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Tame Quivers and Affine Enveloping Algebras

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Abstract: Let $\mathfrak{g}$ be an affine Kac-Moody algebra with symmetric Cartan datum, $\mathfrak{n}^+$ be the maximal nilpotent subalgebra of $\mathfrak{g}$. By the Hall algebra approach, we construct integral bases of the $\mathbb{Z}$-form of the enveloping algebra $U(\mathfrak{n}^+)$. In particular, the representation theory of tame quivers is essentially used in this paper.
Keywords: tame quiver, Hall algebra, affine enveloping algebra.

1. Introduction

1.1. There are remarkable connections between the representation theory of quivers and Lie theory. For a complex simple Lie algebra $\mathfrak{g}$, let Q be the quiver given by orienting the Dynkin diagram of $\mathfrak{g}$. In [Gab], P. Gabriel discovered that the set of isomorphism classes of indecomposable representations of Q is in one-to-one correspondence with the set of positive roots of $\mathfrak{g}$. A direct construction of the Lie algebra $\mathfrak{g}$ using representations of quivers was given by C. M. Ringel. Roughly speaking, he defined the Hall algebra $\mathcal{H}(Q)$ using representations of Q over a finite field and then proved that $\mathcal{H}(Q)$ is isomorphic to the positive part of the quantum group $U_v^+(\mathfrak{g})$, see [R1] [R3]. This result was generalized to the case of Kac-Moody algebras in [Gr], where it was showed that the composition subalgebra $\mathcal{C}(Q)$ provides a realization of $U_v^+(\mathfrak{g})$. Thus when v specializes to 1, the Hall algebra degenerates to the enveloping algebra $U(\mathfrak{n}^+)$, where $\mathfrak{n}^+$ is the maximal nilpotent subalgebra of $\mathfrak{g}$ (see [R2]).

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In [L1], G. Lusztig gave a geometric definition of $\mathcal{H}(Q)$, which is a modified version of A. Schofield [S]. Namely, he used the constructible functions on varieties of $\mathbb{C}(Q)$-modules. The Euler characteristics appeared in the definition of multiplication. Similar to the quantum case, the composition algebra $\mathcal{C}(Q)$ is isomorphic to the enveloping algebra $U(\mathfrak{n}^+)$ (see Theorem 3.1). In the finite and affine case, a Chevalley basis of $\mathfrak{n}^+$ was reconstructed by the approach of Hall algebra in [FMV] (see Proposition 8.3). Thus the structure constants of this basis, which is the Euler characteristics of certain varieties, is given by the cocycles.

1.2. For any complex simple Lie algebra $\mathfrak{g}$, Kostant defined a $\mathbb{Z}$-subalgebra $U_{\mathbb{Z}}$ of the universal enveloping algebra $U(\mathfrak{g})$ using divided powers of the Chevalley basis, which is the well-known Kostant $\mathbb{Z}$-form [Ko]. Then he constructed a $\mathbb{Z}$-basis of $U_{\mathbb{Z}}$. These results were generalized to the affine Kac-Moody algebra by H. Garland, in [Gal]. He defined the root vectors using the loop algebra structure of the affine Kac-Moody algebra. The $\mathbb{Z}$-form is defined as the $\mathbb{Z}$-subalgebra generated by the divided powers of all real root vectors. And he also construct a $\mathbb{Z}$-basis of $U_{\mathbb{Z}}$. For a given order on the set of positive roots, the basis elements given in [Gal] are ordered monomials of the following generators: the divided powers of real root vectors and certain functions of imaginary root vectors. The method in [Gal] is not representation-theoretic and the proof of the integrality of the basis is difficult, using some complicated combinatorial identities.

1.3. In this paper, we will construct $\mathbb{Z}$-bases of $U_{\mathbb{Z}}(\mathfrak{n}^+)$ for affine Kac-Moody algebras by the Hall algebra approach. The representation theory of tame quivers, especially the structure of the Auslander-Reiten-quiver is essentially used in our method. The AR-quiver Γ_Q, whose vertices are isomorphism classes of indecomposable $\mathbb{C}Q$-modules and arrows are irreducible morphisms, gives a nice description of the category of $\mathbb{C}Q$-modules. In [FMV], one can already see that the real root vectors not only come from the preprojective or preinjective components of Γ_Q but also come from the non-homogeneous tubes in the regular component. Furthermore, the behaviors of the imaginary root vectors arising from homogeneous tubes and non-homogeneous tubes are quite different.

Therefore, we construct basis elements from the components of the AR-quiver respectively. More precisely, the basis elements we construct arise from the preprojective component, the preinjective component, each non-homogeneous tube and an embedding of the module category of the Kronecker quiver respectively. Then the ordered monomials of those basis elements form the desired integral basis. In particular, the order is given by the structure of the AR-quiver. In this way, we generalize the main results of Frenkel, Malkin and Vybornov in [FMV] to the level of enveloping algebras.

1.4. One key to prove the integrality of our bases is that the Euler characteristics are always integers. Thus evaluating a product of two characteristic functions of certain constructible sets at any point gives an integer (see 3.4 for details). However, this is not enough since not every basis element can be made as a single characteristic function. Moreover, the supports of two basis elements may have common points. So we have to find suitable constructible functions in order to obtain the basis elements. The most difficult part is the choice in the homogeneous tubes, where our idea comes from both representation theory and the theory of symmetric functions (see 6.5).

1.5. We mention some results in the quantum case. There are several results in constructing $\mathbb{Z}[v, v^{-1}]$-bases of the integral form of U_v^+ using the Hall algebra approach. Ringel has construct an integral PBW-basis of U_v^+ in the case of finite type [R4]. Later a $\mathbb{Q}(v)$-basis of type $A_1^{(1)}$ was given in [Z] and it was improved to be a $\mathbb{Z}[v, v^{-1}]$-basis in [C]. For the affine case, in [LXZ], a PBW-basis of U_v^+ was given as a step to construct the canonical basis by an algebraic method. The method in the present paper obviously stems from that of [LXZ]. However, the basis given there is a $\mathbb{Q}[v, v^{-1}]$-basis. Although it has been proved in [LXZ] that this basis is nicely connected to the canonical basis, hence actually a $\mathbb{Z}[v, v^{-1}]$-basis, it seems difficult to prove it directly only using algebraic methods.

1.6. The paper is organized as follows: In Section 2, we make necessary notations and recall some basic definitions. In Section 3, we recall the definition of Hall algebras. For convenience, we use the geometric version, following Lusztig [L1]. We also calculate the product of characteristic functions in two easy cases. A brief review of the representation theory of tame quivers is given in Section 4, for the details one can see [DR]. In Section 5 we focus on the preprojective and preinjective modules. We define two subalgebras $\mathcal{C}(Q)^{prep}$, $\mathcal{C}(Q)^{prei}$ and construct their $\mathbb{Z}$-bases respectively. The arguments in this section are similar to those in the quantum case, see [R4] for the case of finite type. Moreover, the results in this section are valid for arbitrary quiver without oriented cycles, not only for tame quivers. Sections 6 and 7 are devoted to the construction of basis elements arising from regular components. In Section 6, we consider the Kronecker quiver K and construct $\mathbb{Z}$-bases of $\mathcal{C}_{\mathbb{Z}}(K)$. The most important part of this section is the construction of basis elements from regular components. In Section 7, we consider the cyclic quiver C_r and construct a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(C_r)$. This will provide basis elements arising from non-homogeneous tubes. The method used in this section comes from [DDX]. Finally, in Section 8, we combine the results from Section 5 to 7 to obtain integral bases of $\mathcal{C}_{\mathbb{Z}}(Q)$.

2. Notations and preliminaries

2.1. Cartan datum. Following Lusztig [L2], a *Cartan datum* is a pair $(I, (-,-))$ consisting of a finite set I and a bilinear form $(-,-) : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ which satisfies the following conditions:

$$(i,i) = 2, \text{ for all } i \in I;$$

$$(i,j) \leq 0, \text{ for all } i \neq j;$$

$$(i,j) = 0 \text{ if and only if } (j,i) = 0.$$

Note that if we set $a_{ij} = (i,j)$ then the matrix $A = (a_{ij})$ is a generalized Cartan matrix.

A Cartan datum is said to be *irreducible* if the corresponding Cartan matrix cannot be made block-diagonal by simultaneous permutations of rows and columns. It is said to be *symmetric* if $(i,j) = (j,i)$ for all $i, j \in I$. In this paper we always assume that the Cartan datum is irreducible and symmetric.

A Cartan datum is said to be *of finite type* (resp. *affine*) if the corresponding Cartan matrix is positive definite (resp. positive semi-definite). A Cartan datum is said to be *simply-laced* if $(i,j) \in \{0, -1\}$ for all $i, j \in I$.

2.2. Kac-Moody algebras and their enveloping algebras. For a given Cartan datum there is the corresponding Kac-Moody algebra $\mathfrak{g}$ which has the triangular decomposition $\mathfrak{g} \simeq \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ with $\mathfrak{n}^+$, $\mathfrak{n}^-$ the maximal nilpotent subalgebras and $\mathfrak{h}$ the Cartan subalgebra. The universal enveloping algebra $U = U(\mathfrak{g})$ is the

$\mathbb{C}$-algebra generated by $\{e_i, f_i, h_i | i \in I\}$ with the following relations:

$$[h_i, h_j] = 0, \text{ for all } i, j \in I;$$
$$[e_i, f_j] = \delta_{ij} h_i, \text{ for all } i, j \in I;$$
$$[h_i, e_j] = (i, j) e_j, \text{ for all } i, j \in I;$$
$$[h_i, f_j] = -(i, j) f_j, \text{ for all } i, j \in I;$$
$$\sum_{k=0}^{1-(i,j)} (-1)^k \binom{1-(i,j)}{k} e_i^k e_j e_i^{1-(i,j)-k} = 0, \text{ for } i \neq j;$$
$$\sum_{k=0}^{1-(i,j)} (-1)^k \binom{1-(i,j)}{k} f_i^k f_j f_i^{1-(i,j)-k} = 0, \text{ for } i \neq j.$$

Let U^+ (resp. U^-, U^0) be the subalgebra of U generated by $\{e_i\}_{i \in I}$ (resp. $\{f_i\}_{i \in I}$, $\{h_i\}_{i \in I}$). We know that $U \simeq U^+ \otimes U^0 \otimes U^-$. Actually U^+ (resp. U^-) is the universal enveloping algebra of $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$). The *Kostant $\mathbb{Z}$-form* $U_{\mathbb{Z}}$ is defined as the $\mathbb{Z}$-subalgebra of U generated by $e_i^{(n)}$ and $f_i^{(n)}$, for all $i \in I$ and $n \in \mathbb{N}$, where $e_i^{(n)} = e_i^n / n!$, $f_i^{(n)} = f_i^n / n!$ are called *divided powers.* Let $U_{\mathbb{Z}}^+ = U^+ \cap U_{\mathbb{Z}}$ (resp. $U_{\mathbb{Z}}^- = U^- \cap U_{\mathbb{Z}}$), which is the $\mathbb{Z}$-subalgebra of U generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) .

2.3. Quivers and their representations. A *quiver* is an oriented graph $Q = (I, \Omega, s, t)$ where I is the set of vertices, Ω is the set of arrows; s, t are two maps from Ω to I denoting the starting and terminal vertex respectively. In this paper we consider quivers without loops (i.e. arrows from a vertex to itself).

A *representation* of Q over $\mathbb{C}$ is an I-graded $\mathbb{C}$-vector space $V = \oplus_{i \in I} V_i$ with a collection of linear maps $x = (x_h)_{h \in \Omega} \in \oplus_{h \in \Omega} \mathrm{Hom}_{\mathbb{C}}(V_{s(h)}, V_{t(h)})$. A *morphism* from a representation (V, x) to another one (V', x') is an I-graded $\mathbb{C}$-linear map $\phi = (\phi_i)_{i \in I} : V \to V'$ such that $x'_h \phi_{s(h)} = \phi_{t(h)} x_h$ for any $h \in \Omega$.

The *dimension vector* of a representation $M = (V, x)$ is defined as a vector $\underline{\dim} M = \sum_{i \in I} (\dim_{\mathbb{C}} V_i) i \in \mathbb{N}[I]$. A representation (V, x) is called finite dimensional if V_i is finite dimensional for all i.

A representation (V, x) of Q is called *nilpotent* if there exists $N \in \mathbb{N}$ such that $x_{h_N} \cdots x_{h_1} = 0$ for any sequence $h_1, \cdots, h_N \in \Omega$ with $t(h_i) = s(h_{i+1})$.

Denote by rep(Q) the category of finite dimensional representations of Q. We know that rep(Q) is equivalent to mod$\mathbb{C}Q$, the category of finite dimensional left $\mathbb{C}Q$-modules, where $\mathbb{C}Q$ is the path algebra of Q. For this reason we will use *$\mathbb{C}Q$-modules* or *representations of Q* freely in the sequel, and we will just write modules or representations for short when Q is fixed. Denote by $\text{rep}_0(Q)$ the full subcategory of rep(Q) consisting of all nilpotent representations. Note that if Q has no oriented cycles, we have $\text{rep}(Q) = \text{rep}_0(Q)$.

The isomorphism classes of simple objects in $\text{rep}_0(Q)$ are in one-to-one correspondence with the vertices of Q. Namely, for each $i \in I$, set $V_i = \mathbb{C}$, $V_j = 0$ for $j \neq i$ and $x = 0$. Then the module (V, x) is simple, denoted by S_i.

2.4. Euler forms. Let $(I, (-,-))$ be a Cartan datum, we have the corresponding Dynkin diagram, which is an unoriented graph. Giving any orientation of the graph we obtain a quiver Q. Q is said to be a quiver corresponding to $(I, (-,-))$. Conversely, the Cartan datum $(I, (-,-))$ can be recovered from any quiver corresponding to it.

More precisely, we define a bilinear form $\langle -,- \rangle : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ given by $\langle i, j \rangle = \delta_{ij} - \sharp\{h \in \Omega | s(h) = i, t(h) = j\}$, where δ_{ij} is the Kronecker symbol and $\sharp$ denote the number of elements in a set. This form is called the *Euler form.* We know that for any $M, N \in \text{rep}(Q)$,

$$\langle \underline{\dim} M, \underline{\dim} N \rangle = \dim_{\mathbb{C}} \text{Hom}(M, N) - \dim_{\mathbb{C}} \text{Ext}^1(M, N).$$

The *symmetric Euler form* is defined as $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$, for any $\alpha, \beta \in \mathbb{Z}[I]$. Then $(I, (-,-))$ is a Cartan datum, whose corresponding Dynkin diagram is just the underlying graph of Q.

A quiver is said to be *of finite type* (*tame*) if the corresponding Cartan datum is of finite type (resp. affine). Thus the underlying graph of a tame quiver is of type $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$.

In the sequel, we will write $\mathfrak{g}(Q)$ for the Kac-Moody algebra $\mathfrak{g}$ with Cartan datum corresponding to Q.

3. The Hall algebra

3.1. Constructible functions. Let X be an algebraic variety over $\mathbb{C}$. A subset of X is called *locally closed* if it is the intersection of an open and a closed subset.

A *constructible* set in X is a union of finitely many locally closed subsets of X. A function $f : X \to \mathbb{C}$ is called *constructible* if $f(X)$ is a finite set and $f^{-1}(m)$ is a constructible subset of X for all $m \in \mathbb{C}$. We denote by $\mathcal{M}(X)$ the set of all constructible functions on X with values in $\mathbb{C}$. $\mathcal{M}(X)$ is naturally a $\mathbb{C}$-vector space. Let G be an algebraic group acting on X. We denote by $\mathcal{M}(X)^G$ the subspace of $\mathcal{M}(X)$ consisting of all G-invariant constructible functions.

