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Contracting Pinched Hypersurfaces in Spheres by Their Mean Curvature

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Abstract: In this paper, we study an open problem proposed in [10]. We prove that the mean curvature flow of hypersurfaces in the sphere will contract to a round point in finite time if the initial hypersurface satisfies a curvature pinching condition. Our theorem is a partial improvement of the convergence theorem due to Huisken [7].
Keywords: Mean curvature flow, hypersurface, sphere, curvature pinching.

1. Introduction

Let $F_0 : M^n \to \mathbb{S}^{n+1}$ be a smooth immersion from an n-dimensional Riemannian manifold without boundary to an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. The mean curvature flow is a one-parameter family of smooth immersions $F : M \times [0, T) \to \mathbb{S}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = \overrightarrow{H}(x,t), \\ F(x,0) = F_0(x), \end{cases} \tag{1.1}$$

where $\overrightarrow{H}(x,t)$ is the mean curvature vector of $M_t = F_t(M)$ and $F_t(x) = F(x,t)$. Let $\overrightarrow{H} = -H\nu$, where ν is a unit normal vector field of M in $\mathbb{S}^{n+1}$.

The mean curvature flow was proposed by Mullins [8] to describe the formation of grain boundaries in annealing metals. In [3], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, Huisken [5] showed that if the initial hypersurface in the Euclidean space is compact and uniformly convex, then

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the mean curvature flow converges to a round point in finite time. Later, he generalized this convergence theorem to the mean curvature flow of hypersurfaces in a Riemannian manifold in [6]. He also studied in [7] the mean curvature flow of hypersurfaces satisfying a pinching condition in a sphere.

Theorem 1.1 ([7]). *Let $n \geq 2$ and M_0 be a smooth closed immersed hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Suppose we have on M_0*

$$|A|^2 < \begin{cases} \frac{3}{4}H^2 + \frac{4}{3}, & n = 2, \\ \frac{1}{n-1}H^2 + 2, & n \geq 3. \end{cases} \tag{1.2}$$

Then one of the following holds:

a) Equation (1.1) has a smooth solution M_t on a finite time interval $0 \leq t < T$ and the M_t's converge uniformly to a single round point as $t \to T$.

b) Equation (1.1) has a smooth solution M_t for all $0 \leq t < \infty$ and the M_t's converge in the C^∞-topology to a smooth totally geodesic hypersurface M_∞.

Theorem 1.1 implies that all hypersurfaces of $\mathbb{S}^{n+1}$ satisfying (1.2) are diffeomorphic to $\mathbb{S}^n$. A more general differentiable sphere theorem for complete submanifolds in spheres were proved in [16]. More precisely, let M be an oriented complete submanifold in the unit sphere $\mathbb{S}^{n+q}$ with codimension $q \geq 1$. If $n \geq 2$ and if

$$\lambda(M) := \sup_M \left(|A|^2 - \frac{1}{n-1}|\overrightarrow{H}|^2 - 2\right) < 0, \tag{1.3}$$

then M is diffeomorphic to the standard sphere. Moreover, the pinching condition (1.3) is optimal in the sense that for any $\varepsilon > 0$, there is a positive constant $\mu > 0$ such that the submanifold $\mathbb{S}^1\left(\frac{1}{\sqrt{1+\mu^2}}\right) \times \mathbb{S}^{n-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right) \subset \mathbb{S}^{n+1} \subset \mathbb{S}^{n+q}$ satisfies $|A|^2 < \frac{1}{n-1}|\overrightarrow{H}|^2 + 2 + \varepsilon$.

For any $n \geq 2$ and $\kappa \in \mathbb{R}$, we define

$$\alpha(n,\kappa) = n + \frac{n}{2(n-1)}\kappa^2 - \frac{n-2}{2(n-1)}\sqrt{\kappa^4 + 4(n-1)\kappa^2}.$$

Shiohama-Xu [15] proved the following sphere theorem for submanifolds.

Theorem 1.2 ([15]). *Let M be an oriented complete submanifold in the unit sphere $\mathbb{S}^{n+q}$ with codimension $q \geq 1$. Suppose*

$$\Lambda(M) := \sup_M (|A|^2 - \alpha(n, |\overrightarrow{H}|)) < 0. \tag{1.4}$$

If $n \geq 2$ and $n \neq 3$, then M is homeomorphic to a sphere. If $n = 3$, then M is diffeomorphic to a spherical space form.

The pinching condition (1.4) is optimal since the submanifold $\mathbb{S}^1\Big(\frac{1}{\sqrt{1+\mu^2}}\Big) \times \mathbb{S}^{n-1}\Big(\frac{\mu}{\sqrt{1+\mu^2}}\Big) \subset \mathbb{S}^{n+1} \subset \mathbb{S}^{n+q}$ with some $\mu > 0$ satisfies $|A|^2 = \alpha(n, |\overrightarrow{H}|)$ (see Example 2 in [15]). It is easy to see that (1.4) is implied by (1.3). So it is a natural question that if a complete submanifold in the unit sphere $\mathbb{S}^{n+q}$ satisfying (1.4) is diffeomorphic to the standard sphere. In this paper, we give some partial answers to this problem.

To state our first theorem, we define the following

$$\alpha_\varepsilon(n, H) = (n - 4\varepsilon) + \frac{n}{2(n-1+\varepsilon)}H^2 - \frac{n-2}{2(n-1+\varepsilon)}\sqrt{H^4 + 4(n-1)H^2},$$

where ε is a positive constant.

Theorem 1.3. *Let $n \geq 3$ and M_0 be a smooth closed immersed hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Suppose there is a positive constant $\varepsilon \in (0, \frac{1}{8})$ such that at M_0*

$$|A|^2 \leq \alpha_\varepsilon(n, H) \text{ and } H \geq \frac{n}{\sqrt[4]{\varepsilon}}. \tag{1.5}$$

Then equation (1.1) has a smooth solution M_t on a finite time interval $0 \leq t < T$ and the M_t's converge uniformly to a single round point as $t \to T$.

As a consequence of Theorem 1.3, all smooth closed hypersurfaces in $\mathbb{S}^{n+1}$ satisfying (1.5) are diffeomorphic to the standard sphere.

Similarly, we define

$$\alpha_{s,\varepsilon}(n, H) = (s - 4\varepsilon) + \frac{s}{2(n-1+\varepsilon)}H^2 - \frac{s-2}{2(n-1+\varepsilon)}\sqrt{H^4 + 4(n-1)H^2},$$

where s and ε are positive constants.

Theorem 1.4. *Let $n \geq 3$ and M_0 be a smooth closed immersed hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Suppose there are positive constants $\varepsilon \in (0, \frac{1}{8})$ and $s \in [2, \frac{9}{4}]$ such that at M_0*

(1.6)
$$|A|^2 \leq \alpha_{s,\varepsilon}(n,H) \text{ and } H > \max\Big\{0, \operatorname{sgn}(s-2-\varepsilon)\frac{(n-1)^{1/3}(s-2)^{1/2}}{\varepsilon^{1/2}}\Big\}.$$

Then equation (1.1) has a smooth solution M_t on a finite time interval $0 \leq t < T$ and the M_t's converge uniformly to a single round point as $t \to T$.

As a consequence of Theorem 1.4, all smooth closed hypersurfaces in $\mathbb{S}^{n+1}$ satisfying (1.6) are diffeomorphic to the standard sphere.

The paper is organized as follows. In Section 2, we introduce some basic definitions in submanifold theory, recall some evolution equations along the mean curvature flow, and derive the evolution of the quantity $Q = |A|^2 - \alpha(n,H)$. In Section 3, we give the proof of Theorem 1.3. We first show that a pinching condition is preserved along the mean curvature flow, then we derive a pinching estimate for the tracefree second fundamental form. We also give an estimate of the gradient of the mean curvature in Section 3. These estimates are used to give the proof of Theorem 1.3. Using the similar stages we give the proof of Theorem 1.4 in Section 4.

2. Preliminaries

In this section, we introduce some basic definitions in submanifold theory and recall some evolution equations along the mean curvature flow in spheres. Let $F_0 : M \to \mathbb{S}^{n+1}$ be a smooth immersion of codimension 1 from an n-dimensional closed manifold M into an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. Denote by g and $d\mu$ the induced metric and the volume form on M. Let A and $\overrightarrow{H}$ be the second fundamental form and the mean curvature vector of M in $\mathbb{S}^{n+1}$, respectively. Choose a local orthonormal frame $\{e_i\}$ for the tangent bundle and let ν be the unit normal vector. Let $\{\omega_i\}$ be the dual frame of $\{e_i\}$. Then A and $\overrightarrow{H}$ can be written as

$$A = -\sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j \otimes \nu, \quad \overrightarrow{H} = -H\nu, \quad H = \sum_{i=1}^{n} h_{ii}.$$

The tracefree second fundamental form $\mathring{A}$ is defined by

$$\mathring{A} = -\sum_{i,j=1}^{n} \mathring{h}_{ij}\omega_i \otimes \omega_j \otimes \nu, \quad \mathring{h}_{ij} = h_{ij} - \frac{1}{n}H\delta_{ij}.$$

Let $F : M^n \times [0,T) \to \mathbb{S}^{n+1}$ be a family of immersions satisfying the mean curvature flow equation (1.1) with initial value F_0. We have the following evolution equations [7].

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 + 4H^2 - 2n|A|^2, \tag{2.1}$$

$$\frac{\partial}{\partial t}H = \Delta H + (|A|^2 + n)H. \tag{2.2}$$

The volume form $d\mu_t$ evolves as

$$\frac{\partial}{\partial t}d\mu_t = -H^2 d\mu_t. \tag{2.3}$$

Set $Q = |A|^2 - \alpha(n,H)$. Now we derive the evolution equation of Q.

$$\frac{\partial}{\partial t}Q = \frac{\partial}{\partial t}|A|^2 - \frac{n}{2n-2}\cdot\frac{\partial}{\partial t}H^2 + \frac{n-2}{2n-2}\cdot\frac{H^2\frac{\partial}{\partial t}H^2 + 2(n-1)\frac{\partial}{\partial t}H^2}{\sqrt{H^4+4(n-1)H^2}},$$

$$\begin{aligned}\Delta Q =& \Delta|A|^2 - \frac{n}{2n-2}\Delta H^2 + \frac{n-2}{2n-2}\nabla_i\left(\frac{H^2\nabla^i H^2 + 2(n-1)\nabla^i H^2}{\sqrt{H^4+4(n-1)H^2}}\right)\\ =& \Delta|A|^2 - \frac{n}{2n-2}\Delta H^2\\ &+ \frac{n-2}{2n-2}\cdot\frac{H^2\nabla^i\nabla_i H^2 + \nabla^i H^2\nabla_i H^2 + 2(n-1)\nabla^i\nabla_i H^2}{\sqrt{H^4+4(n-1)H^2}}\\ &- \frac{n-2}{2n-2}\cdot\frac{H^4|\nabla H^2|^2 + 4(n-1)H^2|\nabla H^2|^2 + 4(n-1)^2|\nabla H^2|^2}{(\sqrt{H^4+4(n-1)H^2})^3}.\end{aligned}$$

Hence

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)Q =&\Big(\frac{\partial}{\partial t}-\Delta\Big)|A|^2-\frac{n}{2n-2}\Big(\frac{\partial}{\partial t}-\Delta\Big)H^2\\
&+\frac{n-2}{2n-2}\cdot\frac{H^2\Big(\frac{\partial}{\partial t}-\Delta\Big)H^2+2(n-1)\Big(\frac{\partial}{\partial t}-\Delta\Big)H^2}{\sqrt{H^4+4(n-1)H^2}}\\
&+\frac{n-2}{2n-2}\cdot\frac{H^4|\nabla H^2|^2+4(n-1)H^2|\nabla H^2|^2+4(n-1)^2|\nabla H^2|^2}{(\sqrt{H^4+4(n-1)H^2})^3}\\
&-\frac{n-2}{2n-2}\cdot\frac{|\nabla H^2|^2}{\sqrt{H^4+4(n-1)H^2}}.
\end{aligned}
$$

Inserting (2.1) and (2.2) to the equation above we get

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)Q =&(-2|\nabla A|^2+2|A|^4+4H^2-2n|A|^2)\\
&-\frac{n}{2n-2}(-2|\nabla H|^2+2|A|^2H^2+2nH^2)\\
&+\frac{n-2}{2n-2}\cdot\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\cdot(-2|\nabla H|^2+2|A|^2H^2+2nH^2)\\
&+\frac{n-2}{2n-2}\cdot\frac{4(n-1)^2|\nabla H^2|^2}{(\sqrt{H^4+4(n-1)H^2})^3}.
\end{aligned}
$$

From [5], we have the inequality $|\nabla A|^2\geq\frac{3}{n+2}|\nabla H|^2$. Hence

$$
\Big(\frac{\partial}{\partial t}-\Delta\Big)Q \leq G+R_1+R_2, \tag{2.4}
$$

where

$$
G=\Big(-\frac{6}{n+2}+\frac{n}{n-1}-\frac{n-2}{n-1}\Big(\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\Big)\Big)|\nabla H|^2,
$$

$$
R_1=\frac{n-2}{2n-2}\cdot\frac{4(n-1)^2|\nabla H^2|^2}{(\sqrt{H^4+4(n-1)H^2})^3},
$$

$$
\begin{aligned}
R_2 =&(2|A|^4+4H^2-2n|A|^2)\\
&-\Big(\frac{n}{2n-2}-\frac{n-2}{2n-2}\cdot\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\Big)(2|A|^2H^2+2nH^2).
\end{aligned}
$$

3. Convergence of mean curvature flow under curvature pinching: I

In this section, we will give the proof of Theorem 1.3 by splitting the proof into several theorems. Let $F_t : M^n \to \mathbb{S}^{n+1}$ be a mean curvature flow solution on a maximal existence time interval $[0, T)$.

