]] \in Z_\psi$

$$d\,\hat{\mu}_{\lambda_\psi}([[\tilde{\rho_{it}}, \tilde{\mathcal{V}_{it}}]]) = \frac{(q+1)}{q}\frac{L(1,1+it)L(1,1-it)}{L(1,it)L(1,-it)}\,d\,\nu_0(t).$$

The complement of $Z_\psi \cup \{[[\mathrm{St}]]\}$ *in* $\hat{G}_r(\lambda_\psi)$ *is a set of measure zero. We have* $\hat{\mu}_{\lambda_\psi}(\{[[\mathrm{St}]]\}) = q - 1$.

Remark. If λ is the type for a supercuspidal then the quantities $\dim\lambda, \mu(K_\lambda)$ are known if q is not a power of 2 (see for example [Sh], [KS]). If q is a power of 2 then they can be computed in principle starting from the results in [KP].

Proof. When considering a particular ψ we shall often write λ for λ_ψ; this should not cause confusion. **Further, we note that until §4.8**, each time that we treat a particular set $\hat{G}_r(\lambda_\psi)$ we renormalise Haar measure μ so that $\mu(K_\lambda) = 1$ (*c.f.* Proposition 4.2).

The proof proceeds by an appropriate interpretation of previous results.

(1) Supercuspidal elements. Here $\hat{G}_r(\lambda)$ is a singleton S. Moreover from (0.4) and 4.2(1) we have

$$\hat{\mu}_\lambda(S) = \dim\lambda. \tag{4.5.1}$$

If char $\mathbb{F}_q \neq 2$ then dim λ is well known and can be found in [Sh] for example. If char $\mathbb{F}_q = 2$ the computation for dim λ can be made in principle starting from the results in [KP].

(2) Induced elements. In this case dim $\lambda_\psi = 1$ so that if S is a measurable subset of $\hat{G}_r(\lambda_\psi)$ we have from (4.2.2)

$$\hat{\mu}_\lambda(S) = \hat{\mu}_{\mathcal{H}(G,\lambda)}(\hat{m}_\lambda(S)).$$

We treat the induced cases $\psi^2 = 1, \psi^2 \neq 1$ in §4.6, §4.7 respectively; we complete the proof of Theorem 4.5 in §4.8. In what follows we shall employ the notation and results of Proposition 4.4.

4.6 Induced elements: Case $\psi^2 \neq 1$. From Proposition 4.2(i) there is an isomorphism of normalised Hilbert algebras $\mathcal{H}(G, \lambda_\psi) \to \mathbb{C}[\mathbb{Z}]$. Since $\mathbb{C}[\mathbb{Z}]$ is the (full) Hecke algebra of compactly supported functions for the t.d. group $\mathbb{Z}$ we may apply the Plancherel theorem ([BHK] §3.2):

$$f(0) = \int_{\hat{\mathbb{Z}}} \mathrm{tr}\sigma(f)d\hat{\mu}(\sigma)$$

for $f \in \mathbb{C}[\mathbb{Z}]$. Replacing f by $f * f^*$ as usual we see that

$$|f|^2 = \langle f|f\rangle = \int_{\hat{\mathbb{Z}}} \mathrm{tr}\sigma(f * f^*)\, d\hat{\mu}(\sigma).$$

Now, an irreducible unitary representation of $\mathbb{Z}$ is just a unitary character $\chi_t : \mathbb{Z} \to S^1$ parametrised by $t \in (-\frac{\pi}{\ln q}, \frac{\pi}{\ln q}]$ via $\chi_t(e_1) = q^{-it}$. Thus

$$|f|^2 = \langle f|f\rangle = \int_{\hat{\mathbb{Z}}} \chi_t(f)\chi_t(f^*)\, d\,\hat{\mu}(\chi_t),$$

where

$$\chi_t(f) = \Sigma_{n\in\mathbb{Z}} f(n)\chi_t(n) = \Sigma_{n\in\mathbb{Z}} f(n)q^{-int}.$$

In particular $g : t \mapsto \chi_t(f)$ is a trigonometric polynomial, and the usual Plancherel formula (after making a change of variable) for such a function g says that

$$|\hat{g}|^2 = \frac{\ln q}{2\pi} \int_{-\frac{\pi}{\ln q}}^{\frac{\pi}{\ln q}} |g|^2\, d\,t,$$

where $d\,t$ is Lebesgue measure on $\mathbb{R}$. But $\widehat{\chi_t(f)} = f$. By uniqueness of Plancherel measure, Proposition 4.4(i), (0.1), and (0.4), we deduce that

$$d\,\hat{\mu}([[\tilde{\pi}_{it}, \mathcal{V}_{it}]]) = d\,\nu_0(t), \tag{4.6.1}$$

as claimed.

