On $J(r, n)$-Jacobsthal quaternions

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Abstract: In this paper we introduce the $J(r, n)$-Jacobsthal quaternions and give some of their properties, among others the Binet formula, convolution identity and the generating function.

Keywords: Jacobsthal numbers, quaternions, recurrence relations.

1. Introduction

Let $\mathbb{H}$ be the set of quaternions $q$ of the form

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$.

If $q_1 = a_1 + b_1i + c_1j + d_1k$ and $q_2 = a_2 + b_2i + c_2j + d_2k$ are any two quaternions then equality, addition, subtraction and multiplication by scalar are defined.

Equality: $q_1 = q_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,

addition: $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$,

subtraction: $q_1 - q_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)j + (d_1 - d_2)k$,

multiplication by scalar $s \in \mathbb{R}$: $sq_1 = sa_1 + sb_1i + sc_1j + sd_1k$.

The quaternion multiplication is defined using the rule

$$i^2 = j^2 = k^2 = ijk = -1.$$  

(1)

Note that (1) implies

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$
The conjugate of a quaternion is defined by
\[ q = a + bi + cj + dk = a - bi - cj - dk. \]
The norm of a quaternion is defined by
\[ N(q) = q \cdot \overline{q} = q \cdot q = a^2 + b^2 + c^2 + d^2. \]

For the basics on quaternions theory, see [22].

Numbers of the Fibonacci type are defined by the second-order linear
recurrence relation of the form
\[ a_n = b_1 a_{n-1} + b_2 a_{n-2}, \]
where \( b_i \in \mathbb{N}, i = 1, 2 \).

For special \( b_i, i = 1, 2 \), we obtain the recurrence equation which defines the
Fibonacci numbers and the like (Lucas numbers, Pell numbers, Pell-Lucas
numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers etc.).

The numbers of the Fibonacci type have many applications in distinct
areas of mathematics, also in quaternions theory. In 1963 Horadam [13] in-
troduced \( n \)th Fibonacci and Lucas quaternions. Many interesting properties
of Fibonacci and Lucas quaternions can be found in [11, 16]. In [14] Horadam
mentioned the possibility of introducing Pell quaternions and generalized Pell
quaternions. Interesting results of Pell quaternions, Pell-Lucas quaternions
obtained recently can be found in [8, 19]. In [18] the Authors investigated
Jacobsthal quaternions. There are many generalizations of Fibonacci and Ja-
cobsthal quaternions in the literature, see for example [1, 2, 6, 12, 15, 20].
The another types of generalization of Fibonacci and Jacobsthal quaternions
are octonions and quaternion polynomials, see [4, 5, 7].

In this paper we introduce and study the \( J(r, n) \)-Jacobsthal quaternions.

\section{The \( J(r, n) \)-Jacobsthal numbers}

Let \( n \geq 0 \) be an integer. The \( n \)th Jacobsthal number \( J_n \) is defined recursively
by \( J_n = J_{n-1} + 2 J_{n-2} \), for \( n \geq 2 \) with \( J_0 = 0, J_1 = 1 \). The first ten terms
of the sequence are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171. The direct formula for
\( n \)th Jacobsthal number has the form \( J_n = \frac{2^n - (-1)^n}{3} \), named as the Binet
formula for Jacobsthal numbers. Many authors have generalized the second
order recurrence of the Jacobsthal sequence, see for example [9, 10, 17, 21]. In [3] a one-
parameter generalization of the Jacobsthal numbers was investigated. We recall this generalization.

Let \( n \geq 0, r \geq 0 \) be integers. The \( n \)th \( J(r, n) \)-Jacobsthal number \( J(r, n) \)
is defined as follows
\[ J(r, n) = 2^r J(r, n - 1) + (2^r + 4^r) J(r, n - 2) \text{ for } n \geq 2 \]
with initial conditions \( J(r, 0) = 1, J(r, 1) = 1 + 2^{r+1} \).

It is easily seen that \( J(0, n) = J_{n+2} \). By (2) we obtain

\[
\begin{align*}
J(r, 0) &= 1 \\
J(r, 1) &= 2 \cdot 2^r + 1 \\
J(r, 2) &= 3 \cdot 4^r + 2 \cdot 2^r \\
J(r, 3) &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r \\
J(r, 4) &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r \\
J(r, 5) &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r
\end{align*}
\]

We will now recall some properties of the \( J(r, n) \)-Jacobsthal numbers.

