A global pinching theorem for complete translating solitons of mean curvature flow*

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Abstract: In the present paper, we prove that for a smooth complete translating soliton $M^n (n \geq 3)$ with the mean curvature vector $H$ satisfying $H = VN$ for a unit constant vector $V$ in the Euclidean space $\mathbb{R}^{n+p}$, if the trace-free second fundamental form $\tilde{A}$ satisfies $(\int_M |\tilde{A}|^n d\mu)^{1/n} < K(n)$, $\int_M |\tilde{A}|^n e^{(V,X)} d\mu < \infty$, where $K(n)$ is an explicit positive constant depending only on $n$, then $M$ is a linear subspace.

Keywords: Rigidity theorem, translating soliton, integral curvature pinching.

1. Introduction

Let $X_0 : M \rightarrow \mathbb{R}^{n+p}$ be an $n$-dimensional smooth submanifold isometrically immersed in an $(n + p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. The mean curvature flow with initial value $X_0$ is a smooth family of immersions $X : M \times [0, T) \rightarrow \mathbb{R}^{n+p}$ satisfying

\[
\begin{align*}
\frac{d}{dt} X(x,t) &= H(x,t), \\
X(x,0) &= X_0(x),
\end{align*}
\]

for $x \in M$ and $t \in [0, T)$. Here $H(x,t)$ is the mean curvature vector of $M_t = X_t(M)$ at $X(x,t)$ in $\mathbb{R}^{n+p}$ where $X_t(\cdot) = X(\cdot,t)$.

In the theory of the mean curvature flow, one of the most important fields is the singularity analysis. According to the blow-up rate of the second fundamental form $A$, singularities of the mean curvature flow are divided into two types called Type-I singularity and Type-II singularity. It is well-known that self-shrinkers describe the Type-I singularity models of the mean curvature

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A very important example of Type-II singularities is the translating soliton. A submanifold \( X : M^n \to \mathbb{R}^{n+p} \) is said to be a translating soliton (translator for short) if there exists a constant vector \( V \) with unit length in \( \mathbb{R}^{n+p} \) such that

\[
H = V^N,
\]

where \((\ )^N\) denotes the normal part of a vector field on \( \mathbb{R}^{n+p} \). Let \( V^T \) be the tangent component of vector \( V \), then we have

\[
H + V^T = V.
\]

Translating solitons often occur as Type-II singularities of a mean curvature flow after a rescaling. For instance, Huisken and Sinestrari [7] proved that if the initial hypersurface is mean convex, then the limit hypersurface at Type-II singularity is a convex translating soliton. On the other hand, every translating soliton gives a translating solution \( M_t \) defined by \( M_t = M + tV \) for \( t \in \mathbb{R} \) to the mean curvature flow. That is, it does not change the shape during the evolution, it’s just moving by translation in the direction of \( V \). Similar to self-shrinkers, translating solitons can be regarded as a minimal submanifolds in \( (\mathbb{R}^{n+p}, \bar{g}) \), where \( \bar{g} \) is a conformally flat Riemannian metric due to [9].

There are few examples of translating solitons even in the hypersurface case. The well-known grim reaper \( \Gamma \) is a one-dimensional translating soliton in \( \mathbb{R}^2 \) defined by

\[
y = -\log \cos x, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}).
\]

A trivial generalization is the euclidean product \( \Gamma \times \mathbb{R}^{n-1} \) in \( \mathbb{R}^{n+1} \), which is called the grim reaper cylinder.

Since the geometry of the solution near the Type-II singularity cannot be controlled well, the study of Type-II singularities is more complicated than Type-I. There are some results about the translating solitons, see [1, 10, 14, 15, 18]. For instance, Wang [18] studied the classification of Type-II singularities and proved that for \( n = 2 \) any entire convex translator must be rotationally symmetric in an appropriate coordinate system. Martin, Savas-Halilaj and Smoczyk [14] obtained classification results and topological obstructions for the existence of translating solitons. Xin [20] showed that a smooth complete translating soliton in \( \mathbb{R}^{n+p} \) satisfying \( \left( \int_M |A|^n e^{(V,X)} \right) < \infty \)
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and \( (\int_M |A|^n d\mu)^{1/n} < C \) for certain positive constant \( C \) is a linear space. Here \( A \) denotes the second fundamental form of a submanifold.

Define the trace-free second fundamental form \( \hat{A} \) of a submanifold by \( \hat{A} = A - \frac{1}{n} g \otimes H \). In the present paper, we will prove a rigidity theorem for translating solitons under integral curvature pinching conditions of the trace-free second fundamental form.

**Theorem 1.** Let \( M^n(n \geq 3) \) be a smooth complete translating soliton in the Euclidean space \( \mathbb{R}^{n+p} \). If the trace-free second fundamental form \( \hat{A} \) of \( M \) satisfies

\[
\left( \int_M |\hat{A}|^n d\mu \right)^{1/n} < K(n) \quad \text{and} \quad \int_M |\hat{A}|^{n/2} e(V,X) < \infty,
\]

where \( K(n) \) is an explicit positive constant depending only on \( n \), then \( M \) is a linear subspace.

