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Isomonodromic deformations of logarithmic connections and stable parabolic vector bundles

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Abstract: We consider irreducible logarithmic connections (E, δ) over compact Riemann surfaces X of genus at least two. The underlying vector bundle E inherits a natural parabolic structure over the singular locus of the connection δ; the parabolic structure is given by the residues of δ. We prove that for the universal isomonodromic deformation of the triple (X, E, δ), the parabolic vector bundle corresponding to a generic parameter in the Teichmüller space is parabolically stable. In the case of parabolic vector bundles of rank two, the general parabolic vector bundle is even parabolically very stable.

Keywords: Logarithmic connection, isomonodromic deformation, parabolic bundle, stability, very stability, Teichmüller space.

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1. Introduction

Let $(X,\, D)$ be a compact Riemann surface of genus g with n (ordered) marked points $D = (x_1,\, \cdots,\, x_n)$. The monodromy functor produces an equivalence between the category of holomorphic connections $(E_0\,,\delta_0)$ on $X \setminus D$ and the category of equivalence classes of linear representations of $\pi_1(X\setminus D,\, x_0)$. Here the morphisms are isomorphisms of vector bundles with connections on one side and conjugation of representations on the other side; this is an example of Riemann–Hilbert correspondence. Moreover, given $(E_0,\, \delta_0)$, there exists a logarithmic connection $(E,\, \delta)$ on X, singular over D, which extends $(E_0,\, \delta_0)$. Indeed, one can choose for example a Deligne extension [12].

The classical Riemann-Hilbert problem takes X to be the projective line $\mathbb{CP}^1$ and asks whether it is possible to choose $(E,\,\delta)$ extending $(E_0,\,\delta_0)$ such that E is the trivial holomorphic vector bundle over $X = \mathbb{CP}^1$. The answer to it is no in general; the first counterexample was constructed by Bolibruch in [2]. However, the Riemann-Hilbert problem is known to have a positive answer when $\operatorname{rank}(E_0) = 2$, or when the connection δ_0 is irreducible [27], [11], [9], [22].

An appropriate formulation for the classical Riemann-Hilbert problem in higher genus is to ask whether $(E\,,\delta)$ can be chosen such that E is semistable of degree 0. Indeed, with that formulation, the general negative answer as well as the sufficient conditions for positive answers remain valid, as proven in [14] and [15].

On the other hand, the fundamental group $\pi_1(X \setminus D,\, x_0)$ does not depend on the complex structure of X. Let us consider $(X,\, D)$ as a fiber of the universal family of curves over the Teichmüller space $\mathcal{T}_{g,n}$ of genus g surfaces with n marked points:

$$\begin{CD} (X, D) @>>> (\mathcal{X}, \mathcal{D}) \\ @VVV @VpVV \\ \{t_0\} @>>> \mathcal{T}_{g,n} \end{CD}$$

The fundamental group of each punctured fiber can be identified with the fundamental group $\pi_1(X \setminus D,\, x_0)$, because $\mathcal{T}_{g,n}$ is contractible. Given any $(E,\,\delta)$ on $(X,\,D)$, it extends to a flat logarithmic connection $(\mathcal{E},\,\delta')$ over $\mathcal{X}$, singular over $\mathcal{D}$; this flat logarithmic connection $(\mathcal{E},\,\delta')$ is called the *(universal) isomonodromic deformation* of $(E,\,\delta)$ (see Section 4.2). It is called isomonodromic because with respect to a convenient identification of the fundamental group of the fibers, the corresponding family of monodromy representations is constant.

We are led to another Riemann–Hilbert type problem: *Given any* $(E,\,\delta)$*, is there a parameter* $t \in \mathcal{T}_{g,n}$ *such that for the logarithmic connection* $(\mathcal{E}^t, \delta^t)$ *on* $p^{-1}(t)$ *induced by the isomonodromic deformation, the vector bundle* $\mathcal{E}^t$ *is semistable?* The partial answers in [10] and [19] to this question were generalized in [4] to the following. If the genus g of X is at least 2 and δ is irreducible, then for generic parameters $t \in \mathcal{T}_{g,n}$, the vector bundle $\mathcal{E}^t$ is not only semistable but stable. In case rank two, the general vector bundle is even very stable [5]. This remains valid, for an appropriate generalization of the universal isomonodromic deformation in case δ has irregular singularities [19], [6].

Remark 1.1. Note that the degree of the vector bundle is a topological invariant and thus remains constant along the deformation. If one wishes to investigate the above question in the case (E, δ) is *reducible, i.e.,* there is a subbundle $0 \subsetneq F \subsetneq E$ preserved by δ, then one has to impose that F is not a destabilizing bundle. Under this additional assumption, the proof in [4] still applies.

On the other hand, given a logarithmic connection $(E,\, \delta)$ on a curve, there is a natural parabolic structure on E supported by the singularities of the connection such that the parabolic structure at a singular point of the connection is given by the residue of the logarithmic connection at that point (see Section 2.4). Therefore, underlying the universal isomonodromic deformation is also a family of parabolic vector bundles parametrized by $\mathcal{T}_{g,n}$. Our aim here is to investigate the above questions on stability and very stability of the general underlying bundle in this context of parabolic vector bundles (see Sections 2.3 and 6.1).

We prove the following result in two steps (see Theorem 5.2 and Theorem 6.2).

Theorem 1.2. *Let X be a compact Riemann surface of genus $g \geq 2$, and let D be a divisor on X. Let δ be a logarithmic connection, singular over D, on a holomorphic vector bundle $E \longrightarrow X$. Let $(\mathcal{E}, \delta')$ be its universal isomonodromic deformation, with*

$$\mathcal{E} \longrightarrow \mathcal{X} \stackrel{p}{\longrightarrow} \mathcal{T}_{g,n}\,.$$

Denote $\mathcal{E}^t := \mathcal{E}|_{\mathcal{X}_t}$, where $\mathcal{X}_t := p^{-1}(t)$. Denote by $\mathcal{E}^t_\star$ the corresponding parabolic vector bundle over $\mathcal{X}_t$ with parabolic structure induced by $\delta'|_{\mathcal{E}^t}$. Then there are closed analytic subsets

$$\mathcal{Y} \subset \mathcal{Y}' \subset \mathcal{Y}'' \subset \mathcal{T}_{g,n}$$

such that the following statements hold:

