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Realizing Enveloping Algebras via Moduli Stacks

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Abstract: Let $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ denote the vector space of $\mathbb{Q}$-valued constructible functions on a given stack $\mathfrak{Obj}_{\mathcal{A}}$ for an abelian category $\mathcal{A}$. In [12], Joyce proved that $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is an associative $\mathbb{Q}$-algebra via the convolution multiplication and the subspace $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ of constructible functions supported on indecomposables is a Lie subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. In this paper, we extend Joyce's result to an exact category $\mathcal{A}$ and show that there is a subalgebra $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ isomorphic to the universal enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Moreover we construct a comultiplication on $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ and a degenerate form of Green's theorem. This refines Joyce's result, as well as results of [4].
Keywords: Hall algebra; stack; constructible set; universal enveloping algebra.

1. Introduction

Let Λ be a finite dimensional $\mathbb{C}$-algebra such that it is a representation-finite algebra, i.e., there are finitely many finite dimensional indecomposable Λ-modules up to isomorphism. Let $\mathcal{I}(\Lambda) = \{X_1, \ldots, X_n\}$ be a set of representatives. Let $\mathcal{P}(\Lambda)$ be a set of representatives for all isomorphism classes of Λ-modules. There is a free $\mathbb{Z}$-module $R(\Lambda)$ with a basis $\{u_X \mid X \in \mathcal{P}(\Lambda)\}$. Using the Euler characteristic, $\mathcal{P}(\Lambda)$ can be endowed with a multiplicative structure (see [24] and [15]). The multiplication is defined by

$$u_X \cdot u_Y = \sum_{A \in \mathcal{P}(\Lambda)} \chi(V(X, Y; A)) u_A,$$

where $V(X, Y; A) = \{0 \subseteq A_1 \subseteq A \mid A_1 \cong X, A/A_1 \cong Y\}$ and $\chi(V(X, Y; A))$ is the Euler characteristic of $V(X, Y; A)$. Thus $(R(\Lambda), +, \cdot)$ is a $\mathbb{Z}$-algebra with identity u_0. Let $L(\Lambda)$ be the submodule of $R(\Lambda)$ which is spanned by

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$\{u_X \mid X \in \mathcal{I}(\Lambda)\}$. It follows that $L(\Lambda)$ is a Lie subalgebra of $R(\Lambda)$ with the Lie bracket $[u_X, u_Y] = u_X \cdot u_Y - u_Y \cdot u_X$. Riedtmann studied the universal enveloping algebra of $L(\Lambda)$. Let $R(\Lambda)'$ be the subalgebra of $R(\Lambda)$ generated by $\{u_X \mid X \in \mathcal{I}(\Lambda)\}$. Riedtmann showed that $R(\Lambda)'$ is isomorphic to the universal enveloping algebra of $L(\Lambda)$. These results have been generalized in two ways.

Joyce generalized Riedtmann's work in the context of constructible functions (also stack functions) over moduli stacks. In [11], Joyce defined the Euler characteristics of constructible sets in $\mathbb{K}$-stacks, pushforwards and pullbacks for constructible functions, where $\mathbb{K}$ is an algebraically closed field. Let $\mathcal{A}$ be an abelian category and $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ the vector space of $\mathbb{Q}$-valued constructible functions on $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, where $\mathfrak{Obj}_{\mathcal{A}}$ is the moduli stack of objects in $\mathcal{A}$ and $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ the collection of isomorphism classes of objects in $\mathcal{A}$. Joyce proved that $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is an associative $\mathbb{Q}$-algebra. The algebra $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ can be viewed as a variant of the Ringel-Hall algebra. Let $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ be the subspace of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ satisfying the condition that $f([X]) \neq 0$ implies X is an indecomposable object in $\mathcal{A}$ for every $f \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Then $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ is shown to be a Lie subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ ([12, Theorem 4.9]). Let $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ be the subspace of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ such that

$$\mathrm{supp}(f) = \big\{[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \mid f([X]) \neq 0\big\}$$

is a finite set for every $f \in \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. Let

$$\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) = \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \cap \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}).$$

Assume that a conflation $X \to Y \to Z$ in $\mathcal{A}$ implies that the number of isomorphism classes of Y is finite for all $X, Z \in \mathrm{Obj}\,(A)$. With the assumption, Joyce proved that $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ is an associative algebra and $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ a Lie subalgebra of $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. It follows that $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ is isomorphic to the universal enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. Joyce defined a comultiplication on $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ and proved that $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ is a bialgebra.

In [4], the authors extended Riedtmann's results to algebras of representation-infinite type, i.e., the cardinality of isomorphism classes of indecomposable finite dimensional Λ-modules is infinite. Let $R(\Lambda)$ be the $\mathbb{Z}$-module spanned by $1_{\mathcal{O}}$, where $1_{\mathcal{O}}$ is the characteristic function over a constructible set of stratified Krull-Schmidt $\mathcal{O}$ (see [4, Section 3]). The subspace $L(\Lambda)$ of $R(\Lambda)$ is spanned by $1_{\mathcal{O}}$, where $\mathcal{O}$ are indecomposable constructible sets. The multiplication is defined by

$$1_{\mathcal{O}_1} \cdot 1_{\mathcal{O}_2}(X) = \chi(V(\mathcal{O}_1, \mathcal{O}_2; X)),$$

where X is a Λ-module. Then $R(\Lambda)$ is an associative algebra with identity 1_0 and $L(\Lambda)$ a Lie subalgebra of $R(\Lambda)$ with Lie bracket. The algebra $R(\Lambda) \otimes \mathbb{Q}$ is the universal enveloping algebra of $L(\Lambda) \otimes \mathbb{Q}$. The authors gave the degenerate form of Green's theorem and established the comultiplication of $R(\Lambda)$ in [4].

The goal of this paper is to explicitly construct the enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ by the methods in [4]. Let $\mathcal{A}$ be an exact category satisfying some properties. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$ and $\mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$ the automorphism group of $X \xrightarrow{f} Y \xrightarrow{g} Z$. The key idea in [4] is that $V(X, Y; L)$ has the same Euler characteristic as its fixed point set under the action of $\mathbb{C}^*$. In this paper, we consider exact categories instead of categories of modules. Then as a substitute of the action of $\mathbb{C}^*$, we analyze the action of a maximal torus of $\mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$ on $X \xrightarrow{f} Y \xrightarrow{g} Z$. The universal enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ can be endowed with a comultiplication structure (Definition 4.1). It is compatible with multiplication (Theorem 4.6). The compatibility can be viewed as the degenerate form of Green's theorem on Ringel-Hall algebras (see [5] or [22]).

The paper is organized as follows. In Section 2 we recall the basic concepts about stacks, constructible sets and constructible functions. In Section 3 we define the constructible sets of stratified Krull-Schmidt. We study the the subspace $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ generated by characteristic functions supported on constructible sets of stratified Krull-Schmidt. Then $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ provides a realization of the universal enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. In Section 4 we give the comultiplication Δ in $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ and prove that Δ is an algebra homomorphism.

2. Preliminaries

2.1. Constructible sets and constructible functions

From now on, let $\mathbb{K}$ be an algebraically closed field with characteristic zero. A good introduction to algebraic stacks and 2-categories is [6]. We recall the definitions of constructible sets and constructible functions on $\mathbb{K}$-stacks. These definitions are taken from [11].

Definition 2.1. Let $\mathcal{F}$ be a $\mathbb{K}$-stack. Let $\mathcal{F}(\mathbb{K})$ denote the set of 2-isomorphism classes $[x]$ where $x : \operatorname{Spec}\mathbb{K} \to \mathcal{F}$ are 1-morphisms. Every element of $\mathcal{F}(\mathbb{K})$ is called a geometric point (or $\mathbb{K}$-point) of $\mathcal{F}$. For $\mathbb{K}$-stacks $\mathcal{F}$ and $\mathcal{G}$, let $\phi : \mathcal{F} \to \mathcal{G}$ be a 1-morphism of $\mathbb{K}$-stacks. Then ϕ induces a map $\phi_* : \mathcal{F}(\mathbb{K}) \to \mathcal{G}(\mathbb{K})$ by $[x] \mapsto [\phi \circ x]$.

For any $[x] \in \mathcal{F}(\mathbb{K})$, let $\operatorname{Iso}_{\mathbb{K}}(x)$ denote the group of 2-isomorphisms $x \to x$ which is called a stabilizer group. For ease of notations, $\operatorname{Iso}_{\mathbb{K}}(x)$ is used to denote the group instead of $\operatorname{Iso}_{\mathbb{K}}([x])$. If $\operatorname{Iso}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{K}$-group for each $[x] \in \mathcal{F}(\mathbb{K})$, then we say $\mathcal{F}$ with affine geometric stabilizers. A morphism of algebraic $\mathbb{K}$-groups $\phi_x : \operatorname{Iso}_{\mathbb{K}}(x) \to \operatorname{Iso}_{\mathbb{K}}(\phi_*(x))$ is induced by $\phi : \mathcal{F} \to \mathcal{G}$ for each $[x] \in \mathcal{F}(\mathbb{K})$.

A subset $\mathcal{O} \subseteq \mathcal{F}(\mathbb{K})$ is called a constructible set if $\mathcal{O} = \amalg_{i=1}^{n} \mathcal{F}_i(\mathbb{K})$ for some $n \in \mathbb{N}^+$, where every $\mathcal{F}_i$ is a finite type algebraic $\mathbb{K}$-substack of $\mathcal{F}$. A subset $S \subseteq \mathcal{F}(\mathbb{K})$ is called a locally constructible set if $S \cap \mathcal{O}$ are constructible for all constructible subsets $\mathcal{O} \subseteq \mathcal{F}(\mathbb{K})$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are constructible sets, then $\mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2$ and $\mathcal{O}_1 \setminus \mathcal{O}_2$ are constructible sets by [11, Lemma 2.4].

Let $\Phi : \mathcal{F}(\mathbb{K}) \to \mathcal{G}(\mathbb{K})$ be a map. The set $\Gamma_\Phi = \{(x, \Phi(x)) \mid x \in \mathcal{F}(\mathbb{K})\}$ is called the graph of Φ. Recall that Φ is a pseudomorphism if $\Gamma_\Phi \bigcap (\mathcal{O} \times \mathcal{G}(\mathbb{K}))$ are constructible for all constructible subsets $\mathcal{O} \subseteq \mathcal{F}(\mathbb{K})$. By [11, Proposition 4.6], if $\phi : \mathcal{F} \to \mathcal{G}$ is a 1-morphism then ϕ_* is a pseudomorphism, $\Phi(\mathcal{O})$ and $\Phi^{-1}(y) \cap \mathcal{O}$ are constructible sets for all constructible subset $\mathcal{O} \subseteq \mathcal{F}(\mathbb{K})$ and $y \in \mathcal{G}(\mathbb{K})$. If Φ is a bijection and Φ^{-1} is also a pseudomorphism, we call Φ a pseudoisomorphism.

Then we will recall the definition of the naïve Euler characteristic of a constructible subset of $\mathcal{F}(\mathbb{K})$ in [11].

There is a useful result due to Rosenlicht [23].

Theorem 2.2. *Let G be an algebraic $\mathbb{K}$-group acting on a $\mathbb{K}$-variety X. There exist an open dense G-invariant subset $X_1 \subseteq X$ and a $\mathbb{K}$-variety Y such that there is a morphism of varieties $\phi : X_1 \to Y$ which induces a bijection form $X_1(\mathbb{K})/G$ to $Y(\mathbb{K})$.*

Let X be a separated $\mathbb{K}$-scheme of finite type, the Euler characteristic $\chi(X)$ of X is defined by

$$\chi(X) = \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbb{Q}_p} H^i_{\mathrm{cs}}(X, \mathbb{Q}_p),$$

where p is a prime number, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^r\mathbb{Z}$ is the ring of p-adic integers, $\mathbb{Q}_p$ is its field of fractions and $H^i_{\mathrm{cs}}(X, \mathbb{Q}_p)$ are the compactly-supported p-adic cohomology groups of X for $i \geq 0$.

The following properties of Euler characteristic follow [4] and [11].

Proposition 2.3. *Let X, Y be separated, finite type $\mathbb{K}$-schemes and $\varphi : X \to Y$ a morphism of schemes. Then:*

(1) If Z is a closed subscheme of X, then $\chi(X) = \chi(X \setminus Z) + \chi(Z)$.

(2) $\chi(X \times Y) = \chi(X) \times \chi(Y)$.

(3) Let X be a disjoint union of finitely many subschemes $X_1, \ldots, X_n$, we have

$$\chi(X) = \sum_{i=1}^{n} \chi(X_i).$$

(4) If φ is a locally trivial fibration with fibre F, then $\chi(X) = \chi(F) \cdot \chi(Y)$.

(5) $\chi(\mathbb{K}^n) = 1$, $\chi(\mathbb{KP}^n) = n + 1$ for all $n \geq 0$.

An algebraic $\mathbb{K}$-stack $\mathcal{F}$ is said to be stratified by global quotient stacks if $\mathcal{F}(\mathbb{K}) = \amalg_{i=1}^{s} \mathcal{F}_i(\mathbb{K})$ for finitely many locally closed substacks $\mathcal{F}_i$ where each $\mathcal{F}_i$ is 1-isomorphic to a quotient stack $[X_i/G_i]$, where X_i is an algebraic $\mathbb{K}$-variety and G_i a smooth connected linear algebraic $\mathbb{K}$-group acting on X_i. By [14, Propostion 3.5.9], if $\mathcal{F}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers, then $\mathcal{F}$ is stratified by global quotient stacks.

Let $\mathcal{F} = \amalg_{i=1}^{s} \mathcal{F}_i(\mathbb{K})$ where each $\mathcal{F}_i \cong [X_i/G_i]$ as above. By Theorem 2.2, there exists an open dense G_i-invariant subvariety X_{i1} of X_i for each i such that there exists a morphism of varieties $\phi_{i1} : X_{i1} \to Y_{i1}$, which induces a bijection between $X_{i1}(\mathbb{K})/G_i$ and $Y_{i1}(\mathbb{K})$. Then ϕ_{i1} induces a 1-morphism $\theta_{i1} : \mathcal{G}_{i1} \to Y_{i1}$, where $\mathcal{G}_{i1}$ is 1-isomorphic to $[X_{i1}/G_i]$. Note that

$$\dim(X_{i(j-1)} \setminus X_{ij}) < \dim X_{i(j-1)}$$

for $j = 1, \ldots, k_i$. Using Theorem 2.2 again, we get a stratification

$$\mathcal{F}(\mathbb{K}) = \amalg_{i=1}^{s} \amalg_{j=1}^{k_i} \mathcal{G}_{ij}(\mathbb{K})$$

for $s, k_i \in \mathbb{N}^+$, where $\mathcal{G}_{ij} \cong [X_{ij}/G_i]$ such that $\phi_{ij} : X_{ij} \to Y_{ij}$ is a morphism of $\mathbb{K}$-varieties and $\theta_{ij} : \mathcal{G}_{ij} \to Y_{ij}$ a 1-morphism induced by ϕ_{ij}. Let

$$Y = \amalg_{i=1}^{s} \amalg_{j=1}^{k_i} Y_{ij} \text{ and } \Theta = \amalg_{i=1}^{s} \amalg_{j=1}^{k_i} (\theta_{ij})_* : \mathcal{F}(\mathbb{K}) \to Y(\mathbb{K}).$$

Then Y is a a separated $\mathbb{K}$-scheme of finite type and Θ a pseudoisomorphism (see [11, Proposition 4.4 and Proposition 4.7]).

Definition 2.4. Let $\mathcal{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\mathcal{C} \subseteq \mathcal{F}(\mathbb{K})$ a constructible set. Then $\mathcal{C}$ is pseudoisomorphic to $Y(\mathbb{K})$, where Y is a separated $\mathbb{K}$-scheme of finite type by [11, Proposition 4.7]. The naïve Euler characteristic of $\mathcal{C}$ is defined by $\chi^{\mathrm{na}}(\mathcal{C}) = \chi(Y)$.

The following lemma is a generalization of Proposition 2.3 (4).

Lemma 2.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. If $\mathcal{C} \subseteq \mathcal{F}(\mathbb{K})$, $\mathcal{D} \subseteq \mathcal{G}(\mathbb{K})$ are constructible sets, and $\Phi : \mathcal{C} \to \mathcal{D}$ is a surjective pseudomorphism such that all fibers have the same naïve Euler characteristic χ, then $\chi^{\mathrm{na}}(\mathcal{C}) = \chi \cdot \chi^{\mathrm{na}}(\mathcal{D})$.*

Proof. Because $\mathcal{C}$, $\mathcal{D}$ are constructible sets, there exist separated finite type $\mathbb{K}$-schemes X, Y such that $\mathcal{C}$, $\mathcal{D}$ are pseudoisomorphic to $X(\mathbb{K})$, $Y(\mathbb{K})$ respectively. Therefore $\chi^{\mathrm{na}}(\mathcal{C}) = \chi(X)$, $\chi^{\mathrm{na}}(\mathcal{D}) = \chi(Y)$. Then Φ induces a surjective pseudomorphism between $X(\mathbb{K})$ and $Y(\mathbb{K})$, say $\phi : X(\mathbb{K}) \to Y(\mathbb{K})$. There exist two projective morphisms $\pi_1 : \Gamma_\phi \to X(\mathbb{K})$ and $\pi_2 : \Gamma_\phi \to Y(\mathbb{K})$. Note that π_1 is also a pseudoisomorphism, that is $\chi^{\mathrm{na}}(\Gamma_\phi) = \chi(X)$, and all fibres of π_2 have the same naïve Euler characteristic χ. Then $\chi^{\mathrm{na}}(\Gamma_\phi) = \chi \cdot \chi(Y)$. Hence $\chi(X) = \chi \cdot \chi(Y)$. We finish the proof. □

Definition 2.6. A function $f : \mathcal{F}(\mathbb{K}) \to \mathbb{Q}$ is called a constructible function on $\mathcal{F}(\mathbb{K})$ if the codomain of f is a finite set and $f^{-1}(a)$ is a constructible subset of $\mathcal{F}(\mathbb{K})$ for each $a \in f(\mathcal{F}(\mathbb{K})) \setminus \{0\}$. Let $\mathrm{CF}(\mathcal{F})$ denote the $\mathbb{Q}$-vector space of all $\mathbb{Q}$-valued constructible functions on $\mathcal{F}(\mathbb{K})$.

Let $S \subseteq \mathcal{F}(\mathbb{K})$ be a locally constructible set. The integral of f on S is

$$\int_{x \in S} f(x) = \sum_{a \in f(S) \setminus \{0\}} a\chi^{\mathrm{na}}(f^{-1}(a) \cap S)$$

for each $f \in \mathrm{CF}(\mathcal{F})$.

We recall the pushforwards and pullbacks of constructible functions due to Joyce [11].

