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On the Growth of the Homology of a Free Loop Space

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Abstract: We prove that for a wide class of spaces X the homology of the free loop space $H_*(X^{S^1};\mathbb{Q})$ has a very strong exponential growth. We call this convergence, controlled exponential growth, and we prove the good behavior of the controlled exponential growth with respect to fibrations.
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1 Introduction

In this paper we are concerned with the growth of the homology $H_*(X^{S^1};\mathbb{Q})$ of a free loop space on a simply connected space, X. Gromov conjectured in [11] that this vector space grows exponentially for almost all cases when X is a closed manifold. This would have an important consequence in Riemannian geometry, due to a theorem of Gromov, improved by Ballmann and Ziller:

Theorem. ([11], [2]). *Let $N_g(t)$ denote the number of geometrically distinct closed geodesics of length $\leq t$ on a simply connected closed Riemannian manifold (M,g). Then, for generic metrics g, there are constants $K>0$ and $\beta>0$ such that for k sufficiently large,*

$$N_g(k) \geq K \cdot \max_{\ell \leq \beta k} dim\, H_\ell(M^{S^1};\mathbb{Q})\,.$$

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On the other hand it is known ([4],[9]) that, if X is any finite simply connected CW complex with non-trivial rational homology then, for the classical loop space ΩX, either $H_*(\Omega X;\mathbb{Q})$ grows polynomially, or else it grows exponentially. Since X^{S^1} fibres as $\Omega X \to X^{S^1} \to X$, in the first case $H_*(X^{S^1};\mathbb{Q})$ also grows at most polynomially. In [17] Vigué-Poirrier conjectures that in the second case, $H_*(X^{S^1};\mathbb{Q})$ should also grow exponentially, a conjecture which would give Gromov's conjecture as a special case.

The Vigué-Poirrier conjecture has been proved for a finite wedge of spheres [17], for a non-trivial connected sum of closed manifolds [15] and in the case X is coformal [14].

Here we shall establish this conjecture for a wide class of new spaces, and with a stronger conclusion.

For it, recall that a graded vector space $\{V_i\}_{i\geq 0}$ of finite type grows exponentially if the sequence $r_k = \sum_{i\leq k} \dim V_i$ grows exponentially; i.e., if there are constants $1 < C_1 < C_2 < \infty$ such that for some K,

$$C_1^k \leq r_k \leq C_2^k\,, \qquad k \geq K\,.$$

We set

$$\log\,\text{index}\, V = \limsup_i \frac{\log \dim V_i}{i}\,.$$

In particular, for any path connected topological space X we write $\log\,\text{index}\,\pi_*(X) = \log\,\text{index}\,\pi_{\geq 2}(X)\otimes\mathbb{Q}$. It is straightforward to check that if a graded space V has exponential growth then $0 < \log\,\text{index}\, V < \infty$.

Associated with V is the Hilbert series $V(z) = \sum_i \dim V_i z^i$, and by definition the radius of convergence ρ_V of $V(z)$ is given by $\rho_V = e^{-\log\,\text{index}\, V}$. Thus if V has exponential growth then $0 < \rho_V < 1$. Note as well that $z \mapsto V(z)$ may be regarded as a function from $[0,\infty]$ to $[0,\infty]$.

Here we shall consider a stronger condition than exponential growth

Definition. A graded vector space $V = \{V_i\}_{i\geq 0}$ of finite type has *controlled exponential growth* if $0 < \log\,\text{index}\, V < \infty$, and for each $\lambda > 1$ there is an infinite sequence $n_1 < n_2 < \cdots$ such that $n_{i+1} < \lambda n_i$, $i \geq 0$, and $\dim V_{n_i} = e^{\alpha_I n_i}$ with $\alpha_i \to \log \operatorname{index} V$.

As we shall observe in formula (4) below, for any simply connected space X with rational homology of finite type,

$$\log \operatorname{index} H_*(X^{S^1}) \leq \log \operatorname{index} \pi_*(X) = \log \operatorname{index} H_*(\Omega X; \mathbb{Q})\,.$$

This motivates the following:

Definition: Let X be a simply connected space with rational homology of finite type, and such that $\log \operatorname{index} H_*(\Omega X;\mathbb{Q}) \in (0,\infty)$. Then, X^{S^1} has *good exponential growth* if $H_*(X^{S^1};\mathbb{Q})$ has controlled exponential growth and

$$\log \operatorname{index} H_*(X^{S^1};\mathbb{Q}) = \log \operatorname{index} H_*(\Omega X;\mathbb{Q})\,.$$

Theorem 1. *Let X be a simply connected wedge of spheres of finite type such that* $\log \operatorname{index} H_*(\Omega X;\mathbb{Q}) \in (0,\infty)$. *Then, X^{S^1} has good exponential growth.*

Remark: In section three we prove an even stronger version (Theorem 1') of Theorem 1.

Theorem 2. *Let $F \to Y \to X$ be a fibration between simply connected spaces with rational homology of finite type. If* $\log \operatorname{index} \pi_*(F) < \log \operatorname{index} \pi_*(Y)$ *and Y^{S^1} has good exponential growth, then X^{S^1} has good exponential growth and*

$$\log \operatorname{index} H_*(X^{S^1}) = \log \operatorname{index} H_*(Y^{S^1})\,.$$

Theorem 3. *Let $Y \to X \to B$ be a fibration between simply connected spaces with rational homology of finite type. If* $\log \operatorname{index} \pi_*(B) < \log \operatorname{index} \pi_*(X)$, *then Y^{S^1} has good exponential growth if and only if X^{S^1} does. In this case $H_*(Y^{S^1};\mathbb{Q})$ and $H_*(X^{S^1};\mathbb{Q})$ have the same log index.*

If X is a simply connected space then $\pi_*(\Omega X) \otimes \mathbb{Q}$ equipped with the Samelson product is a graded Lie algebra, the *homotopy Lie algebra* of X. Recall that an element x in a graded Lie algebra, $E = E_{\geq 1}$ of finite type is *inert* if

(i) the ideal I it generates is free as a graded Lie algebra, and

(ii) the $U(E/I)$-module $I/[I,I]$ is free on the single element $\overline{x}$ represented by x.

Theorem 4. *Let X be a simply connected space with rational homology of finite type and homotopy Lie algebra L_X. If L_X contains an inert element x and if the quotient Lie algebra $L_X/(x)$ is finitely generated where (x) is the ideal generated by x, then X^{S^1} has good exponential growth.*

Finally, at the end of the paper we give a number of examples to which these theorems apply.

