A Note on the Heat Flow of Harmonic Maps Whose Gradients Belong to $L^q_t L^p_x$

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Abstract: For any compact $n$-dimensional Riemannian manifold $(M, g)$ without boundary, a compact Riemannian manifold $N \subset \mathbb{R}^k$ without boundary, and $0 < T \leq \infty$, we prove that for $n \geq 3$, if $u : M \times (0, T) \rightarrow N$ is a weak solution to the heat flow of harmonic maps such that $\nabla u \in L^p_t L^q_x (M \times (0, T))$ ($n/p + 2/q = 1$ for some $p > n$), then $u \in C^\infty (M \times (0, T), N)$. For $p = n$, we proved the regularity for the suitable weak solution defined in [1].

Keywords: Heat flow; Suitable solution; Lorentz space; Blow up.

1. Introduction

We adopt the notation and some definitions as in [1] and [2]. For $n \geq 1$, let $(M, g)$ be a smooth, compact $n$-dimensional Riemannian manifold without boundary, and $N \subset \mathbb{R}^k (k \geq 2)$, be a smooth, closed, oriented $m$-dimensional submanifold without boundary. For $0 < T \leq \infty$, a map $u \in C^2 (M \times (0, T), N)$ is a solution to the heat flow of harmonic maps, if

(1.1) $\frac{\partial u}{\partial t} = \Delta_g u + \sum_{\alpha, \beta = 1}^n g^{\alpha\beta} A(u) \left( \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \right)$ in $M \times (0, T),$

where $\Delta_g$ is the Laplace-Beltrami operator of $(M, g)$, $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of $N \subset \mathbb{R}^k$, and $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ is the inverse of $g = (g_{\alpha\beta})$. Let us recall the notation of weak solutions of (1.1).

Definition 1.1. A map $u : M \times [0, T] \rightarrow N$ is a weak solution of (1.1), if

(1) $u_t \in L^p_t L^q_x (M \times [0, T]), \nabla u \in L^p_t L^\infty_x (M \times [0, T]),$

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(2) $u$ satisfies (1.1) in the distribution sense:

$$\int_0^T \int_M u_t \cdot \phi + \nabla u \cdot \nabla \phi = \int_0^T \int_M A(u)(\nabla u, \nabla u) \cdot \phi,$$

for all $\phi \in C_0^\infty(M \times (0, T), R^k)$.

Our goal in this note is to get

**Theorem 1.1.** For $n \geq 3$, let $u : M \times [0, T] \to N$ be a weak solution of (1.1), with $\nabla u \in L^p_x L^q_t (M \times [0, T])$ for some $n < p \leq \infty$ satisfying $\frac{n}{p} + \frac{2}{q} = 1$. Then $u \in C^\infty(M \times (0, T], N)$. Moreover, if $\nabla u \in L^p_x L^q_t$, then there exists a small number $\varepsilon$, such that $\|\nabla u\|_{L^p_x L^q_t(M \times (0, T])} \leq \varepsilon$ which implies that $u \in C^\infty(M \times (0, T], N)$.

By standard parabolic estimate, Theorem 1.1 can be generalized by Theorem 1.3.

**Theorem 1.2.** For $n = 3$, let $u : M \times [0, T] \to N$ be a weak solution of (1.1) which satisfies the monotonicity inequalities (1.5) and energy inequality (1.6), and $\nabla u \in L^n_x L^\infty_t(\mathbb{R}^n \times [0, T])$. Then for any open subset $\omega$ and for any moment of time $t_0 \in (0, T)$, we have

$$N(t_0, \omega) \leq \varepsilon_0^{-3} \limsup_{r \to 0} \frac{1}{r^2} \int_{t_0}^{t_0-r^2} \int_{\omega} |\nabla u|^3(y, s)dyds.$$

Here, $N(t_0, \omega) = \text{card}\{\Sigma(t_0) \cap \omega\}$; i.e. $N(t_0, \omega)$ is the number of points in the set $\Sigma(t_0) \cap \omega$.

