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Regularity for The CR Vector Bundle Problem I

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Dedicated to Professor J.J.Kohn on the occasion of his 75th birthday

Abstract: We give a new solution to the local integrability problem for CR vector bundles over strictly pseudoconvex real hypersurfaces of dimension seven or greater. It is based on a KAM rapid convergence argument and avoids the previous more difficult Nash-Moser methods. The solution is sharp as to Hölder continuity.

Keywords: CR vector bundle, integrability problem, $\bar{\partial}_b$ equation, rapid iteration

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Introduction. The idea of a CR vector bundle $E \rightarrow M$, over a CR manifold M , naturally generalizes that of a holomorphic vector bundle over a complex manifold. Here we shall be concerned with the local integrability problem over a strongly pseudoconvex real hypersurface $M^{2n-1} \subset \mathbf{C}^n$. Therefore, we assume that E is trivial of some rank r , i. e. $E \cong M \times \mathbf{C}^r$, via a local frame field $e = (e_1, \dots, e_r)$, and we phrase the problem in terms of a connection D on E ,

$$De = \omega \otimes e, \quad \Omega = d\omega - \omega \wedge \omega. \quad (0.1)$$

The integrability condition is that the curvature 2-forms $\Omega = (\Omega_i^j)$ belong to the ideal $\mathcal{J}(M)$ generated by the restrictions of $(1,0)$ -forms to M . The problem is to find a new frame e , so that the connection 1-forms $\omega = (\omega_i^j)$ also belong to $\mathcal{J}(M)$. We focus mainly on the regularity of the new frame.

Over complex manifolds the problem goes back to Koszul and Malgrange [5]. Over CR manifolds it was considered in [13], in conjunction with the Nash-Moser theory developed in [12] for the much more difficult CR embedding problem. Ma and Michel [6], [7] have improved the regularity for these results. Here we shall consider the vector bundle problem in its own right. We shall eliminate the difficult Nash-Moser techniques and derive results sharp as to regularity via a very natural KAM rapid convergence argument.

More precisely, we have $d\mathcal{J}(M) \subseteq \mathcal{J}(M)$, and denote by $\bar{\partial}_b$ the reduction of the exterior derivative $d \bmod \mathcal{J}(M)$. In section 1 we choose suitable representatives $\bar{\partial}_M$ for $\bar{\partial}_b$, and ω'' for ω , mod $\mathcal{J}(M)$. Since the other components of ω will be irrelevant in this work, we simplify the notation by setting $\omega = \omega''$. Then, for a frame change $\tilde{e} = Ae$, $\det A \neq 0$, the derivation property of D , and reduction mod $\mathcal{J}(M)$ give

$$\tilde{\omega}A = \bar{\partial}_M A + A\omega. \quad (0.2)$$

We want to make $\tilde{\omega} = 0$. Thus, we seek a solution A in some neighborhood of any given point to

$$-A^{-1}\bar{\partial}_M A = \omega, \quad \text{if } \bar{\partial}_M \omega = \omega \wedge \omega, \quad (0.3)$$

the latter expressing the integrability condition. If the rank $r = 1$, this reduces to the $\bar{\partial}_b$ -problem, which we know to be locally solvable, if $\dim M = 2n - 1 \geq 5$, [4], [1]. If $r > 1$, (0.3) casts the problem in a non-linear light, which reflects our methods.

To measure regularity, we consider both the standard Hölder spaces $C^{k,\alpha}(M)$, and the Folland-Stein Hölder spaces $C_{FS}^{k,\alpha}(M)$ [2], for integer k and $0 \leq \alpha < 1$. We assume that M is of class C^l , $5 \leq l \leq \infty$, and that $\dim M = 2n - 1 \geq 7$, $n \geq 4$ (thus, omitting $\dim M = 5$), and prove the following.

Theorem 0.1. *Suppose that ω is of class $C^k(M)$, $k_1 \leq k \leq l - 4$, with $k_1 = 1$. Then there exists a local solution A of class $C^{k,\alpha}(M)$, for $0 \leq \alpha \leq 1/2$. In particular, if M and ω are of class C^∞ , there is a solution A of class C^∞ .*

The proof given here is complete based on essentially known estimates [4], [11]. Some improvement is still possible. For example, it will follow from [3] that, with appropriate weak definitions, we can get a solution with $l = 3$ and $k = 0$.

Our second result is restricted to a real hyperquadric.

Theorem 0.2. *Let ω be of class $C_{FS}^{k,\alpha}(M)$, $k_2 \leq k < \infty$, with $k_2 = 1$, and $0 < \alpha < 1$, where M is the Heisenberg group. Then there exists a local solution A of class $C_{FS}^{k+1,\alpha}(M)$.*

This can be improved to $k_2 = 0$, and with some effort carried over to vector bundles over general strictly pseudoconvex real hypersurfaces. We shall pursue this in a future work.