It is obvious that the curvature condition in Theorem 1 is weaker than that in the rigidity theorem of Xin [20].

**2. Preliminaries**

Let \( X : M^n \to \mathbb{R}^{n+p} \) be an \( n \)-dimensional immersed submanifold. Denote by \( g \) the induced metric on \( M \). We shall make use of the following convention on the range of indices:

\[ 1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p. \]

Choose a local field of orthonormal frame field \( \{e_A\} \) in \( \mathbb{R}^{n+p} \) such that, restricted to \( M \), the \( e_i \)'s are tangent to \( M^n \). Let \( \{\omega_A\} \) and \( \{\omega_{AB}\} \) be the dual frame field and the connection 1-forms of \( \mathbb{R}^{n+p} \), respectively. Restricting these forms to \( M \), we have

\[
\omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},
\]

\[
A = \sum_{\alpha, i, j} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha = \sum_{ij} h_{ij} \omega_i \otimes \omega_j,
\]

\[
H = \sum_{\alpha, i} h^\alpha_{ii} e_\alpha = \sum_\alpha H^\alpha e_\alpha,
\]

\[
R_{ijkl} = \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}),
\]

\[
R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}).
\]
where $A, H, R_{ijkl}, R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the Riemannian curvature tensor, the normal curvature tensor of $M$, respectively. The trace-free second fundamental form is defined by $\hat{A} = A - \frac{1}{n} g \otimes H$. We have the relations $|\hat{A}|^2 = |A|^2 - \frac{1}{n} |H|^2$ and $|\nabla A|^2 = |\nabla A|^2 - \frac{1}{n} \nabla |H|^2$.

Denoting the first and second covariant derivatives of $h_{ij}$ by $h_{ijk}$ and $h_{ijkl}$ respectively, we have

$$
\sum_k h_{ij\alpha}^k \omega_k = dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_k - \sum_k h_{kj}^\alpha \omega_i - \sum_\beta h_{ij\beta}^k \omega_{\beta\alpha},
$$

$$
\sum_l h_{ijkl}^\alpha \omega_l = dh_{ij}^\alpha - \sum_l h_{ij\alpha}^l \omega_l - \sum_l h_{kl}^\alpha \omega_{ij} - \sum_l h_{lj\alpha}^l \omega_i - \sum_\beta h_{ij\beta}^l \omega_{\beta\alpha}.
$$

Then we have

$$
h_{ij}^\alpha = h_{ij}^\alpha,
$$

$$
h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mijk} - \sum_\beta h_{ij\beta}^l R_{\alpha\beta kl}.
$$

Hence

$$
\Delta h_{ij}^\alpha = \sum_k h_{ij\alpha}^k
$$

$$
= \sum_k h_{kij}^\alpha + \sum_k \left( \sum_m h_{km}^\alpha R_{mjkl} + \sum_m h_{ml}^\alpha R_{mijk} - \sum_\beta h_{k\alpha}^l R_{\alpha\beta kl} \right).
$$

As in [20], we need a linear operator $L_{II}$ on $M$

$$
L_{II} = \Delta + \langle V, \nabla (\cdot) \rangle = e^{-\langle V, X \rangle} \text{div} (e^{\langle V, X \rangle} \nabla (\cdot)),
$$

where $\Delta$, div and $\nabla$ denote the Laplacian, divergence and the gradient operator on $M$, respectively. It can be shown that $L_{II}$ is self-adjoint respect to the measure $e^{\langle V, X \rangle} d\mu$, where $d\mu$ is the volume form of $M$. We denote $\bar{g} = e^{\langle V, X \rangle}$ and $d\mu$ might be omitted in the integrations for notational simplicity.

In order to prove our theorem, we need the following Simons type identities. The first equality has been proved by Xin [20]. For the convention of readers, we also include this part of the proof here.

**Lemma 1.** On a translating soliton $M^\alpha$ in $\mathbb{R}^{n+p}$, we have

$$
L_{II} |A|^2 = 2 |\nabla A|^2 - 2 \sum_{\alpha, \beta} \left( \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 - 2 \sum_{i,j, \alpha, \beta} \left( \sum_p (h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta) \right)^2,
$$

(5)
(6) \[ \mathcal{L}_{II} |H|^2 = 2|\nabla H|^2 - 2 \sum_{i,j} \left( \sum_{\alpha} H^\alpha h_{ij}^\alpha \right)^2, \]

where \( H^\alpha = \sum_i h_{ii}^\alpha. \)

**Proof.** From the translating soliton equation \( H = V^N \), we derive

\[ \nabla_i H^\alpha = -\sum_k \langle V, e_k \rangle h_{ik}^\alpha, \]

and

\[ \nabla_j \nabla_i H^\alpha = -\langle H, h^\alpha_{jk} \rangle h_{ik}^\alpha - \sum_k \langle V, e_k \rangle h_{ikj}^\alpha. \]

