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K-theoretic Donaldson Invariants Via Instanton Counting

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To Friedrich Hirzebruch on the occasion of his eightieth birthday

Abstract: In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wallcrossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wallcrossing of these invariants in terms of elliptic functions and modular forms.
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Introduction

This paper is a sequel to [20]. In [20] we expressed the wallcrossing terms of equivariant Donaldson invariants for a smooth toric surface in terms of the Nekrasov partition function, and then using the solution of the Nekrasov conjecture [41],[48],[4] and its refinement [42] we gave the wallcrossing formula for simply connected projective surfaces with $p_g = 0$ in terms of modular forms, thus recovering the formula in [19] originally proved assuming the Kotschick-Morgan conjecture [29]. The Nekrasov partition function is defined as the generating function of the integrals of the equivariant cohomology class 1 on the Uhlenbeck partial compactifications $M_0(r,n)$ of the moduli spaces of $SU(r)$-instantons on $\mathbb{R}^4$ with $c_2 = n$. (As $M_0(r,n)$ is noncompact, we need a justification of the integration. See [41] for details.)

There is a natural K-theoretic counterpart of the Nekrasov partition function, namely we replace the integration in equivariant cohomology by the character of the coordinate ring of $M_0(r,n)$, where we view $M_0(r,n)$ as an affine algebraic variety via the ADHM description. The coordinate ring itself is infinite dimensional, but the weight spaces are finite dimensional (see [41]), so the character is well-defined. This K-theoretic counterpart is called the 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature [46],[31]. In [43] we proved the analogues of the results obtained in [41] in the K-theoretic version. (The approach in [48] can be applied to the K-theoretic version, while it seems difficult to generalize that of [4].) There is also a mathematical reason why we should study the K-theoretic Nekrasov partition function. By the geometric engineering of Katz, Klemm and Vafa [28], it is (after a parameter is specialized) equal to the generating function of all genus, all degree Gromov-Witten invariants for a certain noncompact toric Calabi-Yau 3-fold. (See [58] for a mathematically rigorous proof). Gromov-Witten invariants for toric Calabi-Yau 3-folds have been studied intensively both in mathematics and physics (see e.g. [39] and the references therein).

On the other hand, the K-theoretic Donaldson invariants have *not* been studied very much in the mathematical literature, as far as the authors know. One of the reasons might be a lack of motivation, as it is unlikely that there is an application to 4-dimensional topology. But another reason seems to lie in technical difficulties in defining the invariants. For example, the dimension counting argument used in the definition of the Donaldson invariants cannot be applied to the K-theoretic situation. Instead of attacking this problem, we restrict our interest to the case when the base 4-manifold is a projective surface X. Then we can use Gieseker-Maruyama moduli spaces of semistable sheaves and define the K-theoretic Donaldson invariants as the holomorphic Euler characteristics of the determinant line bundles. Then the algebro-geometric techniques used in [20] to derive the wallcrossing formula for the ordinary Donaldson invariants can be equally applied to the K-theoretic invariants. We will express the generating function of wallcrossing terms of the K-theoretic Donaldson invariants in terms of elliptic functions, which have a power series development in terms of modular forms. Their lowest order terms are the modular forms which occur in the wallcrossing formula in [20] for the usual Donaldson invariants. If the moduli spaces are smooth of the expected dimension, it is easy to see that this is compatible with the Hirzebruch-Riemann-Roch formula. Our approach is very similar to the one in [20], though the final step identifying invariants with the q-developments of modular forms and elliptic functions is more involved than in [20]. We want to remark that our final answer for the wallcrossing formula strongly suggests that there should exist a definition of K-theoretic Donaldson invariants for any 4-manifold with a $Spin^c$-structure (see §1.3).

The holomorphic Euler characteristics of determinant line bundles are interesting algebro-geometric objects in their own right. They are refinements of the usual Donaldson invariants, which contain a lot of geometrical information about the moduli spaces of stable sheaves on X, their Uhlenbeck compactifications and the linear systems on them. For instance by a result of [34] the morphism associated to certain determinant line bundles defines a projective embedding of the Uhlenbeck moduli spaces. The corresponding Donaldson invariants will determine the degree of the Uhlenbeck compactification and under suitable assumptions one would expect that the K-theoretic Donaldson invariants determine its Hilbert polynomial.

The K-theoretic Donaldson invariant is a natural 2-dimensional analogue of the dimension of the space of conformal blocks (nonabelian theta functions). Another, closely related, analogue is the genuine space of sections of a determinant line bundle, rather than the alternating sum of cohomology groups. Its conjectural formula appeared as four dimensional Verlinde formula in the physics literature [35],[36]. (It is given as the space of sections, but it is not clear to the authors whether the physical approach actually yields the space of sections, not Euler characteristic.) A mathematical formulation was given in [45], where it was called the space of conformal blocks in 4D WZW-Theory. In case the base manifold is the projective plane the strange duality conjecture of Le Potier (see e.g. [6]) gives a duality between the spaces of sections of determinant line bundles for moduli spaces of sheaves of positive rank and their analogues on moduli spaces of pure sheaves of rank 0. This conjecture has been checked in some cases in [5],[6]. An analogue of this conjecture has been proved for some moduli spaces of sheaves on K3-surfaces in [49]. In many examples, the determinant line bundle is ample, or at least nef and big, so we have the vanishing of higher cohomology groups. In those cases there are no difference between the spaces of sections and Euler characteristics. However it is not clear whether we can control the spaces of sections in general.

The paper is organized as follows. In Sect. 1 we collect background material on the holomorphic Euler characteristic of the determinant line bundle and the K-theoretic Nekrasov partition function. We also explain the partition function with $5D$ Chern-Simons terms (see [27, 51]), which naturally appears in our approach. We also calculate the K-theoretic Donaldson invariants for $K3$ surfaces (see §1.5). In Sect. 2 we express the wallcrossing terms in terms of the holomorphic Euler characteristic of some virtual vector bundles on the Hilbert schemes $X_2^{[n]}$ of points on two copies of X. In Sect. 3 we take X a smooth projective toric surface and express the equivariant wallcrossing terms in terms of the K-theoretic Nekrasov partition function. These two sections are parallel to [20, Sect's. 2,3]. Then in Sect. 4 we take the nonequivariant limit and give the formula of wallcrossing terms in terms of modular forms and elliptic functions. We use the solution of the Nekrasov conjecture and its refinement. In particular, we determine the Hilbert series of the determinant line bundles on $M_H^{\mathbb{P}^2}(0,d)$ and $M_H^{\mathbb{P}^2}(H,d)$ for small d in §4.5. In Appendix A we explain the Seiberg-Witten curve for the 5-dimensional supersymmetric Yang-Mills theory. We prove that the Seiberg-Witten prepotential defined via the period of the curve satisfies the contact term equation, which was also satisfied by the nonequivariant limit of the Nekrasov partition function [43]. This completes our proof of Nekrasov's conjecture started in [43], as the solution of the contact term equation is unique.

This paper is dedicated to Friedrich Hirzebruch, one of the founders of K-theory. Among the other subjects of this paper related his work are the Hirzebruch-Riemann-Roch theorem, modular forms and elliptic functions. The first-named author particularly wants to thank him, his teacher, for all the things he learned from him.

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1. Background Material

We will work over $\mathbb{C}$. We usually consider homology and cohomology with rational coefficients and for a variety Y we will write $H_i(Y)$, and $H^i(Y)$ for $H_i(Y,\mathbb{Q})$ and $H^i(Y,\mathbb{Q})$ respectively. If Y is projective and $\alpha \in H^*(Y)$, we denote $\int_Y \alpha$ its evaluation on the fundamental cycle of Y. If Y carries an action of a torus T, α is a T-equivariant class, and $p : X \to pt$ is the projection to a point, we denote $\int_Y \alpha := p_*(\alpha) \in H_T^*(pt)$.

In this whole paper X will be a nonsingular projective surface over $\mathbb{C}$. Later we will specialize X to a smooth projective toric surface. For a class $\alpha \in H^*(X)$, we denote $\langle \alpha \rangle := \int_X \alpha$. If X is a toric surface we use the same notation for the equivariant pushforward to a point.

Let X be simply connected smooth projective surface with $p_g(X) = 0$. Let H be an ample divisor on X. We denote by $M_H^X(r, c_1, c_2)$ the moduli space of rank r

torsion-free H-semistable sheaves (in the sense of Gieseker and Maruyama) with $c_1(E) = c_1$, $c_2(E) = c_2$. Let $M_H^X(r, c_1, c_2)_s$ be the open subset of stable sheaves.

1.1. **Determinant line bundles.** We briefly review the determinant line bundle on the moduli space [11],[32], for more details we refer to [26, Chap. 8].

For a Noetherian scheme Y we denote by $K(Y)$ and $K^0(Y)$ the Grothendieck groups of coherent sheaves and locally free sheaves on Y respectively. Then $K^0(Y)$ is a commutative ring with $1 = [\mathcal{O}_Y]$, with the multiplication given by the tensor product of locally free sheaves. If Y is nonsingular and quasiprojective, then $K(Y) = K^0(Y)$. In particular we have $K(X) = K^0(X)$ for the smooth projective surface X. We will identify $K^0(X)$ with $K(X)$ hereafter. If we want to distinguish a sheaf $\mathcal{F}$ and its class in $K(Y)$, we denote the latter by $[\mathcal{F}]$. But we may also write $\mathcal{F}$ for the class in $K(Y)$. For a proper morphism $f\colon Y_1 \to Y_2$ we have the pushforward homomorphism $f_!\colon K(Y_1) \to K(Y_2)$ defined by $f_!([\mathcal{F}]) = \sum_i (-1)^i [R^i f_* \mathcal{F}]$. When $Y_2 = \mathrm{pt}$, this is the Euler characteristic of $\mathcal{F}$ under the identification of $K(\mathrm{pt}) \cong \mathbb{Z}$: $f_!([\mathcal{F}]) = \chi(Y_1, \mathcal{F}) = \sum_i (-1)^i \dim H^i(Y_1, \mathcal{F})$. We also have a pushforward homomorphism $K^0(Y_1) \to K^0(Y_2)$ when f is a locally complete intersection morphism. (See [1, §4.4].) For any morphism $f\colon Y_1 \to Y_2$ we have the pullback homomorphism $f^*\colon K^0(Y_2) \to K^0(Y_1)$ defined by $f^*[\mathcal{F}] = [f^*\mathcal{F}]$ for a locally free sheaf $\mathcal{F}$ on Y_2.

On $K(X)$ we have a quadratic form $(u, v) \mapsto \chi(X, u \otimes v) \equiv \chi(u \otimes v)$. (We denote $\chi(X, u \otimes v)$ by $\chi(u \otimes v)$ for brevity hereafter.) We say that $u, v \in K(X)$ are numerically equivalent if $u - v$ is in the radical of this quadratic form, and denote $K(X)_{\mathrm{num}}$ the set of numerical equivalence classes. Let $c \in K(X)_{\mathrm{num}}$. Let $\mathcal{E}$ be a flat family of coherent sheaves of class c on X parametrized by a scheme S, and let $p : X \times S \to S$, $q : X \times S \to X$ be the projections. Define $\lambda_{\mathcal{E}} : K(X) \to \mathrm{Pic}(S)$ as the composition

$$K(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{\otimes[\mathcal{E}]} K^0(X \times S) \xrightarrow{p_!} K^0(S) \xrightarrow{\det} \mathrm{Pic}(S), \tag{1.1}$$

(see also [26, (2.1.10), (2.1.11)]). The following elementary facts are important for working with these line bundles:

(1) $\lambda_{\mathcal{E}}$ is a homomorphism, i.e. $\lambda_{\mathcal{E}}(v_1 + v_2) = \lambda_{\mathcal{E}}(v_1) \otimes \lambda_{\mathcal{E}}(v_2)$.
(2) If $\mu \in \mathrm{Pic}(S)$ is a line bundle, then $\lambda_{\mathcal{E}\otimes\mu}(v) = \lambda_{\mathcal{E}}(v) \otimes \mu^{\chi(c\otimes v)}$.
(3) $\lambda_{\mathcal{E}}$ is compatible with base change: if $\phi : S' \to S$ is a morphism, then $\lambda_{\phi^*\mathcal{E}}(v) = \phi^*\lambda_{\mathcal{E}}(v)$.

Let H be a very ample divisor on X. For a class $c \in K(X)_{\mathrm{num}}$ we denote by $K_c := c^\perp = \{v \in K(X) \mid \chi(v \otimes c) = 0\}$. We denote by $K_{c,H} := c^\perp \cap \{1, h, h^2\}^{\perp\perp}$, where $h = [\mathcal{O}_H]$. Now let $c \in K(X)_{\mathrm{num}}$ be the class of an element in $M_H^X(r, c_1, c_2)$. There are homomorphisms $\lambda\colon K_c \to \mathrm{Pic}(M_H^X(r, c_1, c_2)_s)$, and $\lambda\colon K_{c,H} \to \mathrm{Pic}(M_H^X(r, c_1, c_2))$, such that λ commute with the inclusions $K_{c,H} \subset K_c$ and $\mathrm{Pic}(M_H^X(r, c_1, c_2)) \subset \mathrm{Pic}(M_H^X(r, c_1, c_2)_s)$. Note that $\lambda_{\mathcal{E}}(v)$ is independent of the choice of the universal family $\mathcal{E}$ for $v \in K_c$ by the property (2) above, and in fact, we do not need the existence of the universal sheaf to define the map λ. We call H *general* with respect to (r, c_1, c_2) if all the strictly semistable sheaves in $M_H^X(r, c_1, c_2)$ are strictly semistable with respect to all ample divisors on X in a neighbourhood of H (the ample cone has the topology induced from the Euclidean topology on $H^2(X, \mathbb{R})$). In this case any strictly semistable sheaf in $M_H^X(r, c_1, c_2)$ is of type 1 in the sense of [10, 0.3]. Then the stabilizer subgroup $\mathrm{Aut}\, F$ (which appears in the proof of [26, Theorem 8.1.5]) acts trivially on the fiber of the determinant line bundle (on the open subscheme of the quot-scheme). Therefore $\lambda\colon K_{c,H} \to \mathrm{Pic}(M_H^X(r, c_1, c_2))$ can be extended to K_c.

If $\mathcal{E}$ is a flat family of semistable sheaves of rank r and with Chern classes c_1, c_2 on X parametrized by S, then we have $\phi_{\mathcal{E}}^*(\lambda(v)) = \lambda_{\mathcal{E}}(v)$ for all $v \in K_{c,H}$ for $\phi_{\mathcal{E}}^* : \mathrm{Pic}(M_H^X(r, c_1, c_2)) \to \mathrm{Pic}(S)$ the pullback by the classifying morphism. If H is general with respect to (r, c_1, c_2), the same statement holds with $K_{c,H}$ replaced by K_c. If $\mathcal{E}$ is a flat family of stable sheaves, the same statement holds with $K_{c,H}$, $M_H^X(r, c_1, c_2)$ replaced by K_c, $M_H^X(r, c_1, c_2)_s$.

1.2. **K-theoretic Donaldson invariants.** We write $M_H^X(c_1, d)$ for $M_H^X(2, c_1, c_2)$ with $d = 4c_2 - c_1^2 - 3$. Let $v \in K_c$, where c is the class of a coherent rank 2 sheaf with Chern classes c_1, c_2. Assume that H is general with respect to $(2, c_1, c_2)$. The *K-theoretic Donaldson invariant* of X with respect to v, c_1, c_2, H is the holomorphic Euler characteristic $\chi(M_H^X(c_1, d), \lambda(v))$ of the line bundle $\lambda(v)$.

Notation 1.2. We introduce the following notation that we will often use in the paper. For $i \geq 0$, we put $v^{(i)} := [\mathrm{ch}(v) e^{c_1/2}\, \mathrm{Todd}(X)]_i$. Thus $v^{(0)} = \mathrm{rk}(v)$, $v^{(1)} = c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X)$ and $v^{(2)}$ could be interpreted as $\chi(v \otimes \mathcal{O}(c_1/2))$.

By the Riemann-Roch Theorem it follows that

$$\chi(v \otimes c) = 2v^{(2)} - \mathrm{rk}(v)(c_2 - \frac{c_1^2}{4}), \tag{1.3}$$

in particular we see that the condition $v \in K_c$ is independent of d if $\mathrm{rk}(v) = 0$.

An important special case is the following: Let L be a line bundle on X. Assume that $\langle c_1(L), c_1\rangle$ is even (otherwise replace L by $L^{\otimes 2}$). Then for c the class of a rank 2 coherent sheaf with Chern classes c_1, c_2, we put

$$v(L) := -(1 - L^{-1}) - \langle \frac{c_1(L)}{2}, c_1(L) + K_X + c_1\rangle [\mathcal{O}_x] \in K_c. \tag{1.4}$$

Note that $v(L)$ is independent of c_2. The condition that $\langle c_1(L), c_1\rangle$ is even implies that $v(L) \in K(X)$. Assume that H is general with respect to $(2, c_1, c_2)$. Then we denote $\mu(L) := \lambda(v(L)) \in \mathrm{Pic}(M_H^X(c_1, d))$. The *$K$-theoretic Donaldson invariant* of X, with respect to L, c_1, d, H is $\chi(M_H^X(c_1, d), \mathcal{O}(\mu(L)))$. The generating function is

$$\chi_{c_1}^H(L; \Lambda) := \sum_{d\geq 0} \Lambda^d \chi(M_H^X(c_1, d), \mathcal{O}(\mu(L))). \tag{1.5}$$

If $\mathcal{E}$ is a flat family of coherent sheaves parametrized by S, we have $c_1(\mu(L)) = (c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2)/PD(c_1(L)) \in H^2(S)$ by the Riemann-Roch for a smooth morphism ([1, §4.3]). It extends to a class in $H^2(M_H^X(c_1, d))$ by the same argument for $\mu(L)$. This coincides with the definition of $\mu(c_1(L))$ appearing in the usual Donaldson invariant. This is the reason why we denote the line bundle by $\mu(L)$. Thus it follows from the definitions and the singular Riemann-Roch theorem [1] that $\chi(M_H^X(c_1, d), \mathcal{O}(\mu(nL)))$ is a polynomial of degree d in n, whose leading term is the algebraic geometric version of the Donaldson invariants $n^d\Phi_{c_1}^H(c_1(L)^d/d!)$ (in the notations of [20]) when $M_H^X(c_1, d)$ is of the expected dimension.

The above argument also implies that the invariant $\chi(M_H^X(c_1, d), \lambda(v))$ depends only on $\mathrm{ch}(v) \in H^*(X)$. Therefore the invariant is well-defined on $K(X)_{\mathrm{hom}} := K(X)/\sim$ where $v \sim v'$ if and only if $\mathrm{ch}(v) = \mathrm{ch}(v')$.

1.3. **A digression on the definition of the invariants.** The definition of the K-theoretic Donaldson invariants above is only ad hoc and will in general need to be modified, so that the invariants have good properties and so that they might be related to gauge-theoretical invariants.

We expect that for general X, when the moduli space $M_H^X(c_1, d)$ does not have the expected dimension, one needs to use a virtual structure sheaf (see [33]) in the definition. If $M_H^X(c_1, d)$ consists only of stable sheaves, the perfect obstruction theory was constructed in [52, Th. 3.30]. Then we just need to replace

$\chi(M_H^X(c_1,d),\lambda(v))$ by $\chi(M_H^X(c_1,d),\mathcal{O}^{\mathrm{virt}}\otimes\lambda(v))$. If $M_H^X(c_1,d)$ has the expected dimension, then by [33, Prop. 2], the virtual structure sheaf is just the usual structure sheaf, and this definition reduces to our previous definition. When $M_H^X(c_1,d)$ may contain a strictly semistable sheaf, we need to construct a perfect obstruction theory on another moduli space with additional structures and prove that it is independent of the additional structure as in [40], or use the blowup formula as in the definition of the usual Donaldson invariants (see [20, §1.1]). See §1.4 below for the first step in this approach.

Let us examine the possibility to extend our definition of invariants to a C^∞ 4-manifold X. To avoid a technical difficulty, we first assume the moduli space $M_H^X(c_1,d)$ is smooth. Our definition depends on the complex structure of X, and if we have a gauge theoretic definition, it should be independent of the complex structure, and the definition must be modified. Our guess is to consider the index of a Dirac operator instead of the holomorphic Euler characteristic. If X is spin, then we have a square root $K_X^{1/2}$ of K_X, and then $\mu(K_X)$ is a line bundle. It is known that this is isomorphic to half of the canonical bundle of $M_H^X(c_1,d)$ when it is smooth (see e.g. [26, §8.3]). Therefore $\chi(M_H^X(c_1,d),\mathcal{O}(\mu(K_X))$ is equal to the index of the Dirac operator. In this special case, our main result Corollary 4.19 is simplified as $v^{(1)}=-K_X$. In particular, the answer is independent of the complex structure except the term $\sqrt{-1}^{\langle\xi,K_X\rangle}$ which corresponds to the orientation of the moduli space.

More generally the complex structure on X and a line bundle L on X induces the *Spin*^{c}-structure $W^+=(\bigwedge^{0,0}\oplus\bigwedge^{0,2})\otimes L$, $W^-=\bigwedge^{0,1}\otimes L$ on X with the characteristic line bundle $\det W^+=\det W^-=-K_X+2L$. We conjecture that it induces a *Spin*^{c}-structure on the moduli space. The recipe should be somewhat similar to the definition of the orientation of the moduli space induced from the homological orientation on $H^0(X)\oplus H^1(X)^*\oplus H^2_+(X)$, but we do not know how to define it in general (even on the nonsingular part of the moduli space). However in our situation, an obvious candidate for the index is $\chi(M_H^X(c_1,d),\mathcal{O}(\mu(2L))$. This means that the *Spin*^{c}-structure is the one given by the complex structure twisted by the line bundle $\mu(2L)$. The answer given in Corollary 4.19 is written in terms of $v^{(1)}+K_X=K_X-2L$. As this is the negative of the characteristic line bundle of the *Spin*^{c} structure, the candidate seems reasonable.

Now we come to discuss more technical points. For a C^∞ 4-manifold X, we do not have the Gieseker-Maruyama compactification $M_H^X(c_1,d)$ and we need to use the Uhlenbeck compactification $N_H^X(c_1,d)$ of the moduli space of instantons instead. We also need to use its topological K-homology group $K^{\mathrm{top}}(N_H^X(c_1,d))$. When X is a projective surface, we have a homomorphism $\pi_*\colon K^{\mathrm{top}}(M_H^X(c_1,d)) \to K^{\mathrm{top}}(N_H^X(c_1,d))$ given by $\pi\colon M_H^X(c_1,d) \to N_H^X(c_1,d)$ and we can pushforward the virtual structure sheaf on $M_H^X(c_1,d)$ to $N_H^X(c_1,d)$. And it can be shown that the line bundle $\mu(L)$ is a pull-back of a line bundle from $N_H^X(c_1,d)$ under some conditions. (See §1.4. And this assertion, at least for a topological line bundle, is well-known in the gauge theory context.) Therefore the invariants in (1.5) can be defined in terms of $N_H^X(c_1,d)$ and the framework of the topological K-group. However it is not clear, at least to the authors, how to define the K-theoretic fundamental class $[\mathcal{O}_{N_H^X(c_1,d)}] \in K^{\mathrm{top}}(N_H^X(c_1,d))$ for an arbitrary C^∞ 4-manifold X even under the assumption that $N_H^X(c_1,d)$ is of expected dimension.

We have considered the determinant line bundle $\mu(L)$ above. This is the case $\mathrm{rk}(v) = 0$. When $v \in K_c$ is suitably chosen (see [26, §8.1]) with $\mathrm{rk}(v) = 2$, the determinant line bundle $\lambda(v)$ is ample on $M_H^X(c_1,d)$ and does *not* come from $N_H^X(c_1,d)$. This observation seems to suggest that the invariant can be defined *only* for a restricted class v on a C^∞ 4-manifold X. We have discussed $\mathrm{rk}(v) = 0$ is sufficient for the existence of the line bundle $\lambda(v)$ on $N_H^X(c_1,d)$ above, but we do not know whether this is necessary.

Also we do not give the definition of the analog of $\mu(p) \in H^4(M_H^X(c_1,d))$ where p is the point class of $H_0(X)$. It may be defined as

$$\chi(M_H^X(c_1,d), p_!(q^*v \otimes \mathcal{E})),$$

but it is not independent of the choice of the universal bundle $\mathcal{E}$ in general, and may not be defined when we do not have a universal bundle. A possible candidate, which can be defined for any $v \in K(X)$, is given by replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{E}^\vee$, where $^\vee$ is the involution on $K^0(X \times M_H^X(c_1,d))$ defined by taking the dual of a vector bundle. Or more generally, if we have a representation $\rho\colon PGL(2,\mathbb{C}) \to GL(V)$, we may consider $\chi(M_H^X(c_1,d), p_!(q^*v \otimes \rho([\mathcal{E}])))$, or applying ρ (with an appropriate change of $PGL(2,\mathbb{C})$) after the pushforward $p_!$. But we do not study these 'higher' invariants, and stick to our $\chi(M_H^X(c_1,d), \lambda(v))$, which we believe most basic.

1.4. **Blowup formula and the invariants for moduli spaces with strictly semistable sheaves.** As mentioned in the previous subsection, we give a proposal of the definition of invariants when moduli spaces may contain strictly semistable sheaves by using a blowup formula. We assume a kind of smoothness of moduli spaces on the blowup. This allows us to avoid the virtual structure sheaf. However the smoothness assumption is used much more essentially as we use Kawamata-Viehweg vanishing theorem. If we could prove the same vanishing theorem under the assumption that the moduli space is of expected dimension, we could use the blowup formula as the definition of the invariant, as is done in the context of usual Donaldson invariants (see e.g., [20]). Then the invariant is integral, in contrast with the ordinary Donaldson invariants in which we must divide by powers of 2. Moreover we also prove that the pushforward of the structure sheaf of the Gieseker-Maruyama compactification is equal to the structure sheaf of the Uhlenbeck compactification. This seems an evidence of our belief that the K-theoretic Donaldson invariant has a gauge theoretic definition. The material in this subsection is technical, so a reader in hurry can just read the statement of Corollary 1.8 and skip the rest.

Let (X,H) be a polarized rational surface. Let $\widehat{X}$ be the blowup of X in a point and C the exceptional divisor. In the following we always denote a class in $H^*(X,\mathbb{Z})$ and its pullback by the same letter. Write $c := (2,c_1,c_2)$, and $M_H(c) := M_H^X(c)$. Let Q be an open subset of a suitable quot-scheme such that $M_H(c) = Q/GL(N)$.

Let $N_H(c)$ be the Uhlenbeck compactification of the moduli space of slope stable vector bundles on X. The line bundle $\mu(2D)$ is a pull-back of a line bundle from $N_H(c)$ if $D \in \bigcap_{\xi:\langle H,\xi\rangle=0} \xi^\perp$. In fact, the stability is the same for H and $H+\varepsilon D$ with $D\in \cap_{\xi:\langle H,\xi\rangle=0}\xi^\perp$ for a sufficiently small ε. Then $\mu(H+\varepsilon D)$ is nef and big and gives a map to the Uhlenbeck compactification. In particular, $\mu(2D)$ is the pull-back of a line bundle on the Uhlenbeck compactification, which we denote by the same symbol. We further assume H is general with respect to c, then we have $\{\xi \mid \langle H,\xi\rangle = 0\} = \{0\}$. Therefore $\mu(2D)$ is the pull-back of a line bundle on $N_H(c)$ for any D.

We shall study the singularities of $M_H(c)$ and $N_H(c)$.

Lemma 1.6 ([3])**.** *Assume that Q is smooth (e.g. $\langle -K_X,H\rangle > 0$). Then $M_H(c) = Q/GL(N)$ is normal and has only rational singularities.*

We next consider the singularities of $N_H(c)$. Replacing $N_H(c)$ by its normalization, we may assume that $N_H(c)$ is normal.

Lemma 1.7. *Assume that Q is smooth (e.g. $\langle -K_X, H\rangle > 0$).*

(1) *Then $N_H(c)$ has only rational singularities.*

(2) $\mathbf{R}\pi_*(\mathcal{O}_{M_H(c)}) = \mathcal{O}_{N_H(c)}$.

Proof. (1) We first assume that c is primitive. Then there is a resolution of $\pi\colon M_H^\alpha(c) \to N_H(c)$, where $M_H^\alpha(c)$ is the moduli space of α-twisted semi-stable sheaves for suitable α. Then by the Grauert-Riemenschneider vanishing theorem, $\mathbf{R}\pi_*(K_{M_H^\alpha(c)}) = \pi_*(K_{M_H^\alpha(c)})$. Since $K_{M_H^\alpha(c)} \cong \mu(2K_X)$ comes from $N_H(c)$ and $N_H(c)$ is normal, $\mathbf{R}\pi_*(\mathcal{O}_{M_H^\alpha(c)}) = \pi_*(\mathcal{O}_{M_H^\alpha(c)}) = \mathcal{O}_{N_H(c)}$. Thus $N_H(c)$ has only rational singularities.

We next treat $M_H(c)$ with $c = (2, 0, 2n)$. We set $\widehat{c} := (2, C, 2n)$ and $\widehat{M}_H(\widehat{c}) := M^{\widehat{X}}_{H-\varepsilon C}(\widehat{c})$. Then there is a surjective morphism $\widehat{\pi}\colon \widehat{M}_H(\widehat{c}) \to N_H(c)$ which is generically a $\mathbb{P}^1$-bundle. Since $-\mu(C)$ is $\widehat{\pi}$-nef and big, the Kawamata-Viehweg vanishing theorem implies that $R^i\widehat{\pi}_*(K_{\widehat{M}_H(\widehat{c})}(-2\mu(C))) = 0, i > 0$. By our assumption, $\mu(K_X)$ comes from $N_H(c)$. This implies that

$$c_1(K_{\widehat{M}_H(\widehat{c})}) = 2\mu(c_1(K_{\widehat{X}})) = 2\mu(c_1(K_X)) + 2\mu(C) \equiv 2\mu(C) \mod \widehat{\pi}^* H^2(N_H(c), \mathbb{Q}).$$

Hence $R^i\widehat{\pi}_*(\mathcal{O}_{\widehat{M}_H(\widehat{c})}) = 0, i > 0$. Thus $\mathbf{R}\widehat{\pi}_*(\mathcal{O}_{\widehat{M}_H(\widehat{c})}) = \mathcal{O}_{N_H(c)}$. Then $N_H(c)$ has rational singularities by [30, Thm. 1].

(2) It is sufficient to prove the following: For a proper birational map $f\colon Y \to Z$ of normal varieties Y, Z with only rational singularities, $\mathbf{R}f_*(\mathcal{O}_Y) = \mathcal{O}_Z$.

Proof of the claim: Let $g\colon Y' \to Y$ be a resolution of the singularities. Since Y has only rational singularities, $R^ig_*(\mathcal{O}_{Y'}) = 0$, $i > 0$. Then $R^if_*(\mathcal{O}_Y) = R^i(f \circ g)_*(\mathcal{O}_{Y'}) = 0, i > 0$. Hence we get our claim.

□

By the proof, we also get the following.

Corollary 1.8. *Let $\widehat{M}_H(\widehat{c})$ be the moduli space of stable sheaves on $\widehat{X}$ such that $\widehat{c} = (2, c_1 + kC, c_2)$ with $k = 0, 1$. Then*

$$\mathbf{R}\widehat{\pi}_*(\mathcal{O}_{\widehat{M}_H(\widehat{c})}) = \mathcal{O}_{N_H(c)} = \mathbf{R}\pi_*(\mathcal{O}_{M_H(c)}).$$

In particular,

$$\chi(\widehat{M}_H(\widehat{c}), \mu(D)) = \chi(N_H(c), \mu(D)) = \chi(M_H(c), \mu(D))$$

for any line bundle D on X such that $\langle D, c_1\rangle$ is even and $\langle D, \xi\rangle = 0$ for ξ any class of type $(c_1, 4c_1 - c_1^2 - 3)$ on $\widehat{X}$ with $\langle H, \xi\rangle = 0$.

Remark 1.9. Since $M_H(c)$ is normal, the dualizing sheaf $\omega_{M_H(c)}$ is reflexive. If H is a general polarization, then $\mathcal{O}_{M_H(c)}(2\mu(K_X))$ is a line bundle on M which coincides with the dualizing sheaf on the locus of stable sheaves $M_H(c)^s$. If $\dim(M_H(c) \setminus M_H(c)^s) \leq \dim M_H(c) - 2$, then $\omega_{M_H(c)} = \mathcal{O}_{M_H(c)}(2\mu(K_X))$.

1.5. **K-theoretic invariant for $K3$ surfaces and strange duality.** Let X be a projective $K3$ surface. In this subsection we calculate the K-theoretic invariants for X as examples. We also give a formula for the K-theoretic invariants of rank 1 sheaves on abelian surfaces.

For any projective algebraic surface Y and $c \in K(Y)_{\text{hom}}$ we denote by $M_H^Y(c)$ the moduli space of H-stable sheaves E on Y with $\operatorname{ch}(E) = \operatorname{ch}(c)$. This is just a change of notation, but is convenient to see the strange duality. We also define the discriminant by $\Delta(c) = 2\operatorname{rk}(c)c_2(c) - (\operatorname{rk}(c) - 1)c_1(c)^2$, $\Delta(E) = 2\operatorname{rk}(E)c_2(E) - (\operatorname{rk}(E) - 1)c_1(E)^2$.

Proposition 1.10. *Let $c \in K(X)_{\text{hom}}$ with either $\operatorname{rk}(c) > 0$ or $\operatorname{rk}(c) = 0$ and $c_1(c)$ nef and big. Assume that $M_H^X(c)$ consists only of stable sheaves. Then for $v \in K_c$,*

$$\chi(M_H^X(c), \lambda(v)) = \binom{\frac{\Delta(c)}{2} - \operatorname{rk}(c)^2 + \frac{\Delta(v)}{2} - \operatorname{rk}(v)^2 + 2}{\frac{\Delta(c)}{2} - \operatorname{rk}(c)^2 + 1}.$$

Corollary 1.11. *Let $c, v \in K(X)_{\text{hom}}$ with $\chi(v \otimes c) = 0$. Assume that both c and v fulfill the assumptions for c in Proposition 1.10. Then $\chi(M_H^X(c), \lambda(-v)) = \chi(M_H^X(v), \lambda(-c))$.*

Recall that our invariant is well-defined on $K(X)_{\text{hom}}$ (see §1.2). We have $v \in K_c = \{v \mid \chi(v \otimes c) = 0\}$ if and only if $c \in K_v$, therefore the line bundles $\lambda(-v)$, $\lambda(-c)$ exist on $M_H^X(c)$, $M_H^X(v)$ respectively.

Remark 1.12. (1) Corollary 1.11 can be viewed as a weak version of an analogue of the strange duality conjecture, which was formulated by Le Potier for $\mathbb{P}^2$, and which is in turn an analogue of the strange duality (level-rank duality) for moduli spaces of vector bundles on curves (see [2],[7],[44]). Let $c \in K(\mathbb{P}^2)$ with $\mathrm{rk}(c) > 0$ and $v \in K_c$ with $\mathrm{rk}(v) = 0$ and $c_1(v) > 0$ and assume $M^{\mathbb{P}^2}(c) \neq \emptyset \neq M^{\mathbb{P}^2}(v)$. Then the strange duality conjecture of Le Potier (see [5], [6]) predicts an explicit duality between $H^0(M^{\mathbb{P}^2}(c), \lambda(-v))$ and $H^0(M^{\mathbb{P}^2}(v), \lambda(-c))$. It is shown in [6] that the higher cohomology groups $H^i(M^{\mathbb{P}^2}(c), \lambda(-v))$ vanish and in the known cases also the higher cohomology groups $H^i(M^{\mathbb{P}^2}(v), \lambda(-c))$ are zero, thus one has in particular that $\chi(M^{\mathbb{P}^2}(c), \lambda(-v)) = \chi(M^{\mathbb{P}^2}(v), \lambda(-c))$. Thus Corollary 1.11 says that on K3 surfaces this is true more generally for c, v of any nonnegative rank, at least when $M_H^X(c)$ and $M_H^X(v)$ consist only of stable sheaves. It seems natural to conjecture that the condition that the moduli spaces only consist of stable sheaves can be dropped.