As in [13], the main technical tool is the $\bar{\partial}_b$ local homotopy formula of Henkin [4],

$$\phi^{0,q} = \bar{\partial}_M P\phi + Q\bar{\partial}_M\phi, \quad 0 < q < n - 2, \tag{0.4}$$

on suitable subdomains $M_\rho \subset M$. We need it for $q = 1$ as in [12], which is why the 5-dimensional case is omitted. We also refer to the result of Nagel and Rosay [9]. While a linear argument is conceivable, our estimates for the operators P, Q are not adequate (see also Romero [10] in this respect). Therefore, we proceed (somewhat imprecisely) as follows.

We take $A = I + B$, $\|B\| < 1$, apply (0.4) with $\phi = \omega$, and use the integrability condition (0.3). This gives

$$\tilde{\omega}A = \bar{\partial}_M(B + P\omega) + Q(\omega \wedge \omega) + B\omega. \tag{0.5}$$

The choice $B = -P\omega$ gives formally $\|\tilde{\omega}\| \leq c\|\omega\|^2$. This will be repeated on a sequence of shrinking domains $M_\rho \subset M$. The rapid (quadratic) decrease of $\|\omega\|$ will allow us to overcome deficiencies (loss of derivatives and blow-up of coefficients in our estimates) and prove convergence to a solution of the problem.

In section 2 we summarize some previous estimates, which are not sharp but allow us to get directly to the core argument, which is given in section 3. To get the procedure started, we need $\|\omega\|$ to be sufficiently small, initially. This is achieved via a Taylor polynomial argument in section 1. At first we see with this an apparent loss of over half the derivatives in the argument. However this is greatly reduced in section 4, where we show following Moser [8] that, with rapid convergence in some low norm, the higher order derivatives are automatically pulled in. This gives a weaker form of theorem (0.1), where $A \in C^k$. The full theorem follows with the Hölder estimates of section 5.

Another way to avoid this derivative loss on initial shrinking is to use scale invariant Folland-Stein norms, which we do in section 6 on the Heisenberg group. This gives theorem (0.2). The use of scaling also gives another proof of theorem (0.1) on the Heisenberg group.

We point out that the main deficiencies here are the rather imprecise estimates we have for the solution operators P, Q on shrinking domains. This is somewhat addressed in [3].

We add some brief historical remarks. The central result on the local integrability of structures is, of course, the Newlander-Nirenberg theorem (1957). Its importance is reflected in the number of proofs it has since received. The local embedding problem for strictly pseudoconvex CR structures M was first posed by Kohn (1964). Counter examples for $\dim M = 3$ were given by Nirenberg (1973). The first positive results for $\dim M \geq 7$ were given by Kuranishi (1982), and Akahori (1987).

We re-emphasize that the the five-dimensional case is, at this time, still unresolved for both the CR embedding problem and the CR vector bundle problem. Perhaps the case of a CR vector bundle over the 5-dimensional Heisenberg ball would be the easiest problem to resolve.

1. INITIAL NORMALIZATION

Here we achieve the required initial smallness for the connection form ω , by Taylor polynomial arguments.

We take our real hypersurface in the form

$$M^{2n-1} \subset \mathbf{C}^n : r(z) = -y^n + |z'|^2 + h(z', x^n) = 0, \tag{1.1}$$

where $z = (z', z^n)$, $z^n = x^n + iy^n$, and $h = O(|(z', x^n)|^3)$ is of class C^l , $l \geq 3$. For a basis of complex vector fields we take

$$X_\alpha = \partial_\alpha - (r_\alpha/r_n)\partial_n, \quad X_{\bar{\alpha}} = \overline{X_\alpha}, \quad T = \bar{T} = \partial_{x^n}, \tag{1.2}$$

where $r_j = \partial_j r = \partial r / \partial z^j$, and small Greek indices have the range $1 \leq \alpha, \beta \leq n-1$. The dual basis is

$$dz^\alpha, \quad d\bar{z}^\alpha, \quad \theta = \bar{\theta} = -i\partial r, \tag{1.3}$$

and $\mathcal{J}(M)$ is the differential ideal generated by dz^α, θ . With a standard multi-index notation $A = (\alpha_1, \dots, \alpha_q)$, we have representatives for a $(0, q)$ -form ϕ and $\bar{\partial}_b \phi$,

$$\phi^q = \phi^{(0,q)} = \sum_{|A|=q} \phi_{\bar{A}} d\bar{z}^A, \quad \bar{\partial}_M \phi = \sum \bar{\partial}_M \phi_{\bar{A}} \wedge d\bar{z}^A, \tag{1.4}$$

and for function f ,

$$\bar{\partial}_M f = \sum_{\alpha=1}^{n-1} X_{\bar{\alpha}} f d\bar{z}^\alpha. \tag{1.5}$$

For (the (0,1)-part of) the connection one-forms we put

$$\omega = \sum_{\alpha=1}^{n-1} \Gamma_{\bar{\alpha}} d\bar{z}^\alpha. \tag{1.6}$$