3.1. Preserving the curvature pinching condition

We first show that a curvature pinching condition is preserved under the mean curvature flow (1.1).

Theorem 3.1. *Let $n \geq 3$ and suppose we have on M_0*

$$Q < 0 \text{ and } H \geq \frac{1}{\sqrt[3]{3}} n,$$

then these inequalities are preserved under the mean curvature flow (1.1).

Proof. From the evolution equation (2.2) and the maximum principle, we see that the inequality $H \geq \frac{1}{\sqrt[3]{3}} n$ is preserved under the mean curvature flow (1.1). In fact, the inequality is improved along the mean curvature flow.

To prove that $Q < 0$ is preserved, we first show that if $Q = 0$ at a point in space-time, then $\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq 0$ at the same point.

Firstly, we have

$$\begin{aligned} R_2 =& (2|A|^4 + 4H^2 - 2n|A|^2) \\ &- \left(\frac{n}{2n-2} - \frac{n-2}{2n-2} \cdot \frac{H^2 + 2(n-1)}{\sqrt{H^4 + 4(n-1)H^2}}\right)(2|A|^2 H^2 + 2nH^2). \end{aligned}$$

By a direct computation, we see that $\alpha(n, H)$ is a positive solution to $R_2 = 0$ as an equation of $|A|^2$. More precisely, we see that when $|A|^2 = \alpha(n, H)$,

$$\begin{aligned} R_2 =& 2(\alpha^2(n, H) + 2H^2 - n\alpha(n, H)) \\ &- 2\left(\frac{n}{2n-2} - \frac{n-2}{2n-2} \cdot \frac{H^2 + 2(n-1)}{\sqrt{H^4 + 4(n-1)H^2}}\right)(H^2 \alpha(n, H) + nH^2) \\ =& 0. \end{aligned}$$

For $G+R_1$, we have

$$G+R_1 \le \Big(-\frac{6}{n+2}+\frac{n}{n-1}\Big)|\nabla H|^2 - \frac{n-2}{n-1}\cdot\frac{H^3+6(n-1)H}{(\sqrt{H^2+4(n-1)})^3}|\nabla H|^2.$$

Set $\varphi(\tau)=\frac{\tau^3+6(n-1)\tau}{(\sqrt{\tau^2+4(n-1)})^3}$. Then $\varphi\le 1$ and $\lim\limits_{\tau\to+\infty}\varphi(\tau)=1$. When $n\ge 3$, one has

$$-\frac{6}{n+2}+\frac{n}{n-1}-\frac{n-2}{n-1}=\frac{-4n+10}{(n+2)(n-1)}<0.$$

By a direct computation, we have $\varphi(\tau)>1-\frac{4n-10}{n^2-4}$ provided $\tau\ge\frac{n}{\sqrt[3]{3}}$. Hence, if $H\ge\frac{n}{\sqrt[3]{3}}$, one has $G+R_1\le(-\frac{6}{n+2}+\frac{n}{n-1}-\frac{n-2}{n-1}+\frac{n-2}{n-1}\cdot\frac{4n-10}{n^2-4})|\nabla H|^2\le 0$, which implies that $(\frac{\partial}{\partial t}-\Delta)Q\le 0$ at the point where $Q=0$.

Since we assume $Q=|A|^2-\alpha(n,H)<0$ at $t=0$, there is a positive constant ε such that $\tilde{Q}=|A|^2-\alpha_\varepsilon(n,H)\le 0$, where

$$\alpha_\varepsilon(n,H)=(n-4\varepsilon)+\frac{n}{2(n-1+\varepsilon)}H^2-\frac{n-2}{2(n-1+\varepsilon)}\sqrt{H^4+4(n-1)H^2}.$$

By using a similar computation, we can show that if $\tilde{Q}=0$ at a point, then $\Big(\frac{\partial}{\partial t}-\Delta\Big)\tilde{Q}\le 0$ at the same point. This implies that $\tilde{Q}<0$ is preserved under the mean curvature flow. □

3.2. A differential inequality

In this subsection, we assume at $t=0$ that $|A|^2\le\alpha_\varepsilon(n,H)$ for a positive constant $\varepsilon\in(0,\frac{1}{8})$.

Define a function $f_\sigma=\frac{|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}$ for a nonnegative constant σ, where and in the following we write $\alpha(n,H)$ as α for short. Now we compute the evolution of f_σ. Since

$$\begin{aligned}\frac{\partial}{\partial t}f_\sigma =&\frac{\frac{\partial}{\partial t}|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{|\mathring{A}|^2\frac{\partial}{\partial t}(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}}\\ =&\frac{\frac{\partial}{\partial t}|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{f_\sigma\frac{\partial}{\partial t}(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}\end{aligned}$$

and

$$\begin{aligned}\Delta f_\sigma =&\nabla^i\nabla_i f_\sigma\\ =&\nabla^i\bigg(\frac{\nabla_i|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{f_\sigma\nabla_i(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}\bigg)\\ =&\frac{\nabla^i\nabla_i|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{\nabla_i|\mathring{A}|^2\nabla^i(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}}\\ &+(\sigma-1)\frac{\nabla^i|\mathring{A}|^2\nabla_i(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}}+(\sigma-1)\frac{|\mathring{A}|^2\nabla^i\nabla_i(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}}\\ &+(\sigma-2)(\sigma-1)\frac{|\mathring{A}|^2\nabla^i(\alpha-\frac{1}{n}H^2)\nabla_i(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{3-\sigma}}\\ =&\frac{\Delta|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+2(\sigma-1)\frac{\nabla_i|\mathring{A}|^2\nabla^i(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}}\\ &+(\sigma-1)\frac{f_\sigma\Delta(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}+(\sigma-2)(\sigma-1)\frac{f_\sigma|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^2},\end{aligned}$$

we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)f_\sigma =&\frac{\Big(\frac{\partial}{\partial t}-\Delta\Big)|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}-2(\sigma-1)\frac{\nabla_i|\mathring{A}|^2\nabla^i(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}}\\ &+(\sigma-1)\frac{f_\sigma\Big(\frac{\partial}{\partial t}-\Delta\Big)(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}\\ &-(\sigma-2)(\sigma-1)\frac{f_\sigma|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^2}.\end{aligned}$$

Since

$$\nabla f_\sigma=\frac{\nabla|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{f_\sigma\nabla(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2},$$

we have

$$\nabla|\mathring{A}|^2=\Big(\alpha-\frac{1}{n}H^2\Big)^{1-\sigma}\nabla f_\sigma-(\sigma-1)\frac{f_\sigma\nabla(\alpha-\frac{1}{n}H^2)}{(\alpha-\frac{1}{n}H^2)^\sigma}.$$

Then we get

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)f_\sigma =&\frac{\Big(\frac{\partial}{\partial t}-\Delta\Big)|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{f_\sigma\Big(\frac{\partial}{\partial t}-\Delta\Big)(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}\\&-2(\sigma-1)\frac{\nabla_i f_\sigma\nabla^i(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}+2(\sigma-1)^2\frac{f_\sigma|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^2}\\&-(\sigma-2)(\sigma-1)\frac{f_\sigma|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^2}.\end{aligned}$$

We also have

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)\alpha =&\frac{n}{2n-2}\Big(\frac{\partial}{\partial t}-\Delta\Big)H^2\\&-\frac{n-2}{2n-2}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big(\frac{\partial}{\partial t}-\Delta\Big)H^2\\&-\frac{n-2}{n-1}\cdot\frac{8(n-1)^2H^2|\nabla H|^2}{(\sqrt{H^4+4(n-1)H^2})^3}\\=&\Big(\frac{n}{n-1}-\frac{n-2}{n-1}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2\\&+\Big(-\frac{n}{n-1}+\frac{n-2}{n-1}\cdot\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)|\nabla H|^2.\end{aligned}$$

Inserting the evolution equations of $|A|^2$, H and α into the evolution equation of f_σ, we get

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)f_\sigma =&\frac{-2|\nabla A|^2+\frac{2}{n}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}+\frac{2|A|^4+2H^2-2n|A|^2-\frac{2}{n}|A|^2H^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}\\&+(1-\sigma)\frac{f_\sigma\Big(\frac{n^2-2n+2}{n(n-1)}-\frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)|\nabla H|^2}{\alpha-\frac{1}{n}H^2}\\&-(1-\sigma)\frac{f_\sigma\Big(\frac{n^2-2n+2}{n(n-1)}-\frac{n-2}{n-1}\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2}{\alpha-\frac{1}{n}H^2}\\&+2(1-\sigma)\frac{\nabla_i f_\sigma\nabla^i(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2}+\sigma(\sigma-1)\frac{f_\sigma|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^2}.\end{aligned}$$

To estimate the gradient terms, we have

$$
\begin{aligned}
G' :=& \frac{-2|\nabla A|^2 + \frac{2}{n}|\nabla H|^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + (1-\sigma)\frac{f_\sigma\Big(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)|\nabla H|^2}{\alpha - \frac{1}{n}H^2} \\
&+ 2(1-\sigma)\frac{\nabla_i f_\sigma \nabla^i(\alpha - \frac{1}{n}H^2)}{(\alpha - \frac{1}{n}H^2)^{2-\sigma}} + \sigma(\sigma-1)\frac{f_\sigma|\nabla(\alpha - \frac{1}{n}H^2)|^2}{(\alpha - \frac{1}{n}H^2)^2} \\
\leq& \Big(\frac{-4n+4}{n(n+2)}\frac{1}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + \frac{1-\sigma}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}\Big(\frac{n^2-2n+2}{n(n-1)} \\
&- \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)\Big)|\nabla H|^2 + 2(1-\sigma)\frac{\nabla_i f_\sigma \nabla^i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} \\
=& \frac{1}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}\Big(\frac{-4n+4}{n(n+2)} + (1-\sigma)\Big(\frac{n^2-2n+2}{n(n-1)} \\
&- \frac{n-2}{n-1}\frac{H^2+(6n-6)H}{(\sqrt{H^2+4(n-1)})^3}\Big)\Big)|\nabla H|^2 + 2(1-\sigma)\frac{\nabla_i f_\sigma \nabla^i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2}.
\end{aligned}
$$

Here we have used the pinching condition $|A|^2 \leq \alpha_\varepsilon(n,H)$, which implies $\frac{|\mathring{A}|^2}{\alpha - \frac{1}{n}H^2} \leq 1$, and discard the last term in G' since $0 < \sigma < 1$. If we assume that $H \geq \frac{1}{\sqrt[3]{3}}n$, then by a direct computation we see that

$$
\frac{-4n+4}{n(n+2)} + (1-\sigma)\Big(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big) \leq -2\varepsilon_\nabla,
$$

where $\varepsilon_\nabla = \varepsilon_\nabla(n)$ is a positive constant depending only on n. Hence we get

$$
G' \leq 2(1-\sigma)\frac{\nabla_i f_\sigma \nabla^i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} - \frac{2\varepsilon_\nabla}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}|\nabla H|^2.
$$