In the context of Brill-Noether theory of K3 surfaces Markman proposed to put $M_H^X(v) := M_H^X(-v^\vee)$, in case $\mathrm{rk}(v)$ is negative (see [37]). It is easy to see that if $\chi(v \otimes c) = 0$, then also $\chi(-v^\vee \otimes c) = 0$, and $\Delta(-v^\vee) = \Delta(v)$. Thus with this definition Proposition 1.10 also holds if $\mathrm{rk}(v)$ or $\mathrm{rk}(c)$ are negative.

In [56] the proof of an equivalent formulation of Proposition 1.10 in terms of the Mukai vector is sketched. In [49] there is a short sketch of the proof of Proposition 1.10. Furthermore the duality map $H^0(M_H^X(c), \lambda(-v))^\vee \to H^0(M_H^X(v), \lambda(-c))$ is constructed and it is checked in some cases that it is an isomorphism.

We first recall some properties of the moduli spaces $M_H^X(r, c_1, c_2)$. The Mukai lattice of X is $H^*(X, \mathbb{Z})$ with the symmetric bilinear form

$$\langle w, w' \rangle = \int_X (c_1 \wedge c_1' - r \wedge a'\varrho - r' \wedge a\varrho)\,, \tag{1.13}$$

for any $w = (r, c_1, a) \in H^*(X, \mathbb{Z})$ and $w' = (r', c_1', a') \in H^*(X, \mathbb{Z})$. Here the notation $w = (r, c_1, a)$ means $w = r \oplus c_1 \oplus a\varrho$ with $r \in H^0(X, \mathbb{Z})$, $c_1 \in H^2(X, \mathbb{Z})$, $a \in \mathbb{Z}$ and $\varrho \in H^4(X, \mathbb{Z})$ is the fundamental cohomology class of X so that $\int_X \varrho = 1$. We define a weight 2 Hodge structure on $H^*(X, \mathbb{Z})$ by $H^{p,q}(H^*(X, \mathbb{C})) := \oplus_i H^{p+i,q+i}(X)$. We set $H^*(X, \mathbb{Z})_{\mathrm{alg}} := H^*(X, \mathbb{Z}) \cap H^{1,1}(H^*(X, \mathbb{C}))$. Let $\phi : K(X) \to H^*(X, \mathbb{Z})$ be a homomorphism such that

$$\begin{aligned}\phi(E) :=& \left(\mathrm{ch}(E)\sqrt{\mathrm{Todd}(X)}\right)^\vee \\ =&(\mathrm{rk}(E), -c_1(E), (c_1(E)^2)/2 - c_2(E) + \mathrm{rk}(E)).\end{aligned}$$

Then we see that ϕ is injective and the image is $H^*(X, \mathbb{Z})_{\mathrm{alg}}$. We set $w := (r, c_1, (c_1^2)/2 - c_2 + r) \in H^*(X, \mathbb{Z})$. By the definition of the lattice structure, ϕ

induces an isomorphism $\phi : K_c \to w^\perp \cap H^*(X,\mathbb{Z})_{\mathrm{alg}}$. There is a homomorphism θ_w which makes the following diagram commutative:

$$\begin{array}{ccc} K_c & \xrightarrow{\lambda} & \operatorname{Pic}(M_H^X(r,c_1,c_2)) \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ w^\perp & \xrightarrow{\theta_w} & H^2(M_H^X(r,c_1,c_2),\mathbb{Z}) \end{array}$$

If there is a universal family $\mathcal{E}$, then θ_w is given by

$$\theta_w(x) = \left[p_{M_H^X(r,c_1,c_2)*}\left(\operatorname{ch}\mathcal{E}\sqrt{\operatorname{Todd}(X)}x^\vee\right)\right]_1.$$

For $M_H^X(c)$, the following is known (cf. [54], [55]).

Theorem 1.14. *Let $c \in K(X)_{\mathrm{hom}}$ with $\operatorname{rk}(c) > 0$ or $\operatorname{rk}(c) = 0$ and $c_1(c)$ nef and big. Assume that $M_H^X(c)$ consists only of stable sheaves.*

(1) *$M_H^X(c)$ is an irreducible symplectic manifold which is deformation equivalent to $X^{[n]}$, where $n = \Delta(c)/2 - (\operatorname{rk}(c)^2 - 1)$.*

(2) *If $\Delta(c)/2 - (\operatorname{rk}(c)^2 - 1) > 1$, then θ_w is an isomorphism such that θ_w preserves the Hodge structure and the Beauville quadratic form $q_{M_H^X(c)}$ coincides with the quadratic form associated to the Mukai lattice: $\langle x^2\rangle = q_{M_H^X(c)}(\theta_w(x))$. If $\Delta(c)/2 - (\operatorname{rk}(c)^2 - 1) = 1$, then θ_w is surjective with the kernel $\mathbb{Z}w$ and similar properties hold.*

For the Euler characteristic of an irreducible symplectic manifold, we can use the following result due to Fujiki (cf. [22, Corollary 23.18]).

Theorem 1.15. *For an irreducible symplectic manifold M, there is a polynomial $f(x) \in \mathbb{Q}[x]$ such that for all $D \in H^2(M,\mathbb{Z})$,*

$$\int_M e^D \operatorname{Todd}(M) = f(q_M(D)),$$

where q_M is the Beauville quadratic form on $H^2(M,\mathbb{Z})$. Obviously $f(x)$ is deformation invariant.

Thus it is sufficient to compute the Euler characteristic of $\lambda(v)$ for the Hilbert scheme $X^{[n]}$ of n points on a $K3$ surface X. In this case, the Euler characteristic is determined by [14] (cf. [22, Example 23.19]).

$$\chi(X^{[n]},\lambda(v)) = \binom{\frac{q_{X^{[n]}}(\lambda(v))}{2} + 2 + n - 1}{n}. \tag{1.16}$$

Now let $c \in K(X)$ be general. Then for $\phi(v) = (\operatorname{rk} v, -c_1(v), c_1(v)^2/2 - c_2(v) + \operatorname{rk}(v))$ we have

$$q_{M_H^X(c)}(\lambda(v)) = \langle \phi(v)^2 \rangle = -2\operatorname{rk}(v)(c_1(v)^2/2 - c_2(v) + \operatorname{rk}(v)) + c_1(v)^2 = \Delta(v) - 2\operatorname{rk}(v)^2.$$

Therefore Proposition 1.10 follows from (1.16).

If c is a class in $K(Y)$ for a surface Y we want to momentarily introduce the following notation. We write $\overline{M}_H^Y(c)$ for the moduli space of H-semistable sheaves E on Y with $\operatorname{rk}(E) = \operatorname{rk}(c)$, $\det(E) = \det(c)$ and $c_2(E) = c_2(c)$, i.e. the moduli space with fixed determinant. Now let A be an abelian surface, we have a formula very similar to Proposition 1.10.

Remark 1.17. Let $c \in K(A)$ be a class $\operatorname{rk}(c) = 1$. Let $v \in K_c$. Then

$$\chi(\overline{M}_H^A(c), \lambda(v)) = \frac{\Delta(v) + \operatorname{rk}(v)^2\Delta(c)}{\Delta(v) + \Delta(c)} \binom{\frac{\Delta(v)}{2} + \frac{\Delta(c)}{2}}{\frac{\Delta(c)}{2}}.$$

Proof. Put $n := \frac{\Delta(c)}{2}$. Then $\overline{M}_H^A(c) = A^{[n]}$, and if $Z \subset A \times A^{[n]}$ is the universal subscheme, then the universal sheaf is $\mathcal{I}_Z \otimes p_A^*(\det(c))$. Thus for any $v \in K_c$ we get $\lambda(v) = \lambda'(v \otimes \det(c))$, where $\lambda'(w)$ is the determinant bundle on $A^{[n]} = \overline{M}_H^A(c \otimes \det(c)^{-1})$ defined via the universal sheaf $\mathcal{I}_Z$. Thus replacing v by $v \otimes \det(c)$ we can assume that $\det(c) = 0$. We write $r = \operatorname{rk}(v)$. Then we get

$$\begin{aligned}\lambda(v) &= \det(p_{A^{[n]}!}(p_A^*(v) \otimes \mathcal{I}_Z)) = \det(p_{A^{[n]}*}(p_A^*(v) \otimes \mathcal{O}_Z))^{-1} \\ &= \det(p_{A^{[n]}*}(\mathcal{O}_Z))^{\otimes(1-r)} \otimes \det(p_{A^{[n]}*}(p_A^*(\det(v)) \otimes \mathcal{O}_Z))^{-1}.\end{aligned} \tag{1.18}$$

In the last line we use that $\det(p_{A^{[n]}*}(p_A^*(v) \otimes \mathcal{O}_Z))$ depends only on $\operatorname{rk}(v)$ and $\det(v)$, so we can replace v by $\mathcal{O}_A^{\oplus(r-1)} \oplus \det(v)$. Thus by [14, Theorem 5.3] we get

$$\chi(A^{[n]}, \lambda(v)) = \frac{\frac{c_1(v)^2}{2}}{\frac{c_1(v)^2}{2} - (r^2-1)n} \binom{\frac{c_1(v)^2}{2} - (r^2-1)n}{n}.$$

Finally the condition $\chi(c \otimes v)$ gives $c_1(v)^2/2 - c_2(v) = rn$, which is equivalent to $c_1(v)^2/2 = r^2 n + \frac{\Delta(v)}{2}$. The result follows. □

It seems natural to expect that a similar formula also holds for $\operatorname{rk}(c) \geq 0$ arbitrary. The simplest formula possible seems to be

$$\chi(\overline{M}_H^A(c), \lambda(v)) = \frac{\operatorname{rk}(c)^2\Delta(v) + \operatorname{rk}(v)^2\Delta(c)}{\Delta(v) + \Delta(c)} \binom{\frac{\Delta(v)}{2} + \frac{\Delta(c)}{2}}{\frac{\Delta(c)}{2}}.$$

Remark 1.19. Let Y be projective surface and let $c, v \in K(Y)$ with $\mathrm{rk}(v) = \mathrm{rk}(c) = 1$ and $\chi(c \otimes v) = 0$. Then $\chi(\overline{M}^Y_H(c), \lambda(-v)) = \binom{\frac{\Delta(c)}{2}+\frac{\Delta(v)}{2}}{\frac{\Delta(c)}{2}}$, in particular $\chi(\overline{M}^Y_H(c), \lambda(-v)) = \chi(\overline{M}^Y_H(v), \lambda(-c))$.

Proof. Write $c_1(c) = L$, $c_1(v) = M$, $c_2(c) = l = \Delta(c)/2$, $c_2(v) = m = \Delta(c)/2$. Let $Z \subset Y \times Y^{[l]}$ be the universal subscheme. Then the universal sheaf on $Y \times M^Y_H(c) = Y \times Y^{[l]}$ is $\mathcal{I}_Z \otimes p_Y^*(L)$. Thus

$$\lambda(-v) = -\det(p_{Y^{[l]}!}(\mathcal{I}_Z \otimes p_Y^*(L \otimes M))) = \det(p_{Y^{[l]}*}(\mathcal{O}_Z \otimes p_Y^*(L \otimes M))).$$

Thus [14, Lemma 5.1], we get $\chi(\overline{M}^Y_H(c), \lambda(-v)) = \binom{\chi(L\otimes M)}{l}$. By the Riemann-Roch theorem $\chi(c \otimes v) = 0$ is equivalent to $\frac{\Delta(c)}{2} + \frac{\Delta(v)}{2} = \chi(L \otimes M)$. The result follows. $\square$

1.6. **Nekrasov partition function.** We briefly review the K-theoretic Nekrasov partition function in the case of rank 2. For more details see [43, section 1]. Let ℓ_∞ be the line at infinity in $\mathbb{P}^2$. Let $M(n)$ be the moduli space of pairs (E, Φ), where E is a rank 2 torsion-free sheaf on $\mathbb{P}^2$ with $c_2(E) = n$, which is locally free in a neighbourhood of ℓ_∞ and $\Phi : E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus 2}$ is an isomorphism. $M(n)$ is a nonsingular quasiprojective variety of dimension $4n$. The tangent space to $M(n)$ at (E, Φ) is $\mathrm{Ext}^1(E, E(-l_\infty))$.

Let $\Gamma := \mathbb{C}^* \times \mathbb{C}^*$ and $\widetilde{T} := \Gamma \times \mathbb{C}^*$. $\widetilde{T}$ acts on $M(n)$ as follows: For $(t_1, t_2) \in \Gamma$, let F_{t_1,t_2} be the automorphism of $\mathbb{P}^2$ defined by $F_{t_1,t_2}([z_0, z_1, z_2]) \mapsto [z_0, t_1z_1, t_2z_2]$, and for $e_2 \in \mathbb{C}^*$ let G_{e_2} be the automorphism of $\mathcal{O}_{\ell_\infty}^{\oplus 2}$ given by $(s_1, s_2) \mapsto (e_2^{-1}s_1, e_2s_2)$. Then for $(E, \Phi) \in M(n)$ we put $(t_1, t_2, e_2) \cdot (E, \Phi) := \big((F^{-1}_{t_1,t_2})^*E, \Phi'\big)$, where Φ' is the composition

$$(F^{-1}_{t_1,t_2})^*(E)|_{\ell_\infty} \xrightarrow{(F^{-1}_{t_1,t_2})^*\Phi} (F^{-1}_{t_1,t_2})^*\mathcal{O}_{\ell_\infty}^{\oplus 2} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus 2} \xrightarrow{G_{e_2}} \mathcal{O}_{\ell_\infty}^{\oplus 2}$$

where the middle arrow is the homomorphism given by the action.

Notation 1.20. We denote e_2 the one-dimensional $\widetilde{T}$-module given by $(t_1, t_2, e_2) \mapsto e_2$. and similar we write t_i $(i = 1, 2)$ for the 1-dimensional $\widetilde{T}$ modules given by $(t_1, t_2, e_2) \mapsto t_i$. We also write $e_1 := e_2^{-1}$.

Let $\varepsilon_1, \varepsilon_2, a$ be the coordinates on the Lie algebra of $\widetilde{T}$ corresponding to t_1, t_2, e_2. Then $\varepsilon_1, \varepsilon_2, a$ are generators of the equivariant cohomology $H^*_{\widetilde{T}}(pt)$

of a point. We relate the two sets of variables by $t_1 = e^{\boldsymbol{\beta}\varepsilon_1}, t_2 = e^{\boldsymbol{\beta}\varepsilon_2}, e_2 = e^{\boldsymbol{\beta}a}$, where $\boldsymbol{\beta} \in \mathbb{C}$ is a parameter. We write $a_1 := -a$, $a_2 := a$.

The instanton part of the K-theoretic partition function is defined as

$$Z_K^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}) := \sum_{n=0}^{\infty} ((\boldsymbol{\beta}\Lambda)^4 e^{-\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)})^n \sum_i (-1)^i \operatorname{ch} H^i(M(n), \mathcal{O}). \tag{1.21}$$

Here the character ch is a formal sum of weight spaces, which are all finite-dimensional by [42, section 4].

Let x, y be the coordinates on $\mathbb{A}^2 = \mathbb{P}^2 \setminus \ell_\infty$. The fixpoint set $M(n)^{\widetilde{T}}$ is the set of $(\mathcal{I}_{Z_1}, \Phi_1) \oplus (\mathcal{I}_{Z_2}, \Phi_2)$, where the $\mathcal{I}_{Z_\alpha}$ are ideal sheaves of zero dimensional schemes Z_α with support in the origin of $\mathbb{A}^2$ with $\operatorname{len}(Z_1) + \operatorname{len}(Z_2) = n$, and Φ_α $(\alpha = 1, 2)$ are isomorphisms of $\mathcal{I}_{Z_\alpha}|_{\ell_\infty}$ with the α-th factor of $\mathcal{O}_{\ell_\infty}^{\oplus 2}$. Write I_α for the ideal of Z_α in $\mathbb{C}[x, y]$. Then the above is a fixpoint if and only if I_1 and I_2 are generated by monomials in x, y. The fixed point set $M(n)^{\widetilde{T}}$ is parametrized by the pairs of Young diagrams $\vec{Y} = (Y_1, Y_2)$ so that the ideal I_α is generated by the $x^i y^i$ with $(i-1, j-1)$ outside Y_i. The total number of boxes is $|\vec{Y}| := |Y_1| + |Y_2| = n$.

We use the following notations: For a Young diagram Y let λ_i be the length of the i^{th} column. Let Y' be the transpose of Y and let λ'_j be the length of the j^{th} column of Y' (equal to the length of the j^{th} row of Y). For $s = (i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ let

$$a_Y(s) := \lambda_i - j, \quad l_Y(s) = \lambda'_j - i, \quad a'(s) = j - 1, \quad l'(s) = i - 1.$$

Following [43] let, for $\alpha, \beta \in \{1, 2\}$,

$$\begin{aligned} n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, a; \boldsymbol{\beta}) := &\prod_{s \in Y_\alpha} \left(1 - e^{-\boldsymbol{\beta}(-l_{Y_\beta}(s)\varepsilon_1 + (a_{Y_\alpha}(s)+1)\varepsilon_2 + a_\beta - a_\alpha)}\right) \\ &\times \prod_{s \in Y_\beta} \left((1 - e^{-\boldsymbol{\beta}((l_{Y_\alpha}(s)+1)\varepsilon_1 - a_{Y_\beta}(s)\varepsilon_2 + a_\beta - a_\alpha)}\right) \end{aligned} \tag{1.22}$$

be the $\widetilde{T}$-equivariant character of $\left(\operatorname{Ext}^1(\mathcal{I}_{Z_\alpha}, \mathcal{I}_{Z_\beta}(-\ell_\infty))\right)^\vee$. Then by the Atiyah-Bott Lefschetz fixed point formula we have

$$Z_K^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}) = \sum_{\vec{Y}} \frac{((\boldsymbol{\beta}\Lambda)^4 e^{-\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)})^{|\vec{Y}|}}{\bigwedge_{-1} T^*_{\vec{Y}} M(n)} = \sum_{\vec{Y}} \frac{((\boldsymbol{\beta}\Lambda)^4 e^{-\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)})^{|\vec{Y}|}}{\prod_{\alpha,\beta=1,2} n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, a; \boldsymbol{\beta})}, \tag{1.23}$$

where $\bigwedge_{-1}$ is the alternating sum of the exterior powers.

More generally we will consider the *partition function with 5D Chern-Simons term* (see [51]): Let $\mathcal{E}$ be the universal sheaf on $\mathbb{P}^2 \times M(n)$. Consider the line bundle

$$\mathcal{L} := \lambda_{\mathcal{E}}(\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))^{-1} = (\det p_{2!}(\mathcal{E}\otimes p_1^*\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty)))^{-1} = \det R^1 p_{2*}(\mathcal{E}\otimes p_1^*\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty)).$$

For an integer m consider the generating function

(1.24)

$$Z_m^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau) := \sum_{n=0}^{\infty}((\boldsymbol{\beta}\Lambda)^4 e^{-\boldsymbol{\beta}(1+\frac{m}{2})(\varepsilon_1+\varepsilon_2)})^n \sum_i (-1)^i \operatorname{ch} H^i(M(n),\mathcal{L}^{\otimes m}) \times \exp\Big(\tau\Big(-n+\frac{a^2}{\varepsilon_1\varepsilon_2}\Big)\Big).$$

We denote $Z_m^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},0)$ simply by $Z_m^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta})$, in particular $Z_0^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta}) = Z_K^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta})$.

We put

$$C_m^{\vec{Y}}(\varepsilon_1,\varepsilon_2,a;\boldsymbol{\beta},\tau) := \exp\Big(m\boldsymbol{\beta}\sum_{\alpha=1}^{2}\sum_{s\in Y_\alpha}(a_\alpha - l'(s)\varepsilon_1 - a'(s)\varepsilon_2)\Big) \times \exp\Big(\tau\Big(-|\vec{Y}|+\frac{a^2}{\varepsilon_1\varepsilon_2}\Big)\Big).$$

Then we get by localization

(1.25)

$$Z_m^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau) = \sum_{\vec{Y}} \frac{((\boldsymbol{\beta}\Lambda)^4 e^{-\boldsymbol{\beta}(1+\frac{m}{2})(\varepsilon_1+\varepsilon_2)})^{|\vec{Y}|} C_m^{\vec{Y}}(\varepsilon_1,\varepsilon_2,a;\boldsymbol{\beta},\tau)}{\prod_{\alpha,\beta=1,2} n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1,\varepsilon_2,a;\boldsymbol{\beta})}.$$

(see also [51]). We briefly sketch the argument: Let $\vec{Y}=(Y_1,Y_2)$ correspond to a fixpoint $(\mathcal{I}_{Z_1},\Phi_1)\oplus(\mathcal{I}_{Z_2},\Phi_2)$ of $M(n)$. By localization we have to show that

$$H^1(\mathbb{P}^2,(\mathcal{I}_{Z_1}\oplus\mathcal{I}_{Z_2})\otimes\mathcal{O}(-\ell_\infty)) = \sum_{\alpha=1}^{2}\sum_{s\in Y_\alpha} e_\alpha t_1^{-l'(s)} t_2^{-a'(s)},$$

as $\widetilde{T}$ modules. The exact sequence $0\to(\mathcal{I}_{Z_1}\oplus\mathcal{I}_{Z_2})\otimes\mathcal{I}(-\ell_\infty)\to\mathcal{O}(-\ell_\infty)^{\oplus 2}\to\mathcal{O}_{Z_1}\oplus\mathcal{O}_{Z_2}\to 0$ induces an isomorphism $H^1(\mathbb{P}^2,(\mathcal{I}_{Z_1}\oplus\mathcal{I}_{Z_2})\otimes\mathcal{O}(-\ell_\infty)) \simeq H^0(\mathcal{O}_{Z_1})\oplus H^0(\mathcal{O}_{Z_2})$. We have seen that an equivariant basis of $H^0(\mathcal{O}_{Z_\alpha})$ is the set $\{x^{l'(s)}y^{a'(s)} \mid s\in Y_\alpha\}$. By definition $(t_1,t_2)\in\Gamma$ acts by multiplying x by t_1^{-1} and y by t_2^{-1}. Finally by definition e_α acts $H^0(\mathcal{O}_{Z_\alpha})$ by multiplying with e_α. The claim follows.

For a variables τ_0, τ_1 let

(1.26)

$$E^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a; \boldsymbol{\beta}, \tau_0, \tau_1) := \exp\Big(\sum_{\alpha=1}^{2}\sum_{\rho=0}^{1} \tau_\rho \Big[\frac{e^{\boldsymbol{\beta} a_\alpha}}{\boldsymbol{\beta}^2\varepsilon_1\varepsilon_2}\Big(1-(1-e^{-\boldsymbol{\beta}\varepsilon_1})(1-e^{-\boldsymbol{\beta}\varepsilon_2})\sum_{s\in Y_\alpha} e^{-\boldsymbol{\beta}(l'(s)\varepsilon_1 + a'(s)\varepsilon_2)}\Big)\Big]_\rho\Big).$$

Here $[\cdot]_\rho$ means the part of degree ρ, where $a, \varepsilon_1, \varepsilon_2$ have degree 1. This is $\exp(\sum_{\rho=0}^{1}\tau_\rho \operatorname{ch}_{\rho+2}(\mathcal{E})/[\mathbb{C}^2])$. (See [42, p.59].) Then an easy computation gives that

$$(1.27) \quad E^{\vec{Y}}(\varepsilon_1, \varepsilon_2, a; \boldsymbol{\beta}, \tau, m) = C^{\vec{Y}}_{-m}(\varepsilon_1, \varepsilon_2, a; \boldsymbol{\beta}, \tau) \times \exp\Big(m\boldsymbol{\beta}\Big(|\vec{Y}|\frac{\varepsilon_1+\varepsilon_2}{2} + (|Y_2|-|Y_1|)\frac{a^3}{6\varepsilon_1\varepsilon_2}\Big)\Big).$$

As a power series in Λ, $Z^{\text{inst}}_m(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}, \tau)$ starts with 1. Thus

$$F^{\text{inst}}_m(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}, \tau) := \log Z^{\text{inst}}_m(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}, \tau)$$

is well-defined and we put $F^{\text{inst}}_m(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}) := F^{\text{inst}}_m(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}, 0)$, $F^{\text{inst}}_K(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}) := F^{\text{inst}}_0(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta})$.

We define the perturbation part, see [43, section 4.2] for more details. We set

(1.28)

$$\gamma_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda) := \frac{1}{2\varepsilon_1\varepsilon_2}\left(-\frac{\boldsymbol{\beta}}{6}\left(x+\frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3 + x^2\log(\boldsymbol{\beta}\Lambda)\right) + \sum_{n\geq 1}\frac{1}{n}\frac{e^{-\boldsymbol{\beta} n x}}{(e^{\boldsymbol{\beta} n\varepsilon_1}-1)(e^{\boldsymbol{\beta} n\varepsilon_2}-1)},$$

$$\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda) := \gamma_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda) + \frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{\pi^2 x}{6\boldsymbol{\beta}} - \frac{\zeta(3)}{\boldsymbol{\beta}^2}\right) + \frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log(\boldsymbol{\beta}\Lambda) + \frac{\pi^2}{6\boldsymbol{\beta}}\right) + \frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log(\boldsymbol{\beta}\Lambda)$$

for $(x, \boldsymbol{\beta}, \Lambda)$ in a neighbourhood of $\sqrt{-1}\mathbb{R}_{>0}\times\sqrt{-1}\mathbb{R}_{<0}\times\sqrt{-1}\mathbb{R}_{>0}$. We formally expand $\varepsilon_1\varepsilon_2\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda)$ as a power series of $\varepsilon_1, \varepsilon_2$ (around $\varepsilon_1 = \varepsilon_2 = 0$). Expanding

$$(1.29) \qquad \frac{1}{(e^{\varepsilon_1 t}-1)(e^{\varepsilon_2 t}-1)} = \sum_{n\geq 0}\frac{c_n}{n!}t^{n-2},$$

we obtain

$$\sum_{n\geq 1}\frac{1}{n}\frac{e^{-\boldsymbol{\beta} nx}}{(e^{\boldsymbol{\beta} n\varepsilon_1}-1)(e^{\boldsymbol{\beta} n\varepsilon_2}-1)} = \sum_{m\geq 0}\frac{c_m}{m!}\boldsymbol{\beta}^{m-2}\mathrm{Li}_{3-m}(e^{-\boldsymbol{\beta} x}),$$

where Li_{3-m} is the polylogarithm (see [43, Appendix B] for details). Here we choose the branch of log by $\log(r\cdot e^{i\phi}) = \log(r) + i\phi$ with $\log(r)\in\mathbb{R}$ for $\phi \in (-\pi/2, 3\pi/2)$ and $r\in\mathbb{R}$. We define $\gamma_{\varepsilon_1,\varepsilon_2}(-x|\boldsymbol{\beta};\Lambda)$ by analytic continuation along circles in a counter-clockwise way. We then define the perturbation part of the partition function by

$$F_K^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,x;\Lambda,\boldsymbol{\beta}) := -\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(2x|\boldsymbol{\beta};\Lambda) - \widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(-2x|\boldsymbol{\beta};\Lambda), \tag{1.30}$$

Then $F_K^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,x;\Lambda,\boldsymbol{\beta})$ is a formal power series in $\varepsilon_1,\varepsilon_2$ whose coefficients are holomorphic functions in $\Lambda\in\mathbb{C}\setminus\sqrt{-1}\mathbb{R}_{\leq 0}$, $x\in\mathbb{C}\setminus\sqrt{-1}\mathbb{R}_{\leq 0}$, $\boldsymbol{\beta}\in\mathbb{C}$ with $|\boldsymbol{\beta}| < \frac{\pi}{|x|}$.

Finally we define

$$\begin{aligned}
F_m(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau) &:= F_K^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta}) + \log Z_m^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau),\\
F_m(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta}) &:= F_m(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},0),\\
F_K(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta}) &:= F_0(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta}).
\end{aligned}$$

Formally one defines

$$Z_m(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau) := \exp(F_m^{\mathrm{pert}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta}))Z_K^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau),$$

and similarly for $Z_m(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta})$, $Z_K(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta})$.

1.7. **More on the partition function with 5D Chern-Simons term.** We explain how the known properties of the K-theoretic Nekrasov partition function, obtained in [43], can be generalized to the partition function with 5D Chern-Simons term, at least conjecturally. Our explanation is mathematical, so a physical motivation can be found in [27, 51] and the references therein.

This subsection is independent of the rest of this paper, and can be safely skipped. We also keep the notation in [43] except we set $\mathfrak{q} = \Lambda^{1/2r}$.

We consider the general case $r\geq 2$, although we only consider the case $r=2$ in the main part of the paper. Let $M(r,n)$ be the framed moduli space of rank

r torsion free sheaves E on $\mathbb{P}^2$ with $c_2(E) = n$. Let $\mathcal{E}$ be the universal sheaf on $\mathbb{P}^2 \times M(r,n)$. Consider the line bundle

$$\mathcal{L} := \lambda_{\mathcal{E}}(\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))^{-1} = (\det p_{2!}(\mathcal{E} \otimes p_1^* \mathcal{O}_{\mathbb{P}^2}(-\ell_\infty)))^{-1} = \det R^1 p_{2*}(\mathcal{E} \otimes p_1^* \mathcal{O}_{\mathbb{P}^2}(-\ell_\infty)).$$

For an integer m consider the generating function

(1.31)
$$Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}) := \sum_{n=0}^{\infty} ((\boldsymbol{\beta}\Lambda)^{2r} e^{-\boldsymbol{\beta}(r+m)(\varepsilon_1+\varepsilon_2)/2})^n \sum_i (-1)^i \operatorname{ch} H^i(M(r,n), \mathcal{L}^{\otimes m}).$$

By the localization formula we have

(1.32)
$$\begin{aligned} &Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta}) \\ &= \sum_{\vec{Y}} \frac{((\boldsymbol{\beta}\Lambda)^{2r} e^{-\boldsymbol{\beta}(r+m)(\varepsilon_1+\varepsilon_2)/2})^{|\vec{Y}|}}{\prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a}; \boldsymbol{\beta})} \exp\Big(m\boldsymbol{\beta} \sum_\alpha \sum_{s \in Y_\alpha} (a_\alpha - l'(s)\varepsilon_1 - a'(s)\varepsilon_2)\Big), \end{aligned}$$

where $\vec{Y}$ is an r-tuple of Young diagrams. The argument is the same as in the rank 2 case.

We have

$$Z_{-m}^{\text{inst}}(-\varepsilon_1, -\varepsilon_2, -\vec{a}; \Lambda, \boldsymbol{\beta}) = Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta}). \tag{1.33}$$

This is a consequence of Serre duality and the equality $K_{M(r,n)} = e^{-r\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)n}$ ([43, Lemma 3.6]). But it also follows directly from

$$\prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}(-\varepsilon_1, -\varepsilon_2, -\vec{a}; \boldsymbol{\beta}) = e^{\boldsymbol{\beta} r(\varepsilon_1+\varepsilon_2)|\vec{Y}|} \times \prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a}; \boldsymbol{\beta}).$$

1.7.1. *Correlation function on blow-up.* Let X be the blow-up of $\mathbb{P}^2$ at the origin of $\mathbb{C}^2$. Let $\widehat{M}(r,k,\widehat{n})$ be the framed moduli space on X. We define the similar partition function $\widehat{Z}_{m,k,d}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta})$ on X by considering

$$\sum_i (-1)^i \operatorname{ch} H^i(\widehat{M}(r,k,\widehat{n}), \widehat{\mathcal{L}}^{\otimes m} \otimes \mu(C)^{\otimes d}),$$

where $\widehat{\mathcal{L}}$ is defined as in the case of $\mathbb{P}^2$ by taking the universal bundle $\widehat{E}$ over $X \times \widehat{M}(r,k,\widehat{n})$. (See [43, §2.1].) As in [43, §2.2], we can write down $\widehat{Z}_{m,k,d}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta})$

in terms of $Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta})$. The new factor comes from

$$c_1\Big(\bigoplus_\alpha H^1(\mathcal{O}_X(k_\alpha C))e_\alpha\Big) = \sum_\alpha \frac{k_\alpha^3 - k_\alpha}{6}(\varepsilon_1+\varepsilon_2) + \sum_\alpha \frac{k_\alpha(k_\alpha - 1)}{2} a_\alpha$$
$$= \sum_\alpha \frac{k_\alpha^3}{6}(\varepsilon_1+\varepsilon_2) + \sum_\alpha \frac{k_\alpha^2}{2} a_\alpha - \frac{1}{2}(\vec{k}, \vec{a}).$$

Then the blowup formula is a slight modification of [43, (2.2)]:

(1.34)

$$\widehat{Z}_{m,k,d}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta}) = \sum_{\substack{\vec{k}\in\mathbb{Z}^r \\ \sum k_\alpha = k}} \frac{(e^{\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)(d-(r+m)/2)}(\boldsymbol{\beta}\Lambda)^{2r})^{(\vec{k},\vec{k})/2} e^{\boldsymbol{\beta}(\vec{k},\vec{a})(d-m/2)}}{\prod_{\vec{\alpha}\in\Delta} l_{\vec{\alpha}}^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a})}$$
$$\times \exp\left[m\boldsymbol{\beta}\left(\frac{1}{6}(\varepsilon_1+\varepsilon_2)\sum_\alpha k_\alpha^3 + \frac{1}{2}\sum_\alpha k_\alpha^2 a_\alpha\right)\right]$$
$$\times Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1\vec{k}; e^{\boldsymbol{\beta}\varepsilon_1(d-(r+m)/2)/2r)}\Lambda, \boldsymbol{\beta})$$
$$\times Z_m^{\text{inst}}(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2\vec{k}; e^{\boldsymbol{\beta}\varepsilon_2(d-(r+m)/2)/2r}\Lambda, \boldsymbol{\beta}),$$

where $(\vec{k}, \vec{a}) = \frac{1}{2r}\sum_{\alpha,\beta}(k_\alpha - k_\beta)(a_\alpha - a_\beta)$, and similarly for $(\vec{k}, \vec{k})$. Note that we need to normalize a vector $\vec{k} = (k_\alpha)_{\alpha=1}^r$ with $\sum k_\alpha = k$ into $\vec{l} = (k_1 - \frac{k}{r}, \dots, k_r - \frac{k}{r})$, as we assume $\sum a_\alpha = 0$. (We took this normalization in [43] without an explanation. It was explained in [41, §6].) Under this normalization we have $(\vec{k}, \vec{a}) = (\vec{l}, \vec{a})$, $l_{\vec{\alpha}}^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a}) = l_{\vec{\alpha}}^{\vec{l}}(\varepsilon_1, \varepsilon_2, \vec{a})$, $n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1\vec{k}) = n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1\vec{l})$, etc. In particular, we simply replace $\vec{k}$ by $\vec{l}$ in the original partition function with $m = 0$. However the Chern-Simons term requires some care:

$$\exp\left[m\boldsymbol{\beta}\left(\sum_\alpha\sum_{s\in Y_\alpha}\big((a_\alpha + \varepsilon_1 k_\alpha - l'(s)\varepsilon_1 - a'(s)(\varepsilon_2 - \varepsilon_1)\big)\right)\right] = \exp\left[\frac{\boldsymbol{\beta}mk}{r}\varepsilon_1|\vec{Y}|\right]$$
$$\times \exp\left[m\boldsymbol{\beta}\left(\sum_\alpha\sum_{s\in Y_\alpha}\big((a_\alpha + \varepsilon_1 l_\alpha - l'(s)\varepsilon_1 - a'(s)(\varepsilon_2 - \varepsilon_1)\big)\right)\right],$$

$$\sum_\alpha\left(\frac{k_\alpha^3}{6}(\varepsilon_1+\varepsilon_2) + \frac{k_\alpha^2}{2}a_\alpha\right)$$
$$= \sum_\alpha\left(\frac{l_\alpha^3}{6}(\varepsilon_1+\varepsilon_2) + \frac{l_\alpha^2}{2}a_\alpha\right) + \left(\frac{k}{2r}(\vec{l}, \vec{l}) + \frac{k^3}{6r^2}\right)(\varepsilon_1+\varepsilon_2) + \frac{k}{r}(\vec{l}, \vec{a}).$$

We rewrite (1.34) in terms of $\vec{l}$:

(1.35)

$$\begin{aligned}
&\widehat{Z}^{\text{inst}}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) = \exp(\frac{k^3 m\boldsymbol{\beta}}{6r^2}(\varepsilon_1+\varepsilon_2)) \\
&\qquad \times \sum_{\{\vec{l}\}=-k/r} \frac{(\exp\left[\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)(d+m\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2})\right](\boldsymbol{\beta}\Lambda)^{2r})^{(\vec{l},\vec{l})/2}}{\prod_{\vec{\alpha}\in\Delta} l^{\vec{l}}_{\vec{\alpha}}(\varepsilon_1,\varepsilon_2,\vec{a})} \\
&\qquad \times \exp\left[\boldsymbol{\beta}(\vec{l},\vec{a})(d+m(-\frac{1}{2}+\frac{k}{r}))\right] \\
&\qquad \times \exp\left[m\boldsymbol{\beta}\left(\frac{1}{6}(\varepsilon_1+\varepsilon_2)\sum_\alpha l_\alpha^3 + \frac{1}{2}\sum_\alpha l_\alpha^2 a_\alpha\right)\right] \\
&\qquad \times Z^{\text{inst}}_m(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{l};\exp\left[\frac{\boldsymbol{\beta}\varepsilon_1}{2r}\left\{d+m\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2}\right\}\right]\Lambda,\boldsymbol{\beta}) \\
&\qquad \times Z^{\text{inst}}_m(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{l};\exp\left[\frac{\boldsymbol{\beta}\varepsilon_2}{2r}\left\{d+m\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2}\right\}\right]\Lambda,\boldsymbol{\beta}).
\end{aligned}$$

Here $\{\vec{l}\} = -k/r$ means that the fractional part of l_α is independent of α and equal to $-k/r$.