Thus, the $\Gamma_{\bar{\alpha}}$ are $r \times r$ matrices of functions. The integrability condition (0.3) is equivalent to

$$X_{\bar{\alpha}} \Gamma_{\bar{\beta}} - X_{\bar{\beta}} \Gamma_{\bar{\alpha}} = \Gamma_{\bar{\alpha}} \Gamma_{\bar{\beta}} - \Gamma_{\bar{\beta}} \Gamma_{\bar{\alpha}}, \tag{1.7}$$

while the change of frame formula (0.2) is

$$\tilde{\Gamma}_{\bar{\alpha}} A = X_{\bar{\alpha}} A + A \Gamma_{\bar{\alpha}}. \tag{1.8}$$

If ω is of class C^k , $k \leq l-1$, we can arrange that it vanish to order k at 0, by a standard Taylor polynomial argument, as follows. Assume that ω starts with

terms of order s ,

$$\Gamma_{\bar{\alpha}} = \Gamma_{\bar{\alpha}}^{(s)} + \dots, \quad \Gamma_{\bar{\alpha}}^{(s)} = \sum_{0 \leq |B| \leq s} \Gamma_{\bar{\alpha}, \bar{B}}(z', x^n) \bar{z}'^B, \tag{1.9}$$

where $\Gamma_{\bar{\alpha}, \bar{B}}$ is homogeneous of degree $s - |B|$, and symmetric in the indices \bar{B} . We make the change (1.8) with

$$A = I + A^{(s+1)}, \quad A^{(s+1)} = \sum_{0 \leq |B| \leq s+1} A_{\bar{B}}(z', x^n) \bar{z}'^B, \tag{1.10}$$

where the coefficients are symmetric in \bar{B} . We need to make

$$\partial_{\bar{\alpha}} A + \Gamma_{\bar{\alpha}}^{(s)} = 0. \tag{1.11}$$

This defines the $A_{\bar{\alpha}\bar{B}}$ in terms of the $\Gamma_{\bar{\alpha}, \bar{B}}$, consistently, since $\Gamma_{\bar{\alpha}, \bar{B}}$ is symmetric in all its indices by the integrability condition (1.7). Thus we have the following.

Lemma 1.1. *If the connection form ω is of class C^k , $k \leq l - 1$, then we can arrange $\omega = O(k)$ at $0 \in M$, by a preliminary change of frame.*

We have an analogous result relative to Folland-Stein derivatives on the real hyperquadric (Heisenberg group), where $h(z', x^n) = 0$. We say that a function f is of class C_{FS}^k if it has continuous derivatives

$$T^m X^S \bar{X}^R f, \quad 2m + |S| + |R| \leq k, \tag{1.12}$$

where $X^S = X_1^{s_1} \dots X_{n-1}^{s_{n-1}}$, etc.

Lemma 1.2. *Suppose that ω is of class C_{FS}^k on the Heisenberg group M . Then by a preliminary change of frame, we can achieve*

$$T^m X^S \bar{X}^R \Gamma_{\bar{\alpha}}(0) = 0, \quad 2m + |S| + |R| \leq k. \tag{1.13}$$

The proof is similar. We assume that (1.13) holds for $s - 1$, $s \leq k$, in place of k ,

$$\Gamma_{\bar{\alpha}} = \Gamma_{\bar{\alpha}}^{(s)} + \dots, \tag{1.14}$$

where now (s) refers to weight,

$$wt(\bar{z}'^R z'^S (z^n)^m) = |R| + |S| + 2m. \tag{1.15}$$

We then take $A = I + A^{(s+1)}$ with

$$T^m X^S \bar{X}^R X_{\bar{\alpha}} A^{(s+1)}(0) = -T^m X^S \bar{X}^R X_{\bar{\alpha}} \Gamma_{\bar{\alpha}}^{(s)}(0). \tag{1.16}$$

The integrability condition (1.7) guarantees symmetry in the barred coefficients, so that this is consistent. Continuing this, we achieve the lemma.

2. ESTIMATES FOR THE HOMOTOPY FORMULA

Here we give estimates for the operators P, Q which can be readily derived from known results. We refer to Henkin [4], and many references therein, as well as to [11]. For convenience we follow the notation of [11]. This will allow us to get quickly to the main convergence argument, which is given in the next section.

We work with the real hypersurfaces

$$M_\rho = M \cap \{(x^n)^2 + y^n \leq \rho^2\}. \tag{2.1}$$

For $\rho > 0$ sufficiently small, this is a graph over $D_\rho \subset \mathbf{R}^{2n-1}$, which is approximately a ball of radius ρ . We consider two such domains $D_0 = D_{\rho_0} \subset\subset D = D_\rho$ with $\rho_0 = \rho(1 - \sigma)$, $0 < \sigma < 1$. Let $\delta(D) \approx \rho$ denote the diameter, and $\delta(D_0, \partial D) \approx \rho\sigma$ distance to the boundary. In the notation of [11], we have $P = P_0 + P_1$ and $Q = Q_0 + Q_1$, where P_0, Q_0 are integral operators over M_ρ , and P_1, Q_1 are integral operators over the boundary ∂M_ρ .