For the remaining terms in the evolution equation of f_σ, we have

$$
\begin{aligned}
R' :=& \frac{2|A|^4 + 2H^2 - 2n|A|^2 - \frac{2}{n}|A|^2H^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} \\
&- (1-\sigma)\frac{f_\sigma\Big(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2}{\alpha - \frac{1}{n}H^2} \\
=& \frac{2|\mathring{A}|^2(|A|^2-n)}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + (\sigma-1)\frac{f_\sigma\Big(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2}{\alpha - \frac{1}{n}H^2}.
\end{aligned}
$$

If $|A|^2 \leq n$ at a point, then the second term of the second equality in the above equation is nonpositive. Hence we get

$$R' \leq 2(|A|^2 - n)f_\sigma \leq 2\sigma(|A|^2 - n)f_\sigma.$$

Now we suppose $n < |A|^2 \leq \alpha_\varepsilon(n, H) < \alpha$. Then

$$R' = 2(|A|^2 - n)\left(1 - \frac{(1-\sigma)\left(\frac{(n^2-2n+2)H^2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^4+2(n-1)H^2}{\sqrt{H^4+4(n-1)H^2}}\right)}{2n + \frac{(n^2-2n+2)H^2}{n(n-1)} - \frac{n-2}{n-1}\sqrt{H^4+4(n-1)H^2}} \cdot \frac{|A|^2+n}{|A|^2-n}\right)f_\sigma.$$

Since

$$\begin{aligned} C_\sigma :=& \frac{\frac{n^2-2n+2}{n(n-1)}H^2 - \frac{n-2}{n-1}\frac{H^4+2(n-1)H^2}{\sqrt{H^4+4(n-1)H^2}}}{2n + \frac{n^2-2n+2}{n(n-1)}H^2 - \frac{n-2}{n-1}\sqrt{H^4+4(n-1)H^2}} \cdot \frac{|A|^2+n}{|A|^2-n} \\ \geq& \frac{\frac{n^2-2n+2}{n(n-1)}H^2 - \frac{n-2}{n-1}\frac{H^4+2(n-1)H^2}{\sqrt{H^4+4(n-1)H^2}}}{2n + \frac{n^2-2n+2}{n(n-1)}H^2 - \frac{n-2}{n-1}\sqrt{H^4+4(n-1)H^2}} \cdot \frac{\alpha+n}{\alpha-n} \equiv 1. \end{aligned}$$

We get

$$R' \leq 2\sigma(|A|^2 - n)f_\sigma.$$

So, we have proved the following theorem.

Theorem 3.2. *Let $n \geq 3$ and suppose $|A|^2 \leq \alpha_\varepsilon(n, H)$, $\sigma \in (0, \frac{1}{2}\varepsilon)$ and $H \geq \frac{1}{\sqrt[3]{3}}n$ at $t = 0$. Then there is a positive constant ε_∇ depending only on n such that following inequality holds.*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)f_\sigma \leq& -2(\sigma-1)\frac{\nabla_i f_\sigma \nabla^i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} \\ & - \frac{2\varepsilon_\nabla}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}|\nabla H|^2 + 2\sigma(|A|^2 - n)f_\sigma. \end{aligned} \tag{3.1}$$

3.3. The estimate of Z

To carry out the Stampacchia procedure, we have to estimate the term Z in the following Simons identity for hypersurfaces in the unit sphere $\mathbb{S}^{n+1}$.

$$\frac{1}{2}\Delta|A|^2 = \sum_{i,j=1}^{n} h_{ij}\nabla_i\nabla_j H + |\nabla A|^2 + Z + n|\mathring{A}|^2,$$

where $Z = H\mathrm{tr}A^3 - |A|^4$.

At a fixed point, we choose a local frame fields such that $h_{ij} = \lambda_i \delta_{ij}$ at this point. Set $\mathring{\lambda}_i = \lambda_i - \frac{1}{H}$. Then

$$Z = -|\mathring{A}|^4 + \frac{1}{n}|\mathring{A}|^2 H^2 + H \sum_{i=1}^{n} \mathring{\lambda}_i^3.$$

By an inequality in [14], we have for any $\mu > 0$

$$H \sum_{i=1}^{n} \mathring{\lambda}_i^3 \geq -\frac{n-2}{\sqrt{n(n-1)}} H|\mathring{A}|^3 \geq -\frac{\mu}{2}|\mathring{A}|^4 - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}|\mathring{A}|^2 H^2.$$

Hence

$$\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\alpha - \frac{1}{n}H^2)} \geq \frac{n + \left(\frac{1}{n} - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}\right)H^2 - \frac{\mu+2}{2}|\mathring{A}|^2}{n + \frac{n^2-2n+2}{2n(n-1)}H^2 - \frac{n-2}{2n-1}\sqrt{H^4 + 4(n-1)H^2}}.$$

When $|A|^2 \leq \alpha_\varepsilon(n, H)$ for $\varepsilon \in (0, \frac{1}{2})$, we have

$$\begin{aligned} &\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\alpha - \frac{1}{n}H^2)} \\ &\geq \frac{n + \left(\frac{1}{n} - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}\right)H^2 - \frac{\mu+2}{2}(\alpha_\varepsilon(n,H) - \frac{1}{n}H^2)}{n + \frac{n^2-2n+2}{2n(n-1)}H^2 - \frac{n-2}{2n-1}\sqrt{H^4 + 4(n-1)H^2}} \\ &= \frac{n + \frac{H^2}{2(n-1)} - \frac{n}{2}\left(n - 4\varepsilon + \frac{n(n-2)+2-2\varepsilon}{2n(n-1+\varepsilon)}H^2 - \frac{(n-2)\sqrt{H^4+4(n-1)H^2}}{2(n-1+\varepsilon)}\right)}{n + \frac{n^2-2n+2}{2n(n-1)}H^2 - \frac{n-2}{2n-1}\sqrt{H^4 + 4(n-1)H^2}}. \end{aligned}$$

If $H^2 \geq \frac{2(n-1-\frac{\theta}{2})^2}{\theta}$, $\theta > 0$, then

$$H^2 + 2n - 2 - \theta \leq \sqrt{H^4 + 4(n-1)H^2} \leq H^2 + 2n - 2.$$

Now we suppose $H^2 \geq Cn^2$ with $C > 0$ to be determined. Then

$$\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\alpha - \frac{1}{n}H^2)} \geq \frac{n - \frac{n}{2}(n - 4\varepsilon) + \frac{1}{2}\left(\frac{1}{n-1} - \frac{1-\varepsilon}{n-1+\varepsilon}\right)H^2 + \frac{n(n-1)(n-2)(1-\frac{n-1}{Cn^2})}{2(n-1+\varepsilon)}}{n + \frac{n^2-2n+2}{2n(n-1)}H^2 - \frac{n-2}{2n-1}\sqrt{H^4 + 4(n-1)H^2}}.$$

If we pick $C = \frac{1}{\sqrt{\varepsilon}}$, then the right hand side of the inequality above has a positive lower bound of the form $C'n$ for a positive constant C' that is independent of n. That is, we get the following

Theorem 3.3. *If $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$, then there is a positive constant ε_Z independent of n such that we have*

$$\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\alpha - \frac{1}{n}H^2)} \geq \varepsilon_Z n > 0,\ t \in [0,T).$$

3.4. An integral inequality

In this subsection, we want to obtain a lower bound of Δf_σ.

$$\begin{aligned}
\Delta f_\sigma =& \frac{\Delta|\mathring{A}|^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + 2(\sigma-1)\frac{\nabla^i f_\sigma \nabla_i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} \\
&+ (\sigma-1)\frac{f_\sigma \Delta(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} + \sigma(1-\sigma)\frac{f_\sigma|\nabla(\alpha - \frac{1}{n}H^2)|^2}{(\alpha - \frac{1}{n}H^2)^2} \\
\geq& \frac{2(\mathring{h}^{ij}\nabla_i\nabla_j H + Z + n|\mathring{A}|^2 + |\nabla\mathring{A}|^2)}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + 2(\sigma-1)\frac{\nabla^i f_\sigma \nabla_i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} \\
&+ (\sigma-1)\frac{f_\sigma H\Delta H}{\alpha - \frac{1}{n}H^2}\left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\right) \\
&+ \frac{(\sigma-1)f_\sigma}{\alpha - \frac{1}{n}H^2}\left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right)|\nabla H|^2.
\end{aligned}$$

Since $|\nabla\mathring{A}|^2 = |\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \geq \frac{2n-2}{n(n+2)}|\nabla H|^2$, we have

$$\begin{aligned}
&\frac{2|\nabla\mathring{A}|^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + \frac{(\sigma-1)f_\sigma}{\alpha - \frac{1}{n}H^2}\left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right)|\nabla H|^2 \\
\geq& \frac{2(\frac{3}{n+2} - \frac{1}{n})|\nabla H|^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} \\
&- \frac{1}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}\left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right)|\nabla H|^2 \\
=& \frac{|\nabla H|^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}\left(\frac{4n-4}{n(n+2)} - \frac{n^2-2n+2}{n(n-1)} + \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right).
\end{aligned}$$

By a direct computation, we see that if $H \geq \frac{1}{\sqrt[4]{\varepsilon}}n > \frac{1}{\sqrt[3]{3}}n$, then

$$\frac{4n-4}{n(n+2)} - \frac{n^2-2n+2}{n(n-1)} + \frac{n-2}{n-1}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3} > 0.$$

Hence

$$
\begin{aligned}
\Delta f_\sigma \geq & \frac{2(\mathring{h}^{ij}\nabla_i\nabla_j H + Z + n|\mathring{A}|^2)}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + 2(\sigma-1)\frac{\nabla^i f_\sigma \nabla_i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} \\
& + (\sigma - 1)\frac{f_\sigma H \Delta H}{\alpha - \frac{1}{n}H^2}\left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\right).
\end{aligned}
$$

Multiplying both sides of the above inequality by f_σ^{p-1} and integrating on M_t, we get

$$
\begin{aligned}
& \int_{M_t} f_\sigma^{p-1}\Delta f_\sigma \\
\geq & \int_{M_t} \frac{2\mathring{h}^{ij}\nabla_i\nabla_j H f_\sigma^{p-1}}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + \int_{M_t} \frac{2(Z + n|\mathring{A}|^2) f_\sigma^{p-1}}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} \\
& + \int_{M_t} \frac{2(\sigma-1) f_\sigma^{p-1}\nabla^i f_\sigma \nabla_i(\alpha - \frac{1}{n}H^2)}{\alpha - \frac{1}{n}H^2} \\
& + \int_{M_t} \frac{(\sigma-1) f_\sigma^p H \Delta H}{\alpha - \frac{1}{n}H^2}\left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\right) \\
:= & S_1 + S_2 + S_3 + S_4.
\end{aligned}
$$

Now we use integration by parts to handle S_1 and S_4. By a direct computation, we have $\nabla_i \mathring{h}^{ij} = \frac{n-1}{n}\nabla^j H$. Hence

$$
\begin{aligned}
S_1 = & -2\int_{M_t} \nabla_j H\left(\frac{\nabla_i \mathring{h}^{ij} f_\sigma^{p-1} + (p-1)\mathring{h}^{ij} f_\sigma^{p-2}\nabla_i f_\sigma}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}}\right. \\
& \left. - \frac{(1-\sigma)\mathring{h}^{ij} f_\sigma^{p-1}\nabla_i(\alpha - \frac{1}{n}H^2)}{(\alpha - \frac{1}{n}H^2)^{2-\sigma}}\right) \\
= & -2\int_{M_t} \frac{\nabla_j H(\nabla_i \mathring{h}^{ij} f_\sigma^{p-1} + (p-1)\mathring{h}^{ij} f_\sigma^{p-2}\nabla_i f_\sigma)}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} \\
& - 2\int_{M_t} \frac{f_\sigma^{p-1} H(\sigma-1)\mathring{h}^{ij}\nabla_i H \nabla_j H}{(\alpha - \frac{1}{n}H^2)^{2-\sigma}} \\
& \times \left(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1}\frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}\right).
\end{aligned}
$$

$$\begin{aligned}
S_4 =& \int_{M_t} \nabla^i H \nabla_i \bigg(\frac{(1-\sigma) f_\sigma^p H}{\alpha - \frac{1}{n}H^2} \bigg(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1} \frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}} \bigg) \bigg) \\
=& \int_{M_t} \bigg(\frac{(1-\sigma) f_\sigma^p |\nabla H|^2}{\alpha - \frac{1}{n}H^2} \bigg(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1} \frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}} \bigg) \bigg) \\
&+ \int_{M_t} \frac{(1-\sigma) p f_\sigma^{p-1} H}{\alpha - \frac{1}{n}H^2} \bigg(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1} \frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}} \bigg) \nabla_i f_\sigma \nabla^i H \\
&- \int_{M_t} \frac{(1-\sigma) f_\sigma^p H}{(\alpha - \frac{1}{n}H^2)^2} \bigg(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1} \frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}} \bigg) \\
&\times \nabla_i \Big(\alpha - \frac{1}{n}H^2 \Big) \nabla^i H \\
&+ \int_{M_t} \bigg(\frac{8(1-\sigma) f_\sigma^p}{\alpha - \frac{1}{n}H^2} \cdot \frac{n-2}{n-1} \cdot \frac{(n-1)^2 H^2}{(\sqrt{H^4+4(n-1)H^2})^3} \bigg) |\nabla H|^2.
\end{aligned}$$

The third term of the right hand side of the above equation is equal to

$$-\int_{M_t} \frac{(1-\sigma) f_\sigma^p H^2}{(\alpha - \frac{1}{n}H^2)^2} \bigg(\frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1} \frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}} \bigg)^2 |\nabla H|^2.$$

Set $r(n,H) = \frac{n^2-2n+2}{n(n-1)} - \frac{n-2}{n-1} \frac{H^2+2(n-1)}{\sqrt{H^4+4(n-1)H^2}}$. Then if $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$, we have $0 < r(n,H) < \frac{2}{n(n-1)} =: 2a$.