Note that the scaling invariant norm for $\nabla u$ is $\nabla u \in L^p_x L^q_t(M \times [0, T])$ for some $p \in [n, \infty)$ and $q \in [2, \infty)$ satisfying

$$\frac{n}{p} + \frac{2}{q} = 1. \quad (1.2)$$

The scaling invariant space $L^p_x L^q_t$ with $(p, q)$ satisfying (1.2) has played an important role in the regularity issue of Navier-Stokes equation for the Leray-Hopf weak solution. It is well known that both uniqueness and smoothness for the class of weak solutions $v$ of the Navier-Stokes Equation in which $v \in L^p_x L^q_t(\mathbb{R}^3 \times (0, \infty))$ for some $p \in (3, \infty]$ and $q \in [2, \infty)$ satisfying Serrin’s condition (1.2), have been established through works by Prodi [3], Serrin [4], and Ladyzhenskaya [5] in 1960s. On the other hand, for the end
point case $p = 3, q = \infty$, only until recently Escauriaza et al. [5,6] proved the smoothness for weak solutions $v \in L^3_xL^\infty_t, 0 < T < \infty$.

Motivated by these results for the Navier-Stokes equation, Wang [2] considered the class of weak solutions $u : M \times [0, T] \to N$ of (1.1) with $\nabla u \in L^p_xL^q_t(M \times [0, T])$ for some $p \in [n, +\infty]$ and $q \in [2, +\infty]$ satisfying Ser- rin’s condition (1.2). It is stated in [2] that
(i) if $n \geq 4$, and $u$ is a weak solution of (1.1) with $\nabla u \in L^n_xL^\infty_t$, then $u \in C^\infty(M \times (0, T), N)$.
(ii) If $n = 3$, they get the blow up criteria.
(iii) Either $n \geq 4$ or $2 \leq n < 4$ and $p \geq 4$, Theorem 1.1 is true with $\nabla u \in L^p_xL^q_t$ for some $p > n, q \geq 2$ satisfying $n/p + 2/q = 1$.

Our Theorem 1.1 extends their result (iii) to all $p, q$ with $p > n, q \geq 2$ satisfying (1.2).

Since the regularity is a local property, for the sake of simplicity, we will present our proofs in the case where $M = \mathbb{R}^n$. The general case is essentially the same, but technically a little more complicated. Here we shall consider the weak solutions of

\begin{equation}
\frac{\partial u}{\partial t} - \Delta u = A(u)(\nabla u, \nabla u), \text{ in } Q
\end{equation}

where $Q = \Omega \times (0, T)$, $\Omega$ is a domain in $\mathbb{R}^n(n \geq 3)$ with smooth boundary, $0 < T < \infty$. For any weak solution $u : \mathbb{R}^n \times (0, T) \to N$ of (1.3), define

$$
\Sigma = \{z_0 = (x_0, t_0) \in \mathbb{R}^n \times (0, T]; u \text{ is not continuous at } z_0\},
$$

and

$$
\Sigma(t_0) = \Sigma \cap \{t_0\}, \text{ for } t_0 \in (0, T].
$$

The proof in [2] depends on the fact that for $n \geq 4$, $u$ satisfies the monotonicity inequalities ([2, (2.4)]) (which is stronger than (1.5)) and the energy inequality (1.6) under the assumption of $\nabla u \in L^n_xL^\infty_t$ (see [2, Lemma 2.4 and Lemma 2.2]). So the case $n = 3, q = \infty$ and the case $4 > p > n = 3$ are not considered in their paper. Note that in [7], the author consider the interior regularity for the distribution solution of one kind parabolic system. It help us to deal with the case $n = 3, q = \infty$ and the case $4 > p > n = 3$. In Navier-Stokes equation, It is shown in [9], from the assumption $v \in L^{3,\infty}$ one can define the associated pressure $\tilde{p}$ such that $(v, \tilde{p})$ is a suitable weak solution of Navier-Stokes Equation. So the regularity for the weak solution $v \in L^{3,\infty}$ is just the regularity for the suitable weak solution in some sense.
In fact, if we denote \( Q_r(Q_r(x_0,t_0)) \) is a parabolic ball centered at \((x_0,t_0) \in Q:\)
\[
Q_r(x_0,t_0) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}; |x-x_0| < r, -r^2 < t - t_0 < 0\}
\]
such that \( Q_r \subset Q \) and \( B_r(x_0) = \{x \in \mathbb{R}^n; |x-x_0| < r\} \). where \( \Omega \) is a domain in \( \mathbb{R}^n \) with smooth boundary and \( 0 < T < \infty \). Using the results of [7], we have the \( \varepsilon \)-regularity for all \( p,q \) with \( p \geq n, q \geq 2 \) satisfying (1.2).