For $\phi \in C_0^k(D)$, we follow the procedure of sections 3 and 4 of [11]. In particular, we have (3.7) of [11], but without the boundary integral. Any k -th order derivative ∂^k on D can be expressed in terms of the vector fields (1.39) of [11]. We may summarize the procedure by writing

$$\partial^k P_0 \phi = \sum_{|K| \leq k} P_0^{(K)} (\partial^K \phi). \tag{2.2}$$

Here the right hand side is a sum of certain partial derivatives of order up to k , and the $P_0^{(K)}$ are operators of type not worse than P_0 , with kernels involving at most $k + 3$ derivatives of r . The absolute integrability of these kernels follows from [4]. This gives

$$\|P_0 \phi\|_{C^k(D)} \leq c_{k+3} \delta(D) \|\phi\|_{C^k(D)}, \tag{2.3}$$

where c_{k+3} is a constant depending on $k + 3$ derivatives of r . We combine this estimate with a cut-off function $\lambda_0 + \lambda_1 = 1$ on \bar{D} , $\lambda_0 = 1$ on D_0 , $\lambda_0 = 0$ near ∂D . For $\phi \in C^k(\bar{D})$, the product rule and $|\partial \lambda_0| \leq c/\delta(D_0, \partial D)$ give

$$\|\lambda_0 \phi\|_{C^k(D)} \leq c \sum_{j=0}^k \delta(D_0, \partial D)^{j-k} \|\phi\|_{C^j(D)}. \tag{2.4}$$

To estimate $P_0(\lambda_1\phi)$ on D_0 we (crudely) let the derivatives fall on the kernel, increasing the blow-up at the boundary;

$$\|P_0(\lambda_1\phi)\|_{C^k(D_0)} \leq c_{k+3} \frac{\text{vol}(D - D_0)}{\delta(D_0, \partial D)^{2n+2k-1}} \|\phi\|_{C^0(D)}. \tag{2.5}$$

We estimate the boundary integral P_1 similarly as in (3.8), (4.6), and (4.7) of [11], using such bounds as $|r_\zeta \cdot (\zeta - z)| \geq c|\zeta - z|^2$, $|r_\zeta - r_z| \leq c|\zeta - z|$, $|\zeta^n - z^n| \geq c\rho^2\sigma$, and $d\zeta^n \wedge r_z \cdot d\zeta \wedge r_\zeta \cdot d\zeta = d\zeta^n \wedge r_{z'} \cdot d\zeta' \wedge r_{\zeta'} \cdot d\zeta' = O(\rho^2)$, for terms appearing in the kernel (1.31) of [11]. This gives

$$\|P_1(\phi)\|_{C^k(D_0)} \leq c_{k+3} \frac{\text{vol}(\partial D)}{\sigma \delta(D_0, \partial D)^{2n+2k-2}} \|\phi\|_{C^0(D)}. \tag{2.6}$$

We note that $\text{vol}(\partial D) \leq c\rho^{2n-2}$ and $\text{vol}(D - D_0) \leq c(\rho\sigma)\rho^{2n-2}$. Combining and simplifying gives $(\|\cdot\|_{\rho,k} = \|\cdot\|_{C^k(D_\rho)})$

$$\|P(\phi)\|_{\rho(1-\sigma),k} \leq c'_{k+3} \left\{ \rho \sum_{s=0}^k (\rho\sigma)^{s-k} \|\phi\|_{\rho,s} + \eta' \|\phi\|_{\rho,0} \right\} \leq \eta \|\phi\|_{\rho,k}, \tag{2.7}$$

where

$$\eta = \eta(\rho, \sigma, k) = c_{k+3}\eta', \quad \eta' = \sigma^{-2n-2k+1}\rho^{-2k}. \tag{2.8}$$

Notice that for $k = 0$, we don't need the cutoff function, nor (2.4), (2.5) and the factors ρ cancel in (2.6) giving (2.8) with $k = 0$.

A similar estimate holds for Q . These estimates on shrinking domains will be somewhat improved in [3].

3. A KAM RAPID CONVERGENCE ARGUMENT

In this section we construct a sequence of approximate solutions to the problem of theorem (0.1) of the introduction and prove convergence. With the estimates of the previous section, we get a solution with apparent derivative loss in C^k spaces.

We define a sequence of ‘‘radii’’, $\rho > 0$, by

$$\rho_{j+1} = \rho_j(1 - \sigma_j), \quad \sigma_j = 2^{-j-1}, \quad j \geq 0, \tag{3.1}$$

which decrease to a positive limit ρ_∞ depending on $\rho_0 > 0$, which will be chosen later. Then $M_j \equiv M_{\rho_j}$, $0 \leq j \leq \infty$, form a decreasing family of neighborhoods of $0 \in M$.