Let $\phi : X \to Y$ be a morphism of algebraic varieties. We can define two linear maps $\phi^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ and $\phi_! : \mathcal{M}(X) \to \mathcal{M}(Y)$ as follows:

$$\phi^*(g)(x) = g(\phi(x))$$

for any $g \in \mathcal{M}(Y)$ and $x \in X$.

$$\phi_!(f)(y) = \sum_{a \in \mathbb{C}} a\chi(\phi^{-1}(y) \cap f^{-1}(a))$$

for any $f \in \mathcal{M}(X)$ and $y \in Y$, where χ denotes the Euler characteristic with compact support.

3.2. Varieties of representations. Given a quiver Q and a fixed dimension vector $\alpha = \sum_{i \in I} \alpha_i i \in \mathbb{N}[I]$, denote by $\mathbf{E}_\alpha$ the set of all representations of Q with dimension vector α. i.e.

$$\mathbf{E}_\alpha = \prod_{h \in \Omega} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(h)}}, \mathbb{C}^{\alpha_{t(h)}}).$$

Hence $\mathbf{E}_\alpha$ is an affine algebraic variety (In fact, an affine space). Let $G_\alpha = \prod_{i \in I} GL(\alpha_i, \mathbb{C})$. The group G_α acts on $\mathbf{E}_\alpha$ by $g(x_h) = g_{t(h)} x_h g_{s(h)}^{-1}$, for any $g = (g_i)_{i \in I}$, $x = (x_h)_{h \in \Omega}$. Let $\mathbf{E}_\alpha^{nil}$ be the subset of all nilpotent representations in $\mathbf{E}_\alpha$. It is easy to see that $\mathbf{E}_\alpha^{nil}$ is a closed subvariety of $\mathbf{E}_\alpha$. The group G_α also acts on $\mathbf{E}_\alpha^{nil}$.

3.3. The Hall algebra. To simplify the notations, we denote $\mathcal{M}(\mathbf{E}_\alpha^{nil})^{G_\alpha}$ by $\mathcal{H}_\alpha(Q)$. Let $\mathcal{H}(Q) = \oplus_{\alpha \in \mathbb{N}[I]} \mathcal{H}_\alpha(Q)$.

Lusztig [L1] has defined a bilinear map

$$* : \mathcal{H}_\alpha(Q) \times \mathcal{H}_\beta(Q) \to \mathcal{H}_\gamma(Q)$$

for any $\alpha, \beta, \gamma \in \mathbb{N}[I]$ such that $\alpha + \beta = \gamma$. Then an $\mathbb{N}[I]$-graded multiplication can be endowed with $\mathcal{H}(Q)$.

The map $*$ is defined as follows: Consider the following diagram:

$$\mathbf{E}^{nil}_{\alpha} \times \mathbf{E}^{nil}_{\beta} \xleftarrow{p_1} \mathbf{E}'' \xrightarrow{p_2} \mathbf{E}' \xrightarrow{p_3} \mathbf{E}^{nil}_{\gamma}$$

where the notations are as follows:

$\mathbf{E}'$ is the variety of all pairs (x, W) consisting of $x \in \mathbf{E}^{nil}_{\gamma}$ and an x-stable I-graded subspace W of $\mathbb{C}^{\gamma}$ such that $\underline{\dim} W = \beta$;

$\mathbf{E}''$ is the variety of all quadruples $(x, W, R^{\alpha}, R^{\beta})$, where $(x, W) \in \mathbf{E}'$, R^{β} is an isomorphism $\mathbb{C}^{\beta} \simeq W$, R^{α} is an isomorphism $\mathbb{C}^{\alpha} \simeq \mathbb{C}^{\gamma}/W$;

$p_1(x, W, R^{\alpha}, R^{\beta}) = (x^{\alpha}, x^{\beta})$, where $x_h R^{\alpha}_{s(h)} = R^{\alpha}_{t(h)} x^{\alpha}_h$ and $x_h R^{\beta}_{s(h)} = R^{\beta}_{t(h)} x^{\beta}_h$;

$p_2(x, W, R^{\alpha}, R^{\beta}) = (x, W)$; $p_3(x, W) = x$.

Note that p_1 is smooth with connected fibres, p_2 is a principal $G_{\alpha} \times G_{\beta}$-bundle and p_3 is proper.

Now we can define a convolution product of constructible functions

For $f_{\alpha} \in \mathcal{H}_{\alpha}(Q)$, $f_{\beta} \in \mathcal{H}_{\beta}(Q)$, we let f_1 be a constructible function on $\mathbf{E}^{nil}_{\alpha} \times \mathbf{E}^{nil}_{\beta}$ given by $f_1(x_1, x_2) = f_{\alpha}(x_1) f_{\beta}(x_2)$ for any $x_1 \in \mathbf{E}^{nil}_{\alpha}$, $x_2 \in \mathbf{E}^{nil}_{\beta}$. Then there is a unique function $f_2 \in \mathcal{M}(\mathbf{E}')$ such that $p_1^* f_1 = p_2^* f_2$. We define $f_{\alpha} * f_{\beta}$ as $(p_3)_!(f_2)$.

The $\mathbb{C}$-space $\mathcal{H}(Q)$ equipped with the multiplication $*$ is an $\mathbb{N}[I]$-graded associative $\mathbb{C}$-algebra, called the *Hall algebra*. In the sequel we will omit the multiplication symbol $*$.

3.4. Characteristic functions. For a fixed dimension vector α and a G_{α}-invariant constructible subset $\mathcal{O}$ of $\mathbf{E}^{nil}_{\alpha}$, the *characteristic function* of $\mathcal{O}$ is defined as the function taking the value 1 on $\mathcal{O}$ and 0 elsewhere. We denote the function by $\mathbf{1}_{\mathcal{O}}$. It is obvious that $\mathbf{1}_{\mathcal{O}} \in \mathcal{H}_{\alpha}(Q)$.

For any $M \in \mathbf{E}^{nil}_{\alpha}$, the G_{α}-orbit of M is denoted by $\mathcal{O}_M$. In particular, $\mathcal{O}_M$ is a constructible subset of $\mathbf{E}^{nil}_{\alpha}$. In this case we just write $\mathbf{1}_M$ instead of $\mathbf{1}_{\mathcal{O}_M}$.

Let $M \in \mathbf{E}^{nil}_{\alpha}$, $N \in \mathbf{E}^{nil}_{\beta}$. The definition of the multiplication yields

$$\mathbf{1}_M \mathbf{1}_N(L) = \chi(\mathcal{F}(M, N; L)), \text{ for any } L \in \mathbf{E}^{nil}_{\alpha+\beta},$$

where $\mathcal{F}(M, N; L)$ is the variety of all submodules L' of L such that $L' \simeq N$ and $L/L' \simeq M$.

In general, let $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) be a G_α (resp. G_β)-invariant constructible subset of $\mathbf{E}_\alpha^{nil}$ (resp. $\mathbf{E}_\beta^{nil}$), we have

$$\mathbf{1}_{\mathcal{O}_1}\mathbf{1}_{\mathcal{O}_2}(M) = \chi(\mathcal{F}(\mathcal{O}_1, \mathcal{O}_2; M)), \text{ for any } M \in \mathbf{E}_{\alpha+\beta}^{nil},$$

where $\mathcal{F}(\mathcal{O}_1, \mathcal{O}_2; M)$ is the variety of all submodules M' of M such that $M' \in \mathcal{O}_2$ and $M/M' \in \mathcal{O}_1$.

3.5. The composition algebra. For any $i \in I$, the simple module S_i is the unique module with dimension vector i. We simply denote by $\mathbf{1}_i$ the characteristic function $\mathbf{1}_{S_i}$.

Let $\mathcal{C}(Q)$ be the $\mathbb{C}$-subalgebra of $\mathcal{H}(Q)$ generated by $\mathbf{1}_i$, for all $i \in I$. It is called the *composition algebra*. The following theorem is well-known (for example, see [L1]):

Theorem 3.1. *For any quiver Q without loops, the composition algebra $\mathcal{C}(Q)$ is isomorphic to the positive part of the enveloping algebra $U^+ = U^+(\mathfrak{g}(Q))$. This isomorphism is given by $\mathbf{1}_i \mapsto e_i$ for any $i \in I$.*

We can also define the $\mathbb{Z}$-form of the composition algebra (or the integral composition algebra) $\mathcal{C}_\mathbb{Z}(Q)$, which is the $\mathbb{Z}$-subalgebra of $\mathcal{H}(Q)$ generated by the divided powers $\mathbf{1}_i^{(n)}$, for all $i \in I$ and $n \in \mathbb{N}$.

The following corollary can be seen immediately from the theorem.

Corollary 3.2. *The integral composition algebra $\mathcal{C}_\mathbb{Z}(Q)$ is isomorphic to $U_\mathbb{Z}^+$.*

3.6. Some calculations. In general, the calculation of the Euler Characteristic of a variety is difficult. In this subsection, we give two formulas dealing with special cases which we will use later.

For any $M \in \text{rep}(Q)$, we denote by tM the direct sum of t copies of M. A module $M \in \text{rep}(Q)$ is called *exceptional* if $\text{Ext}^1(M, M) = 0$.

Lemma 3.3. *For any indecomposable exceptional module M we have*

$$\mathbf{1}_{tM} = \mathbf{1}_M^{(t)}.$$

Proof. Since M has no self-extensions, we have by definition

$$\mathbf{1}_M^t = \mathbf{1}_M \mathbf{1}_M \cdots \mathbf{1}_M = \chi(\mathcal{F})\mathbf{1}_{tM}$$

where $\mathcal{F}$ is the variety of all filtrations

$$tM = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

with factors isomorphic to M.

It is easy to see that $\chi(\mathcal{F})$ is equal to the Euler characteristic of the variety of complete flags in $\mathbb{C}^t$. Hence $\chi(\mathcal{F}) = t!$ and the lemma holds. □

Lemma 3.4. *For any $M_1, \cdots, M_t \in rep(Q)$ such that $Hom(M_i, M_j) = 0$ and $Ext^1(M_j, M_i) = 0$ for all $i > j$. Then we have*

$$\mathbf{1}_M = \mathbf{1}_{M_1}\mathbf{1}_{M_2}\cdots\mathbf{1}_{M_t}$$

where $M = \oplus_{i=1}^t M_i$.

Proof. It is sufficient to prove the case $t = 2$. The general case follows by induction.

So let $\mathrm{Hom}(M_2, M_1) = 0$ and $\mathrm{Ext}^1(M_1, M_2) = 0$, we need to prove $\mathbf{1}_{M_1 \oplus M_2} = \mathbf{1}_{M_1}\mathbf{1}_{M_2}$. Since $\mathrm{Ext}^1(M_1, M_2) = 0$, we have

$$\mathbf{1}_{M_1}\mathbf{1}_{M_2} = \chi(\mathcal{G})\mathbf{1}_{M_1 \oplus M_2}$$

where

$$\mathcal{G} = \{N \subset M_1 \oplus M_2 | N \simeq M_2 \text{ and } M_1 \oplus M_2/N \simeq M_1\}.$$

As $\mathrm{Hom}(M_2, M_1) = 0$, we know that $\mathcal{G}$ is a single point and hence $\chi(\mathcal{G}) = 1$. □

4. Representation of tame quivers

In this section, we give a brief review of the representation theory of tame quivers. For details one can see [DR], for example.

4.1. The classification of indecomposable modules. Let Q be a quiver without oriented cycles (not necessarily tame). Denote by $\mathrm{ind}(Q)$ the set of isomorphism classes of indecomposable modules. For $M \in \mathrm{rep}(Q)$, denote its isomorphism class by $[M]$.

The objects in $\mathrm{ind}(Q)$ can be classified as follows: $M \in \mathrm{ind}(Q)$ is called *preprojective* (resp. *preinjective*) if there exists a positive integer n such that $\tau^n M = 0$ (resp. $\tau^{-n} M = 0$) where τ denotes the Auslander-Reiten translation. And M

is called *regular* if for any $n \in \mathbb{Z}$, $\tau^n M \neq 0$. We say a decomposable module is preprojective, preinjective or regular if each indecomposable summand is.

Let Prep(Q), Prei(Q) and Reg(Q) denote the full subcategory of rep(Q) consisting of preprojective, preinjective and regular modules respectively. These three subcategories are extension-closed. Moreover, we have the following properties:

Proposition 4.1. *For any $P \in Prep(Q)$,$R \in Reg(Q)$ and $I \in Prei(Q)$, we have*

$$Hom(I,P) = Ext^1(P,I) = 0;$$
$$Hom(I,R) = Ext^1(R,I) = 0;$$
$$Hom(R,P) = Ext^1(P,R) = 0.$$

Roughly speaking, the *Auslander-Reiten-quiver* (AR-quiver, for short) of rep(Q) is the quiver whose vertices are objects in ind(Q) and arrows describe *irreducible morphisms* between modules. We denote it by Γ_Q. Thus in the language of Auslander-Reiten theory, Γ_Q can be divided into three components, called the preprojective, regular and preinjective components respectively.

We remark that if Q is of finite type, there is no regular component in Γ_Q and the preprojective component coincides with the preinjective component.

4.2. Root system and indecomposable modules. We denote by Δ the *root system* of $\mathfrak{g}(Q)$. We identify each $i \in I$ with the simple root α_i. Thus the positive root lattice can be identified with $\mathbb{N}[I]$. And the set of *positive roots* is $\Delta_+ = \Delta \cap \mathbb{N}[I]$.

The set of *real roots* and *imaginary roots* are denoted by Δ^{re} and Δ^{im} respectively. Set $\Delta^{re}_+ = \Delta^{re} \cap \Delta_+$, $\Delta^{im}_+ = \Delta^{im} \cap \Delta_+$.

The following theorem is due to Kac, which is a generalization of Gabriel's theorem.

Theorem 4.2. *(1). For any $[M] \in ind(Q)$, $\underline{dim}M \in \Delta_+$.*

(2). For any $\alpha \in \Delta^{re}_+$, there is a unique $[M] \in ind(Q)$ with $\underline{dim}M = \alpha$.

(3). For any $\alpha \in \Delta^{im}_+$, there are infinitely many $[M] \in ind(Q)$ with $\underline{dim}M = \alpha$.

For any $\alpha \in \Delta^{re}_+$, we denote by $M(\alpha)$ the unique (up to isomorphism) indecomposable module with dimension vector α.

Denote by Δ_+^{prep} (resp. Δ_+^{prei}, Δ_+^{reg}) the set of all positive roots which are dimension vectors of indecomposable preprojective (resp. preinjective, regular) modules. It is known that $\Delta_+^{prep} \cup \Delta_+^{prei} \subset \Delta_+^{re}$, $\Delta_+^{im} \subset \Delta_+^{reg}$ but in general they are not equal.

4.3. The preprojective and preinjective modules. To describe $\mathrm{ind}(Q)$, we need to describe the preprojective, preinjective and the regular component respectively. The case of the preprojective or preinjective is easy. In fact it is similar to the case of finite type. We just recall the following *representation-directed* property:

Lemma 4.3. *(1). There exists a total order (not unique) on* Δ_+^{prep}*:*

$$\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_m \prec \cdots$$

such that

$$Hom(M(\alpha_i), M(\alpha_j)) = 0, \text{ for all } i > j;$$

$$Ext^1(M(\alpha_i), M(\alpha_j)) = 0, \text{ for all } i \leq j.$$

(2). There exists a total order (not unique) on Δ_+^{prei}*:*

$$\cdots \prec \beta_n \prec \cdots \prec \beta_2 \prec \beta_1$$

such that

$$Hom(M(\beta_i), M(\beta_j)) = 0, \text{ for all } i < j;$$

$$Ext^1(M(\beta_i), M(\beta_j)) = 0, \text{ for all } i \geq j.$$

4.4. The Jordan quiver. The description of the regular component is more complicated. We need some preparations in this and the next subsection.