Definition 2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers and $\phi : \mathcal{F} \to \mathcal{G}$ a 1-morphism. For each $f \in \mathrm{CF}(\mathcal{F})$, the naïve pushforward $\phi_!^{\mathrm{na}}(f):\mathcal{F}(\mathbb{K})\to\mathbb{Q}$ of f is

$$\phi_!^{\mathrm{na}}(f)(t)=\sum_{a\in f(\phi_*^{-1}(t))\setminus\{0\}} a\chi^{\mathrm{na}}(f^{-1}(a)\cap\phi_*^{-1}(t))$$

for each $t\in\mathcal{G}(\mathbb{K})$. Then $\phi_!^{\mathrm{na}}(f)$ is a constructible function for each $f\in\mathrm{CF}(\mathcal{F})$ by [11, Theorem 4.9].

Similarly, if $\Phi:\mathcal{F}(\mathbb{K})\to\mathcal{G}(\mathbb{K})$ is a pseudomorphism, the naïve pushforward $\Phi_!^{\mathrm{na}}(f):\mathcal{F}(\mathbb{K})\to\mathbb{Q}$ of $f\in\mathrm{CF}(\mathcal{F})$ is defined by

$$\Phi_!^{\mathrm{na}}(f)(t)=\sum_{a\in f(\Phi^{-1}(t))\setminus\{0\}} a\chi^{\mathrm{na}}(f^{-1}(a)\cap\Phi^{-1}(t))$$

for $t\in\mathcal{G}(\mathbb{K})$. Joyce proved that there is a linear map $\Phi_!^{na}:\mathrm{CF}(\mathcal{F})\to\mathrm{CF}(\mathcal{G})$ and in particular, $\Phi_!^{\mathrm{na}}(f)\in\mathrm{CF}(\mathcal{G})$ [11, Theorem 4.9]. We often apply this result by studying the constructibility of the function $\Phi_!^{\mathrm{na}}(1_{\mathcal{F}(\mathbb{K})})$. The constructibility of the function implies that the set $\{\chi(\Phi^{-1}(t))\mid t\in\mathcal{G}(\mathbb{K})\}$ is a finite set.

If $\phi:\mathcal{F}\to\mathcal{G}$ is a 1-morphism, then we have a long exact sequence of groups

$$1\longrightarrow\mathrm{Ker}(\phi_*)\longrightarrow\mathrm{Iso}_{\mathbb{K}}(x)\xrightarrow{\phi_*}\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))\longrightarrow\mathrm{Coker}(\phi_*)\longrightarrow 1$$

for each $x\in\mathcal{F}(\mathbb{K})$. Note that $\mathrm{Ker}(\phi_*)$ is an affine algebraic $\mathbb{K}$-group and $\mathrm{Coker}(\phi_*)$ is a quasi-projective $\mathbb{K}$-variety. Assume that $\chi(\mathrm{Ker}(\phi_*))\neq 0$, we can define a function $m_\phi:\mathcal{F}(\mathbb{K})\to\mathbb{Q}$ by

$$m_\phi(x)=\frac{\chi(\mathrm{Coker}(\phi_*))}{\chi(\mathrm{Ker}(\phi_*))}$$

for each $x\in\mathcal{F}(\mathbb{K})$. In particular, if ϕ is representable, i.e., for $U\in\mathrm{Sch}_{\mathbb{K}}$, $X\in\mathrm{Obj}(\mathcal{F}(U))$, the map $\phi(U):\mathrm{End}_{\mathcal{F}(U)}(X)\to\mathrm{End}_{\mathcal{G}(U)}(\phi(U)(X))$ is injective, then $\mathrm{Ker}(\phi_*)=\{1\}$ and $m_\phi(x)=\chi(\mathrm{Coker}(\phi_*))$. Here $\mathrm{Sch}_{\mathbb{K}}$ is the 2-category of $\mathbb{K}$-schemes (see Section 2.2 for more details).

For each $f\in\mathrm{CF}(\mathcal{F})$, the pushforward $\phi_!(f):\mathcal{G}(\mathbb{K})\to\mathbb{Q}$ of f is defined by

$$\phi_!(f)=\phi_!^{na}(f\cdot m_\phi),$$

where $(f\cdot m_\phi)(x)=f(x)m_\phi(x)$ for $x\in\mathcal{F}(\mathbb{K})$. Note that $\phi_!(f)\in\mathrm{CF}(\mathcal{G})$ (see [11]).

If ϕ is a 1-morphism of finite type, then $\phi_*^{-1}(\mathcal{D})\subset\mathcal{F}(\mathbb{K})$ is a constructible set for each constructible subset $\mathcal{D}$ of $\mathcal{G}(\mathbb{K})$. Then $g\circ\phi_*\in\mathrm{CF}(\mathcal{F})$ for $g\in$

$\mathrm{CF}(\mathcal{G})$. Recall that the pullback $\phi^* : \mathrm{CF}(\mathcal{G}) \to \mathrm{CF}(\mathcal{F})$ of ϕ is defined by $\phi^*(g) = g \circ \phi_*$ and it is linear.

2.2. Stacks of objects and conflations in $\mathcal{A}$

From now on, let $(\mathcal{A}, \mathcal{S})$ be a Krull-Schmidt exact $\mathbb{K}$-category (see A.1). For simplicity, we write $\mathcal{A}$ instead of $(\mathcal{A}, \mathcal{S})$. Note that $\mathcal{A}$ is idempotent complete (see A.2).

The isomorphism classes of $X \in \mathrm{Obj}\,(\mathcal{A})$ and conflations $X \overset{i}{\to} Y \overset{d}{\to} Z$ in $\mathcal{A}$ are denoted by $[X]$ and $[X \overset{i}{\to} Y \overset{d}{\to} Z]$ (or $[(X, Y, Z, i, d)]$), respectively. Two conflations $X \overset{i}{\to} Y \overset{d}{\to} Z$ and $A \overset{f}{\to} B \overset{g}{\to} C$ are isomorphic if there exist isomorphisms $a : X \to A$, $b : Y \to B$ and $c : Z \to C$ in $\mathcal{A}$ such that the following diagram is communicative

(1)
$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{d} & Z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

The morphism (a, b, c) is called an isomorphism of conflations in $\mathcal{A}$.

Assumption 2.8. Assume that $\dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\dim_{\mathbb{K}} \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ are finite for all $X, Y \in \mathrm{Obj}(\mathcal{A})$. Let $K(\mathcal{A})$ denote the quotient group of the Grothendieck group $K_0(\mathcal{A})$ such that $\widehat{[X]} = 0$ in $K(\mathcal{A})$ implies that X is a zero object in $\mathcal{A}$, where $\widehat{[X]}$ denotes the image of X in $K(\mathcal{A})$.

The following 2-categories are defined in [10].

Let $\mathrm{Sch}_{\mathbb{K}}$ be a 2-category of $\mathbb{K}$-schemes such that objects are $\mathbb{K}$-schemes, 1-morphisms morphisms of schemes and 2-morphisms only the natural transformations id_f for all 1-morphisms f. Let (exactcat) denote the 2-category of all exact categories with 1-morphisms exact functors of exact categories and 2-morphisms natural transformations between the exact functors. If all morphisms of a category are isomorphisms, then the category is called a groupoid. Let (groupoids) be the 2-category with objects groupoids, 1-morphisms functors of groupoids and 2-morphisms natural transformations (see also [10, Definition 2.8]).

In [10, Section 7.1], Joyce defined a stack $\mathcal{F}_{\mathcal{A}} : \mathrm{Sch}_{\mathbb{K}} :\to$ (exactcat) associated to the exact category $\mathcal{A}$ (the original definition is for abelian category, it can be extended to exact categories directly), where $\mathcal{F}_{\mathcal{A}}$ is a contravariant 2-functor and satisfies the condition $\mathcal{F}_{\mathcal{A}}(\mathrm{Spec}(\mathbb{K})) = \mathcal{A}$. Applying $\mathcal{F}_{\mathcal{A}}$, he defined two moduli stacks

$$\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}} : \mathrm{Sch}_{\mathbb{K}} \to (\text{groupoids})$$

which are contravatiant 2-functors ([10, Definition 7.2]). The 2-functor

$$\mathfrak{Obj}_{\mathcal{A}} = F \circ \mathcal{F}_{\mathcal{A}},$$

where $F : (\text{exactcat}) \to (\text{groupoids})$ is a forgetful 2-functor as follows. For an exact category G, $F(G)$ is a groupoid such that $\mathrm{Obj}(F(G)) = \mathrm{Obj}(G)$ and morphisms are isomorphisms in G. For $U \in \mathrm{Sch}_{\mathbb{K}}$, a category $\mathfrak{Exact}_{\mathcal{A}}(U)$ is a groupoid whose objects are conflations in $\mathcal{F}_{\mathcal{A}}(U)$ and morphisms isomorphisms of conflations in $\mathcal{F}_{\mathcal{A}}(U)$.

Let $\eta : U \to V$ and $\theta : V \to W$ be morphisms of schemes in $\mathrm{Sch}_{\mathbb{K}}$. Obviously, the functors $\mathfrak{Obj}_{\mathcal{A}}(\eta) : \mathfrak{Obj}_{\mathcal{A}}(V) \to \mathfrak{Obj}_{\mathcal{A}}(U)$ and $\mathfrak{Exact}_{\mathcal{A}}(\eta) : \mathfrak{Exact}_{\mathcal{A}}(V) \to \mathfrak{Exact}_{\mathcal{A}}(U)$ are induced by $\mathcal{F}_{\mathcal{A}}(\eta) : \mathcal{F}_{\mathcal{A}}(V) \to \mathcal{F}_{\mathcal{A}}(U)$. The natural transformations $\epsilon_{\theta,\eta} : \mathfrak{Obj}_{\mathcal{A}}(\eta) \circ \mathfrak{Obj}_{\mathcal{A}}(\theta) \to \mathfrak{Obj}_{\mathcal{A}}(\theta \circ \eta)$ and $\epsilon_{\theta,\eta} : \mathfrak{Exact}_{\mathcal{A}}(\eta) \circ \mathfrak{Exact}_{\mathcal{A}}(\theta) \to \mathfrak{Exact}_{\mathcal{A}}(\theta \circ \eta)$ are also induced by $\epsilon_{\theta,\eta} : \mathcal{F}_{\mathcal{A}}(\eta) \circ \mathcal{F}_{\mathcal{A}}(\theta) \to \mathcal{F}_{\mathcal{A}}(\theta \circ \eta)$.

Let

$$K'(\mathcal{A}) = \{\widehat{[X]} \in K(\mathcal{A}) \mid X \in \mathrm{Obj}(\mathcal{A})\} \subset K(\mathcal{A}).$$

For each $\alpha \in K'(\mathcal{A})$, Joyce defined $\mathfrak{Obj}^{\alpha}_{\mathcal{A}} : \mathrm{Sch}_{\mathbb{K}} \to (\text{groupoids})$ which is a substack of $\mathfrak{Obj}_{\mathcal{A}}$ in [10, Definition 7.4]. For each $U \in \mathrm{Sch}_{\mathbb{K}}$, $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(U)$ is a full subcategory of $\mathfrak{Obj}_{\mathcal{A}}(U)$. For each object X in $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(U)$, the image of $\mathfrak{Obj}_{\mathcal{A}}(f)(X)$ in $K(\mathcal{A})$ is α for each morphism $f : \mathrm{Spec}(\mathbb{K}) \to U$.

Let $\eta : U \to V$ and $\theta : V \to W$ be morphisms in $\mathrm{Sch}_{\mathbb{K}}$. The functor

$$\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\eta) : \mathfrak{Obj}^{\alpha}_{\mathcal{A}}(V) \to \mathfrak{Obj}^{\alpha}_{\mathcal{A}}(U)$$

is defined by restriction from $\mathfrak{Obj}_{\mathcal{A}}(\eta) : \mathfrak{Obj}_{\mathcal{A}}(V) \to \mathfrak{Obj}_{\mathcal{A}}(U)$. The natural transformation $\epsilon_{\theta,\eta} : \mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\eta) \circ \mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\theta) \to \mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\theta \circ \eta)$ is restricted from $\epsilon_{\theta,\eta} : \mathfrak{Obj}_{\mathcal{A}}(\eta) \circ \mathfrak{Obj}_{\mathcal{A}}(\theta) \to \mathfrak{Obj}_{\mathcal{A}}(\theta \circ \eta)$.

For $\alpha, \beta, \gamma \in K'(\mathcal{A})$ and $\beta = \alpha + \gamma$, $\mathfrak{Exact}^{\alpha,\beta,\gamma}_{\mathcal{A}} : \mathrm{Sch}_{\mathbb{K}} \to (\text{groupoids})$ is defined as follows. For $U \in \mathrm{Sch}_{\mathbb{K}}$, $\mathfrak{Exact}^{\alpha,\beta,\gamma}_{\mathcal{A}}(U)$ is a full subcategory of $\mathfrak{Exact}_{\mathcal{A}}(U)$. The objects of $\mathfrak{Exact}^{\alpha,\beta,\gamma}_{\mathcal{A}}(U)$ are conflations

$$X \xrightarrow{i} Y \xrightarrow{d} Z \in \mathrm{Obj}(\mathfrak{Exact}_{\mathcal{A}}(U)),$$

where $X \in \mathrm{Obj}(\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(U))$, $Y \in \mathrm{Obj}(\mathfrak{Obj}^{\beta}_{\mathcal{A}}(U))$ and $Z \in \mathrm{Obj}(\mathfrak{Obj}^{\gamma}_{\mathcal{A}}(U))$. Similarly, the morphism $\mathfrak{Exact}^{\alpha,\beta,\gamma}_{\mathcal{A}}(\eta)$ and natural transformation $\epsilon_{\theta,\eta}$ are defined by restriction.

Let $\mathcal{TS}$ be a substack of $\mathfrak{Exact}_{\mathcal{A}} \times \mathfrak{Exact}_{\mathcal{A}}$. For each $U \in \mathrm{Sch}_{\mathbb{K}}$, $\mathcal{TS}(U)$ is a full subcategory of $\mathfrak{Exact}_{\mathcal{A}} \times \mathfrak{Exact}_{\mathcal{A}}(U)$ whose objects are $(X \xrightarrow{f} L \xrightarrow{g} Y, L \xrightarrow{l} M \xrightarrow{m} Z)$,

$$\begin{array}{ccccc} X & \xrightarrow{f} & L & \xrightarrow{g} & Y \\ & & \downarrow{\scriptstyle l} & & \\ & & M & & \\ & & \downarrow{\scriptstyle m} & & \\ & & Z & & \end{array}$$

where $X \xrightarrow{f} L \xrightarrow{g} Y$ and $L \xrightarrow{l} M \xrightarrow{m} Z$ are objects in $\mathfrak{Exact}_{\mathcal{A}}(U)$. The morphisms of $\mathcal{TS}(U)$ are (x, a, y, b, z), where $x : X \to X'$, $a : L \to L'$, $y : Y \to Y'$, $b : M \to M'$ and $z : Z \to Z'$ are isomorphisms, such that the following diagrams are commutative

$$\begin{array}{ccccc} X & \xrightarrow{f} & L & \xrightarrow{g} & Y \\ {\scriptstyle x}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow{f'} & L' & \xrightarrow{g'} & Y' \end{array}$$

$$\begin{array}{ccccc} L & \xrightarrow{l} & M & \xrightarrow{m} & Z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle z} \\ L' & \xrightarrow{l'} & M' & \xrightarrow{m'} & Z' \end{array}$$

The morphism $\mathcal{TS}(\eta)$ and natural transformation $\epsilon_{\theta,\eta}$ are defined in a natural way.

The following theorem is taking from [10, Theorem 7.5].

Theorem 2.9. *The 2-functors $\mathfrak{Obj}_{\mathcal{A}}$, $\mathfrak{Exact}_{\mathcal{A}}$ are $\mathbb{K}$-stacks, and $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}$, $\mathfrak{Exact}^{\alpha,\beta,\gamma}_{\mathcal{A}}$ are open and closed $\mathbb{K}$-substacks of them respectively. There are disjoint unions*

$$\mathfrak{Obj}_{\mathcal{A}} = \amalg_{\alpha \in K'(\mathcal{A})} \mathfrak{Obj}^{\alpha}_{\mathcal{A}}, \mathfrak{Exact}_{\mathcal{A}} = \amalg_{\substack{\alpha,\beta,\gamma \in K'(\mathcal{A}) \\ \beta = \alpha+\gamma}} \mathfrak{Exact}^{\alpha,\beta,\gamma}_{\mathcal{A}}.$$

Assume that $\mathfrak{Obj}_{\mathcal{A}}$ and $\mathfrak{Exact}_{\mathcal{A}}$ are locally of finite type algebraic $\mathbb{K}$-stacks with affine algebraic stabilizers. Recall that $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ and $\mathfrak{Exact}_{\mathcal{A}}(\mathbb{K})$ are the collection of isomorphism classes of objects in $\mathcal{A}$ and the collection of isomorphism classes of conflations in $\mathcal{A}$, respectively. For each $\alpha \in K'(\mathcal{A})$,

$\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ is the collection of isomorphism classes of $X \in \mathrm{Obj}(\mathcal{A})$ such that $\widehat{[X]} = \alpha$ (see [12, Section 3.2]).

Example 2.10. Let $Q = (Q_0, Q_1, s, t)$ be a finite connected quiver, where $Q_0 = \{1, \ldots, n\}$ is the set of vertices, Q_1 is the set of arrows and $s : Q_1 \to Q_0$ (resp. $t : Q_1 \to Q_0$) is a map such that $s(\rho)$ (resp. $t(\rho)$) is the source (resp. target) of ρ for $\rho \in Q_1$. Let $A = \mathbb{C}Q$ be the path algebra of Q and mod-A denote the category of all finite dimensional right A-modules.

Let $\underline{d} = (d_j)_{j\in Q_0}$ for all $d_j \in \mathbb{N}$. There is an affine variety

$$\mathrm{Rep}(Q, \underline{d}) = \bigoplus_{\rho\in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(\rho)}}, \mathbb{C}^{d_{t(\rho)}}).$$

For each $x = (x_\rho)_{\rho\in Q_1} \in \mathrm{Rep}(Q, \underline{d})$, there is a $\mathbb{C}$-linear representation $M(x) = (\mathbb{C}^{d_j}, x_\rho)_{j\in Q_0,\rho\in Q_1}$ of Q. Let $\mathrm{rep}(Q)$ denote the category of finite dimensional $\mathbb{C}$-linear representations of Q. Recall that $\mathrm{rep}(Q) \cong$ mod-$\mathcal{A}$. We identify $\mathrm{rep}(Q)$ with mod-$\mathcal{A}$. The linear algebraic group

$$\mathrm{GL}(\underline{d}) = \prod_{j\in Q_0} \mathrm{GL}(d_j, \mathbb{C})$$

acts on $\mathrm{Rep}(Q, \underline{d})$ by $g.x = (g_{t(\rho)} x_\rho g^{-1}_{s(\rho)})_{\rho\in Q_1}$ for $g = (g_j)_{j\in Q_0} \in \mathrm{GL}(\underline{d})$.