- *for every $t \in \mathcal{T}_{g,n} \setminus \mathcal{Y}$, the parabolic vector bundle $\mathcal{E}^t_\star$ is parabolically semistable;*
- *for every $t \in \mathcal{T}_{g,n} \setminus \mathcal{Y}'$, the parabolic vector bundle $\mathcal{E}^t_\star$ is parabolically stable;*
- *for every $t \in \mathcal{T}_{g,n} \setminus \mathcal{Y}''$, the parabolic vector bundle $\mathcal{E}^t_\star$ is parabolically very stable.*

If δ is irreducible, then the analytic subsets $\mathcal{Y}$ and $\mathcal{Y}'$ of $\mathcal{T}_{g,n}$ are proper, and their codimensions are bounded as follows

$$\operatorname{codim}(\mathcal{Y}) \geq g\,; \quad \operatorname{codim}(\mathcal{Y}') \geq g-1\,.$$

If δ is irreducible and E is of rank 2, then the analytic subset $\mathcal{Y}''$ is also proper.

The proof is similar to the non-parabolic case treated in [4] and [5]: the fact that the sets $\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'' \subset \mathcal{T}_{g,n}$ are analytically closed is known from [17]. The main issue is proving that these are proper subsets. We proceed with a deformation-theoretic approach.

This paper is the final one in a series examining the behaviour of "generic properties" such as stability under isomonodromic deformation; the general gist is that isomonodromic deformation is in some sense transversal to the unstable locus. In previous papers, the connection was also allowed to have singularities, but these were basically independent of the structure examined. in the set-up considered here, the parabolic structure and the singularities of the connection are intertwined; the genericity result still holds, however.

2. Logarithmic connections and parabolic bundles

In this section, we recall the definition of the Atiyah bundle for a vector bundle over a pointed curve, and how the Atiyah exact sequence can be used to define logarithmic connections on the vector bundle on the one hand, and infinitesimal deformations of the vector bundle on the pointed curve on the other hand. We further recall that if a vector bundle is endowed with a logarithmic connection, then it has a natural parabolic structure defined by the residues of the connection.

2.1. Logarithmic connections and the Atiyah bundle

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix a finite nonempty subset

$$D = \{x_1, \cdots, x_n\} \subset X$$

of distinct ordered points of cardinality $n \geq 1$. We will employ the convention of denoting by $\mathrm{T}Z$ the holomorphic tangent bundle of a complex manifold Z. Let

$$\mathrm{T}X\,(-\log D) = \mathrm{T}X(-D) := \mathrm{T}X \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$$

be the logarithmic tangent bundle of X.

Take a holomorphic vector bundle E over X of rank r. For any $i \geq 0$, let $\mathrm{Diff}^i\,(E,\, E)$ be the holomorphic vector bundle on X defined by the sheaf of holomorphic differential operators, of order at most i, from the sheaf of holomorphic sections of E to itself. In other words,

$$\mathrm{Diff}^i(E,\, E) \,=\, \mathrm{Hom}(J^i(E),\, E) \,=\, E \otimes J^i(E)^\vee\,,$$

where $J^i(E)$ it the i-th jet bundle for E. Consider the symbol homomorphism

$$\sigma_1 \,:\, \mathrm{Diff}^1(E,\, E) \longrightarrow \mathrm{T}X \otimes \mathrm{End}(E)\,. \tag{2.1}$$

We recall the construction of σ_1. Take any $x \in X$ and any $w \in \mathrm{T}^\vee_x X$. Let f_w be a holomorphic function defined around x such that $f_w(x) = 0$ and $df_w(x) = w$. Let $\mathcal{D}_x$ be a holomorphic section of $\mathrm{Diff}^1(E,\, E)$ defined around x. Then for any $v \in E_x$, we have

$$w\left(\sigma_1\left(\mathcal{D}_x(x)\right)(v)\right) \,=\, \mathcal{D}_x\left(f_w \cdot v'\right)(x)\,, \tag{2.2}$$

where v' is a holomorphic section of E defined around x such that $v'(x) = v$; note that both sides of (2.2) are elements of E_x. The homomorphism σ_1 is evidently surjective. The logarithmic Atiyah bundle is defined as

$$\mathrm{At}_D(E) \,:=\, \sigma_1^{-1}\left(\mathrm{T}X(-D) \otimes \mathrm{Id}_E\right) \,\subset\, \mathrm{Diff}^1(E,\, E)\;.$$

It fits in the logarithmic Atiyah exact sequence

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{At}_D(E) \stackrel{\sigma}{\longrightarrow} \mathrm{T}X(-D) \longrightarrow 0\,, \tag{2.3}$$

where σ is the restriction of the symbol homomorphism σ_1 in (2.1). Therefore, a holomorphic section of $\mathrm{At}_D(E)$ over an open subset $U \subset X$ is a holomorphic differential operator

$$D_U \,:\, E|_U \longrightarrow (E \otimes K_X \otimes \mathcal{O}_U(D))|_U\,, \tag{2.4}$$

where $K_X = (\mathrm{T}X)^*$ is the holomorphic cotangent bundle of X, satisfying the following Leibniz condition:

$$D_U(f \cdot s) \,=\, f \cdot D_U(s) + s \otimes df$$

for every holomorphic function f_U on U and every holomorphic section s of E over U.

We recall that a logarithmic connection on E singular over D is a holomorphic splitting of the exact sequence in (2.3), meaning a holomorphic homomorphism

$$\delta \,:\, \mathrm{T}X(-D) \longrightarrow \mathrm{At}_D(E)$$

such that $\sigma \circ \delta = \mathrm{Id}_{\mathrm{T}X(-D)}$, where σ is the homomorphism in (2.3) [12] (see also [3]).

So a logarithmic connection δ on E singular over D corresponds to a holomorphic differential operator over X

$$D_X : E \longrightarrow E \otimes K_X \otimes \mathcal{O}_U(D)$$

as in (2.4) satisfying the Leibniz condition.