4.7 Induced elements: Case $\psi^2 = 1$. Let

$$Y_\psi = \begin{cases} \{t \in S^1 | -\frac{\pi}{\ln q} < t < 0\}, & \text{for } \psi \neq 1, \\ \{t \in S^1 | -\frac{\pi}{\ln q} < t \leq 0\}, & \text{for } \psi = 1. \end{cases}$$

We identify the space $C_r^*(A(\vartheta))_2\hat{}$ with Y_ψ via the maps induced from $\chi \mapsto \sigma_\chi$, $\chi \mapsto \chi(d)$ as in Propositions 2.10 and 3.15. Then Proposition 3.10 and Theorem 3.14 tell us that $\hat{\mu}_{\mathcal{H}(\vartheta)}$ is zero on the complement of

$$\begin{cases} Y_\psi, & \text{for } \psi \neq 1, \\ Y_\psi \cup \{\rho_1\}, & \text{for } \psi = 1. \end{cases} \tag{4.7.1}$$

Moreover theorem 3.14 implies immediately that the measure associated to ρ_1 is $\frac{q-1}{q+1}$. Applying (0.4) and denoting the equivalence class in $\hat{G}_r$ which corresponds to ρ_1 by [[St]], we see that $\hat{\mu}_\lambda([[\mathrm{St}]]) = \frac{q-1}{q+1}$.

Next, we apply proposition 4.4 (ii), with t in the appropriate interval in (4.7.1): the module (σ_{it}, M_{it}) corresponds to $(\rho_{it}, \mathcal{V}_{it})$. In particular, via the homeomorphism $\hat{m}_\lambda$ of (0.3) we see that the representation $(\rho_{it}, \mathcal{V}_{it})$ is irreducible and pre-unitary: $\mathcal{V}_{it}$ is the space of smooth vectors for an irreducible unitary representation $(\tilde{\rho}_{it}, \tilde{\mathcal{V}}_{it})$ on G. It follows that $Z_\psi = \hat{m}_\lambda^{-1}(Y_\psi)$. Finally, let $d\,\nu_0(\sigma_{it})$ be the measure in Proposition 3.12. Then via the homeomorphism $\hat{\mu}_\lambda$ and the identification in Proposition 3.12 , $d\nu_0(\sigma_{it})$ is just the measure $d\nu_0(t)$, where the last measure is that defined on the unit circle before the statement of theorem 4.5.

For t in the appropriate interval in (4.7.1) put $z = q^{-it} + q^{it}$ and let $f(z) = z^2 - c^2 z - 2(c^2+2)$ where $c = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. With our identifications, Proposition 3.15 implies

$$d\,\hat{\mu}_\psi | Z_\psi([[\tilde{\rho}_{it}, \tilde{\mathcal{V}}_{it}]]) = \begin{cases} d\,\nu_0(t), & \text{if } \psi^2 = 1, \psi \neq 1, \\ \frac{z^2-4}{f(z)} d\,\nu_0(t), & \text{if } \psi = 1, \end{cases} \tag{4.7.2}$$

for $[[\tilde{\rho}_{it}, \tilde{\mathcal{V}}_{it}]] \in Z_\psi$.

The description of $\hat{G}_r(\lambda_\psi)$ in each case follows from Proposition 3.15 and the identifications made above.

4.8 Conclusion of the proof of Theorem 4.5. First we normalize the Haar measure so that $\mu(\mathbb{SL}_2(\mathfrak{o})) = 1$. This has the effect of dividing each formula (4.5.1), (4.6.1), (4.7.2) by $\mu(K_\lambda)$. Moreover if λ is the cover for a type ψ on L then the volume of K_λ is easily computed to be $\frac{q^{1-\mathrm{sw}(\psi)}}{q+1}$.

Thus all that remains is to explain the formula in part (2)(iii) in the statement of Theorem 4.5. We start from the second formula in (4.7.2) above. Set $s = q^{-it}$ so that in (4.7.2) above, $z = s + s^{-1}$. Then we have

$$\frac{z^2-4}{f(z)} = \frac{-(z^2-4)}{-f(z)}$$

$$= \frac{(1-s^2)(1-s^{-2})}{(\gamma - \gamma^{-1}s^{-1})(1+s)(\gamma-\gamma^{-1}s)(1+s^{-1})}$$

$$= \gamma^{-2}\frac{(1-s)(1-s^{-1})}{(1-\gamma^{-2}s)(1-\gamma^{-2}s^{-1})} = q^{-1}\frac{(1-q^{-it})(1-q^{it})}{(1-q^{-1-it})(1-q^{-1+it})}$$

From this we see that

$$\frac{z^2-4}{f(z)} = q^{-1}\frac{L(1,1+it)L(1,1-it)}{L(1,it)L(1,-it)}.$$

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