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Variants of Normality for Noetherian Schemes

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Abstract: This note presents a uniform treatment of normality and three of its variants—topological, weak and seminormality—for Noetherian schemes. The key is to define these notions for pairs $Z \subset X$ consisting of a (not necessarily reduced) scheme X and a closed, nowhere dense subscheme Z. An advantage of the new definitions is that, unlike the usual absolute ones, they are preserved by completions. This shortens some of the proofs and leads to more general results.
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Definition 1. Let X be a scheme and $Z \subset X$ a closed, nowhere dense subscheme. A *finite modification* of X centered at Z is a finite morphism $p: Y \to X$ such that none of the associated primes of Y is contained in $Z_Y := p^{-1}(Z)$ and

$$p|_{Y \setminus Z_Y} : Y \setminus Z_Y \to X \setminus Z \quad \text{is an isomorphism.} \tag{1.1}$$

Let $j : X \setminus Z \hookrightarrow X$ be the natural injection and $J_Z \subset \mathcal{O}_X$ the largest subsheaf supported on Z. There is a one-to-one correspondence between finite modifications and coherent $\mathcal{O}_X$-algebras

$$\mathcal{O}_X / J_Z \subset p_* \mathcal{O}_Y \subset j_* \mathcal{O}_{X \setminus Z}. \tag{1.2}$$

The notion of an *integral modification* of X centered at Z is defined analogously. Let $A \subset j_* \mathcal{O}_{X \setminus Z}$ be the largest subalgebra that is integral over $\mathcal{O}_X$.

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Then $\operatorname{Spec}_X A$ is the maximal integral modification, called the *relative normalization* of the pair $Z \subset X$. We denote it by

$$\pi : (Z^{\mathrm{rn}} \subset X^{\mathrm{rn}}) \to (Z \subset X) \quad \text{or by} \quad \pi : X_Z^{\mathrm{rn}} \to X. \tag{1.3}$$

The relative normalization is the limit of all finite modifications centered at Z. (It would be called the relative normalization of X in $X \setminus Z$ in the terminology of [Sta15, Tag 0BAK].)

If (x, X) is semilocal then a finite modification of X centered at $Z = \{x\}$ is called a *punctual modification* of (x, X). The relative normalization of (x, X) is called the *punctual normalization* and denoted by $\pi : \big(x^{\mathrm{pn}}, X^{\mathrm{pn}}\big) \to (x, X)$.

Definition 2. Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme. We define 4 properties of such pairs, depending on the behavior of all finite modifications $p : Y \to X$ centered at Z.

- N (normality): every $p : Y \to X$ is an isomorphism.
- TN (topological normality): every $p : Y \to X$ is a universal homeomorphism (Definition 48).
- WN (weak normality): if $p : Y \to X$ is a universal homeomorphism then it is an isomorphism.
- SN (seminormality): if $p : Y \to X$ is a universal homeomorphism that preserves residue fields (Definition 48) then it is an isomorphism.

We stress that X is an arbitrary Noetherian scheme and these are all properties of pairs $Z \subset X$. A direct predecessor of these definitions is in the works of S. Kovács who developed the notions of rational and Du Bois pairs [Kov11a, Kov11b], but working with pairs instead of schemes has long been a theme of the Minimal Model Program; see [Kol97, Laz04, KK10, Kol13].

We also say that "$Z \subset X$ is a normal pair" and similarly for the other variants. A semilocal scheme (x, X) is called *punctually normal* if the pair $\{x\} \subset X$ is normal; similarly for the other versions.

We say that a scheme X without isolated points satisfies N (or TN, ...) if the pair $Z \subset X$ satisfies N (or TN, ...) for every closed, nowhere dense subscheme Z. If X has isolated points then in addition we require these points to be reduced for N, WN and SN.

It would have been possible to formulate all these notions for integral morphisms, but I prefer to stay with Noetherian schemes. In all results the integral morphism versions are direct consequences of the finite morphism versions.

In the definition of topological normality it seems artificial to restrict to maps that are isomorphisms over $X \setminus Z$; one should clearly allow universal homeomorphisms over $X \setminus Z$. This leads to the following variant of TN.

Definition 3. Let X be a scheme and $Z \subset X$ a closed, nowhere dense subscheme. A *finite topological modification* of X centered at Z is a finite morphism $p : Y \to X$ such that none of the associated primes of Y is contained in $Z_Y := p^{-1}(Z)$ and

$$p|_{Y \setminus Z_Y} : Y \setminus Z_Y \to X \setminus Z \quad \text{is a universal homeomorphism.} \tag{3.1}$$

In analogy with the notion of topological normality we introduce the following.

STN (strong topological normality): every finite topological modification of X centered at Z is a universal homeomorphism.

This is clearly a topological property. That is, let $f : Y \to X$ be a finite, universal homeomorphism. Then $Z \subset X$ is STN iff $f^{-1}(Z) \subset Y$ is STN. In particular, X is STN iff $\operatorname{red} X$ is STN.

It is clear that STN $\Rightarrow$ TN; eventually we show in Theorem 31 that STN is equivalent to TN.

Comments 4. The main contribution of this note is Definitions 2–3, or rather the assertion the these concepts are natural and useful. To support this claim, we show the following.

- A scheme satisfies N (resp. WN or SN) iff it is normal (resp. weakly or seminormal) (Proposition 5 and Definition 50).
- All 5 properties in Definitions 2–3 are local, even eqivalent to the punctual versions (Proposition 10).
- All the properties are preserved by completion (Corollary 17).
- A scheme X is normal (resp. weakly or seminormal) iff the completion $(\hat{x}, \hat{X})$ is punctually normal (resp. weakly or seminormal) for every $x \in X$ (Corollary 18).
- All the properties descend for faithfully flat morphisms (Proposition 11).

- All the properties ascend for regular morphisms (Theorem 37).
- Properties TN, STN and being geometrically unibranch coincide (Theorem 31 and Corollary 32).
- Weak and seminormality can be described in terms of the conductors of the punctual normalizations (Corollary 28).
- Finiteness of relative normalization is a punctual property (Theorem 19).
- Nagata schemes can be characterized using punctual normalizations or the reducedness of the formal fibers (Theorem 45).

Note, however, that many of the arguments are either classical or can be traced back to one of the main sources [Gro60, Har62, Gro68, Tra70, GT80, Man80]; I try to give more precise references in each section. For many of the topics the best reference is [Sta15].

(4.1) It would be natural to say that "(x, X) is normal" or "X is normal along Z" but this is at variance with standard usage since $Z \subset X$ can be a normal pair even if X is not normal at any point of Z. For instance, the pairs

$$\{(0,0,0)\} \subset X_1 := (xy = 0) \subset \mathbb{A}^3 \quad \text{and} \quad \{(0,0,0)\} \subset X_2 := (z^2 = 0) \subset \mathbb{A}^3$$

are both normal by our definition. This may seem somewhat perverse, but the resulting flexibility is a crucial ingredient in several of our proofs.

(4.2) A key advantage of the above definitions is that the punctual versions are preserved by completion (17). This makes it possible to study several basic results that were known for Nagata schemes and extend them to general Noetherian schemes. In addition, the resulting proofs are shorter and use simpler inductive steps. Note that completions of normal Noetherian rings can be quite complicated; see the examples in [Nag62, A.1] and [Lec86, Hei93, CL04, Nis12].

(4.3) It would seem to be natural to introduce another notion TNR (topological normality preserving residue fields): every finite modification $p : Y \to X$ centered at Z is a universal homeomorphism and preserves residue fields. However, example (36) shows that TNR is not topological. Thus, although the introduction of TNR makes the diagram in (4.4) pleasingly symmetrical, it is probably not a useful concept.

(4.4) For any pair $Z \subset X$ we have the following obvious implications between these notions

$$\begin{array}{ccccc} TN & + & WN & \Leftrightarrow & N \\ \Uparrow & & \Downarrow & & \\ TNR & + & SN & \Leftrightarrow & N. \end{array}$$

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1. Normal pairs

First we show that our definition of normality coincides with the usual one.

Proposition 5. *A Noetherian scheme X without isolated points is normal (resp. weakly or seminormal) iff $Z \subset X$ is a normal (resp. weakly or seminormal) pair for every nowhere dense closed subscheme Z*

Proof. Assume that X is normal. Every finite modification centered at Z is dominated by the normalization, hence necessarily an isomorphism. Similarly, if X is weakly (resp. semi) normal then every birational, universal homeomorphism (resp. one that also preserves residue fields) is an isomorphism.