We consider frames e_j for E over M_j , and the associated connection matrix ω_j of $(0, 1)$ -forms, $0 \leq j < \infty$. They will be chosen so that $e_0 = e$ is the original frame, normalized so that $\omega_0 = \omega$ vanishes to suitably high order at 0, and for $j \geq 1$,

$$e_j = A_j e_{j-1}, \quad A_j = I + B_j, \quad G_l = A_l A_{l-1} \cdots A_1. \quad (3.2)$$

Then on M_l we shall have

$$\omega_l A_l = \bar{\partial}_M A_l + A_l \omega_{l-1}, \quad (3.3)$$

and inductively,

$$\omega_l G_l = \bar{\partial}_M G_l + G_l \omega_0. \quad (3.4)$$

We must show that we have convergence in $C^k(M_\infty)$, $k \geq 1$,

$$\omega_l \rightarrow 0, \quad G_l \rightarrow G_\infty, \quad (3.5)$$

with the matrix of functions G_∞ invertible. Then $A = G_\infty$ will be a solution to our problem.

On each M_j , $0 \leq j < \infty$, we have the homotopy formula (0.4) with operators P_j, Q_j . In (0.5) we take $A_{j+1} = I + B_{j+1}$, with

$$B_{j+1} = -P_j \omega_j, \quad (3.6)$$

giving

$$\omega_{j+1} A_{j+1} = Q_j(\omega_j \wedge \omega_j) + B_{j+1} \omega_j. \quad (3.7)$$

From (3.6), (2.7) we get

$$\|B_{j+1}\|_{\rho_{j+1}, k} \leq \eta_j \|\omega_j\|_{\rho_j, k}, \quad \eta_j \equiv \eta_j^{(k)} = c_{k+3} \sigma_j^{-2n-2k+1} \rho_j^{-2k}. \quad (3.8)$$

For $r \times r$ matrices of functions and a given k , there is a constant $\tilde{c}_k \geq 1$ for which

$$\|AB\|_{\rho, k} \leq \tilde{c}_k \|A\|_{\rho, k} \|B\|_{\rho, k}, \quad (3.9)$$

$$\|(I + B)^{-1}\|_{\rho, k} \leq (1 - \tilde{c}_k \|B\|_{\rho, k})^{-1}, \quad (3.10)$$

provided $\tilde{c}_k \|B\|_{\rho, k} < 1$. (Note that (3.9) holds when A, B are $r \times r$ matrices of forms.) Thus, we shall want to arrange

$$\tilde{c}_k \|B_{j+1}\|_{\rho_{j+1}, k} < 1/2, \quad \|A_{j+1}^{-1}\|_{\rho_{j+1}, k} \leq 2, \quad (3.11)$$

for $0 \leq j < \infty$, at least for $k = 0$, to carry out the procedure.

Given (3.11), we get from (3.7), (2.7)

$$\begin{aligned} \|\omega_{j+1}\|_{\rho_{j+1},k} &\leq 2\{\|Q_j(\omega_j \wedge \omega_j)\|_{\rho_{j+1},k} + \tilde{c}_k \|B_{j+1}\|_{\rho_{j+1},k} \|\omega_j\|_{\rho_{j+1},k}\} \\ &\leq \eta_j \|\omega_j\|_{\rho_j,k}^2, \end{aligned} \tag{3.12}$$

where we have absorbed a factor of $4\tilde{c}_k$ into the constant c_{k+3} .

For $j = 0$ we assume that ω_0 is of class C^m , $m > k$, and normalize as in section 1 so that ω_0 and all its derivatives of order m and less vanish at 0. Then

$$\|\omega_0\|_{\rho_0,k} \leq c\rho_0^{m-k}, \quad \eta_0 \|\omega_0\|_{\rho_0,k} \leq c\rho_0^{m-3k}, \tag{3.13}$$

provided $m > 3k$. Then by shrinking ρ_0 we get (3.11) for $j = 0$.

We set

$$\delta_j \equiv \delta_j^{(k)} = \|\omega_j\|_{\rho_j,k}, \quad \delta_{j+1} \leq \eta_j \delta_j^2. \tag{3.14}$$

We readily see that

$$\eta_{j+1} = \alpha_j \eta_j, \quad \alpha_j \equiv \alpha_j^{(k)} = 2^{2n+2k-1} (1 - \sigma_j)^{-2k}. \tag{3.15}$$

Since the σ_j decrease to 0, the α_j decrease to $2^{2n+2k-1}$. The η_j increase to infinity. Finally we define ζ_j ,

$$\zeta_j \equiv \zeta_j^{(k)} = \alpha_j \eta_j \delta_j, \quad \zeta_{j+1} \leq \zeta_j^2. \tag{3.16}$$

By shrinking ρ_0 , we arrange $\zeta_0 < 1/2\tilde{c}_k$, then the ζ_j decrease rapidly to 0, as also do the δ_j and $\|B_j\|_{\rho_j,k}$, and we have (3.11) for all $j \geq 0$.

