Errata: “Extensions of truncated discrete valuation rings”

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The purposes of these errata are:
(1) to fill in a gap in the proof of Part (ii) of Proposition 2.2 of [I] (= Proposition 2.1 of [R]), and
(2) to explain the current status of, and wrong points in, the preprint [II] (which will never be published) and the survey paper [R].

We thank Shin Hattori for pointing out the gap (1) and discussions on it, and Takeshi Saito for pointing out a fatal error in [II] and for providing a counterexample to Proposition 3.7 of [II].

1. We use the notation of [I]. The proposition in question is the following:

**Proposition.** (i) Let $A$ be a tdvr with residue field $k$ of characteristic $p \geq 0$, and let $a$ be the length of $A$. Then there exists a cdvr $O$ such that $A$ is isomorphic to $O/m^a$, where $m$ is the maximal ideal of $O$. If $pA = 0$, then this $O$ can be taken to be the power series ring $k[[\pi]]$; if $pA \neq 0$, then $O$ as above must be finite over a Cohen $p$-ring ([G], IV, 19.8) with residue field $k$. (If $pA = 0$ and $p \neq 0$, then both types of $O$ are possible.)

(ii) Let $K$ be a cdvf and let $A = O_K/m_K^a$ with $a \geq 1$. For any finite extension $B/A$ of tdvr’s, there exist a finite separable extension $L/K$ and an isomorphism $\psi : O_L/m^a_L O_L \to B$ such that the diagram

\[
\begin{array}{c}
O_L/m^a_L O_L \xrightarrow{\psi} B \\
\uparrow \hspace{1cm} \uparrow \\
O_K/m_K^a \hspace{1cm} A
\end{array}
\]

is commutative, where the left vertical arrow is the one induced by $O_K \hookrightarrow O_L$.

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The proof in [I] has a gap in proving that \( L/K \) can be taken to be separable (the Jacobian criterion applied to the newly taken \( g_1, \ldots, g_n \) should have been considered modulo \( q' = (g_1, \ldots, g_n) \) rather than the original \( q \)). We give here a correct one, including the whole proof (but printing in the tiny font the part which is identical with the original) for the convenience of the reader.

Proof. (i) Let \( W \) be a Cohen \( p \)-ring with residue field \( k \). The reduction map \( W \to k \) lifts by the formal smoothness of \( W \) to a local ring homomorphism \( W \to A \) ([23], 048.6).

If \( A = 0 \), the map \( W \to A \) factors through the residue field \( k \), which makes \( A \) a \( k \)-algebra. Then there exists a surjective \( A \)-algebra homomorphism \( k[x] \to A \) which maps \( \pi \) to \( \pi_A \), where \( \pi_A \) is a uniformizer of \( A \). Hence \( A \) is isomorphic to \( k[x]/(\pi^n) \) ([33], Th. 3.1).

In general, if \( O \) is a valuation ring. It contains \( \mathcal{O} \) by (II, 014Q, 17.1.7), and \( \phi \) extends to a surjective \( \mathcal{O}_K \)-algebra homomorphism \( \psi : \mathcal{O} \to B \). By abuse of notation, we denote also by \( \psi \) the maximal ideal of \( \mathcal{O}_k \). Put \( \pi = \ker(\psi) \). We identify the residue field \( k' \) of \( \mathcal{O}_k \) with that of \( B \) via \( \psi \). Since \( \varphi(m') = m'_B \), the map \( \varphi \) induces a surjective \( k' \)-linear map \( m'/m^2 \to m_B/m_B^2 \) and its kernel is \( (n + m^2)/m^2 \cong n/(n \cap m^2) \). Thus we have an exact sequence

\[
0 \to n/(n \cap m^2) \to m/m^2 \to m_B/m_B^2 \to 0.
\]