A complex $M^\bullet = (M^{(i)}, \partial^i)$, where $M^{(i)} \in \mathrm{Obj}(\text{mod-}\mathcal{A})$ and $\partial^{i+1}\partial^i = 0$, is bounded if there exist some positive integers n_0 and n_1 such that $M^{(i)} = 0$ for $i \le -n_0$ or $i \ge n_1$. Let $\underline{\dim} M^{(i)} = \underline{d}^{(i)}$ be the dimension vector of $M^{(i)}$ for each $i \in \mathbb{Z}$. The vector sequence $(\underline{d}^{(i)})_{i\in\mathbb{Z}}$ of $M^\bullet$ is denoted by $\underline{\mathbf{ds}}(M^\bullet)$.

Let $\mathcal{C}(Q, \underline{\mathbf{d}})$ denote the affine variety consisting of all complexes $M^\bullet$ with $\underline{\mathbf{ds}}(M^\bullet) = \underline{\mathbf{d}}$. The group $G(\underline{\mathbf{d}}) = \prod_{i\in\mathbb{Z}} \mathrm{GL}(\underline{d}^{(i)})$ is a linear algebraic group acting on $\mathcal{C}^b(Q, \underline{\mathbf{d}})$. The action is induced by the actions of $\mathrm{GL}(\underline{d}^{(i)})$ on $\mathrm{Rep}(Q, \underline{d}^{(i)})$ for all $i \in \mathbb{Z}$, that is

$$(g^{(i)})_i.(x^{(i)}, \partial^i)_i = (g^{(i)}.x^{(i)}, g^{(i+1)}\partial^i (g^{(i)})^{-1})_i.$$

Let $\{P_1, \ldots, P_n\}$ be a set of representatives for all isomorphism classes of finite dimensional indecomposable projective A-modules. A complex $P^\bullet =$

$$\ldots \to P^{(i-1)} \xrightarrow{\partial^{i-1}} P^{(i)} \xrightarrow{\partial^i} P^{(i+1)} \to \ldots$$

is projective if $P^{(i)} \cong \bigoplus_{j=1}^{n} m_j^{(i)} P_j$ for $m_j^{(i)} \in \mathbb{N}$ and $i \in \mathbb{Z}$. Let

$$\underline{e}(P^{(i)}) = \underline{m}^{(i)} = (m_1^{(i)}, \ldots, m_n^{(i)})$$

be a vector corresponding to $P^{(i)}$. By the Krull-Schmidt Theorem, $\underline{e}(P^{(i)})$ is unique. The dimension vector of $P^\bullet$ can be defined by

$$\underline{\mathbf{dim}}(P^\bullet) = (\ldots, \underline{m}^{(i-1)}, \underline{m}^{(i)}, \underline{m}^{(i+1)}, \ldots).$$

A dimension vector $\underline{\mathbf{dim}}(P^\bullet)$ is bounded if $P^\bullet$ is bounded.

let $\underline{\mathbf{m}} = (\underline{m}^{(i)})_{i\in\mathbb{Z}}$ be a bounded dimension vector and $\underline{\mathbf{d}}(\underline{\mathbf{m}}) = (\underline{d}^{(i)})_{i\in\mathbb{Z}}$ be the vector sequence of a complex whose dimension vector is $\underline{\mathbf{m}}$. Let $\mathcal{P}^b(Q, \underline{\mathbf{m}})$ be the set of all bounded project complexes $P^\bullet$ with $\underline{\mathbf{dim}}(P^\bullet) = \underline{\mathbf{m}}$ and $\underline{\mathbf{ds}}(P^\bullet) = \underline{\mathbf{d}}(\underline{\mathbf{m}})$. Note that $\mathcal{P}^b(Q, \underline{\mathbf{m}})$ is a locally closed subset of $\mathcal{C}^b(Q, \underline{\mathbf{d}}(\underline{\mathbf{m}}))$. An action of $G(\underline{\mathbf{d}}(\underline{\mathbf{m}}))$ on the variety $\mathcal{P}^b(Q, \underline{\mathbf{m}})$ is induced by the action of $G(\underline{\mathbf{d}}(\underline{\mathbf{m}}))$ on $\mathcal{C}^b(Q, \underline{\mathbf{d}}(\underline{\mathbf{m}}))$.

Let $\mathcal{P}^b(Q)$ denote the exact category with objects bounded project complexes and morphisms $\phi : P^\bullet \to Q^\bullet$ morphisms between bounded projective complexes. The Grothendieck group

$$K_0(\mathcal{P}^b(Q)) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}^n_{(i)},$$

where $\mathbb{Z}^n_{(i)} = \mathbb{Z}^n$. Note that $K(\mathcal{P}^b(Q)) = K_0(\mathcal{P}^b(Q))$ and

$$K'(\mathcal{P}^b(Q)) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{N}^n_{(i)},$$

where $\mathbb{N}^n_{(i)} = \mathbb{N}^n$.

Joyce defined $\mathcal{F}_{\text{mod}-\mathbb{K}Q}$ in [10, Example 10.5]. Similarly, for each $U \in \text{Sch}_\mathbb{K}$, we define $\mathcal{F}_{\mathcal{P}^b(Q)}(U)$ to be the category as follows.

The objects of $\mathcal{F}_{\mathcal{P}^b(Q)}(U)$ are complexes of sheaves $P^\bullet = (P^{(i)}, \partial^i)_{i\in\mathbb{Z}}$, where $P^{(i)} = (\bigoplus_{j\in Q_0} X_j^{(i)}, x^i)$ and $\partial^{i+1}\partial^i = 0$. The data $X_j^{(i)}$ are locally free sheaves of finite rank on U and $x^i = (x^i_\rho)_{\rho\in Q_1}$, where $x^i_\rho : X^{(i)}_{s(\rho)} \to X^{(i)}_{t(\rho)}$ are morphisms of sheaves, such that $P^{(i)} = (\bigoplus_{j\in Q_0} X_j^{(i)}, x^i)$ are projective $\mathbb{C}Q$-modules for all $i \in \mathbb{Z}$. The morphisms of $\mathcal{F}_{\mathcal{P}^b(Q)}(U)$ are morphisms of complexes $\phi^\bullet : (P^{(i)}, \partial^i) \to (Q^{(i)}, d^i)$, where $Q^{(i)} = (\bigoplus_{j\in Q_0} Y_j^{(i)}, y^i)$ and $\phi^\bullet$

is a sequence of morphisms

$$(\phi^i : P^{(i)} \to Q^{(i)})_{i\in\mathbb{Z}}$$

with $\phi^i = (\phi^i_j : X^{(i)}_j \to Y^{(i)}_j)_{j\in Q_0}$ such that $\phi^{i+1}\partial^i = d^i\phi^i$ and $\phi^i_{t(\rho)}x^i_\rho = y^i_\rho\phi^i_{s(\rho)}$ for all $i \in \mathbb{Z}$ and $\rho \in Q_1$. It is easy to see that $\mathcal{F}_{\mathcal{P}^b(Q)}(U)$ is an exact category.

Let $\eta : U \to V$ be a morphism in $\mathrm{Sch}_{\mathbb{K}}$. A functor

$$\mathcal{F}_{\mathcal{P}^b(Q)}(\eta) : \mathcal{F}_{\mathcal{P}^b(Q)}(V) \to \mathcal{F}_{\mathcal{P}^b(Q)}(U)$$

is defined as follows. If $(P^{(i)}, \partial^i)_{i\in\mathbb{Z}} \in \mathrm{Obj}(\mathcal{F}_{\mathcal{P}^b(Q)}(V))$,

$$\mathcal{F}_{\mathcal{P}^b(Q)}(\eta)(P^{(i)}, \partial^i)_{i\in\mathbb{Z}} = (\eta^*(P^{(i)}), \eta^*(\partial^i))_{i\in\mathbb{Z}}$$

for $\eta^*(P^{(i)}) = \big(\bigoplus_{j\in Q_0}\eta^*(X^{(i)}_j), (\eta^*(x^i_\rho))_{\rho\in Q_1}\big)$, where $\eta^*(X^{(i)}_j)$ are the inverse images of $X^{(i)}_j$ by the morphism η, $\eta^*(\partial^i) : \eta^*(P^{(i)}) \to \eta^*(P^{(i+1)})$ with $\eta^*(\partial^{i+1})\eta^*(\partial^i) = 0$ for $i \in \mathbb{Z}$ and

$$\eta^*(x^i_\rho) : \eta^*(X^{(i)}_{s(\rho)}) \to \eta^*(X^{(i)}_{t(\rho)})$$

for $\rho \in Q_1$ are pullbacks of morphisms between inverse images. For a morphism $\phi^\bullet : (P^{(i)}, \partial^i) \to (Q^{(i)}, d^i)$ in $\mathcal{F}_{\mathcal{P}^b(Q)}(V)$, the morphism

$$\mathcal{F}_{\mathcal{P}^b(Q)}(\eta)(\phi^\bullet) : \big(\eta^*(P^\bullet), \eta^*(\partial^i)\big) \to \big(\eta^*(Q^\bullet), \eta^*(d^i)\big)$$

is a sequence of morphisms

$$\Big(\eta^*(\phi^i) : \big(\bigoplus_{j\in Q_0}\eta^*(X^{(i)}_j), (\eta^*(x^i_\rho))_\rho\big) \to \big(\bigoplus_{j\in Q_0}\eta^*(Y^{(i)}_j), (\eta^*(y^i_\rho))_\rho\big)\Big)_{i\in\mathbb{Z}},$$

with $\eta^*(\phi^{i+1})\eta^*(\partial^i) = \eta^*(d^i)\eta^*(\phi^i)$, where $\eta^*(d^i)$ are pullbacks of morphisms between inverse images which satisfy $\eta^*(d^{i+1})\eta^*(d^i) = 0$, and

$$\eta^*(Q^\bullet) = \Big(\bigoplus_{j\in Q_0}\eta^*(Y^{(i)}_j), (\eta^*(y^i_\rho))_{\rho\in Q_1}\Big)_{i\in\mathbb{Z}}$$

such that the pullbacks

$$\eta^*(\phi^i_j) : \eta^*(X^{(i)}_j) \to \eta^*(Y^{(i)}_j)$$

satisfy $\eta^*(\phi^i_{t(\rho)})\eta^*(x^i_\rho) = \eta^*(y^i_\rho)\eta^*(\phi^i_{s(\rho)})$. Because locally free sheaves are flat, $\mathcal{F}_{\mathcal{P}^b(Q)}(\eta)(\phi^\bullet)$ is an exact functor.

Let $\eta : U \to V$ and $\theta : V \to W$ be morphisms in $\mathrm{Sch}_{\mathbb{K}}$. As in [10, Example 9.1], for each $P^\bullet \in \mathrm{Obj}(\mathcal{F}_{\mathcal{P}^b(Q)}(W))$, there is a canonical isomorphism $\epsilon_{\theta,\eta}(P^\bullet) : \mathcal{F}_{\mathcal{P}^b(Q)}(\eta) \circ \mathcal{F}_{\mathcal{P}^b(Q)}(\theta)(P^\bullet) \to \mathcal{F}_{\mathcal{P}^b(Q)}(\theta \circ \eta)(P^\bullet)$. We get a 2-isomorphism of functors

$$\epsilon_{\theta,\eta} : \mathcal{F}_{\mathcal{P}^b(Q)}(\eta) \circ \mathcal{F}_{\mathcal{P}^b(Q)}(\theta) \to \mathcal{F}_{\mathcal{P}^b(Q)}(\theta \circ \eta)$$

by the canonical isomorphisms. Thus we have the 2-functor $\mathcal{F}_{\mathcal{P}^b(Q)}$.

The set $\mathfrak{Obj}_{\mathcal{P}^b(Q)}(\mathbb{C})$ consists of all isomorphism classes of complexes in $\mathcal{P}^b(Q)$.

As in [10, Definition 7.7] and [12, Section 3.2], we have the following 1-morphisms

$$\pi_l : \mathfrak{Exact}_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$$

which induces a map $(\pi_l)_* : \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \to \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ defined by $[X \xrightarrow{i} Y \xrightarrow{d} Z] \mapsto [X]$;

$$\pi_m : \mathfrak{Exact}_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$$

such that the induced map $(\pi_m)_* : \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \to \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ maps $[X \xrightarrow{i} Y \xrightarrow{d} Z]$ to $[Y]$;

$$\pi_r : \mathfrak{Exact}_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$$

inducing the map $(\pi_r)_* : \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \to \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ by $[X \xrightarrow{i} Y \xrightarrow{d} Z] \mapsto [Z]$.

The map $\pi_{l*} \times \pi_{r*} : \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \to \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ is defined by $(\pi_{l*} \times \pi_{r*})([X \xrightarrow{i} Y \xrightarrow{d} Z]) = ([X], [Z])$. Note that $(\pi_l \times \pi_r)_* = \pi_{l*} \times \pi_{r*}$.

3. Hall Algebras

3.1. Constructible sets of stratified Krull-Schmidt

These definitions are related to [4].

Definition 3.1. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, the direct sum of $\mathcal{O}_1$ and $\mathcal{O}_2$ is

$$\mathcal{O}_1 \oplus \mathcal{O}_2 = \big\{[X_1 \oplus X_2] \mid [X_1] \in \mathcal{O}_1, [X_2] \in \mathcal{O}_2 \text{ and } X_1, X_2 \in \mathrm{Obj}(\mathcal{A})\big\}.$$

Let $n\mathcal{O}$ denote the direct sum of n copies of $\mathcal{O}$ for $n \in \mathbb{N}^+$ and $0\mathcal{O} = \{[0]\}$. Similarly, let nX denote the direct sum of n copies of $X \in \mathrm{Obj}(\mathcal{A})$. A constructible subset $\mathcal{O}$ of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ is called indecomposable if $X \in \mathrm{Obj}(\mathcal{A})$ is indecomposable and $X \ncong 0$ for every $[X] \in \mathcal{O}$.

A constructible set $\mathcal{O}$ is called to be of Krull-Schmidt if

$$\mathcal{O} = n_1\mathcal{O}_1 \oplus n_2\mathcal{O}_2 \oplus \ldots \oplus n_k\mathcal{O}_k,$$

where $\mathcal{O}_i$ are indecomposable constructible sets and $n_i \in \mathbb{N}$ for $i = 1, \ldots, k$. If a constructible set $\mathcal{Q} = \amalg_{i=1}^n \mathcal{Q}_i$, where $\mathcal{Q}_i$ are constructible sets of Krull-Schmidt for $1 \le i \le n$, namely $\mathcal{Q}$ is a disjoint union of finitely many constructible sets of Krull-Schmidt, then $\mathcal{Q}$ is said to be a constructible set of stratified Krull-Schmidt.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two indecomposable constructible sets. If $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset$ and $\mathcal{O}_1 \neq \mathcal{O}_2$, we have

$$\mathcal{O}_1 \oplus \mathcal{O}_2 = 2(\mathcal{O}_1 \cap \mathcal{O}_2) \amalg \Big(\big(\mathcal{O}_1 \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)\big) \oplus \big(\mathcal{O}_2 \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)\big) \Big)$$

$$\amalg \Big((\mathcal{O}_1 \cap \mathcal{O}_2) \oplus \big(\mathcal{O}_2 \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)\big) \Big) \amalg \big((\mathcal{O}_1 \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)) \oplus (\mathcal{O}_1 \cap \mathcal{O}_2) \big).$$

If $\mathcal{Q} = m_1\mathcal{O}_1 \oplus \ldots \oplus m_l\mathcal{O}_l$ is a constructible set of Krull-Schmidt, we can write $\mathcal{Q} = \amalg_{i=1}^n \mathcal{Q}_i$ as a constructible set of stratified Krull-Schmidt, where

$$\mathcal{Q}_i = n_{i1}\mathcal{O}_{i1} \oplus n_{i2}\mathcal{O}_{i2} \oplus \ldots \oplus n_{ik_i}\mathcal{O}_{ik_i}$$

for indecomposable constructible sets $\mathcal{O}_{ij}$ which are disjoint each other. Hence we can assume that $\mathcal{O}_1, \ldots, \mathcal{O}_l$ are disjoint each other.

Let $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ be the subspace of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ which is spanned by characteristic functions $1_{\mathcal{O}}$ for constructible sets of stratified Krull-Schmidt $\mathcal{O}$, where each $1_{\mathcal{O}}$ satisfies that $1_{\mathcal{O}}([X]) = 1$ for $[X] \in \mathcal{O}$, and $1_{\mathcal{O}}([X]) = 0$ otherwise.

Example 3.2. Let $\mathbb{P}^1$ be the projective line over $\mathbb{K}$ and $\mathrm{coh}(\mathbb{P}^1)$ denote the category of coherent sheaves on $\mathbb{P}^1$.

Let $O(n)$ denote an indecomposable locally free coherent sheaf whose rank and degree are equal to 1 and n respectively. Let $S_x^{[r]}$ be an indecomposable torsion sheaf such that $\mathrm{rk}(S_x^{[r]}) = 0$, $\deg(S_x^{[r]}) = r$ and the support of $S_x^{[r]}$ is $\{x\}$ for $x \in \mathbb{P}^1$. The Grothendieck group $K_0(\mathrm{coh}(\mathbb{P}^1)) \cong \mathbb{Z}^2$. The data $K(\mathrm{coh}(\mathbb{P}^1))$ and $\mathcal{F}_{\mathrm{coh}(\mathbb{P}^1)}$ are defined in [10, Example 9.1]. The set of isomorphism classes of indecomposable objects in $\mathrm{coh}(\mathbb{P}^1)$ is

$$\{[S_x^{[d]}] \mid x \in \mathbb{P}^1, d \in \mathbb{N}\} \cup \{[O(n)] \mid n \in \mathbb{Z}\}.$$

Recall that a non-trivial subset $U \subset \mathbb{P}^1$ is closed (resp. open) if U is a finite (resp. cofinite) set. Let $\mathcal{O}_d$ be a finite or cofinite subset of $\{[S_x^{[d]}] \mid x \in \mathbb{P}\}$ for each $d \in \mathbb{Z}^+$ and $\mathcal{O}_0$ a finite subset of $\{[O(n)] \mid n \in \mathbb{Z}\}$. Then $\mathcal{O}_d$ and $\mathcal{O}_0$ are indecomposable constructible subsets of $\mathfrak{Obj}_{\mathrm{coh}(\mathbb{P}^1)}(\mathbb{K})$. Note that every indecomposable constructible subset of $\mathfrak{Obj}_{\mathrm{coh}(\mathbb{P}^1)}(\mathbb{K})$ is of the form

$$\mathcal{O}_0 \amalg \mathcal{O}_{i_1} \amalg \ldots \amalg \mathcal{O}_{i_n}$$

for $1 \leq i_1 < \ldots < i_n$. Then the finite direct sum $\oplus(\mathcal{O}_0 \amalg \mathcal{O}_{i_1} \amalg \ldots \amalg \mathcal{O}_{i_n})$ is a constructible set of Krull-Schmidt. Every constructible set of Krull-Schmidt in $\mathfrak{Obj}_{\mathrm{coh}(\mathbb{P}^1)}(\mathbb{K})$ is of the form. A constructible set of stratified Krull-Schmidt is a disjoint union of finitely many constructible sets of Krull-Schmidt.