From (3.11) we clearly have

$$\|A_j\|_{\rho_j,k} \leq 2, \quad \|G_l\|_{\rho_l,k} \leq \tilde{c}_k^{l-1} 2^l. \tag{3.17}$$

Since $G_l - G_{l-1} = B_l G_{l-1}$, we have

$$\|G_l - G_{l-1}\|_{\rho_l,k} \leq (2\tilde{c}_k)^{l-1} \|B_l\|_{\rho_l,k} \leq (2\tilde{c}_k)^{l-1} \zeta_{l-1}. \tag{3.18}$$

It follows by the ratio test that

$$G_l - G_1 = \sum_{j=2}^l (G_j - G_{j-1}) \tag{3.19}$$

converges in $C^k(M_\infty)$ to a limit $G_\infty \in C^k(M_\infty)$. Furthermore, for any $\epsilon > 0$ we may arrange $\zeta_0 < \epsilon/2\tilde{c}_k$, by shrinking ρ_0 as above. Then $\zeta_l < (\epsilon/2\tilde{c}_k)^l$, and

$$\sum_{l=1}^\infty (2\tilde{c}_k)^{l+1} \zeta_l \leq 2\tilde{c}_k \epsilon / (1 - \epsilon). \tag{3.20}$$

Thus, by shrinking ρ_0 a second and final time, we ensure that $G_\infty - G_1 = G_\infty - I - B_1$ and B_1 are so small that G_∞ is invertible.

Thus, we have achieved a solution $A = G_\infty$ of class C^k , in theorem(0.1) provided ω is of class C^{3k+1} . This extra smoothness requirement, which was needed only in (3.16) for the initial smallness, will be removed in the following sections.

4. HIGHER ORDER DERIVATIVES

Now we assume that our initial $\omega = \omega_0$ is of class C^k , $1 \leq k \leq l - 3$. We normalize so that it and all its first order derivatives vanish at 0. By shrinking the initial radius ρ_0 , we can make $\zeta_0^{(s)}$ arbitrarily small for $s = 0$. This allows us to carry out the preceding argument. The constructed sequences B_j, ω_j are of class C^k , but are converging rapidly to 0 only in C^0 -norm, and we cannot yet pass to the limit in (3.4). By modifying an idea in Moser [8], we show that, *without any further change of the sequence*, we have convergence $\omega_l \rightarrow 0, G_l \rightarrow G_\infty$ in $C^k(M_\infty)$.

For this we take any first order partial derivative ∂_x in (3.7),

$$\partial_x \omega_{j+1} A_{j+1} = \partial_x Q_j(\omega_j \wedge \omega_j) + \partial_x B_{j+1} \omega_j + B_{j+1} \partial_x \omega_j - \omega_{j+1} \partial_x B_{j+1}. \tag{4.1}$$

We multiply this by A_{j+1}^{-1} and take $(\|\cdot\|_{\rho_{j+1},0})$ -norms over M_{j+1} , using

$$\begin{aligned} \|\omega_{j+1}\|_{\rho_{j+1},0} &\leq \eta_j^{(0)} \|\omega_j\|_{\rho_j,0}^2, \tag{4.2} \\ \|B_{j+1}\|_{\rho_{j+1},0} &\leq \eta_j^{(0)} \|\omega_j\|_{\rho_j,0}, \quad \|A_{j+1}^{-1}\|_{\rho_{j+1},0} \leq 2, \\ \|\partial_x B_{j+1}\|_{\rho_{j+1},0} &\leq \eta_j^{(1)} \|\omega_j\|_{\rho_j,1}, \\ \|\partial_x Q_j(\omega_j \wedge \omega_j)\|_{\rho_{j+1},0} &\leq \eta_j^{(1)} \|\omega_j \wedge \omega_j\|_{\rho_j,1} \leq 2\tilde{c}_0 \eta_j^{(1)} \|\omega_j\|_{\rho_j,0} \|\omega_j\|_{\rho_j,1}. \end{aligned}$$

We add together all first order derivatives ∂_x . Then (3.14) with $k = 0$, gives

$$\delta_{j+1}^{(1)} \leq 2\tilde{c}_0 \{2\tilde{c}_0 \eta_j^{(1)} \delta_j^{(0)} + \tilde{c}_0 \eta_j^{(1)} (\delta_j^{(0)} + \eta_j^{(0)} (\delta_j^{(0)})^2) + \tilde{c}_0 \eta_j^{(0)} \delta_j^{(0)}\} \delta_j^{(1)} + \eta_j^{(0)} (\delta_j^{(0)})^2. \tag{4.3}$$

We have

$$\eta_j^{(k)} / \eta_j^{(k-1)} \leq \hat{c}_k 4^j, \tag{4.4}$$

where $\hat{c}_k = 4c_{k+3} / (c_{k+2} \rho_\infty^2)$. Taking $k = 1$ in this gives

$$\delta_{j+1}^{(1)} \leq \gamma_j^{(1)} \delta_j^{(1)}, \quad \gamma_j^{(1)} = 2\tilde{c}_0 (4\tilde{c}_0 \hat{c}_1 4^j + \tilde{c}_0 + 1) \eta_j^{(0)} \delta_j^{(0)}. \tag{4.5}$$

Eventually $\gamma_j^{(1)} \rightarrow 0$ rapidly. It follows that from some j_1 onward, $\delta_j^{(1)}$, and $\eta_j^{(1)}\delta_j^{(1)}$ decrease rapidly to zero, and we have the results of the previous section for $k = 1$.