Assume \( n \geq 2 \), as the case \( n = 1 \) can be treated similarly and more easily. Then \( \dim_{\mathcal{O}_K}(m_B/m_B^2) = 1 \) and \( \dim_{\mathcal{O}_K}(n/(n \cap m^2)) = n \). Choose a regular system of parameters \( \{w, f_1, \ldots, f_n\} \) of \( \mathcal{O}_m \) such that \( \varphi(w) \) gives a basis of \( m_B/m_B^2 \) and \( f_1, \ldots, f_n \in n \) give a basis of \( n/(n \cap m^2) \). Let \( \psi \) be the ideal of \( \mathcal{O}_K \) generated by \( f_1, \ldots, f_n \). Then by [23], 041M, 17.1, the quotient ring \( \mathcal{O} = \mathcal{O}_K \to \psi \) is a regular local ring of dimension 1 and hence a discrete valuation ring. It contains \( \mathcal{O}_k \) since \( \varphi \) maps \( \pi_K \) to a non-zero non-unit in \( B \) and is finite over \( \mathcal{O}_K \). Hence it is a cdvr. Since \( n > 0 \), the map \( \varphi \) factors through \( \mathcal{O} \). Thus we see the diagram \( \square \) commutes with \( \mathcal{O} \) in place of \( \mathcal{O}_K \). Since \( \psi \) is flat over \( \mathcal{O} \), the induced homomorphism \( \psi \) is bijective.

To make the fraction field \( L \) of \( \mathcal{O} \) separable over \( K \), we “deform” \( \mathcal{O} \) if necessary. Let \( L_0 \) be the separable closure of \( K \) in \( L \). Then \( L/L_0 \) is purely inseparable and we can find a series of extensions \( L_0 \subset L_1 \subset \cdots \subset L_n = L \) such that

\[
L_{i+1} = L_i(\alpha_i^{1/p}) \quad \text{with some} \quad \alpha_i \in L_i^X \setminus \langle L_i^X \rangle^p.
\]

For each \( i \), the ramification index \( e_{i+1} \) of \( L_{i+1}/L_i \) is either \( p \) or 1. If \( e_{i+1} = p \), then we can take \( \alpha_i \) to be a prime element of \( \mathcal{O}_i := \mathcal{O}_L \). If \( e_{i+1} = 1 \), then \( L_{i+1}/L_i \) has inseparable residual extension of degree \( p \) and hence we can take \( \alpha_i \) to be a unit of \( \mathcal{O}_i \) whose image in the residue field is not a \( p \)-th power. In either case, \( \mathcal{O}_{i+1} \) is then generated by \( \alpha_i^{1/p} \) as an \( \mathcal{O}_i \)-algebra and hence we have

\[
\mathcal{O}_{i+1} \cong \mathcal{O}_i[Y]/(Y^p - \alpha_i).
\]

To deform the \( \mathcal{O}_i \)'s inductively, we adapt the following Recipe: In general, if \( M \) is a finite extension of \( K \) and \( \alpha \in \mathcal{O}_M \) has the same property as \( \alpha_i \) above (i.e. prime or unit which is residually non-\( p \)-th power), then for any non-zero \( \beta \in \mathcal{m}_K^2 \mathcal{O}_M \), the polynomial \( Y^p + \beta Y - \alpha \in \mathcal{O}_M[Y] \) is separable and irreducible over \( M \). In fact, it is Eisenstein if \( \alpha \) is a prime element, and otherwise it gives rise to an inseparable extension of degree \( p \) of the residue field. Hence \( \mathcal{O}_{\alpha, \beta} := \mathcal{O}_M[Y]/(Y^p + \beta Y - \alpha) \) is a complete
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discrete valuation ring whose fraction field is separable over $M$. Note also that the $\mathcal{O}_M$-algebras $\mathcal{O}_{\alpha,\beta} \otimes_{\mathcal{O}_K} \mathcal{A}$ are canonically isomorphic for all $\alpha \in \mathcal{O}_M$ in a fixed class mod $m_K^2 \mathcal{O}_M$ and all $\beta \in m_K^2 \mathcal{O}_M$.