Example 3.3. In Example 2.10, $\mathfrak{Obj}^{\mathbf{m}}_{\mathcal{P}^b(Q)}(\mathbb{C})$ is the set of all isomorphism classes of project complexes in $\mathcal{P}^b(Q, \underline{\mathbf{m}})$. Note that

$$\mathfrak{Obj}_{\mathcal{P}^b(Q)}(\mathbb{C}) = \amalg_{\underline{\mathbf{m}} \in K'(\mathcal{P}^b(Q))} \mathfrak{Obj}^{\mathbf{m}}_{\mathcal{P}^b(Q)}(\mathbb{C}).$$

There is a canonical map

$$p_{\underline{\mathbf{m}}} : \mathcal{P}^b(Q, \underline{\mathbf{m}}) \to \mathfrak{Obj}^{\mathbf{m}}_{\mathcal{P}^b(Q)}(\mathbb{C})$$

which maps $P^\bullet$ to $[P^\bullet]$. A subset $U \subseteq \mathfrak{Obj}^{\mathbf{m}}_{\mathcal{P}^b(Q)}(\mathbb{C})$ is closed (resp. open) if $p^{-1}_{\underline{\mathbf{m}}}(U)$ is closed (resp. open) in $\mathcal{P}^b(Q, \underline{\mathbf{m}})$. A subset $V_{\underline{\mathbf{m}}} \subseteq \mathfrak{Obj}^{\mathbf{m}}_{\mathcal{P}^b(Q)}(\mathbb{C})$ is locally closed if it is an intersection of a closed subset and an open subset of $\mathfrak{Obj}^{\mathbf{m}}_{\mathcal{P}^b(Q)}(\mathbb{C})$. A subset $\mathcal{O} \subseteq \mathfrak{Obj}_{\mathcal{P}^b(Q)}(\mathbb{C})$ is constructible if it is a finite disjoint union of locally closed sets $V_{\underline{\mathbf{m}}}$. Every indecomposable constructible set $\mathcal{O}$ is of the form $\coprod_{\underline{\mathbf{m}} \in S} V_{\underline{\mathbf{m}}}$, where S is a finite set and each complex in $p^{-1}_{\underline{\mathbf{m}}}(V_{\underline{\mathbf{m}}})$ is an indecomposable complex.

3.2. Automorphism groups of conflations

For each $X \in \mathrm{Obj}(\mathcal{A})$, suppose that $X = n_1X_1 \oplus n_2X_2 \oplus \ldots \oplus n_tX_t$, where X_i are indecomposable for $i = 1, \ldots, t$ and $X_i \not\cong X_j$ for $i \neq j$. Then we have

$$\mathrm{Aut}(X) \cong (1 + rad\,\mathrm{End}(X)) \rtimes \sum_{i=1}^{t} \mathrm{GL}(n_i, \mathbb{K}).$$

The rank of maximal torus of $\mathrm{Aut}(X)$ is denoted by $\mathrm{rk}\,\mathrm{Aut}(X)$. Let $n = n_1 + n_2 + \ldots + n_t$. Thus the number of indecomposable direct summands of X is n, which is denoted by $\gamma(X)$. Note that $\gamma(X) = \mathrm{rk}\,\mathrm{Aut}(X)$. Let

$$\gamma(\mathcal{O}) = \max\{\gamma(X) \mid [X] \in \mathcal{O}\}$$

for each constructible set $\mathcal{O}$ in $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$ and $\mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$ denote the group of (a_1, a_2, a_3) for $a_1 \in \mathrm{Aut}(X)$, $a_2 \in \mathrm{Aut}(Y)$ and $a_3 \in \mathrm{Aut}(Z)$ such that the following diagram is commutative

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle a_1}\downarrow & & {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a_3} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

The homomorphism

$$p_1 : \mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \to \mathrm{Aut}(Y)$$

is defined by $(a_1, a_2, a_3) \mapsto a_2$. If $p_1((a_1, a_2, a_3)) = p_1((a_1', a_2, a_3'))$ then $f(a_1 - a_1') = 0$ and $(a_3 - a_3')g = 0$. We have $a_1 = a_1'$ and $a_3 = a_3'$ since f is an inflation and g a deflation. Hence p_1 is an injective homomorphism of affine algebraic $\mathbb{K}$-groups and

$$\mathrm{rk}(\mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)) = \mathrm{rk}\ \mathrm{Im}p_1 \leq \mathrm{rk}\,\mathrm{Aut}(Y) \tag{2}$$

Let

$$p_2 : \mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \to \mathrm{Aut}(X) \times \mathrm{Aut}(Z)$$

be a homomorphism given by $(a_1, a_2, a_3) \mapsto (a_1, a_3)$. If $p_2((a_1, a_2, a_3)) = p_2((a_1, a_2', a_3))$, then $(a_2 - a_2')f = 0$ and $g(a_2 - a_2')$=0, we have

$$a_2 - a_2' \in (\mathrm{Hom}(Z, Y)g) \cap (f\,\mathrm{Hom}(Y, X)).$$

Observe that Kerp_2 is a linear space. It follows that $\chi(\mathrm{Ker}p_2) = 1$ and

$$\text{(3)} \qquad \mathrm{rk}\ \mathrm{Im}(p_2) \leq \mathrm{rk}\,\mathrm{Aut}(X) + \mathrm{rk}\,\mathrm{Aut}(Z).$$

Let $\mathcal{P}(\mathcal{A})$ be a complete set of representatives of all isomorphism classes of objects in $\mathcal{A}$. Let $W(X, Z; Y) = \{(f, g) \mid X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{S}\}$. Note that $W(X, Z; Y)$ is a subset of $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z)$. Let $W(\mathcal{O}_1, \mathcal{O}_2; Y)$ denote the set of $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{S}$, where $X, Y, Z \in \mathcal{P}(\mathcal{A})$ and $[X] \in \mathcal{O}_1$,$[Y] \in \mathcal{O}_2$.

Lemma 3.4. *For $X, Y, Z \in \mathcal{P}(\mathcal{A})$, the set $W(X, Z; Y)$ is a constructible subset of* $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z)$.

Proof. Recall that $\mathrm{Hom}(A, ?)$ and $\mathrm{Hom}(?, A)$ are left exact functors for each $A \in \mathrm{Obj}(\mathcal{A})$. The inflation f induces a monomorphism

$$f^* : \mathrm{Hom}(?, X) \to \mathrm{Hom}(?, Y)$$

in the functor category $\mathrm{Hom}(\mathcal{A}, \mathbf{Ab})$, where $\mathbf{Ab}$ denotes the category of abelian groups. Recall that $\mathrm{Hom}(?, X)$ is a projective object. Because $\mathbf{Ab}$ is an abelian category, $\mathrm{Hom}(\mathcal{A}, \mathbf{Ab})$ is also an abelian category. Let $P(X)$ denote $\mathrm{Hom}(?, X)$ and $inj(P(X), P(Y))$ denote the set of monomorphisms $f^* : P(X) \hookrightarrow P(Y)$. Using $inf(X, Y)$ to denote the set of inflations between X and Y. Note that $inf(X, Y)$ is isomorphic to $inj(P(X), P(Y))$. Because $inj(P(X), P(Y)) = \mathrm{Aut}(P(X))f^*$, $inj(P(X), P(Y))$ is a locally closed subset. Therefore $inf(X, Y)$ is locally closed.

Let $P'(Z) = \mathrm{Hom}(Z, ?)$. Similarly, the deflation g induces a monomorphism

$$g^* : \mathrm{Hom}(Z, ?) \to \mathrm{Hom}(Y, ?),$$

then the set $inj(P'(Z), P'(Y)) = \mathrm{Aut}(Z)g^*$ is locally closed. Hence the set of deflations $g : Y \to Z$ is a locally closed set.

Fixed $X, Y, Z \in \mathcal{P}(\mathcal{A})$, using the facts that f is an inflation and g a deflation, we obtain that $gf = 0$ if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a conflation. Clearly, $(f, g) \in \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z)$ satisfying above conditions if and only if $(f, g) \in W(X, Z; Y)$. Hence $W(X, Z; Y)$ is constructible. □

Two conflations $X \xrightarrow{i} Y \xrightarrow{d} Z$ and $X' \xrightarrow{i'} Y \xrightarrow{d'} Z'$ in $\mathcal{A}$ are said to be equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{d} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle 1_Y}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{i'} & Y & \xrightarrow{d'} & Z' \end{array}$$

where both f and g are isomorphisms. If the two conflations are equivalent, we write $X \xrightarrow{i} Y \xrightarrow{d} Z \sim X' \xrightarrow{i'} Y \xrightarrow{d'} Z'$. The equivalence class of $X \xrightarrow{i} Y \xrightarrow{d} Z$ is denoted by $\langle X \xrightarrow{i} Y \xrightarrow{d} Z\rangle$. Define

$$V(\mathcal{O}_1, \mathcal{O}_2; Y) = \big\{\langle X \xrightarrow{i} Y \xrightarrow{d} Z\rangle \mid X \xrightarrow{i} Y \xrightarrow{d} Z \in \mathcal{S}, [X] \in \mathcal{O}_1, [Z] \in \mathcal{O}_2\big\},$$

where $\mathcal{S}$ is the collection of all conflations of $\mathcal{A}$. Note that $V([X], [Z]; Y)$ is isomorphic to the orbit space $W(X, Z; Y)/(\operatorname{Aut} X \times \operatorname{Aut} Z)$. Note that

$$[W(X, Z; Y)/(\operatorname{Aut} X \times \operatorname{Aut} Z)] = W(X, Z; Y)/(\operatorname{Aut} X \times \operatorname{Aut} Z)$$

since the action of $\operatorname{Aut} X \times \operatorname{Aut} Z$ on $W(X, Z; Y)$ is free. Hence $V([X], [Z]; Y)$ is a quotient stack.

3.3. Associative algebras and Lie algebras

For $f, g \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, define $f \cdot g$ by $(f \cdot g)([X], [Y]) = f([X])g([Y])$ for $([X], [Y]) \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Thus $f \cdot g \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$.

By [10, Theorem 8.4], π_m is representable and $\pi_l \times \pi_r$ is of finite type. The pushforward of π_m is well-defined and p_1 is injective. The following definition of multiplication is taken from [12, Definition 4.1].

Definition 3.5. Using the following diagram

$$\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\pi_l \times \pi_r} \mathfrak{Exact}_{\mathcal{A}} \xrightarrow{\pi_m} \mathfrak{Obj}_{\mathcal{A}},$$

we can define the convolution multiplication

$$\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) \xrightarrow{(\pi_l \times \pi_r)^*} \mathrm{CF}(\mathfrak{Exact}_{\mathcal{A}}) \xrightarrow{(\pi_m)_!} \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}).$$

The multiplication $* : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is a bilinear map defined by

$$f * g = (\pi_m)_![(\pi_l \times \pi_r)^*(f \cdot g)] = (\pi_m)_![\pi_l^*(f) \cdot \pi_r^*(g)].$$

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, the meaning of $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}$ can be understood as follows. The function $m_{\pi_m} : \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \to \mathbb{Q}$, which is defined by

$$m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = \chi\big[\operatorname{Aut}(Y)/p_1\big(\operatorname{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)\big)\big],$$

is a locally constructible function on $\mathfrak{Exact}_{\mathcal{A}}(\mathbb{K})$ by [11, Proposition 4.16], namely $m_{\pi_m}|_{\mathcal{O}}$ is a constructible function on $\mathcal{O}$ for every constructible subset $\mathcal{O} \subseteq \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K})$.

For each $[Y] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$,

$$1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([Y]) = \sum_{c \in \Lambda(\mathcal{O}_1, \mathcal{O}_2; Y)} c\chi^{na}(Q_c(\mathcal{O}_1, \mathcal{O}_2, Y)), \tag{4}$$

where

$$\Lambda(\mathcal{O}_1, \mathcal{O}_2; Y) = \{c = m_{\pi_m}([A \xrightarrow{f} Y \xrightarrow{g} B]) \mid [A] \in \mathcal{O}_1, [B] \in \mathcal{O}_2\} \setminus \{0\}$$

is a finite set, and

$$Q_c(\mathcal{O}_1, \mathcal{O}_2, Y) =$$

$$\{[A \xrightarrow{f} Y \xrightarrow{g} B] \mid [A] \in \mathcal{O}_1, [B] \in \mathcal{O}_2, m_{\pi_m}([A \xrightarrow{f} Y \xrightarrow{g} B]) = c\}$$

are constructible sets for $c \in \Lambda(\mathcal{O}_1, \mathcal{O}_2; Y)$. In fact, the 1-morphism $\pi_l \times \pi_r$ is of finite type by [10, Theorem 8.4]. Hence $(\pi_{l*} \times \pi_{r*})^{-1}(\mathcal{O}_1 \times \mathcal{O}_2)$ is a constructible subset of $\mathfrak{Exact}_{\mathcal{A}}$. Then

$$\Lambda(\mathcal{O}_1, \mathcal{O}_2; Y) = m_{\pi_m}\big[\big((\pi_{l*} \times \pi_{r*})^{-1}(\mathcal{O}_1 \times \mathcal{O}_2)\big) \cap \big((\pi_{m*})^{-1}([Y])\big)\big] \setminus \{0\}$$

is a finite set by [11, Proposition 4.6]. Therefore

$$Q_c(\mathcal{O}_1, \mathcal{O}_2, Y) = m_{\pi_m}^{-1}(c) \cap [(\pi_{l*} \times \pi_{r*})^{-1}(\mathcal{O}_1 \times \mathcal{O}_2)] \cap ((\pi_{m*})^{-1}([Y]))$$

are constructible for all $c \in \Lambda(\mathcal{O}_1, \mathcal{O}_2; Y)$.

For each $([X],[Z]) \in \mathcal{O}_1 \times \mathcal{O}_2$, let

$$\Lambda(X,Z;Y) = \big\{c = m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) \mid [X \xrightarrow{f} Y \xrightarrow{g} Z] \in \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K})\big\}$$

and

$$Q_c(X,Z,Y) = \big\{[X \xrightarrow{f} Y \xrightarrow{g} Z] \,\big|\, m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = c\big\},$$

where $\Lambda(X,Z;Y)$ is a finite set and $Q_c(X,Z,Y)$ are constructible sets for all $c \in \Lambda(X,Z;Y)$. Then

$$(1_{[X]} * 1_{[Z]})([Y]) = \sum_{c\in\Lambda(X,Z;Y)} c\chi^{na}(Q_c(X,Z,Y)). \tag{5}$$

The set consisting of $\chi\Big(\operatorname{Aut}(Y)/p_1\big(\operatorname{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)\big)\Big)$, where

$$[X \xrightarrow{f} Y \xrightarrow{g} Z] \in \bigcup_{c\in\Lambda(\mathcal{O}_1,\mathcal{O}_2;Y)} Q_c(\mathcal{O}_1,\mathcal{O}_2,Y),$$

is finite since $\chi(\operatorname{Aut}(Y)/\operatorname{Im}p_1) = m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z])$.

Let

$$\pi_1 : V(\mathcal{O}_1,\mathcal{O}_2;Y) \to \bigcup_{c\in\Lambda(\mathcal{O}_1,\mathcal{O}_2;Y)} Q_c(\mathcal{O}_1,\mathcal{O}_2,Y)$$

be a morphism given by $\langle X \xrightarrow{f} Y \xrightarrow{g} Z\rangle \mapsto ([X \xrightarrow{f} Y \xrightarrow{g} Z])$. For each fibre of π_1, $\chi^{\mathrm{na}}(\pi_1^{-1}([X \xrightarrow{f} Y \xrightarrow{g} Z])) = \chi\Big(\operatorname{Aut}(Y)/p_1\big(\operatorname{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)\big)\Big)$.

The following result is due to [4, Proposition 6] and [12, Theorem 4.3].

Theorem 3.6. *The $\mathbb{Q}$-space $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is an associative $\mathbb{Q}$-algebra, with convolution multiplication $*$ and identity $1_{[0]}$, where $1_{[0]}$ is the characteristic function of $[0] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$.*

Proof. Let $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ be constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. It suffices to show that $(1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}) * 1_{\mathcal{O}_3}([M]) = 1_{\mathcal{O}_1} * (1_{\mathcal{O}_2} * 1_{\mathcal{O}_3})([M])$ for $M \in \operatorname{Obj}(\mathcal{A})$. Take $X, Y, Z \in \mathcal{P}(\mathcal{A})$ satisfying $[X] \in \mathcal{O}_1, [Y] \in \mathcal{O}_2$ and $[Z] \in \mathcal{O}_3$. Consider

$(f,g,m,l) \in W(X,Y;L) \times W(L,Z;M)$. There is a pushout

$$\begin{array}{ccc} L & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l'} \\ M & \xrightarrow{g'} & L' \end{array}$$

where $L' \in \mathcal{P}(\mathcal{A})$. We obtain an inflation $l' : Y \to L'$ and a deflation $g' : M \to L'$. Let $f' = lf$. Then f' is an inflation and $g'f' = 0$. Hence g' is a cokernel of f' and $X \xrightarrow{f'} M \xrightarrow{g'} L'$ is a conflation.