We have the following:

1. The infinitesimal deformations of the n-pointed compact Riemann surface (X, D) are parametrized by $\mathrm{H}^1(X, \mathrm{T}X(-D))$.
2. The infinitesimal deformations of the above triple
$$(X, D, E)$$
are parametrized by $\mathrm{H}^1(X, \mathrm{At}_D(E))$.
3. The map
$$\mathrm{H}^1(X, \mathrm{T}X(-D)) \longrightarrow \mathrm{H}^1(X, \mathrm{At}_D(E))$$
corresponding to isomonodromic deformation is the one induced by the connection
$$\delta : \mathrm{T}X(-D) \longrightarrow \mathrm{At}_D(E)\,.$$

Here (1) is standard, (2) is a consequence of the results in [20] and (3) is explained in [4].

2.2. Residue of a logarithmic connection

Take any $x_j \in D$. There is a canonical homomorphism

$$\phi_j : \mathrm{At}_D(E)_{x_j} \longrightarrow \mathrm{End}(E)_{x_j} = \mathrm{End}\left(E_{x_j}\right) \tag{2.5}$$

which we will now describe. Consider the commutative diagram of homomorphisms of vector spaces

(2.6)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}\left(E_{x_j}\right) & \xrightarrow{\alpha_j} & \mathrm{At}_D(E)_{x_j} & \xrightarrow{\sigma(x_j)} & \mathrm{T}X(-D)_{x_j} & \longrightarrow & 0 \\
 & & \| & & \downarrow a & & \downarrow b & & \\
0 & \longrightarrow & \mathrm{End}\left(E_{x_j}\right) & \xrightarrow{c_j} & \mathrm{Diff}^1(E, E)_{x_j} & \xrightarrow{\sigma_1(x_j)} & (\mathrm{T}X \otimes \mathrm{End}(E))_{x_j} & \longrightarrow & 0\,,
\end{array}$$

where σ and σ_1 are the homomorphisms in (2.3) and (2.1) respectively, and the top exact row is the restriction of the exact sequence in (2.3) to the point x_j while the bottom exact row is the restriction of the Atiyah exact sequence to the point x_j; both the rows in (2.6) are exact. The homomorphism a in (2.6) is given by the natural inclusion of the coherent sheaf $\mathrm{At}_D(E)$ in $\mathrm{Diff}^1(E, E)$, while b is induced by a. Note that $b = 0$, as x_j is a point of D. This implies that $\sigma_1(x_j) \circ a = b \circ \sigma(x_j) = 0$. Now from the exactness of the bottom row in (2.6) it follows that $\mathrm{image}(a) \subset \mathrm{image}(c_j)$, and hence there is a unique homomorphism

$$\phi_j : \mathrm{At}_D(E)_{x_j} \longrightarrow \mathrm{End}\left(E_{x_j}\right)$$

such that $a = c_j \circ \phi_j$. This produces the homomorphism in (2.5).

From the commutativity of the diagram in (2.6) we conclude that $\phi_j \circ \alpha_j$ coincides with the identity map of $\mathrm{End}\left(E_{x_j}\right)$. From this it follows immediately that the restriction of $\sigma(x_j)$ to

$$\mathrm{kernel}(\phi_j) \subset \mathrm{At}_D(E)_{x_j}$$

is an isomorphism with $\mathrm{T}X(-D)_{x_j}$. Using this isomorphism of $\mathrm{kernel}(\phi_j)$ with $\mathrm{T}X(-D)_{x_j}$ we have a decomposition

$$\mathrm{At}_D(E)_{x_j} = \mathrm{End}\left(E_{x_j}\right) \oplus \mathrm{kernel}(\phi_j) = \mathrm{End}\left(E_{x_j}\right) \oplus \mathrm{T}X(-D)_{x_j}. \tag{2.7}$$

The fiber $\mathrm{T}X(-D)_{x_j}$ is identified with $\mathbb{C}$ using the Poincaré adjunction formula [16, p. 146]. More explicitly, for any holomorphic coordinate z around x_j with $z(x_j) = 0$, the evaluation of the section $z\frac{\partial}{\partial z}$ of $\mathrm{T}X(-D)$ at the point x_j is independent of the choice of the holomorphic coordinate function z; the above identification between $\mathrm{T}X(-D)_{x_j}$ and $\mathbb{C}$ sends this independent element of $\mathrm{T}X(-D)_{x_j}$ to $1 \in \mathbb{C}$.

Let $\delta : \mathrm{T}X(-D) \longrightarrow \mathrm{At}_D(E)$ be a logarithmic connection on E singular over D. For any $x_j \in D$, consider

$$\delta(x_j)(1) \in \mathrm{At}_D(E)_{x_j} = \mathrm{End}\left(E_{x_j}\right) \oplus \mathbb{C}; \tag{2.8}$$

here the above identification $\mathrm{T}X(-D)_{x_j} = \mathbb{C}$ is being used. Let

$$\mathrm{Res}(\delta)(x_j) \in \mathrm{End}\left(E_{x_j}\right) \tag{2.9}$$

be the component of $\delta(x_j)(1)$ in the direct summand $\mathrm{End}\left(E_{x_j}\right)$ in (2.8). This endomorphism $\mathrm{Res}(\delta)(x_j)$ is called the *residue* of δ at the point x_j.

The residue is called *resonant* if it admits two eigenvalues whose difference is a non-zero integer. The connection δ is said to be resonant if it possesses a resonant residue.

Let $D_X : E \longrightarrow E\otimes K_X\otimes\mathcal{O}_X(D)$ be a holomorphic differential operator over X as in (2.4) associated to a logarithmic connection δ on E. For any point $x_j \in D$, consider the composition

$$E \xrightarrow{D_X} E\otimes K_X\otimes\mathcal{O}_X(D) \longrightarrow (E\otimes K_X\otimes\mathcal{O}_X(D))_{x_j} = E_{x_j}\,;$$

the fiber $(K_X\otimes\mathcal{O}_X(D))_{x_j}$ is identified with $\mathbb{C}$ using the Poincaré adjunction formula. This composition is $\mathcal{O}_X$–linear, and hence it produces an endomorphism $R_j \in \mathrm{End}(E_{x_j})$. This endomorphism R_j coincides with the residue $\mathrm{Res}\,(\delta)\,(x_j)$ in (2.9).