**Theorem 1.3.** If \( u \) is a weak solution of (1.1) in \( Q \) with \( u_t \in L^2_xL^2_t(Q), \nabla u \in L^2_xL^\infty_t(Q) \), then there is a positive constant \( \varepsilon < 1 \) such that
\[
\|\nabla u\|_{L^{p,q}(Q_r)} < \varepsilon \text{ which implies}
\]
(a) \( \nabla u \in L^\infty_xL^\beta_t(Q_{r/2}) \) for all \( 2 \leq \beta < \infty \) when \( p > n \).
(b) \( \nabla u \in L^\alpha_xL^\beta_t(Q_{r/2}) \) for all \( 2 \leq \alpha, \beta < \infty \) when \( p = n \).
Here \( \varepsilon = \varepsilon(n,m,p,\beta) \) if \( p > n \) and \( \varepsilon = \varepsilon(n,m,\alpha,\beta) \) if \( p = n \).

We recall the weak \(-L^q\) space for \( 1 < q < \infty \):
\[
L^q(0,T) = \{ f \in L^1(0,T); [f]_{L^q(0,T)} < \infty \},
\]
where
\[
[f]_{L^q(0,T)} = \sup_{s>0} s \mu\{t \in (0,T): |f(t)| > s\}^{1/q}.
\]

By Theorem 1.3, we also can get

**Theorem 1.4.** Let \( u \) be a weak solution of (1.1) in \( Q \) with \( u_t \in L^2_xL^2_t(Q), \nabla u \in L^2_xL^\infty_t(Q) \). Suppose that \( 1 \leq p, q \leq \infty \) satisfies \( n/p + 2/q = 1 \) and \( p > n \). Then there exists a positive constant \( \varepsilon = \varepsilon(n,m,p,\beta) \) such that
\[
(1.4) \quad \|\nabla u\|_{L^{p,q}(Q)} \leq \varepsilon
\]
which implies \( \nabla u \in L^\infty_xL^\beta_t(Q_{r/2}) \) for all \( \beta > 2 \).

**Remark 1.5.** The condition (1.4) is fulfilled if, for example,
\[
\|\nabla u(t)\|_{L^p(B_{r(t_0)})} \leq \frac{\varepsilon}{(t_0 - t)^{1/q}} \text{ for } t \in (-r^2 + t_0, t_0).
\]

**Definition 1.2.** We call a map \( u : M \times (0,T] \rightarrow N \) is a suitable weak solution of (1.1), if it is a weak solution of (1.1), and satisfy the following monotonicity inequalities (1.5) and the energy inequality (1.6).
We adopt the notation as in [1] and [2]. Denote by \( z = (x, t) \) a point in \( M \times \mathbb{R} \). For a distinguished point \( z_0 = (x_0, t_0) \), \( r > 0 \), let

\[
P_r(z_0) = \{ z = (x, t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_0| < r, |t - t_0| < r^2 \}
\]

and

\[
T_r(z_0) = \{ z = (x, t) \in \mathbb{R}^n \times \mathbb{R} : t_0 - 4r^2 < t < t_0 - r^2 \}.
\]

Denote the fundamental solution to the (backward) heat equation \((\frac{\partial}{\partial t} + \Delta)f(x, t) = 0\) on \( \mathbb{R}^m \times \mathbb{R} \) by

\[
G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(\frac{-(x - x_0)^2}{4(t_0 - t)}\right), t < t_0.
\]

We also denote by \( \delta \) the parabolic distance function

\[
\delta((x, t), (y, s)) = \max\{|x - y|, \sqrt{|t - s|}\}.
\]