There is a morphism $m' : L' \to Z$ such that $m = m'g'$ and $m'l' = 0$. It is easy to see that l' is a kernel of m' and (l', m') is a conflation. The following diagram is commutative

$$\begin{array}{ccccc} X & \xrightarrow{f} & L & \xrightarrow{g} & Y \\ \| & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l'} \\ X & \xrightarrow{f'} & M & \xrightarrow{g'} & L' \\ & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m'} \\ & & Z & = & Z \end{array}$$

Note that the rows and columns are conflations. For $L, L' \in \mathcal{P}(\mathcal{A})$, we claim that the morphism

$$\cup_L V([X],[Y];L) \times V([L],[Z];M) \xrightarrow{F} \cup_{L'} V([X],[L'];M) \times V([Y],[Z];L'),$$

which maps $(\langle X \xrightarrow{f} L \xrightarrow{g} Y\rangle, \langle L \xrightarrow{l} M \xrightarrow{m} Z\rangle)$ to $(\langle X \xrightarrow{f'} M \xrightarrow{g'} L'\rangle, \langle Y \xrightarrow{l'} L' \xrightarrow{m'} Z\rangle)$, is a bijection. The proof of this claim is quite similar to the proof of [8, Proposition 2] and so is omitted. The morphism F induces a morphism $T : \mathcal{TS}(\mathbb{K}) \to \mathcal{TS}(\mathbb{K})$ by

$$([X \xrightarrow{f} L \xrightarrow{g} Y], [L \xrightarrow{l} M \xrightarrow{m} Z]) \mapsto ([X \xrightarrow{f'} M \xrightarrow{g'} L'], [Y \xrightarrow{l'} L' \xrightarrow{m'} Z]).$$

The following diagram is commutative

$$\begin{array}{ccc} \cup_L V([X],[Y];L) \times V([L],[Z];M) & \xrightarrow{F} & \cup_{L'} V([X],[L'];M) \times V([Y],[Z];L') \\ \downarrow & & \downarrow \\ \mathcal{TS}(\mathbb{K}) & \xrightarrow{T} & \mathcal{TS}(\mathbb{K}) \end{array}$$

Let $c \in \Lambda(X, Y; L)$, $d \in \Lambda(L, Z; M)$, $c' \in \Lambda(X, L'; M)$, $d' \in \Lambda(Y, Z; L')$. Assume that $m_{\pi_m}([X \xrightarrow{f} L \xrightarrow{g} Y]) = c$, $m_{\pi_m}([L \xrightarrow{l} M \xrightarrow{m} Z]) = d$, $m_{\pi_m}([X \xrightarrow{f'} M \xrightarrow{g'} L']) = c'$ and $m_{\pi_m}([Y \xrightarrow{l'} L' \xrightarrow{m'} Z]) = d'$. Then

$$\chi^{na}\big(T^{-1}([X \xrightarrow{f'} M \xrightarrow{g'} L'], [Y \xrightarrow{l'} L' \xrightarrow{m'} Z])\big) = \frac{c'd'}{cd}.$$

Let $Q_c(X, Y, L)$ be as in Section 3.3. By Lemma 2.5, we have

$$cd\chi^{\mathrm{na}}(Q_c(X, Y, L))\chi^{\mathrm{na}}(Q_d(L, Z, M)) = c'd'\chi^{\mathrm{na}}(Q'_c(X, L', M))\chi^{\mathrm{na}}(Q'_d(Y, Z, L')).$$

It follows that $(1_{[X]} * 1_{[Y]}) * 1_{[Z]}([M]) = 1_{[X]} * (1_{[Y]} * 1_{[Z]})([M])$. Recall that

$$(1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}) * 1_{\mathcal{O}_3}([M]) = \int_{[X]\in\mathcal{O}_1, [Y]\in\mathcal{O}_2, [Z]\in\mathcal{O}_3} (1_{[X]} * 1_{[Y]}) * 1_{[Z]}([M])$$

and

$$1_{\mathcal{O}_1} * (1_{\mathcal{O}_2} * 1_{\mathcal{O}_3})([M]) = \int_{[X]\in\mathcal{O}_1, [Y]\in\mathcal{O}_2, [Z]\in\mathcal{O}_3} 1_{[X]} * (1_{[Y]} * 1_{[Z]})([M]).$$

This completes the proof of Theorem 3.6. □

Joyce defined $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ to be the subspace of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ such that if $f([X]) \neq 0$ then X is an indecomposable object in $\mathcal{A}$ for every $f \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. There is a result of [4, Theorem 13] and [12, Theorem 4.9].

Theorem 3.7. *The* $\mathbb{Q}$*-space* $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ *is a Lie algebra under the Lie bracket* $[f, g] = f * g - g * f$ *for* $f, g \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

Proof. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two indecomposable constructible sets. It suffices to show that $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} - 1_{\mathcal{O}_2} * 1_{\mathcal{O}_1} \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Without loss of generality, we can assume that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. By corollary 3.13, $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} - 1_{\mathcal{O}_2} * 1_{\mathcal{O}_1} \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. □

3.4. The algebra $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$

Lemma 3.8. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be two constructible subsets of* $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. *For any* $Y \in \mathrm{Obj}(\mathcal{A})$, *if* $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([Y]) \neq 0$, *then there exists a conflation* $A \xrightarrow{f} Y \xrightarrow{g} B$ *in* $\mathcal{A}$ *satisfying that* $[A] \in \mathcal{O}_1$, $[B] \in \mathcal{O}_2$ *and* $m_{\pi_m}([A \xrightarrow{f} Y \xrightarrow{g} B]) \neq 0$. *Moreover, there exist* $X, Z \in \mathrm{Obj}(\mathcal{A})$ *such that* $[X] \in \mathcal{O}_1$, $[Z] \in \mathcal{O}_2$ *and* $1_{[X]} * 1_{[Z]}([Y]) \neq 0$.

Proof. Let $Q_c(\mathcal{O}_1, \mathcal{O}_2, Y)$ and $\Lambda(\mathcal{O}_1, \mathcal{O}_2; Y)$ be as in Section 3.3. Let

$$Q = \sqcup_{c \in \Lambda(\mathcal{O}_1, \mathcal{O}_2; Y)} Q_c(\mathcal{O}_1, \mathcal{O}_2, Y) \text{ and } Q_c = Q_c(\mathcal{O}_1, \mathcal{O}_2, Y)$$

for simplicity. Since $\Lambda(\mathcal{O}_1, \mathcal{O}_2, Y)$ is a finite set, Q is constructible.

For each $c \in \Lambda(\mathcal{O}_1, \mathcal{O}_2; Y)$, there exists some conflations $A \xrightarrow{f} Y \xrightarrow{g} B$ in $\mathcal{A}$ such that $[A] \in \mathcal{O}_1$, $[B] \in \mathcal{O}_2$ and $m_{\pi_m}([A \xrightarrow{f} Y \xrightarrow{g} B]) = c$. By equation (4), we know that there exist some $c \neq 0$. This proves the first statement.

Let

$$\pi : Q \to (\pi_{l*} \times \pi_{r*})(Q)$$

be a map which maps $[X \xrightarrow{i} Y \xrightarrow{d} Z]$ to $([X], [Z])$ and

$$m_m = m_{\pi_m}|_Q.$$

It follows that m_m is a constructible function over Q.

Because $\pi_l \times \pi_r$ is a 1-morphism, π is a pseudomorphism by [11, Proposition 4.6]. Thus $\pi(Q)$ is constructible and the naïve pushforward $(\pi)_!^{\text{na}}(m_m)$ of m_m to $\pi(Q)$ exists. Note that $(\pi)_!^{\text{na}}(m_m)$ is a constructible function on $\pi(Q)$. In fact

$$(\pi)_!^{\text{na}}(m_m)([X], [Z]) = 1_{[X]} * 1_{[Z]}([Y])$$

for all $([X], [Z]) \in \pi(Q)$. Therefore

$$\{1_{[X]} * 1_{[Z]}([Y]) \mid ([X], [Z]) \in \pi(Q)\}$$

is a finite set. Note that

$$\pi^{-1}([X], [Z]) = \{[X \xrightarrow{f} Y \xrightarrow{g} Z] \in Q_c\} = Q_c(X, Z, Y)$$

is constructible for $([X], [Z]) \in \pi(Q_c)$ since $\pi_l \times \pi_r$ is of finite type. The set

$$\{1_{[X]} * 1_{[Z]}([Y]) \mid ([X], [Z]) \in \pi(Q)\}$$

is a finite set since $1_{[X]} * 1_{[Z]}$ is a constructible function. Using the equation 5 and the fact that $\Lambda(\mathcal{O}_1, \mathcal{O}_2, Y)$ is a finite set, we know that

$$\{\chi^{\text{na}}(Q_c(X, Z, Y)) \mid ([X], [Z]) \in \pi(Q)\}$$

is a finite set.

Suppose that

$$S_c(X,Z)=\Big\{([A],[B])\in\pi(Q_c)\mid \chi^{\mathrm{na}}(\pi^{-1}([A],[B]))=\chi^{\mathrm{na}}(Q_c(X,Z,Y))\Big\}.$$

Then we have

$$\chi^{\mathrm{na}}(Q_c)=\sum_{([X],[Z])}\chi^{\mathrm{na}}(S_c(X,Z))\chi^{\mathrm{na}}(Q_c(X,Z,Y))$$

for finitely many $([X],[Z])\in\pi(Q_c)$.

For $c\in\Lambda(\mathcal{O}_1,\mathcal{O}_2;Y)$, let $\{([X_1^{(c)}],[Z_1^{(c)}]),\dots,([X_{k_c}^{(c)}],[Z_{k_c}^{(c)}])\}$ be a complete set of representatives for $([X],[Z])\in\pi(Q_c)$ such that

$$\chi^{\mathrm{na}}(Q_c(X_i^{(c)},Z_i^{(c)},Y)\neq\chi^{\mathrm{na}}(Q_c(X_j^{(c)},Z_j^{(c)},Y)$$

for $i\neq j$ and $i,j\in\{1,2,\dots,k_c\}$. It is easy to see that

$$\pi(Q_c)=\sqcup_{i=1}^{k_c}S_c(X_i^{(c)},Z_i^{(c)})\text{ and }\pi(Q)=\cup_c\big(\sqcup_{i=1}^{k_c}S_c(X_i^{(c)},Z_i^{(c)})\big).$$

Assume that $m_m(Q)=\{c_1,c_2,\dots,c_m\}$. Set

$$S(i_1,i_2,\dots,i_n)=S_{c_{i_1}}(X_{l_{i_1}}^{(c_{i_1})},Z_{l_{i_1}}^{(c_{i_1})})\cap\ldots\cap S_{c_{i_n}}(X_{l_{i_n}}^{(c_{i_n})},Z_{l_{i_n}}^{(c_{i_n})})$$

be a non-empty set for $1\le i_1<i_2<\ldots<i_n\le m$ and $1\le l_{i_j}\le k_{c_{i_j}}$, which satisfies the 'minimal' condition, namely $S(i_1,i_2,\dots,i_n)\cap S_c(X_i^{(c)},Z_i^{(c)})=\emptyset$ for any $c\notin\{c_{i_1},\dots,c_{i_n}\}$ or $i\notin\{l_{i_1},\dots,l_{i_n}\}$. The choice of $S(i_1,i_2,\dots,i_n)$ are finite. By definition, $S(i_1,i_2,\dots,i_n)$ are pairwise disjoint. For simplicity, we use $S_1,S_2,\dots,S_r$ to denote sets $S(i_1,i_2,\dots,i_n)$. It follows that

$$S_1\sqcup\ldots\sqcup S_r=\pi(Q).$$

By Lemma 2.5, we obtain that

$$\sum_{c\in\Lambda(\mathcal{O}_1,\mathcal{O}_2,Y)}c\chi^{\mathrm{na}}(Q_c)=\sum_{c\in\Lambda(\mathcal{O}_1,\mathcal{O}_2;Y)}c\sum_{i=1}^{r}\chi^{na}(S_i)\chi^{\mathrm{na}}(Q_c(X_i,Z_i,Y))\delta(i,c),$$

where $([X_i],[Z_i])\in S_i$, $\delta(i,c)=1$ if $S_i\cap\pi(Q_c(X_i,Z_i,Y))\neq\emptyset$ and $\delta(i,c)=0$ otherwise. Then

$$1_{\mathcal{O}_1}*1_{\mathcal{O}_2}([Y])=\sum_{i=1}^{r}\chi^{na}(S_i)\sum_{c\in\Lambda(\mathcal{O}_1,\mathcal{O}_2;Y)}c\chi^{\mathrm{na}}(Q_c(X_i,Z_i,Y))\delta(i,c)$$

$$= \sum_{i=1}^{r} \chi^{na}(S_i)\big(1_{[X_i]} * 1_{[Z_i]}([Y])\big).$$

There exists $([X_i],[Z_i])$ for some $i \in \{1,\ldots,r\}$ such that $1_{[X_k]} * 1_{[Z_k]}([Y]) \neq 0$ since $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([Y]) \neq 0$. □

Let $\mathbf{D}_n(\mathbb{K})$ denote the group of invertible diagonal matrices in $\mathbf{GL}(n,\mathbb{K})$. The following lemma is related to Riedtmann[20, Lemma 2.2].

Lemma 3.9. *Let $X,Y,Z \in \mathrm{Obj}(\mathcal{A})$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$. If $m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) \neq 0$, then $\gamma(Y) \leq \gamma(X)+\gamma(Z)$. In particular, $\gamma(Y)=\gamma(X)+\gamma(Z)$ if and only if $Y \cong X \oplus Z$.*

Proof. Recall that $m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = \chi(\mathrm{Aut}\,Y/\mathrm{Im}(p_1))$.

If $\mathrm{rk}\,\mathrm{Aut}(Y) > \mathrm{rk}\ \mathrm{Im}(p_1)$, then the fibre of the action of a maximal torus of $\mathrm{Aut}(Y)$ on $\mathrm{Aut}\,Y/\mathrm{Im}(p_1)$ is $(\mathbb{K}^*)^k$ for some $k \geq 1$, it forces $\chi(\mathrm{Aut}\,Y/\mathrm{Im}(p_1)) = 0$. Hence we have $\mathrm{rk}\,\mathrm{Aut}(Y) = \mathrm{rk}\ \mathrm{Im}(p_1) \leq \mathrm{rk}\,\mathrm{Aut}(X) + \mathrm{rk}\,\mathrm{Aut}(Z)$.

We prove the second assertion by induction on $\mathrm{rk}\,\mathrm{Aut}(Y)$. First of all, suppose that $X \ncong 0$ and $Z \ncong 0$. If $\mathrm{rk}\,\mathrm{Aut}(Y)=2$ and $Y = Y_1 \oplus Y_2$, then $\mathrm{rk}\,\mathrm{Aut}(X) = \mathrm{rk}\,\mathrm{Aut}(Z) = 1$ since X and Z are not isomorphic to 0. For $t \in \mathbb{K}^* \setminus \{1\}$, $\begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \in \mathrm{Aut}(Y)$ and it is an element of a maximal torus $\mathbf{D}_2(\mathbb{K})$ of $\mathrm{Aut}(Y)$. A maximal torus of $\mathrm{Im}(p_1)$ is also a maximal torus of $\mathrm{Aut}(Y)$ since $\mathrm{rk}\,\mathrm{Aut}(Y) = \mathrm{rk}\ \mathrm{Im}(p_1)$. Because two maximal tori of a connected linear algebraic group are conjugate, there exists $\alpha \in \mathrm{Aut}(Y)$ such that $\alpha \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \alpha^{-1}$ lies in a maximal torus of $\mathrm{Im}(p_1)$. Hence there exist $a \in \mathrm{Aut}(X)$ and $b \in \mathrm{Aut}(Z)$ satisfying $(a, \alpha \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \alpha^{-1}, b) \in \mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$, namely

$$(a, \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix}, b) \in \mathrm{Aut}(X \xrightarrow{\alpha^{-1}f} Y \xrightarrow{g\alpha} Z).$$

Let $f' = \alpha^{-1}f$ and $g' = g\alpha$. Observe $(t, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, t) \in \mathrm{Aut}(X \xrightarrow{f'} Y \xrightarrow{g'} Z)$. Hence $f'(a-t) = \begin{pmatrix} 0 & 0 \\ 0 & t^2-t \end{pmatrix} f'$. Let $s = \frac{1}{t^2-t}(a-t) \in \mathrm{End}(X)$ $(t \neq$

$0, 1)$. Then $f's = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} f'$. Because f' is an inflation and

$$f's^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} f's = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} f' = f's,$$

$s^2 = s$. The category $\mathcal{A}$ is idempotent completion, consequently s has a kernel and an image such that $X = \mathrm{Ker}s \oplus \mathrm{Im}s$. But X is indecomposable, without loss of generality we can assume $X = \mathrm{Ker}s$. Then $s = 0$. Let $f' = \binom{f_1}{f_2}$ and $g' = (g_1, g_2)$. It follows that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = f's = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \end{pmatrix}.$$

We have $f_2 = 0$ and $f' = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. The morphism $Y_1 \oplus Y_2 \xrightarrow{(0,1)} Y_2$ is a deflation by [2, Lemma 2.7]. Because $(0, 1)\binom{f_1}{0} = 0$, there exits $h \in \mathrm{Hom}(Z, Y_1)$ such that $(0, 1) = h(g_1, g_2)$. We have $hg_1 = 0$ and $hg_2 = 1_{Y_2}$. Observe $g_2h \in \mathrm{End}(Z)$ and $(g_2h)(g_2h) = g_2h$, so g_2h has a kernel $k : K \to Z$ and an image $i : I \to Z$. Moreover $Z \cong K \oplus I$. It follows that $Z \cong K$ or $Z \cong I$ since Z is indecomposable. If $Z \cong K$ then $g_2h = 0$. But $hg_2h = h$, $K = 0$. Thus h is an isomorphism and $g_1 = 0$. We have $Z \cong Y_2$. Similarly $X \cong Y_1$. Hence $X \oplus Z \cong Y_1 \oplus Y_2$.

Assume that the assertion is true for $\mathrm{rk}\,\mathrm{Aut}(Y) = n < N$. When $n = N$, we can assume $\mathrm{rk}\,\mathrm{Aut}(X) = n_1$ where $0 < n_1 < N$, then $\mathrm{rk}\,\mathrm{Aut}(Z) = N - n_1 = n_2$. Let $Y = Y' \oplus Y_N$ and $Y' = Y_1 \oplus \ldots \oplus Y_{N-1}$, where Y_i are indecomposable. Observe that $\begin{pmatrix} tI_{N-1} & 0 \\ 0 & t^2 \end{pmatrix}$ lies in a maximal torus of $\mathrm{Aut}(Y)$ for $t \in \mathbb{K}^* \setminus \{1\}$. There exists $(a, c, b) \in \mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$ such that c and $\begin{pmatrix} tI_{N-1} & 0 \\ 0 & t^2 \end{pmatrix}$ are conjugate in $\mathrm{Aut}(Y)$. For simplicity we assume $c = \begin{pmatrix} tI_{N-1} & 0 \\ 0 & t^2 \end{pmatrix}$. So we have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y' \oplus Y_N & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y' \oplus Y_N & \xrightarrow{g} & Z \end{array}$$

where $f = (f_1, f_2, \ldots, f_N)^t$ and $g = (g_1, g_2, \ldots, g_N)$.