2.3. Parabolic bundles and the notion of stability

Let E be a holomorphic vector bundle over X of positive rank. A *quasi-parabolic structure* on E over the divisor D is a strictly decreasing filtration of subspaces

$$E_{x_j} = E^1_j \supsetneq E^2_j \supsetneq \cdots \supsetneq E^{n_j}_j \supsetneq E^{n_j+1}_j = 0 \tag{2.10}$$

for every $1 \le j \le n$. A *parabolic structure* on E over D is a quasiparabolic structure as above together with n decreasing sequences of real numbers

$$0 \le \alpha^1_j < \alpha^2_j < \cdots < \alpha^{n_j}_j < 1\,, \quad 1 \le j \le n\,;$$

the real number α^i_j is called the parabolic weight of the subspace E^i_j in the quasiparabolic filtration. The multiplicity of a parabolic weight α^i_j at x_j is defined to be the dimension of the complex vector space E^i_j/E^{i+1}_j. A parabolic vector bundle is a vector bundle with a parabolic structure. We shall refer to the collection of weights and respective multiplicities at each puncture as the *parabolic data* of a parabolic vector bundle. More details on parabolic bundles can be found in [25], [24].

Let

$$E_\star = \left(E,\, \{E^i_j\},\, \{\alpha^i_j\}\right)$$

be a parabolic bundle as above. The *parabolic degree* of $E_\star$ is defined to be

$$\text{par-deg}\,(E_\star) = \text{degree}(E) + \sum_{j=1}^{n}\sum_{i=1}^{n_j} \alpha^i_j \dim\left(E^i_j/E^{i+1}_j\right)$$

[25, p. 214, Definition 1.11], [24, p. 78].

Take any holomorphic subbundle $F \subset E$. For each $x_j \in D$, the fiber F_{x_j} has a filtration obtained by intersecting the quasiparabolic filtration of E_{x_j} with the subspace F_{x_j}. The parabolic weight of a subspace $S \subset F_{x_j}$ in this filtration is the maximum of the numbers

$$\{\alpha_j^i \mid S \subset E_j^i \cap F_{x_j}\}\,.$$

This way, the parabolic structure on E produces a parabolic structure on the subbundle F. The resulting parabolic bundle will be denoted by $F_\star$.

A parabolic vector bundle $E_\star = \left(E,\, \{E_j^i\},\, \{\alpha_j^i\}\right)$ is called *stable* (respectively, *semistable*) if for all subbundles $F \subsetneq E$ of positive rank the inequality

$$\frac{\text{par-deg}\,(F_\star)}{\text{rank}\,(F_\star)} < \frac{\text{par-deg}\,(E_\star)}{\text{rank}\,(E_\star)} \quad \left(\text{respectively, } \frac{\text{par-deg}\,(F_\star)}{\text{rank}\,(F_\star)} \leq \frac{\text{par-deg}\,(E_\star)}{\text{rank}\,(E_\star)}\right)$$

holds [25].

2.4. Parabolic structure from a logarithmic connection

Let

$$\delta\,:\,\mathrm{T}X(-D) \longrightarrow \mathrm{At}_D(E)$$

be a logarithmic connection on E, singular over D. Using the residues of δ defined in (2.9), we will construct a parabolic structure on E. To each eigenvalue $\boldsymbol{\lambda}$ of $\mathrm{Res}\,(\delta)\,(x_j)$, we associate

$$\lambda := \{\Re\,(\boldsymbol{\lambda})\} := \Re\,(\boldsymbol{\lambda}) - \lfloor \Re\,(\boldsymbol{\lambda}) \rfloor \in [0,1[\,,$$

the fractional part of its real part. Let $x_j \in D$ and let

$$0 \leq \lambda_j^1 < \lambda_j^2 < \cdots < \lambda_j^{n_j} < 1$$

be the fractional parts of the real parts of the eigenvalues of $\mathrm{Res}\,(\delta)\,(x_j)$. Let $F_j^i \subset E_{x_j}$ be the sum of the generalized eigenspaces corresponding to those eigenvalues $\boldsymbol{\lambda}$ of $\mathrm{Res}\,(\delta)\,(x_j)$ such that $\{\Re\,(\boldsymbol{\lambda})\} = \lambda_j^i$. The parabolic weights of E at x_j are the eigenvalues $\{\lambda_j^i\}_{i=1}^{n_j}$. The subspace of E_{x_j} corresponding to the parabolic weight λ_j^i is $\bigoplus_{k\geq i} F_j^k$. Note that according to this definition, the parabolic structure at x_i is determined by the semisimple part $\mathrm{Res}^{ss}\,(\delta)\,(x_j)$ (with respect to the Jordan decomposition) of the residue at x_i. If

$$\Re\,(\boldsymbol{\lambda}) = \{\Re\,(\boldsymbol{\lambda})\} \in [0,1[$$

for each eigenvalue for each residue of δ, then δ is the called the Deligne extension of the restriction of δ to $E|_{X\setminus D}$.

Remark 2.1. We note that $\text{degree}(E)+\sum_{j=1}^{n}\text{trace}\left(\text{Res}\left(\delta\right)\left(x_j\right)\right) = 0$ [26, p. 16, Theorem 3]. Therefore,

$$\text{par-deg}\left(E_\star\right) := \text{degree}(E) + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\lambda_j^i \in \mathbb{Z}\,, \tag{2.11}$$

where $E_\star$ is the parabolic vector bundle constructed from $(E,\,\delta)$.

3. Infinitesimal deformations of parabolic bundles

We shall now establish the space of infinitesimal deformations of parabolic bundles on pointed curves, where the base is allowed to vary. Moreover, we are going to take into account the information of a further subbundle, which shall later be used for testing of parabolic stability.

3.1. Infinitesimal deformations with fixed base curve

Fix a pair $(X,\,D)$ as before. Let $E_\star = \left(E,\,\{E_j^i\},\,\{\alpha_j^i\}\right)$ be a parabolic vector bundle on X with parabolic structure over the divisor D. Let

$$\text{End}_p\left(E_\star\right) \subset \text{End}(E) = E\otimes E^\vee \tag{3.1}$$

denote the coherent subsheaf that preserves the quasiparabolic filtration over every point of D. So, $\text{End}_p\left(E_\star\right)$ coincides with $\text{End}(E)$ over the complement $X\setminus D$. For each point $x_j \in D$, the image of $\text{End}_p(E)_{x_j}$ in

$$\text{End}\left(E_{x_j}\right) = \text{End}(E)_{x_j}$$

consists of all endomorphisms that preserve the quasiparabolic filtration over x_j. In other words, for a section s of $\text{End}_p\left(E_\star\right)$, we have

$$s\left(E_j^i\right) \subseteq E_j^i$$

for all x_j in the domain of definition of s and all $1 \leq i \leq n_j$ (as in (2.10)). Let

$$\text{End}_{p,j}(E) \subset \text{End}\left(E_{x_j}\right) \tag{3.2}$$

be the image of $\text{End}_p(E)_{x_j}$ in $\text{End}\left(E_{x_j}\right)$. We have a short exact sequence of coherent sheaves on X

$$0 \longrightarrow \text{End}_p\left(E_\star\right) \xrightarrow{\beta_0} \text{End}(E) \longrightarrow \bigoplus_{j=1}^{n}\text{End}(E)_{x_j}/\text{End}_{p,j}(E) \longrightarrow 0\,. \tag{3.3}$$

It is known that the infinitesimal deformations of $E_\star$ are parametrized by $\mathrm{H}^1(X,\, \mathrm{End}_p(E_\star))$ [28, Section 5].