Combining (4) and (7), we obtain that

\[
\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \nabla_j \nabla_i H^\alpha \\
+ \sum_{i,j,k,\alpha} h_{ij}^\alpha \left( \sum_m h_{km}^\alpha R_{mijk} + \sum_m h_{mi}^\alpha R_{mkjk} - \sum_\beta h_{k\beta}^\alpha R_{\alpha\beta ij} \right) \\
= -\sum_k \langle V, e_k \rangle h_{ikj}^\alpha h_{ij}^\alpha - \sum_{i,j} \left( \sum_{\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta \right)^2 \\
- \sum_{i,j,\alpha,\beta} \left( \sum_p \left( h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta \right) \right)^2.
\]

Therefore

\[
\mathcal{L}_{II} |A|^2 = \Delta |A|^2 + \langle V, \nabla |A|^2 \rangle \\
= 2 \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha + 2 |\nabla A|^2 + \sum_k \langle V, e_k \rangle h_{ik}^\alpha h_{ij}^\alpha \\
= 2 |\nabla A|^2 - 2 \sum_{\alpha,\beta} \left( \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 - 2 \sum_{i,j,\alpha,\beta} \left( \sum_p \left( h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta \right) \right)^2.
\]

On the other hand, from (7) one has

\[ \Delta |H|^2 = 2|\nabla H|^2 - 2 \sum_{i,j} \left( \sum_{\alpha} H^\alpha h_{ij}^\alpha \right)^2 - 2 \sum_{\alpha,\beta} H^\alpha H^\beta \langle V, e_i \rangle, \]

where \( H^\alpha = \sum_i h_{ii}^\alpha. \)
Then it follows that

\[ L_{II} |H|^2 = \Delta |H|^2 + \langle V, \nabla |H|^2 \rangle = 2|\nabla H|^2 - 2 \sum_{i,j} ( \sum_{\alpha} H^{\alpha} h^{\alpha}_{ij})^2. \]

The following Sobolev inequality for submanifolds in the Euclidean space is very useful in the proof of our theorem.

**Lemma 2** ([21]). Let \( M^n(n \geq 3) \) be a complete submanifold in the Euclidean space \( \mathbb{R}^{n+p} \). Let \( f \) be a nonnegative \( C^1 \) function with compact support. Then for all \( s \in \mathbb{R}^+ \), we have

\[
\|f\|^{2n} \leq D^2(n) \left[ \frac{4(n-1)^2(1+s)}{(n-2)^2} \|\nabla f\|^2 + \left(1 + \frac{1}{s}\right) \frac{1}{n^2} \|H|f|\|^2 \right],
\]

where \( D(n) = 2^n (1 + n) \frac{n+1}{n} (n-1)^{-\frac{1}{n}} \), and \( \sigma_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \).

### 3. Proof of Theorem 1

In this section, we will give several lemmas first to prove Theorem 1.

**Lemma 3.** On a translating soliton \( M^n \) in \( \mathbb{R}^{n+p} \), we have

\[ L_{II}|\hat{A}|^2 \geq 2|\nabla \hat{A}|^2 - \nu |\hat{A}|^4 - \frac{2}{n} |H|^2 |\hat{A}|^2, \]

where

\[ \nu = \begin{cases} 2, & \text{if } p = 1, \\ 4, & \text{if } p \geq 2. \end{cases} \]

**Proof.** Combining (5) and (6), we have

\[ L_{II}|\hat{A}|^2 = L_{II}|A|^2 - \frac{1}{n} L_{II}|H|^2 \]

\[
= 2|\nabla \hat{A}|^2 + \frac{2}{n} \sum_{i,j} \left( \sum_{\alpha} H^{\alpha} h^{\alpha}_{ij} \right)^2 - 2 \sum_{\alpha,\beta} \left( \sum_{i,j} h^{\alpha}_{ij} h^{\beta}_{ij} \right)^2 
- 2 \sum_{i,j,\alpha,\beta} \left( \sum_{p} (h^{\alpha}_{ip} h^{\beta}_{pj} - h^{\alpha}_{jp} h^{\beta}_{pi}) \right)^2.
\]
When the codimension is one, it can be easily obtained that

$$L_{II}|\dot{A}|^2 \geq 2|\nabla|\dot{A}|^2 - 2|\dot{A}|^4 - \frac{2}{n}|H|^2|\dot{A}|^2,$$

where we have used the inequality $|\nabla \dot{A}|^2 \geq |\nabla|\dot{A}||^2$, which is an easy consequence of the Schwartz inequality.