To see the converse, we may assume that X is affine (cf. Proposition 10). First we show that X is reduced. If not then there is an irreducible subscheme Z such that J, the ideal of all nilpotent sections with support in Z, is nonzero. If there is such a Z that is nowhere dense then $\operatorname{Spec}_X \mathcal{O}_X/J \to X$ shows that $Z \subset X$ is not a seminormal pair. Otherwise X has no embedded points hence there is an $r \in \mathcal{O}_X$ such that multiplication by r is injective but not surjective on J. (Here we use that $\dim Z > 0$.) There is a natural algebra structure on the $\mathcal{O}_X$-module $\mathcal{O}_X + \frac{1}{r}J$ where we set $\frac{1}{r}J \cdot \frac{1}{r}J = 0$. The diagonally embedded $\delta : J \hookrightarrow \mathcal{O}_X + \frac{1}{r}J$ is an ideal and $\operatorname{Spec}_X\bigl((\mathcal{O}_X + \frac{1}{r}J)/\delta(J)\bigr) \to X$ shows that $\bigl((r=0) \cap Z \subset X\bigr)$ is not a seminormal pair. Thus X is reduced.

Let $\bar{X} \to X$ be the normalization (resp. weak or seminormalization). We are done if $X = \bar{X}$. Otherwise pick any $\phi \in \mathcal{O}_{\bar{X}} \setminus \mathcal{O}_X$ and set $Z := \operatorname{Supp}\bigl(\mathcal{O}_X[\phi]/\mathcal{O}_X\bigr)$. Then $\operatorname{Spec}_X \mathcal{O}_X[\phi] \to X$ shows that the pair $Z \subset X$ is not normal (resp. weakly or seminormal). $\square$

Lemma 6. *Let (R, m) be a local Noetherian ring. Then exactly one of the following holds.*

1) *(R, m) is Artinian,*
2) *(R, m) is regular of dimension 1,*
3) $\operatorname{depth}_m R \geq 2$ *or*
4) *(R, m) is a non-normal pair.*

Note that (R, m) is a non-normal pair iff there is a ring homomorphism $\phi : R \to S$ (other than an isomorphism) whose kernel and cokernel are killed by a power of m and such that m is not an associated prime of S.

Proof. The proof is essentially the same as that of Serre's criterion of normality. (R, m) is not Artinian iff $V(m) \subset \operatorname{Spec} R$ is nowhere dense; we assume this from now on.

Let $J \subset R$ be the largest ideal killed by a power of m. If $J \neq 0$ then $R \to R/J$ shows that (R, m) is a non-normal pair.

Otherwise $J = 0$ and there is an $r \in m$ that is not a zero-divisor. If m is not an associated prime of $R/(r)$ then $\operatorname{depth}_m R \geq 2$. Thus we are left with the case when there is an $a \in R \setminus (r)$ such that $am \subset rR$.

If $am \subset rm$ then, by the determinantal trick [AM69, Prop.2.4], a/r satisfies a monic polynomial, hence $R \subset R[a/r]$ shows that (R, m) is a non-normal pair.

Otherwise there is a $t_0 \in m$ such that $at_0 = r$; in particular a is not a zero-divisor. For any $t \in m$ we have $at = rt'$ for some $t' \in R$. Thus $a(t - t't_0) = at - t'(at_0) = at - t'r = 0$. Since a is not a zero-divisor this implies that $t = t't_0$ and so $m = (t_0)$. Thus (R, m) is regular of dimension 1. □

This directly implies the following claims.

Corollary 7. *Let X be a Noetherian scheme and $D \subset X$ an effective Cartier divisor. The pair $D \subset X$ is normal iff X is regular at the generic points of D and D has no embedded points.* □

Corollary 8. *A Noetherian scheme X is S_2 iff $Z \subset X$ is a normal pair for every closed subscheme of codimension ≥ 2.* □

9. All of the properties in Definition 2 can be understood in terms of the relative normalization $\pi_Z : X_Z^{\mathrm{rn}} \to X$.

For TN this is clear. Since $\pi_Z : X_Z^{\mathrm{rn}} \to X$ is the limit of finite modifications, $Z \subset X$ satisfies TN iff π is a universal homeomorphism. The only complication is that the relative normalization need not be Noetherian in general. Luckily we will be able to avoid this issue.

A description of WN and SN in terms of the conductor of the relative normalization is given in (28); this is less straightforward.

2. Punctual nature

For a scheme X and point $x \in X$ we set $X_x := \operatorname{Spec}_X \mathcal{O}_{x,X}$. Thus (x, X_x) is a local scheme. We show that our properties can be described in terms of the local schemes of X and their punctual modifications. These arguments are rather standard.

Proposition 10 (Punctual nature). *Let X be a Noetherian scheme, $Z \subset X$ a closed subscheme and $\mathbf{P}$ any of the 5 properties in Definitions 2–3.*

1) *If $\mathbf{P}$ holds for $Z \subset X$ then $\mathbf{P}$ holds for (x, X_x) for every $x \in Z$.*

2) *The converse holds except possibly for TN.*

Complement. We prove in (31) that TN=STN, hence (10.2) also holds for TN.

Proof. Let $f : Y \to X$ be a morphism that shows that $Z \subset X$ does not satisfy $\mathbf{P}$ and let x be a generic point of $\operatorname{Supp}\big(f_*\mathcal{O}_Y/\mathcal{O}_X\big)$. Then localizing at x shows that (x, X_x) does not satisfiy $\mathbf{P}$ in the cases N, SN and WN. For STN, let $g : Z \to X$ be the purely inseparable closure of X in Y (51). Then we take x to be a generic point of $\operatorname{Supp}\big(f_*\mathcal{O}_Y/g_*\mathcal{O}_Z\big)$.

Let $f_x : (y, Y) \to (x, X)$ show that (x, X_x) does not satisfy $\mathbf{P}$. By Corollary 14, it is obtained as the localization of a finite modification $f : Y \to X$ centered at $\bar{x} \subset Z$. If $\mathbf{P}$ is one of N, TN then f shows that $Z \subset X$ does not satisfy $\mathbf{P}$.

If $\mathbf{P}$ is WN (resp. SN) then we have to find such an $f : Y \to X$ that is a universal homeomorphism (resp. also preserves residue fields). These are also guaranteed by (14). For STN we first extend f_x to a finite morphism $f_U : Y_U \to U$ for some open neighborhood $x \in U \subset X$ and then use (13.2) to extend it to $f : Y \to X$. □

Even stronger localization can be obtained using the following result which shows that the properties descend for faithfully flat morphisms.

Proposition 11. *Let $f : Y \to X$ be a faithfully flat morphism of Noetherian schemes, $Z \subset X$ a closed, nowhere dense subscheme and $Z_Y := f^{-1}(Z)$.*

If $Z_Y \subset Y$ satisfies any of the 5 properties **P** *in Definitions 2–3 then so does $Z \subset X$.*

Proof. Let $p : X' \to X$ be a finite modification centered at Z. Then $p_Y := (p \times f) : Y' := X' \times_X Y \to Y$ is a finite morphism that is an isomorphism over $Y \setminus Z_Y$. Since f is flat, none of the associated primes of Y' is contained in $p_Y^{-1}(Z')$. Thus $p_Y : Y' \to Y$ is a finite modification centered at Z_Y. Furthermore, if p is a universal homeomorphism or preserves residue fields then p_Y also has these properties. If f is faithfully flat then p is an isomorphism off p_Y is.

Thus if $p : X' \to X$ shows that $Z \subset X$ does not satisfy the property **P** then $p_Y : Y' \to Y$ shows that $Z_Y \subset Y$ also does not satisfy the property **P**. □

Example 12. If $f : (y, Y) \to (x, X)$ is flat and (y, Y) satisfies the property **P** then (x, X) need not satisfy **P**. For instance, if (x, X) is any local scheme then the pair $\{(x, 0, 0)\} \subset X \times \mathbb{A}^2$ is punctually normal.

The following could have been an exercise in [Har77, Sec.II.5].

Proposition 13 (Extension of finite morphisms). *Let X be a Noetherian scheme, $X^0 \subset X$ an open subscheme and $f^0 : Y^0 \to X^0$ a finite morphism. Then*

1) *f^0 can be extended to a finite, surjective morphism $f : Y \to X$.*
2) *Let $U \subset X$ be an open subset such that f^0 is an isomorphism (resp. partial weak or seminormalization (50)) over $U \cap X^0$. Then we can choose f to be an isomorphism (resp. partial weak or seminormalization) over U.*
3) *If X is Nagata, Y^0 is reduced and $f^0(Y^0)$ is dense in X then there is a unique maximal reduced extension.*

Proof. By adding $X \setminus \overline{X^0}$ both to X^0 and Y^0 we may assume that X^0 is dense in X. Then we would like to take Y to be the relative normalization of X in Y^0; see (51) or [Sta15, Tag 0BAK]. In case (3) this gives a finite extension $Y \to X$ but in general the resulting $\pi : Y^{\max} \to X$ is only integral. (For example, the relative normalization of $k[x, x^{-1}, y]/(y^2)$ in $k[x, y]/(y^2)$ is $k[x, x^{-r}y : r = 0, 1, \dots]/(y^2)$.)