Since $\int_{M_t} f_\sigma^{p-1} \Delta f_\sigma \geq \sum_{i=1}^4 S_i$, we have

$$\begin{aligned}
S_2 \leq& -(p-1) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 + 2(p-1) \int_{M_t} \frac{f_\sigma^{p-2} |\nabla f_\sigma| |\nabla H| |\mathring{A}|}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} \\
&+ \frac{2(n-1)}{n} \int_{M_t} \frac{f_\sigma^{p-1} |\nabla H|^2}{(\alpha - \frac{1}{n}H^2)^{1-\sigma}} + 2(1-\sigma) \int_{M_t} \frac{f_\sigma^{p-1} H r(n,H) |\nabla H|^2 |\mathring{A}|}{(\alpha - \frac{1}{n}H^2)^{2-\sigma}} \\
&+ (1-\sigma)(p-2) \int_{M_t} \frac{f_\sigma^{p-1} H r(n,H) |\nabla f_\sigma| |\nabla H|}{\alpha - \frac{1}{n}H^2} \\
&+ (1-\sigma) \int_{M_t} \frac{f_\sigma^p H^2 r^2(n,H) |\nabla H|^2}{(\alpha - \frac{1}{n}H^2)^2} \\
&- \int_{M_t} \bigg(\frac{8(1-\sigma) f_\sigma^p}{\alpha - \frac{1}{n}H^2} \cdot \frac{n-2}{n-1} \cdot \frac{(n-1)^2 H^2}{(\sqrt{H^4+4(n-1)H^2})^3} \bigg) |\nabla H|^2
\end{aligned}$$

$$\begin{aligned}
\leq &-(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + 2(p-1)\int_{M_t}\frac{f_\sigma^{p-2}|\nabla f_\sigma||\nabla H||\mathring{A}|}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} \\
&+\frac{2(n-1)}{n}\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} + 4a(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}H|\nabla H|^2|\mathring{A}|}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}} \\
&+2a(1-\sigma)(p-2)\int_{M_t}\frac{f_\sigma^{p-1}H|\nabla f_\sigma||\nabla H|}{\alpha-\frac{1}{n}H^2} + 4a^2(1-\sigma)\int_{M_t}\frac{f_\sigma^p H^2|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^2}.
\end{aligned}$$

Also, since

$$\begin{aligned}
\frac{H}{\sqrt{\alpha-\frac{1}{n}H^2}} =& \frac{1}{\sqrt{\frac{n}{H^2}+\frac{n^2-2n+2}{2n(n-1)}-\frac{n-2}{2(n-1)}\sqrt{1+\frac{4(n-1)}{H^2}}}} \\
\leq& \frac{1}{\sqrt{\frac{n}{H^2}+\frac{n^2-2n+2}{2n(n-1)}-\frac{n-2}{2(n-1)}(1+\frac{2(n-1)}{H^2})}} \\
\leq& \frac{1}{\sqrt{\frac{2}{H^2}+\frac{1}{n(n-1)}}} < \frac{1}{\sqrt{a}},
\end{aligned}$$

and

$$\frac{f_\sigma}{\alpha-\frac{1}{n}H^2} = \frac{|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}} \leq \frac{1}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}},$$

we have for any $\eta > 0$,

$$\begin{aligned}
&2(p-1)\int_{M_t}\frac{f_\sigma^{p-2}|\nabla f_\sigma||\nabla H||\mathring{A}|}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} \\
\leq& \frac{p-1}{\eta}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + (p-1)\eta\int_{M_t}\frac{f_\sigma^{p-2}|\nabla H|^2|\mathring{A}|^2}{(\alpha-\frac{1}{n}H^2)^{2(1-\sigma)}} \\
=& \frac{p-1}{\eta}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + (p-1)\eta\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}};
\end{aligned}$$

$$\begin{aligned}
&4a(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}H|\nabla H|^2|\mathring{A}|}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}} \\
\leq& 4\sqrt{a}(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}\frac{\sqrt{\alpha-\frac{1}{n}H^2}|\mathring{A}|}{\alpha-\frac{1}{n}H^2} \\
\leq& 4\sqrt{a}(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}};
\end{aligned}$$

$$
\begin{aligned}
&2a(1-\sigma)(p-2)\int_{M_t}\frac{f_\sigma^{p-1}H|\nabla f_\sigma||\nabla H|}{\alpha-\frac{1}{n}H^2}\\
&\leq (1-\sigma)(p-2)\Big(\frac{1}{\eta}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+\eta a\int_{M_t}\frac{af_\sigma^p H^2|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^2}\Big)\\
&\leq (1-\sigma)(p-2)\Big(\frac{1}{\eta}\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2+\eta a\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}\Big);
\end{aligned}
$$

$$
\begin{aligned}
4a^2(1-\sigma)\int_{M_t}\frac{f_\sigma^p H^2|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^2} \leq& 4a(1-\sigma)\int_{M_t}\frac{f_\sigma^p|\nabla H|^2}{\alpha-\frac{1}{n}H^2}\\
\leq& 4a(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}.
\end{aligned}
$$

Combining these inequalities together we get

$$
\begin{aligned}
S_2 \leq& \Big(\frac{2p-3}{\eta}-p+1\Big)\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2\\
&+\Big(\eta(ap+p-1-2a)+2+4\sqrt{a}+4a\Big)\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}.
\end{aligned}
$$

Also, by the estimate of $Z+n|\mathring{A}|^2$, we have $S_2 \geq 2n\varepsilon_Z\int_{M_t}f_\sigma^p(\alpha-\frac{1}{n}H^2)$. Hence we get the following

Theorem 3.4. *If $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$ and $0<\varepsilon<\frac{1}{8}$, then we have*

$$
\begin{aligned}
&2n\varepsilon_Z\int_{M_t}f_\sigma^p(\alpha-\frac{1}{n}H^2)\\
&\leq \Big(\frac{2p-3}{\eta}-p+1\Big)\int_{M_t}f_\sigma^{p-2}|\nabla f_\sigma|^2\\
&\quad+\Big(\eta(ap+p-1-2a)+2+4\sqrt{a}+4a\Big)\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}. \qquad (3.2)
\end{aligned}
$$

3.5. The L^p-estimate

We multiply both sides of (3.1) by f_σ^{p-1} and integrate it by parts to get

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p \\
&\leq p\int_{M_t} f_\sigma^{p-1}\Delta f_\sigma + 2p(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}\nabla_i f_\sigma \nabla^i(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2} \\
&\quad - 2\varepsilon_\nabla p\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} + 2\sigma p\int_{M_t}(|A|^2-n)f_\sigma^p - \int_{M_t} H^2 f_\sigma^p \\
&\leq -p(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + 2p(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}\nabla_i f_\sigma \nabla^i(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2} \\
&\quad - 2\varepsilon_\nabla p\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} + 2\sigma p\int_{M_t}(|A|^2-n)f_\sigma^p.
\end{aligned}$$

When $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$,

$$\begin{aligned}
\frac{2f_\sigma^{p-1}\nabla_i f_\sigma\nabla^i(\alpha-\frac{1}{n}H^2)}{\alpha-\frac{1}{n}H^2} &\leq \frac{1}{\mu}f_\sigma^{p-2}|\nabla f_\sigma|^2 + \frac{\mu f_\sigma^p|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^2} \\
&\leq \frac{1}{\mu}f_\sigma^{p-2}|\nabla f_\sigma|^2 + \frac{\mu f_\sigma^{p-1}|\nabla(\alpha-\frac{1}{n}H^2)|^2}{(\alpha-\frac{1}{n}H^2)^{2-\sigma}} \\
&\leq \frac{1}{\mu}f_\sigma^{p-2}|\nabla f_\sigma|^2 + \frac{2\mu f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}}.
\end{aligned}$$

By a direct computation, we have $2n\Big(\alpha-\frac{1}{n}H^2\Big) \geq |A|^2$ provided $0<\varepsilon<\frac{1}{8}$. Hence

$$2\sigma p\int_{M_t}(|A|^2-n)f_\sigma^p \leq 4n\sigma p\int_{M_t}(\alpha-\frac{1}{n}H^2)f_\sigma^p - 2n\sigma p\int_{M_t} f_\sigma^p.$$

By integral inequality (3.2), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p \leq& \Big(-p(p-1)+\frac{p}{\mu}+\frac{2p\sigma}{\varepsilon_Z}\Big(\frac{2p-3}{\eta}-p+1\Big)\Big)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \\
&+\Big(2p(\mu-\varepsilon_\nabla)+\frac{2p\sigma}{\varepsilon_Z}(\eta(ap+p-1-2a)+2+4\sqrt{a}+4a)\Big) \\
&\times\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} - 2n\sigma p\int_{M_t} f_\sigma^p.
\end{aligned}$$

We have the following theorem.

Theorem 3.5. *If $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$, and if*

$$0 < \sigma < \frac{\varepsilon_Z \sqrt{\varepsilon_\nabla}}{48\sqrt{p}} \text{ and } p > \max\Big\{2, 1 + \frac{8}{\varepsilon_\nabla}, \Big(\frac{24n}{\varepsilon_Z \sqrt{\varepsilon_\nabla}}\Big)^2\Big\},$$

then

$$\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \leq -2n\sigma p \int_{M_t} f_\sigma^p.$$

Proof. We want to show that

$$-p(p-1) + \frac{p}{\mu} + \frac{2p\sigma}{\varepsilon_Z}\Big(\frac{2p-3}{\eta} - p + 1\Big) \leq 0 \tag{3.3}$$

and

$$2p(\mu - \varepsilon_\nabla) + \frac{2p\sigma}{\varepsilon_Z}(\eta(ap + p - 1 - 2a) + 2 + 4\sqrt{a} + 4a) \leq 0. \tag{3.4}$$

If we pick $\eta = \frac{2p-3}{p-1+(p-1-\frac{1}{\mu})\frac{\varepsilon_Z}{2\sigma}}$, then (3.3) is an equality and (3.4) is equivalent to

$$\mu - \varepsilon_\nabla + \frac{\sigma}{\varepsilon_Z}\Big(\frac{(2p-3)(ap+p-1-2a)}{p-1+(p-1-\frac{1}{\mu})\frac{\varepsilon_Z}{2\sigma}} + 2 + 4\sqrt{a} + 4a\Big) \leq 0.$$

Now we choose $\mu = \frac{4}{p-1}$ and let $p \geq \max\{2, 1 + \frac{8}{\varepsilon_\nabla}\}$. Then $\eta = \frac{2p-3}{p-1+(p-1)\frac{3\varepsilon_Z}{8\sigma}}$. So (3.4) is equivalent to

$$\frac{4}{p-1} - \varepsilon_\nabla + \frac{\sigma}{\varepsilon_Z}\Big(\frac{(2p-3)(ap+p-1-2a)\sigma}{(p-1)\sigma + (p-1)\frac{3\varepsilon_Z}{8}} + 2 + 4\sqrt{a} + 4a\Big) \leq 0. \tag{3.5}$$