2 Growth and log index of Sullivan algebras

A *Sullivan algebra* ([7], §12 for the definition and assertions below) is a differential algebra of the form $(\wedge W, d)$ in which $\wedge W$ is the free graded commutative algebra generated by the graded vector space $W = W^{\geq 1}$, and W is the increasing union of subspaces $W(k)$, $k \geq 0$, with $d : W(0) \to 0$ and $d : W(k) \to \wedge W(k-1)$, $k \geq 1$. It is *minimal* if $d : W \to \wedge^{\geq 2} W$, and it has *finite type* if each W^k is finite dimensional. A *Sullivan extension* is an inclusion $(\wedge V, d) \to (\wedge V \otimes \wedge W, d)$ of Sullivan algebras, the quotient $(\wedge W, \bar{d}) = \Bbbk \otimes_{\wedge V} (\wedge V \otimes \wedge W, d)$ is called the *fibre* of the extension.

Every connected commutative differential graded algebra (A, d) admits a quasi-isomorphism $(\wedge W, d) \xrightarrow{\simeq} (A, d)$ from a minimal Sullivan algebra; $(\wedge W, d)$ is uniquely determined up to isomorphism and is called the *minimal Sullivan model* of (A, d). In particular any Sullivan algebra is isomorphic to one of the form $(\wedge W, d) \otimes \wedge (U \oplus dU)$ in which $(\wedge W, d)$ is its minimal model and $d : U^k \xrightarrow{\cong} (dU)^{k+1}$, $k \geq 1$. Finally, if $(\wedge W, d)$ is a minimal Sullivan model, the graded vector space $(\wedge W)^+ / (\wedge W)^+ (\wedge W)^+$ will be denoted by $\pi^*(\wedge W, d)$; this is isomorphic with W. Then, for any connected commutative differential graded algebra (A, d), we set $\pi^*(A, d) = \pi^*(\wedge W, d)$, where $(\wedge W, d)$ is the minimal model of (A, d).

Let $(\wedge V, d) \to (\wedge V \otimes \wedge W, d)$ be a Sullivan extension of finite type. A basis

$v_1, \cdots, v_r$ of V^1 determines the decomposition of d given by

$$d\Phi = \bar{d}\Phi + \sum v_i \theta_i(\Phi) + \Omega\,, \qquad \Phi \in \wedge V \otimes \wedge W\,,$$

where $\bar{d}\Phi \in \wedge W$, each $\theta_i \Phi$ is in $\wedge V^{\geq 2} \otimes \wedge W$ and $\Omega \in \wedge^{\geq 2} V^1 \cdot (\wedge V \otimes \wedge W)$. The θ_i are then derivations of degree zero in $\wedge V \otimes \wedge W$.

Definition. The Sullivan algebra $(\wedge V, d)$ is π_1-*bounded* if for some p,

$$\theta_i^p : V \to \wedge^{\geq 2} V\,, \qquad 1 \leq i \leq r\,.$$

The extension $\wedge V \to \wedge V \otimes \wedge W$ is π_1-*bounded* if for some p,

$$\theta_i^p : V \oplus W \to \wedge^{\geq 2}(V \oplus W)\,, \qquad 1 \leq i \leq r\,.$$

Lemma 1. *Let $(\wedge V, d) \to (\wedge V \otimes \wedge W, d)$ be a Sullivan extension of finite type, and suppose the extension is π_1-bounded. If $\dim V < \infty$ then $H(\wedge V \otimes \wedge W, d)$ has controlled exponential growth if and only if $H(\wedge W, \bar{d})$ does. Moreover, in this case these graded vector spaces have the same log index.*

Proof: Write $(\wedge V, d) = (\wedge(v_1, \ldots, v_n), d)$ with $d(v_i) \in \wedge(v_1, \ldots, v_{i-1})$. By considering inductively the extension $(\wedge v_i \otimes (\wedge(v_{i+1}, \ldots, v_n) \otimes \wedge W), d)$ we reduce to the case $\wedge V = \wedge v$. Extend $(\wedge v \otimes \wedge W, d)$ to $(\wedge v \otimes \wedge W \otimes \wedge \bar{v}, d)$ by setting $d\bar{v} = v$. Filtering by degree in $\wedge v$ yields a spectral sequence converging to $H(\wedge v \otimes \wedge W, d)$ from $\wedge v \otimes H(\wedge W, \bar{d})$. Similarly, if $\deg v \geq 2$ there is a spectral sequence converging from $H(\wedge v \otimes \wedge W, d) \otimes \wedge \bar{v}$ to $H(\wedge v \otimes \wedge W \otimes \bar{v}, d) \cong H(\wedge W, \bar{d})$. It follows that

$$\dim H^i(\wedge V \otimes \wedge W, d) \leq \sum_{j \leq i} \dim H^j(\wedge W, \bar{d})\,, \qquad i \geq 0 \tag{1}$$

and, if $\deg v \geq 2$,

$$\dim H^i(\wedge W, \bar{d}) \leq \sum_{j \leq i} \dim H^j(\wedge v \otimes \wedge W, d)\,, \quad i \geq 0\,.$$

The lemma is immediate if $\deg v \geq 2$.

If $\deg v = 1$ we write $d(1 \otimes \Phi) = 1 \otimes \bar{d}\Phi + v \otimes \theta\Phi$, $\Phi \in \wedge W$. Then θ is a derivation of degree zero in $\wedge W$, and $\theta\bar{d} = \bar{d}\theta$. In particular, if z is a $\bar{d}$-cycle in

$\wedge W$ and $\theta z = \bar{d}\Phi$ then $d(1 \otimes z + v \otimes \Phi) = 0$, and if z is not a $\bar{d}$-boundary then $1 \otimes z + v \otimes \Phi$ is not a d-boundary. Thus $\ker H(\theta)$ injects into $H(\wedge v \otimes \wedge W)$.