Let C_1 be the quiver with only one vertex and a loop arrow. This is the so-called *Jordan quiver*. Now a module in $\mathrm{rep}_0(C_1)$ is just a pair (V, x) where V is a $\mathbb{C}$-space and x is a nilpotent linear transformation on V.

The unique simple module in $\mathrm{rep}_0(C_1)$ is denoted by S. Any indecomposable module with dimension n is isomorphic to $S[n] = (\mathbb{C}^n, J_n)$, where J_n is the $n \times n$ Jordan block with 0's in the diagonal. And $\tau S[n] = S[n]$ for any n. The AR-quiver of $\mathrm{rep}_0(C_1)$ is called a *homogeneous tube*.

4.5. The cyclic quiver. In this subsection we fix $r \in \mathbb{N}, r \geq 2$. Let $C_r = (I, \Omega, s, t)$ be a *cyclic quiver* with r vertices, i.e. $I = \mathbb{Z}/r\mathbb{Z} = \{1, 2, \cdots, r\}$, $\Omega = \{\rho_i | 1 \leq i \leq r\}$ where $s(\rho_i) = i$, $t(\rho_i) = i + 1$ for all i. Note that underline graph of this quiver is of type $A^{(1)}_{r-1}$, but it has an oriented cycle.

We consider the category $\mathrm{rep}_0(C_r)$. There are r simple modules, denoted by S_i, $i \in I$ (See 2.3). For any $i \in I$ and $l \geq 1$, there is a unique (up to isomorphism) indecomposable module in $\mathrm{rep}_0(C_r)$ with top S_i and length l, denoted by $S_i[l]$. And it is known that the set of isomorphism classes of indecomposable modules in $\mathrm{rep}_0(C_r)$ is just $\{S_i[l] | 1 \leq i \leq r, l \geq 1\}$. Moreover we have $\tau S_i[l] = S_{i+1}[l]$. hence $\tau^r S_i[l] = S_i[l]$ for all i, l. For this reason, the AR-quiver Γ_{C_r} is called a *non-homogeneous tube* or more precisely, a *tube of rank* r (So a homogeneous tube actually means a tube of rank 1).

Let $\delta = (1, 1, \cdots, 1)$ be the minimal imaginary root. We know that $\underline{\dim} S_i[nr] = n\delta$ for any $i \in I$ and $n \geq 1$. And $\underline{\dim} S_i[l] \in \Delta^{re}_+$ for any $i \in I$ and $r \nmid l$.

Denote by $\mathcal{I}(C_r)$ the set of isomorphism classes of all modules in $\mathrm{rep}_0(C_r)$. Let Π be the set of r-tuples of partitions $\pi = (\pi^{(1)}, \pi^{(2)}, \cdots, \pi^{(r)})$ with each components $\pi^{(i)} = (\pi^{(i)}_1 \geq \pi^{(i)}_2 \geq \cdots)$ being a partition of a positive integer. For each $\pi \in \Pi$, we have a module

$$M(\pi) = \bigoplus_{i \in I, j \geq 1} S_i[\pi^{(i)}_j].$$

In this way we obtain a bijection between Π and $\mathcal{I}(C_r)$.

4.6. Tame quivers and the regular modules. In this subsection let $Q = (I, \Omega, s, t)$ be a tame quiver without oriented cycles. Now we can describe the subcategory $\mathrm{Reg}(Q)$. The following lemma shows that the regular component of Γ_Q is a collection of tubes indexed by $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. Moreover, there are only finite many non-homogeneous tubes and all the others are homogeneous.

Lemma 4.4.

$$Reg(Q) \simeq (\coprod_{j=1}^{s} \mathcal{T}_j) \coprod (\coprod_{x \in \mathbb{P}^1 \setminus J} \mathcal{T}_x)$$

as coproduct of abelian categories, where J is a subset of $\mathbb{P}^1$ containing s elements, each $\mathcal{T}_x$ is isomorphic to $rep_0(C_1)$ and each $\mathcal{T}_j$ is isomorphic to $rep_0(C_{r_j})$ for some $r_j > 1$.

According to the lemma, for each $x \in \mathbb{P}^1 - J$ and $1 \le j \le s$, there is an isomorphism functor $F_x : \mathrm{rep}_0(C_1) \xrightarrow{\sim} \mathcal{T}_x$, $F_j : \mathrm{rep}_0(C_{r_j}) \xrightarrow{\sim} \mathcal{T}_j$. In general, the choice of F_j is not unique. We just fix one for each j. Then, for any x and $l \ge 1$, set $S_{x,l} = F_x(S[l])$ (see 4.4). For any $1 \le j \le s$, $1 \le i \le r_j$ and $l \ge 1$, set $S_{j,i,l} = F_j(S_i[l])$ (see 4.5).

The modules $S_{x,1}$ and $S_{j,i,1}$ are called *quasi-simple* for any x, j and i. The module $S_{x,l}$ (resp. $S_{j,i,l}$) is the unique (up to isomorphism) module with *quasi-top* $S_{x,1}$ (resp. $S_{j,i,1}$) and *quasi-length* l. Here the word quasi- means with respect to the subcategory $\mathcal{T}_x$ or $\mathcal{T}_j$. Note that for any x and j, $\mathcal{T}_x$ and $\mathcal{T}_j$ are extension-closed abelian subcategories of $\mathrm{rep}(Q)$.

We know that for an affine Cartan datum, $\Delta_+^{im} = (\mathbb{N} - \{0\})\delta$ where δ is the minimal imaginary root. The following lemma describes all indecomposable modules with dimension vector $n\delta$ and also tells us the difference between Δ_+^{im} and Δ_+^{reg}.

Lemma 4.5. *(1). For any $n \ge 1$, the set of isomorphic classes of indecomposable modules with dimension vector $n\delta$ is the following*

$$\{[S_{x,n}] | x \in \mathbb{P}^1 \setminus J\} \cup \{[S_{j,i,nr_j}] | 1 \le j \le s, 1 \le i \le r_j\}$$

(2). $\underline{dim} S_{j,i,l} \in \Delta_+^{re}$ for any $1 \le j \le s$, $1 \le i \le r_j$ and $r_j \nmid l$.

In fact for each tame quiver Q, the number of non-homogeneous tubes and the rank of each tube can be determined precisely. But we don't need the details in this paper. However, we mention a particular case: the Kronecker quiver K, which is the quiver with two vertices and two arrows pointing from one vertex to the other. In this case all tubes are homogeneous. Later (in section 6) we will discuss the composition algebra $\mathcal{C}(K)$ in details.

At the end of this subsection, we recall a well-known result (see [CB], for example):

Lemma 4.6. *We have:*

$$1 + \sum_{j=1}^{s} (r_j - 1) = |I| - 1$$

and this number is equal to the multiplicity of any imaginary root.

5. Bases arising from preprojective and preinjective components

In this section we assume that $Q = (I, \Omega, s, t)$ is a quiver without oriented cycles (not necessarily tame). We will discuss two subalgebras of the composition algebra $\mathcal{C}(Q)$.

5.1. The subalgebra $\mathcal{C}(Q)^{prep}$ and $\mathcal{C}(Q)^{prei}$. Let $\mathcal{C}(Q)^{prep}$ (resp. $\mathcal{C}(Q)^{prei}$) be the subalgebra of $\mathcal{H}(Q)$ generated by $\mathbf{1}_P$ for all $P \in \mathrm{Prep}(Q)$ (resp. $\mathbf{1}_I$ for all $I \in \mathrm{Prei}(Q)$). In this section we will prove that $\mathbf{1}_P$ and $\mathbf{1}_I$ are actually in $\mathcal{C}_{\mathbb{Z}}(Q)$. Thus $\mathcal{C}(Q)^{prep}$ and $\mathcal{C}(Q)^{prei}$ are subalgebras of the composition algebra $\mathcal{C}(Q)$. Moreover, we can define $\mathcal{C}_{\mathbb{Z}}(Q)^{prep}$ (resp. $\mathcal{C}_{\mathbb{Z}}(Q)^{prei}$) to be the $\mathbb{Z}$-subalgebra of $\mathcal{C}_{\mathbb{Z}}(Q)$ generated by $\mathbf{1}_P$ for all $P \in \mathrm{Prep}(Q)$ (resp. $\mathbf{1}_I$ for all $I \in \mathrm{Prei}(Q)$). At the end of this section we will construct $\mathbb{Z}$-bases of these two subalgebras.

5.2. Reflection functors. To prove that $\mathbf{1}_P$ and $\mathbf{1}_I$ are in the composition algebra we need some tools.

Let i be a *sink* of Q, i.e. there are no arrows starting from i. We define $\sigma_i Q$ to be the quiver obtained from Q by reversing all the arrows connected to i. Following [BGP] we can define the *reflection functors*:

$$\sigma_i^+ : \mathrm{rep}(Q) \to \mathrm{rep}(\sigma_i Q)$$

The action of the functor σ_i^+ on objects is defined by $\sigma_i^+(V, x) = (V', x')$ where

$$V'_k = V_k \text{ if } k \neq i,$$
$$V'_i = \ker(\oplus_{t(h)=i} x_h : \oplus_{t(h)=i} V_{s(h)} \to V_i),$$
$$x'_h = x_h \text{ if } t(h) \neq i,$$
$$x'_h \text{ is the composition } V'_i \to \oplus_{t(h')=i} V_{s(h')} \to V_{s(h)} \text{ if } t(h) = i.$$

The action of σ_i^+ on morphisms is the natural one.

Let $\mathrm{rep}(Q)[i]$ be the subcategory of $\mathrm{rep}(Q)$ consisting of all modules which do not have S_i as direct summands. Note that since i is a sink, S_i is a simple projective module. Hence $\mathrm{rep}(Q)[i]$ is closed under extensions. Then we can define $\mathcal{H}(Q)[i]$ to be the subalgebra of $\mathcal{H}(Q)$ generated by all constructible functions whose support are contained in $\mathrm{rep}(Q)[i]$. The functor σ_i^+ induces an algebra homomorphism

$$\sigma_i : \mathcal{H}(Q)[i] \to \mathcal{H}(\sigma_i Q)[i]$$

defined by

$$\sigma_i(\mathbf{1}_M) = \mathbf{1}_{\sigma_i^+ M}, \text{ for any } M \in \text{rep}(Q)[i]$$

Let $\mathcal{C}(Q)[i] = \mathcal{H}(Q)[i] \cap \mathcal{C}(Q)$, $\mathcal{C}_{\mathbb{Z}}(Q)[i] = \mathcal{H}(Q)[i] \cap \mathcal{C}_{\mathbb{Z}}(Q)$. Note that $\mathcal{C}(Q)$ and $\mathcal{C}(\sigma_i Q)$ (hence $\mathcal{C}_{\mathbb{Z}}(Q)$ and $\mathcal{C}_{\mathbb{Z}}(\sigma_i Q)$) are canonically isomorphic by fixing the Chevalley generators which correspond to the simple modules of rep(Q) and rep$(\sigma_i Q)$ respectively.

We know that (see [BGP]) σ_i^+ restricts to an equivalence of categories:

$$\sigma_i^+ : \text{rep}(Q)[i] \xrightarrow{\sim} \text{rep}(\sigma_i Q)[i].$$

Hence it induces an isomorphism of algebras:

$$\sigma_i : \mathcal{C}(Q)[i] \xrightarrow{\sim} \mathcal{C}(Q)[i].$$

Dually for any *source* $i \in I$ we can define the reflection functor σ_i^-. We have similar results as above.

5.3. Automorphisms of enveloping algebra. Recall for any $i \in I$ we have the following automorphism of the enveloping algebra U (see [Ka]):

$$r_i = \exp(\text{ad}e_i)\exp(\text{ad}(-f_i))\exp(\text{ad}e_i) : U \xrightarrow{\sim} U.$$

And we have

$$\begin{aligned} r_i(e_i) &= -f_i \\ r_i(f_i) &= -e_i \\ r_i(e_j) &= (\text{ad}e_i)^{(-a_{ij})}(e_j), \text{ for } i \neq j \\ r_i(f_j) &= (-\text{ad}f_i)^{(-a_{ij})}(f_j), \text{ for } i \neq j. \end{aligned}$$

Thus r_i is also an automorphism of $U_{\mathbb{Z}}$.

Lemma 5.1. *The restriction of σ_i to $\mathcal{C}_{\mathbb{Z}}(Q)[i]$ is a $\mathbb{Z}$-automorphism which equals to the restriction of r_i.*

Proof. We need a result proved in [XZZ] in which the reflection functors are generalized to the root category. It was proved that the reflection functor σ_i induces an automorphism of the whole Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(Q)$ (and hence an automorphism of the enveloping algebra $U(\mathfrak{g})$) and this automorphism is just the same as r_i.

In our case we goes back to the positive part U^+. When we restrict both automorphisms to $\mathcal{C}_{\mathbb{Z}}(Q)[i] \subset U_{\mathbb{Z}}^+$, we get the result in the lemma. □

5.4. Admissible sequences. Let $i_1, \cdots, i_m$ be an *admissible source sequence* of Q, i.e. i_1 is a source of Q and for any $1 < t \leq m$, the vertex i_t is a source for $\sigma_{i_{t-1}} \cdots \sigma_{i_1} Q$.

Let $M \in \text{Prei}(Q)$ be indecomposable, then there exists an admissible source sequence $i_1, \cdots, i_m$ of Q such that

$$M = \sigma_{i_1}^+ \cdots \sigma_{i_{m-1}}^+ (S_{i_m})$$

where S_{i_m} is the simple $\mathbb{C}(\sigma_{i_{m-1}} \cdots \sigma_{i_1} Q)$-module corresponding to the vertex i (See [BGP]).

Lemma 5.2. *Let $M \in$ Prei(Q) be indecomposable. Then there exists an admissible source sequence $i_1, \cdots, i_m$ of Q such that*

$$\mathbf{1}_M = r_{i_1} \cdots r_{i_{m-1}} \mathbf{1}_{i_m}$$

Proof. It is clear by the definition of σ_i and lemma 5.1. □

Similar results can be proved for indecomposable preprojective modules using *admissible sink sequences* and σ_i^- instead.

Thus by the above lemma we can see that for any indecomposable $M \in \text{Prei}(Q)$ or $\text{Prep}(Q)$, the characteristic function $\mathbf{1}_M \in \mathcal{C}_{\mathbb{Z}}(Q)$.

5.5. $\mathbb{Z}$-Bases of $\mathcal{C}_{\mathbb{Z}}(Q)^{prei}$ and $\mathcal{C}_{\mathbb{Z}}(Q)^{prep}$. We will use the notations in 4.3. Let I be any preinjective $\mathbb{C}Q$-module, then it can be decomposed into a direct sum of indecomposable preinjective modules. We choose an order on Δ_+^{prei} satisfying lemma 4.3.