We repeat this process. We take another first order derivative in (4.1). Similar arguments lead to

$$\delta_{j+1}^{(2)} \leq \gamma_j^{(2)}\delta_j^{(2)}, \tag{4.6}$$

where eventually $\gamma_j^{(2)} \rightarrow 0$ rapidly, and convergence is established with $k = 2$, and so on. Thus, we see that ω_j and B_j tend to zero rapidly with all derivatives up to order k , and that $G_\infty \in C^k(M_\infty)$. This proves theorem (0.1), except that we only have $A = G_\infty \in C^k$.

5. HÖLDER CONTINUITY

We now assume that M is of class C^l , $l \geq 4$, and that our original $\omega = \omega_0$ is of class C^k , $1 \leq k \leq l - 4$, and the constructed sequences B_j, ω_j are converging rapidly to zero in C^k -norm. We want to show that the B_j converge rapidly to zero in the standard Hölder $C^{k,\alpha}$ -norm, $0 \leq \alpha \leq 1/2$,

$$\|B\|_{C^{k,\alpha}(D)} = \|B\|_{C^k(D)} + H_{\alpha,D}(\partial^k B). \tag{5.1}$$

Here $\partial^k B$ stands for all k -th order derivatives of B .

This follows, in principle, from the well known 1/2-estimate [4], applied to the operators $P_0, P_0^{(K)}$ in (2.2), acting on forms with compact support. (A precise version will be given in [3].) Thus we shall assume the following, where the set-up is as in section 2.

Lemma 5.1. *For all $\phi \in C^k(D)$ of compact support, we have*

$$\|P_0\phi\|_{C^{k,\alpha}(D)} \leq c_{k+4}\|\phi\|_{C^k(D)},$$

where $0 \leq \alpha \leq 1/2$, and $k \geq 0$ is an integer.

As in section 2 we use the cutoff function λ_0 . We sacrifice 1/2 derivative for the sake of simplicity to get

$$\|P_0\phi\|_{C^{k,\alpha}(D_0)} \leq c_{k+4}\|\lambda_0\phi\|_{C^k(D)} + \|P_0(\lambda_1\phi)\|_{C^{k+1}(D_0)}. \tag{5.2}$$

Also

$$\|P_1\phi\|_{C^{k,\alpha}(D_0)} \leq \|P_1\phi\|_{C^{k+1}(D_0)}. \tag{5.3}$$

This leads to

$$\|P\phi\|_{\rho(1-\sigma),k,\alpha} \leq \eta\|\phi\|_{\rho,k}, \tag{5.4}$$

for $0 \leq \alpha \leq 1/2$. Replacing k by $k + 1$ in (2.8) gives

$$\eta = c_{k+4}\sigma^{-2n-2(k+1)+1}\rho^{-2(k+1)}. \tag{5.5}$$

With these estimates we readily see that the sequence B_j rapidly decreases in $C^{k,\alpha}(M_\infty)$ in the previous argument, and that the limit $A = G_\infty \in C^{k,\alpha}(M_\infty)$, $0 \leq \alpha \leq 1/2$. This proves theorem (0.1), in full.

6. SCALE INVARIANCE ON THE HEISENBERG GROUP

The difficulties of the last two sections stem mainly from the blow up of the coefficients in our estimates as we initially shrink the domain. This can be largely overcome on the Heisenberg group, or real hyperquadric, (1.1) with $h = 0$,

$$M : y^n = |z'|^2, \quad M_\rho = M \cap \{|z^n| \leq \rho\}, \tag{6.1}$$

by using scale invariance in two ways. Each M_ρ is a graph over the corresponding Heisenberg ball D_ρ , $|z'|^4 + (x^n)^2 \leq \rho^2$. These domains are permuted by the non-isotropic dilations [2],

$$T_\rho(z', z^n) = (\sqrt{\rho}z', \rho z^n), \quad M_\rho = T_\rho(M_1), \tag{6.2}$$

which also preserve the CR structure.

A) By the explicit form of their kernels as given in [11], the operators P_0, Q_0, P_1, Q_1 are easily seen to be invariant under these scalings,

$$P_{(\rho)} = T_{\rho^{-1}}^* P_{(1)} T_\rho^*, \quad Q_{(\rho)} = T_{\rho^{-1}}^* Q_{(1)} T_\rho^*. \tag{6.3}$$

In fact, (modifying somewhat the notations of (1.30) [11]), change-of-variables gives

$$\begin{aligned} (P_{0(\rho)}\phi)(z) &= \int_{M_\rho} \phi(\zeta) \wedge \Omega(\zeta, z) = \int_{M_1} (T_\rho^\zeta)^*(\phi \wedge \Omega) \\ &= (T_{\rho^{-1}}^z)^* \int_{M_1} (T_\rho^\zeta)^*(\phi) \wedge (T_\rho^\zeta \times T_\rho^z)^*\Omega. \end{aligned} \tag{6.4}$$

But $(T_\rho^\zeta \times T_\rho^z)^*\Omega = \Omega$, which also holds for Ω_1 , the kernel (1.31) [11] of the boundary integral P_1 .