3.2. Infinitesimal deformations with varying base curve

Consider the homomorphism ϕ_j constructed in (2.5). The composition

$$\mathrm{At}_D(E)_{x_j} \xrightarrow{\phi_j} \mathrm{End}(E)_{x_j} \longrightarrow \mathrm{End}(E)_{x_j}/\mathrm{End}_{p,j}(E)$$

will be denoted by $\widehat{\phi}_j$; the above map $\mathrm{End}(E)_{x_j} \longrightarrow \mathrm{End}(E)_{x_j}/\mathrm{End}_{p,j}(E)$ is the quotient by the subspace in (3.2). Note that this composition homomorphism is surjective. Let

$$\mathrm{At}_p(E) \subset \mathrm{At}_D(E)$$

be the coherent subsheaf that fits in the following short exact sequence:

$$0 \longrightarrow \mathrm{At}_p(E) \longrightarrow \mathrm{At}_D(E) \xrightarrow{\oplus_j \widehat{\phi}_j} \bigoplus_{j=1}^{n} \mathrm{End}(E)_{x_j}/\mathrm{End}_{p,j}(E) \longrightarrow 0\,. \tag{3.4}$$

Therefore, using (2.3) we have the following commutative diagram with exact rows and columns:

(3.5)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}_p(E_\star) & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \bigoplus_{j=1}^n \mathrm{End}(E)_{x_j}/\mathrm{End}_{p,j}(E) & \longrightarrow & 0 \\
 & & \downarrow \beta & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{At}_p(E) & \longrightarrow & \mathrm{At}_D(E) & \longrightarrow & \bigoplus_{j=1}^n \mathrm{End}(E)_{x_j}/\mathrm{End}_{p,j}(E) & \longrightarrow & 0 \\
 & & \downarrow \sigma' & & \downarrow \sigma & & & & \\
 & & \mathrm{T}X(-D) & = & \mathrm{T}X(-D) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where σ' is the restriction of σ in (2.3). We note that a holomorphic section of $\mathrm{At}_p(E)$ over an open subset $U \subset X$ is a holomorphic differential operator of order one

$$D_U \,:\, E|_U \longrightarrow E|_U$$

satisfying the following conditions:

- the symbol of D_U is a holomorphic section of $\mathrm{T}X(-D)$ over U (so D_U is a section of $\mathrm{At}_D(E)$ over U), and
- for every holomorphic section s of $E|_U$, and every $x_j \in D \cap U$, if $s(x_j) \in E^i_j \subset E_{x_j}$, then $D_U(s)(x_j) \in E^i_j$. Here we used the notation in (2.10).

Lemma 3.1. *The infinitesimal deformations of the triple $(X, D, E_\star)$, with parabolic data of fixed type (fixed parabolic weights and their multiplicities), are parametrized by $\mathrm{H}^1(X, \mathrm{At}_p(E))$. The homomorphism*

$$\beta_* : \mathrm{H}^1(X, \mathrm{End}_p(E_\star)) \longrightarrow \mathrm{H}^1(X, \mathrm{At}_p(E)),$$

induced by β in (3.5), corresponds to the map of infinitesimal deformations where the pair (X, D) is kept fixed. The homomorphism

$$\sigma'_* : \mathrm{H}^1(X, \mathrm{At}_p(E)) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X(-D)),$$

induced by σ' in (3.5), is the forgetful map that sends any infinitesimal deformation of $(X, D, E_\star)$ to the infinitesimal deformation of (X, D) obtained by simply forgetting $E_\star$.

Proof. This lemma is standard. Consider the sheaf of groups on X given by the local automorphisms of E that preserve the parabolic structure (this means that the quasiparabolic structure is preserved, because the parabolic weights do not move). The corresponding sheaf of Lie algebras is $\mathrm{End}_p(E_\star)$. More generally consider the sheaf of groups on X given by the local automorphisms of the pair (X, E) that preserve the parabolic structure. The corresponding sheaf of Lie algebras is $\mathrm{At}_p(E)$. The lemma can be derived from these observations. □

The homomorphism

$$\mathrm{H}^1(X, \mathrm{At}_p(E)) \longrightarrow \mathrm{H}^1(X, \mathrm{At}_D(E))$$

given by the inclusion $\mathrm{At}_p(E) \hookrightarrow \mathrm{At}_D(E)$ in (3.4) is the forgetful map that sends any infinitesimal deformation of $(X, D, E_\star)$ to the infinitesimal deformation of (X, D, E) obtained by simply forgetting the parabolic data.

3.3. Infinitesimal deformations of parabolic bundles with a subbundle

Fix a pair $(X,\, D)$. As before, let $E_\star \,=\, \Big(E,\, \{E^i_j\},\, \{\alpha^i_j\}\Big)$ be a parabolic vector bundle on X with parabolic structure over D. Fix a subbundle $0 \,\neq\, F \,\subsetneq\, E$.

Let

$$\mathrm{End}^F_p(E_\star) \,\subset\, \mathrm{End}_p(E_\star)$$

be the subsheaf that preserves F. The infinitesimal deformations of the pair

$$(E_\star,\, F)$$

(keeping the pair $(X,\, D)$ fixed) are parametrized by

$$\mathrm{H}^1\Big(X,\, \mathrm{End}^F_p(E_\star)\Big)\ .$$

The homomorphism

$$\mathrm{H}^1\Big(X,\, \mathrm{End}^F_p(E_\star)\Big) \,\longrightarrow\, \mathrm{H}^1(X,\, \mathrm{End}_p(E_\star))\ ,$$

given by the inclusion of $\mathrm{End}^F_p(E_\star)$ in $\mathrm{End}_p(E_\star)$, corresponds to the forgetful map of infinitesimal deformations that forgets the subbundle F; recall that $\mathrm{H}^1(X,\, \mathrm{End}_p(E_\star))$ is the space of infinitesimal deformations of $E_\star$. The kernel of this forgetful homomorphism corresponds to infinitesimal deformations of F keeping $E_\star$ fixed.