Lemma 5.3. *For any $I \in Prei(Q)$, if it decomposes as*

$$I = \bigoplus_{k=1}^{m} b_{i_k} M(\beta_{i_k})$$

where $\beta_{i_m} \prec \cdots \prec \beta_{i_2} \prec \beta_{i_1} \in \Delta_+^{prei}$ and $b_{i_k} \neq 0$. Then we have

$$\mathbf{1}_I = \mathbf{1}_{M(\beta_{i_m})}^{(b_{i_m})} \cdots \mathbf{1}_{M(\beta_{i_2})}^{(b_{i_2})} \mathbf{1}_{M(\beta_{i_1})}^{(b_{i_1})}$$

Proof. The key is the representation-directed property. Then the result follows from Lemma 3.3 and Lemma 3.4. □

From this lemma we can see that $\mathbf{1}_I \in \mathcal{C}_{\mathbb{Z}}(Q)$ for all $I \in \text{Prei}(Q)$, which we have claimed at the beginning of this section. Now we can give a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(Q)^{prei}$.

Proposition 5.4. *The set $\{\mathbf{1}_I | I \in Prei(Q)\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(Q)^{prei}$ (hence also a $\mathbb{C}$-basis of $\mathcal{C}(Q)^{prei}$).*

Proof. Since the subcategory $\text{Prei}(Q)$ is extension-closed, we have for any $I_1, I_2 \in \text{Prei}(Q)$,

$$\mathbf{1}_{I_1}\mathbf{1}_{I_2} = \sum_{I \in \text{Prei}(Q); \underline{\dim} I = \underline{\dim} I_1 + \underline{\dim} I_2} \chi(\mathcal{F}(I_1, I_2; I))\mathbf{1}_I$$

where the right hand side is a finite sum because the number of isomorphism classes of modules in $\text{Prei}(Q)$ with fixed dimension vector is finite.

Now our proposition follows as the Euler characteristics are always integers. □

Similarly, we have the following results for preprojective modules.

Lemma 5.5. *For any $P \in Prep(Q)$, if it decomposes as*

$$P = \bigoplus_{k=1}^{m} a_{i_k} M(\alpha_{i_k})$$

where $\alpha_{i_1} \prec \alpha_{i_2} \prec \cdots \prec \alpha_{i_m} \in \Delta_+^{prep}$ and $a_{i_k} \neq 0$. Then we have

$$\mathbf{1}_P = \mathbf{1}_{M(\alpha_{i_1})}^{(a_{i_1})} \mathbf{1}_{M(\alpha_{i_2})}^{(a_{i_2})} \cdots \mathbf{1}_{M(\alpha_{i_m})}^{(a_{i_m})}$$

Proposition 5.6. *The set $\{\mathbf{1}_P | P \in Prep(Q)\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(Q)^{prep}$ (hence a $\mathbb{C}$-basis of $\mathcal{C}(Q)^{prep}$).*

5.6. Remarks. (1). The arguments in this section are essentially the same as in the case of finite type. In fact when Q is of finite type, we have $\mathcal{C}(Q)^{prep} = \mathcal{C}(Q)^{prei} = \mathcal{C}(Q)$. Thus our construction gives a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(Q)$.

(2). The proofs of many results in this section are similar to those in the quantum case. For example, see [R4] (the finite type case).

6. Integral basis: The case of the Kronecker quiver

In this section we consider the simplest tame quiver, namely the Kronecker quiver $K = (I, \Omega, s, t)$ where $I = \{1, 2\}$, $\Omega = \{\rho_1, \rho_2\}$, $s(\rho_1) = s(\rho_2) = 2$ and $t(\rho_1) = t(\rho_2) = 1$. Note that this quiver is the only non-simply-laced tame quiver.

6.1. Some notations. For convenience in this section we will identify $\mathbb{N}^2$ with $\mathbb{N}[I]$ and write the dimension vectors as $(a, b) \in \mathbb{N}^2$. The set of positive roots are

$$\Delta_+ = \{(n, n+1), (m+1, m), (l+1, l+1) | n, m, l \in \mathbb{N}\}.$$

And we have $\Delta_+^{prep} = \{(m+1, m) | m \in \mathbb{N}\}$, $\Delta_+^{prei} = \{(n, n+1) | n \in \mathbb{N}\}$, $\Delta_+^{reg} = \{(l+1, l+1) | l \in \mathbb{N}\}$ respectively. Note that the minimal imaginary root $\delta = (1, 1)$. Hence in this case $\Delta_+^{reg} = \Delta_+^{im}$.

In this case the order on Δ_+^{prei} satisfying lemma 4.3 is unique:

$$\cdots \prec (n, n+1) \prec \cdots \prec (1, 2) \prec (0, 1),$$

and the same for Δ_+^{prep}:

$$(1, 0) \prec (2, 1) \prec \cdots \prec (m+1, m) \prec \cdots .$$

Recall 4.6 that $\text{Reg}(K) \simeq \coprod_{x \in \mathbb{P}^1} \mathcal{T}_x$, where $\mathcal{T}_x \simeq \text{rep}_0(C_1)$ for all x.

6.2. A basis of $\mathfrak{n}^+(K)$. In this section, for simplicity we will denote by $\mathbf{1}_\alpha$ the characteristic function $\mathbf{1}_{M(\alpha)}$ for any $\alpha \in \Delta_+^{re}$. Since $\underline{\dim} S_1 = (1, 0)$, $\underline{\dim} S_2 = (0, 1)$, we write $\mathbf{1}_1 = \mathbf{1}_{(1,0)}$, $\mathbf{1}_2 = \mathbf{1}_{(0,1)}$.

Following [FMV], for any $n \geq 1$, the set of all indecomposable regular modules with dimension vector $n\delta$ is a constructible subset of $\mathbf{E}_{n\delta}$. Let $P_{n\delta}$ be the characteristic function of this set. Hence $P_{n\delta} \in \mathcal{H}(Q)$.

The following results have been proved in [FMV]:

Proposition 6.1. *The set*

$$\{\mathbf{1}_{(m,m+1)}, \mathbf{1}_{(n+1,n)}, P_{k\delta} | m, n \geq 0; k \geq 1\}$$

is a basis of the maximal nilpotent subalgebra $\mathfrak{n}^+(K)$ *of the Lie algebra* $\mathfrak{g}(K)$.

Moreover, the structure constants with respect to the basis are clear:

$$[P_{m\delta}, P_{n\delta}] = 0;$$
$$[\mathbf{1}_{(n,n+1)}, \mathbf{1}_{(m,m+1)}] = 0;$$
$$[\mathbf{1}_{(n+1,n)}, \mathbf{1}_{(m+1,m)}] = 0;$$
$$[P_{n\delta}, \mathbf{1}_{(m+1,m)}] = 2\mathbf{1}_{(m+n+1,m+n)};$$
$$[\mathbf{1}_{(m,m+1)}, P_{n\delta}] = 2\mathbf{1}_{(m+n,m+n+1)};$$
$$[\mathbf{1}_{(m,m+1)}, \mathbf{1}_{(n+1,n)}] = P_{(m+n+1)\delta};$$

for any $m, n \in \mathbb{N}$.

Since $\mathbf{1}_{(m,m+1)}, \mathbf{1}_{(n+1,n)} \in \mathcal{C}_{\mathbb{Z}}(K)$ (see section 5.4), by the last formula in the above proposition we can see that $P_{n\delta} \in \mathcal{C}_{\mathbb{Z}}(K)$ for any $n \geq 1$.

6.3. The function $H_{n\delta}$**.** For $n \geq 1$, the set of all regular modules (may be decomposable) with dimension vector $n\delta$ is also a constructible subset of $\mathbf{E}_{n\delta}$. Let $H_{n\delta}$ be the characteristic function of this set. For convenience we also set $H_{0\delta} = 1$.

Lemma 6.2.

(1). $\mathbf{1}_2^{(n)}\mathbf{1}_1^{(n+1)} = \mathbf{1}_{(n+1,n)} + \sum_{l=1}^{n} \mathbf{1}_{(n+1-l,n-l)} H_{l\delta} + \sum_{P,I,l} \mathbf{1}_P H_{l\delta} \mathbf{1}_I;$

(2). $\mathbf{1}_2^{(n+1)}\mathbf{1}_1^{(n)} = \mathbf{1}_{(n,n+1)} + \sum_{l=1}^{n} H_{l\delta} \mathbf{1}_{(n-l,n+1-l)} + \sum_{P,I,l} \mathbf{1}_P H_{l\delta} \mathbf{1}_I;$

(3). $\mathbf{1}_2^{(n)}\mathbf{1}_1^{(n)} = H_{n\delta} + \sum_{P,I,l} \mathbf{1}_P H_{l\delta} \mathbf{1}_I,$

where in the formulas the last terms sum over all non-zero $P \in$ *Prep*(Q), $I \in$ *Prei*(Q) *and* $1 < l < n-1$ *such that* $\underline{dim}P + \underline{dim}I + (l,l) = (n+1,n), (n,n+1), (n,n)$ *respectively.*

Proof. We just prove (1), the proofs for (2) and (3) are similar.

By lemma 3.3 we know that

$$\mathbf{1}_2^{(n)}\mathbf{1}_1^{(n+1)} = \mathbf{1}_{nS_2}\mathbf{1}_{(n+1)S_1}.$$

We know that $\mathrm{Ext}^1(S_2, S_1) \neq 0$ and $\mathrm{Ext}^1(S_1, S_2) = 0$. So each module with dimension vector $(n+1, n)$ is in the support of $\mathbf{1}_{nS_2}\mathbf{1}_{(n+1)S_1}$. Thus the support of $\mathbf{1}_{nS_2}\mathbf{1}_{(n+1)S_1}$ contains infinitely many orbits of non-isomorphic modules. But for any such module M we have

$$\mathbf{1}_{nS_2}\mathbf{1}_{(n+1)S_1}(M) = \chi(\mathcal{F}(nS_2, (n+1)S_1; M)) = 1,$$

since $\mathrm{Hom}(S_1, S_2) = 0$.

Note that each module can be decomposed into a direct sum of preprojective, regular and preinjective modules. Then using lemma 3.4 and the definition of $H_{n\delta}$ we get the formula (1). □

Corollary 6.3. *$H_{n\delta} \in \mathcal{C}_{\mathbb{Z}}(K)$, for any $n \geq 1$.*

Proof. The left hand sides of the formulas in the above lemma are in $\mathcal{C}_{\mathbb{Z}}(K)$. Also we know that for any $P \in \mathrm{Prep}(K)$, $I \in \mathrm{Prei}(K)$, $\mathbf{1}_P, \mathbf{1}_I \in \mathcal{C}_{\mathbb{Z}}(K)$. Then the corollary follows easily by induction on n. □

By concrete calculations we can find the relation between $H_{n\delta}$ and $P_{n\delta}$:

Lemma 6.4. *For any $n \in \mathbb{N}, n \geq 1$,*

$$H_{n\delta} = \frac{1}{n}\sum_{l=0}^{n-1} H_{l\delta}P_{(n-l)\delta}$$

Proof. By Lemma 6.2 we have

$$\begin{aligned}\mathbf{1}_{(n-1,n)}\mathbf{1}_1 &= \mathbf{1}_2^{(n)}\mathbf{1}_1^{(n-1)}\mathbf{1}_1 - (\sum_{l=1}^{n-1} H_{l\delta}\mathbf{1}_{(n-l-1,n-l)} + \sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I)\mathbf{1}_1 \\ &= n\mathbf{1}_2^{(n)}\mathbf{1}_1^{(n)} - \sum_{l=1}^{n-1} H_{l\delta}\mathbf{1}_{(n-l-1,n-l)}\mathbf{1}_1 - \sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I\mathbf{1}_1 \\ &= nH_{n\delta} + X,\end{aligned}$$

where

$$X = n\sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I - \sum_{l=1}^{n-1} H_{l\delta}\mathbf{1}_{(n-l-1,n-l)}\mathbf{1}_1 - \sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I\mathbf{1}_1.$$

and in the above formula the first (resp. last) term sums over all non-zero preprojective and preinective modules P, I and $1 < l < n-1$ such that $\underline{\dim}P + \underline{\dim}I + (l,l) = (n,n)$ (resp. $(n-1,n)$).

Then by Proposition 6.1 we have

$$P_{n\delta} = \mathbf{1}_{(n-1,n)}\mathbf{1}_1 - \mathbf{1}_1\mathbf{1}_{(n-1,n)} = nH_{n\delta} + X - \mathbf{1}_1\mathbf{1}_{(n-1,n)}.$$

Now we only need to prove

$$X - \mathbf{1}_1\mathbf{1}_{(n-1,n)} = -\sum_{l=1}^{n-1} H_{l\delta}P_{(n-l)\delta}.$$

In fact, in the above formula, the left hand side is

$$\begin{aligned}
&n\sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I - \sum_{l=1}^{n-1} H_{l\delta}\mathbf{1}_{(n-l-1,n-l)}\mathbf{1}_1 - \sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I\mathbf{1}_1 - \mathbf{1}_1\mathbf{1}_{(n-1,n)}\\
&= -\sum_{l=1}^{n-1} H_{l\delta}(\mathbf{1}_{(n-l-1,n-l)}\mathbf{1}_1 - \mathbf{1}_1\mathbf{1}_{(n-l-1,n-l)}) + Y\\
&= -\sum_{l=1}^{n-1} H_{l\delta}P_{(n-l)\delta} + Y,
\end{aligned}$$

where

$$Y = n\sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I - \sum_{P,l,I}\mathbf{1}_P H_{l\delta}\mathbf{1}_I\mathbf{1}_1 - \mathbf{1}_1\mathbf{1}_{(n-1,n)} - \sum_{l=1}^{n-1} H_{l\delta}\mathbf{1}_1\mathbf{1}_{(n-l-1,n-l)}.$$

Thus it remains to prove $Y = 0$. If $Y \neq 0$, it is easy to see that any module M in the support of Y must have a non-zero preinjective summand. But on the other hand,

$$Y = \sum_{l=0}^{n-1} H_{l\delta}P_{(n-l)\delta} - nH_{n\delta},$$

whose support contains only regular modules, which is a contradiction. □

From this lemma we also know that $H_{n\delta}H_{m\delta} = H_{m\delta}H_{n\delta}$ for any $n, m \in \mathbb{N}$.

6.4. The subalgebra $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$. Let $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$ (resp. $\mathcal{C}(K)^{reg}$) be the $\mathbb{Z}$-subalgebra (resp. $\mathbb{C}$-subalgebra) of $\mathcal{C}(Q)$ generated by $\{H_{n\delta}|n \in \mathbb{N}\}$.

For a positive integer n, let $\mathbf{P}(n)$ be the set of all partitions of n. For any $\lambda \in \mathbf{P}(n)$ we also denote by $\lambda \vdash n$ and write $|\lambda| = n$. For $n = 0$, we set $\mathbf{P}(0) = \{0\}$.

For any $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t) \vdash n$, we define

$$H_{\omega\delta} = H_{\omega_1\delta}H_{\omega_2\delta}\cdots H_{\omega_t\delta}.$$

The following lemma is obvious.

Lemma 6.5.

$$\mathcal{C}(K)^{reg} \simeq \mathbb{C}[H_{\delta}, H_{2\delta}, \cdots, H_{n\delta}, \cdots],$$
$$\mathcal{C}_{\mathbb{Z}}(K)^{reg} \simeq \mathbb{Z}[H_{\delta}, H_{2\delta}, \cdots, H_{n\delta}, \cdots].$$

And the set $\{H_{\omega\delta}|\omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$ and a $\mathbb{C}$-basis of $\mathcal{C}(K)^{reg}$.