We define scale-invariant Folland-Stein Hölder norms by

$$\|\phi\|_{\rho,k,\alpha} = \|T_\rho^* \phi\|_{1,k,\alpha}, \quad (6.5)$$

where the right-hand side is the ordinary FS norm on M_1 . As soon as we have fixed a positive lower bound on ρ , these norms are equivalent to the ordinary Folland-Stein norms [2].

As noted in [4], on the Heisenberg group the operators P_0, Q_0 are essentially divergences of the Folland-Stein fundamental solution operators. In fact, their kernels can be constructed from the functions (6.1) in [2] by taking one Folland-Stein derivative. We may appeal to theorem (10.1) of [2]. For compactly supported forms ϕ on the unit Heisenberg ball M_1 , and $k \in \mathbf{Z}$, $0 < \alpha < 1$, this gives

$$\|P_{0(1)}\|_{1,k+1,\alpha} \leq c_{k,\alpha} \|\phi\|_{1,k,\alpha}, \quad (6.6)$$

and similarly for $Q_{0(1)}$.

We then follow the procedure of section 2 using a cutoff function and crudely estimating the boundary integrals, but using Folland-Stein norms, on the unit Heisenberg ball M_1 . We take one extra derivative in (2.5) and (2.6) for the Hölder ratio. The denominators in the coefficients for the estimates over $M_{1-\sigma}$ involve only powers of σ , and this remains so after scaling. Thus we get

$$\|P_{(\rho)}\|_{\rho(1-\sigma),k+1,\alpha} \leq \eta \|\phi\|_{\rho,k,\alpha}, \quad (6.7)$$

$$\eta = c_{k,\alpha} \sigma^{-s(k,n)}, \quad (6.8)$$

where $s(k, n)$ is a positive integer.

To prove theorem (0.2) we apply the procedure of section 3 with the above constructions. We see that we gain one Folland-Stein derivative in passing from ω to B . Initially we normalize ω_0 as in lemma (1.2) to order k . Then $\|\omega_0\|_{\rho_0,k,\alpha}$, and hence $\|B_0\|_{\rho_0,k+1,\alpha}$ tend to zero as we shrink ρ_0 . Thus we need no extra smoothness to get the process of section 3 started. It yields theorem (0.2) in one step.

Strictly speaking, we need $k \geq 2$ so that our forms are at least of class C^1 for the direct derivation of the homotopy formula in [11]. However, on the Heisenberg group we can establish the homotopy formula directly from the Folland-Stein constructions [2]. This only requires $k \geq 1$.

B) We indicate another method using the ordinary norms of sections 2 and 5. This uses the dilations T_κ , $\kappa > 0$, to pull back the (trivial) vector bundle and the connection from M_κ to M_1 ,

$$\omega^\kappa \equiv T_\kappa^* \omega = \sqrt{\kappa} \sum_{\alpha=1}^{n-1} \Gamma_{\bar{\alpha}}(\sqrt{\kappa} z', \kappa x^n) d\bar{z}^\alpha. \tag{6.9}$$

The coefficients, together with any (z', x^n) -derivatives tend to zero uniformly on D_1 , as $\kappa \rightarrow 0$.

We set $\rho_0 = 1$ in (3.1), getting a fixed decreasing sequence of radii $\rho_j \rightarrow \rho_\infty > 0$, and the corresponding sequence of Heisenberg balls D_{ρ_j} , $0 \leq j \leq \infty$. We also choose cut-off functions $\lambda_j = \lambda_j(|z^n|) \geq 0$, $\lambda_j = 1$ on D_{ρ_j} , $\lambda_j = 0$ near $\partial D_{\rho_{j-1}}$. We have $|\partial^{(1)} \lambda_j| \leq c/(\rho_j \sigma_j) \leq \tilde{c}/\sigma_j$, where $\tilde{c} = c/\rho_\infty$, etc.

Since the geometry of Heisenberg balls is different from that of Euclidean balls, the corresponding quantities $\delta(D)$, $\delta(D_0, \partial D)$, and $\text{vol}(D - D_0)$ of section 2 are different. For each j , they depend on ρ_j , σ_j . But since we have the fixed lower bound $\rho_j \geq \rho_\infty$, we can again replace (2.8), (5.5) by

$$\eta_k = c_{k+4} \sigma^{-s(n,k)}, \tag{6.10}$$

where $s(n, k)$ is a positive integer.

Now we take $\omega_0 = \omega^\kappa$, so that by (6.9) $\zeta_0 = O(\sqrt{\kappa})$ in (3.16). By taking $\kappa > 0$ sufficiently small, we can start the argument of sections 3 and 5. It yields convergence on D_∞ . Transforming back by T_κ^{-1} gives theorem (0.1) directly (for k finite), without recourse to sections one or four.

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