By a direct computation, we see that if

$$0 < \sigma < \frac{2\varepsilon_Z(\frac{\varepsilon_\nabla}{4} - \frac{1}{p-1})}{\sqrt{(\frac{1}{2} + \sqrt{a} + a)^2 + (\varepsilon_\nabla - \frac{4}{p-1})\frac{(2p-3)(ap+p-1-2a)}{\frac{3}{2}(p-1)}} + \frac{1}{2} + \sqrt{a} + a}, \tag{3.6}$$

then

$$\frac{4}{p-1} - \varepsilon_\nabla + \frac{\sigma}{\varepsilon_Z}\Big(\frac{(2p-3)(ap+p-1-2a)\sigma}{(p-1)\frac{3\varepsilon_Z}{8}} + 2 + 4\sqrt{a} + 4a\Big) \leq 0.$$

Obviously, this inequality implies (3.5). Also, since for the right hand side of (3.6), we have

$$RHS = O(\frac{1}{\sqrt{p}}) \text{ as } p \to +\infty,$$

we can furthermore choose

$$0 < \sigma < \frac{1}{2} \times \frac{\varepsilon_Z \sqrt{\varepsilon_\nabla}}{24\sqrt{p}} = \frac{\varepsilon_Z \sqrt{\varepsilon_\nabla}}{48\sqrt{p}},$$

where $\frac{\varepsilon_Z \sqrt{\varepsilon_\nabla}}{24\sqrt{p}} > \frac{n}{p}$, i.e., $p > \max\left\{2, 1 + \frac{8}{\varepsilon_\nabla}, \left(\frac{24n}{\varepsilon_Z \sqrt{\varepsilon_\nabla}}\right)^2\right\}$. This completes the proof. □

As a corollary, we have

Corollary 3.6. *If* $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$, *and if*

$$0 < \sigma < \frac{\varepsilon_Z \sqrt{\varepsilon_\nabla}}{48\sqrt{p}} \text{ and } p > \max\left\{2, 1 + \frac{8}{\varepsilon_\nabla}, \left(\frac{24m}{\varepsilon_Z \sqrt{\varepsilon_\nabla}}\right)^2, \left(\frac{24n}{\varepsilon_Z \sqrt{\varepsilon_\nabla}}\right)^2\right\},$$

then

$$\frac{\partial}{\partial t} \int_{M_t} f^p_{\sigma + \frac{m}{p}} \leq 0.$$

3.6. The Stampacchia iteration

For the convenience of readers, we state the theorem of Stampacchia as following.

Theorem 3.7. *Let* $\varphi(t)$, $t_0 \leq t < \infty$, *be a nonnegative and non-increasing function, which satisfies*

$$\varphi(h) \leq \frac{C}{(h-k)^\alpha} |\varphi(k)|^\beta, \; h > k \geq k_0,$$

where $C > 0$, $\alpha > 0$, $\beta > 1$ *are constants. Then*

$$\varphi(k_0 + d) = 0, \; d^\alpha = C|\varphi(k_0)|^{\beta - 1} 2^{\alpha\beta(\beta-1)}.$$

To prove that f_σ is bounded, it is sufficient to show that the function $f_{\sigma,k} := \max\{f_\sigma - k, 0\}$ vanishes for some k. Define the set $A(k,t) =$

$\{M_t|f_{\sigma,k}>0\}$. Notice that on $A(k,t)$ we have$f_\sigma > k$ and $\nabla f_{\sigma,k} = \nabla f_\sigma$. Hence

$$\begin{aligned}\frac{\partial}{\partial t}\int_{A(k,t)} f_{\sigma,k}^p \leq & p\Big(\frac{1}{\mu}-p+1\Big)\int_{A(k,t)} f_{\sigma,k}^{p-2}|\nabla f_{\sigma,k}|^2 \\ & + 2p(\mu-\varepsilon_\nabla)\int_{A(k,t)}\frac{f_{\sigma,k}^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} \\ & + 2\sigma p\int_{A(k,t)}(|A|^2-n)f_{\sigma,k}^p \\ \leq & -\frac{1}{2}p(p-1)\int_{A(k,t)} f_{\sigma,k}^{p-2}|\nabla f_{\sigma,k}|^2 \\ & - p\varepsilon_\nabla\int_{A(k,t)}\frac{f_{\sigma,k}^{p-1}|\nabla H|^2}{(\alpha-\frac{1}{n}H^2)^{1-\sigma}} \\ & + 2\sigma p\int_{A(k,t)}(|A|^2-n)f_{\sigma,k}^p,\end{aligned}$$

where in the second inequality, we have picked $\mu = \frac{4}{p-1}$ with $p \geq \max\{2, 1+\frac{8}{\varepsilon_\nabla}\}$. Set $v_k = f_{\sigma,k}^{\frac{p}{2}}$. We have

$$|\nabla v_k|^2 = |\nabla f_{\sigma,k}^{\frac{p}{2}}|^2 = \frac{p^2}{4}f_{\sigma,k}^{p-2}|\nabla f_{\sigma,k}|^2 \leq \frac{1}{2}p(p-1)f_{\sigma,k}^{p-2}|\nabla f_{\sigma,k}|^2.$$

Hence we get

$$\frac{\partial}{\partial t}\int_{A(k,t)} v_k^2 + \int_{A(k,t)}|\nabla v_k|^2 \leq 2\sigma p\int_{A(k,t)}(|A|^2-n)v_k^2.$$

We need the following Sobolev type inequality for submanifolds in a Riemannian manifold.

Theorem 3.8 ([9]). *Let M^n be a submanifold in a Riemannian manifold N^{n+q}. Suppose the sectional curvature of N satisfies $K_N \leq b^2$ and let $i(N)$ be the injectivity radius of N. If $u \in C_0^{0,1}(M)$ and $u|_{\partial M} = 0$ satisfying*

$$b^2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp}u))^{\frac{2}{n}} \leq 1;$$

$$i(N) \geq 2\rho,$$

where

$$\rho = \begin{cases} b^{-1}\sin^{-1} b(1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp} u))^{\frac{1}{n}}, & b \in R, \\ (1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}\mathrm{Vol}(\mathrm{supp} u))^{\frac{1}{n}}, & \mathrm{Im} b \neq 0, \end{cases}$$

and $0 < \alpha < 1$, *then there is a positive constant* C_S *depending only on* n *such that*

$$\Big(\int_M |u|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq C_S \int_M \Big(|\nabla u| + |H|u\Big).$$

To use the Sobolev inequality, we first estimate the volume of $A(k,t)$.

$$\begin{aligned} \mathrm{Vol}(A(k,t)) &= \int_{A(k,t)} 1 \leq \frac{1}{k}\int_{M_t} f_\sigma \\ &\leq \frac{1}{k}\Big(\mathrm{Vol}(M_t)\Big)^{1-\frac{1}{p}}\Big(\int_{M_t} f_\sigma^p\Big)^{\frac{1}{p}} \\ &\leq \frac{1}{k}\Big(\mathrm{Vol}(M_0)\Big)^{1-\frac{1}{p}}\Big(\int_{M_t} f_\sigma^p\Big)^{\frac{1}{p}}. \end{aligned}$$

Set $p_0 = \max\{2, 1+\frac{8}{\varepsilon_\nabla}, (\frac{24n}{\varepsilon_Z\sqrt{\varepsilon_\nabla}})^2\}$. Then

$$\mathrm{Vol}(A(k,t)) \leq \frac{1}{k}\Big(\mathrm{Vol}(M_0)\Big)^{1-\frac{1}{p_0}}\Big(\int_{M_t} f_\sigma^{p_0}\Big)^{\frac{1}{p_0}} = \frac{1}{k}C_L(M_0).$$

We pick k_0 such that $\mathrm{Vol}(A(k_0,t))$ satisfies the assumption in Theorem 3.9, and let $p \geq p_0$ and $k \geq k_0$. Set $2^* = \frac{2n}{n-2}$. Then by the Hölder inequality, we have
(3.7)

$$\Big(\int_{A(k,t)} |v_k|^{2^*}\Big)^{\frac{2}{2^*}} \leq C_1\int_{A(k,t)} |\nabla v_k|^2 + C_2\Big(\int_{A(k,t)} H^n\Big)^{\frac{2}{n}}\Big(\int_{A(k,t)} |v_k|^{2^*}\Big)^{\frac{2}{2^*}}.$$

Our purpose is to control $\left(\int_{A(k,t)}|v_k|^{2^*}\right)^{\frac{2}{2^*}}$ by $\int_{A(k,t)}|\nabla v_k|^2$. For sufficiently large p and $k>k_0$,

$$\begin{aligned}\left(\int_{A(k,t)}H^n\right)^{\frac{2}{n}}\le&\left(\int_{A(k,t)}H^n\frac{v_k^2}{k^p}\right)^{\frac{2}{n}}\\=&\frac{1}{k^{\frac{2p}{n}}}\left(\int_{A(k,t)}H^nf_\sigma^p\right)^{\frac{2}{n}}\\\le&\frac{1}{k^{\frac{2p}{n}}}\left(\int_{A(k,t)}\left(a^{-1}\sqrt{\alpha_\varepsilon-\frac{1}{n}H^2}\right)^nf_\sigma^p\right)^{\frac{2}{n}}\\\le&\frac{k_0^{\frac{2p}{n}}}{a^2k^{\frac{2p}{n}}}\left(\int_{A(k_0,t)}f^p_{\sigma+\frac{n}{2p}}\right)^{\frac{2}{n}}\\\le&C(k_0,p)\frac{k_0^{\frac{2p}{n}}}{a^2k^{\frac{2p}{n}}}.\end{aligned}$$

Now we pick k sufficiently large such that the second term in the right hand side of (3.7) can be absorbed by the first term. So we get

$$\left(\int_{A(k,t)}|v_k|^{2^*}\right)^{\frac{2}{2^*}}\le C(n,p,k_0)\int_{A(k,t)}|\nabla v_k|^2,$$

which implies that

$$\frac{\partial}{\partial t}\int_{A(k,t)}v_k^2+\left(\int_{A(k,t)}v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\le C(n,p,k_0)\int_{A(k,t)}(|A|^2-n)v_k^2.$$

When $k\ge\sup_{M_0}f_\sigma$, we have for $t>0$

$$\int_{A(k,t)}v_k^2+\int_0^t\left(\int_{A(k,\tau)}v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\le C(n,p,k_0)\int_0^t\int_{A(k,\tau)}(|A|^2-n)v_k^2.$$

Since we assume that the mean curvature is positive, the maximal existence time of the mean curvature flow is finite. Let $[0,T)$ be the maximal existence time interval, $T<\infty$. At $t_0\in[0,T)$, $\int_{A(k,t_0)}v_k^2=\sup_{\tau\in[0,T)}\int_{A(k,\tau)}v_k^2$. Hence

$$\int_{A(k,T)}v_k^2+\int_0^T\left(\int_{A(k,\tau)}v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\le C(n,p,k_0)\int_0^T\int_{A(k,\tau)}(|A|^2-n)v_k^2$$

and

$$\sup_{\tau\in[0,T)}\int_{A(k,\tau)} v_k^2 + \int_0^{t_0}\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C(n,p,k_0)\int_0^{t_0}\int_{A(k,\tau)}(|A|^2-n)v_k^2.$$

Combining these two inequalities we get

$$\sup_{\tau\in[0,T)}\int_{A(k,\tau)} v_k^2 + \int_0^{T}\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq 2C(n,p,k_0)\int_0^{T}\int_{A(k,\tau)}(|A|^2-n)v_k^2.$$

We recall the following interpolation inequality

$$||\varphi||_{q_0} \leq ||\varphi||_1^{\theta}||\varphi||_q^{1-\theta},$$

where $1 \leq q_0 \leq q$ and $0<\theta<1$. We pick $\theta = 1-\frac{1}{q_0}$, then

$$\left(\int_{A(k,\tau)} v_k^{2q_0}\right)^{\frac{1}{q_0}} \leq \left(\int_{A(k,\tau)} v_k^2\right)^{1-\frac{1}{q_0}}\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{nq_0}}.$$