In the codimension $p \geq 2$ case, we need the following estimates. At the point where the mean curvature vector is zero, we have

$$\sum_{\alpha} (\sum_{i,j} H^\alpha h^\alpha_{ij})^2 - 2 \sum_{\alpha,\beta} (\sum_{i,j} h^\alpha_{ij} h^\beta_{ij})^2 = -2 \sum_{\alpha,\beta} N(A^\alpha A^\beta - A^\beta A^\alpha) - 2 \sum_{\alpha,\beta}[\text{tr}(A^\alpha A^\beta)]^2 \geq -3|A|^4,$$

where $A^\alpha = (h^\alpha_{ij})_{n \times n}$ and we have used Theorem 1 in [13] to get the inequality.

At the point where the mean curvature vector is nonzero, we choose $e_{n+1} = \frac{H}{|H|}$. The second fundamental form can be written as $A = \sum_{\alpha} h^\alpha e_{\alpha}$, where $h^\alpha, n+1 \leq \alpha \leq n+p$, are symmetric 2-tensors.

By the choice of $e_{n+1}$, we see that $\text{tr} h^{n+1} = |H|$ and $\text{tr} h^\alpha = 0$ for $\alpha \geq n+2$. The trace-free second fundamental form may be rewritten as $\dot{A} = \sum_{\alpha} \dot{h}^\alpha e_{\alpha}$, where $\dot{h}^{n+1} = h^{n+1} - \frac{|H|}{n} \text{Id}$ and $\dot{h}^\alpha = h^\alpha$ for $\alpha \geq n+2$. We set

$$A_H = h^{n+1} e_{n+1}, \quad A_I = \sum_{\alpha \geq n+2} h^\alpha e_{\alpha},$$

$$\dot{A}_H = \dot{h}^{n+1} e_{n+1}, \quad \dot{A}_I = \sum_{\alpha \geq n+2} \dot{h}^\alpha e_{\alpha}.$$

Then we have

$$|A_I|^2 = \sum_{\alpha \geq n+2} |h^\alpha|^2 = |A|^2 - |A_H|^2,$$

$$|\dot{A}_I|^2 = \sum_{\alpha \geq n+2} |\dot{h}^\alpha|^2 = |\dot{A}|^2 - |\dot{A}_H|^2.$$

Note that $|\dot{A}_H|^2 = |A_H|^2 - \frac{|H|^2}{n}$ and $|\dot{A}_I|^2 = |A_I|^2$. Since $e_{n+1}$ is chosen globally, $|A_H|^2, |\dot{A}_H|^2$ and $|A_I|^2$ are defined globally and independent of the
choice of \( e_i \). Then we have

\[
\sum_{\alpha, \beta} \left( \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 = |\hat{A}_H|^4 + \frac{2}{n} |H|^2|\hat{A}_H|^2 + \frac{1}{n^2} |H|^4
\]

(11)

\[
+ 2 \sum_{\alpha \neq n+1} \left( \sum_{i,j} \hat{h}_{ij}^{n+1} \hat{h}_{ij}^\alpha \right)^2 + \sum_{\alpha, \beta \neq n+1} \left( \sum_{i,j} \hat{h}_{ij}^\alpha \hat{h}_{ij}^\beta \right)^2,
\]

(12)

\[
\sum_{i,j, \alpha, \beta} \left( \sum_{p} (h_{ip}^\alpha h_{pj}^\beta - h_{ip}^\alpha h_{pj}^\beta) \right)^2 = 2 \sum_{\alpha \neq n+1} \sum_{i,j} \left( \sum_{p} (\hat{h}_{ip}^{n+1} \hat{h}_{pj}^\alpha - \hat{h}_{ip}^{n+1} \hat{h}_{pj}^\beta) \right)^2
\]

\[
+ \sum_{\alpha, \beta \neq n+1} \sum_{i,j} \left( \sum_{p} (\hat{h}_{ip}^\alpha \hat{h}_{pj}^\beta - \hat{h}_{ip}^\alpha \hat{h}_{pj}^\beta) \right)^2,
\]

and

\[
\sum_{i,j} \left( \sum_{\alpha} H_{ij}^\alpha \hat{h}_{ij}^\alpha \right)^2 = |H|^2|\hat{A}_H|^2 + \frac{1}{n} |H|^4.
\]

(13)

From (11), (12) and (13), we obtain the following

\[
2 \sum_{\alpha, \beta} \left( \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 + 2 \sum_{i,j, \alpha, \beta} \left( \sum_{p} (h_{ip}^\alpha h_{pj}^\beta - h_{ip}^\alpha h_{pj}^\beta) \right)^2 - \frac{2}{n} \sum_{i,j} \left( \sum_{\alpha} H_{ij}^\alpha \hat{h}_{ij}^\alpha \right)^2
\]

\[
= 2|\hat{A}_H|^4 + \frac{2}{n} |H|^2|\hat{A}_H|^2
\]

\[
+ 4 \sum_{\alpha \neq n+1} \left( \sum_{i,j} \hat{h}_{ij}^{n+1} \hat{h}_{ij}^\alpha \right)^2 + 4 \sum_{\alpha \neq n+1} \sum_{i,j} \left( \sum_{p} (h_{ip}^{n+1} \hat{h}_{pj}^\alpha - h_{ip}^{n+1} \hat{h}_{pj}^\beta) \right)^2
\]

\[
+ 2 \sum_{\alpha, \beta \neq n+1} \left( \sum_{i,j} \hat{h}_{ij}^\alpha \hat{h}_{ij}^\beta \right)^2 + 2 \sum_{\alpha, \beta \neq n+1} \sum_{i,j} \left( \sum_{p} (\hat{h}_{ip}^\alpha \hat{h}_{pj}^\beta - \hat{h}_{ip}^\alpha \hat{h}_{pj}^\beta) \right)^2.
\]