By Serre duality we have

$$\widehat{Z}^{\text{inst}}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) = \widehat{Z}^{\text{inst}}_{-m,k,r-d}(-\varepsilon_1,-\varepsilon_2,-\vec{a};\Lambda,\boldsymbol{\beta})$$

thanks to [43, Lemma 3.6]. It also follows from (1.34) and (1.33) together with

$$\prod_{\vec{\alpha}\in\Delta} l^{\vec{k}}_{\vec{\alpha}}(-\varepsilon_1,-\varepsilon_2,-\vec{a}) = e^{-\boldsymbol{\beta}r(\vec{k},\vec{a})}\prod_{\vec{\alpha}\in\Delta} l^{\vec{k}}_{\vec{\alpha}}(\varepsilon_1,\varepsilon_2,\vec{a}).$$

1.7.2. *The perturbation part.* In [43, Sect. 4.2] one of the reasons for the introduction of the perturbation part was to simplify the the blowup formula. As we have an extra factor

$$\exp\left[m\boldsymbol{\beta}\left(\frac{1}{6}(\varepsilon_1+\varepsilon_2)\sum_\alpha l_\alpha^3 + \frac{1}{2}\sum_\alpha l_\alpha^2 a_\alpha\right)\right],$$

we need to modify the perturbation part so that it is absorbed in the full partition function. The answer is the cubic term:

$$\exp\left[-m\boldsymbol{\beta}\sum_{\alpha=1}^r \frac{a_\alpha^3}{6\varepsilon_1\varepsilon_2}\right].$$

We have the difference equation

$$\left.\frac{x^3}{6\varepsilon_1\varepsilon_2}\right|_{\substack{x\to x+l\varepsilon_1\\ \varepsilon_1\to\varepsilon_1\\ \varepsilon_2\to\varepsilon_2-\varepsilon_1}} + \left.\frac{x^3}{6\varepsilon_1\varepsilon_2}\right|_{\substack{x\to x+l\varepsilon_2\\ \varepsilon_1\to\varepsilon_1-\varepsilon_2\\ \varepsilon_2\to\varepsilon_2}} - \frac{x^3}{6\varepsilon_1\varepsilon_2} = -\frac{l^2x}{2} - \frac{(\varepsilon_1+\varepsilon_2)l^3}{6},$$

We thus define

$$\begin{aligned} F_m(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) &:= \sum_{\vec{\alpha}\in\Delta} -\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(\langle\vec{a},\vec{\alpha}\rangle|\boldsymbol{\beta};\Lambda) - m\boldsymbol{\beta}\sum_{\alpha=1}^{r}\frac{a_\alpha^3}{6\varepsilon_1\varepsilon_2} \\ &\quad + \log Z_m^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}), \\ \widehat{F}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) &:= \sum_{\vec{\alpha}\in\Delta} -\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(\langle\vec{a},\vec{\alpha}\rangle|\boldsymbol{\beta};\Lambda) - m\boldsymbol{\beta}\sum_{\alpha=1}^{r}\frac{a_\alpha^3}{6\varepsilon_1\varepsilon_1} \\ &\quad + \log Z_{m,k,d}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}), \end{aligned}$$

where $\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}$ is as in (1.28). Note that the term $\sum_{\alpha=1}^{r}\frac{a_\alpha^3}{6\varepsilon_1\varepsilon_2}$ disappears when $r=2$ thanks to the condition $a_1+a_2=0$. We formally define

$$\begin{aligned} Z_m(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) &:= \exp(F_m(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta})), \\ \widehat{Z}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) &:= \exp(\widehat{F}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta})). \end{aligned}$$

The blowup formula is

(1.36)

$$\begin{aligned} &\widehat{Z}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) \\ &= \exp\left[\left\{-\frac{\left(4\left(d+m\left(-\frac{1}{2}+\frac{k}{r}\right)\right)-r\right)(r-1)}{48}+\frac{k^3m}{6r^2}\right\}\boldsymbol{\beta}(\varepsilon_1+\varepsilon_2)\right] \\ &\times \sum_{\{\vec{l}\}=-k/r} Z_m(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{l};\exp\left[\frac{\boldsymbol{\beta}\varepsilon_1}{2r}\left\{d+m\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2}\right\}\right]\Lambda,\boldsymbol{\beta}) \\ &\qquad\qquad \times Z_m(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{l};\exp\left[\frac{\boldsymbol{\beta}\varepsilon_2}{2r}\left\{d+m\left(-\frac{1}{2}+\frac{k}{r}\right)-\frac{r}{2}\right\}\right]\Lambda,\boldsymbol{\beta}). \end{aligned}$$

This is exactly the same as [43, (4.9)] with the replacement $d\to d+m(-1/2+k/r)$.

1.7.3. *A conjectural blowup equation.* For the original K-theoretic Nekrasov partition function we have a blowup equation [43, Th. 2.4 and (4.9)], which determines the partition function from its perturbative part. It was derived from vanishing of higher direct image sheaves of a determinant line bundle $\mu(C)$ with respect to the projection $\widehat{\pi}\colon \widehat{M}(r,0,n)\to N(r,n)$, where $N(r,n)$ is the Uhlenbeck compactification of the framed moduli space of locally free sheaves on $\mathbb{P}^2$, denote by $M_0(r,n)$ in [43].

The proof of the vanishing theorem cannot be carried over to the partition functions with Chern-Simons terms. But a numerical computation suggests

$$(1.37)\quad \widehat{Z}^{\text{inst}}_{m,0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) = Z^{\text{inst}}_m(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) \qquad \text{for } 0\le d\le r,\ |m|\le r.$$

This is exactly the same what we have proved for the original K-theoretic partition function, i.e. $m = 0$ in [43]. It seems likely that the left hand side can be always written in terms of the correlation function, which is the holomorphic Euler characteristic of certain (virtual) bundles on $M(r,n)$. But it can be written as above only in the limited range of d and m. In fact, we check the above equation holds in a slightly wider situation when $r = 2$, $m = 1$: it seems to hold for $0 \le d \le 3 = r + m$. But we also check that when $r = 2$, $m = 2$, the above is not true for $d = 4$.

We have the following analogs of [43, Lemma 4.3, Theorem 4.4]:

Proposition 1.38. *Suppose* (1.37) *holds and assume* $|m| < r$. *Then*

(1) $Z^{\text{inst}}_m(\varepsilon_1,-2\varepsilon_1,\vec{a};\Lambda,\boldsymbol{\beta}) = Z^{\text{inst}}_m(2\varepsilon_1,-\varepsilon_1,\vec{a};\Lambda,\boldsymbol{\beta})$.

(2) $\varepsilon_1\varepsilon_2\log Z^{\text{inst}}_m(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta})$ *is regular at* $(\varepsilon_1,\varepsilon_2) = (0,0)$.

We only give the proof of (1), as the proof of (2) is exactly the same as the original.

Proof. By (1.33) we may assume $m \le 0$. By the assumption we have (1.37) for $d = 0$ and $d = r + m$. Note that we have $0 \ne r + m$ as $m \ne -r$.

Let us put $\boldsymbol{\beta} = 1$ for brevity. We take the difference of both sides of (1.34) with $d = r + m$, 0 after setting $\varepsilon_2 = -\varepsilon_1$. We have

$$\begin{aligned}
&(Z_n(\varepsilon_1,-2\varepsilon_1,\vec{a}) - Z_n(2\varepsilon_1,-\varepsilon_1,\vec{a}))\left(e^{(r+m)n\varepsilon_1/2} - e^{-(r+m)n\varepsilon_1/2}\right)\\
&= -\sum_{\substack{(\vec{k},\vec{k})/2+l+l'=n\\ l\ne n,\, l'\ne n}} \frac{e^{r(\vec{k},\vec{a})/2} Z_{l'}(\varepsilon_1,-2\varepsilon_1,\vec{a}+\varepsilon_1\vec{k}) Z_l(2\varepsilon_1,-\varepsilon_1,\vec{a}-\varepsilon_1\vec{k})}{\prod_{\vec{\alpha}\in\Delta} l^{\vec{k}}_{\vec{\alpha}}(\varepsilon_1,-\varepsilon_1,\vec{a})}\\
&\qquad\times \exp\left[\frac{m\boldsymbol{\beta}}{2}\sum_\alpha k_\alpha^2 a_\alpha\right]\\
&\qquad\times\left(e^{(r+m)(\vec{k},\vec{a})/2}e^{(r+m)(l'-l)\varepsilon_1/2} - e^{-(r+m)(\vec{k},\vec{a})/2}e^{-(r+m)(l'-l)\varepsilon_1/2}\right),
\end{aligned}$$

where we expand Z_m^{inst} as

$$Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta} = 1) = \sum_n Z_n(\varepsilon_1, \varepsilon_2, \vec{a})\Lambda^{2rn}.$$

Let us show that $Z_n(\varepsilon_1, -2\varepsilon_1, \vec{a}) = Z_n(2\varepsilon_1, -\varepsilon_1, \vec{a})$ by using the induction on n. It holds for $n = 0$ as $Z_0 = 1$. Suppose that it is true for $l, m < n$. Then the right hand side of the above equation vanishes, as terms with $(\vec{k}, l, l')$ and $(-\vec{k}, l', l)$ cancel thanks to [43, Lemma 4.1(1) and (4.2)], and the term $(0, l, l)$ is 0. Therefore it is also true for n. $\square$

We expand $\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \boldsymbol{\beta})$ as in (4.1). The following can be proved exactly as in [43, (4.11)]:

Proposition 1.39. *Suppose* (1.37) *holds and assume* $|m| < r$. *Then*

$$\exp\left[-\frac{\boldsymbol{\beta}^2}{8r^2}\left(d - \frac{r+m}{2}\right)^2 \frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2}\right] \times \Theta_E\left(-\frac{1}{2\pi\sqrt{-1}}\frac{\boldsymbol{\beta}}{2r}\left(d - \frac{r+m}{2}\right)\frac{\partial^2 \mathcal{F}_0}{\partial \log \Lambda \partial \vec{a}}\middle| \tau(\boldsymbol{\beta})\right)$$

is independent of $d = 0, \ldots, r$. *Here*

$$\tau(\boldsymbol{\beta}) = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial^2 \mathcal{F}_0}{(\partial \vec{a})^2}$$

and Θ_E *is the Riemann theta function with the characteristic* ${}^t\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. (*See* [42, Appendix B] *for convention.*)

We call this the *contact term equation*.

As Θ_E is an even function, the above holds for d if and only if it holds for $r + m - d$. In particular, the above expression is independent of $0 \le d \le r + m$ for $m \ge 0$, and $m \le d \le r$ for $m \le 0$.

We will prove that the Seiberg-Witten prepotential defined via the periods of hyperelliptic curves satisfies the same equation and has the same perturbation part in §A. As the contact term equation determines the instanton part of the prepotential recursively from the perturbation part, we get

Theorem 1.40. *Suppose* (1.37) *holds and assume* $|m| < r$. *Then* $\mathcal{F}_0$ *coincides with the Seiberg-Witten prepotential defined in* (A.5).

As we have (1.37) for the case $m = 0$, we have the assertion without the condition in this case. This is the proof of Nekrasov's conjecture for the K-theoretic partition function [47]. See [48] for another proof.

By Proposition 1.38(1) the next coefficient $H(\vec{a};\Lambda,\boldsymbol{\beta})$ of the expansion (4.1) comes from the perturbation part:

Proposition 1.41. *Suppose* (1.37) *holds and assume* $|m| < r$.

$$H(\vec{a};\Lambda,\boldsymbol{\beta}) = -\pi\sqrt{-1}\langle\vec{a},\rho\rangle.$$

1.7.4. *Genus* 1 *parts.* Next we turn to the genus 1 parts of the expansion (4.1). When $r = 2$, $m = 0$, we determined A, B explicitly as theta constants in [43]. So we assume $r = 2$, $m = 1$. Let $F_1 = A - \frac{2}{3}B$, $G = \frac{1}{3}B$.

We have [43, (4.11)] if we replace d by $d - \frac{m}{2}$. (Note that this is $k = 0$ case.) Taking $d = 1$ (and $r = 2$, $m = 1$) we have

$$\exp(G - F_1) = \exp\left[-\frac{\boldsymbol{\beta}^2}{128}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}\right]\theta_{01}\left(\frac{\boldsymbol{\beta}}{16\pi\sqrt{-1}}\frac{\partial\mathcal{F}_0}{\partial\log\Lambda\partial a}\middle|\tau\right). \tag{1.42}$$

We assume

$$\widehat{Z}^{\text{inst}}_{m,k,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) = 0 \tag{1.43}$$

for $0 < k < r$, $0 < d < r$. Then we have [43, the first displayed equation in p.515] if we replace d by $d + m(-1/2 + k/r)$. We take $k = 1$, $d = 1$ (and $r = 2$, $m = 1$). Then we have exactly the same equation as in [43]. Therefore we get

$$G + F_1 = -\frac{1}{3}\log\left(-2\pi q^{\frac{1}{8}}\prod_{d=1}^{\infty}(1-q^d)^3\right) + C$$

where C is a function on Λ. Here $q = \exp(2\pi\sqrt{-1}\tau) = \exp(-d^2F/da^2)$ and the convention is different from that in [43]. Combining with (1.42), we get

$$\exp F_1 = C' q^{-1/48}\prod_{d=1}^{\infty}(1-q^d)^{-1/2}\exp\left[\frac{\boldsymbol{\beta}^2}{256}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}\right]\theta_{01}\left(\frac{\boldsymbol{\beta}}{16\pi\sqrt{-1}}\frac{\partial\mathcal{F}_0}{\partial\log\Lambda\partial a}\middle|\tau\right)^{-1/2}$$

for $C' = C'(\Lambda)$. By the same argument in [43, p.515] we have $C' \equiv 1$. Let us briefly recall the argument and explain how it is modified in our case. The proof is based on the observation that $\eta(\tau/2)\exp F_1$ depends on Λ in the form $\mathbb{C}[[\zeta_{1,2}\Lambda^4]]$, where $\zeta_{1,2} = \frac{\boldsymbol{\beta}}{1-e^{2\boldsymbol{\beta}a}}$ (see §4.1). There is an extra factor $\exp(m\boldsymbol{\beta}a(|Y^2| - |Y^1|))$ coming from the Chern-Simons terms. Hence the coefficient of Λ^{4n} is divisible by $\exp(mn\boldsymbol{\beta}a)\zeta_{1,2}^n$. Under our assumption $m=1$, we cannot get a term which is constant with respect to a. Therefore

(1.44)

$$\exp F_1 = q^{-1/48}\prod_{d=1}^{\infty}(1-q^d)^{-1/2}\exp\left[\frac{\boldsymbol{\beta}^2}{256}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}\right]\theta_{01}\left(\frac{\boldsymbol{\beta}}{16\pi\sqrt{-1}}\frac{\partial\mathcal{F}_0}{\partial\log\Lambda\partial a}\middle|\tau\right)^{-1/2},$$
$$\exp G = q^{-1/48}\prod_{d=1}^{\infty}(1-q^d)^{-1/2}\exp\left[-\frac{\boldsymbol{\beta}^2}{256}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}\right]\theta_{01}\left(\frac{\boldsymbol{\beta}}{16\pi\sqrt{-1}}\frac{\partial\mathcal{F}_0}{\partial\log\Lambda\partial a}\middle|\tau\right)^{1/2}.$$

2. Computation of the wallcrossing in terms of Hilbert schemes

Let X be a simply connected smooth projective surface with $p_g=0$. In this section we will compute the wallcrossing of the K-theoretic Donaldson invariants of X in terms of the holomorphic Euler characteristic of certain sheaves on Hilbert schemes of points on X. Later we will specialize to the case that X is a smooth toric surface and relate this result to the K-theoretic Nekrasov partition function.

Notation 2.1. Let t be a variable. If Y is a variety and $b\in H^*(Y)[t]$, we denote by $[b]_d$ its part of degree d, where elements in $H^{2n}(Y)$ have degree n and t has degree 1.

If R is a ring, t a variable and $b\in R((t))$, we will denote for $i\in\mathbb{Z}$ by $[b]_{t^i}$ the coefficient of t^i of b.

If E is a vector bundle of rank r on Y, let $\bigwedge_{-t}E := \sum_i(-1)^i\Lambda^i(E)t^i\in K(Y)[t]$, and let $S_t(E):=\sum_i S^i(E)t^i$, where $S^i(E)$ is the i^{th} symmetric power of E. Note that $S_t(E)=\frac{1}{\bigwedge_{-t}(E)}$.

2.1. The wallcrossing term.

Denote by $\mathcal{C}$ the ample cone of X. Then $\mathcal{C}$ has a chamber structure: For a class $\xi\in H^2(X,\mathbb{Z})\setminus\{0\}$ let $W^\xi := \{x\in\mathcal{C}\mid \langle x,\xi\rangle=0\}$. Assume $W^\xi\neq\emptyset$. Then we call ξ a *class of type* (c_1,d) and call W^ξ a *wall of type* (c_1,d) if the following conditions hold

(1) $\xi+c_1$ is divisible by 2 in $H^2(X,\mathbb{Z})$,
(2) $d+3+\xi^2\geq 0$.

We call ξ a *class of type* c_1, if $\xi+c_1$ is divisible by 2 in $H^2(X,\mathbb{Z})$. The *chambers of type* (c_1,d) are the connected components of the complement of the walls of type (c_1,d) in $\mathcal{C}$. Then $M_H^X(c_1,d)$ depends only on the chamber of type (c_1,d) of H.

Let $\xi \in H^2(X,\mathbb{Z})$ be a class of type c_1. We say that ξ *good* and W^ξ is a *good wall* if $D+K_X$ is not effective for any divisor D with $W^{c_1(D)} = W^\xi$. A sufficient condition for ξ to be good is that W^ξ contains an ample divisor H with $H\cdot K_X < 0$. One can show that an ample divisor H is general with respect to $(2,c_1,c_2)$ if and only if H lies in a chamber of type $(c_1, 4c_2 - c_1^2 - 3)$.

Let ξ be a class of type c_1. Let $X^{[n]}$ be the Hilbert scheme of subschemes of length n on X. Let $Z_n(X) \subset X \times X^{[n]}$ be the universal subscheme. Let $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) be the sheaf $p_{1,2}^*(\mathcal{I}_{Z_n(X)})$ (resp. $p_{1,3}^*(\mathcal{I}_{Z_m(X)})$ on $X\times X^{[n]}\times X^{[m]}$. We also denote $\mathcal{F}_1 := \mathcal{I}_1(\frac{c_1+\xi}{2})$ and $\mathcal{F}_2 := \mathcal{I}_{Z_2}(\frac{c_1-\xi}{2},)$. Note that $X^{[n]} = M_H^X(1,\frac{c_1+\xi}{2},n)$ and $X^{[m]} = M_H^X(1,\frac{c_1-\xi}{2},m)$ and $\mathcal{F}_1$, $\mathcal{F}_2$ are the corresponding universal sheaves. Let $f_1, f_2 \in K(X)$ be the classes of elements of $M_H^X(1,\frac{c_1+\xi}{2},n)$ and $M_H^X(1,\frac{c_1-\xi}{2},m)$ respectively.

Let $p : X\times X^{[n]}\times X^{[m]} \to X^{[n]}\times X^{[m]}$, $q : X\times X^{[n]}\times X^{[m]} \to X$ be the projections. Let $\mathcal{A}_{\xi,-} := -p_!(\mathcal{I}_2^\vee \otimes \mathcal{I}_1 \otimes q^!\xi)$, $\mathcal{A}_{\xi,+} := -p_!(\mathcal{I}_1^\vee\otimes\mathcal{I}_2\otimes q^!\xi^\vee) \in K(X^{[n]}\times X^{[m]})$. We also just write $\mathcal{A}_-$ and $\mathcal{A}_+$ instead of $\mathcal{A}_{\xi,-}$, $\mathcal{A}_{\xi,+}$.

Now assume ξ is good. Then $\operatorname{Ext}^0_p(\mathcal{I}_2,\mathcal{I}_1(\xi)) = \operatorname{Ext}^2_p(\mathcal{I}_2,\mathcal{I}_1(\xi)) = 0$ and we will write $\mathcal{A}_{\xi,-}$ for its representative $\operatorname{Ext}^1_p(\mathcal{I}_2,\mathcal{I}_1(\xi))$, which is a locally free sheaf on $X^{[n]}\times X^{[m]}$. Similarly we write $\mathcal{A}_{\xi,+}$ for the locally free sheaf $\operatorname{Ext}^1_p(\mathcal{I}_1,\mathcal{I}_2(-\xi))$, and we put $\mathbb{P}_- := \mathbb{P}(\mathcal{A}_-^\vee)$ and $\mathbb{P}_+ := \mathbb{P}(\mathcal{A}_+^\vee)$ (we use the Grothendieck notation, i.e. this is the bundle of 1-dimensional quotients). Let $\pi_\pm : \mathbb{P}_\pm \to X^{[n]}\times X^{[m]}$ be the projection.

Definition 2.2. Fix $c_1 \in H^2(X,\mathbb{Z})$, and let $v \in K(X)$. Let $\xi \in H^2(X,\mathbb{Z})$ be a class of type c_1. We denote $\chi(f_1\otimes v) = \chi(\mathcal{I}_{Z_1}(\frac{c_1+\xi}{2}\otimes v))$, $\chi(f_2\otimes v) = \chi(\mathcal{I}_{Z_2}(\frac{c_1-\xi}{2}\otimes v))$ for $(Z_1,Z_2)\in X^{[n]}\times X^{[m]}$. By the Riemann-Roch-Theorem we see that

$$\frac{1}{2}(\chi(f_2\otimes v) - \chi(f_1\otimes v)) = -\left\langle \frac{\xi}{2}, v^{(1)}\right\rangle + \frac{\operatorname{rk}(v)}{2}(n-m). \tag{2.3}$$

In particular it only depends on $\operatorname{rk}(v)$ and $c_1(v)$, and it is independent of n,m if $\operatorname{rk}(v)=0$.

The *wallcrossing terms* are

$$
\begin{aligned}
\Delta^X_{\xi,T}(v;\Lambda) &:= \sum_{\substack{n,m\geq 0\\ d=4(n+m)+\xi^2-3}} \frac{\Lambda^d}{T^{\frac{1}{2}(\chi(f_2\otimes v)-\chi(f_1\otimes v))}} \\
&\qquad\times \chi\Big(X^{[n]}\times X^{[m]}, \frac{\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v)}{\bigwedge_{-T}(\mathcal{A}^\vee_{\xi,+})\bigwedge_{-T^{-1}}(\mathcal{A}^\vee_{\xi,-})}\Big), \\
\Delta^X_{\xi}(v;\Lambda) &:= [\Delta^X_{\xi,T}(v;\Lambda)]_{T^0} - [\Delta^X_{\xi,T}(v;\Lambda)]_{(T^{-1})^0}.
\end{aligned}
\tag{2.4}
$$

Here the right hand side of the first equation is understood as a rational function in $T^{1/2}$ as follows, and $[\bullet]_{T^0}$, $[\bullet]_{(T^{-1})^0}$ denote the constant terms of the expansions at $T^{1/2}=0$, $T^{1/2}=\infty$ respectively. We formally apply the Hirzebruch-Riemann-Roch theorem to get

$$
\begin{aligned}
&\chi\Big(X^{[n]}\times X^{[m]}, \frac{\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v)}{\bigwedge_{-T}(\mathcal{A}^\vee_{\xi,+})\bigwedge_{-T^{-1}}(\mathcal{A}^\vee_{\xi,-})}\Big) \\
&\qquad = \int_{X^{[n]}\times X^{[m]}} \frac{\operatorname{ch}(\lambda_{\mathcal{F}_1}(v))\operatorname{ch}(\lambda_{\mathcal{F}_2}(v))}{\operatorname{ch}\bigwedge_{-T}(\mathcal{A}^\vee_{\xi,+})\operatorname{ch}\bigwedge_{-T^{-1}}(\mathcal{A}^\vee_{\xi,-})} \operatorname{Todd}(X^{[n]}\times X^{[m]}).
\end{aligned}
$$

Then we consider $\operatorname{ch}\bigwedge_{-T}(\mathcal{A}^\vee_{\xi,+})$, $\operatorname{ch}\bigwedge_{-T^{-1}}(\mathcal{A}^\vee_{\xi,-})$ as $\operatorname{End}(H^*(X^{[n]}\times X^{[m]}))$-valued Laurent polynomials. Their inverses are defined as their cofactor matrices divided by their determinants (which are equal to $(1-T)^{\operatorname{rk}(\mathcal{A}^\vee_{\xi,+})}$, $(1-T^{-1})^{\operatorname{rk}(\mathcal{A}^\vee_{\xi,-})}$) respectively. Then their inverses are in $\operatorname{End}(H^*(X^{[n]}\times X^{[m]}))\otimes_{\mathbb{Q}}\mathbb{Q}(T)$. Thus the integral is an element of $\mathbb{Q}(T)$. This way of understanding the formula will become more apparent when we will consider the equivariant wallcrossing term in §3. In that case we can interpret the formula so that the computation is done in the localized equivariant K-theory.

The expansions at $T=0$, $T=\infty$ can be also understood differently. Note that for a vector bundle E of rank r we have

$$
\frac{1}{\bigwedge_{-T}(E)} = S_T(E), \quad \frac{1}{\bigwedge_{-T^{-1}}(E)} = \frac{(-T)^r}{\det(E)\otimes\bigwedge_{-T}E^\vee} = (-T)^r\det(E^\vee)\otimes S_T(E^\vee).
\tag{2.5}
$$

Thus $\Delta^X_{\xi,T}(v;\Lambda)$ can be developed as Laurent series both in $T^{1/2}$ and in $\frac{1}{T^{1/2}}$

(2.6)

$$\Delta^X_{\xi,T}(v;\Lambda)=\sum_{\substack{n,m\geq 0\\ d=4(n+m)-\xi^2-3}}\frac{\Lambda^d(-T)^{\mathrm{rk}(\mathcal{A}_-)}}{T^{\frac{1}{2}(\chi(f_2\otimes v)-\chi(f_1\otimes v))}}\chi\Big(X^{[n]}\times X^{[m]},\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v)\otimes \det(\mathcal{A}_{\xi,-})\otimes S_T(\mathcal{A}_{\xi,+}^\vee)\otimes S_T(\mathcal{A}_{\xi,-})\Big)\in\mathbb{Z}((T^{\frac{1}{2}}))[[\Lambda]],$$

(2.7)

$$\Delta^X_{\xi,T}(v;\Lambda)=\sum_{\substack{n,m\geq 0\\ d=4(n+m)-\xi^2-3}}\frac{\Lambda^d(-T^{-1})^{\mathrm{rk}(\mathcal{A}_+)}}{(T^{-1})^{\frac{1}{2}(\chi(f_1\otimes v)-\chi(f_2\otimes v))}}\chi\Big(X^{[n]}\times X^{[m]},\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v)\otimes\det(\mathcal{A}_{\xi,+})\otimes S_{T^{-1}}(\mathcal{A}_{\xi,+})\otimes S_{T^{-1}}(\mathcal{A}_{\xi,-}^\vee)\Big)\in\mathbb{Z}((T^{-\frac{1}{2}}))[[\Lambda]].$$

Then $[\Delta^X_{\xi,T}(v;\Lambda)]_{T^0}$ is equal to the coefficient of T^0 of (2.6) and $[\Delta^X_{\xi,T}(v;\Lambda)]_{(T^{-1})^0}$ is equal to the coefficient of $(\frac{1}{T})^0$ of (2.7). However note that it was not clear the expressions are in $\mathbb{Z}((T))[[\Lambda]]$ or $\mathbb{Z}((T^{-1}))[[\Lambda]]$ in the original formulation in terms of the Hirzebruch-Riemann-Roch theorem.

Remark 2.8. Fix c_1, d and let $c\in K(X)$ be the class of an element in $M^X_H(c_1,d)$. Let $v\in K(X)$. Then either $\frac{1}{2}(\chi(f_2\otimes v)-\chi(f_1\otimes v))\in\mathbb{Z}$ for all $n,m\in\mathbb{Z}_{\geq 0}$ with $4(n+m)-\xi^2-3=d$, or $\frac{1}{2}(\chi(f_2\otimes v)-\chi(f_1\otimes 0))\in\mathbb{Z}+\frac{1}{2}$ for all such n,m. In the second case the coefficients of Λ^d of $[\Delta_{\xi,T}(v,\Lambda)]_{T^0}$ and $[\Delta_{\xi,T}(v,\Lambda)]_{(T^{-1})^0}$ are trivially 0.

On the other hand, if $v\in K_c$, then $\chi(f_2\otimes v)=-\chi(f_1\otimes v)$ and thus $\frac{1}{2}(\chi(f_2\otimes v)-\chi(f_1\otimes v))=\chi(f_2\otimes v)\in\mathbb{Z}$.

Remark 2.9. Let $v\in K(X)$ be a class of rank 0. Let ξ be a wall of type (c_1,d). Let $l=\frac{d+3+\xi^2}{4}$. Fix $l\geq 0$. Write $d:=4l-\xi^2-3$. Note that by [12, Lemma 4.3]

$$\mathrm{rk}(\mathcal{A}_-)=-\frac{\xi(\xi-K_X)}{2}+l-1,\quad \mathrm{rk}(\mathcal{A}_+)=-\frac{\xi(\xi+K_X)}{2}+l-1. \tag{2.10}$$

Note that by definition the coefficient of Λ^d of $\Delta^X_\xi(v;\Lambda)$ is zero if $-\mathrm{rk}(\mathcal{A}_+)<-\langle\xi/2,c_1(v)\rangle<\mathrm{rk}(\mathcal{A}_-)$. By (2.10) it thus follows that the coefficient of Λ^d of $\Delta^X_\xi(L;\Lambda)$ is 0 unless $0\leq l\leq|\langle\frac{\xi}{2},c_1(v)+K_X\rangle|+1+\frac{\xi^2}{2}$, which is equivalent to $-\xi^2-3\leq d\leq\xi^2+|\langle 2\xi,c_1(v)+K_X\rangle|+1$. In particular $\Delta^X_\xi(v;\Lambda)\in\mathbb{C}[\Lambda]$.

The aim of this section is to prove that the wallcrossing for the K-theoretic Donaldson invariants can be expressed as a sum over $\Delta^X_\xi(v;\Lambda)$.

Proposition 2.11. *Fix c_1, d, let $c \in K(X)$ be the class of an element of $M_H^X(c_1,d)$. Let $v \in K_c$. Let H_-, H_+ be ample divisors on X, which do not lie on a wall of type (c_1,d). Let B_+ be the set of classes ξ of type (c_1,d) with $\langle \xi \cdot H_+\rangle > 0 > \langle \xi \cdot H_-\rangle$. Assume that all classes in B_+ are good. Then*

$$\chi(M_{H_+}^X(c_1,d),\lambda(v)) - \chi(M_{H_-}^X(c_1,d),\lambda(v)) = \sum_{\xi\in B_+}\left[\Delta_\xi^X(v;\Lambda)\right]_{\Lambda^d}.$$

In the rest of this section we will show Prop. 2.11.

$M_H^X(c_1,d)$ and thus $\chi(M_H^X(c_1,d),\lambda(v))$ is constant as long as H stays in the same chamber of type (c_1,d) and only changes when H crosses a wall of type (c_1,d). By [12], [16] the change of the moduli spaces can be described as follows. Let B_d be the set of all $\xi \in B_+$ which define a wall of type (c_1,d). For the moment assume for simplicity that B_d consists of a single element ξ. Let $l := (d+3+\xi^2)/4 \in \mathbb{Z}_{\geq 0}$. Write $M_{0,l} := M_{H_-}^X(c_1,d)$. Then successively for all $n = 0,\ldots,l$ write $m := l-n$. Then one has the following: $M_{n,m}$ contains a closed subscheme $E_-^{n,m}$ isomorphic to $\mathbb{P}_-^{n,m}$ and $M_{n,m}$ is nonsingular in a neighbourhood of $E_-^{n,m}$. Let $\widehat{M}_{n,m}$ be the blow up of $M_{n,m}$ along $E_-^{n,m}$. The exceptional divisor is isomorphic to the fibre product $D^{n,m} := \mathbb{P}_-^{n,m}\times_{X^{[n]}\times X^{[m]}}\mathbb{P}_+^{n,m}$. We can blow down $\widehat{M}_{n,m}$ in $D^{n,m}$ in the other fibre direction to obtain a new variety $M_{n+1,m-1}$. The image of $D^{n,m}$ is a closed subset $E_+^{n,m}$ isomorphic to $\mathbb{P}_+^{n,m}$ and $M_{n+1,m-1}$ is smooth in a neighbourhood of $E_+^{n,m}$.