Let

$$\mathrm{At}^F_p(E) \,\subset\, \mathrm{At}_D(E) \tag{3.6}$$

be the coherent subsheaf whose sections over any open subset $U \,\subset\, X$ are all holomorphic differential operators

$$D_U \,:\, E|_U \,\longrightarrow\, E|_U$$

satisfying the following two conditions:

- for every holomorphic section s of $E|_U$, and every $x_j \,\in\, U$, if

$$s\,(x_j) \,\in\, E^i_j \,\subset\, E_{x_j}\ ,$$

  then $D_U\,(s)\,(x_j) \,\in\, E^i_j$, and
- $D_U\,(s)$ is a section of $F|_U$ if s is a holomorphic section of $F|_U$.

Therefore, we actually have

$$\mathrm{At}_p^F(E) \subset \mathrm{At}_p(E)\,. \tag{3.7}$$

We have the following short exact sequence of vector bundles on X:

$$0 \longrightarrow \mathrm{End}_p^F(E_\star) \longrightarrow \mathrm{At}_p^F(E) \longrightarrow \mathrm{T}X(-D) \longrightarrow 0\,. \tag{3.8}$$

Lemma 3.1 has the following straightforward generalization:

Lemma 3.2. *The infinitesimal deformations of the quadruple*

$$(X,\, D,\, E_\star,\, F)$$

with parabolic data of fixed type are parametrized by

$$\mathrm{H}^1\Big(X,\, \mathrm{At}_p^F(E)\Big)\,.$$

The homomorphism

$$\mathrm{H}^1\Big(X,\, \mathrm{At}_p^F(E)\Big) \longrightarrow \mathrm{H}^1(X,\, \mathrm{At}_p(E))$$

given by the inclusion $\mathrm{At}_p^F(E) \hookrightarrow \mathrm{At}_p(E)$ *in* (3.7) *corresponds to the forgetful homomorphism that forgets* F*.*

We note that the homomorphism

$$\mathrm{H}^1\Big(X,\, \mathrm{At}_p^F(E)\Big) \longrightarrow \mathrm{H}^1(X,\, \mathrm{At}_D(E))$$

given by the inclusion $\mathrm{At}_p^F(E) \hookrightarrow \mathrm{At}_D(E)$ in (3.6) corresponds to the forgetful homomorphism that forgets F as well as the parabolic structure on E (recall that the infinitesimal deformations of the triple $(X,\, D,\, E)$ are parametrized by $\mathrm{H}^1(X,\, \mathrm{At}_D(E))$).

4. Isomonodromic deformations

We will now recall the universal isomonodromic deformation of a given initial logarithmic connection, and how it encodes the infinitesimal deformation at the initial parameter of the underlying parabolic vector bundle.

4.1. The initial connection

Take $(X,\, D)$ as before. As in Section 2.4, let E be a holomorphic vector bundle over X of rank r, and let

$$\delta \,:\, \mathrm{T}X(-D) \,\longrightarrow\, \mathrm{At}_D(E)$$

be a logarithmic connection on E, singular over D. Let $E_\star$ be the parabolic vector bundle defined by the parabolic structure on E given by the residues of the logarithmic connection δ (see Section 2.4).

Lemma 4.1. *The image* $\delta\,(\mathrm{T}X(-D)) \,\subset\, \mathrm{At}_D(E)$ *is contained in the subsheaf* $\mathrm{At}_p(E) \,\subset\, \mathrm{At}_D(E)$ *in* (3.4).

Proof. Take any point $x_j \,\in\, D$. From the construction of the parabolic structure using $\mathrm{Res}\,(\delta)\,(x_j)$ it follows that $\mathrm{Res}\,(\delta)\,(x_j)$ preserves the quasiparabolic filtration of $E_\star$ over x_j. This means that

$$\mathrm{Res}\,(\delta)\,(x_j) \,\in\, \mathrm{End}_{p,j}(E) \,\subset\, \mathrm{End}(E)_{x_j}\,.$$

From the definition of residue, given in (2.9), it now follows that

$$\delta\,(\mathrm{T}X(-D)) \,\subset\, \mathrm{At}_p(E)\,.$$

This completes the proof. □

4.2. The universal isomonodromic deformation

For $(X,\, D)$ as before, fix an ordering of the points of D. Let $\mathcal{T}_{g,n}$ be the Teichmüller space for $(X,\, D)$. We briefly recall its construction, details can be found for example in [21]. Let $\mathbf{C}_{g,n}$ denote the space of all complex structures on X, and let Diff denote the group of all diffeomorphisms of X that fix D pointwise. Let

$$\mathrm{Diff}^0 \,\subset\, \mathrm{Diff}$$

be the connected component containing the identity element. Then we have

$$\mathcal{T}_{g,n} \,=\, \mathbf{C}_{g,n}/\mathrm{Diff}^0\,.$$

This $\mathcal{T}_{g,n}$ is a contractible complex manifold of complex dimension $3g-3+n$. Note that there is a base point

$$t_0 \,\in\, \mathcal{T}_{g,n} \tag{4.1}$$

defined by the given complex structure on X.