Let \( \beta > 0 \) be any fixed constant. For any \( z_1 \in \mathbb{R}^n \times \mathbb{R}_+ \), define, for \( R \in (0, \sqrt{t_1/2\beta}) \),

\[
\Psi_\beta(R, u, z_1) = \frac{1}{2} \int_{T_{\beta R}(z_1)} |\nabla u|^2 G_{z_1} \phi_\beta^2 dxdt,
\]

where \( \phi_\beta(x) = \phi((x - x_1)/\beta) \) and \( \phi \in C_0^\infty(B_{1/2}(0)) \) is a cut-off function such that \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on \( B_{1/4}(0) \). It is proved in [6] that the regular solution of (1.1) satisfy:

**Monotonicity inequalities:** There exists a constant \( C > 0 \), depending only on \( m \), such that for any \( z_1 \in \mathbb{R}^n \times \mathbb{R}_+ \) and any \( 0 < R_1 < R_2 \leq \min(1/2, \sqrt{t_1/2\beta}) \),

\[
\Psi_\beta(R_1 u, z_1) \leq e^{C(R_2 - R_1)} \Psi_\beta(R_2, u, z_1) + C(R_2 - R_1)\beta^{-n} \int_{T_{R_2}(z_1)} |\nabla u|^2 dxdt.
\]

**Energy inequality:** For any \( \phi \in C_0^\infty(\mathbb{R}^n) \) and a.e. \( 0 \leq t_1 \leq t_2 < \infty \), it is true that

\[
2\int_{\mathbb{R}^n} \int_{t_1}^{t_2} |u_t|^2 \phi^2 + \int_{\mathbb{R}^n} \phi^2 |\nabla u|^2(x, t_2) \leq \int_{\mathbb{R}^n} \phi^2 |\nabla u|^2(x, t_1) + c(n) \int_{\mathbb{R}^n} \int_{t_1}^{t_2} |\nabla u|^2 |\nabla^2 \phi|^2.
\]

2. **Proof of Theorem 1.3**

**Proof of Theorem 1.3:** Assume that \( N \subset \mathbb{R}^k \) has an orthonormal frame field \( \nu_l \), \( 1 \leq l \leq k - m \), for the normal bundle to \( N \). By [8], the equation
(1.3) may equivalently be written as
\[
(2.1) \quad u_t^i - \Delta u^i = \Omega^{i,j} \cdot \nabla u^j,
\]
where \( \Omega \in L^2(Q; so(k) \times \Lambda^1 \mathbb{R}^k) \), with components locally given by
\[
(2.2) \quad \Omega^{i,j} = \sum_{l=1}^{k-m} \omega_l^i \omega_l^j - \omega_l^j \omega_l^i,
\]
for \( 1 \leq i, j \leq k \) and \( \omega_l = \nu_l \circ u \). If \( u \) is a weak solution of (1.3), then \( v = \nabla u = (v^{i,\alpha})_{i=1,k}^{\alpha=1,n} \in L^2_2 L^2_t(Q; \mathbb{R}^{k+n}) \) satisfying the following parabolic system
\[
(2.3) \quad v_t - \Delta v = \nabla(\Omega v)
\]
in \( Q \) in the distribution sense:
\[
\int \int_Q (\partial_t \phi + \Delta \phi - \text{div} \phi \Omega) v dx dt = 0
\]
for any \( \phi = (\phi^{i,\alpha})_{i=1,k}^{\alpha=1,n} \in C_0^\infty(Q; \mathbb{R}^{k+n}) \). Here
\[
\text{div} \phi \Omega = \left( \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} \phi^{1,\alpha}, \ldots, \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} \phi^{k,\alpha} \right) \begin{pmatrix}
\Omega^{1,1} & \ldots & \Omega^{1,k} \\
\Omega^{2,1} & \ldots & \Omega^{2,k} \\
\vdots & \ddots & \vdots \\
\Omega^{k,1} & \ldots & \Omega^{k,k}
\end{pmatrix}.
\]
Then by [7, Theorem 2.1 and Theorem 2.2 ], we can get Theorem 1.3 and Theorem 1.4.