Thus one needs to prove that a suitable finite subalgebra of $\pi_*\mathcal{O}_Y^{\max}$ works; the proof is very similar to [Har77, Exrc.II.5.15] or see [Sta15, Tag 05K0].

If f^0 is an isomorphism over $U \cap X^0$ then first we extend it to $U \cup Y^0$ (glued along $U \cap X^0$) and then apply (1). Finally, if f^0 is a partial weak (resp. semi) normalization then first we choose any finite extension $f' : Y' \to X$ and then let Y be the relative weak (resp. semi) normalization (50) of X in Y'. □

If $Z' \subset X'$ is a localization of $Z \subset X$ and $Y' \to X'$ is a finite modification then we can first extend it to $Y^0 \to X^0$ for some open subset $X^0 \subset X$ and then use (13) to extend it to $Y \to X$. Thus (13) can be reformulated as follows.

Corollary 14. *The relative normalization X_Z^{rn} commutes with localization. The same holds for the relative weak and seminormalization X_Z^{rwn} and X_Z^{rsn} (51).* □

3. Formal nature

A key advantage of the punctual versions of the 4 properties in Definition 2 is that they are preserved by completions. This is in contrast with the usual notions of normality (resp. weak or seminormality); these are not always preserved by completions and it is frequently not easy to understand the cases when they are preserved.

The following is a special case of formal gluing originated in [Art70]. For a thorough general discussion see [Sta15, Tags 05E5 and 0AEP]. We give a short proof below using only the simpler methods employed there.

Proposition 15. *Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme with ideal sheaf $I \subset \mathcal{O}_X$. Let $\hat{X}$ denote the I-adic completion of X and $Z \cong \hat{Z} \subset \hat{X}$ the corresponding subscheme. Then completion provides an equivalence of categories*

$$\{\textit{finite modifications of } Z \subset X\} \Leftrightarrow \{\textit{finite modifications of } \hat{Z} \subset \hat{X}\}.$$

Proof. The question is local hence we may assume that X is affine. We may also assume that none of the associated primes of X is contained in Z.

Let $\tau : U := X \setminus Z \hookrightarrow X$ be the natural open embedding; similarly we have $\hat{\tau} : \hat{U} := \hat{X} \setminus \hat{Z} \hookrightarrow \hat{X}$. Set $R := H^0(X, \mathcal{O}_X), S := H^0(U, \mathcal{O}_U) = H^0(X, \tau_*\mathcal{O}_U)$ and, similarly $\hat{R} := H^0(\hat{X}, \mathcal{O}_{\hat{X}}), \hat{S} := H^0(\hat{U}, \mathcal{O}_{\hat{U}}) = H^0(\hat{X}, \hat{\tau}_*\mathcal{O}_{\hat{U}})$.

Since $\hat{X} \to X$ is flat, cohomology and base change for $\tau : U \to X$ says that $\hat{S} = S \otimes_R \hat{R}$.

As we noted in (1.2), finite modifications of $Z \subset X$ correspond to subalgebras $R \subset R' \subset S$ that are finite over R.

In general, let R be a Noetherian ring, $I \subset R$ an ideal and $S \supset R$ an R-algebra such that I is nilpotent on S/R. If $R \subset R' \subset S$ is a subalgebra that is finite over R then $I^m R' \subset R$ for some $m > 0$ and hence $R' \subset R : I^m$. Multiplication gives a map

$$\phi : (R : I^m)/I^m \otimes (R : I^m)/I^m \to (R : I^{2m})/R \quad \text{and} \quad \phi\big(R'/I^m, R'/I^m\big) \subset R'/R.$$

Conversely, if $R \subset M \subset (R : I^m)$ is any submodule such that $\phi\big(M/I^m, M/I^m\big) \subset M/R$ then M is a subalgebra.

Take now the I-adic completion $\hat{R}$ and set $\hat{S} := \hat{R} \otimes_R S$. Since I is nilpotent on S/R, we see that $\hat{S}/\hat{R} \cong S/R$ and so $(R : I^m)/I^m \cong (\hat{R} : \hat{I}^m)/\hat{I}^m$ for every $m > 0$. This gives an equivalence of categories between algebras $R \subset R' \subset S$ that are finite over R and algebras $\hat{R} \subset \hat{R}' \subset \hat{S}$ that are finite over $\hat{R}$. □

By passing to the limit on both sides, we get the following.

Corollary 16. *Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme. Then $\hat{X}^{\mathrm{rn}}_{\hat{Z}} = X^{\mathrm{rn}}_Z \times_X \hat{X}$.* □

Corollary 17 (Formal nature). *Using the notation of (15) let* **P** *denote any of the 4 properties in Definition 2. Then* **P** *holds for $Z \subset X$ iff it holds for $\hat{Z} \subset \hat{X}$.*

Complement. We prove in (31) that STN=TN, hence (17) also holds for STN.

Proof. The if part follows from (11) since $\hat{X}$ is faithfully flat over X.

To see the converse, let $\hat{p} : \hat{Y} \to \hat{X}$ be any finite modification. By (15) there is a finite modification $p : Y \to X$ such that $\hat{Y} = Y \times_X \hat{X}$. Since $\hat{X}$ is faithfully flat over X, $\hat{p}$ is an isomorphism (resp. universal homeomorphism) iff p is. □

The above result, together with (5) and (10) gives the following.

Corollary 18. *A Noetherian scheme X is normal (resp. weakly or seminormal) iff the completion $(\hat{x}, \hat{X})$ is punctually normal (resp. weakly or seminormal) for every $x \in X$.* □

4. Finiteness properties

Next we study when a relative normalization is finite over X.

Theorem 19. *Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme. The following are equivalent.*

1) *The relative normalization $\pi_Z : X_Z^{\mathrm{rn}} \to X$ is finite.*
2) *The punctual normalization $\pi_x : \big(x^{\mathrm{pn}}, X^{\mathrm{pn}}\big) \to (x, X)$ is finite for every $x \in Z$.*

Proof. (1) $\Rightarrow$ (2) since $\pi_x : \big(x^{\mathrm{pn}}, X^{\mathrm{pn}}\big) \to (x, X)$ is the localization of $\pi_{\bar{x}} : X_{\bar{x}}^{\mathrm{rn}} \to X$ by (14) and X_Z^{rn} dominates $X_{\bar{x}}^{\mathrm{rn}}$ if $\bar{x} \subset Z$.

Conversely, assume that (2) holds. We may assume that none of the associated points of X are contained in Z.

We repeatedly use the following construction. For any point $x \in X$ take the punctual normalization $\big(x^{\mathrm{pn}}, X^{\mathrm{pn}}\big) \to (x, X)$ and then extend it to a finite modification $p_1 : X_1 \to X$ centered at $\bar{x}$ using (13). As we note in (23), the assumptions of (2) also hold for X_1 and $Z_1 := p_1^{-1}(Z)$.

The canonical choice would seem to be to apply this construction first to the closed points where X is not punctually normal. This works well and we get X_1 that is punctually normal at all closed points. However, when we apply the procedure to X_1, we eliminate the 1-dimensional points where X_1 is not punctually normal but we may generate new punctually non-normal closed points. Thus we have to proceed from the other end. The price we pay is that the procedure itself is non-canonical.

First we apply the construction to the points of Z that have codimension 1 in X. We get $p_1 : X_1 \to X$ such that X_1 is regular at all points of Z_1 that have codimension 1 in X_1. By (20) there are only finitely points $x_1 \in Z_1$ such that the pair (x_1, X_1) is not normal. The closures of these points form a closed subset W_1. Next we repeat the procedure for the generic points of W_1 to get $p_2 : X_2 \to X_1$. After finitely many steps we get a finite partial modification $\pi : X_r \to X$ centered at Z such that X_r is punctually normal at all points of $\pi^{-1}(Z)$. Thus $\pi : X_r \to X$ is the relative normalization of the pair $Z \subset X$ by Proposition 10. □

Lemma 20. *Let X be a Noetherian scheme and $W \subset X$ a closed subscheme that does not contain any of the associated points of X. Then there are only finitely many points $x \in W$ such that the pair (x, X) is not normal.*

Proof. The question is local so we may assume that X is affine. By our assumptions there is a Cartier divisor $(g = 0)$ containing W. By Lemma 6, if a pair (x, X) is not normal then either x has codimension 1 in X (thus x is a generic point of Z) or $\operatorname{depth}_x X = 1$ and $\operatorname{codim}_X x \geq 2$. By Corollary 7 such an x is an associated point of $(g = 0)$. Since X is Noetherian, there are only finitely many such points. □

Definition 21. A Noetherian scheme X is called *punctually N-1* at $x \in X$ if the punctual normalization $\pi_x : \bigl(x^{\mathrm{pn}}, X^{\mathrm{pn}}\bigr) \to (x, X)$ is finite. X is called *punctually N-1* if this holds for every $x \in X$.