**Theorem 1 ([3], Binet formula).** For \( n \geq 0 \) the \( n \)-th \( J(r, n) \)-Jacobsthal number is given by

\[
J(r, n) = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r + 3 \cdot 2^r + 2}}{2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}} \lambda_1^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r - 3 \cdot 2^r - 2}}{2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}} \lambda_2^n,
\]

where

\[
(3) \quad \lambda_1 = 2^{r-1} + \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \lambda_2 = 2^{r-1} - \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}.
\]

**Theorem 2 ([3]).** The generating function of the sequence \( \{J(r, n)\} \) has the following form

\[
f(x) = \frac{1 + (1 + 2^r)x}{1 - 2^r x - (2^r + 4^r)x^2}.
\]

**Proposition 3 ([3]).** Let \( n \geq 4, r \geq 0 \) be integers. Then

\[
J(r, n) = (3 \cdot 8^r + 2 \cdot 4^r)J(r, n - 3) + (2 \cdot 16^r + 3 \cdot 8^r + 4^r)J(r, n - 4).
\]

**Theorem 4 ([3]).** Let \( n \geq 1, r \geq 0 \) be integers. Then

\[
\sum_{l=0}^{n-1} J(r, l) = \frac{J(r, n) + (2^r + 4^r)J(r, n - 1) - 2 - 2^r}{4^r + 2^{r+1} - 1}.
\]

**Theorem 5 ([3], Convolution identity).** Let \( n, m, r \) be integers such that \( m \geq 2, n \geq 1, r \geq 0 \). Then

\[
J(r, m + n) = 2^r J(r, m - 1)J(r, n) + (4^r + 8^r)J(r, m - 2)J(r, n - 1).
\]
3. The $J(r, n)$-Jacobsthal quaternions

For a nonnegative integer $n$, the $n$th $J(r, n)$-Jacobsthal quaternion $JQ^n_r$ is defined as

$$JQ^n_r = J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3),$$

where $J(r, n)$ is given by (2).

**Theorem 6.** Let $n \geq 0$, $r \geq 0$ be integers. Then

(i) $2^r JQ^n_{r+1} + (2^r + 4^r)JQ^n_r = JQ^n_{r+2},$

(ii) $JQ^n_r + \overline{JQ^n_r} = 2J(r, n),$

(iii) $(JQ^n_r)^2 = 2J(r, n)JQ^n_r - N(JQ^n_r).$

**Proof.** (i)

$$2^r JQ^n_{r+1} + (2^r + 4^r)JQ^n_r
= 2^r (J(r, n + 1) + iJ(r, n + 2) + jJ(r, n + 3) + kJ(r, n + 4))
+ (2^r + 4^r)(J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3))
= J(r, n + 2) + iJ(r, n + 3) + jJ(r, n + 4) + kJ(r, n + 5) = JQ^n_{r+2}.$$

(ii) By the definition of the conjugate of the quaternion we have

$$JQ^n_r + \overline{JQ^n_r} = J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3)
+ J(r, n) - iJ(r, n + 1) - jJ(r, n + 2) - kJ(r, n + 3)
= 2J(r, n).$$

(iii) By simple calculations we get

$$(JQ^n_r)^2 = J^2(r, n) - J^2(r, n + 1) - J^2(r, n + 2) - J^2(r, n + 3)
+ 2iJ(r, n)J(r, n + 1) + 2jJ(r, n)J(r, n + 2) + 2kJ(r, n)J(r, n + 3)
+ (ij + ji)J(r, n + 1)J(r, n + 2) + (ik + ki)J(r, n + 1)J(r, n + 3)
+ (jk + kj)J(r, n + 2)J(r, n + 3)
= J^2(r, n) - J^2(r, n + 1) - J^2(r, n + 2) - J^2(r, n + 3)
+ 2iJ(r, n)J(r, n + 1) + jJ(r, n)J(r, n + 2) + kJ(r, n)J(r, n + 3)
= 2J(r, n)(J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3))
- (J^2(r, n) + J^2(r, n + 1) + J^2(r, n + 2) + J^2(r, n + 3))
= 2J(r, n)JQ^n_r - N(JQ^n_r).$$
Theorem 7. Let $n \geq 0$, $r \geq 0$ be integers. Then

$$JQ_r^n - iJQ_{n+1}^r - jJQ_{n+2}^r - kJQ_{n+3}^r = J(r, n) + J(r, n + 2) + J(r, n + 4) + J(r, n + 6).$$