From this lemma we know that $\mathcal{C}(K)^{reg}$ (resp. $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$) is naturally $\mathbb{N}$-graded, namely

$$\mathcal{C}(K)^{reg} = \oplus_{n\in\mathbb{N}}\mathcal{C}(K)^{reg}_n;\ \mathcal{C}_{\mathbb{Z}}(K)^{reg} = \oplus_{n\in\mathbb{N}}\mathcal{C}_{\mathbb{Z}}(K)^{reg}_n,$$

where $\mathcal{C}(K)^{reg}_n$ (resp. $\mathcal{C}_{\mathbb{Z}}(K)^{reg}_n$) is the $\mathbb{C}$-subspace (resp. free $\mathbb{Z}$-submodule) generated by $\{H_{\omega\delta}|\omega \vdash n\}$. Equivalently, $\mathcal{C}(K)^{reg}_n$ (resp. $\mathcal{C}_{\mathbb{Z}}(K)^{reg}_n$) is the $\mathbb{C}$-subspace (resp. free $\mathbb{Z}$-submodule) generated by constructible functions in $\mathcal{C}(K)^{reg}$ whose supports are contained in $\mathbf{E}_{n\delta}$.

Then we know that the dimension of $\mathcal{C}(K)^{reg}_n$ (or the rank of $\mathcal{C}_{\mathbb{Z}}(K)^{reg}_n$) is $|\{\omega|\omega \vdash n\}|$, which is a finite number.

6.5. The functions $M_{\omega\delta}$ and $E_{n\delta}$. For any integer $n \geq 1$ and $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t) \vdash n$, let $\mathcal{S}_{\omega}$ be the constructible subset of $\mathbf{E}_{n\delta}$ consisting of regular modules $R \simeq R_1 \oplus R_2 \oplus \cdots \oplus R_t$ with $\underline{\dim} R_i = \omega_i\delta$ and R_i indecomposable for all i. We define $M_{\omega\delta}$ to be the characteristic function of the set $\mathcal{S}_{\omega}$. We also set $M_{0\delta} = 1$. By definition we have

Lemma 6.6. *For any $n \in \mathbb{N}$,*

$$H_{n\delta} = \sum_{\omega\vdash n} M_{\omega\delta}.$$

We will prove that the set $\{M_{\omega\delta}|\omega \vdash n, n \in \mathbb{N}\}$ is also a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$.

The idea comes from the theory of symmetric functions. Let's recall some notations and results in [M]. Let Λ be the ring of symmetric functions in countably many independent variables with coefficients in $\mathbb{Z}$. For $n \in \mathbb{N}, n \geq 1$, denote by h_n (resp. e_n) the nth complete symmetric function (resp. elementary symmetric function). We know that

$$\Lambda \simeq \mathbb{Z}[h_1, h_2, \cdots, h_n, \cdots] \simeq \mathbb{Z}[e_1, e_2, \cdots, e_n, \cdots]$$

Now we come back to $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$. We use the notations in 4.6. For each $x \in \mathbb{P}^1$, denote by $h_{n,x}$ the characteristic function of all modules in $\mathcal{T}_x$ with dimension vector $n\delta$, and let $e_{n,x}$ be the characteristic function of the module $nS_{x,1}$. Note that $S_{x,1}$ is the unique quasi-simple module in $\mathcal{T}_x$. Let $\mathcal{H}(\mathcal{T}_x)$ be the subalgebra of $\mathcal{H}(Q)$ generated by all characteristic functions $\mathbf{1}_M$ with $M \in \mathcal{T}_x$. The following lemma also comes from [M]:

Lemma 6.7. *For any $x \in \mathbb{P}^1$, there exists an isomorphism*

$$\psi_x : \mathcal{H}(\mathcal{T}_x) \xrightarrow{\sim} \Lambda$$

where $\psi_x(h_{n,x}) = h_n$ and $\psi_x(e_{n,x}) = e_n$ for any $n \geq 1$.

For any $n \geq 1$, let $E_{n\delta}$ be the characteristic function of the set $\mathcal{S}_{(1^n)}$ where $(1^n) = (1, 1, \cdots, 1) \vdash n$. So $E_{n\delta} = M_{(1^n)\delta}$. For convenience, set $E_{0\delta} = 1$. We also define $E_{\omega\delta} = E_{\omega_1\delta}E_{\omega_2\delta} \cdots E_{\omega_t\delta}$ for $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t) \vdash n$.

Lemma 6.8. *The set $\{E_{\omega\delta}|\omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$ and a $\mathbb{C}$-basis of $\mathcal{C}(K)^{reg}$.*

Proof. First it is easy to see that the elements in the set $\{E_{\omega\delta}|\omega \vdash n, n \in \mathbb{N}\}$ are $\mathbb{Z}$-linear independent.

Let $E(t) = 1 + \sum_{n\geq 1} E_{n\delta}t^n$, $H(t) = 1 + \sum_{n\geq 1} H_{n\delta}t^n$ be the generating functions. Also for each $x \in \mathbb{P}^1$ let $E_x(t) = 1 + \sum_{n\geq 1} e_{n,x}t^n$ and $H_x(t) = 1 + \sum_{n\geq 1} h_{n,x}t^n$.

By the definitions we can see that

$$E(t) = \prod_{x\in\mathbb{P}^1} E_x(t), \ H(t) = \prod_{x\in\mathbb{P}^1} H_x(t).$$

By Lemma 6.7 and results in [M] (section I.2), we have $H_x(t)E_x(-t) = 1$ for any $x \in \mathbb{P}^1$. Thus

$$H(t)E(-t) = 1.$$

Equivalently, we have

$$\sum_{k=0}^{n}(-1)^k E_{k\delta}H_{(n-k)\delta} = 0$$

for all $n \geq 1$.

Now by induction we can see that for any n, $H_{n\delta}$ is in the $\mathbb{Z}$-span of $\{E_{\omega\delta}|\omega \vdash n\}$ and vice versa. Since the set $\{H_{\omega\delta}|\omega \vdash n\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(K)_n^{reg}$, we see that $\{E_{\omega\delta}|\omega \vdash n\}$ is also a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(K)_n^{reg}$. Hence the lemma holds. □

For any partition $\lambda \vdash n$, let λ' be the *conjugate* of λ. By definition $\lambda' \vdash n$ and the Young diagram of λ' is the transpose of the one of λ. Recall that for any positive integer n, the *dominance order* on the set $\mathbf{P}(n)$ is defined as follows: $\lambda \leq \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$.

Lemma 6.9. *For any $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t) \vdash n$, we have*

$$E_{\omega\delta} = M_{\omega'\delta} + \sum_{\mu<\lambda'} a_{\omega\mu}M_{\mu\delta},$$

where $a_{\omega\mu} \in \mathbb{Z}$.

Proof. Note that $M_{\omega\delta} \in \mathcal{C}(K)_n^{reg}$ for any fixed $\omega \vdash n$. Further, $\{M_{\omega\delta}|\omega \vdash n\}$ is a linearly independent set. Hence it is a $\mathbb{C}$-basis of $\mathcal{C}(K)_n^{reg}$. So $E_{\omega\delta}$ is a $\mathbb{C}$-linear combination of $M_{\mu\delta}$, $\mu \vdash n$.

By definition

$$E_{\omega\delta} = E_{\omega_1\delta}E_{\omega_2\delta}\cdots E_{\omega_t\delta}.$$

For any $N \in \mathrm{Reg}(K)$, Let $\mathcal{F}(\omega; N)$ be the set of all filtrations

$$0 = N_t \subset N_{t-1} \subset \cdots \subset N_0 = N$$

such that N_{i-1}/N_i is isomorphic to a direct sum of ω_i quasi-simples. So we have

$$E_{\omega\delta}(N) = \chi(\mathcal{F}(\omega; N)).$$

Suppose that $\omega' = (\omega'_1, \omega'_2, \cdots, \omega'_{t'})$. It is not difficult to see that N is in the support of $E_{\omega\delta}$ if and only if $N \in \mathcal{S}_\mu$ for some $\mu \leq \omega'$. Thus

$$E_{\omega\delta} = \sum_{\mu \leq \omega'} a_{\omega\mu} M_{\mu\delta}.$$

Choosing any $N_\mu \in \mathcal{S}_\mu$, we have

$$E_{\omega\delta}(N_\mu) = \chi(\mathcal{F}(\omega; N_\mu)) = a_{\omega\mu} \in \mathbb{Z}.$$

Now it remains to prove $a_{\omega\omega'} = 1$. This is equivalent to prove that for any $N_{\omega'} \in \mathcal{S}_{\omega'}$, $\chi(\mathcal{F}(\omega; N_{\omega'}) = 1$. But the only filtration of $N_{\omega'}$ in $\mathcal{F}(\omega; N_{\omega'})$ is

$$0 = \mathrm{qrad}^{\lambda_t}(N) \subset \cdots \subset \mathrm{qrad}^1(N) \subset \mathrm{qrad}^0(N) = N,$$

where qrad denote the quasi-radical i.e. the radical in the subcategory $\mathrm{Reg}(K)$. Hence $\mathcal{F}(\omega; N_{\omega'})$ is a single point and we are done. □

Finally we can prove the following:

Lemma 6.10. *The set* $\{M_{\omega\delta} | \omega \vdash n, n \in \mathbb{N}\}$ *is a* $\mathbb{Z}$*-basis of* $\mathcal{C}_\mathbb{Z}(K)^{reg}$ *and a* $\mathbb{C}$*-basis of* $\mathcal{C}(K)^{reg}$.

Proof. For any fixed $n \in \mathbb{N}$, $\{E_{\omega\delta} | \omega \vdash n\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_\mathbb{Z}(K)^{reg}_n$. By the lemma above the transition matrix from $\{M_{\omega\delta} | \omega \vdash n)\}$ to $\{E_{\omega\delta} | \omega \vdash n\}$ is upper triangular with 1's in the diagonal. Thus $\{M_{\omega\delta} | \omega \vdash n)\}$ is also a $\mathbb{Z}$-basis of $\mathcal{C}_\mathbb{Z}(K)^{reg}_n$. It follows that $\{M_{\omega\delta} | \omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_\mathbb{Z}(K)^{reg}$. □

6.6. Integral bases of $\mathcal{C}_\mathbb{Z}(K)$. The main result of this section is the following:

Proposition 6.11. *The set*

$$\{\mathbf{1}_P M_{\omega\delta} \mathbf{1}_I | P \in \mathit{Prep}(K), I \in \mathit{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

is a $\mathbb{Z}$*-basis of the algebra* $\mathcal{C}_\mathbb{Z}(K)$.

Proof. First we prove that the above set is a $\mathbb{C}$-basis of the algebra $\mathcal{C}(K)$.

By Proposition 6.1 and the PBW-basis theorem. the set

$$\{\mathbf{1}_P P_{\omega\delta} \mathbf{1}_I | P \in \mathrm{Prep}(K), I \in \mathrm{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

is a $\mathbb{C}$-basis of $\mathcal{C}(K)$, where $P_{\omega\delta} = P_{\omega_1\delta} \cdots P_{\omega_t\delta}$. But from lemma 6.4 we can see that $\{P_{\omega\delta}|\omega \vdash n, n \in \mathbb{N}\}$ and $\{H_{\omega\delta}|\omega \vdash n, n \in \mathbb{N}\}$ can be $\mathbb{C}$-linearly expressed by each other (actually the coefficients are in $\mathbb{Q}$). So the following set

$$\{\mathbf{1}_P H_{\omega\delta} \mathbf{1}_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

is a $\mathbb{C}$-basis of $\mathcal{C}(K)$.

By Lemma 6.5 and Lemma 6.10 we can see that the set in the proposition is also a $\mathbb{C}$-basis of $\mathcal{C}(K)$.

Now consider the $\mathbb{Z}$-subalgebra of $\mathcal{C}(K)$ generated by

$$\{\mathbf{1}_P, M_{\omega\delta}, \mathbf{1}_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}.$$

We claim that this $\mathbb{Z}$-subalgebra is $\mathcal{C}_{\mathbb{Z}}(K)$. First, by Lemma 5.3 and Lemma 5.5 the divided powers $\mathbf{1}_1^{(l)}$ and $\mathbf{1}_2^{(m)}$, for any $l, m \in \mathbb{N}$, are contained in the above generators. Second, from results in section 5.5 and 6.5 we know that the generators above are all in $\mathcal{C}_{\mathbb{Z}}(K)$. Thus our claim holds.

It remains to prove that any product of the generators is in the $\mathbb{Z}$-span of the elements in the set. For the case $\mathbf{1}_P \mathbf{1}_{P'}$ with $P, P' \in \text{Prep}(K)$ and $\mathbf{1}_I \mathbf{1}_{I'}$ with $I, I' \in \text{Prei}(K)$, we have already done in 5.5. And Lemma 6.10 shows that for any $\lambda \vdash n, \mu \vdash m$, $M_{\lambda\delta} M_{\mu\delta}$ also has the desired property.

For any $P \in \text{Prep}(K), I \in \text{Prei}(K)$, since the set is a $\mathbb{C}$-basis of $\mathcal{C}(K)$, we have

$$\mathbf{1}_I \mathbf{1}_P = \sum_{P', \omega, I'} a_{P',\omega,I'} \mathbf{1}_{P'} M_{\omega\delta} \mathbf{1}_{I'},$$

where $a_{P',\omega,I'} \in \mathbb{C}$ and

$$\underline{\dim} P' + |\omega|\delta + \underline{\dim} I' = \underline{\dim} P + \underline{\dim} I.$$

We need to prove all the coefficients $a_{P',\omega,I'}$ are integers. For any P', ω, I', the function $\mathbf{1}_{P'} M_{\omega\delta} \mathbf{1}_{I'}$ is the characteristic function of the following set (recall the definition of $\mathcal{S}_\omega$ in 6.5):

$$\{M \simeq P' \oplus R \oplus I' | R \in \mathcal{S}_\omega\}.$$

Denote the set by $\mathcal{S}_{P',\omega,I'}$. It is easy to see that $\mathcal{S}_{P',\omega,I'} \cap \mathcal{S}_{P'',\mu,I''} \neq \emptyset$ if and only if $P' = P''$, $\omega = \mu$, and $I' = I''$.

Thus for any fixed P', ω, I', and any module $N_{P',\omega,I'} \in \mathcal{S}_{P',\omega,I'}$ we have

$$\mathbf{1}_I \mathbf{1}_P(N_{P',\omega,I'}) = a_{P',\omega,I'}.$$

But on the other hand by the definition of the multiplication in the Hall algebra,

$$\mathbf{1}_I \mathbf{1}_P(N_{P',\omega,I'}) = \chi(\mathcal{F}(I, P; N_{P',\omega,I'})).$$

Hence $a_{P',\omega,I'} = \chi(\mathcal{F}(I, P; N_{P',\omega,I'})) \in \mathbb{Z}$.

Next we consider $M_{\omega\delta}\mathbf{1}_P$ for any $\omega \vdash n$ and $P \in \text{Prep}(K)$. Since preinjective modules do not occur in the direct summands of the extension of a regular module by a preprojective module, so first we have

$$M_{\omega\delta}\mathbf{1}_P = \sum_{P',\mu} b_{P',\mu}\mathbf{1}_{P'}M_{\mu\delta},$$

where $b_{P',\mu} \in \mathbb{C}$ and $\underline{\dim}P' + |\mu|\delta = \underline{\dim}P + n\delta$.

For any module N, let $\mathcal{F}(\omega, P; N)$ be the set consisting of all submodules L of N such that $L \simeq P$ and $N/L \in \mathcal{S}_\omega$. For any P' and μ, let $\mathcal{S}_{P',\mu}$ be the set of all modules N such that $N \simeq P' \oplus R$, $R \in \mathcal{S}_\mu$.