But since $\wedge v \to \wedge v \otimes \wedge W$ is π_1-bounded, for some p, $\theta^p : W \to \wedge^{\geq 2} W$. Thus $\theta^{ps} : \wedge^s W \to \wedge^{\geq s+1} W$, $s \geq 1$, and so

$$\theta^{p\frac{k(k+1)}{2}} : (\wedge W)^k \to (\wedge^{\geq k+1} W)^k = 0\,.$$

It follows that for $k > 0$,

$$\frac{2}{pk(k+1)} \dim H^k(\wedge W, \bar{d}) \leq \dim\,[\ker H(\theta)]^k \leq \dim H^k(\wedge v \otimes \wedge W)\,.$$

The lemma follows from this and formula (1) above. □

Proposition 1. *Let $(\wedge V, d) \to (\wedge V \otimes \wedge W, d)$ be a π_1-bounded Sullivan extension of finite type. Suppose that*

$$\textit{log index}\, H(\wedge W, \bar{d}) < \textit{log index}\, H(\wedge V \otimes \wedge W, d) < \infty\,,$$

$$\textit{log index}\, H(\wedge V, d) \leq \textit{log index}\, H(\wedge V \otimes \wedge W, d)\,,$$

and that $H(\wedge V \otimes \wedge W, d)$ has controlled exponential growth. Then

$$\textit{log index}\, H(\wedge V, d) = \textit{log index}\, H(\wedge V \otimes \wedge W, d)$$

and $H(\wedge V, d)$ has controlled exponential growth.

proof : Write $V = U \oplus Y$ so that $\dim U < \infty$, $U^1 = V^1$, and $\wedge U$ is preserved by d. Apply Lemma 1 to the Sullivan extensions

$$(\wedge U, d) \to (\wedge U \otimes \wedge Y, d) \quad \text{and} \quad (\wedge U, d) \to (\wedge U \otimes \wedge(Y \oplus W), d)$$

to conclude that it is sufficient to prove the proposition for the extension

$$(\mathbb{k} \otimes_{\wedge U} \wedge V, d) \to \mathbb{k} \otimes_{\wedge U} (\wedge V \otimes \wedge W, d)\,,$$

and that this extension does satisfy the hypotheses of the proposition. Thus, we restrict to the case $V^1 = 0$.

Filtering by the subspaces $(\wedge V)^{\geq p} \otimes \wedge W$ yields a spectral sequence converging from $H(\wedge V, d) \otimes H(\wedge W, \bar{d})$ to $H(\wedge V \otimes \wedge W, d)$. Setting $b_i^V = \dim H^i(\wedge V, d)$, $b_i^W = \dim H^i(\wedge W, \bar{d})$ and $b_i^{V \oplus W} = \dim H^i(\wedge V \otimes \wedge W, d)$, we have

$$b_k^{V \oplus W} \leq \sum_{i=0}^{k} b_i^V \, b_{k-i}^W \, , \qquad k \geq 0 \, . \tag{2}$$

In particular $\log \operatorname{index} H(\wedge V \otimes \wedge W, d) \leq \max\{\log \operatorname{index} H(\wedge V, d), \log \operatorname{index} H(\wedge W, \bar{d})\}$ and so our hypotheses imply

$$\log \operatorname{index} H(\wedge V \otimes \wedge W, d) = \log \operatorname{index} H(\wedge V, d) \, .$$

Next, denote $\log \operatorname{index} H(\wedge W, \bar{d})$ and $\log \operatorname{index} H(\wedge V, d)$ respectively by β and α. Then, $\beta < \alpha$. By definition of log index, given any $\varepsilon > 0$ there is a $C = C(\varepsilon)$ such that for all $k \geq 0$,

$$b_k^V \leq C \, e^{(\alpha+\varepsilon)k} \quad \text{and} \quad b_k^W \leq C \, e^{(\beta+\varepsilon)k} \, . \tag{3}$$

Now, fix $\lambda > 1$ and choose $\gamma, \varepsilon > 0$ so that $\gamma + \varepsilon < \frac{\lambda - 1}{\lambda}(\alpha - \beta)$.

Lemma 2. *For any integer $k > 0$ and any integer $m > 0$ such that $\lambda m \leq k$ it follows that*

$$\sum_{i=k-m}^{k} b_{k-i}^V \, b_i^W \leq (m+1) C^2 \, e^{(\alpha - \gamma)k} \, .$$

Proof: From (3) we have

$$\sum_{i=k-m}^{k} b_{k-i}^V \, b_i^W \leq \sum_{i=k-m}^{k} C^2 e^{(\beta+\varepsilon)i} e^{(\alpha+\varepsilon)(k-i)} \, .$$

Now

$$\begin{aligned}
(\beta+\varepsilon)i + (\alpha+\varepsilon)(k-i) &= \alpha k + (\beta - \alpha)i + \varepsilon k \\
&\leq \alpha k + (\beta - \alpha)(k - m) + \varepsilon k \\
&= k \left[\alpha + (\beta - \alpha)(1 - \tfrac{m}{k}) + \varepsilon\right] \\
&\leq k \left[\alpha + (\beta - \alpha)(1 - \tfrac{m}{\lambda m}) + \varepsilon\right] \\
&= k \left[\alpha + (\beta - \alpha)(1 - \tfrac{1}{\lambda}) + \varepsilon\right]
\end{aligned}$$

But we have chosen γ and ε so that

$$\gamma + \varepsilon \leq \frac{\lambda - 1}{\lambda}(\alpha - \beta)\,.$$

It follows that

$$\left(\frac{\lambda-1}{\lambda}\right)(\beta - \alpha) \leq -\gamma - \varepsilon$$

and so

$$k\left[\alpha + (\beta-\alpha)(1-\frac{1}{\lambda}) + \varepsilon\right] \leq k\left[\alpha - \gamma - \varepsilon + \varepsilon\right] = (\alpha-\gamma)k\,.$$

□

Lemma 3. *Let $k > 0$ and m be integers such that $\lambda m \leq k < \lambda(m+1)$. If*

$$b_i^V \leq e^{(\alpha-\delta)i} \qquad m+1 \leq i \leq k\,,$$

then,

$$\sum_{i=0}^{k} b_{k-i}^V b_i^W \leq Ce^{\alpha k}\left[(m+1)Ce^{-\gamma k} + ke^{-(\delta/\lambda)\,k}\right]\,.$$

Proof: We first observe that $(m+1) > k/\lambda$ since $k < \lambda(m+1)$. Thus, for any $\delta > 0$,

$$\sum_{i=0}^{k-m-1} b_{k-i}^V\, b_i^W \leq \sum_{i=0}^{k-m-1} Ce^{(\beta+\varepsilon)i}e^{(\alpha-\delta)(k-i)} < \sum_{i=0}^{k-m-1} Ce^{\alpha i}e^{(\alpha-\delta)(k-i)}$$
$$= Ce^{\alpha k}\sum_{i=0}^{k-m-1} e^{-\delta(k-i)} < kCe^{\alpha k}e^{-\delta(m+1)} < kCe^{\alpha k}e^{-(\delta/\lambda)\,k}\,.$$