Proof.

$$JQ_r^n - iJQ_{n+1}^r - jJQ_{n+2}^r - kJQ_{n+3}^r = J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3)$$

$$- i(J(r, n + 1) + iJ(r, n + 2) + jJ(r, n + 3) + kJ(r, n + 4))$$

$$- j(J(r, n + 2) + iJ(r, n + 3) + jJ(r, n + 4) + kJ(r, n + 5))$$

$$- k(J(r, n + 3) + iJ(r, n + 4) + jJ(r, n + 5) + kJ(r, n + 6)).$$

By simple calculations we get

$$JQ_r^n - iJQ_{n+1}^r - jJQ_{n+2}^r - kJQ_{n+3}^r = J(r, n) + J(r, n + 2) + J(r, n + 4) + J(r, n + 6).$$

Theorem 8. Let $n \geq 1$, $r \geq 0$ be integers. Then

$$\sum_{l=0}^{n-1} JQ_l^r = \frac{JQ_n^r + (2^r + 4^r)JQ_{n-1}^r - (2 + 2^r)(1 + i + j + k)}{4^r + 2^{r+1} - 1} - (i + j(2 + 2^{r+1}) + k(2^{r+2} + 3 \cdot 4^r + 2)).$$

Proof. By the definition of the $J(r, n)$-Jacobsthal quaternions we have

$$\sum_{l=0}^{n-1} JQ_l^r = JQ_0^r + JQ_1^r + \ldots + JQ_{n-1}^r$$

$$= J(r, 0) + iJ(r, 1) + jJ(r, 2) + kJ(r, 3)$$

$$+ J(r, 1) + iJ(r, 2) + jJ(r, 3) + kJ(r, 4) + \ldots$$

$$+ J(r, n - 1) + iJ(r, n) + jJ(r, n + 1) + kJ(r, n + 2)$$

$$= J(r, 0) + J(r, 1) + \ldots + J(r, n - 1)$$

$$+ i(J(r, 1) + J(r, 2) + \ldots + J(r, n) + J(r, 0) - J(r, 0))$$

$$+ j(J(r, 2) + J(r, 3) + \ldots + J(r, n + 1) + J(r, 0) + J(r, 1))$$
\[-J(r, 0) - J(r, 1) \]
\[+ k \left(J(r, 3) + J(r, 4) + \ldots + J(r, n + 2) + J(r, 0) + J(r, 1) + J(r, 2) - J(r, 0) - J(r, 1) - J(r, 2)\right).\]

Using Theorem 4, we obtain
\[
\sum_{l=0}^{n-1} JQ_r^n = \frac{1}{4^r + 2^{r+1} - 1} \left[ J(r, n) + (2^r + 4^r)J(r, n - 1) - 2 - 2^r \right.
\]
\[+ i(J(r, n + 1) + (2^r + 4^r)J(r, n) - 2 - 2^r) \]
\[+ j(J(r, n + 2) + (2^r + 4^r)J(r, n + 1) - 2 - 2^r) \]
\[+ k(J(r, n + 3) + (2^r + 4^r)J(r, n + 2) - 2 - 2^r) \]
\[- i - j(2 + 2^{r+1}) - k(2^{r+2} + 3 \cdot 4^r + 2) \]
\[
= \frac{JQ_n^r + (2^r + 4^r)JQ_{n-1}^r - (2 + 2^r)(1 + i + j + k)}{4^r + 2^{r+1} - 1} - \left( i + j(2 + 2^{r+1}) + k(2^{r+2} + 3 \cdot 4^r + 2) \right). \]
\]

We will give the Binet formula for the \(J(r, n)\)-Jacobsthal quaternions.