By the same argument as in the case $\mathbf{1}_I\mathbf{1}_P$ we can see that

$$b_{P',\mu} = \chi(\mathcal{F}(\omega, P; N_{P',\mu})) \in \mathbb{Z},$$

where $N_{P',\mu}$ is a module in $\mathcal{S}_{P',\mu}$.

The case $\mathbf{1}_I M_{\omega\delta}$ is completely similar.

Thus the set

$$\{\mathbf{1}_P M_{\omega\delta}\mathbf{1}_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(K)$. □

Corollary 6.12. *The following two sets*

$$\{\mathbf{1}_P H_{\omega\delta}\mathbf{1}_I | P \in \textit{Prep}(K), I \in \textit{Prei}(K), \omega \vdash n, n \in \mathbb{N}\},$$

$$\{\mathbf{1}_P E_{\omega\delta}\mathbf{1}_I | P \in \textit{Prep}(K), I \in \textit{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

are also $\mathbb{Z}$-bases of the algebra $\mathcal{C}_{\mathbb{Z}}(K)$.

Proof. Note that the elements in the above two sets are different from the one in Proposition 6.11 only in the regular part. However, they can be $\mathbb{Z}$-linearly expressed by each other, see Lemma 6.5, 6.8 and 6.10. So the corollary holds. □

The results above immediately implies that the algebra $\mathcal{C}_{\mathbb{Z}}(K)$ has an integral triangular decomposition:

Corollary 6.13.

$$\mathcal{C}_{\mathbb{Z}}(K) \simeq \mathcal{C}_{\mathbb{Z}}(K)^{prep} \otimes \mathcal{C}_{\mathbb{Z}}(K)^{reg} \otimes \mathcal{C}_{\mathbb{Z}}(K)^{prei}.$$

6.7. Remarks. (1). The proofs of lemma 6.2, 6.4 are similar to the quantum case [Z]. However, in our case the calculation is easier and we have avoid using some complicated combinatorial formulas in [Z].

(2). The relation between $P_{k\delta}$ and $H_{k\delta}$ given by lemma 6.4 is equivalent to the following:

$$\sum_{i\geq 0} H_{i\delta} t^i = \exp(\sum_{j\geq 1} \frac{P_{j\delta}}{j} t^j).$$

This relation also appeared in the basis elements corresponding to imaginary roots in [Gal]. However, the bases we constructed in 6.6 are different from [Gal].

7. Integral basis: The case of cyclic quivers

In this section we consider the cyclic quiver C_r. We will construct a $\mathbb{Z}$-basis of the integral composition algebra $\mathcal{C}_{\mathbb{Z}}(C_r)$. We use the notations in 4.5.

7.1. Generic extensions. Given any two modules M, N in $\mathrm{rep}_0(C_r)$, there exists a unique (up to isomorphism) extension L of M by N with maximal $\dim_{\mathbb{C}}\mathcal{O}_L$ (or equivalently, minimal $\dim_{\mathbb{C}}\mathrm{End}(L)$), see [Re]. This extension module L is called the *generic extension* of M by N, denoted by $L = M \diamond N$. We can define $[M] \diamond [N] = [M \diamond N]$ then it is known that the operator $\diamond$ is associative and $(\mathcal{I}(C_r), \diamond)$ is a monoid with identity $[0]$.

An n-tuple of partitions $\pi = (\pi^{(1)}, \pi^{(2)}, \cdots, \pi^{(n)})$ in Π is called *aperiodic* or *separated* if for each $l \geq 1$ there is some $i = i(l) \in I$ such that $\pi^{(i)}_j \neq l$ for all $j \geq 1$. We denote by Π^a the set of aperiodic n-tuples of partitions. A module M in $\mathrm{rep}_0(C_r)$ is called *aperiodic* if $M \simeq M(\pi)$ for some $\pi \in \Pi^a$. For any dimension vector $\alpha \in \mathbb{N}[I]$, set $\Pi_\alpha = \{\lambda \in \Pi | \underline{\dim} M(\lambda) = \alpha\}$ and $\Pi^a_\alpha = \Pi^a \cap \Pi_\alpha$.

Let $\mathcal{W}$ be the set of all words on the alphabet I. For each $\omega = i_1 i_2 \cdots i_m \in \mathcal{W}$, set

$$M(\omega) = S_{i_1} \diamond S_{i_2} \diamond \cdots \diamond S_{i_m}.$$

Then there is a unique $\pi \in \Pi$ such that $M(\pi) \simeq M(\omega)$ and we set $\wp(\omega) = \pi$. It has been proved that $\pi = \wp(\omega) \in \Pi^a$ and $\wp$ induces a surjection $\wp : \mathcal{W} \twoheadrightarrow \Pi^a$.

7.2. Distinguished words. For $\omega \in \mathcal{W}$, we write ω in *tight form*: $\omega = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$ with $j_r \neq j_{r+1}$ for all r. A word ω is called *distinguished* if $M(\wp(\omega))$ has a unique filtration

$$M(\wp(\omega)) = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

with $M_{r-1}/M_r \simeq e_r S_r$.

For $\lambda \in \Pi$ and $\omega = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \mathcal{W}$, let $\tilde{\chi}_\omega^\lambda$ denote the Euler characteristic of the variety consisting of all filtrations of $M(\lambda)$:

$$M(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

with $M_{r-1}/M_r \simeq e_r S_r$. Thus if ω is distinguished then $\tilde{\chi}_\omega^{\wp(\omega)} = 1$.

The following proposition was proved in [DDX]:

Proposition 7.1. *For any $\pi \in \Pi^a$, there exists a distinguished word*

$$\omega_\pi = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \wp^{-1}(\pi).$$

For each $\pi \in \Pi^a$, we fix a distinguished word $\omega_\pi \in \wp^{-1}(\pi)$. The set $\mathcal{D} = \{\omega_\pi | \pi \in \Pi^a\}$ is called a section of distinguished words of $\wp$ over Π^a.

7.3. Monomial bases. For $\lambda \in \Pi$ and $\omega = i_1 i_2 \cdots i_m \in \mathcal{W}$, we denote by χ_ω^λ the Euler characteristic of the variety consisting of all filtrations of $M(\lambda)$

$$M(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_m = 0$$

with $M_{r-1}/M_r \simeq S_{i_r}$.

For each word $\omega = i_1 i_2 \cdots i_m \in \mathcal{W}$ we define a monomial

$$\mathfrak{m}_\omega = \mathbf{1}_{i_1} \mathbf{1}_{i_2} \cdots \mathbf{1}_{i_m} \in \mathcal{C}(C_r).$$

The following result was proved in [DD]:

Proposition 7.2. *Fix a distinguished section $\mathcal{D} = \{\omega_\pi \in \wp^{-1}(\pi) | \pi \in \Pi^a\}$ over Π^a. The set $\{\mathfrak{m}_\pi | \pi \in \Pi^a\}$ is a $\mathbb{C}$-basis of $\mathcal{C}(C_r)$.*

Note that in [DD] the above result was derived from the quantum case. However, we can also obtain a proof independent of the quantum case by a similar method, which we omit here.

7.4. A geometric order on Π. We can define a partial order on the set Π as follows: For $\lambda, \mu \in \Pi$, set $\lambda \leq \mu$ if and only if $\mathcal{O}_{M(\lambda)} \subset \overline{\mathcal{O}}_{M(\mu)}$. Of course this order can be endowed in $\mathcal{I}(C_r)$ by setting $M(\lambda) \leq M(\mu)$ if and only if $\lambda \leq \mu$.

The following lemma asserts that the order is compatible with the generic extension, see [DD].

Lemma 7.3. *$M' \leq M$, $N' \leq N$ implies $M' \diamond N' \leq M \diamond N$.*

Lemma 7.4. *For each $\omega = i_1 i_2 \cdots i_m \in \mathcal{W}$, we have*

$$\mathfrak{m}_\omega = \sum_{\lambda \leq \wp(\omega)} \chi_\omega^\lambda \mathbf{1}_{M(\lambda)}$$

Proof. By the definition of $\mathfrak{m}_\omega$, we just need to prove that $\chi_\omega^\lambda \neq 0$ implies $\lambda \leq \wp(\omega)$. We prove by induction on m.

If $m = 1$, there is nothing to prove. So let $m > 1$ and set $\omega' = i_2 \cdots i_m$. Then

$$M(\omega) = S_{i_1} \diamond (S_{i_2} \diamond \cdots \diamond S_{i_m}) = S_{i_1} \diamond M(\omega').$$

Since $\chi_\omega^\lambda \neq 0$, $M(\lambda)$ has a submodule M' with $M(\lambda)/M' \simeq S_{i_1}$ and M' has a composition series of type ω'.

By the inductive hypothesis, we have $M' \leq M(\omega')$. Hence

$$M(\lambda) \leq S_{i_1} \diamond M' \leq S_{i_1} \diamond M(\omega') = M(\omega) = M(\wp(\omega)).$$

That is, $\lambda \leq \wp(\omega)$. □

7.5. A $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(C_r)$. For each $\omega = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \mathcal{W}$ in tight form, define

$$\mathfrak{m}^{(\omega)} = \mathbf{1}_{j_1}^{(e_1)} \mathbf{1}_{j_2}^{(e_2)} \cdots \mathbf{1}_{j_t}^{(e_t)}.$$

Then we have

$$\mathfrak{m}^{(\omega)} = \sum_{\lambda \leq \wp(\omega)} \tilde{\chi}_\omega^\lambda \mathbf{1}_{M(\lambda)}.$$

In particular, for a distinguished word $\omega_\pi \in \wp^{-1}(\pi)$ with $\pi \in \Pi^a$, since $\tilde{\chi}_{\omega_\pi}^\pi = 1$, we have

$$\mathfrak{m}^{(\omega_\pi)} = \mathbf{1}_{M(\pi)} + \sum_{\lambda < \pi} \tilde{\chi}_{\omega_\pi}^\lambda \mathbf{1}_{M(\lambda)}.$$

Lemma 7.5. *Let $\mathcal{P}(C_r)$ be the $\mathbb{C}$-subspace of $\mathcal{H}(C_r)$ spanned by all $\mathbf{1}_{M(\lambda)}$ with $\lambda \in \Pi \setminus \Pi^a$. Then as a vector space, $\mathcal{H}(C_r) = \mathcal{C}(C_r) \oplus \mathcal{P}(C_r)$.*

Proof. Since $\mathcal{H}(C_r)$ and $\mathcal{C}(C_r)$ are $\mathbb{N}[I]$-graded, it suffices to prove that for each $\alpha \in \mathbb{N}[I]$, $\mathcal{H}(C_r)_\alpha = \mathcal{C}(C_r)_\alpha \oplus \mathcal{P}(C_r)_\alpha$, where $\mathcal{P}(C_r)_\alpha$ is the $\mathbb{C}$-subspace of $\mathcal{H}(C_r)_\alpha$ spanned by all $\mathbf{1}_{M(\lambda)}$ with $\lambda \in \Pi_\alpha \setminus \Pi^a_\alpha$.

Now we show $\mathcal{C}(C_r)_\alpha \cap \mathcal{P}(C_r)_\alpha = \{0\}$. Once this is done, a dimension comparison forces $\mathcal{H}(C_r)_\alpha = \mathcal{C}(C_r)_\alpha \oplus \mathcal{P}(C_r)_\alpha$.

Take an $x \in \mathcal{C}(C_r)_\alpha \cap \mathcal{P}(C_r)_\alpha$ and suppose $x \neq 0$. Then we can write

$$x = \sum_{\pi \in \Pi^a_\alpha} a_\pi \mathfrak{m}_{\omega_\pi}$$

for some $a_\pi \in \mathbb{C}$. Let $\mu \in \Pi^a_\alpha$ be maximal such that $a_\mu \neq 0$. We can rewrite $x = \sum_{\lambda \in \Pi_\alpha} b_\lambda \mathbf{1}_{M(\lambda)}$. By the maximality of μ, we have $b_\mu = a_\mu \chi^\mu_{\omega_\mu}$, which contradicts the fact that $x \in \mathcal{P}(C_r)_\alpha$. □

Now we fix a section of distinguished words $\mathcal{D} = \{\omega_\pi | \pi \in \Pi^a\}$, define inductively the elements E_π as follows:

For any $\alpha \in \mathbb{N}[I]$ and $\pi \in \Pi^a_\alpha$, if π is minimal, let

$$E_\pi = \mathfrak{m}^{(\omega_\pi)} \in \mathcal{C}_\mathbb{Z}(C_r)_\alpha.$$

In general, assume that $E_\lambda \in \mathcal{C}_\mathbb{Z}(C_r)_\alpha$ has been defined for all $\lambda \in \Pi^a_\alpha$ with $\lambda < \pi$, then we define

$$E_\pi = \mathfrak{m}^{(\omega_\pi)} - \sum_{\lambda < \pi, \lambda \in \Pi^a_\alpha} \tilde{\chi}^\lambda_{\omega_\pi} E_\lambda \in \mathcal{C}_\mathbb{Z}(C_r)_\alpha.$$

Lemma 7.6. *Let $\{\omega_\pi | \pi \in \Pi^a\}$ be a given distinguished section. For each $\pi \in \Pi^a_\alpha$, we have*

$$E_\pi = \mathbf{1}_{M(\pi)} + \sum_{\lambda \in \Pi_\alpha \setminus \Pi^a_\alpha, \lambda < \pi} g^\pi_\lambda \mathbf{1}_{M(\lambda)}$$

for some $g^\pi_\lambda \in \mathbb{Z}$, and

$$\mathfrak{m}^{(\omega_\pi)} = E_\pi + \sum_{\lambda < \pi, \lambda \in \Pi^a_\alpha} \tilde{\chi}^\lambda_{\omega_\pi} E_\lambda.$$

Proof. The second formula follows immediately from the definition. The first one follows from induction and Lemma 7.5. □

Proposition 7.7. *For each distinguished section $\mathcal{D} = \{\omega_\pi | \pi \in \Pi^a\}$ of $\wp$ over Π^a, the set $\{E_\pi | \pi \in \Pi^a\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_\mathbb{Z}(C_r)$.*

Proof. It is easy to see that the elements in the set are $\mathbb{Z}$-linearly independent. So it suffices to prove that for any $\alpha \in \mathbb{N}[I]$, the $\mathbb{Z}$-module $\mathcal{C}_{\mathbb{Z}}(C_r)_\alpha$ is spanned by $\{E_\lambda | \lambda \in \Pi^a_\alpha\}$.

Let $\mathcal{W}_\alpha = \{\omega \in \mathcal{W} | \underline{\dim} M(\wp(\omega)) = \alpha\}$. It is clear that $\mathcal{C}_{\mathbb{Z}}(C_r)_\alpha$ is spanned by $\mathfrak{m}^{(\pi)}$, $\pi \in \mathcal{W}_\alpha$. Thus it remains to prove that each $\mathfrak{m}^{(\pi)}$ is a $\mathbb{Z}$-linear combination of E_π, $\pi \in \Pi^a_\alpha$.

Take arbitrary $\omega \in \mathcal{W}_\alpha$, and set $\pi = \wp(\omega) \in \Pi^a_\alpha$. We have

$$\mathfrak{m}^{(\pi)} = \sum_{\lambda \le \pi} \tilde{\chi}^\lambda_\omega \mathbf{1}_{M(\lambda)},$$

hence

$$\mathfrak{m}^{(\pi)} - \sum_{\lambda \in \Pi^a_\alpha, \lambda \le \pi} \tilde{\chi}^\lambda_\omega \mathbf{1}_{M(\lambda)} = \sum_{\lambda \in \Pi_\alpha \setminus \Pi^a_\alpha, \lambda \le \pi} a^\pi_\lambda \mathbf{1}_{M(\lambda)},$$

for some $a^\pi_\lambda \in \mathbb{Z}$.