### 3. Proof of Theorem 1.2

First, we have the following lemma,

**Lemma 3.1.** [2, Lemma 2.1] For \( n \geq 2 \) and \( 0 < T \leq +\infty \), suppose that \( u : M \times [0, T] \rightarrow N \) is a weak solution of (1.1) with \( \nabla u \in L^2_2 L^\infty_t(M \times [0, T]) \). Then \( u \in C([0, T], L^n(M)) \), and
\[
(3.1) \quad ||\nabla u(t)||_{L^n(M)} \leq ||\nabla u(t)||_{L^2_2 L^\infty_t(M \times [0, T])}, \quad \forall t \in [0, T].
\]
For \( z \in \mathbb{R}^n \times \mathbb{R}_+ \) and \( 0 < r < \sqrt{t} \) define

\[
E(r, u, z) = r^{-n} \int_{P_r(z)} |\nabla u|^2 dy ds.
\]

In fact, if we let \( \omega \subset \mathbb{R}^n \) be an open domain, we have the following Lemma which is given in [10] for stability solution of (1.3).

**Lemma 3.2.** Let \( u \in H^1_{loc}(\omega \times (0, \infty), N) \) be a weak solution of (1.3) satisfying the (1.5) and (1.6). Then for any parabolic cylinder \( P_{r_0}(z_0) \subset \omega \times (0, \infty) \) and for any \( z_1 = (x_1, t_1) \in P_{ar_0}(z_0) \) and \( 0 < r < br_0 \), where \( a \) and \( b \) are two positive constants satisfying \( a + 2b \leq 1/2 \), we have

\[
r^2 - n \left( \int_{P_r(z_1)} |\partial_t u|^2 dz + \int_{B_r(x_1) \times \{t_1\}} |\nabla u|^2 dx \right) \\
\leq C r_0^{-2} \int_{P_{r_0}(z_0)} |\nabla u|^2 dz.
\]

**Proof:** As the argument in [1, Lemma 2.2, Lemma 2.3](Although only the case \( N = S^k \) is considered there, these two lemmas are true without this restriction), we can show there exists \( K > 0 \), such that

\[
E(r, u, z_1) \leq KE(r_0, u, z_0)
\]

for any \( z_1 = (x_1, t_1) \in P_{ar_0}(z_0) \) and \( 0 < r < br_0 \), where \( a \) and \( b \) are two positive constants satisfying \( a + 2b \leq 1/2 \). By Fubini’s theorem we may choose \( \alpha \in (1/2, 7/8) \) such that

\[
\int_{B_r(x_1)} |
abla u|^2(y, t_1 - \alpha^2 r^2) dy \leq C r^{-2} \int_{P_r(z_1)} |\nabla u|^2 dy ds.
\]

Choose a smooth function \( \phi \in C^\infty_0(\mathbb{R}^n) \) such that \( \phi = 1 \) in \( B_{\alpha r}(x) \), \( \phi = 0 \) outside \( B_r(x_0) \), \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq C/r \). It follows from (1.6) and (3.2) that

\[
r^{2-n} \int_{P_r(z_1)} |\partial_t u|^2 dy ds \leq CE(r_0, u, z_0)
\]

for any \( z_1 = (x_1, t_1) \in P_{ar_0}(z_0) \) and \( 0 < r < br_0 \). On the other hand, use (1.6) with \( t_2 = t_1 \) and \( t_1 = t_0 - \alpha r_0^2 \), we can also have

\[
r^{2-n} \int_{B_r(x_1) \times \{t_1\}} |\nabla u|^2 dx \leq CE(r_0, u, z_0).
\]

By Lemma 3.2, [10, Theorem 2], and the argument in [10], we can get
Lemma 3.3. Let $n = 3$ or $n = 4$, and let $u \in H^1(P_r(z_0), N)$ be a weak solution of (1.3) satisfying (1.5) and (1.6). If
\[
 r^{-n} \int_{P_r(z_0)} |\nabla u|^2 \leq \varepsilon_0^2
\]
for a sufficiently small number $\varepsilon_0 > 0$, then $u$ is smooth in $P_{r/2}(z_0)$.