If X is reduced then the normalization of a local ring dominates its punctual normalization, hence if the local rings are N-1 (44) then X is also punctually N-1. (The converse probably does not hold; see (22) and (54).)

The following is closely related to [Sta15, Tag 0333].

Proposition 22. *A Noetherian integral scheme X is N-1 iff it is punctually N-1 and there are only finitely many points $x_i \in X$ such that (x_i, X) is not punctually normal.*

Proof. Let $X^{\mathrm{n}} \to X$ denote the normalization with structure sheaf $\mathcal{O}^{\mathrm{n}}_X$ and $X^{\mathrm{pn}}_i \to X$ the punctual normalization at x_i with structure sheaf $\mathcal{O}^{\mathrm{pn}}_{X_i}$. Since X^{n} dominates X^{pn}_i we get injections

$$\mathcal{O}^{\mathrm{pn}}_{X_i}/\mathcal{O}_X \hookrightarrow \mathcal{O}^{\mathrm{n}}_X/\mathcal{O}_X.$$

If $\mathcal{O}^{\mathrm{n}}_X$ is a coherent $\mathcal{O}_X$-sheaf then $\mathcal{O}^{\mathrm{pn}}_{X_i}$ is also coherent and there are only finitey many of them since each x_i is an associated point of $\mathcal{O}^{\mathrm{n}}_X/\mathcal{O}_X$.

Conversely, assume that X is punctually N-1 and there are only finitely many points $x_i \in X$ such that (x_i, X) is not punctually normal. Apply (19) with $Z := \cup_i \bar{x}_i$. The relative normalization $\pi_Z : X^{\mathrm{rn}}_Z \to X$ is finite and punctually normal along the preimage of Z. Since π_Z is an isomorphism over $X \setminus Z$, we conclude that X^{rn}_Z is punctually normal everyhere hence normal by Propositions 5 and 10. □

Lemma 23. *Let $g : Y \to X$ be a finite modification of Noetherian schemes centered at some $W \subset X$. Assume that none of the associated points of X is contained in W. Pick a point $x \in X$ and set $y := \operatorname{red} g^{-1}(x)$. Then X is punctually N-1 at x iff Y is punctually N-1 at y.*

Proof. If $x \notin W$ then $X_x \cong Y_y$ so the claim is interesting only if $x \in W$.

$Y^{\rm pn}$ dominates $X^{\rm pn}$, so the only if part is clear. To see the converse, we may assume that X is affine. Then there is a non-zerodivisor $\phi \in \mathcal{O}_X$ such that $\phi \cdot \mathcal{O}_Y \subset \mathcal{O}_X$. Thus $\phi \cdot \mathcal{O}_Y^{\rm pn} \subset \mathcal{O}_X^{\rm pn}$ hence if $\mathcal{O}_X^{\rm pn}$ is coherent then so is $\mathcal{O}_Y^{\rm pn}$. □

Next we study the effect of nilpotents on the relative normalization.

24 (Structure of the relative normalization). Let X be a Noetherian scheme, $Z \subset X$ a closed, nowhere dense subscheme and $j : U := X \setminus Z \hookrightarrow X$ the natural open embedding. Assume for notational simplicity that none of the associated primes of X is contained in Z. Let $J \subset \mathcal{O}_X$ be a nilpotent ideal. Note that $j_*(J|_U) \subset j_*\mathcal{O}_U$ is nilpotent, hence its sections are integral over X. Thus we have exact sequences

$$\begin{array}{ccccccccc} 0 & \to & j_*(J|_U) & \to & j_*\mathcal{O}_U & \stackrel{\rho}{\to} & j_*(\mathcal{O}_U/J|_U) & \to & R^1 j_*(J|_U) \\ 0 & \to & j_*(J|_U) & \to & \mathcal{O}_X^{\rm rn} & \stackrel{\rho}{\to} & (\mathcal{O}_X/J)^{\rm rn} & \to & R^1 j_*(J|_U) \end{array} \tag{24.1}$$

where $\mathcal{O}_X^{\rm rn}$ and $(\mathcal{O}_X/J)^{\rm rn}$ denote the structure sheaves of the relative normalizations of the pairs $Z \subset X$ and $Z \subset \operatorname{Spec}_X(\mathcal{O}_X/J)$. It is rather unclear when ρ is surjective; see (36) for an example. However, if $R^1 j_*(J|_U) = 0$ then ρ is surjective and we obtain the following.

Claim 24.2. If $\dim \operatorname{Supp} J = 1$ then every finite modification of the pair $Z \subset \operatorname{Spec}_X(\mathcal{O}_X/J)$ lifts to a finite modification of $Z \subset X$. In particular, $Z \subset X$ satisfies TN iff $Z \subset \operatorname{Spec}_X(\mathcal{O}_X/J)$ does. □

The second sequence in (24.1) shows that if $\mathcal{O}_X^{\rm rn}$ is coherent then so is $j_*(J|_U)$. By the easy direction of [Gro60, IV.5.11.1] (or by the argument in Proposition 5) this implies the following.

Claim 24.3. Assume that $X_Z^{\rm rn} \to X$ is finite and let $J \subset \mathcal{O}_X$ be a nilpotent ideal. Then either $\operatorname{Supp} J \subset Z$ or $Z \cap \operatorname{Supp} J$ has codimension ≥ 2 in $\operatorname{Supp} J$.

Thus if $X_Z^{\rm rn} \to X$ is finite for every Z then the nilradical of $\mathcal{O}_X$ has zero-dimensional support. □

Next let $I \subset \mathcal{O}_X$ be the nilradical and set $\mathcal{O}_X^{\rm nil} := \rho^{-1}\big(\mathcal{O}_{\operatorname{red} X}\big)$. We have an exact sequence

$$0 \to j_*(I|_U) \to \mathcal{O}_X^{\rm nil} \stackrel{\rho}{\to} \mathcal{O}_{\operatorname{red} X} \tag{24.4}$$

and $\pi : \operatorname{Spec}_X \mathcal{O}_X^{\rm nil} \to X$ is a limit of residue field preserving universal homeomorphisms. Note that π is an isomorphism iff $I = j_*(I|_U)$; equivalently, if $\operatorname{depth}_Z I \geq 2$. Thus we obtain the following.

Claim 24.5. Let $Z \subset X$ be a seminormal pair and $I \subset \mathcal{O}_X$ the nilradical. Then $\operatorname{depth}_Z I \geq 2$ and so $\operatorname{codim}_W(Z \cap W) \geq 2$ for every associated prime W of I. □

The implication of (24.3) becomes an equivalence for Nagata schemes (44).

Proposition 25. *Let X be a Nagata scheme and $Z \subset X$ a closed, nowhere dense subscheme. The following are equivalent.*

1) *The relative normalization $X_Z^{\mathrm{rn}} \to X$ is finite over X.*
2) *Let $J \subset \mathcal{O}_X$ be a nilpotent ideal. Then either* $\operatorname{Supp} J \subset Z$ *or* $Z \cap \operatorname{Supp} J$ *has codimension* ≥ 2 *in* $\operatorname{Supp} J$.

Proof. (1) ⇒ (2) follows from (24.3).

To see the converse, let $I \subset \mathcal{O}_X$ be the nilradical and $U := X \setminus Z$. (24.1) gives the exact sequence

$$0 \to j_*(I|_U) \to \mathcal{O}_X^{\mathrm{rn}} \stackrel{\rho}{\to} \mathcal{O}_{\operatorname{red} X}^{\mathrm{n}}.$$

By assumption, if $W \subset X$ is any associated prime of I then either $W \subset Z$ or $Z \cap W$ has codimension ≥ 2 in W. Thus $j_*(I|_U)$ is coherent by [Gro60, IV.5.11.1]. The normalization $(\operatorname{red} X)^{\mathrm{n}}$ is finite over $\operatorname{red} X$ since X is Nagata. Thus $\mathcal{O}_X^{\mathrm{rn}}$ is coherent. □

A complete semilocal ring is Nagata [Nag62, 32.1] and, again using [Gro60, IV.5.11.1], we see that $j_*\mathcal{O}_U$ is coherent iff X does not have 1-dimensional irreducible components. Hence we have proved the following.

Corollary 26. *Let (x, X) be a Noetherian, semilocal, complete scheme without isolated points and $j : U := X \setminus \{x\} \hookrightarrow X$ the natural embedding.*

1) *The punctual normalization $\pi : (x^{\mathrm{pn}}, X^{\mathrm{pn}}) \to (x, X)$ is finite iff $\mathcal{O}_X$ has no nilpotent ideals with 1-dimensional support.*
2) *The punctual hull $j_*\mathcal{O}_U$ is integral over $\mathcal{O}_X$ iff X does not have 1-dimensional irreducible components.*
3) *The punctual hull $j_*\mathcal{O}_U$ is finite over $\mathcal{O}_X$ iff X does not have 1-dimensional associated primes.* □

Combining (26) and (15) gives a characterization of punctually N-1 local schemes. The 1-dimensional case is essentially in [Kru30]; see also [Kol07, Sec.1.13].