The transformation from $M_{n,m}$ to $M_{n+1,m-1}$ does not have to be birational. It is possible that $E_+^{n,m} = \emptyset$, i.e. $\mathcal{A}_+ = 0$. As $\mathrm{rk}(\mathcal{A}_-) + \mathrm{rk}(\mathcal{A}_+) + 2l = d+1$, this happens if and only if $E_-^{n,m}$ has dimension d and thus by the smoothness of $M_{n,m}$ near $E_-^{n,m}$, we get that $E_-^{n,m}$ is a connected component of $M_{n,m}$. Then blowing up along $E_-^{n,m}$ just means deleting $E_-^{n,m}$. Thus in this case $M_{n+1,m-1} = M_{n,m}\setminus E_-^{n,m}$. Similarly we have $E_-^{n,m} = \emptyset$, i.e. $\mathcal{A}_- = 0$, if and only if $E_+^{n,m}$ is a connected component of $M_{n+1,m-1}$ and $M_{n+1,m-1} = M_{n,m}\sqcup E_+^{n,m}$. Below, if the transformation from $M_{n,m}$ to $M_{n+1,m-1}$ is birational, we say we are in case (1), otherwise in case (2). Finally we have $M_{l+1,-1} = M_{H_+}^X(c_1,d)$. If B_d consists of more than one element, one obtains $M_{H_+}(c_1,d)$ from $M_{H_-}(c_1,d)$ by iterating this procedure in a suitable order over all $\xi \in B_+$.

Fix ξ in B_d. Fix $n, m \in \mathbb{Z}_{\geq 0}$ with $n+m = l := (d+3+\xi^2)/4$. We write $M_- := M_{n,m}$, $M_+ := M_{n+1,m-1}$. Let $\overline{\mathcal{E}}_\pm$ be universal sheaves on $X\times M_\pm$ respectively. Let $E_- := E_-^{n,m}$, $E_+ = E_+^{n,m}$. Let $\widetilde{M}$ be the blowup of M_- along

E_-, and denote by D the exceptional divisor (which is also the exceptional divisor of the blowup of M_+ along E_+). Write $D' := X \times D$ and let $j : D \to \widetilde{M}$, $j' : X \times D \to X \times \widetilde{M}$ be the embeddings. Let $\mathcal{E}_-$, $\mathcal{E}_+$ be the pullbacks of $\overline{\mathcal{E}}_-$, $\overline{\mathcal{E}}_+$ to $X \times \widetilde{M}$.

Notation 2.12. We denote by T_- (resp. T_+) the universal quotient line bundle on $\mathbb{P}_- = \mathbb{P}(\mathcal{A}_-^\vee)$ (resp. on $\mathbb{P}_+ = \mathbb{P}(\mathcal{A}_+^\vee)$). For a class $a \in H^*(X)$ we also denote by a its pullback to $X \times Y$ for a variety Y. We write $\mathcal{I}_1, \mathcal{I}_2$ also for the pullback of $\mathcal{I}_1$, $\mathcal{I}_2$ to D' and we write T_+, T_- also for their pullbacks to D and D'.

By the condition $\chi(c \otimes v) = 0$, we can replace $\frac{1}{2}(\chi(f_2 \otimes v) - \chi(f_1 \otimes v))$ by $\chi(f_2 \otimes v)$. We will show

(2.13)

$$\begin{aligned}
&\chi(M_+, \lambda_{\overline{\mathcal{E}}_+}(v)) - \chi(M_-, \lambda_{\overline{\mathcal{E}}_-}(v)) \\
&= \chi\Bigg(X^{[n]} \times X^{[m]}, \lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v) \otimes \Bigg(\Big[\frac{(-t)^{\mathrm{rk}(\mathcal{A}_-)} S_t(\mathcal{A}_+^\vee) \otimes S_t(\mathcal{A}_-) \otimes \det(\mathcal{A}_-)}{t^{\chi(f_2 \otimes v)}} \Big]_{t^0} \\
&\qquad\qquad - \Big[\frac{(-t)^{\mathrm{rk}(\mathcal{A}_+)} S_t(\mathcal{A}_-^\vee) \otimes S_t(\mathcal{A}_+) \otimes \det(\mathcal{A}_+)}{t^{-\chi(f_2 \otimes v)}} \Big]_{t^0} \Bigg) \Bigg).
\end{aligned}$$

Formula (2.13) implies Proposition 2.11 by summing over all $\xi \in B_+$, and over all n, m with $n + m = (d + \xi^2 + 3)/4$.

Assume first that we are in case (1). Note that this is equivalent to both $\mathrm{rk}(\mathcal{A}_-)$ and $\mathrm{rk}(\mathcal{A}_+)$ strictly positive, and then it is evident that the first (resp. second) summand on the left hand side of (2.13) vanishes if $\chi(f_2 \otimes v) \le 0$ (resp. if $\chi(f_2 \otimes v) \ge 0$). Let $\pi_\pm : \widetilde{M} \to M_\pm$ be the blowup morphisms. By [17, Prop. VI.4.1] and its proof, $R^i \pi_{\pm *} \mathcal{O}_{\widetilde{M}} = 0$ for $i > 0$, and $\pi_{\pm *} \mathcal{O}_{\widetilde{M}} = \mathcal{O}_{M_\pm}$. Thus the projection formula gives $\chi(M_\pm, L) = \chi(\widetilde{M}, \pi_\pm^* L)$ for any line bundle L on $M_\pm$. Therefore it is enough to prove (2.13) with the left-hand side replaced by $\chi(\widetilde{M}, \lambda_{\mathcal{E}_+}(v)) - \chi(\widetilde{M}, \lambda_{\mathcal{E}_-}(v))$.

Lemma 2.14. *In $K(\widetilde{M})$ we have*

$$\lambda_{\mathcal{E}_+}(v) - \lambda_{\mathcal{E}_-}(v) = j_* \Bigg(\Big(\frac{t^{\chi(f_2 \otimes v)} - s^{-\chi(f_2 \otimes v)}}{1 - st} \pi^* (\lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v) \Big) \Big|_{\substack{s = T_- \\ t = T_+}} \Bigg).$$

Proof. By [12, section 5] there exists a line bundle μ on D such that

$$(2.15)\qquad \begin{aligned} (j')^*\mathcal{E}_- &= \mathcal{F}_1\otimes\mu + \mathcal{F}_2\otimes T_-^{-1}\otimes\mu \text{ in } K(D'),\\ \mathcal{E}_+ &= \mathcal{E}_- - j'_*(\mathcal{F}_2\otimes T_-^{-1}\otimes\mu) \text{ in } K(X\times\widetilde{M}). \end{aligned}$$

Thus we get $\lambda_{\mathcal{E}_+}(v) = \lambda_{\mathcal{E}_-}(v)\otimes\det\big(p_!(v\otimes j'_*(\mathcal{F}_2\otimes T_-^{-1}\otimes\mu))\big)^{-1}$. Note that $p_!(v\otimes j'_*(\mathcal{F}_2\otimes T_-^{-1}\otimes\mu))$ is a coherent sheaf of rank $\chi(v\otimes f_2)$ on D. As D is a Cartier divisor, it follows that

$$\lambda_{\mathcal{E}_+}(v) = \lambda_{\mathcal{E}_-}(v)\otimes\det(\chi(f_2\otimes v)[\mathcal{O}_D])^{-1} = \lambda_{\mathcal{E}_-}(v)\otimes\det(\mathcal{O}_{\widetilde{M}}(-D))^{\chi(f_2\otimes v)}.$$

For the second equality we have used that $\mathcal{O}_D = \mathcal{O}_{\widetilde{M}} - \mathcal{O}_{\widetilde{M}}(-D)$ in $K(\widetilde{M})$ and thus $\det(\mathcal{O}_D) = \det(\mathcal{O}_{\widetilde{M}}(-D))^{-1}$. Thus we get in $K(\widetilde{M})$ that

$$\begin{aligned} \lambda_{\mathcal{E}_+}(v) - \lambda_{\mathcal{E}_-}(v) &= (\mathcal{O}_{\widetilde{M}}(-D)^{\chi(f_2\otimes v)} - 1)\otimes\lambda_{\mathcal{E}_-}(v)\\ &= j_*\Big(\Big(\frac{t^{\chi(f_2\otimes v)}-1}{1-t}j^*(\lambda_{\mathcal{E}_-}(v))\Big)|_{t=T_+\otimes T_-}\Big). \end{aligned}$$

In the last step we have used that for a locally free sheaf $\mathcal{G}$ on $\widetilde{M}$ we have $(1-\mathcal{O}_{\widetilde{M}}(-D))\otimes\mathcal{G} = j_*(j^*\mathcal{G})$ in $K(\widetilde{M})$. As the determinant bundles are compatible with pullback we obtain by (2.15) that

$$(2.16)\qquad \begin{aligned} j^*(\lambda_{\mathcal{E}_-}(v)) &= \lambda_{\pi^*(\mathcal{F}_1)\otimes\mu}(v)\otimes\lambda_{\pi^*(\mathcal{F}_2)\otimes T_-^{-1}\otimes\mu}(v)\\ &= \pi^*(\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v))\otimes\mu^{\chi(f_1\otimes v)+\chi(f_2\otimes v)}T_-^{-\chi(f_2\otimes v)}\\ &= \pi^*(\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v))\otimes T_-^{-\chi(f_2\otimes v)}. \end{aligned}$$

In the last step we use that $\chi(f_1\otimes v)+\chi(f_2\otimes v) = 0$. The result follows. □

In case $\chi(f_2\otimes v) = 0$ the left hand side of Lemma 2.14 is obviously 0. Thus we only need to show (2.13) in the cases $\chi(f_2\otimes v) > 0$ and $\chi(f_2\otimes v) < 0$.

(a) $\chi(f_2\otimes v) < 0$: We apply the formula

$$(2.17)\qquad \frac{y^{-c}-x^c}{1-xy} = y^{-1}\frac{x^c-y^{-c}}{x-y^{-1}} = \sum_{\substack{a+b=c\\ a\ge 0,\ b>0}} x^a y^{-b},\quad c\in\mathbb{Z}_{>0}$$

for $x = T_-$, $y = T_+$ to Lemma 2.14 to obtain

$$\lambda_{\mathcal{E}_+}(v) - \lambda_{\mathcal{E}_-}(v) = j_*\Big(\sum_{\substack{a+b=-\chi(f_2\otimes v)\\ a\ge 0,\ b>0}} T_-^a\otimes T_+^{-b}\otimes\pi^*(\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v))\Big)$$

in $K(\widetilde{M})$. Let $\mathcal{E}$ be a vector bundle of rank e on a variety Y and let $p : \mathbb{P}(\mathcal{E}^\vee) \to Y$ be the projection and $\mathcal{O}(1)$ the universal quotient line bundle on $\mathbb{P}(\mathcal{E}^\vee)$. Then by [23, Ex. III.8.4]

$$p_!(\mathcal{O}(n)) = \begin{cases} S^n(\mathcal{E}^\vee) & n \geq 0, \\ (-1)^{e-1} S^{-n-e}(\mathcal{E}) \otimes \det(\mathcal{E}) & n \leq -e, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi : D = \mathbb{P}(\mathcal{A}_+^\vee) \times_{X^{[n]} \times X^{[m]}} \mathbb{P}(\mathcal{A}_+^\vee) \to X^{[n]} \times X^{[m]}$ be the projection. Then we get using the projection formula

(2.18)

$$\chi(\widetilde{M}, \lambda_{\mathcal{E}_+}(v)) - \chi(\widetilde{M}, \lambda_{\mathcal{E}_-}(v)) = \sum_{\substack{a+b=-\chi(f_2\otimes v)\\ a\geq 0,\ b>0}} \chi\big(D, T_-^a \otimes T_+^{-b} \otimes \pi^*(\lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v))\big)$$

$$= -\sum_{\substack{a+b=-\chi(f_2\otimes v)\\ a\geq 0,\ b>0}} (-1)^{\operatorname{rk}(\mathcal{A}_+)} \chi\Big(X^{[n]} \times X^{[m]}, \lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v)$$

$$\otimes S^a(\mathcal{A}_-^\vee) \otimes S^{b-\operatorname{rk}(\mathcal{A}_+)}(\mathcal{A}_+) \otimes \det(\mathcal{A}_+)\Big)$$

$$= \chi\Big(X^{[n]} \times X^{[m]}, \lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v)$$

$$\otimes \Big[-\frac{(-t)^{\operatorname{rk}(\mathcal{A}_+)} \otimes S_t(\mathcal{A}_-^\vee) \otimes S_t(\mathcal{A}_+) \otimes \det(\mathcal{A}_+)}{t^{-\chi(f_2\otimes v)}}\Big]_{t^0}\Big).$$

(b) $\chi(f_2 \otimes v) > 0$: The formula (2.17) for $x = T_+$, $y = T_-$, gives

$$\lambda_{\mathcal{E}_+}(v) - \lambda_{\mathcal{E}_-}(v) = -j_*\Big(\sum_{\substack{a+b=\chi(f_2\otimes v)\\ a\geq 0,\ b>0}} T_+^a \otimes T_-^{-b} \otimes \pi^*(\lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v))\Big).$$

Then the same arguments as in the case $\chi(f_2 \otimes v) < 0$ show that

$$-\sum_{\substack{a+b=\chi(f_2\otimes v)\\ a\geq 0,\ b>0}} \chi\Big(D, \pi^*(\lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v)) \otimes T_-^a \otimes T_+^{-b})\Big)$$

$$= \chi(X^{[n]} \times X^{[m]}, \lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v)$$

$$\otimes \Big[\frac{(-t)^{rk(\mathcal{A}_-)} \otimes S_t(\mathcal{A}_+^\vee) \otimes S_t(\mathcal{A}_-) \otimes \det(\mathcal{A}_-))}{t^{\chi(f_2\otimes v)}}\Big]_{t^0}\Big).$$

In case (2), we can assume by symmetry that $\mathbb{P}_+ = \emptyset$, thus $\mathcal{A}_+ = 0$ and $\mathcal{A}_-$ has rank $d + 1 - 2(n + m)$. Then we have

$$\chi(M_+, \lambda_{\overline{\mathcal{E}}_+}(v)) - \chi(M_-, \lambda_{\overline{\mathcal{E}}_-}(v)) = -\chi(\mathbb{P}_-, \lambda_{(j')^*(\overline{\mathcal{E}}_-)}(v))$$

where $j' : X \times \mathbb{P}_- \to X \times M_-$ is the inclusion. The same argument as in the proof of (2.16) shows that

$$\lambda_{(j')^*(\overline{\mathcal{E}}_-)}(v) = \pi_-^*(\lambda_{\mathcal{F}_1(v)} \otimes \lambda_{\mathcal{F}_2}(v)) \otimes T_-^{-\chi(f_2 \otimes v)}.$$

Note that differently from case (1) this is not zero when $\chi(f_2 \otimes v) = 0$. Now the same arguments as in the proof of (2.18) show that $-\chi\big(\mathbb{P}_-, \lambda_{(j')^*(\overline{\mathcal{E}}_-)}(v)\big)$ is equal to

$$-\chi\big(X^{[n]} \times X^{[m]}, \lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v) \otimes S^{-\chi(f_2 \otimes v)}(\mathcal{A}_-^\vee)\big)$$

in case $\chi(f_2 \otimes v) \leq 0$, and to

$$(-1)^{\mathrm{rk}(\mathcal{A}_-)}\chi(X^{[n]} \times X^{[m]}, \lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v) \otimes S^{\chi(f_2 \otimes v) - \mathrm{rk}(\mathcal{A}_-)}(\mathcal{A}_-) \otimes \det(\mathcal{A}_-))$$

in case $\chi(f_2 \otimes v) > 0$. As $S_t(\mathcal{A}_+) = 1$, this shows (2.13) also in case (2) and thus finishes the proof of Proposition 2.11.

3. Comparison with the partition function

For the next two sections (except in §4.6) let X be a smooth projective toric surface over $\mathbb{C}$, in particular X is simply connected and $p_g(X) = 0$. X carries an action of $\Gamma := \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, which we will denote by $p_1, \ldots, p_\chi$, where χ is the Euler number of X. Let $w(x_i)$, $w(y_i)$ the weights of the Γ-action on $T_{p_i}X$. Then there are local coordinates x_i, y_i at p_i, so that $(t_1, t_2)x_i = e^{-w(x_i)}x_i$ and $(t_1, t_2)y_i = e^{-w(y_i)}y_i$. By definition $w(x_i)$ and $w(y_i)$ are linear forms in ε_1 and ε_2. For $\beta \in H^*_\Gamma(X)$ or $\beta \in H_*^\Gamma(X)$, we denote by $\iota_{p_i}^*\beta$ its pullback to the fixpoint p_i. More generally, if Γ acts on a nonsingular variety Y and $W \subset Y$ is invariant under the Γ-action, we denote by $\iota_W^* : H^*_\Gamma(Y) \to H^*_\Gamma(W)$ the pullback homomorphism.

Note that T_X and the canonical bundle are canonically equivariant. Thus any polynomial in the Chern classes $c_i(X)$ and K_X is canonically an element of $H^*_\Gamma(X)$.

3.1. **Equivariant K-theoretic Donaldson invariants and equivariant wall-crossing.** For $t \in \Gamma$ denote by F_t the automorphism $X \to X; x \mapsto t \cdot x$. Then Γ acts on $X^{[n]} \times X^{[m]}$ by $t \cdot (\mathcal{I}_{Y_1}, \mathcal{I}_{Y_2}) = ((F_t^{-1})^*\mathcal{I}_{Y_1}, (F_t^{-1})^*\mathcal{I}_{Y_2})$ and on $X \times X^{[n]} \times X^{[m]}$ by $t \cdot (x, \mathcal{I}_{Y_1}, \mathcal{I}_{Y_2}) = (F_t(x), (F_t^{-1})^*\mathcal{I}_{Y_1}, (F_t^{-1})^*\mathcal{I}_{Y_2})$ and the sheaves $\mathcal{I}_1$, $\mathcal{I}_2$ are Γ-equivariant. If we choose an equivariant lifting of c_1 and ξ, then also $\mathcal{F}_1$, $\mathcal{F}_2$ are Γ-equivariant sheaves.

We write $X_2 := X \sqcup X$ and $X_2^{[l]} := \coprod_{n+m=l} X^{[n]} \times X^{[m]}$. The fixpoints of the Γ-action on $X_2^{[l]}$ are the pairs (Z_1, Z_2) of zero-dimensional subschemes with support in $\{p_1, \dots, p_\chi\}$ with $\operatorname{len}(Z_1) + \operatorname{len}(Z_2) = l$ and such that each I_{Z_α, p_i} is generated by monomials in x_i, y_i. We associate to (Z_1, Z_2) the χ-tuple $(\vec{Y}^1, \dots, \vec{Y}^\chi)$ with $\vec{Y}^i = (Y_1^i, Y_2^i)$, where

$$Y_\alpha^i = \left\{(n, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \;\middle|\; x_i^{n-1} y_i^{m-1} \notin I_{Z_\alpha, p_i}\right\}.$$

We write $|Y_\alpha^i|$ for the number of elements of Y_α^i and $|\vec{Y}^i| := |Y_1^i| + |Y_2^i|$. This gives a bijection from the fixpoint set $(X_2^{[l]})^\Gamma$ to the set of the χ-tuples of pairs of Young diagrams $(\vec{Y}^1, \dots, \vec{Y}^\chi)$, with $\sum_i |\vec{Y}^i| = l$.

Similarly Γ acts on $X \times M_X^H(c_1, c_2)$ by $t \cdot (x, E) = (F_t(x), (F_t^{-1})^* E)$. Assume for the moment that there exist a universal sheaf $\mathcal{E}$ over $X \times M_X^H(c_1, d)$, then one can show that $\mathcal{E}$ has a lifting to a Γ-equivariant sheaf, unique up to twist by a character.

The definition of the determinant bundles and the K-theoretic Donaldson invariants is easily generalized to the equivariant case. If Y is a variety with an action of Γ, we denote by $K^\Gamma(Y)$, $K^{0\Gamma}(X)$ the Grothendieck groups of Γ-equivariant coherent sheaves and Γ-equivariant locally free sheaves respectively. $\chi(u \otimes v) : K^\Gamma(X)^2 \to \mathbb{Z}$ is still a quadratic form. The formula (1.1) defines a homomorphism $K^\Gamma(X) \to \operatorname{Pic}^\Gamma(S)$, where now S is a scheme with a Γ-action, and $\mathcal{E}$ a flat family of Γ-equivariant coherent sheaves of class $c \in K(X)_{\mathrm{num}}$ on X, flat over S. For $c \in K(X)_{\mathrm{num}}$ we define $K_c^\Gamma, K_{c,H}^\Gamma \subset K^\Gamma(X)$ by the same formula as in section 1.1. In the same way as in 1.1, there are homomorphisms $\lambda\colon K_c^\Gamma \to \operatorname{Pic}^\Gamma(M_H^X(r, c_1, c_2)_s)$, $\lambda\colon K_{c,H}^\Gamma \to \operatorname{Pic}^\Gamma(M_H^X(r, c_1, c_2))$, which commute with the inclusions $K_{c,H}^\Gamma \subset K_c^\Gamma$ and $\operatorname{Pic}^\Gamma(M_H^X(r, c_1, c_2)_s) \subset \operatorname{Pic}^\Gamma(M_H^X(r, c_1, c_2))$. If H is general with respect to (r, c_1, c_2), then $\lambda\colon K_{c,H}^\Gamma \to \operatorname{Pic}^\Gamma(M_H^X(r, c_1, c_2))$ can be extended to K_c. For a flat family $\mathcal{E}$ of equivariant stable sheaves on X parametrized by S, λ and $\lambda_{\mathcal{E}}$ commute with the pullback $\phi_{\mathcal{E}}^* : \operatorname{Pic}^\Gamma(M_H^X(r, c_1, c_2)) \to \operatorname{Pic}^\Gamma(S)$ by the classifying morphism.

Let $v \in K_c^\Gamma$, where c is the class of an element of $M_H^X(c_1, d)$, where $d = 4c_2 - c_1^2 - 3$. Assume that H is general with respect to $(2, c_1, c_2)$. If Y is a variety with a Γ-action and $w \in K^{0\Gamma}(Y)$, we denote

$$\widetilde{\chi}(Y, w) := \pi_!(w) \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}], \tag{3.1}$$

where $\pi : Y \to pt$ is the projection to a point. The *equivariant K-theoretic Donaldson invariant* of X with respect to v, c_1, d, H is $\widetilde{\chi}(M^X_H(c_1,d), \lambda(v))$. If L is a Γ equivariant line bundle on X with $\langle c_1(L), c_1\rangle$ even, let $v(L) \in K^\Gamma_c$ be an equivariant lift of the class defined by (1.4) and $\mu(L) := \lambda(v(L)) \in \mathrm{Pic}^\Gamma(M^X_H(c_1,d))$. We put $\widetilde{\chi}(M^X_H(c_1,d), \mu(L))$ and $\widetilde{\chi}^H_{c_1}(L;\Lambda) := \sum_{d\geq 0}\Lambda^d\widetilde{\chi}(M^X_H(c_1,d),\mu(L))$.

Definition 3.2. Let $v \in K(X)$. Let $\xi \in H^2(X,\mathbb{Z})$ be an equivariant lifting of a class of type c_1. Then $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{F}_1$, $\mathcal{F}_2$. $\mathcal{A}_{\xi,+}$ and $\mathcal{A}_{\xi,-}$ are in a natural way equivariant sheaves on $X^{[n]}\times X^{[m]}$ (resp. elements in $K^\Gamma(X^{[n]}\times X^{[m]})$), and the equivariant wallcrossing terms $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)$, $\widetilde{\Delta}^X_{\xi}(v;\Lambda)$ are defined by the right-hand side of formulas (2.4), with the holomorphic Euler characteristic χ replaced by the equivariant pushforward $\widetilde{\chi}$ to a point. Now $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)$ can be understood by localization in equivariant K-theory on $X^{[n]}\times X^{[m]}$. Then $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda) \in \Lambda^{-\xi^2-3}\mathbb{Q}(t_1,t_2,T^{\frac{1}{2}})[[\Lambda]]$. Then using (2.6) we can view $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)$ as an element of $\Lambda^{-\xi^2-3}\mathbb{Q}[t_1^{\pm1},t_2^{\pm1}]((T^{\frac{1}{2}}))[[\Lambda]]$, and $[\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)]_{T^0}$ is its coefficient of T^0. Similarly using (2.7), $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)$ is an element of $\Lambda^{-\xi^2-3}\mathbb{Q}[t_1^{\pm1},t_2^{\pm1}]((T^{-\frac{1}{2}}))[[\Lambda]]$, and $[\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)]_{(T^{-1})^0}$ is its coefficient of $(T^{-1})^0$. In particular $\widetilde{\Delta}^X_{\xi}(v;\Lambda) \in \mathbb{Q}[t_1^{\pm1},t_2^{\pm1}][[\Lambda]]$, and $\widetilde{\Delta}^X_{\xi}(v;\Lambda)|_{t_1=t_2=1} = \Delta^X_\xi(v;\Lambda)$.

Let $c \in K(X)$ be the class of an element of $M^X_H(c_1,d)$. In the same way as in Remark 2.8, we see that the coefficient of Λ^d of $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)$ is either in $T^{\frac{1}{2}}\mathbb{Q}(t_1,t_2,T)$ (and the coefficient of Λ^d of $\widetilde{\Delta}^X_{\xi}(v;\Lambda)$ is 0) or in $\mathbb{Q}(t_1,t_2,T)$. If $v \in K^\Gamma_c$, then the coefficient is in $\mathbb{Q}(t_1,t_2,T)$.

Let $v \in K^\Gamma_c$. Under the assumptions of Proposition 2.11 let $\widetilde{B}_+$ be a set consisting of one equivariant lift ξ for each class of type (c_1,d) with $\langle \xi\cdot H_+\rangle > 0 > \langle \xi\cdot H_-\rangle$. Then the same proof as before (with all sheaves and classes replaced by their equivariant versions) shows that

$$\widetilde{\chi}(M^X_{H_+}(c_1,d),\lambda(v)) - \widetilde{\chi}(M^X_{H_-}(c_1,d),\lambda(v)) = \sum_{\xi\in B_+}\left[\widetilde{\Delta}^X_\xi(v;\Lambda)\right]_{\Lambda^d}.$$

Now we want to give a formula expressing $\widetilde{\Delta}^X_{\xi,T}(v;\Lambda)$ in terms of the K-theoretic Nekrasov partition function Z_K. For the rest of this section let ξ be an equivariant lift of a class of type c_1, and let $v \in K^\Gamma(X)$. We first give, up to a correction term, an expression for $\widetilde{\Delta}^X_{\xi,T}(v,\Lambda)$ in terms of the instanton part. Then we show that this correction term is given by the perturbation part.

Theorem 3.3. *Let $v \in K^\Gamma(X)$.*

$$\widetilde{\Delta}^X_{\xi,e^{-\boldsymbol{\beta} t}}(v;\boldsymbol{\beta}\Lambda)\big|_{\substack{t_1\to e^{\boldsymbol{\beta}\varepsilon_1}\\ t_2\to e^{\boldsymbol{\beta}\varepsilon_2}}}$$
$$= \frac{1}{\boldsymbol{\beta}\Lambda}\exp\Bigg(\boldsymbol{\beta}\Big(\frac{\langle K_X^3\rangle}{48} - \frac{\langle \operatorname{Todd}_2(X) K_X\rangle}{2} + \langle [2\operatorname{ch}(v)\exp(c_1/2)\operatorname{Todd}(X)]_3\rangle\Big)$$
$$+ \sum_{i=1}^{\chi} F_{-\operatorname{rk}(v)}\big(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{-\boldsymbol{\beta}\iota_{p_i}^* K_X/4}, \boldsymbol{\beta}\iota_{p_i}^*(c_1(v) + \tfrac{\operatorname{rk}(v)}{2}(c_1 - K_X))\big)\Bigg).$$

Note that the left-hand side lies in $(\boldsymbol{\beta}\Lambda)^{-\xi^2-3}\mathbb{Q}(e^{\varepsilon_1}, e^{\varepsilon_2}, e^{t\boldsymbol{\beta}})[[\boldsymbol{\beta}\Lambda]]$. In the course of the proof we will also have to show how one can interpret the right-hand side, so that both sides lie in the same ring.

Lemma 3.4. *Let M be a Γ-equivariant line bundle on X, with $c_1(M) = \xi$. Then in $(\boldsymbol{\beta}\Lambda)^{-\xi^2-3}\mathbb{Q}(e^{\boldsymbol{\beta}\varepsilon_1}, e^{\boldsymbol{\beta}\varepsilon_2}, e^{\boldsymbol{\beta}t})[[\boldsymbol{\beta}\Lambda]]$ we have*

$$\widetilde{\Delta}^X_{\xi,e^{-\boldsymbol{\beta} t}}(v,\boldsymbol{\beta}\Lambda)\big|_{\substack{t_1\to e^{\boldsymbol{\beta}\varepsilon_1}\\ t_2\to e^{\boldsymbol{\beta}\varepsilon_2}}} = \exp(2\boldsymbol{\beta}\langle v^{(3)}\rangle)$$
$$\cdot \frac{\prod_{i=1}^{\chi} Z^{\mathrm{inst}}_{-\operatorname{rk}(v)}\big(w(x_i), w(y_i), \frac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{-\boldsymbol{\beta}\iota_{p_i}^*(K_X)/4}, \boldsymbol{\beta}, \boldsymbol{\beta}\iota_{p_i}^*(c_1(v) + \frac{\operatorname{rk}(v)}{2}(c_1 - K_X)\big)}{(\boldsymbol{\beta}\Lambda)^{\xi^2+3}\bigwedge_{-e^{-\boldsymbol{\beta} t}}(-\widetilde{\chi}(X, M^\vee)^\vee)\bigwedge_{-e^{\boldsymbol{\beta} t}}(-\widetilde{\chi}(X, M)^\vee)}.$$

Proof. Following [20], we denote $C(0) := \operatorname{ch}(\mathcal{I}_1)e^{\xi/2} + \operatorname{ch}(\mathcal{I}_2)e^{-\xi/2}$ on $X\times X^{[n]}\times X^{[m]}$, and $C_i(0) := [C(0)]_i$. The Grothendieck-Riemann-Roch theorem implies that

$$\begin{aligned}\operatorname{ch}(\lambda_{\mathcal{F}_1}(v)\otimes\lambda_{\mathcal{F}_2}(v)) &= \exp([p_*(q^*(\operatorname{ch}(v))\operatorname{ch}(\mathcal{F}_1\oplus\mathcal{F}_2)\operatorname{Todd}(X)]_1)\\ &= \exp([p_*(q^*(\operatorname{ch}(v))C(0)e^{c_1/2}\operatorname{Todd}(X))]_1)\\ &= \exp(C_3(0)/\operatorname{rk}(v) + C_2(0)/v^{(1)} + 2/v^{(3)}).\end{aligned} \tag{3.5}$$

Let $(Z_1, Z_2) \in (X^{[n]} \times X^{[m]})^\Gamma$ correspond to $(\vec{Y}^1, \dots, \vec{Y}^\chi)$. By [20, Lemma 3.4] the cotangent space and the fibres of $\mathcal{A}_\pm^\vee$ at (Z_1, Z_2) are

$$
\begin{aligned}
\textstyle\bigwedge_{-1} T^*_{(Z_1,Z_2)} X^{[n]} \times X^{[m]}|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} &= \prod_{i=1}^{\chi} \prod_{\gamma=1}^{2} n^{\vec{Y}_i}_{\gamma,\gamma}(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}\xi}{2}; \boldsymbol{\beta}), \\
\textstyle\bigwedge_{-e^{-t\boldsymbol{\beta}}} \mathcal{A}_+^\vee(Z_1, Z_2)|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} &= \textstyle\bigwedge_{-e^{-t\boldsymbol{\beta}}} -\widetilde{\chi}(X, M^\vee)^\vee|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} \\
&\quad \times \prod_{i=1}^{\chi} n^{\vec{Y}_i}_{1,2}\big(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}\xi}{2}; \boldsymbol{\beta}\big), \\
\textstyle\bigwedge_{-e^{t\boldsymbol{\beta}}} \mathcal{A}_-^\vee(Z_1, Z_2)|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} &= \textstyle\bigwedge_{-e^{t\boldsymbol{\beta}}} -\widetilde{\chi}(X, M)^\vee|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} \\
&\quad \times \prod_{i=1}^{\chi} n^{\vec{Y}_i}_{2,1}\big(w(x_i), w(y_i), \tfrac{t-\iota_{p_i}\xi}{2}; \boldsymbol{\beta}\big).
\end{aligned} \tag{3.6}
$$

By (1.27) and [20, (3.11)] we get

(3.7)

$$
\begin{aligned}
&\prod_{i=1}^{\chi} \exp\Big(\boldsymbol{\beta}\,\mathrm{rk}(v)|\vec{Y}_i|\frac{w(x_i)+w(y_i)}{2}\Big) C^{\vec{Y}}_{-\mathrm{rk}(v)}(w(x_i), w(y_i), \tfrac{t-\iota^*_{p_i}\xi}{2}; \boldsymbol{\beta}, \boldsymbol{\beta}\iota^*_{p_i} v^{(1)}) \\
&= \prod_{i=1}^{\chi} E^{\vec{Y}}(w(x_i), w(y_i), \tfrac{t-\iota^*_{p_i}\xi}{2}; \boldsymbol{\beta}, \boldsymbol{\beta}\iota^*_{p_i} v^{(1)}, \mathrm{rk}(v)) \\
&= \iota^*_{(Z_1,Z_2)} \exp\Big(\big[\mathrm{ch}(\mathcal{I}_1) e^{\frac{\xi-\boldsymbol{\beta}t}{2}} \oplus \mathrm{ch}(\mathcal{I}_2) e^{\frac{\boldsymbol{\beta}t-\xi}{2}}\big]_2 / v^{(1)} \\
&\qquad\qquad + \big[\mathrm{ch}(\mathcal{I}_1) e^{\frac{\xi-\boldsymbol{\beta}t}{2}} \oplus \mathrm{ch}(\mathcal{I}_2) e^{\frac{\boldsymbol{\beta}t-\xi}{2}}\big]_3 / \mathrm{rk}(v)\Big)|_{\substack{\varepsilon_1 \to \boldsymbol{\beta}\varepsilon_1 \\ \varepsilon_2 \to \boldsymbol{\beta}\varepsilon_2}} \\
&= \iota^*_{(Z_1,Z_2)}\Big(\exp\big(C_2(0)/v^{(1)} + C_3(0)/\mathrm{rk}(v)\big)\Big)_{\substack{\varepsilon_1 \to \boldsymbol{\beta}\varepsilon_1 \\ \varepsilon_2 \to \boldsymbol{\beta}\varepsilon_2}} (e^{\boldsymbol{\beta}t})^{\frac{1}{2}(\chi(f_2 \otimes v) - \chi(f_1 \otimes v))} \\
&= \exp(-2\boldsymbol{\beta}\langle v^{(3)}\rangle)\iota^*_{(Z_1,Z_2)}\Big(\mathrm{ch}(\lambda_{\mathcal{F}_1}(v) \otimes \lambda_{\mathcal{F}_2}(v))\Big)_{\substack{\varepsilon_1 \to \boldsymbol{\beta}\varepsilon_1 \\ \varepsilon_2 \to \boldsymbol{\beta}\varepsilon_2}} (e^{\boldsymbol{\beta}t})^{\frac{1}{2}(\chi(f_2 \otimes v) - \chi(f_1 \otimes v))}.
\end{aligned}
$$

In the fourth line we use that $\mathrm{ch}_0(\mathcal{I}_\alpha) = 1$, $\mathrm{ch}_1(\mathcal{I}_\alpha) = 0$ for $\alpha = 1, 2$, and that $\mathrm{ch}_2(\mathcal{I}_1)/1 = -n$, $\mathrm{ch}_2(\mathcal{I}_2)/1 = -m$ and thus

$$
\begin{aligned}
&\iota^*_{(Z_1,Z_2)}\Big(\big[\mathrm{ch}(\mathcal{I}_1) e^{\frac{\xi-\boldsymbol{\beta}t}{2}} \oplus \mathrm{ch}(\mathcal{I}_2) e^{\frac{\boldsymbol{\beta}t-\xi}{2}}\big]_2 / v^{(1)}\Big) = \iota^*_{(Z_1,Z_2)}\big(C_2(0)/v^{(1)}\big) - \langle \xi/2, v^{(1)}\rangle \boldsymbol{\beta}t, \\
&\iota^*_{(Z_1,Z_2)}\Big(\big[\mathrm{ch}(\mathcal{I}_1) e^{\frac{\xi-\boldsymbol{\beta}t}{2}} \oplus \mathrm{ch}(\mathcal{I}_2) e^{\frac{\boldsymbol{\beta}t-\xi}{2}}\big]_3 / \mathrm{rk}(v)\Big) \\
&\qquad\qquad = \iota^*_{(Z_1,Z_2)}\big(C_3(0)/\mathrm{rk}(v)\big) + (n-m)\frac{\mathrm{rk}(v)}{2}\boldsymbol{\beta}t,
\end{aligned}
$$

and formula (2.3). In the last line of (3.7) we use (3.5).