The left hand side in the above formula is in $C_{\mathbb{Z}}(C_r)_\alpha$. Hence by Lemma 7.5, it must be zero. That yields

$$\mathfrak{m}^{(\pi)} = \sum_{\lambda \in \Pi^a_\alpha, \lambda \le \pi} \tilde{\chi}^\lambda_\omega E_\lambda.$$

□

7.6. Connection with a basis of $\mathfrak{n}^+(C_r)$. We investigate the relation between the $\mathbb{Z}$-basis we constructed in Proposition 7.7 and the basis of $\mathfrak{n}^+(C_r)$ constructed in [FMV]:

Proposition 7.8. *The union of the following two sets*

$$\{\mathbf{1}_{S_i[l]} | 1 \le i \le r, r \nmid l\}$$

$$\{\mathbf{1}_{S_i[nr]} - \mathbf{1}_{S_{i+1}[nr]} | n \ge 1, 1 \le i \le r-1\}$$

is a basis of $\mathfrak{n}^+(C_r)$.

Note that in the proposition the elements in the first set are real root vectors while those in the second one are imaginary root vectors.

Lemma 7.9. *(1). For any i, l with $1 \le i \le r, r \nmid l$, we have*

$$\mathbf{1}_{S_i[l]} = E_\pi,$$

where $\pi \in \Pi^a$ and $M(\pi) = S_i[l]$.

(2). For any i, n with $n \geq 1, 1 \leq i \leq r-1$, we have

$$\mathbf{1}_{S_i[nr]} - \mathbf{1}_{S_{i+1}[nr]} = E_{\pi_1} - E_{\pi_2},$$

where $\pi_1, \pi_2 \in \Pi^a$ and $M(\pi_1) = S_i[nr]$, $M(\pi_2) = S_{i+1}[nr]$.

Proof. We just prove (1), the proof of (2) is completely similar.

By lemma 7.6,

$$E_\pi = \mathbf{1}_{S_i[l]} + \sum_{\lambda \in \Pi_\alpha \setminus \Pi_\alpha^a, \lambda < \pi} g_\lambda^\pi \mathbf{1}_{M(\lambda)}.$$

We have known that $\mathbf{1}_{S_i[l]} \in \mathfrak{n}^+(C_r) \subset \mathcal{C}(C_r)$. But the second term in the formula above is in $\mathcal{P}(C_r)$. Thus by Lemma 7.5 it must be zero. □

8. Integral bases: The general affine case

Now we consider general tame quivers. In this section let $Q = (I, \Omega, s, t)$ be a tame quiver without oriented cycles. We will use the notations in 4.6.

8.1. Embedding of the module category of Kronecker quiver. Let K be the Kronecker quiver (see 4.6 and section 6). If $Q \neq K$, the main difference between rep(K) and rep(Q) is that the regular component of rep(K) only consists of homogeneous tubes, while rep(Q) has s non-homogeneous tubes. A well-known result in representation theory of tame quivers is that rep(K) can be embedded into rep(Q).

To make it more precise, we need more notations. In the rest of this section δ denotes the minimal positive imaginary root of Q, and the minimal positive imaginary root of K is denoted by δ_K. For the modules in rep(K) and in rep(Q) we distinguish them by putting different superscripts K and Q respectively.

Lemma 8.1. *There exists a fully faithful, exact functor F : rep(K) $\hookrightarrow$ rep(Q) which satisfies*

(1). $F(P^K) \in$ Prep(Q), $F(I^K) \in$ Prei(Q) for all $P^K \in$ Prep(K), $I^K \in$ Prei(K).

(2). $F(S^K_{x,l}) = S^Q_{x,l}$ for all $x \in \mathbb{P}^1 \setminus J$ and $l \geq 1$.

(3). For each $1 \leq j \leq s$ there exists $1 \leq k_j \leq r_j$ such that $F(S^K_{j,l}) = S^Q_{j,k_j,lr_j}$ for all $l \geq 1$.

The embedding functor $F : \text{rep}(K) \hookrightarrow \text{rep}(Q)$ gives rise to an injective morphism between the corresponding Hall algebras $\mathcal{H}(K) \hookrightarrow \mathcal{H}(Q)$, which we still denote by F. Namely $F(\mathbf{1}_{M^K}) = \mathbf{1}_{F(M^K)}$ for any $M^K \in \text{rep}(K)$.

Note that by (1) in the above lemma, $F(S^K) \in \text{Prep}(Q)$ or $\text{Prei}(Q)$ for each simple module S^K in $\text{rep}(K)$. Hence $F(\mathbf{1}_{S^K}) \in \mathcal{C}_{\mathbb{Z}}(Q)$.

So we have proved the following:

Lemma 8.2. *$F : \mathcal{H}(K) \hookrightarrow \mathcal{H}(Q)$ restricts to an injective morphism $F : \mathcal{C}(K) \hookrightarrow \mathcal{C}(Q)$ and also $F : \mathcal{C}_{\mathbb{Z}}(K) \hookrightarrow \mathcal{C}_{\mathbb{Z}}(Q)$.*

Recall that the sets $\{M_{\omega\delta_K} | \omega \vdash n, n \in \mathbb{N}\}$, $\{H_{\omega\delta_K} | \omega \vdash n, n \in \mathbb{N}\}$, $\{E_{\omega\delta_K} | \omega \vdash n, n \in \mathbb{N}\}$ are $\mathbb{Z}$-bases of $\mathcal{C}_{\mathbb{Z}}(K)^{reg}$ (Lemma 6.5, 6.8 and 6.10). Set $M_{\omega\delta} = F(M_{\omega\delta_K})$, $H_{\omega\delta} = F(H_{\omega\delta_K})$ and $E_{\omega\delta} = F(E_{\omega\delta_K})$ for all $\omega \vdash n, n \in \mathbb{N}$. We also define $P_{n\delta}$ to be $F(P_{n\delta_K})$ for any $n \in \mathbb{N}$. By the above lemma, $M_{\omega\delta}, H_{\omega\delta}, E_{\omega\delta} \in \mathcal{C}_{\mathbb{Z}}(Q)$.

8.2. A basis of $\mathfrak{n}^+(Q)$. In [FMV], a basis of $\mathfrak{n}^+(Q)$ has been given:

Proposition 8.3. *The union of the following sets*

$$\{\mathbf{1}_P, \mathbf{1}_I | P \in \mathit{Prep}(Q), I \in \mathit{Prei}(Q) \text{ and } P, I \text{ indecomposable}\},$$

$$\{\mathbf{1}_{S_{j,i,l}} | 1 \le j \le s, 1 \le i \le r_j, r_j \nmid l\},$$

$$\{\mathbf{1}_{S_{j,i,kr_j}} - \mathbf{1}_{S_{j,i+1,kr_j}} | 1 \le j \le s, 1 \le i \le r_j - 1, k \ge 1\},$$

$$\{P_{n\delta} | n \ge 1\};$$

forms a $\mathbb{Z}$-basis of $\mathfrak{n}^+(Q)$.

Note that in this proposition, $\mathbf{1}_P$, $\mathbf{1}_I$ and $\mathbf{1}_{S_{j,i,l}}$ are real root vectors while $\mathbf{1}_{S_{j,i,kr_j}} - \mathbf{1}_{S_{j,i+1,kr_j}}$ and $P_{n\delta}$ are imaginary root vectors. One can check the multiplicities of imaginary roots, recall Lemma 4.6.

8.3. Basis elements arising from non-homogeneous tubes. For any non-homogeneous tube $\mathcal{T}_j$ $(1 \le j \le s)$, which is an extension-closed abelian subcategory of $\text{rep}(Q)$ (see 4.6), we can define a subalgebra $\mathcal{H}(\mathcal{T}_j)$ of $\mathcal{H}(Q)$. Namely, it is the $\mathbb{C}$-subalgebra generated by all constructible functions whose supports are contained in $\mathcal{T}_j$. Then define $\mathcal{C}(\mathcal{T}_j)$ to be the $\mathbb{C}$-subalgebra of $\mathcal{H}(\mathcal{T}_j)$ generated by

$\mathbf{1}_{S_{j,i,1}}$ for all $1 \leq i \leq r_j$. And $\mathcal{C}_{\mathbb{Z}}(\mathcal{T}_j)$ is defined to be the $\mathbb{Z}$-subalgebra of $\mathcal{H}(\mathcal{T}_j)$ generated by $\mathbf{1}^{(t)}_{S_{j,i,1}}$ for all $1 \leq i \leq r_j$ and $t \geq 1$.

We have the following result:

Lemma 8.4. *For any $1 \leq j \leq s$, the subalgebra $\mathcal{C}(\mathcal{T}_j)$ is contained in $\mathcal{C}(Q)$. And $\mathcal{C}_{\mathbb{Z}}(\mathcal{T}_j)$ is a $\mathbb{Z}$-subalgebra of $\mathcal{C}_{\mathbb{Z}}(Q)$.*

Proof. We claim that $\mathbf{1}^{(m)}_{S_{j,i,1}} \in \mathcal{C}_{\mathbb{Z}}(Q)$ for all $1 \leq i \leq r_j$ and $m \in \mathbb{N}$.

By Proposition 8.3 we see that $\mathbf{1}_{S_{j,i,1}}$ is a real root vector of the Lie algebra $\mathfrak{g}(Q)$, thus it can be obtained from $\mathbf{1}_i$ for some i by a series of automorphisms r_k (see 5.3). This forces $\mathbf{1}_{S_{j,i,1}} \in \mathcal{C}_{\mathbb{Z}}(Q)$ and so for the divided powers.

The lemma follows immediately. ☐

Note that for any homogeneous tube $\mathcal{T}_x$, $x \in \mathbb{P}^1 \setminus J$, we can define the subalgebras $\mathcal{H}(\mathcal{T}_x)$, $\mathcal{C}(\mathcal{T}_x)$ similarly. But $\mathcal{C}(\mathcal{T}_x)$ is not contained in $\mathcal{C}(Q)$ any more.

Recall 4.6 that we have the isomorphic functor $F_j : \mathrm{rep}_0(C_{r_j}) \xrightarrow{\sim} \mathcal{T}_j$. This induces an isomorphism of the corresponding Hall algebras $\mathcal{H}(C_{r_j}) \simeq \mathcal{H}(\mathcal{T}_j)$, which we still denote by F_j. Obviously F_j restricts to an isomorphism $\mathcal{C}_{\mathbb{Z}}(C_{r_j}) \simeq \mathcal{C}_{\mathbb{Z}}(\mathcal{T}_j)$.

We have to introduce more notations to distinguish objects in various $\mathrm{rep}_0(C_{r_j})$ and $\mathcal{H}(C_{r_j})$. Let Π^a_j denote the set of aperiodic r_j-tuples of partitions. So for $\pi \in \Pi^a_j$, $M(\pi) \in \mathrm{rep}_0(C_{r_j})$, let $M(\pi)_j = F_j(M(\pi)) \in \mathcal{T}_j$.

By Proposition 7.7 the set $\{E_\pi | \pi \in \Pi^a_j\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(C_{r_j})$. For $1 \leq j \leq s$ and $\pi \in \Pi^a_j$, let $E_{\pi,j} = F_j(E_\pi)$. Thus for any j, $\{E_{\pi,j} | \pi \in \Pi^a_j\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(\mathcal{T}_j)$.

8.4. Main result: $\mathbb{Z}$-bases of $\mathcal{C}_{\mathbb{Z}}(Q)$. Now we can state the main result in this paper. Let $\mathcal{J}$ be the set of quadruples $\mathbf{c} = (P_{\mathbf{c}}, I_{\mathbf{c}}, \pi_{\mathbf{c}}, \omega_{\mathbf{c}})$ where $P_{\mathbf{c}} \in \mathrm{Prep}(Q)$, $I_{\mathbf{c}} \in \mathrm{Prei}(Q)$, $\pi_{\mathbf{c}} = (\pi_{\mathbf{c}1}, \pi_{\mathbf{c}2}, \cdots, \pi_{\mathbf{c}s})$, $\pi_{\mathbf{c}j} \in \Pi^a_j$ and $\omega_{\mathbf{c}} \vdash n, n \in \mathbb{N}$.

For each $\mathbf{c} \in \mathcal{J}$ we define

$$B_{\mathbf{c}} = \mathbf{1}_{P_{\mathbf{c}}} E_{\pi_{\mathbf{c}1},1} E_{\pi_{\mathbf{c}2},2} \cdots E_{\pi_{\mathbf{c}s},s} M_{\omega_{\mathbf{c}}\delta} \mathbf{1}_{I_{\mathbf{c}}}.$$

Theorem 8.5. *The set $\{B_{\mathbf{c}} | \mathbf{c} \in \mathcal{J}\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(Q)$.*

Note that for any $\mathbf{c} \in \mathcal{J}$, the modules in the support of $B_\mathbf{c}$ have the same dimension vector. So we define

$$\underline{\dim} B_\mathbf{c} = \underline{\dim} P_\mathbf{c} + \sum_{j=1}^{s} \underline{\dim} M(\pi_{\mathbf{c}j})_j + |\omega_\mathbf{c}|\delta + \underline{\dim} I_\mathbf{c}.$$

Once this theorem is proved, we have the following corollary. The proof is similar to Corollary 6.12.

Corollary 8.6. *The following two sets*

$$\{B'_\mathbf{c} = \mathbf{1}_{P_\mathbf{c}} E_{\pi_{\mathbf{c}1},1} E_{\pi_{\mathbf{c}2},2} \cdots E_{\pi_{\mathbf{c}s},s} H_{\omega_\mathbf{c}\delta} \mathbf{1}_{I_\mathbf{c}} | \mathbf{c} \in \mathcal{J}\}$$
$$\{B''_\mathbf{c} = \mathbf{1}_{P_\mathbf{c}} E_{\pi_{\mathbf{c}1},1} E_{\pi_{\mathbf{c}2},2} \cdots E_{\pi_{\mathbf{c}s},s} E_{\omega_\mathbf{c}\delta} \mathbf{1}_{I_\mathbf{c}} | \mathbf{c} \in \mathcal{J}\}$$

are also $\mathbb{Z}$-bases of $\mathcal{C}_\mathbb{Z}(Q)$.

Define $\mathcal{C}(Q)^{reg}$ (resp. $\mathcal{C}_\mathbb{Z}(Q)^{reg}$) to be the $\mathbb{C}$-subalgebra (resp. $\mathbb{Z}$-subalgebra) generated by $\{E_{\pi,j} | \pi \in \Pi_j^a, 1 \leq j \leq s\}$ and $\{M_{\omega\delta} | \omega \vdash n, n \in \mathbb{N}\}$. As in Corollary 6.13, we have a triangular decomposition of the integral composition algebra $\mathcal{C}_\mathbb{Z}(Q)$:

Corollary 8.7.

$$\mathcal{C}_\mathbb{Z}(Q) \simeq \mathcal{C}_\mathbb{Z}(Q)^{Prep} \otimes \mathcal{C}_\mathbb{Z}(Q)^{Reg} \otimes \mathcal{C}_\mathbb{Z}(Q)^{Prei}.$$

The rest of the paper is devoted to the proof of Theorem 8.5.