There is a universal n-pointed Riemann surface $(\mathcal{X},\, (s_1,\, \cdots, s_n))$ over $\mathcal{T}_{g,n}$. This means that

$$p\,:\,\mathcal{X}\,\longrightarrow\,\mathcal{T}_{g,n} \tag{4.2}$$

is a holomorphic family of Riemann surfaces such that any fiber $p^{-1}(t)$ is the Riemann surface associated to t, and $s_i\,:\,\mathcal{T}_{g,n}\,\longrightarrow\,\mathcal{X}$, for $1\,\leq\,i\,\leq\,n$, are disjoint sections of the projection p in (4.2). The n-pointed Riemann surface $(p^{-1}(t)\,,\,(s_1(t),\,\cdots,\,s_n(t)))$ is represented by the point $t\,\in\,\mathcal{T}_{g,n}$. Moreover, if t_0 denotes the base point in (4.1), we have the following identification of n-pointed Riemann surfaces:

$$\left(p^{-1}(t_0)\,,\,(s_1(t_0)\,,\,\cdots,s_n(t_0))\right) = (X,\,(x_1,\,\cdots,x_n))\,=\,(X,\,D)$$

(recall that we have fixed an ordering of the points of D).
Since $\mathcal{T}_{g,n}$ is contractible, the inclusion map

$$X\setminus D\,\hookrightarrow\,\mathcal{X}\setminus\mathcal{D} \tag{4.3}$$

as the fiber over t_0, where $\mathcal{D}\,:=\,(\sqcup_{i=1}^n s_i(\mathcal{T}_{g,n}))$, is a homotopy equivalence.

As in Section 2.4, let E be a holomorphic vector bundle on X or rank r, and let

$$\delta\,:\,\mathrm{T}X(-D)\,\longrightarrow\,\mathrm{At}_D(E) \tag{4.4}$$

be a logarithmic connection on E, singular over D. There exists a vector bundle $\mathcal{E}$ on $\mathcal{X}$, endowed with a flat logarithmic connection $\widetilde{\delta}$, singular over $\mathcal{D}$, such that the restriction of $\left(\mathcal{E},\,\widetilde{\delta}\right)$ to $p^{-1}(t_0)\,=\,X$ is identified with $(E,\,\delta)$, where t_0 is the base point in (4.1). Let us briefly recall the construction (see [18, Section 3] for details).

Let

$$\rho\,:\,\pi_1(X\setminus D,\,x_0)\,\longrightarrow\,\mathrm{GL}(E_{x_0})$$

be the monodromy representation for the flat connection δ; here $x_0\,\in\,X\setminus D$ is a fixed base point. Since the inclusion map in (4.3) is a homotopy equivalence, we have a homomorphism

$$\pi_1(\mathcal{X}\setminus\mathcal{D},\,x_0)\,=\,\pi_1(X\setminus D,\,x_0)\,\stackrel{\rho}{\longrightarrow}\,\mathrm{GL}(E_{x_0})$$

which will be denoted by $\widetilde{\rho}$. This $\widetilde{\rho}$ produces a holomorphic vector bundle $\widetilde{\mathcal{E}}$ over the complement $\mathcal{X}\setminus\mathcal{D}$ equipped with a flat holomorphic connection $\widetilde{\delta}$

[12]. Using an argument of Malgrange [23] generalizing Deligne extensions in this context, this holomorphic vector bundle $\widetilde{\mathcal{E}}$ admits an extension $\mathcal{E}$ to $\mathcal{X}$ as a holomorphic vector bundle such that

- the connection $\widetilde{\delta}$ extends to a logarithmic connection δ' on $\mathcal{E}$, and
- the restriction of $(\mathcal{E},\, \delta')$ to $p^{-1}(t_0) = X$ is identified with $(E,\, \delta)$, where t_0 is the point in (4.1).

The pair $(\mathcal{E},\, \delta')$ is unique and admits a universal property with respect to germs of isomonodromic deformations of the same initial connection. It is therefore called the *universal isomonodromic deformation* in [18]. In the current work, we will refer to the pair $(\mathcal{E},\, \delta')$ simply as *the isomonodromic deformation* of the logarithmic connection $(E,\, \delta)$ on X.

For any $t \in \mathcal{T}_{g,n}$, the Riemann surface $p^{-1}(t)$ will be denoted by $\mathcal{X}_t$. The restriction of the holomorphic vector bundle $\mathcal{E}$ to $\mathcal{X}_t$ will be denoted by $\mathcal{E}^t$. The restriction of the logarithmic connection δ' to $\mathcal{E}^t$ will be denoted by δ^t.

4.3. The underlying infinitesimal deformation of the parabolic bundle

We adopt the notation of Section 4.2. As shown in Section 2.4, the logarithmic connection δ^t produces a parabolic structure on $\mathcal{E}^t$. The resulting parabolic vector bundle on $\mathcal{X}_t$ will be denoted by $\mathcal{E}^t_\star$. Let

$$\mathcal{E}_\star \longrightarrow \mathcal{X} \xrightarrow{\;p\;} \mathcal{T}_{g,n} \tag{4.5}$$

be the above family of parabolic vector bundles constructed from δ' (which in turn is constructed from δ).

Lemma 4.2. *Let $(\mathcal{E}, \delta)$ be the isomonodromic deformation of (E, δ). Then for each $1 \le i \le n$, the collection of parabolic weights and their multiplicities of $\mathcal{E}^t_\star$ at the parabolic point $s_i(t) \in \mathcal{X}_t$ is independent of t.*

Proof. For any x_0^t in $\mathcal{X}_t \setminus (\sqcup_{i=1}^n s_i(t))$ and any path from x_0 to x_0^t in $\mathcal{X} \setminus \mathcal{D}$, the holonomy of δ' yields an isomorphism $E_{x_0} \simeq \mathcal{E}^t_{x_0^t}$ identifying the monodromy ρ of δ with the monodromy of δ^t. Different choices of paths yield conjugated monodromy representations. However, the conjugacy class of the local monodromy of δ^t around $s_i(t)$ does not depend on $t \in \mathcal{T}_{g,n}$. On the other hand, the parabolic data at $s_i(t)$ is entirely encoded by the conjugacy class of the local monodromy of δ^t around $s_i(t)$. Indeed, the semisimple part of the local monodromy at $s_i(t)$ is conjugated to $\exp\left(\mathrm{Res}^{ss}\left(\delta^t\right)\left(s_i(t)\right)\right)$ (see for example [8, Theorem 1]). ∎

In Lemma 3.1 we saw that the infinitesimal deformations of the triple $(X, D, E_\star)$, with parabolic data of fixed type (fixed parabolic weights and their multiplicities), are parametrized by $\mathrm{H}^1(X, \mathrm{At}_p(E))$. On the other hand, for any $t \in \mathcal{T}_{g,n}$, we have

$$\mathrm{T}_t\mathcal{T}_{g,n} = \mathrm{H}^1\left(\mathcal{X}_t, \mathrm{T}\mathcal{X}_t \otimes \mathcal{O}_{\mathcal{X}_t}\left(-\sum_{j=1}^{n} s_j(t)\right)\right).$$

In particular, we have $\mathrm{T}_{t_0}\mathcal{T}_{g,n} = \mathrm{H}^1(X, \mathrm{T}X(-D))$, where t_0 is the base point in (4.1). Let

$$\gamma : \mathrm{H}^1(X, \mathrm{T}X(-D)) = \mathrm{T}_{t_0}\mathcal{T}_{g,n} \longrightarrow \mathrm{H}^1(X, \mathrm{At}_p(E)) \tag{4.6}$$

be the classifying homomorphism corresponding to the family of parabolic vector bundles in (4.5) constructed from δ' (which in turn is constructed from δ).