Write $|Y| := |\vec{Y}_1| + \ldots + |\vec{Y}_\chi|$, and write (Z_1^Y, Z_2^Y) for the point of $X^{[n]} \times X^{[m]}$ with $n+m = |Y|$ determined by an χ-tuple $Y = (\vec{Y}_1, \ldots, \vec{Y}_\chi)$ of pairs of Young diagrams. Using that $\iota_{p_i}^* K_X = -w(x_i) - w(y_i)$, we get by localization, (3.7) and (1.25)

$$
\begin{aligned}
&(\boldsymbol{\beta}\Lambda)^{-\xi^2-3} \frac{\prod_{i=1}^{\chi} Z^{\mathrm{inst}}_{-\operatorname{rk}(v)}\big(w(x_i), w(y_i), \frac{t-\iota_{p_i}^*\xi}{2}; \Lambda e^{-\boldsymbol{\beta}\iota_{p_i}^* K_X/4}, \boldsymbol{\beta}, \boldsymbol{\beta}\iota_{p_i}^* v^{(1)}\big)}{\bigwedge_{e^{-\boldsymbol{\beta}t}} -\widetilde{\chi}(X, M^\vee)^\vee \bigwedge_{-e^{\boldsymbol{\beta}t}} -\widetilde{\chi}(X, M)^\vee})\Big|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} \\
&= \sum_{Y = (\vec{Y}_1, \ldots, \vec{Y}_\chi)} (\boldsymbol{\beta}\Lambda)^{4|Y| - \xi^2 - 3} \\
&\quad \times \frac{\prod_{i=1}^{\chi} \exp\Big(\boldsymbol{\beta}\operatorname{rk}(v)|\vec{Y}_i| \frac{w(x_i)+w(y_i)}{2}\Big) C^{\vec{Y}}_{-\operatorname{rk}(v)}(w(x_i), w(y_i), \frac{t-\iota_{p_i}^*\xi}{2}; \boldsymbol{\beta}, \boldsymbol{\beta}\iota_{p_i}^* v^{(1)})}{\big(\bigwedge_{-1}(T^*_{(Z_1^Y, Z_2^Y)} X_2^{[|Y|]}) \bigwedge_{-e^{-\boldsymbol{\beta}t}} \mathcal{A}_+^\vee(Z_1^Y, Z_2^Y) \bigwedge_{-e^{\boldsymbol{\beta}t}} \mathcal{A}_-^\vee(Z_2^Y, Z_1^Y))\big)\Big|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}}} \\
&= \exp(-2\boldsymbol{\beta}\langle v^{(3)}\rangle) \sum_{\substack{n,m \ge 0 \\ d = 4(n+m) - \xi^2 - 3}} (\boldsymbol{\beta}\Lambda)^d \\
&\qquad \times \widetilde{\chi}\Big(X^{[n]} \times X^{[m]}, \frac{\lambda_{\mathcal{F}_1}(v) \otimes \Lambda_{\mathcal{F}_2}(v)}{e^{-\boldsymbol{\beta}t(\frac{1}{2}\chi(f_2 \otimes v) - \chi(f_1 \otimes v)))} \bigwedge_{-e^{-\boldsymbol{\beta}t}} \mathcal{A}_+^\vee \bigwedge_{-e^{\boldsymbol{\beta}t}} \mathcal{A}_-^\vee}\Big)\Big|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} \\
&= \widetilde{\Delta}^X_{\xi, e^{-\boldsymbol{\beta}t}}(v, \boldsymbol{\beta}\Lambda)\Big|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}} \exp(-2\boldsymbol{\beta}\langle v^{(3)}\rangle).
\end{aligned}
$$

In the third line we use (3.7) and equivariant localization. □

Now we identify the contribution of the perturbation part. Let $\widetilde{\mathcal{O}}$ be the ring of holomorphic functions in $(\Lambda, \boldsymbol{\beta}, t)$ in a neighborhood of $\sqrt{-1}\mathbb{R}_{>0} \times \sqrt{-1}\mathbb{R}_{<0} \times \sqrt{-1}\mathbb{R}_{>0}$.

Lemma 3.8.

$$
\begin{aligned}
&\sum_{i=1}^{\chi} F^{\mathrm{pert}}(w(x_i), w(y_i), \tfrac{t - i_{p_i}^*\xi}{2}; \Lambda e^{-\boldsymbol{\beta}\iota_{p_i}^* K_X/4}, \boldsymbol{\beta}) \\
&\qquad = \big(-(\chi(M) + \chi(M^\vee)\big) \log(\boldsymbol{\beta}\Lambda) - \frac{\boldsymbol{\beta}\langle K_X^3\rangle}{48} + \frac{\boldsymbol{\beta}}{2}\langle \mathrm{Todd}_2(X) K_X\rangle \\
&\qquad\quad + \log\Big(\frac{1}{\bigwedge_{-e^{\boldsymbol{\beta}t}} -\widetilde{\chi}(X, M)^\vee \bigwedge_{-e^{-\boldsymbol{\beta}t}} -\widetilde{\chi}(X, M^\vee)^\vee}\Big)\Big|_{\substack{t_1 \to e^{\boldsymbol{\beta}\varepsilon_1} \\ t_2 \to e^{\boldsymbol{\beta}\varepsilon_2}}}.
\end{aligned}
$$

holds in $\widetilde{\mathcal{O}}[[\varepsilon_1, \varepsilon_2]][\prod_i (w(x_i) w(y_i))^{-1}]$.

Proof. By [20, (3.17)] we get that $\sum_{i=1}^{\chi} F_K^{\mathrm{pert}}(w(x_i), w(y_i), \frac{t-i_{p_i}^*\xi}{2}; \Lambda, \boldsymbol{\beta})$ is given by the same formula with $\frac{\boldsymbol{\beta}}{2}\langle \mathrm{Todd}_2(X)K_X\rangle$ replaced by $-\frac{\boldsymbol{\beta}}{4}\langle \xi^2 K_X\rangle + \frac{\boldsymbol{\beta}}{2}\langle \xi K_X\rangle t$. Note that by (1.28), when changing Λ to $\Lambda e^{-\boldsymbol{\beta} K_X/4}$, the result changes by adding

$$-\boldsymbol{\beta}\sum_{i=1}^{\chi}\frac{w(x_i)+w(y_i)}{4w(x_i)w(y_i)}\Big((t-\iota_{p_i}^*\xi)^2+\frac{w(x_i)^2+w(y_i)^2+3w(x_i)w(y_i)}{6}\Big)$$
$$=\Big(\frac{\boldsymbol{\beta}}{4}\langle \xi^2K_X\rangle-\frac{\boldsymbol{\beta}}{2}\langle \xi K_X\rangle t+\frac{\boldsymbol{\beta}}{2}\langle \mathrm{Todd}_2(X)K_X\rangle\Big).$$

The result follows. □

Writing $\mathrm{ch}(-\widetilde{\chi}(X,M))=\sum_{i=0}^{\ell} e^{\alpha_j}$, $\mathrm{ch}(-\widetilde{\chi}(X,M^\vee))=\sum_{i=0}^{\ell'} e^{\alpha'_k}$, we see that

$$\log\Big(\frac{\boldsymbol{\beta}^{-(\chi(M)+\chi(M^\vee))}}{\bigwedge_{-e^{\boldsymbol{\beta}t}}-\widetilde{\chi}(X,M)^\vee\bigwedge_{-e^{-\boldsymbol{\beta}t}}-\widetilde{\chi}(X,M^\vee)^\vee}\Big)\Big|_{\substack{t_1\to e^{\boldsymbol{\beta}\varepsilon_1}\\ t_2\to e^{\boldsymbol{\beta}\varepsilon_2}}}$$
$$(3.9)\qquad =\sum_{j=1}^{\ell}\log\Big(\frac{\boldsymbol{\beta}}{1-e^{-(\alpha_j-t)\boldsymbol{\beta}}}\Big)+\sum_{k=1}^{\ell'}\log\Big(\frac{\boldsymbol{\beta}}{1-e^{-(\alpha'_k+t)\boldsymbol{\beta}}}\Big).$$

Let $\overline{\mathcal{O}}$ denote the ring of holomorphic functions in $(t,\Lambda,\boldsymbol{\beta})$ in an open subset of $\mathbb{C}^3$ which contains for any $(t,\Lambda)\in(\mathbb{C}\setminus\mathbb{R}_{\le 0})^2$ an open neighbourhood of $\boldsymbol{\beta}=0$. Then (3.9) shows that the left hand side of Lemma 3.8 lies in $\overline{\mathcal{O}}[[\varepsilon_1,\varepsilon_2]]$. Thus we can view also the left hand side of Lemma 3.8 to lie in $\overline{\mathcal{O}}[[\varepsilon_1,\varepsilon_2]]$, and we can take the exponential of both sides of the equation. Note that the exponential of the right hand side lies in $\mathbb{Q}(e^{\boldsymbol{\beta}\varepsilon_1},e^{\boldsymbol{\beta}\varepsilon_2},e^{\boldsymbol{\beta}t})[[\boldsymbol{\beta}\Lambda]]$. With this remark Theorem 3.3 follows from Lemma 3.4 and Lemma 3.8.

Now we express $\widetilde{\Delta}^X_{\xi,T}(v,\Lambda)$ in terms of the $Z_{-\mathrm{rk}(v)}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta})$. The Nekrasov conjecture determines the lowest order terms in $\varepsilon_1,\varepsilon_2$ of $F_{-\mathrm{rk}(v)}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta})$, but not of $F_{-\mathrm{rk}(v)}(\varepsilon_1,\varepsilon_2,a;\Lambda,\boldsymbol{\beta},\tau)$.

Corollary 3.10. *Let $v\in K^\Gamma(X)$. Then*

$$\widetilde{\Delta}^X_{\xi,e^{-\boldsymbol{\beta}t}}(v,\boldsymbol{\beta}\Lambda)=\frac{1}{\boldsymbol{\beta}\Lambda}\exp\Big(\boldsymbol{\beta}\Big(\frac{\langle K_X^3\rangle}{48}$$
$$-\frac{1}{2}\Big\langle \mathrm{Todd}_2(X)(K_X+c_1(v)+\frac{\mathrm{rk}(v)}{2}(c_1-K_X))\Big\rangle+2\langle[\mathrm{ch}(v)e^{c_1/2}\,\mathrm{Todd}(X)]_3\rangle\Big)\Big)$$
$$\times\Big(\sum_{i=1}^{\chi}F_{-\mathrm{rk}(v)}\big(w(x_i),w(y_i),\tfrac{t-\iota_{p_i}^*\xi}{2};\Lambda e^{-\frac{\boldsymbol{\beta}}{4}\iota_{p_i}^*(K_X+c_1(v)+\frac{\mathrm{rk}(v)}{2}(c_1-K_X))}\big)\Big).$$

Proof. Let τ, σ be variables. In the same way as in [42, section 4.5], we see that

$$Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda e^{-\sigma/4}, \boldsymbol{\beta}, \tau) = \exp\left(\frac{\tau a^2}{\varepsilon_1\varepsilon_2}\right) Z_m^{\text{inst}}(\varepsilon_1, \varepsilon_2, a; \Lambda e^{-(\tau+\sigma)/4}, \boldsymbol{\beta}).$$

On the other hand, by [43, formula after (4.12)], we get that

$$F_m^{\text{pert}}(\varepsilon_1, \varepsilon_2, a; \Lambda e^{-\sigma/4}, \boldsymbol{\beta}, \tau) = F_m^{\text{pert}}(\varepsilon_1, \varepsilon_2, a; \Lambda e^{-(\tau+\sigma)/4}, \boldsymbol{\beta}) - \frac{\tau a^2}{\varepsilon_1\varepsilon_2} - \frac{\tau(\varepsilon_1^2 + \varepsilon_2^2 + 3\varepsilon_1\varepsilon_2)}{24\varepsilon_1\varepsilon_2}.$$

The result follows by by localization and Theorem 3.3. □

4. Explicit formulas in terms of modular forms

The result of [43] together with §A implies that the following solution of Nekrasov's conjecture and its refinement are true for the K-theoretic partition function when $m = 0$:

(1) $\varepsilon_1\varepsilon_2 F_m(\varepsilon_1, \varepsilon_2, a; \Lambda)$ is regular at ε_1, $\varepsilon_2 = 0$,
(2) $\mathcal{F}_0(a; \Lambda)$ is the Seiberg-Witten prepotential associated with the Seiberg-Witten curve $Y^2 = P(X)^2 - 4(-X)^{2+m}(\boldsymbol{\beta}\Lambda)^4$,
(3) H comes only from the perturbation part, i.e. $H(a, \Lambda) = \pi\sqrt{-1}a$,
(4) $\exp A = \left(\frac{2}{\theta_{00}\theta_{10}}\right)^{1/2}$, $\exp B = \theta_{01} \exp A$, where the θ_{**} are theta functions with variable $q = e^{2\pi\sqrt{-1}\tau}$, where τ is the period of the above Seiberg-Witten curve, i.e. $\tau = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial^2\mathcal{F}_0}{\partial a^2}$.

Here $\mathcal{F}_0$, H, A, B are given by the expansion

$$(4.1) \quad \varepsilon_1\varepsilon_2 F_m(\varepsilon_1, \varepsilon_2, a; \Lambda, \boldsymbol{\beta}) = \mathcal{F}_0(a; \Lambda, \boldsymbol{\beta}) + (\varepsilon_1 + \varepsilon_2)H(a; \Lambda) + \varepsilon_1\varepsilon_2 A(a; \Lambda, \boldsymbol{\beta}) + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3}B(a; \Lambda, \boldsymbol{\beta}) + \cdots .$$

When $|m| < 2$, the above (1)–(3) follow from a conjectural blowup equation (1.37) as we explained in §1.7. The analogue of the statement (4) is (1.44) which follows from the conjecture (1.43). In the above we implicitly assume $|m| \le 2$ as the Seiberg-Witten curve changes the genus otherwise. According to a physical argument [27, 51], the remaining case $m = \pm 2$ is similar to the case $|m| < 2$, in particular (1),(2) should be true. (These probably follow from the approach in [48].) But we believe that the blowup equation must be modified, and (3) is probably *not* true.

In the following we *assume* the above (1)–(3) and (1.44) are also true for $m = \pm 1$.

Once we have the above (1)–(3), then the same argument as in [20, proof of Thm. 4.2, in particular of (4.12)] gives

Corollary 4.2.

$$\Delta^X_{\xi, e^{-\boldsymbol{\beta} t}}(v, \boldsymbol{\beta}\Lambda)\Big|_{t=2a} = \frac{1}{\boldsymbol{\beta}\Lambda}\sqrt{-1}^{\langle \xi, K_X\rangle} q^{-\frac{1}{2}\left(\frac{\xi}{2}\right)^2}$$
$$\times \exp\left[\frac{\boldsymbol{\beta}}{8}\frac{\partial^2 \mathcal{F}_0}{\partial a \partial \log\Lambda}\left\langle \xi(K_X + c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X))\right\rangle\right.$$
$$\left. + \frac{\boldsymbol{\beta}^2}{32}\frac{\partial^2 \mathcal{F}_0}{(\partial \log\Lambda)^2}\left\langle (K_X + c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X))^2\right\rangle + \chi A + \sigma B\right].$$

We have expressed the wallcrossing $\Delta^X_{\xi, e^{-\boldsymbol{\beta} t}}(v, \boldsymbol{\beta}\Lambda)$ in terms of the partition function with 5D Chern-Simons term. As in [20, §4] we use the Nekrasov conjecture to give an explicit formula in terms of q-development of modular forms.

We identify $t/2$ with a hereafter.

Theorem 4.3. (1) *Let* $\Delta^X_\xi(v, \boldsymbol{\beta}\Lambda) = \sum_{n\ge 0}\Delta_n \Lambda^{4n - \xi^2 - 3}$. *Then* Δ_n *is equal to* 0 *if* $\langle \xi, c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X)\rangle + \mathrm{rk}(v)n$ *is odd, and equal to the coefficient in*

$$2 \operatorname*{Coeff}_{(q^{1/8})^0}\left[\Delta^X_{\xi, e^{-2\boldsymbol{\beta} a}}(v, \Lambda)\frac{a^2}{\Lambda}\Big|_{a = a(q^{1/8}, \Lambda)} q^{1/8}\frac{\partial\left(\frac{\Lambda}{a}\right)}{\partial(q^{1/8})}\right]$$

otherwise.

(2) *Suppose* $\mathrm{rk}(v) = -m = 0$. *Then the terms in* [ ] *above are given in explicit modular forms in* $\mathbb{C}((q^{1/8}))[[\Lambda]]$.

Here the change of variable from $\frac{\Lambda}{a}$ to $q^{1/8}$ will be explained later during the proof. It will be done in several steps in §§4.1,4.2,4.3. The explicit forms stated in (2) will be given in §4.4.

For $\mathrm{rk}(v) = \pm 1$, the terms are written in terms of the Seiberg-Witten prepotential $\mathcal{F}_0$, but we do not know how to write them *explicitly* in terms of $q^{1/8}$ and Λ at this moment. This is a problem about elliptic integrals and modular forms.

4.1. **From the residues at $e^{\boldsymbol{\beta}a} = 0, \infty$ to the residue at $e^{\boldsymbol{\beta}a} = 1$.** Let $\Delta^X_{\xi, e^{-2\boldsymbol{\beta}a}}(v, \boldsymbol{\beta}\Lambda) = \sum_{n\ge 0} \Delta_n \Lambda^{4n-\xi^2-3}$.

Proposition 4.4. (1) *The coefficient Δ_n is a rational function in $e^{\boldsymbol{\beta}a}$, which is regular on $\mathbb{P}^1 \setminus \{0, \infty, 1, -1\}$.*

(2) *Δ_n is multiplied by $(-1)^{\langle \xi, c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X)\rangle + \mathrm{rk}(v)n}$ under the replacement $e^{\boldsymbol{\beta}a} \mapsto -e^{\boldsymbol{\beta}a}$.*

Corollary 4.5. *Assume* $\mathrm{rk}(v)$ *and* $\langle \xi, c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X)\rangle$ *are even. Then*

$$\begin{aligned}\Delta^X_\xi(v; \boldsymbol{\beta}\Lambda) &= \operatorname*{Res}_{e^{-\boldsymbol{\beta}a}=0} \Delta^X_{\xi, e^{-2\boldsymbol{\beta}a}}(v; \boldsymbol{\beta}\Lambda) \frac{de^{-\boldsymbol{\beta}a}}{e^{-\boldsymbol{\beta}a}} + \operatorname*{Res}_{e^{-\boldsymbol{\beta}a}=\infty} \Delta^X_{\xi, e^{-2\boldsymbol{\beta}a}}(v; \boldsymbol{\beta}\Lambda) \frac{de^{-\boldsymbol{\beta}a}}{e^{-\boldsymbol{\beta}a}} \\ &= -2 \operatorname*{Res}_{e^{-\boldsymbol{\beta}a}=1} \Delta^X_{\xi, e^{-2\boldsymbol{\beta}a}}(v; \Lambda) \frac{de^{-\boldsymbol{\beta}a}}{e^{-\boldsymbol{\beta}a}}.\end{aligned}$$

The first equality follows from (1) (and $T = e^{-\boldsymbol{\beta}t} = e^{-2\boldsymbol{\beta}a}$). The second equality follows from (1) and the residue theorem, together with (2). This corollary means that we can move the position taking residues from 0, ∞ to 1.

When either $\mathrm{rk}(v)$ or $\langle \xi, c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X)\rangle$ is not even, the coefficient of $\Lambda^{4n-\xi^2-3}$ in the left hand side is 0 or equal to the coefficient in the right hand side, depending on the parity of $(-1)^{\langle \xi, c_1(v) + \frac{\mathrm{rk}(v)}{2}(c_1 - K_X)\rangle + \mathrm{rk}(v)n}$. We assume that both are even for brevity in the above corollary, but it is clear that we have a statement like in Theorem 4.3(1).

Before starting the proof of Proposition 4.4 we give new variables so that the partition function becomes homogeneous.

Recall we set $a_1 = -a$, $a_2 = a$. Following [43, §5], we set

$$\zeta_{\alpha,\beta} := \frac{\boldsymbol{\beta}}{1 - e^{-(a_\alpha - a_\beta)\boldsymbol{\beta}}}.$$

We first consider the case when the 5D Chern-Simons term is *not* included.

By [43, (5.3)] we have

$$\varepsilon_1 \varepsilon_2 F^{\mathrm{inst}}_K \in \mathbb{C}[\zeta_{1,2}, \zeta_{2,1}, \boldsymbol{\beta}][[\varepsilon_1, \varepsilon_2, \zeta_{1,2}\Lambda^4]].$$

We assign degrees as $\deg \varepsilon_1 = \deg \varepsilon_2 = \deg \Lambda = 1$ and $\deg \boldsymbol{\beta} = \deg \zeta_{\alpha,\beta} = -1$. Then Z^{inst}_K is homogeneous of degree 0, and hence $\varepsilon_1\varepsilon_2 F^{\mathrm{inst}}_K$ is of degree 2. Let

$$\mathcal{F}^{\mathrm{inst}}_0 := \varepsilon_1 \varepsilon_2 F^{\mathrm{inst}}_K \big|_{\varepsilon_1=\varepsilon_2=0} = \sum_{n\ge 1} \mathcal{F}^{\mathrm{inst}}_n (\boldsymbol{\beta}\Lambda)^{4n}.$$

Then the coefficient $\mathcal{F}_n^{\text{inst}}$ is a homogeneous polynomial of $\boldsymbol{\beta}$ and $\zeta_{\alpha,\beta}$ of degree $2-4n$. When we exchange a_1 and a_2, $\zeta_{2,1}$ and $\zeta_{1,2}$ are exchanged accordingly. Since $\mathcal{F}_n^{\text{inst}}$ is symmetric in a_1, a_2, $\mathcal{F}_n^{\text{inst}}$ is symmetric in $\zeta_{1,2}$ and $\zeta_{2,1}$. By the equality $\zeta_{2,1} = \boldsymbol{\beta} - \zeta_{1,2}$, we see that there is a weighted homogeneous polynomial $A_{4n-2}(x,y) \in \mathbb{C}[x,y]$ of degree $4n-2$ with $\deg x = 1$ and $\deg y = 2$ such that

$$\mathcal{F}_n^{\text{inst}} = A_{4n-2}(\boldsymbol{\beta}, \zeta_{1,2}\zeta_{2,1}).$$

Moreover, as F_K^{inst} is a formal power series in $\zeta_{1,2}\Lambda^4$ by [43, (5.3)], $\mathcal{F}_n^{\text{inst}}$ is divisible by $(\zeta_{1,2}\zeta_{2,1})^n$.

We further introduce

$$z := \frac{-\sqrt{-1}\boldsymbol{\beta}\Lambda}{e^{\boldsymbol{\beta}a_1} - e^{\boldsymbol{\beta}a_2}}.$$

We have $z^2 = \zeta_{1,2}\zeta_{2,1}\Lambda^2$. From the above consideration we have

$$\mathcal{F}_0^{\text{inst}} \in z^2\Lambda^2\mathbb{C}[\boldsymbol{\beta},\Lambda][[z^2]]. \tag{4.6}$$

As $\frac{\partial}{\partial a} z = -z(\zeta_{1,2} - \zeta_{2,1})$, and $\frac{\partial}{\partial a}(\zeta_{1,2} - \zeta_{2,1}) = 4(z/\Lambda)^2$, we have

$$\begin{aligned} \frac{\partial \mathcal{F}_0^{\text{inst}}}{\partial a} &\in (\zeta_{1,2} - \zeta_{2,1}) z^2\Lambda^2\mathbb{C}[\boldsymbol{\beta},\Lambda][[z^2]], \\ \frac{\partial \mathcal{F}_0^{\text{inst}}}{\partial a^2} &\in z^2\mathbb{C}[\boldsymbol{\beta},\Lambda][[z^2]]. \end{aligned} \tag{4.7}$$

Even when we include the 5d Chern-Simons term (1.25), we can repeat the above proof. We only need to replace $\mathbb{C}[\boldsymbol{\beta},\Lambda]$ by $\mathbb{C}[\boldsymbol{\beta},\Lambda, e^{\pm \operatorname{rk}(v)\boldsymbol{\beta}a}]$.

Proof of Proposition 4.4. Let us look at the expression of $\Delta^X_{\xi,e^{-2\boldsymbol{\beta}a}}(v, \boldsymbol{\beta}\Lambda)$ given in Corollary 4.2. We will write it as a multiple of an explicit rational function in $e^{\boldsymbol{\beta}a}$ and a formal power series in z. The explicit function comes from the perturbation part of the partition function.

First note that $\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2}$ consists only of the instanton part. Therefore (4.6) implies

$$\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2} \in z^2\Lambda^2\mathbb{C}[\boldsymbol{\beta},\Lambda, e^{\pm \operatorname{rk}(v)\boldsymbol{\beta}a}][[z^2]].$$

Next we have

(4.8)
$$q^{1/8} = \left(\frac{-\sqrt{-1}\boldsymbol{\beta}\Lambda}{e^{-\boldsymbol{\beta}a} - e^{\boldsymbol{\beta}a}}\right) \exp\left(-\frac{1}{8}\frac{\partial^2 \mathcal{F}^{\text{inst}}}{\partial a^2}\right) \in z\left(1 + z^2\mathbb{C}[\boldsymbol{\beta},\Lambda, e^{\pm \operatorname{rk}(v)\boldsymbol{\beta}a}][[z^2]]\right).$$

from (4.7). Here we have used

$$\overline{\gamma}_0''(x|\boldsymbol{\beta};\Lambda) = 2\log\left(\frac{-\sqrt{-1}\boldsymbol{\beta}\Lambda}{e^{\boldsymbol{\beta}x/2} - e^{-\boldsymbol{\beta}x/2}}\right)$$

(cf. (A.8)) to calculate the first term coming from the perturbation part.

Next consider the genus 1 parts. When $\operatorname{rk}(v) = 0$, we have

$$\exp(\chi A + \sigma B) = \left(\frac{2}{\theta_{00}\theta_{10}}\right)^2 \theta_{01}^{\sigma} \in 4z^{-2}\left(1 + z^2\mathbb{C}[\boldsymbol{\beta},\Lambda,e^{\pm\operatorname{rk}(v)\boldsymbol{\beta}a}][[z^2]]\right).$$

The case $\operatorname{rk}(v) = \pm 1$ is similar thanks to (1.44).

Finally again by (4.7) we have

$$\exp\left(\frac{\boldsymbol{\beta}}{8}\frac{\partial^2\mathcal{F}_0}{\partial a\partial\log\Lambda}\langle\xi, K_X + c_1(v) + \frac{\operatorname{rk}(v)}{2}(c_1 - K_X)\rangle\right) = \left(e^{-\boldsymbol{\beta}a}\right)^N$$
$$\times\exp\left(N\frac{\boldsymbol{\beta}}{8}\frac{\partial^2\mathcal{F}_0^{\mathrm{inst}}}{\partial a\partial\log\Lambda}\right) \in \left(e^{-\boldsymbol{\beta}a}\right)^N\mathbb{C}[\boldsymbol{\beta},\Lambda,e^{\pm\operatorname{rk}(v)\boldsymbol{\beta}a}][[(\zeta_{1,2}-\zeta_{2,1})z^2\Lambda^2, z^2]],$$

with $N = \langle\xi, K_X + c_1(v) + \frac{\operatorname{rk}(v)}{2}(c_1 - K_X)\rangle$. Note that $\langle\xi, c_1 - K_X\rangle \equiv \langle\xi,\xi - K_X\rangle \equiv 0 \mod 2$, where the first equality follows from the assumption (§2.1(2)), and the second from the Riemann-Roch theorem. Therefore N is an integer.

As $z = \frac{-\sqrt{-1}\boldsymbol{\beta}\Lambda}{e^{-\boldsymbol{\beta}a} - e^{\boldsymbol{\beta}a}}$, $\zeta_{1,2} - \zeta_{2,1} = -\boldsymbol{\beta}\frac{e^{2\boldsymbol{\beta}a}+1}{e^{2\boldsymbol{\beta}a}-1}$, the statement (1) becomes clear now.

Let us check the statement (2). We substitute $e^{\boldsymbol{\beta}a}$ by $-e^{\boldsymbol{\beta}a}$. Then z changes the sign and $\zeta_{1,2}$, $\zeta_{2,1}$ are invariant. Therefore the change of the instanton part of $\Delta^X_{\xi,e^{-2\boldsymbol{\beta}a}}(v,\boldsymbol{\beta}\Lambda)$ comes only from $C^{\vec{Y}}_{-\operatorname{rk}(v)}(\varepsilon_1,\varepsilon_2,a;\boldsymbol{\beta},\tau)$ in (1.25). It is multiplied by $(-1)^{\operatorname{rk}(v)(|Y^1|+|Y^2|)}$. The perturbation part of $\Delta^X_{\xi,e^{-2\boldsymbol{\beta}a}}(v,\boldsymbol{\beta}\Lambda)$ is multiplied by

$$(-1)^{N+\langle\xi^2\rangle} = (-1)^{\langle\xi,c_1(v)+\frac{\operatorname{rk}(v)}{2}(c_1-K_X)\rangle}.$$

Altogether the coefficient of $\Lambda^{4n-\xi^2-3}$ in $\Delta^X_{\xi,e^{-2\boldsymbol{\beta}a}}(v,\boldsymbol{\beta}\Lambda)$ is multiplied by

$$(-1)^{\langle\xi,c_1(v)+\frac{\operatorname{rk}(v)}{2}(c_1-K_X)\rangle+\operatorname{rk}(v)n}.$$

□

4.2. **From the expansion at $a = 0$ to $a = \infty$.** We set $\boldsymbol{\beta} = 1$ hereafter.

We expand $\Delta^X_{\xi,e^{-2a}}(v,\Lambda)$ at $a = 0$:

$$\Delta^X_{\xi,e^{-2a}}(v;\Lambda) = \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\Delta_{m,n}a^m\Lambda^{4n-\xi^2-3} \in \Lambda^{-\xi^2-3}\mathbb{C}((a))[[\Lambda]]. \tag{4.9}$$

Then

$$\operatorname*{Res}_{e^a=1} \Delta^X_{\xi,e^{-2a}}(v;\Lambda)\frac{de^a}{e^a} = \operatorname*{Coeff}_{(a)^0}\left[\Delta^X_{\xi,e^{-2a}}(v;\Lambda)\times a\right] = \sum_n \Delta_{-1,n}\Lambda^{4n-\xi^2-3}.$$

Proposition 4.10. *$\Delta^X_{\xi,e^{-2a}}(v;\Lambda)$ is in $\Lambda^{-\xi^2-3}\mathbb{C}[[\frac{\Lambda}{a},a]]$, i.e. $\Delta_{m,n}=0$ unless $m\geq -n$ in* (4.9).

This is a consequence of the proof of Proposition 4.4. The key observation is that z, $(\zeta_{1,2}-\zeta_{2,1})\Lambda\in\frac{\Lambda}{a}\mathbb{C}[[a]]$.

We rewrite the above expansion as

$$\begin{aligned}\Delta^X_{\xi,e^{-2a}}(v;\Lambda)\times a &= \sum_{\substack{n\geq 0\\ m+n\geq 0}} \Delta_{m,n}a^{m+1}\Lambda^{4n-\xi^2-3}\\ &= \sum_{\substack{n\geq 0\\ m+n\geq 0}} \Delta_{m,n}\left(\frac{\Lambda}{a}\right)^{-m-1}\Lambda^{4n+m+1-\xi^2-3}.\end{aligned}$$

The last expression is an element in $\Lambda^{-\xi^2-2}\mathbb{C}((\frac{\Lambda}{a}))[[\Lambda]]$, and $\sum_n \Delta_{-1,n}\Lambda^{4n-\xi^2-3}$ is equal to its coefficient of $\left(\frac{\Lambda}{a}\right)^0$. Thus we get

Corollary 4.11.

$$2\operatorname*{Res}_{e^a=1} \Delta^X_{\xi,e^{-2a}}(v;\Lambda)\frac{de^a}{e^a} = 2\operatorname*{Coeff}_{(\frac{\Lambda}{a})^0}\left[\Delta^X_{\xi,e^{-2a}}(v;\Lambda)\times\frac{a}{\Lambda}\Lambda\right].$$

4.3. **From $a=\infty$ to $q=0$.** By (4.8) we have the following expansion in $\mathbb{C}[[\frac{\Lambda}{a},a]]$:

$$q^{1/8} = \frac{\sqrt{-1}\Lambda}{2a}\left(1+O(a,\frac{\Lambda}{a})\right).$$

As in the previous subsection, we consider this as an element in $\mathbb{C}((\frac{\Lambda}{a}))[[\Lambda]]$. Then we have

$$\begin{aligned}q^{1/8} &= q_0(\frac{\Lambda}{a}) + q_1(\frac{\Lambda}{a})\Lambda+\cdots,\\ q_0(\frac{\Lambda}{a}) &= \frac{\sqrt{-1}\Lambda}{2a} + b_2\left(\frac{\Lambda}{a}\right)^2 + b_3\left(\frac{\Lambda}{a}\right)^3+\cdots, \qquad b_i\in\mathbb{C}.\end{aligned}$$

From this we see that $\mathbb{C}((\frac{\Lambda}{a}))[[\Lambda]]\cong\mathbb{C}((q_0))[[\Lambda]]\cong\mathbb{C}((q^{1/8}))[[\Lambda]]$. We now change the variable from $\frac{\Lambda}{a_2}$ to $q^{1/8}$ by the following lemma:

Lemma 4.12. *Let us consider the change of the variable from x to y given by $y = y(x,\Lambda) = y_0(x) + y_1(x)\Lambda + \cdots \in \mathbb{C}((x))[[\Lambda]]$. Assume $y_0(x) = x + a_2x^2 + \cdots \in x(1 + x\mathbb{C}[[x]])$. Let $f(y,\Lambda) \in \mathbb{C}((y))[[\Lambda]] \cong \mathbb{C}((x))[[\Lambda]]$. Then*

$$\operatorname*{Coeff}_{y^0}[yf(y,\Lambda)] = \operatorname*{Coeff}_{x^0}\left[xf(y(x,\Lambda),\Lambda)\frac{dy}{dx}\right].$$

This lemma just means the invariance of the residue under the change of variables. As we have an extra parameter Λ which does not appear in the usual setting, we give a proof.