Corollary 27. *Let (x, X) be a Noetherian, semilocal scheme without isolated points and $j : U := X \setminus \{x\} \hookrightarrow X$ the natural embedding.*

1) *The punctual normalization $\pi : \big(x^{\mathrm{pn}}, X^{\mathrm{pn}}\big) \to (x, X)$ is finite iff the completion $\hat{X}$ has no nilpotent ideals with 1-dimensional support.*
2) *The punctual hull $j_*\mathcal{O}_U$ is integral over $\mathcal{O}_X$ iff $\hat{X}$ does not have 1-dimensional irreducible components.*
3) *The punctual hull $j_*\mathcal{O}_U$ is finite over $\mathcal{O}_X$ iff $\hat{X}$ does not have 1-dimensional associated primes.* □

These give a characterization of weakly and seminormal schemes in terms of the conductor of the normalization. This is well known if the normalization is finite (53), but in general the (global) conductor ideal could be trivial. We go around this problem by working punctually.

Corollary 28. *For a Noetherian scheme X the following are equivalent.*

1) *X is semi (resp. weakly) normal.*
2) *X is reduced and for every $x \in X$, the punctual normalization $\pi_x : X_x^{\mathrm{pn}} \to X$ is finite and its conductor $C_x^{\mathrm{pn}} \subset X_x^{\mathrm{pn}}$ is reduced (resp. $\{x\}$ is purely inseparably closed in C_x^{pn}).*

Note that in (2) the conductor C_x^{pn} is 0-dimensional, so these are very simple conditions.

Proof. If X is seminormal then the punctual normalization $(x^{\mathrm{pn}}, X^{\mathrm{pn}}) \to (x, X)$ is finite by (24.5), (17) and (27). (This should also follow from a theorem of Chevalley and Mori; see [Nag62, 33.10].) Then (2) holds by (53).

Conversely, if (2) holds then (x, X) satisfies SN (resp. WN) for every $x \in X$ by (53) hence X is semi (resp. weakly) normal by (10). □

Observe that if X is a Noetherian, integral, seminormal scheme whose normalization is not finite then, by (22), there are infinitely many distinct points $x_i \in X$ such that (x_i, X) is not punctually normal.

5. Factoring topological modifications

Theorem 29. *Let X be a Noetherian scheme, $Z \subset X$ a closed, nowhere dense subscheme and $g : Y \to X$ a finite, topological modification centered* *at Z. Then g can be factored as*

$$g : Y \stackrel{g'}{\to} X' \stackrel{\pi}{\to} X$$

where g' is a finite, universal homeomorphism and π is a finite modification centered at Z.

Remark 30. The factorization is not unique and there does not seem to be a canonical choice for it, except when $\dim Z = 0$ (33). For example, $\mathbb{C}[x^2, xy, y] \to \mathbb{C}[x, y]$ can be factored as

$$\mathbb{C}[x^2, xy, y] \to \mathbb{C}[x^2, x^{2n+1}, xy, y] \to \mathbb{C}[x, y]$$

for any $n \geq 0$, thus there is no minimal choice for X'. Similarly, $\mathbb{C}[x, y]/(x^2y^2) \to \mathbb{C}[x] + \mathbb{C}[y]$ can be factored as

$$\mathbb{C}[x, y]/(x^2y^2) \to \mathbb{C}[x, x^{-n}\epsilon] + \mathbb{C}[y, y^{-n}\epsilon] \to \mathbb{C}[x] + \mathbb{C}[y]$$

for any $n \geq 0$, thus there is no maximal choice for X'.

Corollary 31. *Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme. Then $Z \subset X$ satisfies TN iff it satisfies STN.*

Proof. It is clear that STN $\Rightarrow$ TN. Next assume that we have $g : Y \to X$ showing that STN fails for $Z \subset X$. Using (29) we get $\pi : X' \to X$ which shows that TN fails for $Z \subset X$. □

Corollary 32. *For a Noetherian scheme X the following are equivalent.*

1) *X is topologically normal,*
2) *X is geometrially unibranch (see [Gro60, Sec.IV.6.15] or [Sta15, Tag 06DJ]),*
3) *the normalization $X^{\mathrm{n}} \to X$ is a universal homeomorphism,*
4) *the weak normalization X^{wn} is normal.* □

33 (Punctual topological normalization). Let (x, X) be a complete local scheme and $p : (y, Y) \to (x, X)$ a finite morphism. Let $A \subset k(y)$ be the separable closure of $k(x)$. There is a (unique) finite, étale morphism $(x', X') \to (x, X)$ whose fiber over x is isomorphic to $\operatorname{Spec} A$. Furthermore, p lifts to $p_Y : (y, Y) \to (x', X')$ since $(x', X') \to (x, X)$ is formally smooth and (y, Y) is complete.

Assume next that $p : Y \setminus \{y\} \to X \setminus \{x\}$ is a universal homeomorphism. Then $\operatorname{red}(p_Y(Y))$ is a union of some of the irreducible components of $\operatorname{red}(X')$. Thus there is a largest subscheme $X^u \subset X'$ such that $\operatorname{red}(X^u) = \operatorname{red}(p_Y(Y))$ and x is not an associated point of X^u. Then $X^u \setminus \{x'\} \to X \setminus \{x\}$ is étale and an isomorphism on the underlying reduced subschemes, hence an isomorphism.

We have thus factored p as

$$p : (y, Y) \stackrel{p'}{\to} (x^u, X^u) \stackrel{p^u}{\to} (x, X) \tag{33.1}$$

where p^u is an unramified punctual modification and p' is a universal homeomorphism. By (15), this factorization also exists even if (x, X) is not complete. This proves (29) in the special case when Z is a closed point.

Applying this to $p : \big(\hat{x}^{\mathrm{pn}}, (\operatorname{red}\hat{X})^{\mathrm{pn}}\big) \to (\hat{x}, \operatorname{red}\hat{X}) \to (\hat{x}, \hat{X})$ and then descending to X we obtain the following.

Claim 33.2. Let (x, X) be a semilocal, Noetherian scheme. Then there is a unique punctual modification

$$\pi^{\mathrm{ptn}} : \big(x^{\mathrm{ptn}}, X^{\mathrm{ptn}}\big) \to (x, X)$$

such that π^{ptn} is unramified and $\big(x^{\mathrm{ptn}}, X^{\mathrm{ptn}}\big)$ is the smallest topologically normal punctual modification of (x, X). □

Note that if (x, X) is Henselian with perfect residue field then X^{ptn} is simply the disjoint union of the closures of the connected components of $X \setminus \{x\}$; cf. (43). De Jong pointed out that (33.2) can also be obtained from this by Galois descent.

34 (Proof of Theorem 29). We use Noetherian induction on $g^{-1}(Z) \subset Y$.

Let $x \in X$ be the generic points of Z. After localizing at x we obtain a punctual, topological modification $p : (y, Y) \to (x, X)$. By (33.1) it can be factored as

$$p : (y, Y) \stackrel{p'}{\to} (x', X') \stackrel{p''}{\to} (x, X), \tag{34.1}$$

where p'' is an unramified punctual modification and p' is a universal homeomorphism. We extend (34.1) first to an open neighborhood and then, using (13), we get morphisms

$$g : Y \stackrel{g_1}{\to} X_1 \stackrel{p_1}{\to} X, \tag{34.2}$$

where p_1 is a finite modification centered at Z and $g_1 : Y \to X_1$ is a finite morphism that is a universal homeomorphism outside $p_1^{-1}(Z)$ and also over the generic points of $p_1^{-1}(Z)$. By Noetherian induction, g_1 can be factored as

$$g_1 : Y \stackrel{g_1'}{\to} X_1' \stackrel{\pi_1}{\to} X_1 \tag{34.3}$$

where g_1' is a finite, universal homeomorphism and π_1 is a finite modification centered at $p^{-1}(Z)$. Thus we can set $X' := X_1'$, $g' := g_1'$ and

$$g : Y \stackrel{g'}{\to} X' \stackrel{p_1\circ\pi_1}{\longrightarrow} X \tag{34.4}$$

gives the required factorization for g. □

Arguing as in (19) and using (33.2), this implies the following.

Corollary 35. *Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme. Then there is a finite modification $\pi : Y \to X$ centered at Z such that the pair $\pi^{-1}(Z) \subset Y$ is topologically normal.* □

The following example shows that there does not seem to be a good notion of strong topological normality with residue field preservation.

Example 36. Start with the function field $K = k(t)$ with $\operatorname{char} k = 2$. Set

$$R := K[x, y, \sqrt{t}x, \sqrt{t}y] \subset K(\sqrt{t})[x, y].$$

Note that the normalization of R is $K(\sqrt{t})[x, y]$ and the conductor ideal is $(x, y) \subset K(\sqrt{t})[x, y]$. Thus R is seminormal but not weakly normal and does not satisfy TNR.