The lemma now follows from Lemma 2. □

We return to the proof of the proposition. Fix $\mu > 1$ and choose λ so that $1 < \lambda^2 < \mu$. Since $H(\wedge V \otimes \wedge W, d)$ has controlled exponential growth and (as observed above) its log index is α, there is an infinite sequence $q_1 < q_2 < \cdots$ such that $q_{j+1} < \lambda q_j$ and

$$\frac{\log\ b_{q_j}^{V\oplus W}}{q_j} \to \alpha\,.$$

Write $b^{V\oplus W}_{q_j} = e^{(\alpha+\varepsilon_j)q_j}$. Then, $\varepsilon_j \to 0$.

Next, we extend any integer p_1 to an infinite sequence $p_1 < p_2 < \cdots$ as follows. Assuming $p_1 < \cdots < p_{i-1}$ are constructed, note that for some $j = j(i)$, q_j must satisfy $\lambda p_{i-1} < q_j < \lambda^2 p_{i-1}$. Then choose p_i so that $\lambda p_{i-1} \le p_i \le q_j$ and

$$\frac{\log b^V_{p_i}}{p_i} \ge \frac{\log b^V_s}{s} \qquad \text{for } \lambda p_{i-1} \le s \le q_j \,.$$

Clearly, $p_i \ge \lambda p_{i-1} > p_{i-1}$ and $p_i < \lambda^2 p_{i-1} < \mu p_{i-1}$.

Write

$$b^V_{p_i} = e^{(\alpha-\delta_i)p_i},$$

and apply Lemma 3 with $k = q_j$ and m the largest integer such that $\lambda m \le q_j$. This, together with (2) yields

$$e^{(\alpha+\varepsilon_j)q_j} = b^{V\oplus W}_{q_j} \le \sum_{i=0}^{q_j} b^V_{q_j-i}\, b^W_i \;\le C e^{\alpha q_j}\; \left[C(m+1)e^{-\gamma q_j} + q_j e^{(-\delta_i/\lambda)\, q_j}\right] .$$

By hypothesis, $\log \text{index}\, H(\wedge V, d) = \alpha$ and so $\limsup_i(-\delta_i) \le 0$. On the other hand, if $-\delta_i < -\tau < 0$ for infinitely many i we may choose σ so that $-\gamma < -\sigma$, and $-\delta_i/\lambda < -\sigma$ for those i. Then, the inequality above reduces to

$$\varepsilon_j \le \frac{\log(C^2(m+1) + Cq_j)}{q_j} - \sigma$$

and this must hold for infinitely many $j = j(i)$, which is impossible because $\varepsilon_j \to 0$. It follows that $\delta_i \to 0$ and so $H(\wedge V, d)$ has controlled exponential growth. □

Lemma 4. *Let $W = W^{>0}$ be a graded vector space of finite type and let $L = L_{>0}$ be a graded Lie algebra of finite type. Then,*

$$\log \text{index} \wedge W = \log \text{index}\, W \quad \textit{and} \quad \log \text{index}\, UL = \log \text{index}\, L\,.$$

In particular, if $(\wedge W, d)$ is a Sullivan algebra then $\log \text{index}\, H(\wedge W, d) \le \log \text{index}\, W$.

Proof: The first equality follows from the second: let L be the abelian Lie algebra defined by $L_k \cong W^k$, and note that $(UL)_k \cong (\wedge W)^k$. For the second, let $\mu = \log \operatorname{index} L$. Then, for any $h > 0$, there is an integer $n(h)$ such that $\dim(UL_{>n(h)})_k \leq e^{(\mu+h)k}$ ([9], Lemma 1). Now $UL \cong \wedge L_{\leq n(h)} \otimes UL_{>n(h)}$. The first factor grows polynomially, and so we have

$$\log \operatorname{index} UL = \log \operatorname{index} UL_{>n(h)} \leq \mu + h\,.$$

Since this holds for any $h > 0$, $\log \operatorname{index} UL \leq \mu = \log \operatorname{index} L$. □

Proposition 2. *Let* $(\wedge V, d) \to (\wedge V \otimes \wedge W, d)$ *be a* π_1*-bounded Sullivan extension of finite type in which*

$$\log \operatorname{index} V < \log \operatorname{index} H(\wedge W, \bar{d}) < \infty\,.$$

Then, $\log \operatorname{index} H(\wedge V \otimes \wedge W, d) = \log \operatorname{index} H(\wedge W, \bar{d})$*, and* $H(\wedge V \otimes \wedge W, d)$ *has controlled exponential growth if and only if* $H(\wedge W, \bar{d})$ *does.*

Proof: As in Proposition 1 we may assume $V^1 = 0$. Then, from the spectral sequence converging from $H(\wedge V, d) \otimes H(\wedge W, \bar{d})$ to $H(\wedge V \otimes \wedge W, d)$, we deduce $\log \operatorname{index} H(\wedge V \otimes \wedge W, d) \leq \log \operatorname{index} H(\wedge W, \bar{d})$. Next, recall from ([7], Lemma 12.5) that there is a Sullivan extension $(\wedge V, d) \to (\wedge V \otimes \wedge \bar{V}, d)$ with $H^+(\wedge V \otimes \wedge \bar{V}, d) = 0$ and $\bar{V}^k \cong V^{k+1}$, $k \geq 1$. This gives the Sullivan extension

$$(\wedge V \otimes \wedge W, d) \to (\wedge V \otimes \wedge W, d) \otimes_{\wedge V} (\wedge V \otimes \wedge \bar{V}, d) = (\wedge V \otimes \wedge W \otimes \wedge \bar{V}, d)\,.$$

Since $d : \bar{V} \to \wedge V \otimes \wedge \bar{V}$ and $\wedge V$ is simply connected it follows that the spectral sequence for this extension converges from $H(\wedge V \otimes \wedge W, d) \otimes H(\wedge \bar{V}, \bar{d})$ to $H(\wedge V \otimes \wedge W \otimes \wedge \bar{V}, d) \cong H(\wedge W, \bar{d})$. It follows that $\log \operatorname{index} H(\wedge W, \bar{d}) \leq \log \operatorname{index} H(\wedge V \otimes \wedge W, d)$.