Proof of Theorem 1.2: For simplicity, we can assume that $M = \mathbb{R}^3$. Suppose that $\Sigma(t_0) \neq \emptyset$. We can follow the blow up argument in [2] to get a map $v : \mathbb{R}^n_{-1} \to N$ is a weak solution of (1.1), with $\nabla v \in L^3_xL^\infty_t(\mathbb{R}^n_{-1})$, and
\[
(3.3) \quad R^{-3} \int_{P_R} |\nabla v|^2 \geq \varepsilon_0^2, \quad \forall R > 0.
\]
For the completeness of our theorem, we show it here. By Lemma 3.3, we have that $x_0 \in \Sigma(t_0)$ implies that
\[
(3.4) \quad r^{-3} \int_{P_r(x_0,t_0)} |\nabla u|^2 \geq \varepsilon_0^2, \quad \forall r > 0.
\]
For $r_i \downarrow 0$, define $v_i(x,t) = u(x_0 + r_i x, t_0 + r_i^2 t) : \mathbb{R}^3 \times (-r_i^2, 0] \to N$. Then we can show $v_i(x,t)$ is a weak solution of (1.3), and $v_i$ satisfies
\[
(3.5) \quad \|\nabla v_i\|_{L^3_xL^\infty_t(\mathbb{R}^3 \times [-r_i^{-2} t_0, 0])} = \|\nabla u\|_{L^3_xL^\infty_t(\mathbb{R}^3 \times [0, t_0])} < \infty,
\]
and
\[
(3.6) \quad R^{-n} \int_{P_R} |\nabla v_i|^2 = (Rr_i)^{-3} \int_{P_{Rr_i}(x_0,t_0)} |\nabla u|^2 \geq \varepsilon_0^2, \quad \forall R > 0.
\]
Moreover, by (3.5), we have
\[
\sup_i \int_{P_R} |\nabla v_i|^2 \leq \sup_i R^{5/3} \left[ \int_{P_R} |\nabla v_i|^3(x,t)dxdt \right]^{2/3}
\leq \sup_i R^3 \|\nabla v_i\|_{L^3_xL^\infty_t(P_R)}
\leq \sup_i R^3 \sup_{t_0 - (Rr_i)^2 < s < t_0} \int_{B_R(x_0)} |\nabla u|^3(y,s)dy
\leq CR^3, \quad \forall R > 0.
\]
From Lemma 3.2, we have
\[
(3.8) \quad \sup_i \int_{P_R} |(v_i)_t|^2 \leq CR^{-2} \sup_i \int_{P_{2R}} |\nabla v_i|^2 \leq CR, \quad \forall R > 0.
\]
It follows from (3.7) and (3.8) that \( \{v_i\} \subset H^1_{loc}(\mathbb{R}^{n+1}_{-}) \) is a bounded sequence. Thus there exists a map \( v : \mathbb{R}^{n+1}_{-} \rightarrow N \) such that \( \nabla v_i \rightarrow \nabla v \) weakly in \( L^2_{loc}(\mathbb{R}^{n+1}_{-}) \), and \( v_i \rightarrow v \) strongly in \( L^2_{loc}(\mathbb{R}^{n+1}_{-}) \). Note that

\[(v)_t - \Delta v_i = A(v_i)(\nabla v_i, \nabla v_i) \text{ in } \mathbb{R}^n \times (-r^{-2}t_0, 0], \]

and

\[(3.10) |A(v_i)(\nabla v_i, \nabla v_i)| \leq C|\nabla v_i|^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}^{n+1}_{-}) \]

Thus

\[(3.11) r^{-2} \int_{P_r(x_0, t_0)} |\nabla v|^3 \geq \varepsilon^3_0, \forall r > 0. \]

For any finite subset \( \{x_1, ..., x_l\} \subset \Sigma(t_0) \cap \omega \), let \( r_0 > 0 \) be small enough so that \( \{B_r(x_j)\}_{j=1}^l \) are mutually disjoint for any \( 0 < r \leq r_0 \) and \( B_r(x_j) \subset \omega \) for all \( j = 1, ..., l \). By (3.11), for any \( 0 < r \leq r_0 \), we have

\[l \varepsilon^3_0 \leq r^{-2} \sum_{i=1}^l \int_{P_r(x_i, t_0)} |\nabla v|^3 \leq r^{-2} \int_{t_0}^{t_0 - r^{-2}} \int_\omega |\nabla u|^3.\]

Then the proof is over.

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