**Theorem 9.** Let \(n \geq 0, \ r \geq 0\) be integers. Then
\[
JQ_n^r = C_1 \lambda_1^n \left(1 + i \lambda_1 + j \lambda_1^2 + k \lambda_1^3\right) + C_2 \lambda_2^n \left(1 + i \lambda_2 + j \lambda_2^2 + k \lambda_2^3\right),
\]
where
\[
C_1 = \sqrt{\frac{4 \cdot 2^r - 5 \cdot 4^r + 3 \cdot 2^r + 2}{2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}}}, \quad C_2 = \sqrt{\frac{4 \cdot 2^r - 3 \cdot 2^r - 2}{2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}}}
\]
and \(\lambda_1, \lambda_2\) were defined by (3), i.e.
\[
\lambda_1 = 2^{r-1} + \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \lambda_2 = 2^{r-1} - \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}.
\]
Proof. By Theorem 1 we get

\[ J(r, n) = C_1\lambda_1^n + C_2\lambda_2^n \]

and

\[
JQ_n^r = J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3)
\]

\[
= C_1\lambda_1^n + C_2\lambda_2^n + i(C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1})
\]

\[
+ j(C_1\lambda_1^{n+2} + C_2\lambda_2^{n+2}) + k(C_1\lambda_1^{n+3} + C_2\lambda_2^{n+3})
\]

\[
= C_1\lambda_1^n \left(1 + i\lambda_1 + j\lambda_2 + k\lambda_3\right) + C_2\lambda_2^n \left(1 + i\lambda_1 + j\lambda_2 + k\lambda_3\right),
\]

which ends the proof. \qed

**Proposition 10.** Let \( n \geq 4, r \geq 0 \). Then

\[
JQ_n^r = (3 \cdot 8^r + 2 \cdot 4^r)JQ_{n-3}^r + (2 \cdot 16^r + 3 \cdot 8^r + 4^r)JQ_{n-4}^r.
\]

Proof. Let \( A = 3 \cdot 8^r + 2 \cdot 4^r, B = 2 \cdot 16^r + 3 \cdot 8^r + 4^r \). Using Proposition 3, we obtain

\[
JQ_n^r = J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3)
\]

\[
= A \cdot J(r, n - 3) + B \cdot J(r, n - 4) + i(A \cdot J(r, n - 2) + B \cdot J(r, n - 3))
\]

\[
+ j(A \cdot J(r, n - 1) + B \cdot J(r, n - 2)) + k(A \cdot J(r, n) + B \cdot J(r, n - 1))
\]

\[
= A(J(r, n - 3) + iJ(r, n - 2) + jJ(r, n - 1) + kJ(r, n))
\]

\[
+ B(J(r, n - 4) + iJ(r, n - 3) + jJ(r, n - 2) + kJ(r, n - 1)).
\]

Hence we have \( JQ_n^r = A \cdot JQ_{n-3}^r + B \cdot JQ_{n-4}^r \), which ends the proof. \qed

**Theorem 11.** Let \( n, m, r \) be integers such that \( m \geq 2, n \geq 1, r \geq 0 \). Then

\[
2JQ_{m+n}^r = 2rJQ_{m-1}^rJQ_m^r + (4^r + 8^r)JQ_{m-2}^rJQ_{m+1}^r
\]

\[
+ J(r, m + n) + J(r, m + n + 2) + J(r, m + n + 4) + J(r, m + n + 6).
\]

Proof. By (4) we have

\[
2rJQ_{m-1}^rJQ_m^r + (4^r + 8^r)JQ_{m-2}^rJQ_{m+1}^r
\]

\[
= 2r(J(r, m - 1) + iJ(r, m) + jJ(r, m + 1) + kJ(r, m + 2))
\]

\[
\cdot (J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3))
\]

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By simple calculations and using Theorem 5 we get

\[
2^r JQ_{m-1}^r JQ_n^r + (4^r + 8^r) JQ_{m-2}^r JQ_{n-1}^r = 2^r (J(r, m - 1) J(r, n) + iJ(r, m - 1) J(r, n + 1) + jJ(r, m) J(r, m + 1) \cdot (J(r, n - 1) + iJ(r, n) + jJ(r, n + 1) + kJ(r, n + 2))
\]

+ \( (4^r + 8^r)(J(r, m - 2) J(r, n - 1) + iJ(r, m - 2) J(r, n)) + jJ(r, m - 2) J(r, n + 1) + kJ(r, m - 1) J(r, n + 1) - jJ(r, m - 1) J(r, n + 2) \)