Proof. It is enough to check the case $f(y,\Lambda) = y^{m-1}$ for $m \in \mathbb{Z}$. First suppose $m \neq 0$. Then the left hand side is equal to 0. On the other hand,

$$y(x,\Lambda)^{m-1}\frac{dy}{dx} = \frac{1}{m}\frac{d}{dx}\left(y(x,\Lambda)^m\right)$$

does not contain the term x^{-1}, as it is a derivative of a formal power series in x. Therefore the right hand side is also 0.

Next suppose $m = 0$. Then the left hand side is 1. Let us consider

$$\log\frac{y(x,\Lambda)}{y_0(x)} = \log\left(1 + \frac{y_1(x)}{y_0(x)}\Lambda + \cdots\right).$$

This is well-defined in $\mathbb{C}((x))[[\Lambda]]$. Then we have

$$\begin{aligned}\frac{1}{y(x,\Lambda)}\frac{dy}{dx} &= \frac{1}{y_0(x)}\frac{dy_0(x)}{dx} + \frac{d}{dx}\left\{\log\left(1 + \frac{y_1(x)}{y_0(x)}\Lambda + \cdots\right)\right\}\\ &= \frac{1}{x}\left(1 + a_2x + \cdots\right)^{-1}\left(1 + 2a_2x + \cdots\right) + \frac{d}{dx}\left\{\log\left(1 + \frac{y_1(x)}{y_0(x)}\Lambda + \cdots\right)\right\}.\end{aligned}$$

The second term does not contain the term x^{-1} by the same reason as above. Therefore we get x^{-1} only from the first term. Hence we have found that the right hand side is also equal to 1. □

Applying this to the right hand side of Corollary 4.11 we get

$$2\operatorname*{Coeff}_{(\frac{\Lambda}{a})^0}\left[\Delta^X_{\xi,e^{2a}}(u;\Lambda) \times a\right] = 2\operatorname*{Coeff}_{(q^{1/8})^0}\left[\Delta^X_{\xi,e^{2a}}(u;\Lambda) \times \left(\frac{a}{\Lambda}\right)^2 \Lambda\Big|_{\frac{\Lambda}{a} = \frac{\Lambda}{a}(q^{1/8},\Lambda)} q^{1/8}\frac{d(\frac{\Lambda}{a})}{d(q^{1/8})}\right].$$

This completes the proof of Theorem 4.3(1).

4.4. **Explicit expressions.** Our remaining task is to express the terms in [ ] of the right hand side of Theorem 4.3 in explicit forms in $\mathbb{C}((q^{1/8}))[[\Lambda]]$. We suppose $m = -\operatorname{rk}(v) = 0$ in this subsection.

By [43, §5] $\exp(\chi A + \sigma B)$ can be written explicitly in terms of $q^{1/8}$. So we only need to express $q^{1/8}\frac{\partial(\Lambda/a)}{\partial(q^{1/8})}$, $\frac{\partial^2\mathcal{F}_0}{\partial a \partial \log\Lambda}$, and $\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}$. The expressions will be given in (4.14), (4.15), (4.16) respectively.

4.4.1. *The term* $q^{1/8}\frac{\partial(\Lambda/a)}{\partial(q^{1/8})}$. We consider a defined as a period of the Seiberg-Witten curve as in §A. In particular, we are in the region D^* such that $\sqrt{-1}a$ has a large real part and $0 < |\Lambda| \ll 1$. We will compute $q^{1/8}\frac{\partial(\Lambda/a)}{\partial(q^{1/8})}$ first in this region and then see later that the computation holds in $\mathbb{C}((q^{1/8}))[[\Lambda]]$.

For simplicity we introduce a variable u by

$$u := -\frac{\theta_{00}^4 + \theta_{10}^4}{\theta_{00}^2\theta_{10}^2}\boldsymbol{\beta}^2\Lambda^2 \in \boldsymbol{\beta}^2\Lambda^2\mathbb{C}((q^{1/8}))$$

where θ-functions are evaluated at $(0,\tau)$. This definition is motivated by a fundamental variable in the homological version (see [20, (4.1)]). By (A.34) we have

$$U_1 = \pm 2\sqrt{1 + u + \boldsymbol{\beta}^4\Lambda^4}.$$

By a certain standard equality for θ-functions (cf. [20, p.29]) we have

$$\frac{du}{d\tau} = -\frac{\boldsymbol{\beta}^2\Lambda^2\pi}{2\sqrt{-1}}\frac{\theta_{01}^8}{\theta_{00}^2\theta_{10}^2}.$$

Combining this with (A.35), we get

$$\frac{da}{d\tau} = \frac{da}{dU_1}\frac{dU_1}{d\tau} = \pm\frac{\pi\Lambda}{4}\frac{\theta_{01}^8}{\theta_{00}\theta_{10}}\frac{1}{\sqrt{1 + u + \boldsymbol{\beta}^4\Lambda^4}}.$$

Therefore we have

$$\left(\frac{d\tau}{da}\right)^2 = \frac{16}{\pi^2\Lambda^2}\frac{\theta_{00}^2\theta_{10}^2}{\theta_{01}^{16}}\left(1 + u + \boldsymbol{\beta}^4\Lambda^4\right). \tag{4.13}$$

This is *a priori* an equality on D^*. However both sides extend to $\Lambda = 0$: The right hand side is a function in $q^{1/8}$ and we have $q^{1/8} \sim \frac{-\sqrt{-1}\boldsymbol{\beta}\Lambda}{e^{\boldsymbol{\beta}a_1} - e^{\boldsymbol{\beta}a_2}}$. Here $\sim$ means the equality up to the instanton part.

Therefore θ_{10}/Λ and u are regular at $\Lambda = 0$, hence so is the right hand side. The left hand side is a triple derivative of the prepotential with respect to a, and hence has no perturbation part. Thus it is regular at $\Lambda = 0$. Therefore (4.13) holds even at $\Lambda = 0$.

Next we consider the coefficients of Λ^k for both sides of (4.13). The equality holds *a priori* for a such that $\sqrt{-1}a$ has a large real part. However both sides are rational functions in $e^{\boldsymbol{\beta} a}$: This claim can be checked as above. The left hand side has no perturbation part, so the claim was proved during the proof of Proposition 4.4. The right hand side is a function in $q^{1/8}$, hence the claim was again proved during the proof of Proposition 4.4. Considering the expansion at $a = 0$, we conclude that (4.13) holds in $\mathbb{C}((a))[[\Lambda]]$.

From the discussion in §4.2 we see that both sides of (4.13) are in $\frac{1}{a^2}\mathbb{C}[[a, \frac{\Lambda}{a}]]$. Therefore (4.13) holds in $\frac{1}{a^2}\mathbb{C}[[a, \frac{\Lambda}{a}]]$, and hence in $\mathbb{C}((\frac{\Lambda}{a}))[[\Lambda]]$. We now change the variable from Λ/a to $q^{1/8}$ as in §4.3 and use the composition law to get

$$\left(\frac{a^2}{\Lambda} q^{1/8} \frac{d\left(\frac{\Lambda}{a}\right)}{d(q^{1/8})} \right)^2 = \left(-\sqrt{-1} \frac{\theta_{01}^8 \Lambda}{\theta_{00}\theta_{10}} \right)^2 \frac{1}{1 + u + \boldsymbol{\beta}^4 \Lambda^4}.$$

As $a \sim \frac{\sqrt{-1}}{2} \frac{\Lambda}{q^{1/8}}$, we can determine the branch of the square root to get

$$\frac{a^2}{\Lambda} q^{1/8} \frac{d\left(\frac{\Lambda}{a}\right)}{d(q^{1/8})} = \sqrt{-1} \frac{\theta_{01}^8 \Lambda}{\theta_{00}\theta_{10}} \sum_{n \geq 0} \binom{-\frac{1}{2}}{n} (u + \boldsymbol{\beta}^4 \Lambda^4)^n. \tag{4.14}$$

This is an equality in $\mathbb{C}((q^{1/8}))[[\Lambda]]$.

4.4.2. *The term* $\frac{\partial^2 \mathcal{F}_0}{\partial a \partial \log \Lambda}$. Let $h := -\frac{1}{4} \frac{\partial^2 \mathcal{F}_0}{\partial a \partial \log \Lambda} = \frac{\pi\sqrt{-1}}{2} \frac{\partial a^D}{\partial \log \Lambda}$. Let us rewrite (A.36) in terms of sn associated with the elliptic curve with period τ. (Be aware that we have used sn with period $-2/\tau$ before.) We get

$$-\frac{\theta_{10}}{\theta_{00}} \operatorname{sn}(\theta_{00}^2 \frac{\boldsymbol{\beta} h}{2\sqrt{-1}}, \kappa(\tau)) = \frac{\theta_{11}(\frac{\boldsymbol{\beta} h}{2\pi\sqrt{-1}})}{\theta_{01}(\frac{\boldsymbol{\beta} h}{2\pi\sqrt{-1}})} = -\boldsymbol{\beta}\Lambda.$$

Therefore

$$\theta_{00}^2 \frac{\boldsymbol{\beta} h}{2\sqrt{-1}} = \int_0^{\frac{\theta_{00}}{\theta_{10}}\boldsymbol{\beta}\Lambda} \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} = \frac{\theta_{00}}{\theta_{10}} \int_0^{\boldsymbol{\beta}\Lambda} \frac{dx}{\sqrt{1 + \frac{u}{\boldsymbol{\beta}^2\Lambda^2} x^2 + x^4}}.$$

Therefore

$$h = \frac{2\sqrt{-1}}{\boldsymbol{\beta}\theta_{00}\theta_{10}} \int_0^{\boldsymbol{\beta}\Lambda} \frac{dx}{\sqrt{1 + \frac{u}{\boldsymbol{\beta}^2\Lambda^2} x^2 + x^4}}.$$

Using $\frac{1}{\sqrt{1+\frac{u}{\boldsymbol{\beta}^2\Lambda^2}x^2+x^4}} = \sum_{n\geq 0, n\geq k\geq 0} \binom{-\frac{1}{2}}{n}\binom{n}{k}\left(\frac{u}{\boldsymbol{\beta}^2\Lambda^2}\right)^k x^{4n-2k}$, we get

$$h = \frac{2\sqrt{-1}}{\boldsymbol{\beta}\theta_{00}\theta_{10}} \sum_{\substack{n\geq 0\\ n\geq k\geq 0}} \binom{-\frac{1}{2}}{n}\binom{n}{k} \frac{u^k(\boldsymbol{\beta}\Lambda)^{4(n-k)+1}}{4n-2k+1}. \tag{4.15}$$

This gives us an explicit expression in terms of $q^{1/8}$ as, e.g.

$$\boldsymbol{\beta}\theta_{00}\theta_{10}h = 2\sqrt{-1}\left(\boldsymbol{\beta}\Lambda - \frac{u}{6}\boldsymbol{\beta}\Lambda + \cdots\right).$$

4.4.3. *The term* $\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}$. We use

$$\frac{\theta_{11}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)}{\theta_{01}(0,\tau)} = \frac{\theta_{11}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)}{\theta_{01}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)}\frac{\theta_{01}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)}{\theta_{01}(0,\tau)} = -\boldsymbol{\beta}\Lambda\exp\left(\frac{\boldsymbol{\beta}^2}{32}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}\right),$$

where the second equality follows from (A.36) and (A.23). We use the formula (see [53, 21·43]):

$$\frac{\theta_{11}(z,\tau)}{\theta'_{11}(0,\tau)} = z\exp\left(-\sum_{k=1}^{\infty}\frac{G_{2k}(\tau)}{2k}z^{2k}\right),$$

where $G_{2k} = 2\zeta(2k)E_{2k}$ are Eisenstein series, and E_{2k} are normalized Eisenstein series. Using Jacobi's derivative formula ([53, 21·41]), we get

$$\frac{\boldsymbol{\beta}^2}{32}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2} = \log\left[\frac{\theta_{00}\theta_{10}h}{2\sqrt{-1}\Lambda}\right] - \sum_{k=1}^{\infty}\frac{G_{2k}(\tau)}{2k}\left(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}}\right)^{2k}. \tag{4.16}$$

Combining with (4.15), we get an explicit formula of $\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}$ in terms of $q^{1/8}$. For example, we have

$$\frac{\boldsymbol{\beta}^2}{32}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2} = -\frac{u}{6} + \frac{h^2}{24}E_2\boldsymbol{\beta}^2 + \cdots.$$

4.5. **Explicit computations: the case of $\mathbb{P}^2$.** Let H be the hyperplane bundle on $\mathbb{P}^2$, we denote by the same letter its first Chern class. As an illustration of our results we compute the holomorphic Euler characteristics of determinant line bundles on $M_H^{\mathbb{P}_2}(0,d)$ and $M_H^{\mathbb{P}_2}(H,d)$, and write the corresponding Hilbert series explicitly for small d.

The determinant line bundles $\mu(H^{\otimes n})$ are by (1.4) defined on $M_H^{\mathbb{P}_2}(0,d)$ for all n and on $M_H^{\mathbb{P}_2}(H,d)$ for n even. Let Y be the blowup of $\mathbb{P}^2$ in a point, and let E be the exceptional divisor. Denote by H also its pullback to Y, and write $F = H-E$. Then for ϵ sufficiently small $M^Y_{F+\epsilon H}(E,d+1)=\emptyset$, $M^Y_{F+\epsilon H}(H,d)=\emptyset$, and thus $\chi(M^Y_{F+\epsilon H}(E,d+1),\mathcal{O}(\mu(H^{\otimes n})))=0$, $\chi(M^Y_{F+\epsilon H}(H,d),\mathcal{O}(\mu(H^{\otimes 2n})))=0$ for all n. On the other hand we get by Corollary 1.8 $\chi(M^{\mathbb{P}^2}_H(0,d),\mathcal{O}(\mu(H^{\otimes n})))=\chi(M^Y_{H-\epsilon E}(E,d+1),\mathcal{O}(\mu(H^{\otimes n})))$, $\chi(M^{\mathbb{P}^2}_H(H,d),\mathcal{O}(\mu(H^{\otimes 2n})))=\chi(M^Y_{H-\epsilon E}(H,d),\mathcal{O}(\mu(H^{\otimes 2n})))$. Thus we only have to sum the wallcrossing over all the classes ξ of type E (respectively of type H) with $\langle \xi H\rangle>0>\langle\xi F\rangle$. These are $\{2mH-(2l+1)E \mid l\ge m>0\}$ for type E and $\{(2m-1)H-2lE \mid l\ge m>0\}$ for type H.

Putting this into Theorem 4.3 and using the results of subsection §4.4, and putting $\boldsymbol{\beta}=1$, we obtain the following.

$$\begin{aligned}\sum_{d\ge 0}&\chi(M^{\mathbb{P}^2}_H(0,d),\mathcal{O}(\mu(H^{\otimes n})))\Lambda^d\\ &=\operatorname*{Coeff}_{q^0}\Bigg[\sum_{l\ge m>0}(-1)^{l+m+1}q^{\frac12((l+\frac12)^2-m^2)}e^{(m(n+3)-l-1/2)h}\\ &\qquad\Big(-\frac{\theta_{11}(\frac{h}{2\pi\sqrt{-1}})}{\Lambda\theta_{01}}\Big)^{n^2+6n+8}\frac{8\theta_{01}^8}{\Lambda\theta_{00}^3\theta_{10}^3}\frac{1}{\sqrt{1+u+\Lambda^4}}\Bigg],\end{aligned}$$

$$\begin{aligned}\sum_{d\ge 0}&\chi(M^{\mathbb{P}^2}_H(H,d),\mathcal{O}(\mu(H^{\otimes 2n})))\Lambda^d\\ &=\operatorname*{Coeff}_{q^0}\Bigg[\sum_{l\ge m>0}(-1)^{l+m}q^{\frac12(l^2-(m-\frac12)^2)}e^{((m-\frac12)(2n+3)-l)h}\\ &\qquad\Big(-\frac{\theta_{11}(\frac{h}{2\pi\sqrt{-1}})}{\Lambda\theta_{01}}\Big)^{4n^2+12n+8}\frac{8\theta_{01}^8}{\theta_{00}^3\theta_{10}^3}\frac{1}{\sqrt{1+u+\Lambda^4}}\Bigg].\end{aligned}$$

It is straightforward to write a maple program which computes the lower order terms in Λ. This computation can be extended to much higher degrees in Λ, in principle up to any given power. We get

$$\sum_{n\ge 0}\chi(M^{\mathbb{P}^2}_H(0,d),\mathcal{O}(\mu(H^{\otimes n}))t^n=\frac{P_d(t)}{(1-t)^{d+1}},\quad 5\le d\le 21,$$
$$\sum_{n\ge 0}\chi(M^{\mathbb{P}^2}_H(H,d),\mathcal{O}(\mu(H^{\otimes 2n}))t^n=\frac{Q_d(t)}{(1-t)^{d+1}},\quad 0\le d\le 24,$$

with $P_d(t)\in\mathbb{Z}[t]$ of degree $d-5$ with $t^{d-5}P_d(1/t)=P_d$ and $Q_d(t)\in\mathbb{Z}[t]$ of degree $d-2$ with $t^{d-2}Q_d(1/t)=Q_d$ for $d\ge 4$. In particular

$$P_5=1,\ P_9=1+t^2+t^4,\ P_{13}=1+t+7t^2+7t^3+22t^4+7t^5+7t^6+t^7+t^8,$$

$$P_{17}=1+3t+27t^2+83t^3+312t^4+504t^5+680t^6+504t^7+312t^8+83t^9+27t^{10}+3t^{11}+t^{12},$$
$$P_{21}=1+6t+77t^2+484t^3+2877t^4+10374t^5+27027t^6+46992t^7+57532t^8 + 46992t^9+27027t^{10}+10374t^{11}+2877t^{12}+484t^{13}+77t^{14}+6t^{15}+t^{16};$$
$$Q_0=1,\ Q_4=1+t+t^2,\ Q_8=1+12t+57t^2+92t^3+57t^4+12t^5+t^6,$$
$$Q_{12}=1+43t+751t^2+5301t^3+16598t^4+24137t^5+\dots$$
$$Q_{16}=1+109t+5149t^2+103820t^3+976685t^4+4609643t^5+11476395t^6 + 15506676t^7+\dots$$
$$Q_{20}=1+231t+25026t^2+1189860t^3+26750979t^4+308439936t^5 + 1946037411t^6+7038264246t^7+15046564512t^8+19347012191t^9+\dots$$
$$Q_{24}=1+437t+97958t^2+9845240t^3+467190310t^4+11368550417t^5 + 152640855877t^6+1196951395072t^7+5716465354180t^8+17128652740280t^9 + 32841892687972t^{10}+40750517543272t^{11}+\dots,$$

where $\dots$ stands for terms of degree larger than $\deg(Q_d)/2$. One checks that $P_d(1)=\Phi_0^{\mathbb{P}^2}(H^d)$, $Q_d(1)=2^d\Phi_H^{\mathbb{P}^2}(H^d)$, by comparing with [13], as required by the Hirzebruch-Riemann-Roch theorem. In [5],[6] the $\chi(M_H^{\mathbb{P}^2}(0,d),\mathcal{O}(\mu(H^{\otimes n}))$ were determined for $d\le 13$ and all n and for $d=17, n=2,3$.

4.6. **Generalization to non-toric surfaces.** In this section we will generalize our results to arbitrary simply connected surfaces. We extend Corollary 4.2 and Theorem 4.3 for the wallcrossing terms to any good wall ξ on any simply connected projective surface X with $p_g=0$. More generally let X be a smooth projective surface (not necessarily connected), and let $\xi\in \mathrm{Pic}(X)$ and $v\in K(X)$. We define the wallcrossing terms $\Delta^X_{\xi,T}(v,\Lambda)$, $\Delta^X_{\xi}(v,\Lambda)$ by the formulas (2.4),(2.6),(2.7), where we replace in the summation index $d=4(n+m)-\xi^2-3$ by $d=4(n+m)-\xi^2-3\chi(\mathcal{O}_X)$. Then we show that these are computed by a suitable generalization of Corollary 4.2 and Theorem 4.3. This is done by adapting the corresponding argument of [20] for the wallcrossing of the usual Donaldson invariants, which is based on the fact that intersection numbers on Hilbert schemes of points on X are given by universal formulas in terms of intersection numbers on X.

If X is a simply connected with $p_g=0$ and ξ is a *good* class, then Proposition 2.11 shows that the wallcrossing of the K-theoretic Donaldson invariants for the wall defined by ξ is given by the wallcrossing terms, thus we get a formula for the wallcrossing in terms of modular forms and elliptic functions. In the future we plan to adapt the arguments of [40] to show that Proposition 2.11 and thus our wallcrossing formula also holds in case ξ is not good.

We start by sketching a proof of the following result:

Lemma 4.17. *Fix $r \in \mathbb{Z}$. There exist universal power series $A_i \in \Lambda\mathbb{Q}((T))[[\Lambda]]$, ($i = 1, \dots, 7$), such that for all projective surfaces X, $\xi \in \operatorname{Pic}(X)$ and all $v \in K(X)$ of rank r*

$$\frac{(-T)^{\xi(\xi-K_X)/2+\chi(\mathcal{O}_X)}\Lambda^{\xi^2+3\chi(\mathcal{O}_X)}}{T^{\xi v^{(1)}/2}(1-T)^{\xi^2+2\chi(\mathcal{O}_X)}}\Delta^X_{\xi,T}(v,\Lambda) =$$
$$\exp(\xi^2 A_1 + \xi K_X A_2 + K_X^2 A_3 + c_2(X) A_4 + \xi v^{(1)} A_5 + K_X v^{(1)} A_6 + (v^{(1)})^2 A_7).$$

Here, as before $v^{(1)} = c_1(v) + \frac{\operatorname{rk}(v)}{2}(c_1 - K_X)$.

A simple modification of the proof of [20, Lemma 5.5] shows the following.

Lemma 4.18. *Fix $n, m \geq 0$. Let P be any polynomial in $\operatorname{ch}_{i_1}(\mathcal{A}_+)$, $\operatorname{ch}_{i_2}(\mathcal{A}_-)$, $\operatorname{ch}_{i_3}(\mathcal{I}_1)\xi^{i_4}/(v^{(1)})^{i_5}$ $\operatorname{ch}_{i_6}(\mathcal{I}_2)\xi^{i_7}/(v^{(1)})^{i_8}, c_{i_9}(X^{[n]}\times X^{[n]})$ for $i_1, \dots, i_9 \in \mathbb{Z}_{\geq 0}$. Then there exists a universal polynomial Q (depending only on P) in ξ^2, ξK_X, K_X^2, $c_2(X)$, $\xi v^{(1)}$, $K_X v^{(1)}$, $(v^{(1)})^2$, such that $\int_{X^{[n]}\times X^{[m]}} P = Q$.*

The statement is very similar to [20, Lemma 5.5]. The only differences are that we replaced $X_2^{[l]}$ by $X^{[n]} \times X^{[m]}$, and that we also allow the $c_i(X^{[n]} \times X^{[m]})$ in P. However looking at the proof of [20, Lemma 5.5] it obviously also works for $X^{[n]} \times X^{[m]}$, and in [14] the argument is also made for the $c_i(X^{[n]})$. It readily generalizes to $X^{[n]} \times X^{[m]}$.

Denote the left-hand-side of Lemma 4.17 by $\overline{\Delta}^X_{\xi,T}(v,\Lambda)$. By applying the Riemann-Roch theorem to definition (2.6), we obtain that

$$\overline{\Delta}^X_{\xi,T}(v,\Lambda) = \sum_{n,m\geq 0}\sum_{i\in\mathbb{Z}} \Lambda^{4(n+m)} T^i \int_{X^{[n]}\times X^{[m]}} S_{n,m,i},$$

where $S_{n,m,i}$ is a polynomial in the Chern characters of $\mathcal{A}_+$, $\mathcal{A}_-$, $\lambda_{\mathcal{F}_1}(v)$, $\lambda_{\mathcal{F}_2}(v)$ and the $c_j(X^{[n]} \times X^{[n]})$, which is zero for $i \ll 0$. By (3.5) the Chern characters of the $\lambda_{\mathcal{F}_j}(v)$ are polynomials in the $\operatorname{ch}_{i_1}(\mathcal{I}_j)\xi^{i_2}/(v^{(1)})^{i_3}$. Thus by Lemma 4.18 we see that $\overline{\Delta}^X_{\xi,T}(v,\Lambda) = \sum_{l\geq 0}\sum_{i\in\mathbb{Z}} \Lambda^{4l} P_{l,i} T^i$, where $P_{l,i}$ is a universal polynomial in ξ^2, ξK_X, K_X^2, $c_2(X)$, $\xi v^{(1)}$, $K_X v^{(1)}$, $(v^{(1)})^2$, which is zero for $i \ll 0$. From the definition (2.6), one readily computes that the coefficient of Λ^0 of $\overline{\Delta}^X_{\xi,T}(v,\Lambda)$ as a power series in Λ is 1. Now the proof of Lemma 4.17 is finished by the same arguments as that of [20, Theorem 5.1].

Corollary 4.19. (1) *Corollary 4.2 and Theorem 4.3 hold for any simply connected smooth projective surface with $p_g = 0$ and any $\xi \in \mathrm{Pic}(X)$.*

(2) *More generally for any smooth projective surface X and any $\xi \in \mathrm{Pic}(X)$ we have*

$$\Delta^X_{\xi, e^{-2\boldsymbol{\beta} a}}(v, \boldsymbol{\beta}\Lambda) = \sqrt{-1}^{\langle \xi, K_X\rangle} \frac{q^{-\frac{1}{2}(\frac{\xi}{2})^2}}{(\boldsymbol{\beta}\Lambda)^{\chi(\mathcal{O}_X)}} \exp\left(\frac{\boldsymbol{\beta}}{8}\frac{\partial^2 \mathcal{F}_0}{\partial a \partial \log\Lambda}\langle \xi, v^{(1)} + K_X\rangle \right.$$
$$\left. + \frac{\boldsymbol{\beta}^2}{32}\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2}\langle (v^{(1)} + K_X)^2\rangle\right) \exp(A)^{4\chi(\mathcal{O}_X)} \exp(B-A)^{\sigma}$$

Proof. It is enough to show part (2). We put $T^{\frac{1}{2}} := e^{-\boldsymbol{\beta} a}$, and as above write $z = \frac{-\sqrt{-1}\boldsymbol{\beta}\Lambda}{e^{-\boldsymbol{\beta} a} - e^{\boldsymbol{\beta} a}} = \frac{\sqrt{-1}\boldsymbol{\beta}\Lambda T^{\frac{1}{2}}}{1-T}$. Then by (4.8) we have $q^{\frac{1}{8}} = z\exp(l_1)$, with $l_1 \in z^2\mathbb{C}[\boldsymbol{\beta}, T^{\pm\,\mathrm{rk}(v)/2}, \Lambda][[z^2]] \subset \Lambda^2\mathbb{C}[\boldsymbol{\beta}, \Lambda]((T^{\frac{1}{2}}))$. Similarly (4.7) implies $\frac{\partial^2\mathcal{F}_0}{(\partial\log\Lambda)^2} \in \Lambda^2\mathbb{C}[\boldsymbol{\beta},\Lambda]((T^{\frac{1}{2}}))$, $\frac{\partial^2\mathcal{F}_0^{inst}}{\partial a\partial\log\Lambda} \in \Lambda^2\mathbb{C}[\boldsymbol{\beta},\Lambda]((T^{\frac{1}{2}}))$, and from the definition we see that $\frac{\partial^2\mathcal{F}_0^{pert}}{\partial a\partial\log\Lambda} = -8a$. Finally by (1.44) we have $\exp(A) = q^{\frac{1}{16}}\exp(l_2)$, $\exp(B-A) = \exp(l_3)$, with $l_2, l_3 \in \Lambda\mathbb{C}[\boldsymbol{\beta},\Lambda]((T))$. Thus we see that the left hand side of Corollary 4.19 can be rewritten as $M\exp(\xi^2 B_1 + \xi(v^{(1)}+K_X)B_2 + (v^{(1)}+K_X)^2 B_3 + c_2(X)B_4 + K_X^2 B_5)$, with $B_i \in \Lambda\mathbb{C}((T^{\frac{1}{2}}))[[\Lambda]]$ and

$$M = \sqrt{-1}^{\langle\xi,K_X\rangle}\left(\frac{\sqrt{-1}\boldsymbol{\beta}\Lambda T^{\frac{1}{2}}}{1-T}\right)^{-\xi^2 - 2\chi(\mathcal{O}_X)} \frac{T^{\langle\xi(v^{(1)}+K_X)\rangle/2}}{\Lambda^{\chi(\mathcal{O}_X)}}$$
$$= \frac{T^{\xi v^{(1)}/2}(1-T)^{\xi^2+2\chi(\mathcal{O}_X)}}{(-T)^{\xi(\xi-K_X)/2+\chi(\mathcal{O}_X)}\Lambda^{3\chi(\mathcal{O}_X)}}.$$

As the A_i, $i = 1,\dots,7$ of Lemma 4.17 are determined by the $\Delta^X_{\xi,T}(v,\Lambda)$ for toric surfaces, Corollary 4.2 implies the result. □

When $v = v(2L)$, we have $v^{(1)} + K_X = K_X - 2L$, which is equal to the negative of the characteristic line bundle $\det W^{\pm}$ of the $Spin^c$ structure $W^{\pm}$ induced from the complex structure of X and the line bundle L (see §1.3). Then $\sqrt{-1}^{-\langle\xi,K_X\rangle}\left[\Delta^X_\xi(v,\boldsymbol{\beta}\Lambda)\right]_{\Lambda^d}$ is a polynomial in $\langle\xi, c_1(\det W^{\pm})\rangle$ and $\langle c_1(\det W^{\pm})^2\rangle$ whose coefficients depend only on $\langle\xi^2\rangle$, d and the homotopy type of X. This statement is a natural analogue of the Kotschick-Morgan conjecture [29] in the context of the K-theoretic Donaldson invariants. Thus our formula above supports our belief that the K-theoretic Donaldson invariants have a gauge theoretic definition.

Appendix A. Seiberg-Witten curves for K-theoretic version

The purpose of this appendix is to prove some results on Seiberg-Witten curves for the K-theoretic version with Chern-Simons terms. In particular, we show

a) the perturbation part of the Seiberg-Witten prepotential coincides with the genus 0 part of the perturbation part introduced in §1.7.2,
b) the Seiberg-Witten prepotential satisfies the contact term equation in Proposition 1.39.

The corresponding results of the Seiberg-Witten curves for the homological version have been known [25, 38, 50, 18], and were reproduced in [43, §2]. Our proofs go along the same line, while we need to consider the cases $r+m$ even and odd separately. The adaptation might be standard to experts, but we cannot find the statements or proofs in the literature.

A.1. **Seiberg-Witten curves.** We consider a family of curves parametrized by $\vec{U} = (U_1, \dots, U_{r-1})$:

$$C_{\vec{U},m} : (-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r X^{(r+m)/2}\left(w + \frac{1}{w}\right) = P(X),$$
$$P(X) = X^r + U_1 X^{r-1} + U_2 X^{r-2} + \cdots + U_{r-1}X + (-1)^r$$

for $|m| \le r$, $m \in \mathbb{Z}$. We call them *Seiberg-Witten curves*. When $r+m$ is odd, we should understand this expression formally, and the rigorous definition will be given soon below. The projection $C_{\vec{U},m} \ni (w, X) \mapsto X \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution ι is given by $\iota(w) = 1/w$.

We introduce a new variable $Y = (-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r X^{(r+m)/2}(w - \frac{1}{w})$. Thus we have

$$Y^2 = P(X)^2 - 4(-X)^{r+m}(\boldsymbol{\beta}\Lambda)^{2r}.$$

This *does* make sense for $r+m$ odd also.

Note that $|m| \le r$ guarantees that the curve has genus $r-1$. Later we further assume $|m| \neq r$.

If we replace the coordinate X (near 0) by $1/X$ (near ∞), then the equation of the curve becomes

$$Y^2 = X^{2r}\left(P(1/X)^2 - 4(-1/X)^{r+m}(\boldsymbol{\beta}\Lambda)^{2r}\right) = \widetilde{P}(X)^2 - 4(-X)^{r-m}(\boldsymbol{\beta}\Lambda)^{2r},$$

where $\widetilde{P}(X) = X^r + (-1)^r U_{r-1}X^{r-1} + \cdots + (-1)^r$. Thus the curves for m and $-m$ are essentially the same (exactly the same when $r = 2$), once written in a coordinate near 0 and once in a coordinate near infinity.

Let us define the *Seiberg-Witten differential* by

$$\begin{aligned} dS &= \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\log X\frac{dw}{w} = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\log X\,\frac{X^{(r+m)/2}(X^{-(r+m)/2}P(X))'dX}{Y} \\ &= \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\log X\,\frac{2XP'(X)-(r+m)P(X)}{2XY}dX, \end{aligned}$$

where we have used

$$X^{-(r+m)/2}Y\frac{dw}{w} = (-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r\left(w - \frac{1}{w}\right)\frac{dw}{w} = \left(X^{-(r+m)/2}P(X)\right)' dX.$$

This is a multi-valued meromorphic differential on $C_{\vec{U},m}$. The last expression makes sense even in the case $r+m$ odd.

Let $X_1,\ldots, X_r$ be the zeroes of $P(X) = 0$. We have $\prod X_i = 1$.

A.2. **Homological limit $\boldsymbol{\beta} \to 0$.** We move $\boldsymbol{\beta}$ in a small disk around the origin. We see that the Seiberg-Witten curve becomes the Seiberg-Witten curve for the homological version (i.e. the 4-dimensional gauge theory in the physics terminology) at $\boldsymbol{\beta} = 0$.

We choose z_i with $X_i = e^{-\sqrt{-1}\boldsymbol{\beta}z_i}$. We consider $\vec{z} = (z_i)$ is a parameter for the curve. Let $X = \frac{2-\sqrt{-1}\boldsymbol{\beta}z}{2+\sqrt{-1}\boldsymbol{\beta}z}$. Then

$$\begin{aligned} &(-\sqrt{-1}\boldsymbol{\beta})^{-r}X^{-(r+m)/2}P(X) \\ &= \left(1+\frac{\boldsymbol{\beta}^2z^2}{4}\right)^{-\frac{r+m}{2}}\left(1+\frac{\sqrt{-1}\boldsymbol{\beta}}{2}\right)^m\prod_{i=1}^r\left[\frac{e^{-\sqrt{-1}\boldsymbol{\beta}z_i}+1}{2}z - \frac{e^{-\sqrt{-1}\boldsymbol{\beta}z_i}-1}{-\sqrt{-1}\boldsymbol{\beta}}\right]. \end{aligned}$$

If we introduce a new variable $y = (-\sqrt{-1}\boldsymbol{\beta})^{-r}Y(1+\frac{\sqrt{-1}}{2}\boldsymbol{\beta}z)^r$, we have

$$y^2 = \prod_{i=1}^r\left[\frac{e^{-\sqrt{-1}\boldsymbol{\beta}z_i}+1}{2}z - \frac{e^{-\sqrt{-1}\boldsymbol{\beta}z_i}-1}{-\sqrt{-1}\boldsymbol{\beta}}\right]^2 - 4\Lambda^{2r}\left(1+\frac{\boldsymbol{\beta}^2z^2}{4}\right)^{r-m}\left(1-\frac{\sqrt{-1}\boldsymbol{\beta}}{2}z\right)^{2m}.$$

Therefore in the limit $\boldsymbol{\beta} \to 0$, the Seiberg-Witten curve converges to

$$\Lambda^r(w + \frac{1}{w}) = \prod_{i=1}^r (z - z_i) \text{ or } y^2 = \prod_{i=1}^r (z - z_i)^2 - 4\Lambda^{2r}.$$

This is the Seiberg-Witten curve for the homological version. (The variable w is the same.) The Seiberg-Witten differential converges to that of the homological version, i.e. $-\frac{1}{2\pi} z \frac{dw}{w}$.