This implies

$$\begin{aligned}
&\left(\int_0^T\int_{A(k,\tau)} v_k^{2q_0}\right)^{\frac{1}{q_0}}\\
&\leq \left\{\int_0^T\left[\left(\int_{A(k,\tau)} v_k^2\right)^{q_0-1}\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\right]\right\}^{\frac{1}{q_0}}\\
&\leq \sup_{\tau\in[0,T)}\left(\int_{A(k,\tau)} v_k^2\right)^{\frac{q_0-1}{q_0}}\left[\int_0^T\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\right]^{\frac{1}{q_0}}\\
&\leq \sup_{\tau\in[0,T)}\int_{A(k,\tau)} v_k^2 + \frac{1}{q_0}\Big(\frac{q_0-1}{q_0}\Big)^{q_0-1}\int_0^T\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&\leq \sup_{\tau\in[0,T)}\int_{A(k,\tau)} v_k^2 + \int_0^T\left(\int_{A(k,\tau)} v_k^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.
\end{aligned}$$

Hence we get

$$\left(\int_0^T \int_{A(k,\tau)} v_k^{2q_0}\right)^{\frac{1}{q_0}} \le 2C(n,p,k_0)\int_0^T \int_{A(k,\tau)} (|A|^2-n)v_k^2.$$

Set $||A(k,t)|| := \int_0^t (\int_{A(k,\tau)} 1)d\tau$. By using the Hölder inequality, we have

$$\left(\int_0^T \int_{A(k,\tau)} v_k^{2q_0}\right)^{\frac{1}{q_0}} \ge ||A(k,T)||^{\frac{1}{q_0}-1}\int_0^T \int_{A(k,\tau)} v_k^2$$

and

$$\begin{aligned}&\int_0^T \int_{A(k,\tau)} (|A|^2-n)v_k^2\\ &\le \Big(||A(k,T)||\Big)^{1-\frac{1}{r}}\left\{\int_0^T \int_{A(k,\tau)} \left[2n\Big(\alpha-\frac{1}{n}H^2\Big)\right]^r v_k^{2r}\right\}^{\frac{1}{r}}.\end{aligned}$$

So we have for $h \ge k$

$$\begin{aligned}&|h-k|^p||A(h,T)||\\ &\le \int_0^T \int_{A(h,\tau)} v_k^2 \le \int_0^T \int_{A(k,\tau)} v_k^2\\ &\le \overline{C}(n,p,k_0)||A(k,T)||^{2-\frac{1}{q_0}-\frac{1}{r}}\left\{\int_0^T \int_{A(k,\tau)} \left[2n\Big(\alpha-\frac{1}{n}H^2\Big)\right]^r v_k^{2r}\right\}^{\frac{1}{r}}.\end{aligned}$$

Now we first choose r sufficiently large such that $2-\frac{1}{q_0}-\frac{1}{r}>1$, then pick p and σ such that

$$\int_{A(k,\tau)} \left[2n\Big(\alpha-\frac{1}{n}H^2\Big)\right]^r f_\sigma^{pr} = \int_{A(k,\tau)} (2n)^r f_{\sigma+\frac{1}{p}}^{pr}$$

satisfies the assumptions of Theorem 3.5 and Corollary 3.6. This implies that

$$\left\{\int_0^T \int_{A(k,\tau)} \left[2n\Big(\alpha-\frac{1}{n}H^2\Big)\right]^r v_k^{2r}\right\}^{\frac{1}{r}}$$

is bounded. Furthermore, choose k_0 and k sufficiently large and let $\varphi(k) = ||A(k,T)||$. Then by Theorem 3.7, there is a constant d with

$$d^p = \bar{C}(n,p,k_0)2^{p(\frac{2rq_0-r-q_0}{3rq_0-r-q_0})}||A(k_0,T)||^{1-\frac{1}{q_0}-\frac{1}{r}}$$

such that $\varphi(k+d)=0$. Since $T<\infty$, we have $||A(k_0,T)||\leq\int_0^T\int_{M_\tau}1<\infty$ which is bounded independently of t. So $f_\sigma\leq C<\infty$, $t\in[0,T)$, with a uniform C for chosen p and σ.

Theorem 3.9. *If $H\geq\frac{n}{\sqrt[4]{\varepsilon}}$ at $t=0$, there are positive constants $C\in(0,\infty)$ and $\sigma>0$ that depend on M_0, such that for $t\in[0,T)$*

$$|\mathring{A}|^2\leq C\Big(\alpha-\frac{1}{n}H^2\Big)^{1-\sigma}.$$

3.7. Gradient estimate

In this subsection, we derive the gradient estimate of H, which is used to obtain a Harnack type inequality of H.

We know that $|\nabla H|^2$ satisfies the following evolution equation

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)|\nabla H|^2=-2|\nabla^2H|^2+A*A*\nabla A*\nabla A.$$

Since $|A|^2\leq\alpha_\varepsilon$, $H\geq C(n)>0$ and $|\nabla A|^2\geq\frac{3}{n+2}|\nabla H|^2$, there are constants λ and μ depending on M_0 such that

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)|\nabla H|^2\leq(\lambda H^2+\mu)|\nabla A|^2.$$

By using similar computations as in [1, 2, 11], we have

Proposition 3.10. *Let N_i, $i=1,2$, be arbitrary constants. Then there exist constants $C_1=C_1(N_2)$ and $C_2=C_2(N_1,N_2)$ such that*

(i) $\Big(\frac{\partial}{\partial t}-\Delta\Big)H^4\geq-12H^2|\nabla H|^2+\frac{4}{n}H^6$,

(ii) $\Big(\frac{\partial}{\partial t}-\Delta\Big)((N_1+N_2H^2)|\mathring{A}|^2)\leq-\frac{4(n-1)}{3n}\Big((N_2-1)H^2|\nabla A|^2+(N_1-C_1)|\nabla A|^2\Big)+C_2|\mathring{A}|^2(1+H^4)$.

Now we prove the following

Theorem 3.11. *For any $\eta>0$, there is constant $C(\eta)$ such that*

$$|\nabla H|^2\leq\eta H^4+C(\eta).$$

Proof. Define the function $f = |\nabla H|^2 + (N_1 + N_2 H^2)|\mathring{A}|^2$. Then

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) f \leq (\lambda H^2 + \mu)|\nabla A|^2 - \frac{4(n-1)}{3n}(N_2 - 1)H^2|\nabla A|^2 - \frac{4(n-1)}{3n}(N_1 - C_1)|\nabla A|^2 + C_2|\mathring{A}|^2(1 + H^4).$$

We can pick N_1 and N_2 large enough such that the first term on the right hand side of the above inequality can be absorbed by two negative terms on the right hand side. So we get

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) f \leq -\frac{4(n-1)}{3n}(N_2 - 1)H^2|\nabla A|^2 - \frac{4(n-1)}{3n}(N_1 - C_1)|\nabla A|^2 + C_2|\mathring{A}|^2(1 + H^4).$$

Let $g_\eta = f - \eta H^4$. Then

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) g_\eta \leq -\frac{4(n-1)}{3n}(N_2 - 1)H^2|\nabla A|^2 - \frac{4(n-1)}{3n}(N_1 - C_1)|\nabla A|^2 + C_2(N_1, N_2)|\mathring{A}|^2(1 + H^4) - \eta(-12H^2|\nabla H|^2 + \frac{4}{n}H^6).$$

We first choose N_2 large enough such that the coefficient of $H^2|\nabla A|^2$ is nonpositive, then choose N_1 large enough such that the coefficient of H^2 is nonpositive. Discard these two terms and get

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) g_\eta \leq C_2(N_1, N_2)|\mathring{A}|^2(1 + H^4) - \frac{4\eta}{n}H^6.$$

Since $|\mathring{A}|^2 \leq M^*(\alpha_\varepsilon - \frac{1}{n}H^2)^{1-\sigma} \leq C^*_{M_0}(1 + H^2)^{1-\sigma}$, we get

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) g_\eta \leq C_3(N_1, N_2).$$

Hence there exists $\overline{C}(N_1, N_2)$ such that $g_\eta \leq \overline{C}(N_1, N_2)$. This completes the proof. □

3.8. The convergence of mean curvature flow

Now we prove the following convergence theorem.

Theorem 3.12. *If at $t = 0$, there holds $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$ and $|A|^2 \leq \alpha_\varepsilon(n, H)$ with $\varepsilon \in (0, \frac{1}{8})$, then the mean curvature flow equation has a smooth solution on a finite time interval $[0, T)$ and M_t's shrink to a round point as $t \to T$.*

Proof. Since for any $\eta > 0$, there is $C(\eta)$ such that $|\nabla H| \leq \eta H^2 + c(\eta)$, and $\max_{M_t} H^2 \to \infty$ as $t \to T$, there is a time $\tau(\eta)$ such that $C(\frac{\eta}{2}) \leq \frac{1}{2\eta} \max_{M_t} H^2$. Hence $|\nabla H| \leq \eta \max_{M_t} H^2$.

Next we estimate the Ricci curvature of M_t and wish to show that when $t \to T$ the Ricci curvature will approach to infinity.

Since $|A|^2 \leq \alpha_\varepsilon(n, H)$, by an inequality in [15], we have

$$\begin{aligned}
\mathrm{Ric}_{M_t} \geq & \frac{n-1}{n}\left(n + \frac{2}{n}H^2 - |A|^2 - \frac{n-2}{\sqrt{n(n-1)}}H|\mathring{A}|\right) \\
= & -\frac{n-1}{n}\left(|\mathring{A}| + \frac{(n-2)H}{2\sqrt{n(n-1)}} + \frac{\sqrt{n(n-1)H^2 + 4n(n-1)^2}}{2(n-1)}\right) \\
& \times \left(|\mathring{A}| + \frac{n-2}{2\sqrt{n(n-1)}}H - \frac{\sqrt{n(n-1)H^2 + 4n(n-1)^2}}{2(n-1)}\right) \\
= & \frac{n-1}{n}\left(\frac{4n-4+4\varepsilon-n^2}{2n(n-1+\varepsilon)}H^2 + \frac{n(n-2)\sqrt{H^4 + 4(n-1)H^2}}{2n(n-1+\varepsilon)}\right) \\
& - \frac{n-1}{n}\frac{n-2}{\sqrt{n(n-1)}}H \\
& \times \sqrt{n + \frac{n^2-2n+2-2\varepsilon}{2n(n-1+\varepsilon)}H^2 - \frac{n(n-2)\sqrt{H^4+4(n-1)H^2}}{2n(n-1+\varepsilon)}}.
\end{aligned}$$

When $H^2 \geq \frac{(n-1+\varepsilon)^2}{2(n-1+\varepsilon)}$, we have $\sqrt{1 + \frac{4(n-1)}{H^2}} \geq 1 + \frac{n-1+\varepsilon}{H^2}$. So,

$$\begin{aligned}
\mathrm{Ric}_{M_t} \geq & \frac{n-1}{n}\left(\frac{n-2+2\varepsilon}{n(n-1+\varepsilon)}H^2 + \frac{n-2}{2}\right) \\
& - \frac{n-1}{n}\frac{n-2}{\sqrt{n(n-1)}}H\sqrt{\frac{n+2}{2} + \frac{1-\varepsilon}{n(n-1+\varepsilon)}H^2}.
\end{aligned}$$

Hence

$$\frac{\mathrm{Ric}_{M_t}}{H^2} \geq \frac{n-1}{n^2\sqrt{n-1-\varepsilon}}\left(\frac{n-2+2\varepsilon}{\sqrt{n-1+\varepsilon}} - (n-2)\sqrt{\frac{1-\varepsilon}{n-1}}\right) := C_{Ric}(n,\varepsilon) > 0.$$

So if $H \geq \frac{n}{\sqrt[4]{\varepsilon}}$, then

$$\mathrm{Ric}_{M_t} \geq \frac{n-1}{2n}C_{Ric}(n,\varepsilon)H^2 \geq \frac{n-1}{2n}C_{Ric}(n,\varepsilon)C^2(n,\varepsilon).$$

Then by the Myers-Bonnet theorem, the diameter of M_t has a uniform upper bound. For any $x,\ y \in M_t$, there is a geodesic with length less than a constant $D(n,\varepsilon)$ depending only on n, ε connecting x and y. Choose η sufficiently small and integrate $|dH|$ along the geodesic, then we see that there is a constant $\zeta(n,\varepsilon) > 0$ such that $H(x,t) \geq \zeta \max_{M_t} H$, $t \in [0,T)$. Hence $\lim_{t\to T} \min_{M_t} H = \infty$. Hence the Myers-Bonnet theorem implies that $\lim_{t\to T} \operatorname{diam}(M_t) = 0$.