We can embed $\operatorname{Spec} R$ into $\mathbb{A}^4_K$ by $(x, y, \sqrt{t}x, \sqrt{t}y) \mapsto (u_1, v_1, u_2, v_2)$. The image is defined by the obvious equations

$$u_1v_2 - u_2v_1 = tu_1v_1 - u_2v_2 = tu_1^2 - u_2^2 = tv_1^2 - v_2^2 = 0.$$

Let $X \subset \mathbb{A}^4_K$ be defined by the last 2 equations. It is a complete intersection, hence S_2 and so $\{\mathbf{0}\} \subset X$ is a normal pair which also satisfies TNR.

On the other hand, the last 2 equations imply that

$$(u_1v_2 - u_2v_1)^2 = (tu_1v_1 - u_2v_2)^2 = 0,$$

hence $\operatorname{red} X \cong \operatorname{Spec} R$. Thus $\{\mathbf{0}\} \subset X$ satisfies TNR but $\{\mathbf{0}\} \subset \operatorname{red} X$ does not.

6. Flat families

All 5 properties behave well for flat families.

Theorem 37. *Let $f : Y \to X$ be a flat morphism of Noetherian schemes with (geometrically) reduced fibers and* **P** *any of the 5 properties in Definitions 2–3. Assume that X and the geometric generic fibers satisfy* **P**. *Then Y also satisfies* **P**.

For normality this is classical; see for instance [Mat86, 23.9]. For seminormality this is proved in [GT80], for weak normality in [Man80]; in both cases for N-1 schemes (called Mori schemes in these papers). The argument below follows the method of [GT80].

Proof. Pick any point $y \in Y$ and set $x := f(y)$. If Y is contained in a generic fiber then (y, Y) satisfies **P** by assumption. If y is a generic point of its fiber Y_x then, as we prove in (38), (y, Y) satisfies **P**.

X is reduced by Proposition 5 in cases N, SN, WN and for every other point we have

$$\operatorname{depth}_y Y = \operatorname{depth}_x X + \operatorname{depth}_y Y_x \geq 1 + 1 = 2, \tag{37.1}$$

hence (y, Y) is even punctually normal by Lemma 6 and it satisfies **P** for every **P**.

As we noted in (4.4), in case TN it is enough to show that $\operatorname{red} Y$ satisfies TN. We may replace X and Y by $\operatorname{red} X$ and $\operatorname{red} Y$. We do not actually need that $\operatorname{red} X$ satisfies TN. It is S_1 and this is all we used in (37.1). □

Proposition 38. *Let $f : Y \to X$ be a flat, regular morphism of Noetherian schemes. If X satisfies any of the 5 properties* **P** *in Definitions 2–3 then so does Y.*

Proof. Pick a point $y \in Y$ and set $x := f(y)$. By (17), $\hat{X}_x$ satisfies **P**. Note that $Y \times_X \hat{X}_x \to Y$ is faithfully flat. Thus if $\big(y, Y \times_X \hat{X}_x\big)$ satisfies **P** then so does (y, Y) by Proposition 11. Thus it is sufficient to prove (38) in case (x, X) is local and complete. We need slightly different arguments in the various cases.

If (x, X) is punctually normal then either X is regular (of dimension 1) hence Y is also regular, or $\operatorname{depth}_x X \geq 2$ hence $\operatorname{depth}_y Y \geq 2$.

For TN we may assume, using (24.2), that $\mathcal{O}_X$ has no nilpotent ideals with 1-dimensional support, and then the punctual normalization

$(x^{\rm pn}, X^{\rm pn}) \to (x, X)$ is finite by (26). It is also a universal homeomorphism by our assumptions. Thus $Y \times_X X^{\rm pn} \to Y$ is also a universal homeomorphism (resp. also preserves residue fields). By the already established normal case the pair $Y \times_X x^{\rm pn} \subset Y \times_X X^{\rm pn}$ is normal. Therefore, as we noted in Paragraph 9, Y satisfies TN.

For SN and WN we use (28), thus the punctual normalization $\pi : X^{\rm pn} \to X$ is finite and its conductor $C_X^{\rm pn} \subset X^{\rm pn}$ is reduced (resp. $\{x\}$ is purely inseparably closed in $C_X^{\rm pn}$). Therefore the conductor $C_Y^{\rm pn}$ of $Y \times_X X^{\rm pn} \to Y$ is also reduced and $Y_x := Y \times_X x$ is purely inseparably closed in it in the weakly normal case by (52). Thus Y satisfies WN (resp. SN) by (53). □

Example 39. It is natural to ask what happens if in (37) we only assume that the fibers are topologically normal in codimension m for some m. There are easy counter examples.

The fibers of $\operatorname{Spec} k[x, y]/(y^2 - x^2) \to \operatorname{Spec} k[x]$ are 0-dimensional, hence topologically normal yet $\operatorname{Spec} k[x, y]/(y^2 - x^2)$ is not topologically normal.

Similarly, let $R \subset k[x, y]$ consist of those polynomials for which $p(0, 0) = p(0, 1)$. Then the fibers of $\operatorname{Spec} R \to \operatorname{Spec} k[x]$ are 1-dimensional hence topologically normal in codimension 2 yet $\operatorname{Spec} R$ is not topologically normal in codimension 2.

Nonetheless, these are essentially the only counter examples. We can even keep track of the codimension as in (41).

Proposition 40. *Fix $n, m \geq 1$. Let $g : X \to S$ be a morphism of pure relative dimension n for some n and $W \subset X$ a closed subset. Assume that*

1) *$X \setminus W$ is topologically normal in codimension m.*
2) *every fiber X_s is topologically normal in codimension m and*
3) $\operatorname{codim}_{X_s}(W \cap X_s) \geq m$ *for every $s \in S$.*

Then X is topologically normal in codimension m.

Proof. Let $f : Y \to X$ be a finite modification centered at $Z \subset X$ that is a putative counter example. We need to show that f is a universal homeomorphism. By assumption (1) this holds over $X \setminus W$, we can thus assume that $Z \subset W$.

We claim that Y_s has pure dimension n for every $s \in S$. It is enough to prove the claim when S is local and irreducible. If $s \in S$ has codimension r then s is set-theoretically the complete intersection of r Cartier divisors, thus Y_s is also set-theoretically the complete intersection of r Cartier divisors, hence every irreducible component of Y_s has dimension $\geq n$. Since X_s has pure dimension n and $Y_s \to X_s$ is finite, this implies that Y_s has pure dimension n.

In particular, none of the irreducible components of Y_s is contained in $f_s^{-1}(W \cap X_s)$ hence $f_s : Y_s \to X_s$ is a finite topological modification centered at $W \cap X_s$. Then f_s is a universal homeomorphism by assumption (2) and so is f. □

7. Connectedness properties

A version of topological normality first appeared in [Har62] where it is proved that $\operatorname{depth}_x X \geq 2$ implies that (x, X) is punctually TN. More generally, one can understand topological normality in codimension $\geq m$ in terms of connectedness properties of étale covers of X. Many parts of the following theorem are discussed in [Har62] and [Gro68, III.3 and XIII.2].

Theorem 41. *Fix a natural number $m \geq 1$. For a Noetherian scheme X the following are equivalent.*

1) *$Z \subset X$ is topologically normal for every subscheme of codimension $\geq m$.*
2) *For every point $x \in X$ of codimension $\geq m$ any of the following holds:*
   a) *(x, X) is topologically normal,*
   b) *the Henselization $\left(x^{\mathrm{h}}, X^{\mathrm{h}}\right)$ is topologically normal,*
   c) *the strict Henselization $\left(x^{\mathrm{sh}}, X^{\mathrm{sh}}\right)$ is topologically normal,*
   d) *$X^{\mathrm{sh}} \setminus \{x^{\mathrm{sh}}\}$ is connected or*
   e) *the completion $\left(\hat{x}, \hat{X}\right)$ is topologically normal.*
3) *For every quasi-finite étale morphism $g : Y \to X$ with Y connected, the complement of any closed subset $W \subset Y$ of codimension $\geq m$ is connected.*

Proof. The equivalence of (1) and (2.a) follows by noting that the arguments of Proposition 10 preserve the codimension.

(2.a) is equivalent to (2.b) and (2.c) by (11) and Proposition 38 and to (2.e) by (17). (2.c) and (2.d) are equivalent by (43).