8.5. A $\mathbb{C}$-basis of $\mathcal{C}(Q)$. In this subsection we prove the set $\{B_\mathbf{c} | \mathbf{c} \in \mathcal{J}\}$ is a $\mathbb{C}$-basis of $\mathcal{C}(Q)$, which is the first step to prove Theorem 8.5. We need the following lemma:

Lemma 8.8. *(1). Fix $1 \leq j \leq s$. For any $1 \leq i \leq r_j$ and $r_j \nmid l$ we have*

$$\mathbf{1}_{S_{j,i,l}} = E_{\pi,j}$$

where $\pi \in \Pi_j^a$ such that $M(\pi)_j = S_{j,i,l}$.

(2). Fix $1 \leq j \leq s$. For any $1 \leq i \leq r_j - 1$, $n \geq 1$ we have

$$\mathbf{1}_{S_{j,i,nr_j}} - \mathbf{1}_{S_{j,i+1,nr_j}} = E_{\pi_1,j} - E_{\pi_2,j}$$

where $\pi_1, \pi_2 \in \Pi_j^a$ such that $M(\pi_1)_j = S_{j,i,nr_j}$, $M(\pi_2)_j = S_{j,i+1,nr_j}$.

Proof. It follows immediately from Lemma 7.9. □

By the PBW-theorem, the monomials in a fixed order on the basis elements of $\mathfrak{n}^+(Q)$ given in Proposition 8.3 form a $\mathbb{C}$-basis of $\mathcal{C}(Q)$.

Note that $\mathcal{C}(Q)$ is $\mathbb{N}[I]$-graded: $\mathcal{C}(Q) = \oplus_{\alpha\in\mathbb{N}[I]}\mathcal{C}(Q)_\alpha$, where $\mathcal{C}(Q)_\alpha$ is the subspace spanned by constructible functions in $\mathcal{C}(Q)$ whose supports are in $\mathbf{E}_\alpha$. The PBW-basis elements are of course homogeneous. By construction $B_{\mathbf{c}}$ is also homogeneous for any $\mathbf{c} \in \mathcal{J}$.

Now by the results in 5.5, 6.6 and lemma 8.8, we can see that for any $\alpha \in \mathbb{N}[I]$, the PBW-basis of $\mathcal{C}(Q)_\alpha$ can be expressed by $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$. Moreover, by definition the elements in $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ are $\mathbb{C}$-linearly independent. Hence $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ is a $\mathbb{C}$-basis of $\mathcal{C}(Q)$.

8.6. Commutation relations. We have known that the elements in the set $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ are all in $\mathcal{C}_{\mathbb{Z}}(Q)$. And the divided powers of $\mathbf{1}_S$ for any simple module S are in the set $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$. Thus the $\mathbb{Z}$-subalgebra generated by $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ is equal to $\mathcal{C}_{\mathbb{Z}}(Q)$.

Therefore, to prove Theorem 8.5 we have to check the product of any two elements in $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ is still a $\mathbb{Z}$-combination of elements in $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$. So the procedure is similar to the proof of Proposition 6.11. However, it is more complicated here since we have basis elements $E_{\pi,j}$ arising from non-homogeneous tubes. Moreover, the support of $M_{\omega\delta}$ contains modules not only in the homogeneous tubes but also non-homogeneous tubes.

For the case $\mathbf{1}_P\mathbf{1}_{P'}$ and $\mathbf{1}_I\mathbf{1}_{I'}$ with $P, P' \in \mathrm{Prep}(Q)$; $I, I' \in \mathrm{Prei}(Q)$, we have done in 5.5. And we have the case $M_{\lambda\delta}M_{\omega\delta}$ for any $\lambda \vdash n, \omega \vdash m$ done in 6.4.

Consider $E_{\pi_1,j}E_{\pi_2,k}$ for any $1 \le j, k \le s$, $\pi_1 \in \Pi_j^a, \pi_2 \in \Pi_k^a$. Since there are no non-trivial extensions between different tubes, $E_{\pi_1,j}E_{\pi_2,k} = E_{\pi_2,k}E_{\pi_1,j}$ for $j \neq k$. When $j = k$, we know that $E_{\pi_1,j}E_{\pi_2,j}$ must be a $\mathbb{Z}$-combination of $\{E_{\pi,j}|j \in \Pi_j^a\}$, see 7.5.

Lemma 8.9. *For any fixed $1 \le j \le s$, $\pi \in \Pi_j^a$ and $P \in Prep(Q)$,*

$$E_{\pi,j}\mathbf{1}_P = \sum_{P'\in Prep(Q),\pi'\in\Pi_j^a} a_{P',\pi',j}\mathbf{1}_{P'}E_{\pi',j}$$

where $\underline{dim}M(\pi')_j + \underline{dim}P' = \underline{dim}P + \underline{dim}M(\pi)_j$ and $a_{P',\pi',j} \in \mathbb{Z}$.

Proof. For any module M in the support of $E_{\pi,j}\mathbf{1}_P$, the direct summand of M only contains preprojective modules and regular modules in $\mathcal{T}_j$. So

$$E_{\pi,j}\mathbf{1}_P = \sum_{P'\in \mathrm{Prep}(Q),\pi'\in\Pi_j^a} a_{P',\pi',j}\mathbf{1}_{P'}E_{\pi',j}$$

for some $a_{P',\pi',j} \in \mathbb{C}$. And by a comparison of the dimension vectors in both sides, we have $\underline{\dim} M(\pi')_j + \underline{\dim} P' = \underline{\dim} P + \underline{\dim} M(\pi)_j$.

By Lemma 7.6, we have

$$E_{\pi',j} = \mathbf{1}_{M(\pi')_j} + \sum_{\lambda\in\Pi_j\setminus\Pi_j^a,\lambda<\pi'} g_{\lambda,j}^{\pi'}\mathbf{1}_{M(\lambda)_j}$$

with $g_{\lambda,j}^{\pi'} \in \mathbb{Z}$.

For any fixed P' and π', let $M_{P',\pi',j} = P' \oplus M(\pi')_j$. We can see that $M_{P',\pi',j}$ is not contained in the support of any other $\mathbf{1}_{P''}E_{\pi'',j}$. Thus

$$E_{\pi,j}\mathbf{1}_P(M_{P',\pi',j}) = a_{P',\pi',j}\mathbf{1}_{P'}E_{\pi',j}(M_{P',\pi',j}) = a_{P',\pi',j}.$$

Again by Lemma 7.6,

$$E_{\pi,j} = \mathbf{1}_{M(\pi)_j} + \sum_{\lambda\in\Pi_j\setminus\Pi_j^a,\lambda<\pi} g_{\lambda,j}^{\pi}\mathbf{1}_{M(\lambda)_j}$$

with $g_{\lambda,j}^{\pi} \in \mathbb{Z}$.

This yields

$$a_{P',\pi',j} = \chi(\mathcal{F}(M(\pi)_j, P; M_{P',\pi',j})) + \sum_{\lambda\in\Pi_j\setminus\Pi_j^a,\lambda<\pi} g_{\lambda,j}^{\pi}\chi(\mathcal{F}(M(\lambda)_j, P; M_{P',\pi',j})).$$

Hence $a_{P',\pi',j} \in \mathbb{Z}$. □

Similarly we can prove

Lemma 8.10. *For any fixed* $1 \le j \le s$, $\pi \in \Pi_j^a$ *and* $I \in Prei(Q)$,

$$\mathbf{1}_I E_{\pi,j} = \sum_{\pi',I'} b_{\pi',I',j}E_{\pi',j}\mathbf{1}_{I'}$$

where $\underline{\dim} M(\pi')_j + \underline{\dim} I' = \underline{\dim} I + \underline{\dim} M(\pi)_j$ *and* $b_{\pi',I',j} \in \mathbb{Z}$.

For $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_t) \vdash n$, denote by $\mathcal{S}_\lambda$ the set of all regular modules $M \simeq \oplus_{i=1}^t R_i \in \mathrm{rep}(Q)$ such that R_i indecomposable homogeneous and $\underline{\dim} R_i = \lambda_i\delta$ (note that this definition coincides with the one in 6.5 when $Q = K$).

Lemma 8.11. *For any fixed* $1 \leq j \leq s$, $\pi \in \Pi_j^a$ *and* $\omega \vdash n, n \in \mathbb{N}$,

$$M_{\omega\delta} E_{\pi,j} = \sum_{\pi' \in \Pi_j^a, \lambda \vdash k} c_{\pi',\lambda,j} E_{\pi',j} M_{\lambda\delta}$$

where $\underline{dim} M(\pi')_j + k\delta = n\delta + \underline{dim} M(\pi)_j$ *and* $c_{\pi',\lambda,j} \in \mathbb{Z}$.

Proof. Since there are no non-trivial extensions between different tubes, $M_{\omega\delta} E_{\pi,j}$ has the desired expression with $c_{\pi',\lambda,j} \in \mathbb{C}$.

We prove $c_{\pi',\lambda,j} \in \mathbb{Z}$ by inverse induction.

We can find a positive integer m such that for any $k > m$ and any $\lambda \vdash k$, $\pi' \in \Pi_j^a$, the coefficient $c_{\pi',\lambda,j} = 0$. Now fix $\pi' \in \Pi_j^a$ and $\lambda \vdash m$, let $N_{\pi',j,\lambda}$ be a module isomorphic to the direct sum of $M(\pi')_j$ and R where $R \in \mathcal{S}_\lambda$. It is not difficult to see that $N_{\pi',j,\lambda}$ is not contained in the support of $E_{\pi'',j} M_{\lambda'\delta}$ unless $\pi'' = \pi'$ and $\lambda' = \lambda$.

Hence we have

$$M_{\omega\delta} E_{\pi,j}(N_{\pi',j,\lambda}) = c_{\pi',\lambda,j} E_{\pi',j} M_{\lambda\delta}(N_{\pi',j,\lambda}) = c_{\pi',\lambda,j}.$$

Since

$$E_{\pi,j} = \mathbf{1}_{M(\pi)_j} + \sum_{\lambda \in \Pi_j \setminus \Pi_j^a, \lambda < \pi} g_{\lambda,j}^{\pi} \mathbf{1}_{M(\lambda)_j}$$

with $g_{\lambda,j}^{\pi} \in \mathbb{Z}$, we have

$$c_{\pi',\lambda,j} = \chi(\mathcal{F}(\omega, M(\pi)_j; N_{\pi',j,\lambda})) + \sum_{\lambda \in \Pi_j \setminus \Pi_j^a, \lambda < \pi} g_{\lambda,j}^{\pi} \chi(\mathcal{F}(\omega, M(\lambda)_j; N_{\pi',j,\lambda})) \in \mathbb{Z}.$$

Now we assume that $c_{\pi',\lambda,j} \in \mathbb{Z}$ for all $\pi' \in \Pi_j^a$ and $\lambda \vdash k$, $n+1 \leq k \leq m$. Consider $\lambda' \vdash n$ and $\pi' \in \Pi^a$. Again we choose a module $N_{\pi',j,\lambda'} \simeq M(\pi')_j \oplus R$ where $R \in \mathcal{S}_{\lambda'}$. We can see that $N_{\pi',j,\lambda'}$ is in the support of $E_{\pi'',j} M_{\lambda''\delta}$ only if $\pi'' = \pi'$ and $\lambda'' = \lambda'$ or $\lambda'' \vdash k$ for some $k > n$.

Hence we have

$$\begin{aligned} & M_{\omega\delta} E_{\pi,j}(N_{\pi',j,\lambda'}) \\ &= c_{\pi',\lambda',j} E_{\pi',j} M_{\lambda'\delta}(N_{\pi',j,\lambda'}) + \sum_{\pi'' \in \Pi_j^a, |\lambda''| > |\lambda'|} c_{\pi'',\lambda'',j} E_{\pi'',j} M_{\lambda''\delta}(N_{\pi',j,\lambda'}) \\ &= c_{\pi',\lambda',j} + \sum_{\pi'' \in \Pi_j^a, |\lambda''| > |\lambda'|} c_{\pi'',\lambda'',j} E_{\pi'',j} M_{\lambda''\delta}(N_{\pi',j,\lambda'}) \end{aligned}$$

On the other hand, $M_{\omega\delta}E_{\pi,j}(N_{\pi',j,\lambda'})$ and $E_{\pi'',j}M_{\lambda''\delta}(N_{\pi',j,\lambda'})$ are integers.

By the inductive hypothesis, $c_{\pi'',\lambda'',j} \in \mathbb{Z}$ for all $|\lambda''| > |\lambda'|$ and $\pi'' \in \Pi^a_j$. Thus we know that $c_{\pi',\lambda',j} \in \mathbb{Z}$.

Finally, by induction we can see all the coefficients $c_{\pi',\lambda,j} \in \mathbb{Z}$. □

Let $\mathcal{J}(\hat{I})$ (resp. $\mathcal{J}(\hat{P})$) be the subset of $\mathcal{J}$ consisting of $\mathbf{c} = (P_\mathbf{c}, 0, \pi_\mathbf{c}, \omega_\mathbf{c})$ (resp. $\mathbf{c} = (0, I_\mathbf{c}, \pi_\mathbf{c}, \omega_\mathbf{c})$).

Let $\mathcal{S}_\mathbf{c}$ be the set of all modules $N \simeq P_\mathbf{c} \oplus M(\pi_{\mathbf{c}1})_1 \oplus \cdots \oplus M(\pi_{\mathbf{c}s})_s \oplus R \oplus I_\mathbf{c}$, where $R \in \mathcal{S}_{\omega_\mathbf{c}}$.

Lemma 8.12. *For any $\omega \vdash n, n \in \mathbb{N}$ and $P \in Prep(Q)$, we have*

$$M_{\omega\delta}\mathbf{1}_P = \sum_{\mathbf{c}\in\mathcal{J}(\hat{I})} d_\mathbf{c} B_\mathbf{c}$$

with $\underline{dim}B_\mathbf{c} = n\delta + \underline{dim}P$ and $d_\mathbf{c} \in \mathbb{Z}$.

Proof. The extension of a regular module by a preprojective contains no direct summands of preinjective modules. Note that the support of $M_{\omega\delta}$ contains modules not only in the homogeneous tubes but also non-homogeneous tubes. So the terms $E_{\pi_{\mathbf{c}j},j}$ $(1 \leq j \leq s)$ occur in the right hand side. Hence $\mathbf{c} \in \mathcal{J}(\hat{I})$.

We can prove $d_\mathbf{c} \in \mathbb{Z}$ by completely similar arguments as in the proof of Lemma 8.11. □

By similar methods we can prove:

Lemma 8.13. *For any $\omega \vdash n, n \in \mathbb{N}$ and $I \in Prei(Q)$, we have*

$$\mathbf{1}_I M_{\omega\delta} = \sum_{\mathbf{c}\in\mathcal{J}(\hat{P})} e_\mathbf{c} B_\mathbf{c}$$

with $\underline{dim}B_\mathbf{c} = \underline{dim}I + n\delta$ and $e_\mathbf{c} \in \mathbb{Z}$.

Lemma 8.14. *For any $P \in Prep(Q)$ and $I \in Prei(Q)$, we have*

$$\mathbf{1}_I\mathbf{1}_P = \sum_{\mathbf{c}\in\mathcal{J}} f_\mathbf{c} B_\mathbf{c}$$

with $\underline{dim}B_\mathbf{c} = \underline{dim}I + \underline{dim}P$ and $f_\mathbf{c} \in \mathbb{Z}$.

Now by all the lemmas above, we see that any monomial of $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ is still in the $\mathbb{Z}$-span of the set $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$. Therefore $\{B_{\mathbf{c}}|\mathbf{c} \in \mathcal{J}\}$ is a $\mathbb{Z}$-basis of $\mathcal{C}_{\mathbb{Z}}(Q)$.

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