Finally, the argument of Proposition 1, interchanging the roles of $H(\wedge V, d)$ and $H(\wedge W, \bar{d})$, shows that if $H(\wedge V \otimes \wedge W, d)$ has controlled exponential growth then $H(\wedge W, \bar{d})$ has also controlled exponential growth. The same argument, applied to the Sullivan extension $(\wedge V \otimes \wedge W, d) \to (\wedge V \otimes \wedge W \otimes \wedge \bar{V}, d)$ shows that $H(\wedge V \otimes \wedge W, d)$ has controlled exponential growth if $H(\wedge W, \bar{d})$ does. □

3 Growth and log index of free loop spaces

Let X be a simply connected space with rational homology of finite type. Then, the free loop space X^{S^1} is the total space of a fibration

$$X \leftarrow X^{S^1} \leftarrow \Omega X\,.$$

In particular, from Lemma 4 we deduce that

$$\text{(4)} \qquad \text{log index}\, H_*(X^{S^1};\mathbb{Q}) \leq \text{log index}\, \pi_*(X) = \text{log index}\, H_*(\Omega X;\mathbb{Q})\,.$$

Now suppose that $(\wedge W, d)$ is a Sullivan model for X. Then, in [19], Vigué-Poirrier and Sullivan showed that the fibration above corresponds to a Sullivan extension

$$(\wedge W, d) \to (\wedge W \otimes \wedge \bar{W}, d) \to (\wedge \bar{W}, 0)$$

where $\bar{W}^k = W^{k+1}$ (the identification being denoted by $\bar{w} \leftrightarrow w$, $w \in W$) and the differential in $\wedge W \otimes \wedge \bar{W}$ is defined as follows: let δ be the derivation in $\wedge W \otimes \wedge \bar{W}$ given by $\delta w = \bar{w}$ and $\delta \bar{w} = 0$ and set $d\bar{w} = -\delta dw$. In particular $(\wedge W \otimes \wedge \bar{W}, d)$ is a Sullivan model for X^{S^1} (minimal if $(\wedge W, d)$ is) and the morphism $H(\wedge W \otimes \wedge \bar{W}, d) \to \wedge \bar{W}$ is dual to the morphism $H_*(\Omega X;\mathbb{Q}) \to H_*(X^{S^1};\mathbb{Q})$.

We now turn our attention to Theorem 1. The first step in the proof is an analysis of the case of a Sullivan algebra $(\wedge W, d)$, 1-connected and of finite type, in which $d : W \to \wedge^2 W$. In this case ([7],§24, Example 7), $(\wedge W, d) = \mathcal{C}^*(L)$ where L is the homotopy Lie algebra of $(\wedge W, d)$. Moreover, in $(\wedge W \otimes \wedge \bar{W}, d)$ we have $d : \wedge \bar{W} \to W \otimes \wedge \bar{W}$, and so ([7], §23(e)), $(\wedge W \otimes \wedge \bar{W}, d) = \mathcal{C}^*(L; \wedge \bar{W})$ for a representation of L in $\wedge \bar{W}$.

On the other hand, W is the dual of sL, where $(sL)_q = L_{q-1}$ ([7], §22(e)). It follows that $\bar{W}$ is the dual of L and so $\wedge^k \bar{W}$ is dual to $\wedge^k L$ via a pairing described for $\wedge^k W$ and $\wedge^k sL$ in ([7], §21(e)). Thus $(\wedge W \otimes \wedge \bar{W}, d)$ is dual to $C_*(L, \wedge L)$ and a straightforward computation shows that the representation of L in $\wedge L$ is just the adjoint representation. But the Poincaré Birkoff Witt isomorphism ([7], Prop. 21.2) commutes with the adjoint representations of L in UL and in $\wedge L$, and so $(\wedge W \otimes \wedge \bar{W}, d)$ is dual to $C_*(L; UL)$. This identifies the morphism $H_*(\Omega X;\mathbb{Q}) \to H_*(X^{S^1};\mathbb{Q})$ with the morphism $UL \to \text{Tor}_0^{UL}(\mathbb{Q}, UL)$, whose image is $\text{Tor}_0^{UL}(\mathbb{Q}, UL)$.

Theorem 1'. *Let X be a simply connected wedge of spheres of finite type such that* $\log \text{index}\, H_*(\Omega X;\mathbb{Q}) = \alpha \in (0,\infty)$. *Then, the image* Im *of* $H_*(\Omega X;\mathbb{Q})$ *in* $H_*(X^{S^1};\mathbb{Q})$ *satisfies*

$$\log \text{index Im} = \log \text{index}\, H_*(X^{S^1};\mathbb{Q}) = \log \text{index}\, H_*(\Omega X;\mathbb{Q})\,.$$

Moreover for some d and any $\varepsilon > 0$ there is a $K' = K'(\varepsilon)$ such that for $n \geq K'$

$$e^{(\alpha-\varepsilon)n} \leq \sum_{i=n}^{n+d} \dim\, (\text{Im})_i \leq \sum_{i=n}^{n+d} e^{(\alpha+\varepsilon)n}\,.$$

In particular, Im *has controlled exponential growth.*

Corollary. *With X as in Theorem 1', given $\varepsilon > 0$, there is a $K'' = K''(\varepsilon)$ such that, for $n \geq K''$,*

$$e^{(\alpha-\varepsilon)n} \leq \sum_{i=n}^{n+d} \dim H_i(X^{S^1};\mathbb{Q}) \leq e^{(\alpha+\varepsilon)n}\,.$$

In particular, $H_*(X^{S^1};\mathbb{Q})$ has controlled exponential growth.

Proof of Theorem 1': In this case ([7], Theorem 24.5), L is a free Lie algebra, UL is a tensor algebra TU, and the differential in the Sullivan model $(\wedge W, d)$ maps W to $\wedge^2 W$. In view of the discussion preceding the statement of Theorem 1' we may identify $\text{Im} \cong \text{Tor}_0^{TU}(\mathbb{Q}, TU)$.