To see that the limit point is round, we can use the same argument as in [13] since the Bishop-Gromov volume comparison theorem holds under the condition on the Ricci curvature. □

4. Convergence of mean curvature flow under curvature pinching: II

In this section, we give the proof of Theorem 1.4. We follow the procedure in Section 3. Set

$$\alpha_{s,\varepsilon}(n,H) = (s-4\varepsilon) + \frac{s}{2(n-1+\varepsilon)}H^2 - \frac{s-2}{2(n-1+\varepsilon)}\sqrt{H^4 + 4(n-1)H^2},$$

for $\varepsilon \in (0, 1/8)$ and let $Q_{s,\varepsilon} = |A|^2 - \alpha_{s,\varepsilon}(n,H)$.

4.1. Preserving the curvature pinching condition

We first show that a curvature pinching condition is preserved under the mean curvature flow (1.1).

Theorem 4.1. *Let $n \geq 3$ and suppose we have on M_0*

$$Q_{s,\varepsilon} < 0, \text{ where } 2 \leq s \leq \begin{cases} 3, & n \geq 4 \\ \frac{12}{5}, & n = 3 \end{cases},$$

then it is preserved under the mean curvature flow (1.1).

Proof. For a positive constant s, define

$$\alpha_s(n,H) = s + \frac{s}{2(n-1)}H^2 - \frac{s-2}{2(n-1)}\sqrt{H^4 + 4(n-1)H^2},$$

and set $Q_s = |A|^2 - \alpha_s(n,H)$. We first show that if $Q_s = 0$ at a point, then $(\frac{\partial}{\partial t} - \Delta)Q_s \leq 0$. As in Section 3, we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)Q_s \leq G_s + R_{1,s} + R_{2,s},$$

where

$$G_s = \Big(- \frac{6}{n+2} + \frac{s}{n-1} - \frac{s-2}{n-1} \frac{H^2 + 2(n-1)}{\sqrt{H^4 + 4(n-1)H^2}} \Big) |\nabla H|^2,$$

$$R_{1,s} = (2|A|^4 + 4H^2 - 2n|A|^2) - \Big(\frac{s}{n-1} - \frac{s-2}{n-1} \frac{H^2 + 2(n-1)}{\sqrt{H^4 + 4(n-1)H^2}} \Big) (|A|^2 H^2 + nH^2),$$

$$R_{2,s} = \frac{s-2}{n-1} \frac{2(n-1)^2 |\nabla H^2|^2}{(\sqrt{H^4 + 4(n-1)H^2})^3}.$$

For $R_{1,s}$, we have

$$\begin{aligned} \frac{1}{2} R_{1,s} =& (|A|^4 + 2H^2 - n|A|^2) \\ & - \Big(\frac{s}{2(n-1)} - \frac{s-2}{2(n-1)} \frac{H^2 + 2(n-1)}{\sqrt{H^4 + 4(n-1)H^2}} \Big) (|A|^2 H^2 + nH^2) \\ =& |A|^4 + 2H^2 - \Big(n + \frac{sH^2}{2(n-1)} - \frac{s-2}{2(n-1)} \frac{H^2 + 2(n-1)}{\sqrt{H^4 + 4(n-1)H^2}} \\ & + \frac{(s-2)H^2}{\sqrt{H^4 + 4(n-1)H^2}} \Big) |A|^2 - n \Big(\frac{sH^2}{2(n-1)} \\ & - \frac{(s-2)\sqrt{H^4 + 4(n-1)H^2}}{2(n-1)} + \frac{(s-2)H^2}{\sqrt{H^4 + 4(n-1)H^2}} \Big). \end{aligned}$$

If $|A|^2 - \alpha_{s,\varepsilon}(n, H) = 0$ at a point, then for $s \in [2, 3]$,

$$\begin{aligned} \frac{1}{2} R_{1,s} \leq & (n-s) \Big(\frac{s-2}{n-1} \Big(\frac{H^4}{\sqrt{H^4 + 4(n-1)H^2}} - H^2 \Big) \\ & + \frac{3(s-2)H^2 - s\sqrt{H^4 + 4(n-1)H^2}}{\sqrt{H^4 + 4(n-1)H^2}} \Big) \leq 0. \end{aligned}$$

For other two terms, we have

$$G_s + R_{2,s} \leq \Big(\frac{(s-6)n + 2s + 6}{(n+2)(n-1)} - \frac{s-2}{n-1} \frac{H^2 (H^4 + 6(n-1)H^2)}{(\sqrt{H^4 + 4(n-1)H^2})^3} \Big) |\nabla H|^2.$$

Under the condition $2 \leq s \leq \begin{cases} 3, & n \geq 4 \\ \frac{12}{5}, & n = 3 \end{cases}$, we have $G_s + R_{2,s} \leq 0$. Hence $(\frac{\partial}{\partial t} - \Delta) Q_s \leq 0$.

By using a similar computation, we can show that if $Q_{s,\varepsilon} = |A|^2 - \alpha_{s,\varepsilon}(n,H) = 0$ at a point for $\varepsilon > 0$, then $(\frac{\partial}{\partial t} - \Delta)Q_{s,\varepsilon} \le 0$ at the same point. This implies that $Q_{s,\varepsilon} \le 0$ is preserved under the mean curvature flow. Since M_0 is compact, we can find such positive constant ε. This completes the proof. □

4.2. The differential inequality

In this subsection, we assume that $|A|^2 \le \alpha_{s,\varepsilon} := \alpha_{s,\varepsilon}(n,H)$. Define $f_\sigma = \frac{|\mathring{A}|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}}$, $\sigma \ge 0$, where

$$\begin{aligned}\bar{\alpha}_{s,\varepsilon} :=& \bar{\alpha}_{s,\varepsilon}(n,H)\\ =&(s-4\varepsilon) + \left(\frac{s}{2(n-1+\varepsilon)} + \frac{\varepsilon}{n(n-1+\varepsilon)}\right)H^2\\ &- \frac{s-2}{2(n-1+\varepsilon)}\sqrt{H^4 + 4(n-1)H^2}.\end{aligned}$$

We compute the evolution equation of f_σ as in Section 3.

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)f_\sigma =& \frac{-2|\nabla A|^2 + \frac{2}{n}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}} + \frac{2|A|^4 + 2H^2 - 2n|A|^2 - \frac{2}{n}|A|^2H^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}}\\ &+ (1-\sigma)\frac{f_\sigma\left(\frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon}\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right)|\nabla H|^2}{\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2}\\ &- (1-\sigma)\frac{f_\sigma\left(\frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon}\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right)(|A|^2+n)H^2}{\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2}\\ &+ 2(1-\sigma)\frac{\nabla_i f_\sigma \nabla^i(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{2-\sigma}} + \sigma(\sigma-1)\frac{f_\sigma|\nabla(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^2}.\end{aligned}$$

$$\begin{aligned}\Delta f_\sigma =& \nabla^i\nabla_i f_\sigma\\ =& \nabla^i\left(\frac{\nabla_i|\mathring{A}|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}} + (\sigma-1)\frac{|\mathring{A}|^2\nabla_i(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{2-\sigma}}\right)\\ =& \frac{\Delta|\mathring{A}|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}} + 2(\sigma-1)\frac{\nabla_i|\mathring{A}|^2\nabla^i(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{2-\sigma}}\\ &+ (\sigma-1)\frac{f_\sigma\Delta(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2} + (\sigma-2)(\sigma-1)\frac{f_\sigma|\nabla(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^2}.\end{aligned}$$

Hence

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)f_\sigma =&\frac{\Big(\frac{\partial}{\partial t}-\Delta\Big)|\mathring{A}|^2}{(\alpha_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}-2(\sigma-1)\frac{\nabla_i|\mathring{A}|^2\nabla^i(\alpha_{s,\varepsilon}-\frac{1}{n}H^2)}{(\alpha_{s,\varepsilon}-\frac{1}{n}H^2)^{2-\sigma}}\\
&+(\sigma-1)\frac{f_\sigma\Big(\frac{\partial}{\partial t}-\Delta\Big)(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
&-(\sigma-2)(\sigma-1)\frac{f_\sigma|\nabla(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^2}\\
=&\frac{\Big(\frac{\partial}{\partial t}-\Delta\Big)|\mathring{A}|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}+(\sigma-1)\frac{f_\sigma\Big(\frac{\partial}{\partial t}-\Delta\Big)(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
&-2(\sigma-1)\frac{\nabla_i f_\sigma\nabla^i(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
&+\sigma(\sigma-1)\frac{f_\sigma|\nabla(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^2}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)\bar{\alpha}_{s,\varepsilon}\\
&=\left(\frac{s}{n-1+\varepsilon}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right)(|A|^2+n)H^2\\
&\quad+\left(\frac{-s}{n-1+\varepsilon}+\frac{s-2}{n-1+\varepsilon}\cdot\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right)|\nabla H|^2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2\Big)\\
&=\left(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right)(|A|^2+n)H^2\\
&\quad+\left(\frac{-s+2n-2}{n(n-1+\varepsilon)}+\frac{s-2}{n-1+\varepsilon}\cdot\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\right)|\nabla H|^2.
\end{aligned}
$$

Since $\sigma\in(0,1)$ and $f_\sigma\le(\alpha_{s,\varepsilon}-\frac{1}{n}H^2)^\sigma$, we have

$$
\Big(\frac{\partial}{\partial t}-\Delta\Big)f_\sigma =\frac{-2|\nabla A|^2+\frac{2}{n}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}+\frac{2|A|^4+2H^2-2n|A|^2-\frac{2}{n}|A|^2H^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}
$$

$$
\begin{aligned}
&+(1-\sigma)\frac{f_\sigma\Big(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)|\nabla H|^2}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
&-(1-\sigma)\frac{f_\sigma\Big(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
&+2(1-\sigma)\frac{\nabla_i|\mathring{A}|^2\nabla^i(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{2-\sigma}}+\sigma(\sigma-1)\frac{f_\sigma|\nabla(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^2}.
\end{aligned}
$$

We discard the last term since it is nonpositive.

To estimate the gradient terms, we have

$$
\begin{aligned}
G' :=&\frac{-2|\nabla A|^2+\frac{2}{n}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}\\
&+(1-\sigma)\frac{f_\sigma\Big(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)|\nabla H|^2}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
\le&\frac{-\frac{6}{n+2}|\nabla H|^2+\frac{2}{n}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}\\
&+(1-\sigma)\frac{\Big(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H(H^2+(6n-6))}{(\sqrt{H^2+4(n-1)})^3}\Big)|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}\\
\le&\Big(-\frac{4n-4}{n(n+2)}+\frac{sn-2n+2}{n(n-1+\varepsilon)}\Big)\frac{|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}\le-\frac{2\varepsilon_\nabla|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}
\end{aligned}
$$

for some positive constant ε_∇ under the assumptions $s\in[2,\frac{9}{4}]$ and $\varepsilon\in(0,\frac{1}{8})$.