There is another commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{(f^*,f_N)^t} & Y'\oplus Y_N & \xrightarrow{(g^*,g_N)} & Z \\ \downarrow{\scriptstyle tI_{n_1}} & & \downarrow{\scriptstyle tI_N} & & \downarrow{\scriptstyle tI_{n_2}} \\ X & \xrightarrow{(f^*,f_N)^t} & Y'\oplus Y_N & \xrightarrow{(g^*,g_N)} & Z \end{array}$$

where $f^* = (f_1, f_2, \ldots, f_{N-1})^T$ and $g^* = (g_1, g_2, \ldots, g_{N-1})$. Then $f = (f^*, f_N)^t$, $g = (g^*, g_N)$ and $f(a - tI_{n_1}) = \begin{pmatrix} 0I_{N-1} & 0 \\ 0 & t^2 - t \end{pmatrix} f$. Let

$$s_N = \frac{1}{t^2 - t}(a - tI_{n_1}).$$

Then $fs_N = \mathrm{diag}\{0, \ldots, 0, 1\}f$. It follows $f^*s_N = 0$, $f_Ns_N = f_N$ and $g_Nf_N = g\begin{pmatrix} 0I_{N-1} & 0 \\ 0 & 1 \end{pmatrix} f = gfs_N = 0$. Moreover s_N is an idempotent, we know that $X = \mathrm{Ker}s_N \oplus \mathrm{Im}s_N$. If $f_N \neq 0$ then $\mathrm{Im}s_N$ is not isomorphic to 0. Similarly we can define s_1, s_2, …, $s_{N-1} \in \mathrm{End}(X)$ with the property that $fs_i = \mathrm{diag}\{0, \ldots, 0, 1, 0, \ldots, 0\}f = (0, \ldots, 0, f_i, 0, \ldots, 0)^t$. Hence s_i is idempotent and if $f_i \neq 0$ then $\mathrm{Im}s_i$ is not isomorphic to 0 for each i. Note that $s_1 + s_2 + \ldots + s_N = 1_X \in \mathrm{Aut}(X)$, it follows

$$X = \mathrm{Im}s_1 \oplus \ldots \oplus \mathrm{Im}s_N.$$

Hence $f_i = 0$ for some i since $\mathrm{rk}\,\mathrm{Aut}(X) < N$. Without loss of generality, we assume $f_N = 0$. Let $(0, \ldots, 0, 1) : Y_1 \oplus \ldots \oplus Y_N \to Y_N$, then

$$(0, \ldots, 0, 1)(f_1, \ldots, f_N)^t = 0$$

Hence there exists $h \in \mathrm{Hom}(Z, Y_N)$ such that $h(g_1, \ldots, g_N) = (0, \ldots, 0, 1)$, namely $hg_1 = 0, \ldots, hg_{N-1} = 0$ and $hg_N = 1$. Therefore Y_N is isomorphic to a direct summand of Z. Assume that $Z = Z' \oplus Y_N$ where $\gamma(Z') = \gamma(Z) - 1$. The morphism $(1, 0) : Z' \oplus Y_N \to Z'$ is a deflation, so $g' = g^*(1, 0) : Y' \to Z'$ is a deflation by Definition A.1. Obviously, $(f_1, \ldots, f_{N-1})^t : X \to Y_1 \oplus \ldots \oplus Y_{N-1}$ is a kernel of g'. Thus

$$X \xrightarrow{(f_1,\ldots,f_{N-1})^t} Y_1 \oplus \ldots \oplus Y_{N-1} \xrightarrow{g'} Z'$$

is a conflation. By hypothesis, $Y_1 \oplus \ldots \oplus Y_{N-1} \cong X \oplus Z'$. Hence $Y = Y_1 \oplus \ldots \oplus Y_N \cong X \oplus Z$. The proof is completed. □

Remark 3.10. If $1_{[X]} * 1_{[Z]}([Y]) \neq 0$, then $\gamma(Y) \leq \gamma(X) + \gamma(Z)$, where the equality holds if and only if $Y \cong X \oplus Z$.

Lemma 3.11. *Let $X, Y, Z \in \mathrm{Obj}\,(\mathcal{A})$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$. If $m_{\pi_m}([X \xrightarrow{f} Y \xrightarrow{g} Z]) \neq 0$, $\gamma(Y) < \gamma(X) + \gamma(Z)$ and $Y = Y_1 \oplus Y_2$, then there exist two conflations $X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1$ and $X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2$ in $\mathcal{A}$ such that $X \cong X_1 \oplus X_2$, $Z \cong Z_1 \oplus Z_2$ and $f = diag\{f_1, f_2\}, g = diag\{g_1, g_2\}$.*

Proof. Suppose that $\mathrm{rk}\,\mathrm{Aut}(X) = n_1$, $\mathrm{rk}\,\mathrm{Aut}(X) = N$ and $\mathrm{rk}\,\mathrm{Aut}(Z) = n_2$. Then $N < n_1 + n_2$. For simplicity, we use the notation as above. Let $Y = Y_1 \oplus \ldots \oplus Y_N$, $f = (f_1, f_2, \ldots, f_N)^t$, $g = (g_1, g_2, \ldots, g_N)$ and the isomorphisms $(a, c, b), (tI_{n_1}, tI_N, tI_{n_2}) \in \mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$, where $c = \begin{pmatrix} tI_{N-1} & 0 \\ 0 & t^2 \end{pmatrix}$. Recall that

$$s_N = \frac{1}{t^2 - t}(a - tI_{n_1}) \in \mathrm{End}(X)$$

is an idempotent such that

$$fs_N = (0, \ldots, 0, f_N)^t$$

and $X = \mathrm{Ker}s_N \oplus \mathrm{Im}s_N$. Similarly, there exists an idempotent

$$r_N = \frac{1}{t - t^2}(b - tI_{n_2})$$

in $\mathrm{End}(Z)$ such that $r_N g = (0, \ldots, 0, g_N)$ and $Z = \mathrm{Ker}r_N \oplus \mathrm{Im}r_N$. Without loss of generality, we assume that $f_N \neq 0$ and $g_N \neq 0$. Because $f_N s_N = f_N$ and $r_N g_N = g_N$,

$$g_N f_N = r_N g_N f_N s_N = r_N (g_1, \ldots, g_N)(f_1, \ldots, f_N)^t s_N = 0.$$

It is clear that $i : \mathrm{Ker}s_N \hookrightarrow X$ is a kernel of $f_N : X \to Y_N$. There exists a morphism $f'_N : \mathrm{Im}s_N \to Y_N$ which is an image of f_N since $X = \mathrm{Ker}s_N \oplus \mathrm{Im}s_N$. Similarly we can find a morphism $g'_N : Y_N \to \mathrm{Im}r_N$ which is a coimage of g_N such that $g_N = jg'_N$, where $j : \mathrm{Im}(r_N) \hookrightarrow Z$ is an image of g_N. It is easy to check that f'_N is an inflation, g'_N a deflation and $g'_N f'_N = 0$. Let

$h : Y_N \to A$ be a morphism in $\mathcal{A}$ such that $hf'_N = 0$. The morphism

$$(0, \dots, 0, h) : Y_1 \oplus \dots \oplus Y_N \to A$$

satisfies $(0, \dots, 0, h)f = 0$. There exists $k \in \mathrm{Hom}_{\mathcal{A}}(Z, A)$ such that

$$(0, \dots, 0, h) = kg$$

since g is a cokernel of f. It follows that $h = kg_N = kjg'_N$. Hence g'_N is a cokernel of f'_N. Therefore $\mathrm{Im} s_N \xrightarrow{f'_N} Y_N \xrightarrow{g'_N} \mathrm{Im} r_N$ is a conflation. By induction, every indecomposable direct summand of Y is extended by the direct summands of X and Z. The proof is finished. □

Lemma 3.12. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two indecomposable constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Let $A \in \mathrm{Obj}(\mathcal{A})$ and $\gamma(A) \geq 2$. If $[A] \notin \mathcal{O}_1 \oplus \mathcal{O}_2$, then $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([A]) = 0$.*

Proof. If $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([A]) \neq 0$, then there exist $X, Y \in \mathrm{Obj}(\mathcal{A})$ such that $[X] \in \mathcal{O}_1$, $[Y] \in \mathcal{O}_2$ and $1_{[X]} * 1_{[Y]}(A) \neq 0$ by Lemma 3.8. It follows that $\gamma(A) = 2$ and $A \cong X \oplus Y$ by Lemma 3.9 (also see [12, Theorem 4.9]). This leads to a contradiction. □

Corollary 3.13. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be indecomposable constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. If $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, then*

$$1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} = 1_{\mathcal{O}_1 \oplus \mathcal{O}_2} + \sum_{i=1}^{m} a_i 1_{\mathcal{P}_i}$$

where $\mathcal{P}_i$ are indecomposable constructible subsets and $a_i = 1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([X])$ for $[X] \in \mathcal{P}_i$.

Proof. Let $[M] \in \mathcal{O}_1$ and $[N] \in \mathcal{O}_2$. Then M is not isomorphic to N since $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Using the fact that $m_{\pi_m}([M \xrightarrow{(1,0)^t} M \oplus N \xrightarrow{(0,1)} N]) = 1$, we obtain

$$\begin{aligned} &1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([M \oplus N]) \\ = &m_{\pi_m}([M \xrightarrow{(1,0)^t} M \oplus N \xrightarrow{(0,1)} N]) \cdot \chi^{\mathrm{na}}([M \xrightarrow{(1,0)^t} M \oplus N \xrightarrow{(0,1)} N]) = 1. \end{aligned}$$

By Lemma 3.12, we know that if $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}([X]) \neq 0$ and $[X] \notin \mathcal{O}_1 \oplus \mathcal{O}_2$, then X is an indecomposable object. Note that

$$\big(1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}(\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \setminus \mathcal{O}_1 \oplus \mathcal{O}_2)\big) \setminus \{0\} = \{a_1, a_2, \dots, a_m\}.$$

Then $\mathcal{P}_i = (1_{\mathcal{O}_1} * 1_{\mathcal{O}_2})^{-1}(a_i) \setminus \mathcal{O}_1 \oplus \mathcal{O}_2$ for $1 \leq i \leq m$. We complete the proof. □

Using Lemma 3.9 and Lemma 3.11, one easily obtains the following corollary:

Corollary 3.14. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two constructible sets. There exist finitely many constructible sets $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ such that*

$$1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} = \sum_{i=1}^{n} a_i 1_{\mathcal{Q}_i}$$

where $\gamma(\mathcal{Q}_i) \leq \gamma(\mathcal{O}_1) + \gamma(\mathcal{O}_2)$ and $a_i = (1_{\mathcal{O}_1} * 1_{\mathcal{O}_2})([X])$ for any $[X] \in \mathcal{Q}_i$.

For indecomposable constructible sets $\mathcal{O}_1, \ldots, \mathcal{O}_k$ and $X \in \mathrm{Obj}(\mathcal{A})$, $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} * \ldots * 1_{\mathcal{O}_k}([X]) \neq 0$ implies that $\gamma(X) \leq k$. In particular, $\gamma(X) = k$ means $X = X_1 \oplus \ldots \oplus X_k$ with $[X_i] \in \mathcal{O}_i$ for $1 \leq i \leq k$.

Let $X_1, \ldots, X_m \in \mathrm{Obj}(\mathcal{A})$ and there be r isomorphic classes, we can assume that $X_1, \ldots, X_{m_1}$ are isomorphic, $X_{m_1+1}, \ldots, X_{m_2}$ are isomorphic, …, and $X_{m_{r-1}+1}, \ldots, X_{m_r}$ are isomorphic, where $m_1 + \ldots + m_r = m$. By [12], we have

$$\begin{aligned} &\mathrm{Aut}(X_1 \oplus \ldots \oplus X_m)/\,\mathrm{Aut}(X_1) \times \ldots \times \mathrm{Aut}(X_m) \\ &\qquad \cong \mathbb{K}^l \times \prod_{i=1}^{r} (\mathrm{GL}(m_i, \mathbb{K})/(\mathbb{K}^*)^{m_i}), \end{aligned} \tag{6}$$

$$\chi(\mathrm{Aut}(X_1 \oplus X_2 \oplus \ldots \oplus X_m)/\,\mathrm{Aut}(X_1) \times \ldots \times \mathrm{Aut}(X_m)) = \prod_{i=1}^{r} m_i!. \tag{7}$$

Proposition 3.15. *Let $\mathcal{O}$ be an indecomposable constructible set. Then*

$$1_{\mathcal{O}}^{*k} = k! 1_{k\mathcal{O}} + \sum_{i=1}^{t} m_i 1_{\mathcal{P}_i}$$

where $\gamma(\mathcal{P}_i) < k$ for each i and $m_i = 1_{\mathcal{O}}^{*k}([X])$ for $[X] \in \mathcal{P}_i$.

Proof. We prove the proposition by induction on k. When $k = 1$, it is easy to see that the formula is true. If $k = 2$, then

$$1_{\mathcal{O}}^{*2}([X \oplus X]) = 1_{\mathcal{O}}([X]) \cdot 1_{\mathcal{O}}([X]) \cdot \chi(\operatorname{Aut}(X \oplus X)/\operatorname{Aut}(X) \times \operatorname{Aut}(X)) = 2$$

for $[X] \in \mathcal{O}$ and

$$1_{\mathcal{O}}^{*2}([X \oplus Y]) =$$
$$\big(1_{\mathcal{O}}([X])1_{\mathcal{O}}([Y]) + 1_{\mathcal{O}}([Y])1_{\mathcal{O}}([X])\big) \cdot \chi\big(\operatorname{Aut}(X \oplus Y)/\operatorname{Aut}(X) \times \operatorname{Aut}(Y)\big)$$
$$= 2,$$

where $[X], [Y] \in \mathcal{O}$ and $X \ncong Y$. If $[X] \notin \mathcal{O} \oplus \mathcal{O}$ and $\gamma(X) \geq 2$ then $1_{\mathcal{O}}^{*2}([X]) = 0$ by Lemma 3.12. Hence $1_{\mathcal{O}}^{*2} = 2 \cdot 1_{\mathcal{O} \oplus \mathcal{O}} + \sum_i m_i \mathcal{P}_i$ where $\mathcal{P}_i$ are indecomposable constructible sets by Corollary 3.14.

Now we suppose that the formula is true for $k \leq n$. When $k = n + 1$, we have

$$1_{\mathcal{O}}^{*(n+1)} = 1_{\mathcal{O}}^{*(n)} * 1_{\mathcal{O}} = (n! 1_{n\mathcal{O}} + \sum c_{\mathcal{P}'} 1_{\mathcal{P}'}) * 1_{\mathcal{O}},$$

where $\mathcal{P}'$ are constructible sets with $\gamma(\mathcal{P}') < n$. If the formula is true for $k = n + 1$, then

$$n! 1_{n\mathcal{O}} * 1_{\mathcal{O}} = (n + 1)! 1_{(n+1)\mathcal{O}} + \sum c_{\mathcal{Q}} 1_{\mathcal{Q}},$$

where $\mathcal{Q}$ are constructible sets with $\gamma(\mathcal{Q}) < n + 1$. Hence it suffices to show that the initial term of $1_{n\mathcal{O}} * 1_{\mathcal{O}}$ is $(n + 1)1_{(n+1)\mathcal{O}}$, namely $(1_{n\mathcal{O}} * 1_{\mathcal{O}})([X]) = n + 1$ for all $[X] \in (n + 1)\mathcal{O}$.

Assume that $X = m_1 X_1 \oplus m_2 X_2 \oplus \ldots \oplus m_r X_r$, where $X_1, \ldots, X_r \in \operatorname{Obj}(\mathcal{A})$ which are not isomorphic to each other, $[X_i] \in \mathcal{O}$ for $1 \leq i \leq r$, $m_1, \ldots, m_r$ are positive integers and $m_1 + m_2 + \ldots + m_r = n + 1$.

$$(1_{n\mathcal{O}} * 1_{\mathcal{O}})([X]) = (1_{[(m_1-1)X_1 \oplus m_2 X_2 \oplus \ldots \oplus m_r X_r]} * 1_{[X_1]})([X])$$
$$+(1_{[m_1 X_1 \oplus (m_2-1)X_2 \oplus \ldots \oplus m_r X_r]} * 1_{[X_2]})([X])$$
$$+\ldots$$
$$+(1_{[m_1 X_1 \oplus \ldots \oplus m_{r-1} X_{r-1} \oplus (m_r-1)X_r]} * 1_{[X_r]})([X])$$

Using Equation (7), it follows that

$$1_{[X_1]}^{*(m_1-1)} * 1_{[X_2]}^{*m_2} * \ldots * 1_{[X_r]}^{*m_r}$$

$$= (m_1-1)!m_2!\dots m_r! 1_{[(m_1-1)X_1\oplus m_2X_2\oplus\dots\oplus m_rX_r]} + \dots,$$

$$1_{[X_1]}^{*(m_1-1)} * 1_{[X_2]}^{*m_2} * \dots * 1_{[X_r]}^{*m_r} * 1_{[X_1]} = (\prod_{i=1}^{r} m_i!) 1_{[m_1X_1\oplus m_2X_2\oplus\dots\oplus m_rX_r]} + \dots$$

Compare the initial monomials of the two equations, it follows that

$$1_{[(m_1-1)X_1\oplus m_2X_2\oplus\dots\oplus m_rX_r]} * 1_{[X_1]} = m_1 1_{[m_1X_1\oplus m_2X_2\oplus\dots\oplus m_rX_r]} + \dots$$

Thus $1_{[(m_1-1)X_1\oplus m_2X_2\oplus\dots\oplus m_rX_r]} * 1_{[X_1]}([X]) = m_1$.

Similarly, we have $1_{[m_1X_1\oplus\dots\oplus(m_i-1)X_i\oplus\dots\oplus m_rX_r]} * 1_{[X_i]}([X]) = m_i$ for $i = 2,\dots,r$. Hence $(1_{n\mathcal{O}} * 1_{\mathcal{O}})([X]) = \sum_{i=1}^{r} m_i = n+1$ which completes the proof. □

By induction, we have the following corollary.

Corollary 3.16. *Let $\mathcal{O}_1,\mathcal{O}_2,\dots,\mathcal{O}_k$ be indecomposable constructible sets which are pairwise disjoint. Then we have the following equations*

$$1_{\mathcal{O}_1}^{*n_1} * 1_{\mathcal{O}_2}^{*n_2} \dots * 1_{\mathcal{O}_k}^{*n_k} = n_1!n_2!\dots n_k! 1_{n_1\mathcal{O}_1\oplus\dots\oplus n_k\mathcal{O}_k} + \dots,$$

$$1_{m_1\mathcal{O}_1\oplus\dots\oplus m_k\mathcal{O}_k} * 1_{n_1\mathcal{O}_1\oplus\dots\oplus n_k\mathcal{O}_k}$$
$$= \prod_{i=1}^{k} \frac{(m_i+n_i)!}{m_i!n_i!} 1_{(m_1+n_1)\mathcal{O}_1\oplus\dots\oplus(m_k+n_k)\mathcal{O}_k} + \dots,$$

where k is a positive integer and $m_1,\dots,m_k,n_1,\dots,n_k \in \mathbb{N}$.