Let v_i be a basis of U and let $r \geq 1$ be an integer. In the tensor algebra TU, the subspace T^rU spanned by the monomials of length r admits an automorphism σ defined by

$$\sigma(v_{i_1} \otimes \cdots \otimes v_{i_r}) = (-1)^{|v_{i_1}|\,|v_{i_2}\otimes\cdots\otimes v_{i_r}|} v_{i_2} \otimes \cdots \otimes v_{i_r} \otimes v_{i_1}\,.$$

Note that σ preserves degrees and that $\sigma^r = id$. Moreover, denoting the adjoint representation of TU in itself by $\Phi \otimes \Psi \mapsto \Phi \circ \Psi$ we have that

$$v_{i_1} \circ (v_{i_2} \otimes \cdots \otimes v_{i_r}) = (id - \sigma)(v_{i_1} \otimes \cdots \otimes v_{i_r})\,.$$

It follows that

(5)
$$\sum_{j=0}^{r-1} \sigma^j\, [v_{i_1} \circ (v_{i_2} \otimes \cdots \otimes v_{i_r})] = \sum_{j=0}^{r-1} \sigma^j (v_{i_1} \otimes \cdots \otimes v_{i_r}) - \sum_{j=1}^{r} \sigma^j (v_{i_1} \otimes \cdots \otimes v_{i_r}) = 0\,.$$

The isomorphisms σ^i $(0 \le i < r)$ give an action of the cyclic group of order r on the set of monomials of length r, so that each orbit has at most r elements. Let $W(\tau)$ be the linear span of the monomials in an orbit τ, so that

$$T^r U = \bigoplus_{\tau} W(\tau)$$

and for each degree k,

$$(T^r U)_k = \bigoplus_{\tau} W_k(\tau)\,.$$

Write $I = \sum_{j=0}^{r-1} \sigma^j$ and fix a single monomial $w(\tau) = x_1 \otimes \cdots \otimes x_r$ in each $W_k(\tau)$. If $I(w(\tau)) = 0$ then for some $\ell < r$, $\sigma^\ell(w(\tau)) = -w(\tau)$. This shows that

$$w(\tau) = (x_1 \otimes \cdots \otimes x_\ell) \otimes \cdots \otimes (x_1 \otimes \cdots \otimes x_\ell)\,.$$

Since $\sigma^\ell(w(\tau)) = -w(\tau)$, $x_1 \otimes \cdots \otimes x_\ell$ has odd degree and the number of tensorands $(x_1 \otimes \cdots \otimes x_\ell)$ is even. Therefore, r and k are even and $w(\tau)$ is the square of a monomial in $T^{r/2}U$.

Let $\mathcal{U}_k^{(r)}$ be the linear span of the $w(\tau)$ that are not in the kernel of I. Since $I((TU)_+ \circ TU) = 0$ it follows that

$$\mathcal{U}_k^{(r)} \cap ((TU)_+ \circ TU) = 0\,. \tag{6}$$

Moreover, since each orbit has at most r monomials, there are at least $\dfrac{m(k,r)}{r}$ orbits where $m(k,r) = \dim(T^r U)_k$. Since the number of $w(\tau)$ in $\ker I$ is $\le m(k/2, r/2)$, it follows that $(T^r U)_k \cap [(TU)_+ \circ TU]$ has codimension at least $\dfrac{m(k,r)}{r} - m(k/2, r/2)$. Hence,

$$\dim \left[\mathrm{Tor}_0^{TU}(\mathbb{Q}, TU)\right]_k \ge \sum_{r \ge 1} \left(\frac{m(k,r)}{r} - m(k/2, r/2) \right)\,.$$

On the other hand, since $U = U_{>0}$, $(T^r U)_k = 0$ for $r > k$. Thus,

$$\sum_r \frac{m(k,r)}{r} = \sum_{r \le k} \frac{m(k,r)}{r} \ge \frac{1}{k} \sum_{r \le k} m(k,r) = \frac{1}{k} \sum_{r \le k} \dim(T^r U)_k = \frac{1}{k} \dim(TU)_k\,.$$

From ([9], Theorem 4) we deduce that there is a positive integer d such that, for each $\varepsilon > 0$, there is an integer K for which

$$e^{(\alpha-\varepsilon)n} \leq \sum_{i=n}^{n+d} \dim\,(TU)_i \leq e^{(\alpha+\varepsilon)n}\,, \qquad n \geq K\,.$$

Therefore, for $n \geq K$,

$$\frac{e^{(\alpha-\varepsilon)n}}{n+d} - e^{\frac{(\alpha+\varepsilon)n}{2}} \leq \sum_{i=n}^{n+d} \dim\left[\mathrm{Tor}_0^{TU}(\mathbb{Q},TU)\right]_i \leq \sum_{i=n}^{n+d} \dim(TU)_i \leq e^{(\alpha+\varepsilon)n}\,.$$

It follows that, given $\varepsilon > 0$, there is a K' such that, for $n \geq K'$,

$$e^{(\alpha-\varepsilon)n} \leq \sum_{i=n}^{n+d} \dim \mathrm{Tor}_0^{TU}(\mathbb{Q},TU) \leq e^{(\alpha+\varepsilon)n}\,.$$

□

We next consider a general map $Y \to X$ of simply connected spaces in which Y and X have rational homology of finite type. There is then ([7], Proposition 15.5) a Sullivan extension

$$(\wedge V, d) \to (\wedge V \otimes \wedge W, d)$$

in which $(\wedge V, d)$ is a minimal Sullivan model for X, the fibre $(\wedge W, \bar{d})$ is a minimal Sullivan model for the homotopy fibre F of φ, and $(\wedge V \otimes \wedge W, d)$ is a Sullivan model for Y. This yields the Sullivan extension

$$(\wedge V \otimes \wedge \bar{V}, d) \to (\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}, d)$$

from a Sullivan model of X^{S^1} to a Sullivan model of Y^{S^1} with fibre a Sullivan model of F^{S^1}.