Now choose any non-zero $\beta \in m_K^2 \mathcal{O}_0$. Set $\mathcal{O}_0' := \mathcal{O}_0$. For $i \geq 0$, suppose that we have a finite extension of complete discrete valuation rings $\mathcal{O}'_i/\mathcal{O}_0'$ such that Frac($\mathcal{O}'_i$)/$K$ is separable, and also an isomorphism of $\mathcal{O}_K$-algebras $\mathcal{O}'_i \otimes_{\mathcal{O}_K} \mathcal{A} \simeq \mathcal{O}_i \otimes_{\mathcal{O}_K} \mathcal{A}$. Choose $\alpha'_i \in \mathcal{O}'_i$ such that the images of $\alpha'_i$ and $\alpha_i$ in these rings correspond via this isomorphism. Note that $\alpha'_i$ is a prime element (resp. unit which is residually non-$p$-th power) if $\alpha_i$ is so. Then the ring

$$\mathcal{O}'_{i+1} := \mathcal{O}_i'[Y]/(Y^p + \beta Y - \alpha'_i).$$

is a finite extension of complete discrete valuation rings over $\mathcal{O}'_i$, the extension Frac($\mathcal{O}'_{i+1}$)/$K$ is separable and we also have an isomorphism of $\mathcal{O}_K$-algebras $\mathcal{O}'_{i+1} \otimes_{\mathcal{O}_K} \mathcal{A} \simeq \mathcal{O}_{i+1} \otimes_{\mathcal{O}_K} \mathcal{A}$. Repeating this, we obtain a desired lift of $B$ whose fraction field is separable over $K$. □

2. The theorem numbers of this section are those of [II]. The purpose of [II] was to show that, for a truncated discrete valuation ring $\mathcal{A}$ of length $\geq m$, the category $\mathcal{FFP}^{<m}_\mathcal{A}$ of finite flat principal $\mathcal{A}$-algebras with “ramification bounded by $m$” can be constructed with no reference to a particular lift of $\mathcal{A}$ to a complete discrete valuation ring (in particular, it is independent of such a lift). After [II] was posted in the arXiv, however, Takeshi Saito found that there was a counterexample to Proposition 3.7 and that there was a serious error in the proof of Lemma 3.10, which was used in the proof of Proposition 3.7.

The counterexample is as follows: Let $S = k[[X,Y]]$, where $k$ is an algebraically closed field of characteristic $\neq 2$, and let $p$ be the height 1 prime ideal $(Y^2 - X)$ of $S$. Then $S$ is normal, integral and $p$-adically complete. Let $\mathcal{B} := S[Z]/(Z^2 - X)$, which is $p$-adically complete and flat over $S$. The residue field $\kappa(p)$ of $p$ can be identified with the power series field $k((Y))$, and we have $\mathcal{B} \otimes_S \kappa(p) \simeq \kappa(p) \times \kappa(p)$ (so that $\pi_0(\mathcal{B} \otimes_S \kappa(p))$ consists of two points). On the other hand, the fraction field $C$ of $S$ is $k((X,Y))$ and $\mathcal{B} \otimes_S C = k((Y,Z))$ (so that $\pi_0(\mathcal{B} \otimes_S C)$ consists of one point).

The error in the proof of Lemma 3.10 is that, in applying the Henselian property, we did not (and in fact cannot) check that $s^k g(x/s)$ and $s^k h(x/s)$ are coprime modulo $I$.

*In [I], we used the notation $\mathcal{FFP}^{\leq m}_\mathcal{A}$ to denote this category. It was pointed out by M. Yoshida that the strict inequality “$< m$” was more suitable in view of the meaning of the category, and we adopted the notation $\mathcal{FFP}^{\leq m}_\mathcal{A}$ in [II] and [R].
Thus the main “results” of [II], as well as Corollary 1.2 of [R], remain to be a “conjecture”, while Theorem 1.1 of [R] is correct as long as the category $\mathcal{FFP}_A^{<m}$ is defined by using a lift $\mathcal{O}_K \to A$ (Note that Corollary 1.2 follows from Theorem 1.1 only if the category $\mathcal{FFP}_A^{<m}$ is independent of the choice of such a lift.)

A large part of the “conjecture” (in the case where $A$ is of $p$-torsion) has been proved by Hattori [H] by using the theory of perfectoid spaces.

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