(14)

Choose \( \{e_i\} \) such that \( h_{ij}^{n+1} = \lambda_i \delta_{ij} \). Then \( \hat{h}_{ij}^{n+1} = \hat{\lambda}_i \delta_{ij} \), where \( \hat{\lambda}_i = \lambda_i - \frac{|H|}{n} \). We have the following estimates.

\[
4 \sum_{\alpha \neq n+1} \left( \sum_{i,j} \hat{h}_{ij}^{n+1} \hat{h}_{ij}^\alpha \right)^2
\]

\[
= 4 \sum_{\alpha \neq n+1} \left( \sum_{i} \hat{\lambda}_i \hat{h}_{ii}^\alpha \right)^2
\]

\[
\leq 4 \left( \sum_{i} \hat{\lambda}_i^2 \right) \left( \sum_{\alpha \neq n+1} \sum_{i} (\hat{h}_{ii}^\alpha)^2 \right)
\]
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\[ -4|\hat{A}_H|^2 \sum_{\alpha \neq n+1} \sum_i (\hat{h}^\alpha_{ii})^2, \]

where we have used the Cauchy-Schwarz inequality. We also have

\[ 4 \sum_{\alpha \neq n+1} \sum_{i,j} \left( \sum_p (\hat{h}^\alpha_{ip} \hat{h}^\alpha_{pj} - \hat{h}^{n+1}_{jp} \hat{h}^\alpha_{pi}) \right)^2 \]
\[ = 4 \sum_{\alpha \neq n+1} \sum_{i \neq j} (\lambda_i - \lambda_j)^2 (\hat{h}^\alpha_{ij})^2 \]
\[ = 4 \sum_{\alpha \neq n+1} \sum_{i \neq j} (\hat{\lambda}_i - \hat{\lambda}_j)^2 (\hat{h}^\alpha_{ij})^2 \]
\[ \leq 8 \sum_{\alpha \neq n+1} \sum_{i \neq j} (\hat{\lambda}_i^2 + \hat{\lambda}_j^2) (\hat{h}^\alpha_{ij})^2 \]
\[ \leq 8|\hat{A}_H|^2 \sum_{\alpha \neq n+1} \sum_{i \neq j} (\hat{h}^\alpha_{ij})^2 \]
\[ = 8|\hat{A}_H|^2 (|\hat{A}|^2 - \sum_{\alpha \neq n+1} \sum_i (\hat{h}^\alpha_{ii})^2). \]

By using Theorem 1 in [13], we obtain that

\[ 2 \sum_{\alpha, \beta \neq n+1} \left( \sum_{i,j} \hat{h}^\alpha_{ij} \hat{h}^\beta_{ij} \right)^2 + 2 \sum_{\alpha, \beta \neq n+1} \sum_{i,j} \left( \sum_p (\hat{h}^\alpha_{ip} \hat{h}^\beta_{pj} - \hat{h}^{n+1}_{jp} \hat{h}^\alpha_{pi}) \right)^2 \leq 3|\hat{A}|^4. \]

Hence, we have the following estimate

\[ \frac{2}{n} \sum_{i,j} \left( \sum_{\alpha} H^\alpha h^\alpha_{ij} \right)^2 - 2 \sum_{\alpha, \beta} \left( \sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \right)^2 \]
\[ - 2 \sum_{i,j, \alpha, \beta} \left( \sum_p (h^\alpha_{ip} h^\beta_{pj} - h^{n+1}_{jp} h^\alpha_{pi}) \right)^2 \]
\[ \geq - 4|\hat{A}|^4 - \frac{2}{n} |H|^2 |\hat{A}|^2. \]

Combining (10) and (15), we have

\[ \frac{2}{n} \sum_{i,j} \left( \sum_{\alpha} H^\alpha h^\alpha_{ij} \right)^2 - 2 \sum_{\alpha, \beta} \left( \sum_{i,j} h^\alpha_{ij} h^\beta_{ij} \right)^2 \]
\[ - 2 \sum_{i,j, \alpha, \beta} \left( \sum_p (h^\alpha_{ip} h^\beta_{pj} - h^{n+1}_{jp} h^\alpha_{pi}) \right)^2 \]
\[ \geq - 4|\hat{A}|^4 - \frac{2}{n} |H|^2 |\hat{A}|^2. \]