Let $\mathrm{Ind}(\alpha)$ be the subset of $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ such that X are indecomposable for all $[X] \in \mathrm{Ind}(\alpha)$.

Lemma 3.17. *For each $\alpha \in K'(\mathcal{A})$, $\mathrm{Ind}(\alpha)$ is a locally constructible set.*

Proof. Assume $\alpha,\beta,\gamma \in K'(\mathcal{A})\setminus\{0\}$. The map

$$f: \coprod_{\substack{\beta,\gamma;\\ \beta+\gamma=\alpha}} \mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\gamma}(\mathbb{K}) \to \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$$

is defined by $([B],[C]) \mapsto [B\oplus C]$. It is clear that f is a pseudomorphism. Every $\mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\gamma}(\mathbb{K})$ is a locally constructible set. For any constructible set $\mathcal{C} \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, there are finitely many $\mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\gamma}(\mathbb{K})$ such that $\mathcal{C} \cap (\mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\gamma}(\mathbb{K})) \neq \emptyset$. Hence

$\amalg_{\beta,\gamma;\beta+\gamma=\alpha}\mathfrak{Obj}^{\beta}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}^{\gamma}_{\mathcal{A}}(\mathbb{K})$ is locally constructible. Then $\mathrm{Im} f$ is a locally constructible set. It follows that $\mathrm{Ind}(\alpha)=\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})\setminus\mathrm{Im} f$ is locally constructible. □

The following proposition is due to [4, Proposition 11].

Proposition 3.18. *Let $\mathcal{O}_1,\mathcal{O}_2$ be two constructible sets of Krull-Schmidt. It follows that*

$$1_{\mathcal{O}_1}*1_{\mathcal{O}_2}=\sum_{i=1}^{c}a_i1_{\mathcal{Q}_i}$$

for some $c\in\mathbb{N}^+$, where $a_i=1_{\mathcal{O}_1}*1_{\mathcal{O}_2}([X])$ for each $[X]\in\mathcal{Q}_i$ and $\mathcal{Q}_i$ are constructible sets of stratified Krull-Schmidt such that $\gamma(\mathcal{Q}_i)\le\gamma(\mathcal{O}_1)+\gamma(\mathcal{O}_2)$.

Proof. Because $\mathcal{O}_1,\mathcal{O}_2$ are constructible sets, the equation holds for some constructible sets $\mathcal{Q}_i$ with $\gamma(\mathcal{Q}_i)\le\gamma(\mathcal{O}_1)+\gamma(\mathcal{O}_2)$ by Corollary 3.14.

For every $[Y_i]\in\mathcal{Q}_i$, $1_{\mathcal{O}_1}*1_{\mathcal{O}_2}([Y_i])\neq 0$. By Lemma 3.8, there exist $X_i,Z_i\in\mathrm{Obj}(\mathcal{A})$ such that $[X_i]\in\mathcal{O}_1$, $[Z_i]\in\mathcal{O}_2$ and $1_{[X_i]}*1_{[Z_i]}([Y_i])\neq 0$ since $1_{\mathcal{O}_1}*1_{\mathcal{O}_2}([Y_i])\neq 0$. Thanks to Lemma 3.9, we have that $\gamma(Y_i)\le\gamma(X_i)+\gamma(Z_i)$. According to Lemma 3.11, all indecomposable direct summands of Y_i are extended by the direct summands of X_i and Z_i since $1_{[X_i]}*1_{[Z_i]}([Y_i])\neq 0$.

By the discussion in Section 3.1, we can suppose that $\mathcal{O}_1=\bigoplus_{i=1}^{t}a_i\mathcal{C}_i$ and $\mathcal{O}_2=\bigoplus_{j=1}^{t}b_j\mathcal{C}_j$, where $a_i,b_j\in\{0,1\}$ for all i,j and $\mathcal{C}_i$ are indecomposable constructible sets such that $\mathcal{C}_i\cap\mathcal{C}_j=\emptyset$ or $\mathcal{C}_i=\mathcal{C}_j$ for all $i\neq j$. Let $1\le r\le t$, the set

$$\{A_1,A_2,\ldots,A_r\mid\emptyset\neq A_i\subseteq\{1,\ldots,n\}\text{ for }i=1,\ldots,r\}$$

is called an r-partition of $\{1,2,\ldots,t\}$ if $A_1\cup A_2\cup\ldots\cup A_r=\{1,2,\ldots,t\}$ and $A_i\cap A_j=\emptyset$ for all $i\neq j$. Obviously, the cardinal number of all partitions of $\{1,2,\ldots,t\}$ is finite. Let $\{A_1,A_2,\ldots,A_r\}$, $\{B_1,B_2,\ldots,B_r\}$ be two r-partitions of $\{1,2,\ldots,t\}$ and $c_k\in\mathbb{Q}\setminus\{0\}$ for $k=1,2,\ldots,r$. Set $\mathcal{O}_{A_k}=\bigoplus_{i\in A_k}a_i\mathcal{C}_i$ and $\mathcal{O}_{B_k}=\bigoplus_{j\in B_k}b_j\mathcal{C}_j$ for $1\le k\le r$. Then we have

$$\mathcal{R}_{A_k,B_k,c_k}=\{[X]\in\mathcal{O}_{A_k}\oplus\mathcal{O}_{B_k}\mid 1_{\mathcal{O}_{A_k}}*1_{\mathcal{O}_{B_k}}([X])=c_k\},$$

$$\mathcal{I}_{A_k,B_k,c_k}=\{[X]\mid X\text{ indecomposable},1_{\mathcal{O}_{A_k}}*1_{\mathcal{O}_{B_k}}([X])=c_k\}.$$

This means that for each $[X] \in \mathcal{R}_{A_k,B_k,c_k}$, there exist $[A] \in \mathcal{O}_{A_k}$ and $[B] \in \mathcal{O}_{B_k}$ such that $X \cong A \oplus B$. For each $[Y] \in \mathcal{I}_{A_k,B_k,c_k}$, there exist $[C] \in \mathcal{O}_{A_k}$ and $[D] \in \mathcal{O}_{B_k}$ such that $C \to Y \to D$ is a non-split conflation in $\mathcal{A}$. Note that

$$\mathcal{R}_{A_k,B_k,c_k} = ((1_{\mathcal{O}_{A_k}} * 1_{\mathcal{O}_{B_k}})^{-1}(c_k)) \cap (\mathcal{O}_{A_k} \oplus \mathcal{O}_{B_k}).$$

By Corollary 3.16, $\mathcal{R}_{A_k,B_k,c_k} = \emptyset$ or $\mathcal{O}_{A_k} \oplus \mathcal{O}_{B_k}$. Hence $\mathcal{R}_{A_k,B_k,c_k}$ is a constructible set of Krull-Schmidt. There exist $\alpha_1, \ldots, \alpha_s \in K'(\mathcal{A})$ such that $\mathcal{I}_{A_k,B_k,c_k} = (\amalg_{i=1}^{s} \mathrm{Ind}(\alpha_i)) \cap ((1_{\mathcal{O}_{A_k}} * 1_{\mathcal{O}_{B_k}})^{-1}(c_k))$. By Lemma 3.17, $\mathcal{I}_{A_k,B_k,c_k}$ is an indecomposable constructible set.

Finally, $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}$ is a $\mathbb{Q}$-linear combination of finitely many $1_{\oplus_{k=1}^{r} \mathcal{O}_{A_k,B_k,c_k}}$, where $\mathcal{O}_{A_k,B_k,c_k}$ run through $\mathcal{R}_{A_k,B_k,c_k}$ and $\mathcal{I}_{A_k,B_k,c_k}$ for all r-partitions and $r = 1, 2, \ldots, t$. We finish the proof. □

Thus we summarize what we have proved as the following theorem which is due to [4, Theorem 12].

Theorem 3.19. *The* $\mathbb{Q}$*-space* $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ *is an associative* $\mathbb{Q}$*-algebra with convolution multiplication* $*$ *and identity* $1_{[0]}$.

3.5. The universal enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$

From now on, let $U(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}))$ denote the universal enveloping algebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ over $\mathbb{Q}$. The multiplication in $U(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}))$ will be written as $(x, y) \mapsto xy$. There is a $\mathbb{Q}$-algebra homomorphism

$$\Phi : U(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})) \to \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$$

defined by $\Phi(1) = 1_{[0]}$ and $\Phi(f_1 f_2 \ldots f_n) = f_1 * f_2 * \ldots * f_n$, where $f_1, f_2, \ldots, f_n$ belong to $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

The following theorem is related to [4, Theorem 15].

Theorem 3.20. $\Phi : U(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})) \to \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ *is an isomorphism.*

Proof. For simplicity of presentation, let

$$U = U(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})) \text{ and } \mathrm{CF} = \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}).$$

Assume that $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{k-1}$ and $\mathcal{O}_k$ are indecomposable constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ which are pairwise disjoint. It follows that $1_{\mathcal{O}_1}, 1_{\mathcal{O}_2}, \ldots, 1_{\mathcal{O}_k}$ are linearly independent in $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

Let $U_{\mathcal{O}_1\ldots\mathcal{O}_k}$ denote the subspace of U which is spanned by all $1_{\mathcal{O}_1}^{n_1}1_{\mathcal{O}_2}^{n_2}\ldots 1_{\mathcal{O}_k}^{n_k}$ for $n_i\in\mathbb{N}$ and $i=1,\ldots,k$.

Define $\mathrm{CF}_{\mathcal{O}_1\ldots\mathcal{O}_n}$ to be the subalgebra of CF which is generated by the elements $1_{n_1\mathcal{O}_1\oplus n_2\mathcal{O}_2\oplus\ldots\oplus n_k\mathcal{O}_k}$ of CF, where $n_i\in\mathbb{N}$ for $i=1,2,\ldots,k$.

The homomorphism Φ induces a homomorphism

$$\Phi_{\mathcal{O}_1\ldots\mathcal{O}_k}:U_{\mathcal{O}_1\ldots\mathcal{O}_k}\to \mathrm{CF}_{\mathcal{O}_1\ldots\mathcal{O}_k}$$

which maps $1_{\mathcal{O}_1}^{n_1}1_{\mathcal{O}_2}^{n_2}\ldots 1_{\mathcal{O}_k}^{n_k}$ to $1_{\mathcal{O}_1}^{*n_1}*1_{\mathcal{O}_2}^{*n_2}*\ldots*1_{\mathcal{O}_k}^{*n_k}$.

First of all, we want to show that $\Phi_{\mathcal{O}_1\ldots\mathcal{O}_k}$ is injective.

For $m\in\mathbb{N}$, let $U^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ be the subspace of U which is spanned by

$$\big\{1_{\mathcal{O}_1}^{n_1}1_{\mathcal{O}_2}^{n_2}\ldots 1_{\mathcal{O}_k}^{n_k}\mid \sum_{i=1}^{k}n_i\le m, n_i\ge 0 \text{ for } i=1,\ldots,k\big\}$$

Using the PBW Theorem, we obtain that

$$\big\{1_{\mathcal{O}_1}^{n_1}1_{\mathcal{O}_2}^{n_2}\ldots 1_{\mathcal{O}_k}^{n_k}\mid \sum_{i=1}^{k}n_i= m, n_i\ge 0 \text{ for } i=1,\ldots,k\big\}$$

is a basis of the $\mathbb{Q}$-vector space $U^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}/U^{(m-1)}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ for $m\ge 1$.

Similarly, we define $\mathrm{CF}^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ to be a subspace of $\mathrm{CF}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ such that each $f\in \mathrm{CF}^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ is of the form $\sum_{i=1}^{l}c_i1_{\mathcal{C}_i}$, where $l\in\mathbb{N}^+$, $c_i\in\mathbb{Q}$, $1_{\mathcal{C}_i}\in \mathrm{CF}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ and $\mathcal{C}_i$ are constructible sets of Krull-Schmidt such that $\gamma(\mathcal{C}_i)\le m$.

In $\mathrm{CF}^{(m)}\,/\,\mathrm{CF}^{(m-1)}$, the set

$$\{1_{n_1\mathcal{O}_1\oplus n_2\mathcal{O}_2\oplus\ldots\oplus n_k\mathcal{O}_k}\mid \sum_{i=1}^{k}n_i= m, n_i\ge 0 \text{ for } i=1,\ldots,k\}$$

is linearly independent by the Krull-Schmidt Theorem.

For each $m\ge 1$, $\Phi_{\mathcal{O}_1\ldots\mathcal{O}_k}$ induce a map

$$\Phi^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}:U^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}/U^{(m-1)}_{\mathcal{O}_1\ldots\mathcal{O}_k}\to \mathrm{CF}^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}\,/\,\mathrm{CF}^{(m-1)}_{\mathcal{O}_1\ldots\mathcal{O}_k}$$

which maps $1_{\mathcal{O}_1}^{n_1}1_{\mathcal{O}_2}^{n_2}\ldots 1_{\mathcal{O}_k}^{n_k}$ to $n_1!n_2!\ldots n_k!1_{n_1\mathcal{O}_1\oplus n_2\mathcal{O}_2\oplus\ldots\oplus n_k\mathcal{O}_k}$ (also see Corollary 3.16), where $\sum_{i=1}^{k}n_i=m$ and $m_i\ge 0$. From this we know that $\Phi^{(m)}_{\mathcal{O}_1\ldots\mathcal{O}_k}$ is injective for all $m\in\mathbb{N}$. Obviously, both $U_{\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n}$ and $\mathrm{CF}_{\mathcal{O}_1\ldots\mathcal{O}_n}$

are filtered. From the properties of filtered algebra, we know that $\Phi_{\mathcal{O}_1\ldots\mathcal{O}_k}$ is injective. Hence $\Phi : U \to \mathrm{CF}$ is injective.

Finally, we show that Φ is surjective by induction on m. When $m = 1$, the statement is trivial. Then we assume that every constructible function $f = \sum_{i=1}^{t} a_i 1_{\mathcal{Q}_i}$ lies in $\mathrm{Im}(\Phi)$, where $a_i \in \mathbb{Q}$ and $\mathcal{Q}_i$ are constructible sets of stratified Krull-Schmidt with $\gamma(\mathcal{Q}_i) < m$.

Let $n_1 + n_2 + \ldots + n_k = m$ and $n_i \in \mathbb{N}$ for $1 \le i \le k$. Then

$$\Phi(1_{\mathcal{O}_1}^{n_1} 1_{\mathcal{O}_2}^{n_2} \ldots 1_{\mathcal{O}_k}^{n_k}) = 1_{\mathcal{O}_1}^{*n_1} * 1_{\mathcal{O}_2}^{*n_2} * \ldots * 1_{\mathcal{O}_k}^{*n_k}$$

$$= n_1!n_2!\ldots n_n! 1_{n_1\mathcal{O}_1 \oplus n_2\mathcal{O}_2 \oplus \ldots \oplus n_k\mathcal{O}_k} + \sum_{j=1}^{s} b_j 1_{\mathcal{P}_j},$$

where $b_j \in \mathbb{Q}$ and $\mathcal{P}_j$ are constructible sets of stratified Krull-Schmidt with $\gamma(\mathcal{P}_j) < m$. By the hypothesis, $\sum_{j=1}^{s} b_j 1_{\mathcal{P}_j} \in \mathrm{Im}(\Phi)$. Hence $1_{n_1\mathcal{O}_1 \oplus n_2\mathcal{O}_2 \oplus \ldots \oplus n_k\mathcal{O}_k}$ lies in $\mathrm{Im}(\Phi)$. The algebra CF is generated by all $1_{n_1\mathcal{O}_1 \oplus \ldots \oplus n_k\mathcal{O}_k}$, which proves that Φ is surjective, the proof is finished. □

4. Comultiplication and Green's theorem

4.1. Comultiplication

We now turn to define a comultiplication on the algebra $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$. For $f, g \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, $f \otimes g$ is define by $f \otimes g([X],[Y]) = f([X])g([Y])$ for $([X],[Y]) \in (\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})(\mathbb{K}) = \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ (see [12, Difinition 4.1]). Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{A}$. Recall that the map p_2 : $\mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z) \to \mathrm{Aut}(X) \times \mathrm{Aut}(Z)$ is defined by $(a_1, a_2, a_3) \mapsto (a_1, a_3)$ and $\chi(\mathrm{Ker} p_2) = 1$.

The following definitions are related to [4, Section 6] and [12, Defintion 4.16].

Definition 4.1. From now on, assume that $\pi_m : \mathfrak{Exact}_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$ is of finite type and $\pi_l \times \pi_r$ is representable. Then we have the following diagram

$$\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) \xleftarrow{(\pi_l \times \pi_r)_!} \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Exact}_{\mathcal{A}}) \xleftarrow{(\pi_m)^*} \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}).$$

The comultiplication

$$\Delta : \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$$

is defined by $\Delta = (\pi_l \times \pi_r)_! \circ (\pi_m)^*$, where $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$ is regarded as a topological completion of $\mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$.

The counit $\varepsilon : \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathbb{Q}$ maps f to $f([0])$.

Note that Δ is a $\mathbb{Q}$-linear map since $(\pi_l \times \pi_r)_!$ and $(\pi_m)^*$ are $\mathbb{Q}$-linear map.

Definition 4.2. Let $\alpha = [A], \beta = [B] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ and $\mathcal{O} \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ be a constructible set of stratified Krull-Schmidt, define

$$h_{\mathcal{O}}^{\beta\alpha} = \Delta(1_{\mathcal{O}})([A], [B]).$$

Let $\mathcal{O}_1$ and $\mathcal{O}_2 \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ be constructible sets, define

$$g_{\mathcal{O}_2\mathcal{O}_1}^{\alpha} = 1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}(\alpha).$$

Because $\Delta(1_{\mathcal{O}})$ is a constructible function, $\Delta(1_{\mathcal{O}}) = \sum_{i=1}^{n} h_{\mathcal{O}}^{\beta_i\alpha_i} 1_{\mathcal{O}_i}$ for some $\alpha_i, \beta_i \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ and $n \in \mathbb{N}$, where $\mathcal{O}_i$ are constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$.

Lemma 4.3. *Let X, Y, $Z \in \mathrm{Obj}(\mathcal{A})$. If $X \oplus Z$ is not isomorphic to Y, then $\Delta(1_{[Y]})([X], [Z]) = 0$.*

Proof. If $\Delta(1_{[Y]})([X], [Z]) \neq 0$, there exists a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ such that $m_{\pi_l \times \pi_r}([X \xrightarrow{f} Y \xrightarrow{g} Z]) \neq 0$. Recall that

$$m_{\pi_l \times \pi_r}([X \xrightarrow{f} Y \xrightarrow{g} Z]) = \chi\big((\mathrm{Aut}(X) \times \mathrm{Aut}(Z))/\mathrm{Im} p_2\big).$$

If $\mathrm{rk}\ \mathrm{Im} p_2 < \mathrm{rk}\big(\mathrm{Aut}(X) \times \mathrm{Aut}(Z)\big)$, the fibre of the action of a maximal torus of $\mathrm{Aut}(X) \times \mathrm{Aut}(Z)$ on $(\mathrm{Aut}(X) \times \mathrm{Aut}(Z))/\mathrm{Im} p_2$ is $(\mathbb{K}^*)^l$ for some $l > 0$. Then $\chi\big((\mathrm{Aut}(X) \times \mathrm{Aut}(Z))/\mathrm{Im} p_2\big) = 0$, which is a contradiction. Hence $\mathrm{rk}\big(\mathrm{Aut}(X) \times \mathrm{Aut}(Z)\big) = \mathrm{rk}\ \mathrm{Im} p_2$.