Finally (42) shows that (2.c) ⇒ (3); here we also use that quasifinite morphisms do not decrease the codimension. Since the strict Henselization is the limit of quasi-finite étale morphisms, (3) ⇒ (2.d) is clear. □

Lemma 42. *If $Z \subset X$ is topologically normal then X and $X \setminus Z$ have the same connected components.*

Proof. We may assume that X is connected. If $X \setminus Z$ is disconnected, write it as $Y_1 \cup Y_2$ where the Y_i are disjoint, closed subschemes. Let $\bar{Y}_i \subset X$ denote the closure of Y_i. Then the natural map $\bar{Y}_1 \amalg \bar{Y}_2 \to X$ shows that $Z \subset X$ is not a topologically normal pair. □

Proposition 43. *Let (x, X) be a Henselian local scheme with separably closed residue field $k(x)$. The following are equivalent.*

1) (x, X) *is strongly topologically normal,*
2) (x, X) *is topologically normal and*
3) $X \setminus \{x\}$ *is connected.*

Proof. (1) $\Rightarrow$ (2) is clear and (2) $\Rightarrow$ (3) follows from (42). To see (3) $\Rightarrow$ (1), let $g : X' \to X$ be a finite morphism that is a universal homeomorphism over $X \setminus \{x\}$. If X is Henselian, there is a one–to–one correspondence between connected components of X' and connected components of $x' := g^{-1}(x)$; see [Mil80, I.4.2]. If $k(x')/k(x)$ is not purely inseparable then x' is not connected hence $X' \setminus \{x'\}$ is also not connected. The latter is, however, homeomorphic to $X \setminus \{x\}$. □

8. Nagata schemes

Definition 44. An integral Noetherian scheme X is called *N-1* if its normalization in $k(X)$ is finite over X and *N-2* if its normalization in any finite field extension of $k(X)$ is finite over X. The latter is equivalent to the following: every integral scheme with a finite, dominant morphism $X' \to X$ is N-1.

A Noetherian scheme X is *Nagata* if every integral subscheme $W \subset X$ is N-2; see for instance [Mat86, p.264] or [Sta15, Tag 033R].

We obtain a characterization of local Nagata schemes in terms of punctual normalizations and formal fibers. Most parts of this have been known. A local Nagata domain is analytically unramified and an analytically unramified domain is N-1; see [Nag62, 32.2] and [Sta15, Tag 0331]. (Recall that a reduced, semilocal scheme (x, X) is called *analytically unramified* iff its completion $\hat{X}$ is reduced.) The connection with formal fibers is essentially part of the argument that shows that a quasi-excellent scheme is Nagata, see [Mat80, 33.H] or [Sta15, Tag 07QV].

For a scheme X let $\mathrm{SiS}(X)$ denote the set of all semilocal, integral X-schemes obtained by localizing an X-scheme that is finite over X.

Theorem 45. *For a Noetherian scheme the following are equivalent.*

1) *All local rings of X are Nagata.*
2) *For every $(y, Y) \in \mathrm{SiS}(X)$ the normalization $Y^{\mathrm{n}} \to Y$ is finite.*
3) *For every $(y, Y) \in \mathrm{SiS}(X)$ the punctual normalization $\left(y^{\mathrm{pn}}, Y^{\mathrm{pn}}\right) \to (y, Y)$ is finite.*
4) *Every $(y, Y) \in \mathrm{SiS}(X)$ is analytically unramified.*
5) *For every $(y, Y) \in \mathrm{SiS}(X)$ the geometric generic fibers of $(\hat{y}, \hat{Y}) \to (y, Y)$ are reduced.*
6) *For every $x \in X$ the morphism $(\hat{x}, \hat{X}) \to (x, X)$ has (geometrically) reduced fibers.*

Remark 46. This suggests that one should call a Noetherian scheme *locally Nagata* if it satsifies the condition (45.2). By the theorem, this is equivalent to assuming that all local rings of X are Nagata.

The condition (45.5) should be compared with the defining property of G-rings: For every $x \in X$ the morphism $(\hat{x}, \hat{X}) \to (x, X)$ has (geometrically) regular fibers; see [Mat86, p.256] or [Sta15, Tag 07GG]. Hence every G-ring is locally Nagata.

47 (Proof of Theorem 45). We may assume that X is integral and affine. By induction we may assume that the equivalence holds over any integral X-scheme such that $Z \to X$ is finite and not dominant.

First note that (1) $\Rightarrow$ (2) by definition and (2) $\Rightarrow$ (3) since Y^{n} dominates $\left(y^{\mathrm{pn}}, Y^{\mathrm{pn}}\right)$.

Assume that (3) holds and fix a point $x \in X$. We aim to show that $(\hat{x}, \hat{X}) \to (x, X)$ has reduced geometric generic fibers. Pick any $g \in m_x$. As in the proof of Theorem 19 (and using Corollary 7), after replacing X by a suitable finite partial normalization, we may assume that X is regular at the generic points of $(g = 0)$ and the latter has no embedded primes. By induction, we already know that $\mathrm{red}(g = 0)$ is analytically unramified. Under these conditions, [Sta15, Tag 0330] says that X is analytically unramified. Thus (3) $\Rightarrow$ (4).

In order to see the equivalence of (5) and (6) fix $\pi : Y \to X$ and set $x := \pi(y)$. Note that $\hat{Y} = Y \times_X \hat{X}$ where we complete Y at y and X at x. If $\hat{X} \to X$ has geometrically reduced fibers then so does $\hat{Y} \to Y$. Conversely, pick any point $p \in X$ and let $L \supset k(p)$ be a finite field extension. We can realize $\operatorname{Spec} L \to p$ as the generic fiber of a suitable $(y, Y) \in \operatorname{SiS}(X)$. Thus if (5) holds for Y then $\operatorname{Spec} L \times_p \hat{X}$ is reduced, hence (6) holds. (It is sufficient to check that the fibers stay reduced after finite field extensions; see [Sta15, Tag 030V].)

It is clear that (4) implies (5) and if $\hat{X} \to X$ has geometrically reduced fibers then so does $\hat{Y} \to Y$. Thus $\hat{Y}$ is reduced and Y is analytically unramified, proving (6) $\Rightarrow$ (4).

Finally a strictly increasing chain of partial normalizations $\cdots \to Y_{i+1} \to Y_i \to \cdots \to Y$ gives a a strictly increasing chain of morphisms

$$\cdots \to \hat{Y} \times_Y Y_{i+1} \to \hat{Y} \times_Y Y_i \to \cdots \to \hat{Y}$$

and these are partial normalizations if $\hat{Y}$ is reduced. This is impossible since a complete semilocal ring is Nagata [Nag62, 32.1], thus (4) implies (1). $\square$

9. Universal homeomorphisms

We recall the basic propeties of universal homeomorphisms that we use.

Definition 48. A morphism of schemes $g : U \to V$ is a *universal homeomorphism* if for every $W \to V$ the induced morphism $U \times_V W \to W$ is a homeomorphism. This notion is called "radiciel" in [Gro71, I.3.7], "radicial" in [Sta15, Tag 01S2] and a "purely inseparable morphism" by some authors. The reason for the latter is the following observation.

Pick a point $v \in V$ and base-change to the algebraic closure of $k(v)$. We obtain that the set-theoretic fiber $g^{-1}(v)$ is a single point v' and $k(v')$ is a purely inseparable field extension of $k(v)$. We say that g is *residue field preserving* if $k(v') = k(v)$ for every $v \in V$.

A universal homeomorphism is affine and integral; see [Sta15, Tag 04DF]. For integral morphisms the notion of universal homeomorphism is pretty much set theoretic since a continuous proper map of topological spaces which is injective and surjective is a homeomorphism.

The following summarizes the basic characterizations.

Lemma 49. *[Gro71, I.3.7–8] For an integral morphism $g : U \to V$ the following are equivalent.*

1) *g is a universal homeomorphism.*

2) *g is surjective and universally injective.*

3) $k\big(\operatorname{red} g^{-1}(v)\big)/k(v)$ *is a purely inseparable for every* $v \in V$.

4) *g is surjective and injective on geometric points.*

In concrete situations it is usually easiest to check (49.3). Note that for a finite type morphism $g : U \to V$ the points $v \in V$ that satisfy (49.3) form a constructible set.

Using property (49.3) we also obtain that if $p : V' \to V$ is surjective then an integral morphism $g : U \to V$ is a universal homeomorphism iff the pull back $p^* g : V' \times_V U \to V'$ is.

10. Weakly normal and seminormal schemes

We recall the definitions and basic properties of weakly normal and seminormal schemes, following [AN67, AB69, GT80, Man80].

Definition 50. A morphism of schemes $g : X' \to X$ is called a *partial normalization* if X' is reduced, g is integral and $\operatorname{red}(g) : X' \to \operatorname{red}(X)$ is birational [Kol13, 1.11]. Thus a partial normalization is dominated by the normalization $(\operatorname{red} X)^{\mathrm{n}}$ of $\operatorname{red} X$ which is the limit of all finite, partial normalizations. A scheme is normal iff every finite, partial normalization $g : X' \to X$ is an isomorphism.

A partial normalization $g : X' \to X$ is called a *partial seminormalization* if, in addition, $k\big(\operatorname{red} g^{-1}(x)\big) = g^* k(x)$ for every $x \in X$ and a *partial weak normalization* if $k\big(\operatorname{red} g^{-1}(x)\big)/g^* k(x)$ is a purely inseparable field extension for every $x \in X$. (If X has residue characteristic 0 then these 2 notions coincide.)