For the remaining terms in the evolution equation of f_σ, we have

$$
\begin{aligned}
R' :=&\frac{2|A|^4+2H^2-2n|A|^2-\frac{2}{n}|A|^2H^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}\\
&-(1-\sigma)\frac{f_\sigma\Big(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\\
&=2f_\sigma(|A|^2-n)\\
&\quad\times\Bigg(1-\frac{(1-\sigma)\Big(\frac{sn-2n+2}{n(n-1+\varepsilon)}-\frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\Big)(|A|^2+n)H^2}{2(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)(|A|^2-n)}\Bigg).
\end{aligned}
$$

If $|A|^2 \le n$, then $R' \le 0$. Now we assume $\bar{\alpha}_{s,\varepsilon} \ge |A|^2 > n$.

$$
\begin{aligned}
C_\sigma :=& \frac{\left(\frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right)(|A|^2+n)H^2}{2(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)(|A|^2-n)} \\
=& \frac{\frac{1}{2}\left(\frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right)(|A|^2+n)H^2}{\left(\frac{sn-2n+2}{2n(n-1+\varepsilon)}H^2 - \frac{s-2}{2(n-1+\varepsilon)}\sqrt{H^4+4(n-1)H^2}+s-4\varepsilon\right)(|A|^2-n)} \\
=& \left(1 - \frac{s-4\varepsilon - \frac{(s-2)(n-1)}{n-1+\varepsilon}\cdot\frac{H^2}{\sqrt{H^4+4(n-1)H^2}}}{\frac{sn-2n+2}{2n(n-1+\varepsilon)}H^2 - \frac{(s-2)\sqrt{H^4+4(n-1)H^2}}{2(n-1+\varepsilon)}+s-4\varepsilon}\right)\left(1+\frac{2n}{|A|^2-n}\right) \\
\ge& \left(1 - \frac{s-4\varepsilon - \frac{(s-2)(n-1)}{n-1+\varepsilon}\cdot\frac{H^2}{\sqrt{H^4+4(n-1)H^2}}}{\frac{sn-2n+2}{2n(n-1+\varepsilon)}H^2 - \frac{(s-2)\sqrt{H^4+4(n-1)H^2}}{2(n-1+\varepsilon)}+s-4\varepsilon}\right)\left(1+\frac{2n}{\alpha_{s,\varepsilon}-n}\right).
\end{aligned}
$$

We wish to show that $C_\sigma \ge 1$. We assume that $s \in [2, \frac{9}{4}]$, $\varepsilon \in (0, \frac{1}{8})$ and $\sigma \in (0, \frac{1}{2}\varepsilon)$. By a long but direct computation, we see that

$$
\begin{aligned}
&\frac{2n}{\alpha_{s,\varepsilon}-n} - \frac{s-4\varepsilon - \frac{(s-2)(n-1)}{n-1+\varepsilon}\cdot\frac{H^2}{\sqrt{H^4+4(n-1)H^2}}}{\frac{sn-2n+2}{2n(n-1+\varepsilon)}H^2 - \frac{s-2}{2(n-1+\varepsilon)}\sqrt{H^4+4(n-1)H^2}+s-4\varepsilon} \\
\ge& \frac{2n}{\alpha_{s,\varepsilon}-n}\cdot\frac{s-4\varepsilon - \frac{(s-2)(n-1)}{n-1+\varepsilon}\cdot\frac{H^2}{\sqrt{H^4+4(n-1)H^2}}}{\frac{sn-2n+2}{2n(n-1+\varepsilon)}H^2 - \frac{s-2}{2(n-1+\varepsilon)}\sqrt{H^4+4(n-1)H^2}+s-4\varepsilon}.
\end{aligned}
$$

This implies $C_\sigma \ge 1$. Hence we have proved the following

Theorem 4.2. *Let $n \ge 3$ and suppose $|A|^2 \le \alpha_{s,\varepsilon}(n,H)$, $s \in [2, \frac{9}{4}]$, $\varepsilon \in (0, \frac{1}{8})$ and $\sigma \in (0, \frac{1}{2}\varepsilon)$ at $t = 0$. Then there is a positive constant ε_∇ depending only on n such that following inequality holds.*

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) f_\sigma \le& -2(\sigma-1)\frac{\nabla_i f_\sigma \nabla^i(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2} \\
&- \frac{2\varepsilon_\nabla}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}}|\nabla H|^2 + 2\sigma(|A|^2-n)f_\sigma. \qquad (4.1)
\end{aligned}
$$

4.3. The estimate of Z

As before,

$$Z = -|\mathring{A}|^4 + \frac{1}{n}|\mathring{A}|^2 H^2 + H \sum_{i=1}^{n} \mathring{\lambda}_i^3$$

and

$$H \sum_{i=1}^{n} \mathring{\lambda}_i^3 \geq -\frac{n-2}{\sqrt{n(n-1)}} H |\mathring{A}|^3 \geq -\frac{\mu}{2}|\mathring{A}|^4 - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}|\mathring{A}|^2 H^2.$$

Hence

$$\begin{aligned} &\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)} \\ &\geq \frac{n + \left(\frac{1}{n} - \frac{1}{2\mu}\frac{(n-2)^2}{n(n-1)}\right)H^2 - \frac{\mu+2}{2}|\mathring{A}|^2}{\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2} \\ &\geq \frac{\frac{1}{2(n-1)}H^2 + n - \frac{n}{2}\left(s - \varepsilon + \frac{(s-2)n+2-2\varepsilon}{2n(n-2+\varepsilon)}H^2 - \frac{(s-2)\sqrt{H^4+4(n-1)H^2}}{2(n-1+\varepsilon)}\right)}{\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2}. \end{aligned}$$

Here we have used $|A|^2 \leq \bar{\alpha}_{s,\varepsilon}(n,H)$, and assumed $\varepsilon \in (0, \frac{1}{2})$ and $\mu = n-2$.

We first assume $s < 2 + \varepsilon$. Since $\sqrt{H^4 + 4(n-1)H^2} \leq H^2 + 2(n-1)$, we have

$$\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\alpha_{s,\varepsilon} - \frac{1}{n}H^2)} \geq \frac{\frac{1}{2}\left(\frac{1}{n-1} - \frac{1-\varepsilon}{n-1+\varepsilon}\right)H^2 + n - \frac{n}{2}(s-\varepsilon)}{\alpha_{s,\varepsilon} - \frac{1}{n}H^2} \geq \varepsilon_Z n > 0.$$

Secondly, if $s \geq 2 + \varepsilon$, we assume $H^2 \geq \frac{2(n-1-\theta/2)^2}{\theta}$, $\theta > 0$. Then

$$\sqrt{H^4 + 4(n-1)H^2} \geq H^2 + 2n - 2 - \theta.$$

Hence

$$\frac{Z + n|\mathring{A}|^2}{|\mathring{A}|^2(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)} \geq \frac{\frac{1}{2}\left(\frac{1}{n-1} - \frac{1-\varepsilon}{n-1+\varepsilon}\right)H^2 + n - \frac{n}{2}(s-\varepsilon) + \frac{n(n-2)(2n-2-\theta)}{4(n-1+\varepsilon)}}{\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2}.$$

If we pick $\theta = \frac{4(n-1)\varepsilon}{s-2}$, then the right hand side of the inequality above has a positive uniform lower bound. Hence we have proved

Theorem 4.3. *If* $H^2 \geq \mathrm{sgn}(s-2-\varepsilon)\frac{(n-1)^{3/2}(s-2)^{1/2}}{\sqrt{\varepsilon}}$ *at* $t=0$, *then there is a positive constant* ε_Z *independent of* n *such that we have*

$$\frac{Z+n|\mathring{A}|^2}{|\mathring{A}|^2(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)} \geq \varepsilon_Z n > 0,\ \ t \in [0,T).$$

4.4. The integral inequality

As in Section 3, we have

$$\begin{aligned}\Delta f_\sigma \geq & \frac{2(\mathring{h}^{ij}\nabla_j\nabla_j H + Z + n|\mathring{A}|^2 + |\nabla\mathring{A}|^2)}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}} + 2(\sigma-1)\frac{\nabla^i f_\sigma \nabla_i(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2} \\ & + (\sigma-1)\frac{f_\sigma H\Delta H}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\left(\frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right).\end{aligned}$$

Multiplying both sides of the above inequality by f_σ^{p-1} and integrating on M_t, we get

$$\begin{aligned}&\int_{M_t} f_\sigma^{p-1}\Delta f_\sigma \\ &\geq \int_{M_t}\frac{2\mathring{h}^{ij}\nabla_i\nabla_j H f_\sigma^{p-1}}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}} + \int_{M_t}\frac{2(Z+n|\mathring{A}|^2)f_\sigma^{p-1}}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}} \\ &\quad + \int_{M_t}\frac{2(\sigma-1)f_\sigma^{p-1}\nabla^i f_\sigma\nabla_i(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2} \\ &\quad + \int_{M_t}\frac{(\sigma-1)f_\sigma^p H\Delta H}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\left(\frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon}\cdot\frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}\right).\end{aligned}$$

The second term (we denote it by S_2) of the right hand side of the inequality above is not bigger than

$$\begin{aligned}&-(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 + 2(p-1)\int_{M_t}\frac{f_\sigma^{p-2}|\nabla f_\sigma||\nabla H||\mathring{A}|}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}} \\ &+ \frac{2(n-1)}{n}\int_{M_t}\frac{f_\sigma^{p-1}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{1-\sigma}} + 2(1-\sigma)\int_{M_t}\frac{f_\sigma^{p-1}H|r_s||\nabla H|^2|\mathring{A}|}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^{2-\sigma}} \\ &+ (1-\sigma)(p-2)\int_{M_t}\frac{f_\sigma^{p-1}H|r_s||\nabla f_\sigma||\nabla H|}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2} + (1-\sigma)\int_{M_t}\frac{f_\sigma^p H^2|r_s||\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2)^2} \\ &- \int_{M_t}\left(\frac{8(1-\sigma)f_\sigma^p}{\bar{\alpha}_{s,\varepsilon}-\frac{1}{n}H^2}\frac{s-2}{n-1+\varepsilon}\frac{H^2(n-1)^2}{\sqrt{H^4+4(n-1)H^2}^3}\right)|\nabla H|^2,\end{aligned}$$

where $r_s = r_s(n, H) = \frac{sn-2n+2}{n(n-1+\varepsilon)} - \frac{s-2}{n-1+\varepsilon} \cdot \frac{H^2+2n-2}{\sqrt{H^4+4(n-1)H^2}}$. Since $H > 0$ on M_0, there is a positive constant θ such that $H \geq \theta > 0$ for all t. It is easy to see that $|r_s| \leq C < \infty$. We may assume $C \geq 1 > \frac{2}{n(n-1+\varepsilon)}$. Hence

$$S_2 \leq \Big(\frac{C}{a} \cdot \frac{p-2}{\eta} + \frac{p-1}{\eta} - p + 1\Big) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 + \Big(2 + \frac{4C}{\sqrt{a}} + \frac{4C^2}{a} + \eta(p-1+cp-2c)\Big) \int_{M_t} \frac{f_\sigma^{p-1}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}}.$$

As in Section 3, we have

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \\ &\leq \Big(-p(p-1) + \frac{p}{\mu} + \frac{2p\sigma}{\varepsilon_Z}\Big(\frac{C}{a} \cdot \frac{p-2}{\eta} + \frac{p-1}{\eta} - p + 1\Big)\Big) \int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \\ &\quad + \Big(2p(\mu - \varepsilon_\nabla) + \frac{2p\sigma}{\varepsilon_Z}\Big(2 + \frac{4C}{\sqrt{a}} + \frac{4C^2}{a} + \eta(p-1+cp-2c)\Big)\Big) \\ &\quad \times \int_{M_t} \frac{f_\sigma^{p-1}|\nabla H|^2}{(\bar{\alpha}_{s,\varepsilon} - \frac{1}{n}H^2)^{1-\sigma}}. \end{aligned}$$

We pick $\mu = \frac{4}{p-1}$, $p \geq \max\{2, 1 + \frac{8}{\varepsilon_\nabla}, \frac{9C4^2}{\varepsilon_\nabla}\}$ and

$$0 < \sigma < \frac{-3C^2 + \sqrt{9C^4 + (\varepsilon_\nabla - \frac{4}{p-1})\frac{8C^2p^2}{3\varepsilon_Z(p-1)}}}{\frac{16C^2p^2}{3\varepsilon_Z(p-1)}} = O\Big(\frac{1}{\sqrt{p}}\Big).$$

Then we get

$$-p(p-1) + \frac{p}{\mu} + \frac{2p\sigma}{\varepsilon_Z}\Big(\frac{C}{a} \cdot \frac{p-2}{\eta} + \frac{p-1}{\eta} - p + 1\Big) \leq 0,$$

$$2p(\mu - \varepsilon_\nabla) + \frac{2p\sigma}{\varepsilon_Z}\Big(2 + \frac{4C}{\sqrt{a}} + \frac{4C^2}{a} + \eta(p-1+cp-2c)\Big) \leq 0.$$

So we have proved the follwing

Theorem 4.4. *If*

$$p > \max\Big\{2, 1 + \frac{8}{\varepsilon_\nabla}, \frac{9C4^2}{\varepsilon_\nabla}\Big\} \text{ and } 0 < \sigma < \sigma + \frac{m}{p} < \frac{\varepsilon_Z\sqrt{\varepsilon_\nabla}}{16\sqrt{Cp}},$$

then

$$\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p \leq 0 \text{ and } \frac{\partial}{\partial t}\int_{M_t} f_{\sigma+\frac{m}{p}}^p \leq 0.$$

Now we can carry out the Stampacchia iteration process and the gradient estimate to complete the proof of the smooth convergence of the mean curvature flow.

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