Assume that $\mathrm{rk}\,\mathrm{Aut}(X) = n_1$, $\mathrm{rk}\,\mathrm{Aut}(Z) = n_2$ and $\mathrm{rk}\,\mathrm{Aut}(Y) = n$ for some positive integers n_1, n_2 and n. Note that $\mathbf{D}_{n_1} \times \mathbf{D}_{n_2}$ is a maximal torus of $\mathrm{Aut}(X) \times \mathrm{Aut}(Z)$. Because $\mathrm{rk}\big(\mathrm{Aut}(X) \times \mathrm{Aut}(Z)\big) = \mathrm{rk}\,\mathrm{Im}(p_2)$, each maximal torus of $\mathrm{Im} p_2$ is also a maximal torus of $\mathrm{Aut}(X) \times \mathrm{Aut}(Z)$. Therefore every maximal torus of $\mathrm{Im} p_2$ and $\mathbf{D}_{n_1} \times \mathbf{D}_{n_2}$ are conjugate. For simplicity, we can assume that $\mathbf{D}_{n_1} \times \mathbf{D}_{n_2}$ is a maximal torus of $\mathrm{Im} p_2$. For $(t_1 I_{n_1}, t_2 I_{n_2}) \in \mathbf{D}_{n_1} \times \mathbf{D}_{n_2}$, where $t_1 \neq t_2$, there exists $\tau \in \mathrm{Aut}(Y)$ such that

$(t_1 I_{n_1}, \tau, t_2 I_{n_2}) \in \mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$. Then we have the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle t_1 I_{n_1}}\downarrow & & {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle t_2 I_{n_2}} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

The morphism $(t_2 I_{n_1}, t_2 I_n, t_2 I_{n_2})$ is also in $\mathrm{Aut}(X \xrightarrow{f} Y \xrightarrow{g} Z)$. The following diagram is commutative

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle t_2 I_{n_1}}\downarrow & & {\scriptstyle t_2 I_n}\downarrow & & \downarrow{\scriptstyle t_2 I_{n_2}} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

Consequently $g(\tau - t_2 I_n) = 0$. Because f is a kernel of g, there exists $h \in \mathrm{Hom}(Y, X)$ such that $\tau - t_2 I_n = fh$. Then $\tau = fh + t_2 I_n$. We have

$$f(t_1 I_{n_1}) = \tau f = (fh + t_2 I_n))f,$$

it follows that

$$fhf = f(t_1 I_{n_1}) - (t_2 I_n)f = f(t_1 I_{n_1} - t_2 I_{n_1}).$$

Then $hf = (t_1 - t_2) I_{n_1}$ since f is an inflation. Let $f' = \frac{1}{t_1 - t_2} h$, then $f'f = 1_X$. Hence X is isomorphic to a direct summand of Y. The proof is completed. □

For an indecomposable object $X \in \mathrm{Obj}(\mathcal{A})$, direct summands of X are only X and 0. Thus $\Delta(1_{[X]}) = 1_{[X]} \otimes 1_{[0]} + 1_{[0]} \otimes 1_{[X]}$. It follows that $\Delta(f) = f \otimes 1_{[0]} + 1_{[0]} \otimes f$ for $f \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

By Lemma 4.3, $h_{\mathcal{O}}^{\beta\alpha} = 1$ if $\alpha \oplus \beta \in \mathcal{O}$, and $h_{\mathcal{O}}^{\beta\alpha} = 0$ otherwise. Let $\mathcal{O} = n_1 \mathcal{O}_1 \oplus \ldots \oplus n_m \mathcal{O}_m$ be a constructible set of Krull-Schmidt, where $\mathcal{O}_i$ are indecomposable constructible sets for all $1 \leq i \leq m$. By Lemma 4.3, the formula $\Delta(1_{\mathcal{O}}) = \sum\limits_{i=1}^{n} h_{\mathcal{O}}^{\beta_i \alpha_i} 1_{\mathcal{O}_i}$ can be written as

$$\Delta(1_{\mathcal{O}}) = \sum_{1 \leq i \leq m; 0 \leq k_i \leq n_i} 1_{k_1 \mathcal{O}_1 \oplus \ldots \oplus k_m \mathcal{O}_m} \otimes 1_{(n_1 - k_1)\mathcal{O}_1 \oplus \ldots \oplus (n_m - k_m)\mathcal{O}_m}.$$

Hence we have the following proposition.

Proposition 4.4. *Let $\mathcal{O}$ be a constructible set of stratified Krull-Schmidt, then $\Delta(1_{\mathcal{O}}) \in \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$, i.e., the map*

$$\Delta : \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}))$$

is well-defined.

4.2. Green's theorem on stacks

Recall that

$$\int_{x\in S} f(x) = \sum_{a\in f(S)\setminus\{0\}} a\chi^{\mathrm{na}}(f^{-1}(a)\cap S),$$

where f is a constructible function and S a locally constructible set.

Let $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_\rho, \mathcal{O}_\sigma, \mathcal{O}_\epsilon, \mathcal{O}_\tau, \mathcal{O}_\lambda$ be constructible sets and $\alpha \in \mathcal{O}_1, \beta \in \mathcal{O}_2, \rho \in \mathcal{O}_\rho, \sigma \in \mathcal{O}_\sigma, \epsilon \in \mathcal{O}_\epsilon, \tau \in \mathcal{O}_\tau, \lambda \in \mathcal{O}_\lambda$ such that $\mathcal{O}_\rho \oplus \mathcal{O}_\sigma = \mathcal{O}_1$ and $\mathcal{O}_\epsilon \oplus \mathcal{O}_\tau = \mathcal{O}_2$.

The following theorem is the degenerate form of Green's theorem which is related to [4, Theorem 22].

Theorem 4.5. *Let $\mathcal{O}_1, \mathcal{O}_2$ be constructible subsets of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ and $\alpha', \beta' \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, then we have*

$$g^{\alpha'\oplus\beta'}_{\mathcal{O}_2\mathcal{O}_1} = \int_{\rho,\sigma,\epsilon,\tau\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K});\rho\oplus\sigma\in\mathcal{O}_1,\epsilon\oplus\tau\in\mathcal{O}_2} g^{\alpha'}_{\epsilon\rho} g^{\beta'}_{\tau\sigma}.$$

Proof. By the proof of Lemma 3.8, $g^{\alpha'\oplus\beta'}_{\mathcal{O}_2\mathcal{O}_1} = \int_{\alpha\in\mathcal{O}_1,\beta\in\mathcal{O}_2} g^{\alpha'\oplus\beta'}_{\beta\alpha}$. It suffices to prove the following formula

$$g^{\alpha'\oplus\beta'}_{\beta\alpha} = \int_{\rho,\sigma,\epsilon,\tau\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K});\rho\oplus\sigma=\alpha,\epsilon\oplus\tau=\beta} g^{\alpha'}_{\epsilon\rho} g^{\beta'}_{\tau\sigma}.$$

Suppose that $[A]=\alpha$, $[B]=\beta$, $[A']=\alpha'$, $[B']=\beta'$, $[C]=\rho$, $[D]=\sigma$, $[E]=\epsilon$ and $[F]=\tau$ for $A,B,C,D,E,F \in \mathrm{Obj}(\mathcal{A})$. There are finitely many (ρ,σ) and (ϵ,τ) such that $\rho\oplus\sigma=\alpha$ and $\epsilon\oplus\tau=\beta$. Take

$$V = \bigcup_{\substack{[C],[D],[E],[F];\\ [C\oplus D]=[A],[E\oplus F]=[B]}} V([C],[E];A') \times V([D],[F];B').$$

The map

$$i : V \to V([A],[B];A'\oplus B')$$

is defined by

$$(\langle C \xrightarrow{f_1} A' \xrightarrow{g_1} E\rangle, \langle D \xrightarrow{f_2} B' \xrightarrow{g_2} F\rangle) \mapsto \langle C \oplus D \xrightarrow{f} A' \oplus B' \xrightarrow{g} E \oplus F\rangle,$$

where $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$. Because both $C \xrightarrow{f_1} A' \xrightarrow{g_1} E$ and $D \xrightarrow{f_2} B' \xrightarrow{g_2} F$ are conflations, $C \oplus D \xrightarrow{f} A' \oplus B' \xrightarrow{g} E \oplus F$ is a conflation by [2, Proposition 2.9]. Hence the morphism is well-defined. Note that i is injective.

There is a map $\Omega_1 : V(A, B, A' \oplus B') \to \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K})$ which maps $\langle A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B\rangle$ to $[A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B]$. Recall that

$$\chi(\Omega_1^{-1}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B])) = m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B]).$$

Take

$$Q(A, B, A' \oplus B') = \sqcup_{a \in \Lambda(A,B;A' \oplus B')} Q_a(A, B, A' \oplus B')$$

which is the image of Ω_1.

A map

$$\Omega_2 : V \to \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K})$$

is defined by

$$(\langle C \xrightarrow{f_1} A' \xrightarrow{g_1} E\rangle, \langle D \xrightarrow{f_2} B' \xrightarrow{g_2} F\rangle) \mapsto ([C \xrightarrow{f_1} A' \xrightarrow{g_1} E], [D \xrightarrow{f_2} B' \xrightarrow{g_2} F]).$$

The Euler characteristic of $\Omega_2^{-1}\big(([C \xrightarrow{f_1} A' \xrightarrow{g_1} E], [D \xrightarrow{f_2} B' \xrightarrow{g_2} F])\big)$ is $m_{\pi_m}([C \xrightarrow{f_1} A' \xrightarrow{g_1} E])m_{\pi_m}([D \xrightarrow{f_2} B' \xrightarrow{g_2} F])$. Let

$$Q(c, d, C, D, E, F) = Q_c(C, E, A') \times Q_d(D, F, B')$$

for $c \in \Lambda(C, E; A')$, $d \in \Lambda(D, F; B')$ and

$$Q(A', B') = \sqcup_{c,d,[C],[D],[E],[F]} Q(c, d, C, D, E, F),$$

where $C \oplus E \cong A$ and $D \oplus F \cong B$.

There is a morphism

$$\bar{i} : Q(A', B') \to Q(A, B, A' \oplus B')$$

by $([C \xrightarrow{f_1} A' \xrightarrow{g_1} E], [D \xrightarrow{f_2} B' \xrightarrow{g_2} F]) \mapsto [C \oplus D \xrightarrow{f} A' \oplus B' \xrightarrow{g} E \oplus F]$. Then there is a commutative diagram

$$\begin{array}{ccc} \Omega_2^{-1}(Q(A', B')) & \xrightarrow{i} & \Omega_1^{-1}(Q(A, B, A' \oplus B')) \\ \Omega_2 \downarrow & & \downarrow \Omega_1 \\ Q(A', B') & \xrightarrow{\bar{i}} & Q(A, B, A' \oplus B') \end{array}$$

According to Lemma 3.11, if $m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B]) \neq 0$, then there exist two conflations $C \xrightarrow{f_1} A' \xrightarrow{g_1} E$ and $D \xrightarrow{f_2} B' \xrightarrow{g_2} F$ in $\mathcal{A}$ such that $A \cong C \oplus D$, $B \cong E \oplus F$, $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$. If

$$m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B]) = 0,$$

then $[A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B] \in \mathfrak{Exact}_{\mathcal{A}}(\mathbb{K}) \setminus Q(A, B, A' \oplus B')$. Hence $\bar{i}$ is surjective. For each $[A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B] \in Q(A, B, A' \oplus B')$,

$$\begin{aligned} &\chi\big(\bar{i}^{-1}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B])\big) \\ &= \frac{m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B])}{m_{\pi_m}([C \xrightarrow{f_1} A' \xrightarrow{g_1} E]) m_{\pi_m}([D \xrightarrow{f_2} B' \xrightarrow{g_2} F])}. \end{aligned}$$

By Lemma 2.5, it follows that

$$cd\chi^{\mathrm{na}}(Q_c(C, E; A'))\chi^{\mathrm{na}}(Q_d(D, F; B')) = a\chi^{\mathrm{na}}(Q_c(A, B; A' \oplus B'),$$

where $c = m_{\pi_m}([C \xrightarrow{f_1} A' \xrightarrow{g_1} E])$, $d = m_{\pi_m}([D \xrightarrow{f_2} B' \xrightarrow{g_2} F])$, $a = m_{\pi_m}([A \xrightarrow{f} A' \oplus B' \xrightarrow{g} B])$ and $acd \neq 0$. This completes the proof. □

For all $f_1, f_2, g_1, g_2 \in \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$, define $(f_1 \otimes g_1) * (f_2 \otimes g_2) = (f_1 * f_2) \otimes (g_1 * g_2)$. Using Green's theorem, we have the following theorem due to [4, Theorem 24].

Theorem 4.6. *The map* $\Delta : \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}^{\mathrm{KS}}(\mathfrak{Obj}_{\mathcal{A}})$ *is an algebra homomorphism.*

Proof. The proof is similar to the one in [4, Theorem 24]. Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ be constructible sets of stratified Krull-Schmidt. Then

$$\Delta(1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}) = \Delta(\sum_{\lambda} g^{\lambda}_{\mathcal{O}_2\mathcal{O}_1} 1_{\mathcal{O}_\lambda}) = \sum_{\lambda} g^{\lambda}_{\mathcal{O}_2\mathcal{O}_1} \Delta(1_{\mathcal{O}_\lambda})$$
$$= \sum_{\lambda} g^{\lambda}_{\mathcal{O}_2\mathcal{O}_1} (\sum_{\alpha',\beta'} h^{\beta'\alpha'}_{\mathcal{O}_\lambda} 1_{\mathcal{O}_{\alpha'}} \otimes 1_{\mathcal{O}_{\beta'}}) = \sum_{\alpha',\beta'} g^{\alpha'\oplus\beta'}_{\mathcal{O}_2\mathcal{O}_1} 1_{\mathcal{O}_{\alpha'}} \otimes 1_{\mathcal{O}_{\beta'}},$$

$$\Delta(1_{\mathcal{O}_1}) * \Delta(1_{\mathcal{O}_2}) = (\sum_{\rho,\sigma} h^{\sigma\rho}_{\mathcal{O}_1} 1_{\mathcal{O}_\rho} \otimes 1_{\mathcal{O}_\sigma}) * (\sum_{\epsilon,\tau} h^{\tau\epsilon}_{\mathcal{O}_2} 1_{\mathcal{O}_\epsilon} \otimes 1_{\mathcal{O}_\tau})$$
$$= \sum_{\rho,\sigma,\epsilon,\tau} h^{\sigma\rho}_{\mathcal{O}_1} h^{\tau\epsilon}_{\mathcal{O}_2} (1_{\mathcal{O}_\rho} * 1_{\mathcal{O}_\epsilon}) \otimes (1_{\mathcal{O}_\sigma} * 1_{\mathcal{O}_\tau})$$
$$= \sum_{\rho,\sigma,\epsilon,\tau} h^{\sigma\rho}_{\mathcal{O}_1} h^{\tau\epsilon}_{\mathcal{O}_2} (\sum_{\alpha',\beta'} g^{\alpha'}_{\mathcal{O}_\epsilon\mathcal{O}_\rho} g^{\beta'}_{\mathcal{O}_\tau\mathcal{O}_\sigma} 1_{\mathcal{O}_{\alpha'}} \otimes 1_{\mathcal{O}_{\beta'}})$$
$$= \sum_{\alpha',\beta'} (\sum_{\rho,\sigma,\epsilon,\tau} h^{\sigma\rho}_{\mathcal{O}_1} h^{\tau\epsilon}_{\mathcal{O}_2} g^{\alpha'}_{\mathcal{O}_\epsilon\mathcal{O}_\rho} g^{\beta'}_{\mathcal{O}_\tau\mathcal{O}_\sigma} 1_{\mathcal{O}_{\alpha'}} \otimes 1_{\mathcal{O}_{\beta'}}).$$

According to Theorem 4.5, it follows that

$$\sum_{\rho,\sigma,\epsilon,\tau} h^{\sigma\rho}_{\mathcal{O}_1} h^{\tau\epsilon}_{\mathcal{O}_2} g^{\alpha'}_{\mathcal{O}_\epsilon\mathcal{O}_\rho} g^{\beta'}_{\mathcal{O}_\tau\mathcal{O}_\sigma} = g^{\alpha'\oplus\beta'}_{\mathcal{O}_2\mathcal{O}_1}.$$

Therefore $\Delta(1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}) = \Delta(1_{\mathcal{O}_1}) * \Delta(1_{\mathcal{O}_2})$. We have thus proved the theorem. □

Appendix A. Exact categories

We recall the definition of an exact category (see [13, Appendix A]).

Definition A.1. Let $\mathcal{A}$ be an additive category. A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{A}$ is called exact if f is a kernel of g and g is a cokernel of f. The morphisms f and g are called inflation and deflation respectively. The short exact sequence is called a conflation. Let $\mathcal{S}$ be the collection of conflations closed under isomorphism and satisfying the following axioms

A0 $1_0 : 0 \to 0$ is a deflation.

A1 The composition of two deflations is a deflation.

A2 For every $h \in \mathrm{Hom}(X, X')$ and every inflation $f \in \mathrm{Hom}(X, Y)$ in $\mathcal{A}$, there exists a pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

where $f' \in \mathrm{Hom}(X', Y')$ is an inflation.

A3 For every $l \in \mathrm{Hom}(Z', Z)$ and every deflation $g \in \mathrm{Hom}(Y, Z)$ in $\mathcal{A}$, there exists a pullback

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Z' \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} \\ Y & \xrightarrow{g} & Z \end{array}$$

where $g' \in \mathrm{Hom}(Y', Z')$ is an deflation. Then $(\mathcal{A}, \mathcal{S})$ is called an exact category.

The definition of idempotent complete is taken from[2, Definition 6.1].

Definition A.2. Let $\mathcal{A}$ be an additive category. The category $\mathcal{A}$ is idempotent complete if for every idempotent morphism $s : A \to A$ in $\mathcal{A}$, s has a kernel $k : K \to A$ and a image $i : I \to A$ (a kernel of a cokernel of s) such that $A \cong K \oplus I$. We write $A \cong \mathrm{Ker}s \oplus \mathrm{Im}s$, for simplicity.

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