Substituting (16) into (9), we obtain that
\[ \mathcal{L}_{II} |\hat{A}|^2 \geq 2|\nabla |\hat{A}|^2 - 4|\hat{A}|^4 - \frac{2}{n} |H|^2 |\hat{A}|^2. \]
Thus, we complete the proof. \(\Box\)

**Lemma 4.** For any smooth function \(\eta\) with compact support on \(M\) and any \(0 < \varepsilon < n - 1\), we have
\[
\int_M |\nabla |\hat{A}|^2| \hat{A}|^{n-2}\eta^2 \varrho \leq \frac{1}{n - 1 - \varepsilon} \left( \frac{t}{2} \int_M |\hat{A}|^{n+2}\eta^2 \varrho + \frac{1}{n} \int_M |\hat{A}|^n|H|^2\eta^2 \varrho + \frac{1}{\varepsilon} \int_M |\hat{A}|^n|\nabla \eta|^2 \varrho \right).
\]

**Proof.** Multiplying \(|\hat{A}|^{n-2}\eta^2\) on both sides of the (8) and integrating by parts with respect to the measure \(\varrho d\mu\) on \(M\) yield
\[
0 \geq 2 \int_M |\nabla |\hat{A}|^2| \hat{A}|^{n-2}\eta^2 \varrho - t \int_M |\hat{A}|^{n+2}\eta^2 \varrho - \frac{2}{n} \int_M |\hat{A}|^n|H|^2\eta^2 \varrho
- \int_M |\hat{A}|^{n-2}\eta^2 \mathcal{L}_{II} |\hat{A}|^2 \varrho.
\]
Since \(\eta\) has compact support on \(M\), by the Stokes theorem, we obtain that
\[
- \int_M |\hat{A}|^{n-2}\eta^2 \mathcal{L}_{II} |\hat{A}|^2 \varrho
= - \int_M |\hat{A}|^{n-2}\eta^2 \text{div}(\varrho \cdot \nabla |\hat{A}|^2)
= 2 \int_M \varrho |\hat{A}| |\nabla| \hat{A}| \cdot \nabla(|\hat{A}|^{n-2}\eta^2)
= 2(n - 2) \int_M |\nabla |\hat{A}|^2| \hat{A}|^{n-2}\eta^2 \varrho + 4 \int_M (\nabla |\hat{A}| \cdot \nabla \eta) |\hat{A}|^{n-1}\eta \varrho.
\]
Combining (18) and (19), we get
\[
0 \geq 2(n - 1) \int_M |\nabla |\hat{A}|^2| \hat{A}|^{n-2}\eta^2 \varrho - t \int_M |\hat{A}|^{n+2}\eta^2 \varrho - \frac{2}{n} \int_M |\hat{A}|^n|H|^2\eta^2 \varrho
+ 4 \int_M (\nabla |\hat{A}| \cdot \nabla \eta) |\hat{A}|^{n-1}\eta \varrho.
\]
By the Cauchy inequality, for any $0 < \varepsilon < n - 1$, we obtain that
\[
\int_M |\mathcal{A}|^{n+2} \eta^2 \varrho + \frac{2}{n} \int_M |\mathcal{A}|^n |H|^2 \eta^2 \varrho + \frac{2}{\varepsilon} \int_M |\mathcal{A}|^n |\nabla \eta|^2 \varrho \\
\geq 2(n - 1 - \varepsilon) \int_M \nabla |\mathcal{A}|^2 |\mathcal{A}|^{-2} \eta^2 \varrho.
\]

Lemma 5. Setting $f = |\mathcal{A}|^{n/2} \varrho^{1/2} \eta$, we have
\[(20) \quad \int_M |\nabla f|^2 = \int_M |\nabla (|\mathcal{A}|^{n/2} \eta)|^2 \varrho - \frac{1}{2} \int_M |\mathcal{A}|^n \eta^2 \varrho + \frac{1}{4} \int_M |\mathcal{A}|^n |V^T|^2 \eta^2 \varrho,
\]
where $\eta$ is a smooth function with compact support on $M$ and $V^T$ is the tangent component of vector $V$.

Proof. Integrating by parts, one obtain
\[
\int_M |\nabla f|^2 = \int_M |\nabla (|\mathcal{A}|^{n/2} \eta)|^2 \varrho + \frac{1}{2} \int_M \nabla (|\mathcal{A}|^n \eta^2) \nabla \varrho + \int_M |\mathcal{A}|^n \eta^2 |\nabla \varrho^1|^2 \\
= \int_M |\nabla (|\mathcal{A}|^{n/2} \eta)|^2 \varrho - \frac{1}{2} \int_M |\mathcal{A}|^n \eta^2 \Delta \varrho + \int_M |\mathcal{A}|^n \eta^2 |\nabla \varrho^1|^2.
\]

By direct computations, we have
\[
\nabla \varrho = \nabla e^{(V, X)} = \varrho V^T,
\]
and
\[
\nabla \varrho^1 = \frac{1}{2} \varrho^{-1/2} \nabla \varrho = \frac{1}{2} \varrho^1 V^T.
\]