The points $X = 0, \infty$ correspond to $z = \frac{2\sqrt{-1}}{\boldsymbol{\beta}}, -\frac{2\sqrt{-1}}{\boldsymbol{\beta}}$. Therefore in the limit $\boldsymbol{\beta} \to 0$, both points go to a common point $z = \infty$.

We find $X_i^\pm$ near X_i such that

$$P(X_i^\pm) = \pm 2(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r (X_i^\pm)^{(r+m)/2}.$$

When $r + m$ is odd, we take the branch of $(X_i^\pm)^{1/2}$ so that it is the same branch as $X_i^{1/2} = e^{-\sqrt{-1}\boldsymbol{\beta} z_i/2}$. Let us choose $z_i^\pm$ so that $\prod(z_i^\pm - z_i) = \pm 2\Lambda^r$. Then $X_i^\pm \to z_i^\pm$ (more precisely after moving to the z-coordinates).

The correspondence between the coefficients is more tricky, as U_i is the i^{th} elementary symmetric function in $e^{-\sqrt{-1}\boldsymbol{\beta} z_i}$ while u_i is the i^{th} elementary symmetric function in z_i, up to sign. For example, $r = 2$

$$U_1 = -(e^{-\sqrt{-1}\boldsymbol{\beta} z_1} + e^{-\sqrt{-1}\boldsymbol{\beta} z_2}) \approx -2 + \frac{\boldsymbol{\beta}^2}{2}(z_1^2 + z_2^2) = -2 - \boldsymbol{\beta}^2 u_2.$$

A.3. a_i, a_i^D **and the prepotential** $\mathcal{F}_0$**.** We first work in the region containing $z_1, \dots, z_r \in \mathbb{R}$ and $z_1 > z_2 > \cdots > z_r$. Then we will analytically continue to the whole region. The curve itself is parametrized by $\vec{U}$, but its homology basis introduced below depends on $\vec{z} = (z_i)$. We also first suppose that Λ is a sufficiently small positive real number and then will analytically continue to a small punctured disk.

We take cycles A_i, B_j $(i = 1, \dots, r,\ j = 2, \dots, r)$ so that it gives the cycles for the Seiberg-Witten curves for the homological version given in [42, §2] at $\boldsymbol{\beta} = 0$. Let us explain a little bit more precisely: Our curve $C_{\vec{U}}$ is hyperelliptic and is made up of two copies of the Riemann sphere, glued along the r cuts between X_i^- and X_i^+. We then define A_i as the cycle encircling the cut between X_i^- and X_i^+. Note that we have $\sum_i A_i = 0$. We choose cycles B_j $(j = 2, \dots, r)$ as in [42, Figure 1], i.e. B_j is the sum $\sum_{k=2}^j C_k$ where C_k is a cycle starting from $X_{k-1}^\pm$, passing through $X_k^\pm$, and then returning back to $X_{k-1}^\pm$ in the another sheet. Here the sign is $+$ for i odd, $-$ for i even. Then A_i, B_i $(i = 2, \dots, r)$ form a symplectic basis of $H_1(C_{\vec{U}}, \mathbb{Z})$.

We define a_i, a_j^D by

$$a_i = \int_{A_i} dS, \qquad a_j^D = \int_{B_j} dS, \qquad i = 1, \dots, r, \ j = 2, \dots, r.$$

We consider a region disjoint from a segment from ∞ to 0 which does not pass $e^{-\sqrt{-1}\boldsymbol{\beta} z_i}$. Therefore $\log X$ is single-valued in the region. We take the branch of $\log X$ so that it is given by $-\sqrt{-1}\boldsymbol{\beta} z_i$ at $X_i = e^{-\sqrt{-1}\boldsymbol{\beta} z_i}$. The A_i, B_i cycles are taken from the region.

We have the following expansion:

(A.1)

$$\begin{aligned} a_i &= \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \int_{A_i} \log X d \left[\log \left(\prod_{j=1}^{r} (X^{\frac{1}{2}} - e^{-\sqrt{-1}\boldsymbol{\beta} z_j} X^{-\frac{1}{2}}) \right) - \frac{m}{2} \log X \right] + O(\Lambda) \\ &= -\sqrt{-1} z_i + O(\Lambda). \end{aligned}$$

We invert the roles of a_i and U_p, so we consider a_i as variables and U_p are functions in a_i.

Let us differentiate the defining equation of $C_{\vec{U}}$ with respect to U_p by setting w to be constant:

$$0 = \left(X^{-(r+m)/2} P(X) \right)' \frac{\partial X}{\partial U_p} + X^{(r-m)/2-p}. \tag{A.2}$$

Therefore the differential of the Seiberg-Witten differential dS is

(A.3)

$$\left. \frac{\partial}{\partial U_p} dS \right|_{w=\text{const}} = -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \frac{X^{(r-m)/2-p-1}}{(X^{-(r+m)/2}P(X))'} \frac{dw}{w} = -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \frac{X^{r-p-1} dX}{Y}.$$

It is well-known that these form a basis of holomorphic differentials on $C_{\vec{U}}$ for $p = 1, \dots, r-1$ (see e.g., [21, §2.3]). In other words, the Seiberg-Witten differential is a '*potential*' for holomorphic differentials.

Let (σ_{ip}) be the matrix given by

$$\sigma_{ip} = \frac{\partial a_i}{\partial U_p} = -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \int_{A_i} \frac{X^{r-p-1} dX}{Y} \qquad i = 2, \dots, r, \ p = 1, \dots, r-1.$$

If (σ^{pi}) is the inverse matrix, the normalized holomorphic 1-forms

$$\omega_j = -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \sum_p \sigma^{pj} \frac{X^{r-p-1}dX}{Y} = \frac{\partial}{\partial a_j} dS \Big|_{w=\mathrm{const}}$$

satisfies $\int_{A_i} \omega_j = \delta_{ij}$. Therefore the *period matrix* $\tau = (\tau_{ij})$ of the curve $C_{\vec{U}}$ is given by

$$\tau_{ij} = \int_{B_i} \omega_j = \frac{\partial a_i^D}{\partial a_j}. \tag{A.4}$$

Since (τ_{ij}) is symmetric (see e.g., [21, §2.2]), there exists a locally defined function $\mathcal{F}_0$ such that

$$a_i^D = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial \mathcal{F}_0}{\partial a_j}. \tag{A.5}$$

It is unique up to a function independent of a_i. The ambiguity will be fixed later. This function $\mathcal{F}_0$ is called the *Seiberg-Witten prepotential*. We may also write $\mathcal{F}_0(\vec{a})$ or $\mathcal{F}_0(\vec{a};\Lambda)$.

A.4. **Perturbative part.** We determine the perturbative part of the prepotential $\mathcal{F}_0$ in this subsection.

Let

$$\overline{\gamma}_0(x|\boldsymbol{\beta};\Lambda) = 2\left(\frac{1}{\boldsymbol{\beta}^2}(\mathrm{Li}_3(e^{-\boldsymbol{\beta}x}) - \zeta(3)) + \frac{x^2}{2}\log(\boldsymbol{\beta}\Lambda) + \frac{\pi^2}{6\boldsymbol{\beta}}x\right) - \frac{x^2\pi\sqrt{-1}}{2} - \frac{\boldsymbol{\beta}x^3}{6},$$

where Li_3 is the trilogarithm. See [42, App. B] for the definition and properties of polylogarithms. The relation to the perturbative part $\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda)$ in §1.6 is the following: We have defined $\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda)$ first when $\boldsymbol{\beta}x > 0$ and then analytically continued it to the whole plane. Then we considered $\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x|\boldsymbol{\beta};\Lambda) + \widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(-x|\boldsymbol{\beta};\Lambda)$. The coefficient of $1/\varepsilon_1\varepsilon_2$ is equal to $\overline{\gamma}_0(x|\boldsymbol{\beta};\Lambda)$. See [43, p. 510, the second displayed formula from the bottom]. This becomes regular and its value is $-x^2\left(\log\frac{\sqrt{-1}x}{\Lambda}\right) + \frac{3}{2}x^2$ at $\boldsymbol{\beta} = 0$ ([loc.cit., p.510, the last displayed formula]).

Proposition A.6.

$$\mathcal{F}_0(\vec{a};\Lambda) = -\sum_{i<j} \overline{\gamma}_0(a_i - a_j|\boldsymbol{\beta};\Lambda) - \frac{m\boldsymbol{\beta}}{6}\sum_{i=1}^r a_i^3 + O(\Lambda).$$

The term $-\sum_{i<j}\overline{\gamma}_0(a_i - a_j|\boldsymbol{\beta};\Lambda) - \frac{m\boldsymbol{\beta}}{6}\sum_{i=1}^r a_i^3$ is called the *perturbative part* of $\mathcal{F}_0$. Recall that $\mathcal{F}_0$ was defined up to a function (in Λ) independent of a_i. We, in fact, prove

$$
\begin{aligned}
& -\partial\mathcal{F}_0/\partial a_i = 2\pi\sqrt{-1}a_i^D \\
\text{(A.7)} \quad & = -\sum_{j>1}\overline{\gamma}_0'(a_1 - a_j|\boldsymbol{\beta};\Lambda) + \sum_{j:i<j}\overline{\gamma}_0'(a_i - a_j|\boldsymbol{\beta};\Lambda) - \sum_{j:j<i}\overline{\gamma}_0'(a_j - a_i|\boldsymbol{\beta};\Lambda) \\
& \qquad + \frac{m\boldsymbol{\beta}}{2}\left(a_i^2 - a_1^2\right) + O(\Lambda).
\end{aligned}
$$

Then we take a function so that the above formula holds. The remained ambiguity in $O(\Lambda)$ will be fixed later.

Note that in the $r = 2$ case the term $\frac{m\boldsymbol{\beta}}{6}\sum_{i=1}^r a_i^3$ vanishes as $a_1 + a_2 = 0$. Therefore this does not show up in §1.6.

Let us describe the branch of $\overline{\gamma}_0$. As our $\boldsymbol{\beta}$ is in a small disk around the origin, it is enough for us to fix the branch at $\boldsymbol{\beta} = 0$. Then the ambiguity occurs only at $\log\left(\frac{\sqrt{-1}x}{\Lambda}\right)$. When $z_1 > \cdots > z_r$ and $\Lambda \in \mathbb{R}_{>0}$, a_i is pure imaginary and $\sqrt{-1}(a_i - a_j) \in \mathbb{R}_{>0}$ for $i < j$. We then choose $\log\left(\frac{\sqrt{-1}x}{\Lambda}\right) \in \mathbb{R}$. Therefore we have $a_i^D \in \mathbb{R}$.

Note that

$$
\text{(A.8)} \quad \overline{\gamma}_0'(x|\boldsymbol{\beta};\Lambda) = -2\left(\frac{1}{\boldsymbol{\beta}}(\mathrm{Li}_2(e^{-\boldsymbol{\beta}x}) - \frac{\pi^2}{6}) - x\log(\boldsymbol{\beta}\Lambda)\right) - x\pi\sqrt{-1} - \frac{\boldsymbol{\beta}x^2}{2}.
$$

We denote $\mathrm{Li}_2(e^{-\boldsymbol{\beta}x}) - \frac{\pi^2}{6} - \boldsymbol{\beta}x\log(\boldsymbol{\beta}\Lambda)$ by $\widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}x})$ for brevity.

Our proof is given so that it reduces to the proof of [42, Prop. 2.2] when $\boldsymbol{\beta} \to 0$. (The proof of [42, Prop. 2.2] was based on [25] in turn.)

Proof of Proposition A.6. In the proof we move Λ in a punctured disk by analytic continuation, starting from positive real numbers. Then a_i^D is a multi-valued holomorphic function in Λ.

Let C_i be a cycle starting from $e^{-\sqrt{-1}\boldsymbol{\beta}z_{i-1}^{\pm}}$, passing through $e^{-\sqrt{-1}\boldsymbol{\beta}z_i^{\pm}}$, and then returning back to $e^{-\sqrt{-1}\boldsymbol{\beta}z_{i-1}^{\pm}}$ in the another sheet. Here the sign is $+$ for i odd, $-$ for i even. Then $B_i = \sum_{k=2}^i C_k$.

We note that $\int_{C_i} dS$ is a local function of Λ^{2r}. Since C_i changes to $C_i + A_i - A_{i-1}$ under the analytic continuation along $\Lambda^{2r} \to e^{2\pi\sqrt{-1}}\Lambda^{2r}$, $\int_{C_i} dS - (a_i - a_{i-1})\log\Lambda^{2r}$ is a single valued function on the punctured disk $0 < |\Lambda^{2r}| \ll 1$.

We take a small positive real number δ with $|\Lambda| \ll \delta$ and rewrite the integral as

$$\begin{aligned}\int_{C_i} dS &= 2\int_{e^{-\sqrt{-1}\boldsymbol{\beta} z_{i-1}^{\pm}}}^{e^{-\sqrt{-1}\boldsymbol{\beta} z_i^{\pm}}} dS \\ &= 2\int_{e^{-\sqrt{-1}\boldsymbol{\beta} z_{i-1}^{\pm}}}^{e^{-\sqrt{-1}\boldsymbol{\beta}(z_{i-1}-\delta)}} dS + 2\int_{e^{-\sqrt{-1}\boldsymbol{\beta}(z_{i-1}-\delta)}}^{e^{-\sqrt{-1}\boldsymbol{\beta}(z_i+\delta)}} dS + 2\int_{e^{-\sqrt{-1}\boldsymbol{\beta}(z_i+\delta)}}^{e^{-\sqrt{-1}\boldsymbol{\beta} z_i^{\pm}}} dS.\end{aligned}$$

We first compute the second term. Let us write $\boldsymbol{\beta}' = -\sqrt{-1}\boldsymbol{\beta}$ for brevity. Then

$$\begin{aligned}&-2\pi\int_{e^{\boldsymbol{\beta}'(z_{i-1}-\delta)}}^{e^{\boldsymbol{\beta}'(z_i+\delta)}} dS \\ &= -\int_{z_{i-1}-\delta}^{z_i+\delta}\frac{m\boldsymbol{\beta}' t}{2}dt + \left[\sum_j \frac{\log X}{\boldsymbol{\beta}'}\log\left(\frac{X^{\frac12} - e^{\boldsymbol{\beta}' z_j}X^{-\frac12}}{\boldsymbol{\beta}'\Lambda}\right)\right]_{e^{\boldsymbol{\beta}'(z_{i-1}-\delta)}}^{e^{\boldsymbol{\beta}'(z_i+\delta)}} \\ &\qquad - \int_{z_{i-1}-\delta}^{z_i+\delta}\sum_j\left(\frac{\boldsymbol{\beta}' t}{2} + \log\left(\frac{1-e^{-\boldsymbol{\beta}'(t-z_j)}}{\boldsymbol{\beta}'\Lambda}\right)\right)dt + O(\delta) \\ &= \left[-\frac{m\boldsymbol{\beta}'}{4}z_i^2 + \sum_{j\neq i} z_i\log\left(\frac{1-e^{-\boldsymbol{\beta}'(z_i-z_j)}}{\boldsymbol{\beta}'\Lambda}\right) + \frac{r}{2}\boldsymbol{\beta}' z_i^2 + z_i\log\left(\frac{1-e^{-\boldsymbol{\beta}'\delta}}{\boldsymbol{\beta}'\Lambda}\right) - \frac{r}{4}\boldsymbol{\beta}' z_i^2\right. \\ &\qquad \left. - \frac{1}{\boldsymbol{\beta}'}\sum_{j>i}\widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}'(z_i-z_j)}) + \frac{1}{\boldsymbol{\beta}'}\sum_{j<i}\widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}'(z_j-z_i)}) + \sum_{j<i}\frac{\boldsymbol{\beta}'(z_j-z_i)^2}{2}\right] \\ &\qquad - \Big[\text{the same term with } z_i \to z_{i-1}\Big] + O(\delta).\end{aligned}$$

Here we have determined the branch of log so that this is real-valued when $\boldsymbol{\beta} = 0$ and z_j's are all real with $z_1 > \cdots > z_r$. As $\boldsymbol{\beta}$ is small, we have $\log\left(\frac{1-e^{-\boldsymbol{\beta}'(t-z_j)}}{\boldsymbol{\beta}'\Lambda}\right) \approx \log(t-z_j)$. We may also suppose t is real. Then the branch of $\log(t-z_j)$ is given so that it is a real number, i.e. $\log|t-z_j|$. Therefore when $t < z_j$ (i.e. when we are integrating the summand $j < i$), we have $\log\left(\frac{1-e^{-\boldsymbol{\beta}'(t-z_j)}}{\boldsymbol{\beta}'\Lambda}\right) = \log\left(\frac{1-e^{\boldsymbol{\beta}'(t-z_j)}}{\boldsymbol{\beta}'\Lambda}\right) - \boldsymbol{\beta}'(t-z_j)$, with the branch of log in the right hand is determined so that it is approximated by $\log|z_j - t| = \log(t - z_j)$. Similarly we have $\int \log\left(\frac{1-e^{-\boldsymbol{\beta}'(t-z_j)}}{\boldsymbol{\beta}'\Lambda}\right) dt = -\frac{1}{\boldsymbol{\beta}'}\widehat{\mathrm{Li}}_2(e^{\boldsymbol{\beta}'(t-z_j)}) - \frac{\boldsymbol{\beta}'(t-z_j)^2}{2}$, and the branch of $\widehat{\mathrm{Li}}$ is given by the same way.

Let us turn to the third term:

$$
\begin{aligned}
-2\pi\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} dS &= \frac{1}{\boldsymbol{\beta}'}\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} \log X \frac{dw}{w} \\
&= \frac{1}{\boldsymbol{\beta}'}\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} \log e^{\boldsymbol{\beta}' z_i}\frac{dw}{w} + \frac{1}{\boldsymbol{\beta}'}\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} (\log X - \log e^{\boldsymbol{\beta}' z_i})\frac{dw}{w}.
\end{aligned}
$$

We take a positive number $N_\delta < \delta$ such that

$$
N_\delta^r \left(\frac{P(e^{\boldsymbol{\beta}'(z_i+\delta)})}{\boldsymbol{\beta}'^r e^{r\boldsymbol{\beta}'(z_i+\delta)/2}}\right)^{-1} \ll \delta.
$$

Then for $|\Lambda| < N_\delta$, we have

$$
\sqrt{1 - 4\Lambda^{2r}\left(\frac{P(e^{\boldsymbol{\beta}'(z_i+\delta)})}{\boldsymbol{\beta}'^r e^{r\boldsymbol{\beta}'(z_i+\delta)/2}}\right)^{-2}} = 1 + O(\delta).
$$

We note that $w_{|e^{\boldsymbol{\beta}' z_i^{\pm}}} = \pm 1$.

$$
\begin{aligned}
&\frac{1}{\boldsymbol{\beta}'}\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} \log e^{\boldsymbol{\beta}' z_i}\frac{dw}{w} = z_i[\log w]_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} = z_i\left[\log\frac{Y+P(X)}{2(\boldsymbol{\beta}'\Lambda)^r X^{(r+m)/2}}\right]_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} \\
&= -z_i\log\left[\frac{1}{2(\boldsymbol{\beta}'\Lambda)^r}\left(\frac{P(e^{\boldsymbol{\beta}'(z_i+\delta)})}{w_{|e^{\boldsymbol{\beta}' z_i^{\pm}}} e^{(r+m)\boldsymbol{\beta}'(z_i+\delta)/2}}\right)\right. \\
&\qquad\qquad \left.\times\left(1+\sqrt{1-4\Lambda^{2r}\left(\frac{P(e^{\boldsymbol{\beta}'(z_i+\delta)})}{\boldsymbol{\beta}'^r e^{r\boldsymbol{\beta}'(z_i+\delta)/2}}\right)^{-2}}\right)\right] \\
&= -z_i\left(\sum_{j\neq i}\log\left(\frac{1-e^{-\boldsymbol{\beta}'(z_i-z_j)}}{\boldsymbol{\beta}'\Lambda}\right) + \log\left(\frac{1-e^{-\boldsymbol{\beta}'\delta}}{\boldsymbol{\beta}'\Lambda}\right) + \frac{(r-m)\boldsymbol{\beta}'}{2}z_i\right) + O(\delta),
\end{aligned}
$$

where the branch of log is the same as before.

Claim.

$$
\frac{1}{\boldsymbol{\beta}'}\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} (\log X - \log e^{\boldsymbol{\beta}' z_i})\frac{dw}{w} = O(\delta).
$$

Proof. If $X = e^{\boldsymbol{\beta}' t}$ and $|t - z_i| \approx \delta$, we have

$$\frac{\log X - \log e^{\boldsymbol{\beta}' z_i}}{X - e^{\boldsymbol{\beta}' z_i}} = e^{-\boldsymbol{\beta}' z_i} + O(\delta),$$

$$\frac{X - e^{\boldsymbol{\beta}' z_i}}{\prod_{j\neq i}(e^{\boldsymbol{\beta}' z_i} - e^{\boldsymbol{\beta}' z_j})^{-1} P(X)} = 1 + O(\delta).$$

Thus we get

$$\log X - \log e^{\boldsymbol{\beta}' z_i} = e^{-\boldsymbol{\beta}' z_i + \frac{r}{2}\boldsymbol{\beta}' z_i} \frac{\prod_{j\neq i}(e^{\boldsymbol{\beta}' z_i} - e^{\boldsymbol{\beta}' z_j})^{-1} P(X)}{X^{\frac{r}{2}}} + E(X)$$

with $E(X) = O(\delta^2)$. The integration of $E(X)$ yields $O(\delta^2) O(\log\delta) = O(\delta)$.

For the main part we have

$$\begin{aligned}
\int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} X^{-\frac{r+m}{2}} P(X) \frac{dw}{w} &= \int_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} X^{-\frac{r+m}{2}} P(X) \frac{(X^{-\frac{r+m}{2}} P(X))'}{X^{-\frac{r+m}{2}} Y} dX \\
&= \left[X^{-\frac{r+m}{2}} Y \right]_{e^{\boldsymbol{\beta}'(z_i+\delta)}}^{e^{\boldsymbol{\beta}' z_i^{\pm}}} \\
&= \boldsymbol{\beta}'^{r} O(\delta).
\end{aligned}$$

Since

$$\prod_{j\neq i}(e^{\boldsymbol{\beta}' z_i} - e^{\boldsymbol{\beta}' z_j}) \approx \boldsymbol{\beta}'^{r-1} \prod_{j\neq i}(z_i - z_j),$$

we get the assertion. □

The computation of the first term is similar. Since $O(\Lambda)\log\Lambda = O(\delta)$ for $\Lambda \ll \delta$, we have the following:

(A.9)

$$\begin{aligned}
&-2\pi \int_{C_i} dS - 2 \Bigg[-\frac{r\boldsymbol{\beta}'}{4}(z_i^2 - z_{i-1}^2) - \frac{1}{\boldsymbol{\beta}'} \sum_{j>i} \widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}'(z_i - z_j)}) \\
&\quad + \frac{1}{\boldsymbol{\beta}'} \sum_{j<i} \widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}'(z_j - z_i)}) + \frac{1}{\boldsymbol{\beta}'} \sum_{j>i-1} \widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}'(z_{i-1} - z_j)}) \\
&\quad - \frac{1}{\boldsymbol{\beta}'} \sum_{j<i-1} \widehat{\mathrm{Li}}_2(e^{-\boldsymbol{\beta}'(z_j - z_{i-1})}) + \sum_{j<i} \frac{\boldsymbol{\beta}'(z_j - z_i)^2}{2} - \sum_{j<i-1} \frac{\boldsymbol{\beta}'(z_j - z_{i-1})^2}{2} \Bigg] = O(\delta).
\end{aligned}$$

We now replace $\boldsymbol{\beta}' z_i$ by $\boldsymbol{\beta} a_i$. As $a_i + \sqrt{-1} z_i = O(\Lambda)$ by (A.1), the left hand side is still $O(\delta)$ after the replacement. Since the LHS is a single valued holomorphic function of Λ on $0 < |\Lambda| < N_\delta$, it is extended to a holomorphic function on

$|\Lambda| < N_\delta$. Since the LHS does not depend on δ, it is 0 at $\Lambda = 0$. Thus the left hand side of (A.9) is $O(\Lambda)$.

Therefore we have

(A.10)

$$\begin{aligned}
&2\pi\sqrt{-1}a_i^D = 2\pi\sqrt{-1}\sum_{k=2}^{i}\int_{C_k} dS \\
&= \Bigg[-\frac{(r-m)\boldsymbol{\beta}}{2}(a_i^2 - a_1^2) - \frac{2}{\boldsymbol{\beta}}\sum_{j>i}\left(\mathrm{Li}_2(e^{-\boldsymbol{\beta}(a_i-a_j)}) - \frac{\pi^2}{6}\right) \\
&\qquad + \frac{2}{\boldsymbol{\beta}}\sum_{j<i}\left(\mathrm{Li}_2(e^{-\boldsymbol{\beta}(a_j-a_i)}) - \frac{\pi^2}{6}\right) + \frac{2}{\boldsymbol{\beta}}\sum_{j>1}\left(\mathrm{Li}_2(e^{-\boldsymbol{\beta}(a_1-a_j)}) - \frac{\pi^2}{6}\right) \\
&\qquad\quad + \sum_{j<i}\boldsymbol{\beta}(a_j - a_i)^2 + 2r(a_i - a_1)\log(\boldsymbol{\beta}\Lambda) - r\pi\sqrt{-1}(a_i - a_1)\Bigg] + O(\Lambda).
\end{aligned}$$

By (A.8) we get (A.7). □

A.5. **A renormalization group equation.** We assume $m \neq \pm r$ hereafter.

We give an analogue of the renormalization group equation for the homological version (see [42, §2.4]).

We set w to be constant and differentiate the defining equation of $C_{\vec{U}}$ with respect to $\log\Lambda$ to get

$$\frac{\partial X}{\partial \log\Lambda} = \frac{rX^{-(r+m)/2}P(X)}{(X^{-(r+m)/2}P(X))'} - \sum_{p=1}^{r-1}\frac{\partial U_p}{\partial \log\Lambda}\frac{X^{(r+m)/2-p}}{(X^{-(r+m)/2}P(X))'}.$$

Therefore

(A.11)

$$\begin{aligned}
&\frac{\partial}{\partial\log\Lambda}dS\Big|_{w=\mathrm{const}} = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\frac{\partial X}{\partial\log\Lambda}\frac{dw}{Xw} \\
&\qquad = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\left[\frac{rX^{-(r+m)/2}P(X)}{(X^{-(r+m)/2}P(X))'} - \sum_{p=1}^{r-1}\frac{\partial U_p}{\partial\log\Lambda}\frac{X^{(r+m)/2-p}}{(X^{-(r+m)/2}P(X))'}\right]\frac{dw}{Xw} \\
&\qquad = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\left[\frac{rP(X)dX}{XY} - \sum_{p=1}^{r-1}\frac{\partial U_p}{\partial\log\Lambda}\frac{X^{r-p-1}dX}{Y}\right].
\end{aligned}$$

We thus have

$$0 = \frac{\partial a_i}{\partial \log\Lambda} = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\int_{A_i}\frac{rP(X)dX}{XY} + \sum_{p=1}^{r-1}\frac{\partial U_p}{\partial\log\Lambda}\frac{\partial a_i}{\partial U_p}, \tag{A.12}$$

$$\frac{\partial a_i^D}{\partial \log\Lambda} = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\int_{B_i}\frac{rP(X)dX}{XY} + \sum_{p=1}^{r-1}\frac{\partial U_p}{\partial\log\Lambda}\frac{\partial a_i^D}{\partial U_p}. \tag{A.13}$$

Combining these equalities we get

$$\frac{\partial a_i^D}{\partial\log\Lambda} = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\left[\int_{B_i}\frac{rP(X)dX}{XY} - \sum_{j=2}^{r}\frac{\partial a_i^D}{\partial a_j}\int_{A_j}\frac{rP(X)dX}{XY}\right].$$

From (A.12) the meromorphic differential $\frac{2\pi\sqrt{-1}\boldsymbol{\beta}}{r}\left.\frac{\partial}{\partial\log\Lambda}dS\right|_{w=\mathrm{const}}$ has vanishing A-periods. Its poles are inverse images of $X = 0, \infty$. As they are not branch points, we have four points. Let us denote them by 0_+, ∞_- $(w=\infty)$, 0_-, ∞_+ $(w=0)$. This convention is taken so that their residues are given by

$$0_\pm : \pm 1, \quad \infty_\pm : \pm 1.$$

The assumption $m \neq \pm r$ is used here, otherwise $X = 0$, ∞ may not correspond to $w = 0, \infty$.

By the Riemann bilinear relation (see e.g., [21, §2.2]) we have

$$-\frac{1}{2\pi\sqrt{-1}}\frac{\partial^2\mathcal{F}_0}{\partial a_i\partial\log\Lambda} = \frac{\partial a_i^D}{\partial\log\Lambda} = \frac{r}{\boldsymbol{\beta}}\int_{0_-+\infty_-}^{0_++\infty_+}\omega_i = \frac{2r}{\boldsymbol{\beta}}\int_{\infty_-}^{0_+}\omega_i, \tag{A.14}$$

where we have used the hyperelliptic involution ι in the second equality. The path of the integral is taken disjoint from the cycles A_i, B_i.

When $\boldsymbol{\beta}\to 0$, two points $X = 0$, ∞ converge to a single point $z=\infty$ as we observed in §A.2. Here more precisely, 0_+, ∞_- go to $z=\infty$, $w=\infty$ and 0_-, ∞_+ goes to $z=\infty$, $w=0$.

As $\omega_i = \frac{\partial}{\partial a_i}dS$, this equation suggests $\frac{\partial\mathcal{F}_0}{\partial\log\Lambda} = -\frac{4\pi\sqrt{-1}r}{\boldsymbol{\beta}}\int_{\infty_-}^{0_+}dS$. However the integral does not make sense as dS has singularities at 0_+ and ∞_-. We overcome the difficulty by introducing a new differential

$$dS' = \frac{Y}{P(X)}dS = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\log X\,\frac{X^{(r+m)/2}(X^{-(r+m)/2}P(X))'dX}{P(X)}.$$

Then $dS - dS'$ can be integrated from 0_+ to ∞_-. From (A.3, A.11) we have

(A.15)
$$\begin{aligned} \frac{\partial}{\partial a_i} dS' \Big|_{w=\text{const}} &= -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \sum_p \frac{\partial U_p}{\partial a_i} \frac{X^{r-p-1} dX}{P(X)}, \\ \frac{\partial}{\partial \log \Lambda} dS' \Big|_{w=\text{const}} &= \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \left[\frac{r dX}{X} - \sum_{p=1}^{r-1} \frac{\partial U_p}{\partial \log \Lambda} \frac{X^{r-p-1} dX}{P(X)} \right]. \end{aligned}$$

Differentiating $P(X) = \prod(X - e^{-\sqrt{-1}\boldsymbol{\beta} z_i})$ by U_p, we get

$$\frac{X^{r-p-1}}{P(X)} = \sqrt{-1}\boldsymbol{\beta} \sum_i \frac{e^{-\sqrt{-1}\boldsymbol{\beta} z_i}}{X(X - e^{-\sqrt{-1}\boldsymbol{\beta} z_i})} \frac{\partial z_i}{\partial U_p} = \sqrt{-1}\boldsymbol{\beta} \sum_i \frac{1}{X - e^{-\sqrt{-1}\boldsymbol{\beta} z_i}} \frac{\partial z_i}{\partial U_p},$$

where we have used $\sum_i z_i = 0$. Therefore we have

(A.16)
$$\begin{aligned} \int_{\infty_-}^{0_+} \frac{X^{r-p-1}}{P(X)} dX &= \sqrt{-}\boldsymbol{\beta} \left[\sum_i \frac{\partial z_i}{\partial U_p} \log(X - e^{-\sqrt{-1}\boldsymbol{\beta} z_i}) \right]_{X=\infty}^{X=0} \\ &= \sqrt{-1}\boldsymbol{\beta} \sum_i \frac{\partial z_i}{\partial U_p} \log(-e^{-\sqrt{-1}\boldsymbol{\beta} z_i}) - \sum_i \frac{\partial z_i}{\partial U_p} \log(1 - e^{-\sqrt{-1}\boldsymbol{\beta} z_i}/X)_{|X=\infty} \\ &= \boldsymbol{\beta}^2 \sum_i \frac{\partial z_i}{\partial U_p} z_i = \frac{\boldsymbol{\beta}^2}{2} \frac{\partial}{\partial U_p} \sum_i z_i^2 = \frac{\boldsymbol{\beta}^2}{2r} \frac{\partial}{\partial U_p} \sum_{i<j} (z_i - z_j)^2, \end{aligned}$$

where we take a path in the upper half plane and we also used $\sum_i z_i = 0$. Therefore

$$\int_{\infty_-}^{0_+} \frac{\partial}{\partial a_i} dS' \Big|_{w=\text{const}} = -\frac{\boldsymbol{\beta}}{4\pi\sqrt{-1}r} \frac{\partial}{\partial a_i} \sum_{j<k} (z_j - z_k)^2$$

Combining with (A.14), we get

$$-\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}_0}{\partial a_i \partial \log \Lambda} = \frac{2r}{\boldsymbol{\beta}} \frac{\partial}{\partial a_i} \left[\int_{\infty_-}^{0_+} \left(dS - dS'\right) - \frac{\boldsymbol{\beta}}{4\pi\sqrt{-1}r} \sum_{j<k} (z_j - z_k)^2 \right].$$

Therefore we have

$$\text{(A.17)} \qquad -\frac{1}{2\pi\sqrt{-1}} \frac{\partial \mathcal{F}_0}{\partial \log \Lambda} = \frac{2r}{\boldsymbol{\beta}} \int_{\infty_-}^{0_+} \left(dS - dS'\right) - \frac{1}{2\pi\sqrt{-1}} \sum_{i<j} (z_i - z_j)^2$$

up to a function of Λ independent of a_α. The right hand side has a perturbative expansion as

$$\frac{1}{2\pi\sqrt{-1}} \sum_{i<j} (a_i - a_j)^2 + O(\Lambda).$$

This is exactly equal to the one given in Proposition A.6. Therefore we finally fix the ambiguity of $\mathcal{F}_0$ in $O(\Lambda)$ so that (A.17) holds.

When $\boldsymbol{\beta} \to 0$, both points 0_+, ∞_- converge to $z = \infty$, $w = \infty$. We have $dS = dS'$ at the limit point. Therefore the first integral disappears in the limit and we get

$$\left.\frac{\partial \mathcal{F}_0}{\partial \log \Lambda}\right|_{\boldsymbol{\beta}=0} = \sum_{i<j} (z_i - z_j)^2.$$

This is nothing but the renormalization group equation [42, 2.3] in the homological version. On the other hand, if $\boldsymbol{\beta}$ stays nonzero, $\frac{\partial \mathcal{F}_0}{\partial \log \Lambda}$ could not be expressed as a simple function in U_p.

We differentiate (A.17) by $\log \Lambda$:

$$\begin{aligned}
&-\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2} = \frac{2r}{\boldsymbol{\beta}} \int_{\infty_-}^{0_+} \frac{\partial}{\partial \log \Lambda} \left(dS - dS' \right) - \frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial \log \Lambda} \sum_{i<j} (z_i - z_j)^2 \\
&= \frac{r}{\pi\sqrt{-1}\boldsymbol{\beta}^2} \int_{\infty_-}^{0_+} \left[\frac{r(P(X) - Y)dX}{XY} - \sum_{p=1}^{r-1} \frac{\partial U_p}{\partial \log \Lambda} \left(\frac{X^{r-p-1}}{Y} - \frac{X^{r-p-1}}{P(X)} \right) dX \right] \\
&\qquad - \frac{1}{2\pi\sqrt{-1}} \sum_{i<j} \sum_{p=1}^{r-1} \frac{\partial U_p}{\partial \log \Lambda} \frac{\partial}{\partial U_p} (z_i - z_j)^2 \\
&= \frac{r}{\pi\sqrt{-1}\boldsymbol{\beta}^2} \int_{\infty_-}^{0_+} \left[\frac{r(P(X) - Y)dX}{XY} - \sum_{p=1}^{r-1} \frac{\partial U_p}{\partial \log \Lambda} \frac{X^{r-p-1}dX}{Y} \right],
\end{aligned}$$

where we have used (A.16) in the last equality.