Lemma 5. *This Sullivan extension is π_1-bounded.*

Proof: Let $\bar{v}_1, \dots, \bar{v}_s$ be a basis for $\bar{V}^1$ and let $\theta_1, \dots, \theta_s$ be the corresponding derivations in $\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}$. From the definition of the differential it follows that each θ_i vanishes in V and W and that each θ_i maps $\bar{V}$ into $\wedge V$ and $\bar{W}$ into $\wedge V \otimes \wedge W$. Thus θ_i^2 vanishes in $V, W, \bar{V}$ and $\bar{W}$. □

Proposition 3. *Let X be a simply connected space with rational homology of finite type and denote by $X(k)$ its k-connected cover. Then, X^{S^1} has good exponential growth if and only if $X(k)^{S^1}$ does.*

Proof: We may suppose X is a CW complex, in which case we have a fibration $X(k) \to X \to Z$ in which $\pi_*(Z) = \{\pi_i(X)\}_{i\leq k}$. In particular $\pi_*(Z)\otimes\mathbb{Q}$ is finite dimensional. This ([7], Proposition 15.5), gives rise to a π_1-bounded Sullivan extension

$$(\wedge V^{\leq k}\otimes\wedge\bar{V}^{<k}, d) \to (\wedge V^{\leq k}\otimes\wedge\bar{V}^{<k}\otimes\wedge V^{>k}\otimes\wedge\bar{V}^{\geq k}, d)\,,$$

in which $H(\wedge V^{>k}\otimes\wedge\bar{V}^{\geq k},\bar{d}) \cong H^*(X(k)^{S^1};\mathbb{Q})$. The proposition follows from Lemma 1.

□

Proof of Theorem 2: The map φ induces a map $\varphi(2) : Y(2) \to X(2)$ between the 2-connected covers of Y and X. Denote its homotopy fibre by F'. Clearly, $\log \operatorname{index} \pi_*(F') = \log \operatorname{index} \pi_*(F)$ and $\log \operatorname{index} H_*(\Omega Y(2);\mathbb{Q}) = \log \operatorname{index} \pi_*(Y(2)) = \log \operatorname{index} \pi_*(Y) = \log \operatorname{index} H_*(\Omega Y;\mathbb{Q})$. By Proposition 3, $H_*(Y^{S^1};\mathbb{Q})$ (respectively $H_*(X^{S^1};\mathbb{Q})$) has good exponential growth if and only if $H_*(Y(2)^{S^1};\mathbb{Q})$ (respectively $H_*(X(2)^{S^1};\mathbb{Q})$ does. Thus, we lose no generality in assuming that Y and X are 2-connected, in which case F is simply connected.

Next, note that, since $\log \operatorname{index} \pi_*(-)\otimes\mathbb{Q} = \log \operatorname{index} H_*(\Omega -;\mathbb{Q})$ (cf (4)), the long exact homotopy sequence for $F \to Y \to X$ gives

$$\log \operatorname{index} H_*(\Omega Y;\mathbb{Q}) = \log \operatorname{index} H_*(\Omega X;\mathbb{Q})\,.$$

It is thus sufficient to show that $H_*(X^{S^1};\mathbb{Q})$ has good exponential growth and that

$$\log \operatorname{index} H_*(X^{S^1};\mathbb{Q}) = \log \operatorname{index} H_*(Y^{S^1};\mathbb{Q})\,.$$

But, as described above, the map $\varphi : Y \to X$ determines a π_1-bounded Sullivan extension of finite type,

$$(\wedge V\otimes\wedge\bar{V}, d) \to (\wedge V\otimes\wedge\bar{V}\otimes\wedge W\otimes\wedge\bar{W}, d)\,.$$

Now according to formula (4),

$$\log \operatorname{index} H_*(F^{S^1};\mathbb{Q}) \leq \log \operatorname{index} \pi_*(F^{S^1}) = \log \operatorname{index} H_*(\Omega F;\mathbb{Q})\,.$$

Thus, by hypothesis,

$$\log \operatorname{index} H^*(\wedge W \otimes \wedge \bar{W}, d) < \log \operatorname{index} H^*(\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}, d)\,.$$

On the other hand, also by hypothesis,

$$\begin{aligned}\log \operatorname{index} H(\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}) &= \log \operatorname{index} H_*(Y^{S^1};\mathbb{Q}) = \log \operatorname{index} \pi_*(Y)\\ &= \log \operatorname{index} \pi_*(X) = \log \operatorname{index} V\\ &\geq \log \operatorname{index} H(\wedge V \otimes \wedge \bar{V}, d)\,.\end{aligned}$$

Since $H_*(\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}, d)$ has good exponential growth, also by hypothesis, the theorem follows from Proposition 1. □

Proof of Theorem 3: As in Theorem 2 we lose no generality in assuming B is 2-connected. The fibration $Y^{S^1} \to X^{S^1} \to B^{S^1}$ yields a Sullivan extension of the form

$$(\wedge V \otimes \wedge \bar{V}, d) \to (\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}, d)$$

in which the fibre is a Sullivan model for Y^{S^1}, and $\log \operatorname{index} W > \log \operatorname{index} V$. If Y^{S^1} has good exponential growth then

$$\begin{aligned}\log \operatorname{index} H(\wedge W \otimes \wedge \bar{W}, \bar{d}) &= \log \operatorname{index} H_*(Y^{S^1};\mathbb{Q}) = \log \operatorname{index} \pi_*(Y)\\ &= \log \operatorname{index} W > \log \operatorname{index} V\,.\end{aligned}$$

In this case it follows at once from Proposition 2 that X^{S^1} has good exponential growth.

Conversely, if X^{S^1} has good exponential growth, then,

$$\log \operatorname{index} \pi_*(X) = \log \operatorname{index} W = \log \operatorname{index} H(\wedge V \otimes \wedge \bar{V} \otimes \wedge W \otimes \wedge \bar{W}, d)$$

$$\leq \max \left\{\log \operatorname{index} H(\wedge V \otimes \wedge \bar{V}, d)\,, \log \operatorname{index} H(\wedge W \otimes \wedge \bar{W}, \bar{d})\right\}\,.$$

But $\log \operatorname{index} V < \log \operatorname{index} W$ and so Lemma 4 gives

$$\log \operatorname{index} W \leq \log \operatorname{index} H(\wedge W \otimes \wedge \bar{W}, \bar{d}) \leq \log \operatorname{index} W\,.$$

It now follows from Proposition 2 (with $\wedge V \otimes \wedge \bar{V}$ and $\wedge W \otimes \wedge \bar{W}$ playing the roles of $\wedge V$ and $\wedge W$ respectively) that $H(\wedge W \otimes \wedge \bar{W}, \bar{d})$ has controlled exponential growth and so Y^{S^1} has good exponential growth. □

We now turn our attention to Theorem 4.

Lemma 6. *The ideal I generated by an inert element x in a graded Lie algebra $E = E_{\geq 1}$ satisfies*

$$\textit{log index}\, UE/I < \textit{log index}\, UI\,,$$

if E/I is finitely generated.