By the translating soliton equation $H = V^N$, we get
\[
\Delta \varrho = \sum_i \nabla_i \varrho \langle V, e_i \rangle + \sum_i \varrho \langle V, \nabla_i e_i \rangle = \varrho (|V^T|^2 + |V^N|^2) = \varrho.
\]

Hence, it follows that
\[
\int_M |\nabla f|^2 = \int_M |\nabla (|\mathcal{A}|^{n/2} \eta)|^2 \varrho - \frac{1}{2} \int_M |\mathcal{A}|^n \eta^2 \varrho + \frac{1}{4} \int_M |\mathcal{A}|^n |V^T|^2 \eta^2 \varrho.
\]

Now we will give the proof of Theorem 1.
Proof. Combining the Sobolev inequality in Lemma 2 and (20) in Lemma 5, we have

\[
\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
\leq D^2(n) \cdot \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \left( \int_M |\nabla (|\hat{A}|^{n/2})|^2 \varrho - \frac{1}{4} \int_M |\hat{A}|^n |V^T|^2 \varrho^2 \right) \\
- \frac{1}{2} \int_M |\hat{A}|^n |H|^2 \varrho \right\} + \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\hat{A}|^n |H|^2 \varrho^2 \right\} \\
= D^2(n) \cdot \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \left( \int_M \frac{n^2}{4} |\nabla |\hat{A}||^2 |\hat{A}|^{n-2} \varrho \\
+ \int_M n|\hat{A}|^{n-2} \nabla |\hat{A}| \cdot \nabla \varrho + \int_M |\hat{A}|^n |\nabla \varrho|^2 \varrho - \frac{1}{4} \int_M |\hat{A}|^n |V^T|^2 \varrho^2 \right) \\
- \frac{1}{2} \int_M |\hat{A}|^n |H|^2 \varrho \right\} + \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\hat{A}|^n |H|^2 \varrho^2 \right\}.
\]

Note that

\[ |V^T|^2 + |V^N|^2 = |V^T|^2 + |H|^2 = 1. \]

We deduce that

\[
\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
\leq D^2(n) \cdot \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \left( \int_M |\nabla (|\hat{A}|^{n/2})|^2 \varrho - \frac{1}{4} \int_M |\hat{A}|^n |V^T|^2 \varrho^2 \right) \\
- \frac{1}{2} \int_M |\hat{A}|^n |H|^2 \varrho \right\} + \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\hat{A}|^n |H|^2 \varrho^2 \right\}.
\]

By the Cauchy inequality, we have for any \( \delta > 0 \)

(21)

\[
\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
\leq \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2} \left\{ (1 + \delta) \frac{n^2}{4} \int_M |\nabla |\hat{A}||^2 |\hat{A}|^{n-2} \varrho \\
+ \left( 1 + \frac{1}{\delta} \right) \int_M |\hat{A}|^n |\nabla \varrho|^2 \varrho - \frac{1}{4} \int_M |\hat{A}|^n |V^T|^2 \varrho^2 \right) \\
+ D^2(n) \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\hat{A}|^n |H|^2 \varrho^2 \right\}.
\]
Substituting \((17)\) into \((21)\), we get
\[
\left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2} \left\{ \frac{n^2(1+\delta)}{4(n-1-\varepsilon)} \left( \frac{\ell}{2} \int_M |\dot{A}|^{n+2}\eta^2 \rho \right) \\
+ \frac{1}{n} \int_M |\dot{A}|^n |H|^2 \eta^2 \rho + \frac{1}{\varepsilon} \int_M |\dot{A}|^n |\nabla \eta|^2 \rho \right\}
\]
\[
+ \left( 1 + \frac{1}{\delta} \right) \int_M |\dot{A}|^n |\nabla \eta|^2 \rho - \frac{1}{2} \int_M |\dot{A}|^n |H|^2 \eta^2 \rho \right\}
\]
\[
+ D^2(n) \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\dot{A}|^n |H|^2 \eta^2 \rho.
\]