Let us consider

$$\frac{(P(X) - Y)dX}{2XY} - \frac{1}{2r} \sum_{p=1}^{r-1} \frac{\partial U_p}{\partial \log \Lambda} \frac{X^{r-p-1}dX}{Y}. \tag{A.18}$$

From (A.12) and $\int_{A_\alpha} \frac{dX}{X} = 0$, it also has the vanishing A-periods. Its poles are 0_- and ∞_+ with residues -1 and 1 respectively. These properties characterize the meromorphic differential form uniquely. Let us denote it by $\omega_{\infty_+ - 0_-}$ as customary. Substituting this into above, we get

$$\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2} = \frac{4r^2}{\boldsymbol{\beta}^2} \int_{0_+}^{\infty_-} \omega_{\infty_+ - 0_-}. \tag{A.19}$$

A.6. **Case $r+m$ even.** We assume that $r+m$ is even in this subsection.

Recall that we set $X_1, \dots, X_r$ be the zeroes of $P(X)=0$. For small Λ, we can find $X_i^{\pm}$ near X_i such that $P(X_i^{\pm}) = \pm 2(-\sqrt{-1}\beta\Lambda)^r (X_i^{\pm})^{(r+m)/2}$. These are branch points of the Seiberg-Witten curve $C_{\vec{U},m}$. We have a natural partition of them as $\{X_i^+\} \sqcup \{X_i^-\}$, which corresponds to the even half-integer characteristic E. It is the same as one in the homological version, i.e. ${}^t(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots)$. This is true regardless of the parity of r.

Recall that the Szegö kernel of the hyperelliptic curve is explicitly given by

$$\begin{aligned}\Psi_E(X_1, X_2) &= \frac{\Theta_E(\int_{X_1}^{X_2} \vec{\omega}|\tau)}{\Theta_E(0) E(X_1, X_2)} = \frac{1}{2}\left(\sqrt[4]{\frac{\psi_E(X_1)}{\psi_E(X_2)}} + \sqrt[4]{\frac{\psi_E(X_2)}{\psi_E(X_1)}}\right) \frac{\sqrt{dX_1 dX_2}}{X_2 - X_1} \\ &= \frac{Y_2 \prod (X_1 - X_\alpha^+) + Y_1 \prod (X_2 - X_\alpha^+)}{2(X_2 - X_1)} \sqrt{\frac{dX_1 dX_2}{Y_1 Y_2 \prod (X_1 - X_\alpha^+)(X_2 - X_\alpha^+)}},\end{aligned}$$

where E is the prime form and

$$\psi_E(X) = \frac{\prod (X - X_\alpha^+)}{\prod (X - X_\alpha^-)} = \frac{P(X) - 2(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r X^{(r+m)/2}}{P(X) + 2(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r X^{(r+m)/2}}.$$

See [15, p.12 Example]. We have $\psi_E(0_\pm) = \psi_E(\infty_\pm) = 1$. Therefore

$$E(0_-, \infty_+)^2 \, dX_1|_{X_1 = 0_-} \left(\frac{dX_2}{X_2^2}\right)\bigg|_{X_2 = \infty_+} = \frac{\Theta_E(\int_{0_-}^{\infty_+} \vec{\omega}|\tau)^2}{\Theta_E(0)^2}. \tag{A.20}$$

On the other hand, [15, p.17, Remark v)] we have

$$\begin{aligned}&E(0_-, \infty_+)^2 \, dX_1|_{X_1 = 0_-} \left(\frac{dX_2}{X_2^2}\right)\bigg|_{X_2 = \infty_+} \\ &\qquad = \exp\left\{\int_{0_+}^{\infty_-} \omega_{\infty_+ - 0_-} + \sum_{i=2}^{r} m_i \int_{0_+}^{\infty_-} \omega_i, \right\}\end{aligned} \tag{A.21}$$

where $m_i = \frac{1}{2\pi}\int_{A_i} d\arg \frac{X - X(\infty_+)}{X - X(0_-)}$. In our situation, this is equal to 0. We thus get

$$\frac{\Theta_E(\int_{0_-}^{\infty_+} \vec{\omega})^2}{\Theta_E(0)^2} = \exp\left\{\int_{0_+}^{\infty_-} \omega_{\infty_+ - 0_-}\right\}. \tag{A.22}$$

By (A.19) and (A.14) we get

$$\frac{\Theta_E(\frac{\boldsymbol{\beta}}{2r}\frac{\partial a_\alpha^D}{\partial \log \Lambda}|\tau)}{\Theta_E(0|\tau)} = \exp\left\{\frac{\boldsymbol{\beta}^2}{8r^2}\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2}\right\}. \tag{A.23}$$

Thus we get the contact term equation [43, (4.12) with $d = \frac{r}{2}$]. More precisely, the above holds up to sign. However both sides go to 1 when $\boldsymbol{\beta} \to 0$, so the above holds without the sign ambiguity.

A.6.1. *A differential equation for U_p.* By [15, Prop. 2.10 (38)] we have

(A.24)
$$\frac{\Psi_E(X, 0_-)\Psi_E(X, \infty_+)}{\Psi_E(0_-, \infty_+)} = \omega_{\infty_+ - 0_-} + \sum_{i=2}^{r} \left[\frac{\partial \log \Theta_E}{\partial \xi_i} (\int_{0_-}^{\infty_+} \vec{\omega}) - \frac{\partial \log \Theta_E}{\partial \xi_i}(0) \right] \omega_i(X).$$

The left hand side is equal to

$$\frac{(P(X) - Y)dX}{2XY}.$$

As E is an even characteristic, $\frac{\partial \log \Theta_E}{\partial \xi_\alpha}(0) = 0$. Looking at (A.18) we have

$$\frac{1}{2r} \sum_{p=1}^{r-1} \frac{\partial U_p}{\partial \log \Lambda} \frac{X^{r-p-1} dX}{Y} = \sum_i \frac{\partial \log \Theta_E}{\partial \xi_i} (\int_{0_-}^{\infty_+} \vec{\omega}) \omega_i(X).$$

In other words,

$$\text{(A.25)} \qquad \frac{1}{2r} \frac{\partial U_p}{\partial \log \Lambda} = -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}} \sum_i \left. \frac{\partial \log \Theta_E}{\partial \xi_i} \right|_{\vec{\xi} = -\frac{\boldsymbol{\beta}}{4\pi\sqrt{-1}r} \frac{\partial^2 \mathcal{F}_0}{\partial \log \Lambda \partial \vec{a}}} \frac{\partial U_p}{\partial a_i}.$$

This is an analog of the equation in [42, Th. 2.4]. This equation suggests that it is possible to define U_p in terms of the instanton counting as in the homological version.

A.6.2. *Higher order equations.* By [15, Cor. 2.19 (43)] we have

$$\frac{\Theta_E(\sum_{i=1}^d y_i - \sum_{i=1}^d x_i)}{\Theta_E(0)} \frac{\prod_{i<j} E(x_i, x_j) E(y_j, y_i)}{\prod_{i,j} E(x_i, y_j)} = \det \left(\frac{\Theta_E(y_j - x_i)}{\Theta_E(0) E(x_i, x_j)} \right)$$
$$= \det \left(\Psi_E(x_i, y_j) \right).$$

Let us study the limit of this equation when all x_i (resp. y_j) goes to 0_- (resp. ∞_+). As $E(x_i, x_j) = \frac{(x_i - x_j)}{\sqrt{dx_i}\sqrt{dx_j}} \left(1 + O(x_i - x_j)^2\right)$, we have

$$\frac{\det \left(\Psi_E(x_i, y_j) \right)}{\prod_{i<j} E(x_i, x_j) E(y_j, y_i)} \to (-1)^{d(d-1)/2} \det \left(\frac{1}{i!j!} \partial_x^i \partial_y^j (\Psi_E)(x, y) \Big|_{\substack{x = 0_- \\ y = \infty_+}} \right)_{0 \le i,j \le d-1}.$$

Therefore the answer depends only on the differentials of Ψ_E up to order $d-1$. Note that

$$\psi_E(X) = \frac{P(X) - 2X^{(r+m)/2}(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r}{P(X) + 2X^{(r+m)/2}(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r}$$

$$= 1 - \frac{4X^{(r+m)/2}(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r}{P(X) + 2X^{(r+m)/2}(-\sqrt{-1}\boldsymbol{\beta}\Lambda)^r} = \begin{cases} 1 + O(X^{(r+m)/2}) & \text{as } X \to 0, \\ 1 + O(X^{-(r-m)/2}) & \text{as } X \to \infty. \end{cases}$$

Therefore we can replace either $\psi_E(x_i)$ or $\psi_E(y_i)$ by 1 when we compute the limit if $0 \le d \le \max(r+m, r-m)/2$. We may assume $m \le 0$ without loss of generality. Then we can replace $\psi_E(y_i)$ by 1. Thus

(A.26)

$$\frac{\Theta_E(d\int_{0_-}^{\infty_+}\vec{\omega})}{\Theta_E(0)} = \left.\frac{\det \Psi_E(x_i, y_j) \prod_{i,j} E(x_i, y_j)}{\prod_{i<j} E(x_i, x_j) E(y_j, y_i)}\right|_{\substack{x_i = 0_- \\ y_j = \infty_+}}$$

$$= \det\left(\frac{1}{2}\left(\sqrt[4]{\psi_E(x_i)} + \frac{1}{\sqrt[4]{\psi_E(x_i)}}\right)\frac{\sqrt{dx_i}\sqrt{dy_j}}{y_j - x_i}\right)\left.\frac{\prod_{i,j} E(x_i, y_j)}{\prod_{i<j} E(x_i, x_j) E(y_j, y_i)}\right|_{\substack{x_i = 0_- \\ y_j = \infty_+}}$$

$$= \prod_{i=1}^{d}\left(\frac{1}{2}\left(\sqrt[4]{\psi_E(x_i)} + \frac{1}{\sqrt[4]{\psi_E(x_i)}}\right)\sqrt{dx_i}\right)\prod_{j=1}^{d}\sqrt{dy_j}$$

$$\times \det\left(\frac{1}{y_j - x_i}\right)\left.\frac{\prod_{i,j} E(x_i, y_j)}{\prod_{i<j} E(x_i, x_j) E(y_j, y_i)}\right|_{\substack{x_i = 0_- \\ y_j = \infty_+}}$$

$$= \left(E(0_-, \infty_+)\left.\sqrt{dX_1}\right|_{X_1 = 0_-}\left.\left(\frac{\sqrt{dX_2}}{X_2}\right)\right|_{X_2 = \infty_+}\right)^{d^2}$$

$$= \exp\left(\frac{d^2}{2}\int_{0_+}^{\infty_-}\omega_{\infty_+ - 0_-}\right),$$

where we have used (A.20, A.22) in the last equality. Hence we get

$$\text{(A.27)} \qquad \frac{\Theta_E(\frac{d\boldsymbol{\beta}}{2r}\frac{\partial a_\alpha^D}{\partial \log \Lambda}|\tau)}{\Theta_E(0|\tau)} = \exp\left\{\frac{d^2\boldsymbol{\beta}^2}{8r^2}\frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2}\right\}$$

for $0 \le d \le \max(r+m, r-m)/2$. This is the same equation derived in Proposition 1.39 under the assumption (1.37).

A.7. **Case $r+m$ odd.** We assume that $r+m$ is odd in this subsection.

Let us introduce a new variable $W = \sqrt{X}$ and consider the branched double covering $p\colon \widehat{C}_{\vec{U},m} \to C_{\vec{U},m}$ given by

$$Y^2 = P(W^2)^2 - 4(-W^2)^r(\boldsymbol{\beta}\Lambda)^{2r} \\ = \left(P(W^2) - 2(\sqrt{-1}W\boldsymbol{\beta}\Lambda)^r\right)\left(P(W^2) + 2(\sqrt{-1}W\boldsymbol{\beta}\Lambda)^r\right). \tag{A.28}$$

The branched points are $X = 0_\pm$, $\infty_\pm$. The genus of $\widehat{C}_{\vec{U}}$ is $2r-1$.

For the new curve $\widehat{C}_{\vec{U},m}$ the calculation of the previous section can be applied. We then use formulas in [15, §5] relating the theta functions for $\widehat{C}_{\vec{U},m}$ and those for $C_{\vec{U},m}$. This is our strategy to prove the contact term equation for the $r+m$ odd case.

Let us fix notations. See [loc. cit.] for more detail. Let ϕ be the involution $W \mapsto -W$ corresponding to the projection p. We choose a symplectic basis A_2, B_2,…, A_r, B_r, A_*, B_*, A'_2, B'_2,…, A'_r, B'_r of $H_1(\widehat{C}_{\vec{U}}, \mathbb{Z})$ as in Figure 1, where the involution ϕ is the rotation by π about the vertical axis passing through $0_\pm$, $\infty_\pm$. They satisfy

(1) A_i, B_i $i = 2, \dots r$ are taken so that they are in a single sheet of p and mapped to the corresponding cycles in the original curve $C_{\vec{U}}$,
(2) $A'_i = -\phi(A_i)$, $B'_i = -\phi(B_i)$,
(3) $A_* + \phi(A_*) = 0 = B_* + \phi(B_*)$.

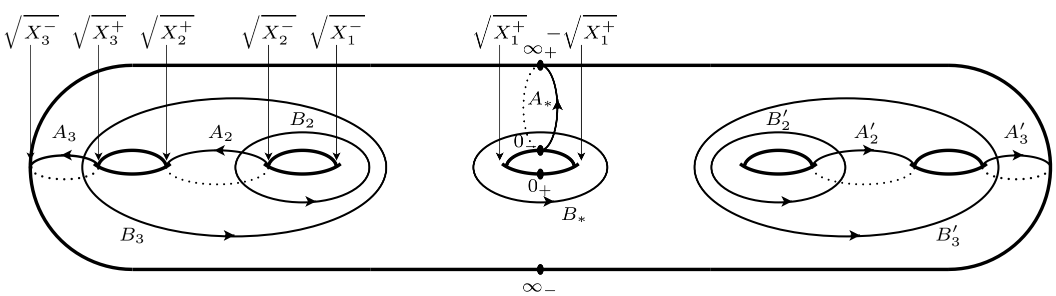

FIGURE 1. Double cover of the Seiberg-Witten curve for $r = 3$, m: even

The normalized holomorphic differentials $\hat{\omega}_i$, $\hat{\omega}_*$, $\hat{\omega}'_i$ on $\widehat{C}_{\vec{U}}$ satisfy

$$\phi^*\hat{\omega}_i = -\hat{\omega}'_i, \qquad \phi^*\hat{\omega}_* = -\hat{\omega}_*$$

and are related to those on $C_{\vec{U}}$ as

$$p^*\omega_i = \hat{\omega}_i - \hat{\omega}'_i.$$

We denote a vector in $\mathbb{C}^{2r-1}$ by $[\xi, \eta, \xi']$ with $\xi, \xi' \in \mathbb{C}^{r-1}$, $\eta \in \mathbb{C}$. Let $\pi^*\colon J_0(C_{\vec{U}}) \to J_0(\widehat{C}_{\vec{U}})$ be the pull-back homomorphism of the divisor classes. It lifts to a map $\mathbb{C}^{r-1} \to \mathbb{C}^{2r-1}$ by

$$\pi^*(\xi) = [\xi, 0, -\xi]\,.$$

Let us choose two points S, T from four branched points $0_\pm$, $\infty_\pm$. Let S', T' be the remaining two points. Let $\xi_0 = \frac{1}{4}\int_{S+T}^{S'+T'} \vec{\omega}$ where $\vec{\omega} = (\omega_2, \dots, \omega_r)$ is the vector of the normalized holomorphic differentials. Then [15, p.91 (102)] says that there exists a unique half-period $[0, c_*, 0] \in J_0(\widehat{C}_{\vec{U}})$ such that

$$k_0 := \frac{\widehat{\Theta}_{[c, c_*, -c]}(\pi^*\xi)}{\Theta_c(\xi + \xi_0)\Theta_c(\xi - \xi_0)} \tag{A.29}$$

is independent of $\xi \in \mathbb{C}^{r-1}$ and a half-integer characteristic c for the curve $C_{\vec{U}}$. We choose $S, T = 0_-, \infty_-$, so

$$\xi_0 = \frac{1}{4}\int_{0_- + \infty_-}^{0_+ + \infty_+} \vec{\omega} = \frac{1}{2}\int_{0_-}^{\infty_+} \vec{\omega}. \tag{A.30}$$

The double cover $\widehat{C}_{\vec{U}}$ is also a hyperelliptic curve by the involution $\widehat{\iota}\colon Y \mapsto -Y$. In Figure 1 the involution $\widehat{\iota}$ is the rotation by π about the horizontal axis. Note that 0_- and ∞_+ lie in the *same* sheet of the covering $\widehat{C}_{\vec{U}} \to \widehat{C}_{\vec{U}}/\widehat{\iota} = \mathbb{P}^1$ as we have $P(X) \approx Y$ at both points. (We have $P(X) \approx -Y$ in another sheet.) The sheet is the upper part of $\widehat{C}_{\vec{U}}$ in Figure 1.

The branched points are $W = \sqrt{X_i^\pm}, -\sqrt{X_i^\pm}$. (Recall that we have fixed the branch of $\sqrt{X_i^\pm}$ so that $\sqrt{X_i^\pm} \approx \sqrt{X_i} = e^{-\sqrt{-1}\boldsymbol{\beta} z_i/2}$. We have a natural partition of them as $\{\sqrt{X_i^+}, -\sqrt{X_i^-}\} \sqcup \{\sqrt{X_i^-}, -\sqrt{X_i^+}\}$. It corresponds to the factorization of the right hand side of (A.28). Let $\widehat{E}$ be the corresponding even theta characteristic. We now repeat the argument in §A.6. We do not determine the characteristic $\widehat{E}$ explicitly at this moment, as the argument goes through if $\widehat{E}$ corresponds to the above partition. We need to take the path $0_+ \to \infty_-$ disjoint from A, B-cycles. This can be accomplished if we shift A_* a little bit. For this choice, m_i appeared in (A.21) is also 0. The remaining arguments are unchanged, and by (A.26) we get

$$\frac{\widehat{\Theta}_{\widehat{E}}(2d\int_{0_-}^{\infty_+} \vec{\omega}\hat{})}{\widehat{\Theta}_{\widehat{E}}(0)} = \exp\left\{2d^2 \int_{0_+}^{\infty_-} \widehat{\omega}_{\infty_+ - 0_-}\right\}, \tag{A.31}$$

for $2d \le \max(r+m, r-m)$ (i.e. $d \le (\max(r+m,r-m)-1)/2$), where $\vec{\omega}\hat{}$ is the vector of the normalized holomorphic differentials of $\widehat{C}_{\vec{U}}$, and $\widehat{\omega}_{\infty_+ - 0_-}$ is the meromorphic differential with $\operatorname{Res}_{\infty_+} = +1$, $\operatorname{Res}_{0_-} = -1$ having the vanishing A-periods.

Lemma A.32. *The characteristic* $\widehat{E}$ *is of the form* $[E, c_*, -E]$ *where the half-period* $[0, c_*, 0]$ *corresponds to the partition* $\{0_+, \infty_+\} \sqcup \{0_-, \infty_-\}$ *as above.*

Proof. We took the idea of proof from that of [15, Prop. 5.3]. We pinch two cycles in $\widehat{C}_{\vec{U}}$ as in Figure 2. The limit is the union of a genus 1 curve C_* (containing A_*, B_*) and two copies of $C_{\vec{U}}$. These curves are glued at P and Q as in Figure 2, i.e., two points P, Q in C_* are identified with a point in $C_{\vec{U}}$ and its copy in another $C_{\vec{U}}$ respectively.

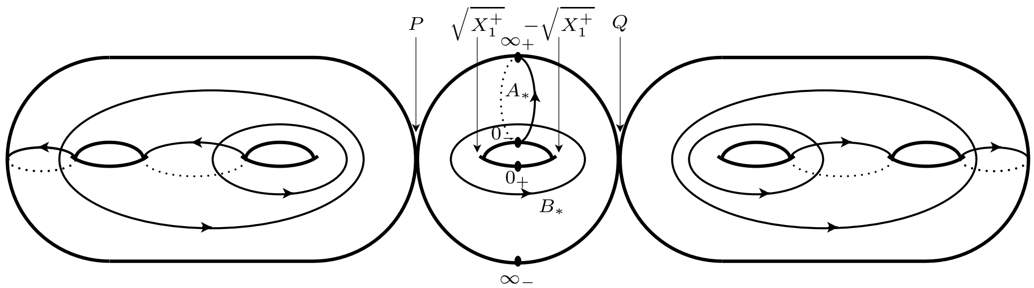

Figure 2. Degenerate curve

Then it is enough to calculate the characteristic in the limit. It is clear that the $C_{\vec{U}}$-parts have characteristic E and $-E$ respectively.

Let us concentrate on the genus 1 part. Among the original branched points, $\pm\sqrt{X_1^+}$ are contained in C_*, and P, Q are new branched points. As the limit of the partition corresponding to $\widehat{E}$, we get the partition $\{\sqrt{X_1^+}, Q\} \sqcup \{-\sqrt{X_1^+}, P\}$. This can be seen by pinching only one of the two cycles, say one corresponding to Q. As each part has the equal number of branched points, we must have $\{Q, \sqrt{X_1^+}, \sqrt{X_2^+}, \dots\} \sqcup \{-\sqrt{X_1^+}, \sqrt{X_1^-}, \sqrt{X_2^-}, \dots\}$. Pinching the remaining cycle corresponding to P, we get the assertion. On the other hand, the partition $\{0_+, \infty_+\} \sqcup \{0_-, \infty_-\}$ of the branched points of ϕ is clearly preserved under the degeneration.

Thus the elliptic curve C_* has two hyperelliptic involutions $\widehat{\iota}$ and ϕ, and we have the corresponding partitions of branched points $\{\sqrt{X_1^+}, Q\} \sqcup \{-\sqrt{X_1^+}, P\}$

and $\{0_+, \infty_+\} \sqcup \{0_-, \infty_-\}$. It is clear from the picture that both give rise to the same characteristic of the theta function (in fact, it is θ_{00}). $\square$

The denominator of the left hand side of (A.31) is

$$k_0 \Theta_E(\xi_0)^2 = k_0 \Theta_E(\frac{1}{2}\int_{0_-}^{\infty_+} \vec{\omega})^2.$$

On the other hand, we have

$$2d \int_{0_-}^{\infty_+} \vec{\omega}\hat{} = \left[d \int_{0_-}^{\infty_+} \vec{\omega}, d, -d \int_{0_-}^{\infty_+} \vec{\omega} \right].$$

To evaluating the value of the theta function at this point, we can replace d by 0 as d is an integer. Therefore the numerator of the left hand side of (A.31) is equal to

$$\widehat{\Theta}_{\widehat{E}} \left(d \int_{0_-}^{\infty_+} \vec{\omega}, 0, -d \int_{0_-}^{\infty_+} \vec{\omega} \right) = k_0 \Theta_E((d + \frac{1}{2}) \int_{0_-}^{\infty_+} \vec{\omega}) \Theta_E((d - \frac{1}{2}) \int_{0_-}^{\infty_+} \vec{\omega}).$$

On the other hand, we have $\widehat{\omega}_{\infty_+ - 0_-} = \frac{1}{2} p^*(\omega_{\infty_+ - 0_-})$. Therefore the right hand side of (A.31) is

$$\exp\left\{ d^2 \int_{0_+}^{\infty_-} \omega_{\infty_+ - 0_-} \right\}.$$

Thus we have

$$\frac{\Theta_E((d + \frac{1}{2}) \int_{0_-}^{\infty_+} \vec{\omega})}{\Theta_E(\frac{1}{2} \int_{0_-}^{\infty_+} \vec{\omega})} = \exp\left\{ \frac{d(d+1)}{2} \int_{0_+}^{\infty_-} \omega_{\infty_+ - 0_-} \right\},$$

i.e.

$$\frac{\Theta_E((d + \frac{1}{2}) \frac{\boldsymbol{\beta}}{2r} \frac{\partial a_\alpha^D}{\partial \log \Lambda} | \tau)}{\Theta_E(\frac{\boldsymbol{\beta}}{4r} \frac{\partial a_\alpha^D}{\partial \log \Lambda} | \tau)} = \exp\left\{ \frac{d(d+1)}{2} \frac{\boldsymbol{\beta}^2}{4r^2} \frac{\partial^2 \mathcal{F}_0}{(\partial \log \Lambda)^2} \right\} \tag{A.33}$$

for $0 \leq d \leq (\max(r + m, r - m) - 1)/2$. This is the same equation derived in Proposition 1.39 under the assumption (1.37).

A.8. **rank** 2 **case.** We assume $r = 2$, $m = 0$ in this subsection.

We have $P(X) = X^2 + U_1 X + 1$. Then

$$\begin{aligned} Y^2 &= P(X)^2 - 4X^2 \boldsymbol{\beta}^4 \Lambda^4 \\ &= \left\{ X^2 + U_1 X + 1 + 2X\boldsymbol{\beta}^2 \Lambda^2 \right\} \left\{ X^2 + U_1 X + 1 - 2X\boldsymbol{\beta}^2 \Lambda^2 \right\} \\ &= \left\{ \alpha_+ (X+1)^2 - \beta_+ (X-1)^2 \right\} \left\{ \alpha_- (X+1)^2 - \beta_- (X-1)^2 \right\} \end{aligned}$$

where

$$\alpha_\pm = \frac{1}{2} + \frac{U_1}{4} \pm \frac{\boldsymbol{\beta}^2\Lambda^2}{2}, \quad \beta_\pm = -\frac{1}{2} + \frac{U_1}{4} \pm \frac{\boldsymbol{\beta}^2\Lambda^2}{2}.$$

Then the solutions of $P(X)^2 - 4X^2\boldsymbol{\beta}^4\Lambda^4 = 0$ are

$$\frac{-\sqrt{\frac{\beta_+}{\alpha_+}}+1}{-\sqrt{\frac{\beta_+}{\alpha_+}}-1}, \quad \frac{-\sqrt{\frac{\beta_-}{\alpha_-}}+1}{-\sqrt{\frac{\beta_-}{\alpha_-}}-1}, \quad \frac{\sqrt{\frac{\beta_-}{\alpha_-}}+1}{\sqrt{\frac{\beta_-}{\alpha_-}}-1}, \quad \frac{\sqrt{\frac{\beta_+}{\alpha_+}}+1}{\sqrt{\frac{\beta_+}{\alpha_+}}-1}.$$

Here we choose the branch of $\sqrt{\beta_\pm/\alpha_\pm}$ so that the above are X_1^+, X_1^-, X_2^-, X_2^+ in sequence. (Recall the A-cycle encircles X_2^-, X_2^+, and B-cycles encircles X_1^-, X_2^-.) We introduce new variables

$$x = \sqrt{\frac{\alpha_+}{\beta_+}}\frac{X+1}{X-1}, \quad y = \frac{1}{\sqrt{\beta_+\beta_-}}\frac{Y}{(X-1)^2}$$

Then the Seiberg-Witten curve is

$$y^2 = (1-x^2)(1-\kappa^2x^2),$$

where

$$\kappa = \sqrt{\frac{\alpha_-\beta_+}{\alpha_+\beta_-}} = \sqrt{1 + \frac{\boldsymbol{\beta}^2\Lambda^2}{\frac{U_1^2}{16} - \left(\frac{\boldsymbol{\beta}^2\Lambda^2}{2} + \frac{1}{2}\right)^2}}.$$

In the x-coordinates, the A-cycle encircles 1, $1/\kappa$, and the B-cycle encircles $\pm 1/\kappa$. Note that the A-cycle encircles ± 1 usually, so A, B-cycles are interchanged in our convention. Note also that the curve has period 2τ instead of τ usually. Therefore when we use various formulas in textbooks (e.g. [53]), we need to replace τ by $-2/\tau$. From [53, 22 · 11] we have

$$\sqrt{\frac{\alpha_-\beta_+}{\alpha_+\beta_-}} = \kappa = \frac{\theta_{10}(-2/\tau)^2}{\theta_{00}(-2/\tau)^2}.$$

Therefore

$$\begin{aligned} \frac{U_1^2}{16} &= \frac{\boldsymbol{\beta}^2\Lambda^2}{\kappa^2-1} + \left(\frac{\boldsymbol{\beta}^2\Lambda^2}{2}+\frac{1}{2}\right)^2 = -\boldsymbol{\beta}^2\Lambda^2\frac{\theta_{00}(-2/\tau)^4}{\theta_{01}(-2/\tau)^4} + \left(\frac{\boldsymbol{\beta}^2\Lambda^2}{2}+\frac{1}{2}\right)^2 \\ &= \frac{1}{4}\left(1 - \frac{\theta_{00}(\tau)^4+\theta_{10}(\tau)^4}{\theta_{00}(\tau)^2\theta_{10}(\tau)^2}\boldsymbol{\beta}^2\Lambda^2 + \boldsymbol{\beta}^4\Lambda^4\right). \end{aligned} \tag{A.34}$$

We also have

$$\frac{\partial a}{\partial U_1} = -\frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}}\int_A \frac{dX}{Y} = \frac{1}{2\pi\sqrt{-1}\boldsymbol{\beta}\sqrt{\alpha_+\beta_-}}\int_1^{1/\kappa}\frac{dx}{y} = \frac{K'(-2/\tau)}{2\pi\boldsymbol{\beta}\sqrt{\alpha_+\beta_-}}.$$

Note that $\alpha_-\beta_+ = \alpha_+\beta_- + \boldsymbol{\beta}^2\Lambda^2$. Therefore

$$\alpha_+\beta_- = -\boldsymbol{\beta}^2\Lambda^2\frac{\theta_{00}(-2/\tau)^4}{\theta_{01}(-2/\tau)^4}, \quad \alpha_-\beta_+ = -\boldsymbol{\beta}^2\Lambda^2\frac{\theta_{10}(-2/\tau)^4}{\theta_{01}(-2/\tau)^4}.$$

Substituting $K'(-2/\tau) = \sqrt{-1}\pi\theta_{00}(-2/\tau)^2/\tau$ ([53, 22 · 32]) we get

$$\frac{\partial a}{\partial U_1} = \frac{\theta_{01}(-2/\tau)^2}{2\boldsymbol{\beta}^2\Lambda\tau} = \sqrt{-1}\frac{\theta_{00}(\tau)\theta_{10}(\tau)}{2\boldsymbol{\beta}^2\Lambda}. \tag{A.35}$$

Here we fix the sign so that it coincides with the formula for the homological version when $\boldsymbol{\beta}\to 0$, i.e. $da/du = -\sqrt{-1}\theta_{00}(\tau)\theta_{10}(\tau)/2\Lambda$.

Let $\operatorname{sn}(\bullet,\kappa(-2/\tau))$, $\operatorname{cn}(\bullet,\kappa(-2/\tau))$, $\operatorname{dn}(\bullet,\kappa(-2/\tau))$ be Jacobi's elliptic functions for the period $-2/\tau$. From (A.14) we have

$$\boldsymbol{\beta}\frac{\partial a^D}{\partial\log\Lambda} = 4\int_{0_-}^{\infty_+}\omega = -\frac{2\sqrt{-1}}{K'(-2/\tau)}\int_{-\sqrt{\frac{\alpha_+}{\beta_+}}}^{\sqrt{\frac{\alpha_+}{\beta_+}}}\frac{dx}{y} = -\frac{4\sqrt{-1}}{K'(-2/\tau)}\operatorname{sn}^{-1}(\sqrt{\frac{\alpha_+}{\beta_+}}).$$

Here we have used that ω is normalized so that $\int_A\omega = 2\int_1^{1/\kappa}\omega = 1$, and hence $\omega = \frac{dx}{2\sqrt{-1}K'y}$.

Let $h := -\frac{1}{4}\frac{\partial^2\mathcal{F}_0}{\partial a\partial\log\Lambda} = \frac{\pi\sqrt{-1}}{2}\frac{\partial a^D}{\partial\log\Lambda}$. Then by using addition theorem for theta functions and the definition of Jacobi's elliptic functions,

$$\begin{aligned}\frac{\theta_{11}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)}{\theta_{01}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)} &= \frac{\theta_{10}(\frac{\boldsymbol{\beta}h}{4\pi\sqrt{-1}},\frac{\tau}{2})\theta_{11}(\frac{\boldsymbol{\beta}h}{4\pi\sqrt{-1}},\frac{\tau}{2})}{\theta_{00}(\frac{\boldsymbol{\beta}h}{4\pi\sqrt{-1}},\frac{\tau}{2})\theta_{01}(\frac{\boldsymbol{\beta}h}{4\pi\sqrt{-1}},\frac{\tau}{2})}\\ &= \sqrt{-1}\frac{\theta_{01}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}\tau},-\frac{2}{\tau})\theta_{11}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}\tau},-\frac{2}{\tau})}{\theta_{00}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}\tau},-\frac{2}{\tau})\theta_{10}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}\tau},-\frac{2}{\tau})}\\ &= -\sqrt{-1}\kappa'(-\frac{2}{\tau})\frac{\operatorname{sn}(\frac{K\boldsymbol{\beta}h}{\pi\sqrt{-1}\tau},\kappa(-\frac{2}{\tau}))}{\operatorname{cn}(\frac{K\boldsymbol{\beta}h}{\pi\sqrt{-1}\tau},\kappa(-\frac{2}{\tau}))\operatorname{dn}(\frac{K\boldsymbol{\beta}h}{\pi\sqrt{-1}\tau},\kappa(-\frac{2}{\tau}))},\end{aligned}$$

where $K = K(-2/\tau)$. As $\frac{K\boldsymbol{\beta}h}{\pi\sqrt{-1}\tau} = -\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}}\sqrt{-1}K' = -\operatorname{sn}^{-1}(\sqrt{\frac{\alpha_+}{\beta_+}})$, the above is equal to

$$-\sqrt{-1}\sqrt{1-\frac{\alpha_-\beta_+}{\alpha_+\beta_-}}\sqrt{\frac{\alpha_+}{\beta_+}}\sqrt{\frac{\beta_+}{\beta_+-\alpha_+}}\sqrt{\frac{\beta_-}{\beta_--\alpha_-}} = \pm\boldsymbol{\beta}\Lambda,$$

where we have used $\alpha_\pm - \beta_\pm = 1$. Hence we get

$$\frac{\theta_{11}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)}{\theta_{01}(\frac{\boldsymbol{\beta}h}{2\pi\sqrt{-1}},\tau)} = -\boldsymbol{\beta}\Lambda. \tag{A.36}$$

Here the sign was fixed by considering the limit $\boldsymbol{\beta} \to 0$:

$$\frac{1}{\boldsymbol{\beta}} \frac{\theta_{11}(\frac{\boldsymbol{\beta} h}{2\pi\sqrt{-1}}, \tau)}{\theta_{01}(\frac{\boldsymbol{\beta} h}{2\pi\sqrt{-1}}, \tau)} \xrightarrow{\boldsymbol{\beta}\to 0} -\frac{\theta_{11}'(0,\tau)}{\theta_{01}} \frac{1}{8\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}_0}{\partial a \partial \log \Lambda} = -\Lambda$$

The equation (A.36) can be also derived from the blowup formula [43, Prop. 3.2(1)] for $c_1 =$ odd together with the argument in [43, §4.3].

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