Proof: Since E/I is finitely generated, it follows from [1] that the Hilbert series $U(E/I)(z)$ satisfies $\lim_{z\to\rho} U(E/I)(z) = \infty$, where ρ is the radius of convergence. Moreover, since I is a free Lie algebra, $UI \cong T\,(I/[I,I])$. Recall now that x is an inert element, so that $I/[I,I] \cong U(E/I)\cdot\overline{x}$. Thus it is sufficient to show that log index $W <$ log index TW for any graded vector space $W = W_{\geq 1}$ of finite type where the Hilbert series $W(z)$ has a radius of convergence ρ and satisfies $\lim_{z\to\rho} W(z) = \infty$.

But the Hilbert series $(TW)(z)$ is just $\frac{1}{1-W(z)}$. Thus, for some unique $r < \rho$, we have $W(r) = 1$ and clearly r is the radius of convergence of $TW(z)$. Since the respective log indexes are $-\log\rho$ and $-\log r$ the lemma follows. □

Proof of Theorem 4: Let $I \subset L_X$ be the ideal generated by x. Since I is a free graded Lie algebra we can find a wedge of spheres Y and a map $\varphi : Y \to X$ such that $\pi_*(\Omega Y) : L_Y \xrightarrow{\cong} I$. For the homotopy fibre F of φ we then have

$$(L_F)_k \cong (L_X/I)_{k+1}\,.$$

Since x is inert and L_X/I is finitely generated, by Lemma 6,

$$\text{log index}\, U(L_X/I) < \text{log index}\, UI \leq \text{log index}\, UL_X\,.$$

Recall that a graded Lie algebra and its enveloping algebra have the same log index (Lemma 2). This gives

$$\text{log index}\, H_*(\Omega F;\mathbb{Q}) < \text{log index}\, H_*(\Omega Y;\mathbb{Q})\,.$$

Now, by Theorem 1, Y satisfies the hypotheses of Theorem 2, Theorem 4 follows.
□

4 Examples

Example 1. Deleted manifolds.

Let M be a closed simply connected manifold whose rational cohomology algebra is not generated by a single class, and set $X = M - \{a\}$ for some point $a \in M$. Then according to [12], L_X contains an inert element x and $L_X/I \cong L_M$, where I is the ideal generated by x. Thus, by Theorem 4, if L_M is finitely generated then X^{S^1} has good exponential growth.

Example 2. Strongly separated manifolds.

Let N be a hypersurface in a closed simply connected manifold M, separating M into two manifolds M_0 and M_1 with boundary N. Suppose $\omega_0 \in H^*(M_0;\mathbb{Q})$ and $\omega_1 \in H^*(M_1;\mathbb{Q})$ are cohomology classes of odd degrees which restrict to zero in $H^*(N;\mathbb{Q})$ and for which there are maps $\varphi_0 : S^{m_0} \to M_0\backslash N$ and $\varphi_1 : S^{m_1} \to M_1\backslash N$ with $0 \neq H^*(\varphi_i)\omega_i$, $i = 0, 1$.

Then, the map $S^{m_0} \vee S^{m_1} \to M$ admits a rational retraction and so L_M contains a free Lie algebra on two generators and $H_*(M^{S^1};\mathbb{Q})$ grows exponentially.

In fact, let S be the simplicial set of singular simplices in M whose images are either in M_0 or M_1, and denote by $A_{PL}(-)$ the Sullivan functor ([7], §12b, [18]) from simplicial sets to commutative graded differential algebras over $\mathbb{Q}$. Then, a minimal Sullivan model for $A_{PL}(S)$ is also a minimal Sullivan model for M. Moreover, ω_0 and ω_1 are represented by cycles z_0 and z_1 in $A_{PL}(S)$ such that z_i vanishes on the simplices of M_{1-i}. In particular $z_0 \wedge z_1 = 0$.

The resulting morphism $H^*(S^{m_0} \vee S^{m_1};\mathbb{Q}) \to A_{PL}(S)$ determines a morphism $\varphi : (\wedge W, d) \to (\wedge V, d)$ from the minimal model of $S^{m_0} \vee S^{m_1}$ to the minimal Sullivan model of M, and hence a continuous map $M_{\mathbb{Q}} \to S^{m_0}_{\mathbb{Q}} \vee S^{m_1}_{\mathbb{Q}}$ between the rationalizations ([7], §17c). Evidently, φ_0 and φ_1 define a reverse inclusion. This exhibits $(S^{m_0}_{\mathbb{Q}} \vee S^{m_1}_{\mathbb{Q}})^{S^1}$ as a retract of $M^{S^1}_{\mathbb{Q}}$ and so Theorem 1 implies that $H_*(M^{S^1};\mathbb{Q})$ grows exponentially. This example generalizes the case of connected sums [15] □

Example 3. Let X be the configuration space $F(M, 2)$ of ordered pairs of distinct points in a 2-connected closed manifold M whose cohomology algebra is not generated by a single element, and such that L_M is finitely generated.

Here [3], there is a fibration

$$M - \{ pt \} \to F(M, 2) \to M \,,$$

which we write as $Y \to X \to M$ to simplify notation. As observed in Example 1, $\log \operatorname{index} L_Y > \log \operatorname{index} L_M$. Thus, by Theorem 3, X^{S^1} also has good exponential growth.

□

Example 4. High skeleta of finite Postnikov towers.

A simply connected space Y is rationally a finite Postnikov tower if $\dim \pi_*(Y) \otimes \mathbb{Q}$ is finite. In [13] it is shown that for such a space there is some N such that, for $n \geq N$, the homotopy fibre F of the inclusion of the n-skeleton $Y(n) \to Y$ is rationally a wedge of spheres. If F is rationally a single sphere then it follows from the Gysin sequence that the Betti numbers, $\dim H_k(Y; \mathbb{Q})$, are bounded. Thus, if these Betti numbers are unbounded, there must be at least two spheres in the fibre, and so F^{S^1} has good exponential growth. Since log index $H_*(\Omega Y; \mathbb{Q}) \neq 0$, Theorem 3 asserts that $Y(n)^{S^1}$ has good exponential growth.

In particular, if Y is rationally a finite product of Eilenberg MacLane spaces and dimension of $\pi_{2*}(Y) \otimes \mathbb{Q} \geq 2$, then $Y(n)^{S^1}$ has good exponential growth. □

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