Put
\[
\delta = \delta(s, \varepsilon) = \frac{2sn^2(n-1)^2 - (n-2)^2(n-1-\varepsilon)}{sn^3(n-1)^2} - 1 > 0,
\]
for some positive constant \(s\) satisfies
\[
s > \frac{(n-2)^2(n-1-\varepsilon)}{n^2(n-1)^2(n-2-2\varepsilon)} \in \mathbb{R}^+
\]
and some \(\varepsilon \in (0, \frac{n-2}{2})\) to be defined later. Then we conclude that
\[
\kappa^{-1} \left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{n^2(1+s)(1+\delta)}{4(n-1-\varepsilon)} \left( \frac{\ell}{2} \int_M |\dot{A}|^{n+2}\eta^2 \rho + \frac{1}{\varepsilon} \int_M |\dot{A}|^n |\nabla \eta|^2 \rho \right)
\]
\[
+ \left( 1 + s \right) \left( 1 + \frac{1}{\delta} \right) \int_M |\dot{A}|^n |\nabla \eta|^2 \rho
\]
\[
= \frac{(1+s)\ell(2sn^2(n-1)^2 - (n-2)^2)}{8sn(n-1)^2} \int_M |\dot{A}|^{n+2}\eta^2 \rho
\]
\[
+ C(s, \varepsilon, n) \int_M |\dot{A}|^n |\nabla \eta|^2 \rho,
\]
where \(C(s, \varepsilon, n)\) is an explicit positive constant depending on \(s, \varepsilon\) and \(n\), and
\[
\kappa = \frac{4D^2(n)(n-1)^2}{(n-2)^2}.
\]
By the Hölder inequality, we have
\[
\int_M |\dot{A}|^{n+2}\eta^2 \rho \leq \left( \int_M |\dot{A}|^{2\frac{n}{n-2}} \right)^{\frac{n-2}{n}} \left( \int_M (|\dot{A}|^n \eta^2 \rho)^{\frac{n}{n-2}} \right)^{\frac{n-2}{n}}.
\]
Hence
\[
\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{-\frac{n-2}{n}} \leq \left(1 + s\right)\kappa\left[2sn^2(n-1)^2 - (n-2)^2\right] \left(\int_M |\hat{A}|^n\right)^{\frac{2}{n}} \cdot \left(\int_M |f|^{\frac{2n}{n-2}}\right)^{-\frac{n-2}{n}}
\]
\[+ C(s, \varepsilon, n) \int_M |\hat{A}|^n|\nabla \eta|^2 \varrho.\]

Put
\[
K(n, s) = \sqrt{\frac{8sn(n-1)^2}{(1 + s)\kappa\left[2sn^2(n-1)^2 - (n-2)^2\right]^n}}.
\]

For simplicity, we choose
\[
s = s(\varepsilon) = \frac{(n-2)^2}{n^2(n-1)(n-2-2\varepsilon)}
\]
such that
\[
K(n, \varepsilon) = K(n, s(\varepsilon)) = \sqrt{\frac{2n(n-2)^2}{iD^2(n)(n+2\varepsilon)[(n-2)^2/(n-2-2\varepsilon) + n^2(n-1)]}}.
\]

Set
\[
K(n) = \sup_{\varepsilon \in (0, \frac{n-2}{2})} K(n, \varepsilon) = \sqrt{\frac{2(n-2)^2}{iD^2(n)[n-2 + n^2(n-1)]}},
\]
where
\[
t = \begin{cases} 
2, & \text{if } p = 1, \\
4, & \text{if } p \geq 2.
\end{cases}
\]

Since we have the assumption
\[
\left(\int_M |\hat{A}|^n d\mu\right)^{1/n} < K(n),
\]
there exists a positive constant \(\tilde{K}\) such that
\[
\left(\int_M |\hat{A}|^n d\mu\right)^{1/n} < \tilde{K} < K(n). \tag{24}
\]
Thus, there exists \(\varepsilon = \varepsilon_0 > 0\) such that
\[
\tilde{K} < K(n, \varepsilon_0) < K(n).
\]
That is to say
\[
(1 + s)\left[2sn^2(n-1)^2 - (n-2)^2\right] = \kappa^{-1} \cdot K(n, \varepsilon_0)^{-2},
\]
where
\[
s = s(\varepsilon_0) = \frac{(n-2)^2}{n^2(n-1)(n-2-2\varepsilon_0)}.
\]
Combining (23), (24) and (25), it implies that there exists $0 < \epsilon < 1$ such that
\[
\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
\leq \kappa^{-1} \cdot K(n, \varepsilon_0)^{-2} \cdot \tilde{K}^2 \left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \tilde{C}(n, \varepsilon_0) \int_M |\hat{A}|^n|\nabla \eta|^2 \varrho \\
\leq \frac{1 - \epsilon}{\kappa} \left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \tilde{C}(n, \varepsilon_0) \int_M |\hat{A}|^n|\nabla \eta|^2 \varrho,
\]
namely,
\[
\frac{\epsilon}{\kappa} \left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \tilde{C}(n, \varepsilon_0) \int_M |\hat{A}|^n|\nabla \eta|^2 \varrho.
\]
Let $\eta(X) = \eta_r(X) = \varphi\left(\frac{|X|}{r}\right)$ for any $r > 0$, where $\varphi$ is a nonnegative function on $[0, +\infty)$ satisfying
\[
\varphi(x) = \begin{cases} 
1, & \text{if } x \in [0, 1), \\
0, & \text{if } x \in [2, +\infty),
\end{cases}
\]
and $|\varphi'| \leq C$ for some absolute constant.
Since $\int_M |\hat{A}|^n \varrho$ and the constant $\tilde{C}(n, \varepsilon_0)$ are bounded, the right hand side of (26) approaches to zero as $r \to +\infty$, which implies $|\hat{A}| \equiv 0$. Therefore, $M$ is a linear subspace. This completes the proof of Theorem 1. \(\square\)

References


A global pinching theorem for translating solitons


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