There is a unique largest partial weak (resp. semi) normalization $X^{\mathrm{wn}} \to X$ (resp. $X^{\mathrm{sn}} \to X$) that dominates every other partial weak (resp. semi) normalization of X. It is called the *weak normalization* (resp. *seminormalization*) of X.

The weak normalization $X^{\mathrm{wn}} \to X$ is a universal homeomorphism and it is the maximal universal homeomorphism that is dominated by the normalization.

A scheme X is called *weakly normal* (resp. *seminormal*) iff every finite, partial weak (resp. semi) normalization $g : X' \to X$ is an isomorphism.

Note that seminormalization is a functor; that is, every morphism $f : Y \to X$ lifts to the seminormalizations $f^{\mathrm{sn}} : Y^{\mathrm{sn}} \to X^{\mathrm{sn}}$. By contrast,

$f : Y \to X$ lifts to the weak normalizations $f^{\mathrm{wn}} : Y^{\mathrm{wn}} \to X^{\mathrm{wn}}$ if every irreducible component of Y dominates an irreducible component of X but not otherwise.

It is easy to see using (13) that an open subscheme of a weakly (resp. semi) normal scheme is also weakly (resp. semi) normal and being weakly (resp. semi) normal is a local property. For excellent schemes weak (resp. semi) normality a formal-local property; see [GT80, Man80].

Definition 51 (Constrained normalization). More generally, let $\pi : Y \to X$ be any morphism. The *π-constrained normalization* (resp. *weak or seminormalization*) of X is the unique largest partial normalization (resp. weak or semi normalization) $X' \to X$ such that $\mathrm{red}(\pi)$ factors as $\mathrm{red}(Y) \to X' \to X$. (Note that we do not require π to be dominant.)

The weak or semi normalization constrained by the relative normalization $X^{\mathrm{rn}}_Z \to X$ is called the *relative weak or seminormalization* of the pair $Z \subset X$. These are denoted by $X^{\mathrm{rwn}}_Z \to X$ and $X^{\mathrm{rsn}}_Z \to X$.

As in [Sta15, Tag 0BAK], let $A^{\mathrm{int}} \subset \pi_* \mathcal{O}_Y$ denote the maximal subalgebra that is integral over $\mathcal{O}_X$ and $A^{\mathrm{pi}} \subset \pi_* \mathcal{O}_Y$ the maximal subalgebra that is purely inseparable over $\mathcal{O}_X$. (That is, $\mathrm{Spec}_X A^{\mathrm{pi}} \to X$ is a universal homeomorphism and maximal with this property.) Then $\mathrm{Spec}_X A^{\mathrm{int}}$ is called the *relative normalization* of X in Y and $\mathrm{Spec}_X A^{\mathrm{pi}}$ the *purely inseparable closure* of X in Y. (The relative normalization frequently agrees with the constrained normalization but the notions are quite different if π is not birational.)

Example 52. Let k be a field and $i : k \hookrightarrow A$ a finite, semisimple k-algebra; that is, a sum of finite field extensions. If A is separable then k is purely inseparably closed in A but the converse does not hold. For instance, $k(s^p, t^p)$ is purely inseparably closed in $A := k(s, t^p) + k(s^p, t)$ where $p = \mathrm{char}\, k$.

In order to get a characterization, [Man80] considers the complex

$$k \xrightarrow{(i,i)} A + A \xrightarrow{\delta} A \otimes_k A \tag{52.1}$$

where $\delta(a_1, a_2) = a_1 \otimes 1 - 1 \otimes a_2$. Let $\mathrm{red}(A \otimes_k A)$ denote the quotient of $A \otimes_k A$ by its nil-radical. We claim that

$$k \xrightarrow{(i,i)} A + A \xrightarrow{\delta} \mathrm{red}(A \otimes_k A) \tag{52.2}$$

is exact iff $\operatorname{Spec} k$ is weakly normal in $\operatorname{Spec} A$. To see this, pick $(a_1, a_2) \in A + A$ and set $q = (\operatorname{char} k)^m$ for some $m \geq 1$. Then

$$(a_1 \otimes 1 - 1 \otimes a_2)^q = (a_1 \otimes 1)^q - (1 \otimes a_2)^q = a_1^q \otimes 1 - 1 \otimes a_2^q,$$

hence $\delta(a_1, a_2)$ is nilpotent iff $a_1^q = a_2^q \in k$ for some $q = (\operatorname{char} k)^m$.

Let Y_k be a geometrically regular k-scheme. Then

$$\operatorname{red}(A \otimes_k A) \otimes_k \mathcal{O}_{Y_k} = \operatorname{red}\big(\mathcal{O}_{Y_A} \otimes_{\mathcal{O}_{Y_k}} \mathcal{O}_{Y_A}\big),$$

thus, if the sequence (52.2) is exact, then so is the tensored sequence

$$\mathcal{O}_X \stackrel{(i,i)}{\longrightarrow} \mathcal{O}_{Y_A} + \mathcal{O}_{Y_A} \stackrel{\delta}{\longrightarrow} \operatorname{red}\big(\mathcal{O}_{Y_A} \otimes_{\mathcal{O}_{Y_k}} \mathcal{O}_{Y_A}\big). \qquad (52.3)$$

Therefore, if k is purely inseparably closed in A then Y_k is purely inseparably closed in Y_A.

The next result connecting the absolute and relative versions of weak and seminormality is very useful in inductive treatments. The proof follows [GT80, Man80].

Proposition 53. *Let $f : Y \to X$ be a finite modification with conductor subschemes $C_Y \subset Y$ and $C_X \subset X$. Assume that the pair $C_Y \subset Y$ satisfies WN (resp. SN).*

Then the pair $C_X \subset X$ satisfies WN (resp. SN) iff C_Y (and hence also C_X) are reduced and C_X is its own purely inseparable closure in C_Y (resp. its own $f|_{C_Y}$-constrained seminormalization).

Proof: If C_Y is not reduced then $\operatorname{Spec}_X\big(\mathcal{O}_X + \sqrt{\operatorname{cond}_Y}\big) \to X$ is a universal homeomorphism that is not an isomorphism, where $\operatorname{cond}_Y \subset \mathcal{O}_Y$ denotes the conductor ideal. Let $C_Y \to \tilde{C} \to C_X$ denote the purely inseparable closure (resp. $f|_{C_Y}$-constrained seminormalization) and $F \subset f_*\mathcal{O}_Y$ the preimage of $\mathcal{O}_{\tilde{C}}$. Then $\operatorname{Spec}_X F \to X$ is a universal homeomorphism (res. also preserves residue fields) that is not an isomorphism. Thus the conditions are necessary.

Conversely, assume that $p : X' \to X$ is a universal homeomorphism (resp. also preserves residue fields) that is an isomorphism over $X \setminus C_X$. Let $\mathcal{O}_{Y'}$ be the composite of $f_*\mathcal{O}_Y$ and $p_*\mathcal{O}_{X'}$ inside $j_*\mathcal{O}_U$ where $U := X \setminus C_X$. Then $Y' \to Y$ is a universal homeomorphism (resp. also preserves residue fields) hence an isomorphism by assumption. Thus $p_*\mathcal{O}_{X'} \subset f_*\mathcal{O}_Y$ and hence $p_*\mathcal{O}_{X'}/\operatorname{cond}_X \subset \mathcal{O}_{C_Y}$. The corresponding map $\operatorname{Spec}_X\big(p_*\mathcal{O}_{X'}/\operatorname{cond}_X\big) \to$

C_X is a universal homeomorphism (resp. partial seminormalization), hence an isomorphism if C_Y is reduced and C_X is its own purely inseparable closure in $\operatorname{red} C_Y$ (resp. $f|_{C_Y}$-constrained seminormalization). Thus $p : X' \to X$ is an isomorphism. □

11. Open problems

Let R be an integral domain with normalization R^{n}. Then R is not punctually normal at a prime ideal $\mathfrak{p} \subset R$ iff $\mathfrak{p}$ is an associated prime of R^{n}/R. It is not hard to construct 1-dimensional, Noetherian, seminormal domains with infinitely many prime ideals, all of which are associated primes of R^{n}/R.

The situation seems more complicated in higher dimensions. By (20), for every $\mathfrak{q} \neq (0)$, only fintely many of the primes $\mathfrak{p} \supset \mathfrak{q}$ can be associated primes of R^{n}/R. This leads to the following problems.

Question 54. Are there integral domains of every dimension such that every height 1 prime is an associated prime of R^{n}/R?

Question 55. Are there seminormal integral domains of every dimension such that every height 1 prime is an associated prime of R^{n}/R?

Question 56. What can one say about the height ≥ 2 associated primes of R^{n}/R?

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