+ \( jJ(r, m) J(r, n - 1) - kJ(r, m) J(r, n) - J(r, m) J(r, n + 1) + iJ(r, m) J(r, n + 2) + jJ(r, m + 1) J(r, n - 1) + jJ(r, m + 1) J(r, n) - jJ(r, m + 1) J(r, n + 2)))

+ \( 2^r J(r, m - 1) J(r, n) + (4^r + 8^r)(J(r, m - 2) J(r, n - 1)) + i(2^r J(r, m - 1) J(r, n + 1) + (4^r + 8^r) J(r, m - 2) J(r, n)) + j(2^r J(r, m - 1) J(r, n + 2) + (4^r + 8^r) J(r, m - 2) J(r, n + 1)) + k(2^r J(r, m - 1) J(r, n + 3) + (4^r + 8^r) J(r, m - 2) J(r, n + 2)) + i(2^r J(r, m) J(r, n) + (4^r + 8^r) J(r, m - 1) J(r, n - 1)) + j(2^r J(r, m + 1) J(r, n) + (4^r + 8^r) J(r, m) J(r, n - 1)) + k(2^r J(r, m) J(r, n + 2) + (4^r + 8^r) J(r, m - 1) J(r, n + 1)) - 2^r J(r, m) J(r, n + 1) - (4^r + 8^r) J(r, m - 1) J(r, n) - 2^r J(r, m + 1) J(r, n + 2) - (4^r + 8^r) J(r, m) J(r, n + 1) \)
Using Theorem 5 again, we obtain
\[
2^r JQ_{m-1}^r JQ_n^r + (4^r + 8^r) JQ_{m-2}^r JQ_{n-1}^r
\]
\[
= 2(J(r, m + n) + iJ(r, m + n + 1) + jJ(r, m + n + 2)
+ kJ(r, m + n + 3)) - (J(r, m + n) + J(r, m + n + 2)
+ J(r, m + n + 4) + J(r, m + n + 6))
\]
\[
= 2JQ_{m+n}^r - (J(r, m + n) + J(r, m + n + 2)
+ J(r, m + n + 4) + J(r, m + n + 6)),
\]
which ends the proof. \(\square\)

At the end we shall give the ordinary generating functions for the \(J(r, n)\)-Jacobsthal quaternions.

**Theorem 12.** The generating function for the \(J(r, n)\)-Jacobsthal quaternion sequence \(\{JQ_n^r\}\) is
\[
G(x) = \frac{JQ_0^r + (JQ_1^r - 2^r JQ_0^r)x}{1 - 2^r x - (2^r + 4^r)x^2}.
\]

**Proof.** Assuming that the generating function of the \(J(r, n)\)-Jacobsthal quaternion sequence \(\{JQ_n^r\}\) has the form \(G(x) = \sum_{n=0}^{\infty} JQ_n^r x^n\), we obtain
\[
(1 - 2^r x - (2^r + 4^r)x^2)G(x)
\]
\[
= (1 - 2^r x - (2^r + 4^r)x^2) \cdot (JQ_0^r + JQ_1^r x + JQ_2^r x^2 + \ldots)
\]
\[
= JQ_0^r + JQ_1^r x + JQ_2^r x^2 + \ldots
- 2^r JQ_0^r x - 2^r JQ_1^r x^2 - 2^r JQ_2^r x^3 - \ldots
- (2^r + 4^r)JQ_0^r x^2 - (2^r + 4^r)JQ_1^r x^3 - (2^r + 4^r)JQ_2^r x^4 - \ldots
\]
\[
= JQ_0^r + (JQ_1^r - 2^r JQ_0^r)x,
\]
since \(JQ_n^r = 2^r JQ_{n-1}^r + (2^r + 4^r) JQ_{n-2}^r\) and the coefficients of \(x^n\) for \(n \geq 2\) are equal to zero.

Moreover,
\[
JQ_0^r = 1 + i(2 \cdot 2^r + 1) + j(3 \cdot 4^r + 2 \cdot 2^r) + k(5 \cdot 8^r + 5 \cdot 4^r + 2^r)
\]
and

$$JQ_1^r - 2^r JQ_0^r = 2^r + 1 + i(4^r + 2^r) + j(2 \cdot 8^r + 3 \cdot 4^r + 2^r) + k(3 \cdot 16^r + 5 \cdot 8^r + 2 \cdot 4^r).$$

\(\square\)

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