In Lemma 4.1 we saw that $\delta(\mathrm{T}X(-D)) \subset \mathrm{At}_p(E)$. Let

$$\delta_* : \mathrm{H}^1(X, \mathrm{T}X(-D)) \longrightarrow \mathrm{H}^1(X, \mathrm{At}_p(E)) \tag{4.7}$$

be the homomorphism induced by $\delta : \mathrm{T}X(-D) \longrightarrow \mathrm{At}_p(E)$.

Lemma 4.3. *The homomorphism γ in (4.6) coincides with the homomorphism δ_* in (4.7).*

Proof. Lemma 4.3 is straightforward to prove; the case without parabolic structure is dealt with in [4, p. 131]. In the presence of parabolic structure it remains valid after appropriate modifications. □

5. The isomonodromic deformation contains stable parabolic bundles

We are now ready to prove the first main result: if the initial connection is irreducible, the vector bundle corresponding to a generic fiber of the parameter space in its (universal) isomonodromic deformation is parabolically stable.

5.1. A criterion for extending a subbundle to the isomonodromy family

Let δ be a logarithmic connection on E as in (4.4). Assume that δ is *irreducible* in the sense that no nonzero subbundle $E' \subsetneq E$ is preserved by δ.

Let $F \subset E$ be a subbundle. We have the commutative diagram of sheaves on X:

(5.1)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}_p^F(E_\star) & \longrightarrow & \mathrm{At}_p^F(E) & \longrightarrow & \mathrm{T}X(-D) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{End}_p(E_\star) & \xrightarrow{\beta} & \mathrm{At}_p(E) & \xrightarrow{\sigma'} & \mathrm{T}X(-D) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma_0} & & \downarrow{\scriptstyle\gamma_1} & & & & \\
 & & \mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star) & = & \mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the top short exact sequence is the one in (3.8) and the short exact sequence at the bottom is the one in (3.5). Consider the composition homomorphism

$$\mathrm{T}X(-D) \xrightarrow{\delta} \mathrm{At}_p(E) \xrightarrow{\gamma_1} \mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star) \tag{5.2}$$

(Lemma 4.1 says that the image of δ is in $\mathrm{At}_p(E)$); this composition homomorphism will be denoted by f_0. Since F is not preserved by the connection δ by the irreducibility assumption, we have

$$f_0 \neq 0 .$$

Let

$$\mathcal{L} \subset \mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star) \tag{5.3}$$

be the holomorphic line subbundle generated by the image $f_0(\mathrm{T}X(-D))$. We note that $\mathcal{L}$ coincides with the inverse image, in $\mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star)$, of the torsion part

$$\Big(\Big(\mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star)\Big)/f_0(\mathrm{T}X(-D))\Big)_{\mathrm{torsion}}$$
$$\subset \Big(\mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star)\Big)/f_0(\mathrm{T}X(-D))$$

under the quotient map

$$\mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star) \longrightarrow \Big(\mathrm{End}_p(E_\star)/\mathrm{End}_p^F(E_\star)\Big)/f_0(\mathrm{T}X(-D)) .$$

Now define

(5.4)

$$\mathrm{End}_p^\delta(E_\star) := \gamma_0^{-1}(\mathcal{L}) \subset \mathrm{End}_p(E_\star) \quad \text{and} \quad \mathrm{At}_p^\delta(E) := \gamma_1^{-1}(\mathcal{L}) \subset \mathrm{At}_p(E) ,$$

where $\mathcal{L}$ is the line subbundle in (5.3), and γ_0, γ_1 are the homomorphisms in (5.1). Note that from (5.1) we have the following commutative diagram of sheaves on X:

(5.5)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}_p^F(E_\star) & \longrightarrow & \mathrm{At}_p^F(E) & \longrightarrow & \mathrm{T}X(-D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mu & & \| & & \\
0 & \longrightarrow & \mathrm{End}_p^\delta(E_\star) & \longrightarrow & \mathrm{At}_p^\delta(E) & \xrightarrow{\sigma''} & \mathrm{T}X(-D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \gamma' & & & & \\
 & & \mathcal{L} & = & \mathcal{L} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where σ'' and γ' respectively are the restrictions of the homomorphisms σ' and γ_1 constructed in (5.1).

From the definition of $\mathrm{At}_p^\delta(E)$ in (5.4) it follows immediately that the image of the connection homomorphism $\mathrm{T}X(-D) \xrightarrow{\delta} \mathrm{At}_p(E)$ is contained in the subbundle $\mathrm{At}_p^\delta(E)$. Let

(5.6)
$$\xi : \mathrm{T}X(-D) \longrightarrow \mathcal{L}$$

be the homomorphism given by the composition f_0 in (5.2).

Consider the family of parabolic bundles

$$\mathcal{E}_\star \longrightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}_{g,n}$$

constructed in (4.5) using δ' (which is constructed from δ). From the commutative diagram in (5.5) we can now deduce the following proposition.

Proposition 5.1. *If the subbundle $F \subset E$ extends to a subbundle $\mathcal{F}$ of $\mathcal{E}$ over the first order infinitesimal neighborhood of the point $t_0 \in \mathcal{Y}$, where $\mathcal{Y}$ is a closed analytic subset of $\mathcal{T}_{g,n}$, then the homomorphism defined by the composition*

$$\mathrm{T}_{t_0}\mathcal{Y} \hookrightarrow \mathrm{T}_{t_0}\mathcal{T}_{g,n} = \mathrm{H}^1(X,\, \mathrm{T}X(-D)) \xrightarrow{\xi_*} \mathrm{H}^1(X,\, \mathcal{L}) \ ,$$

induced by ξ in (5.6)*, vanishes identically.*

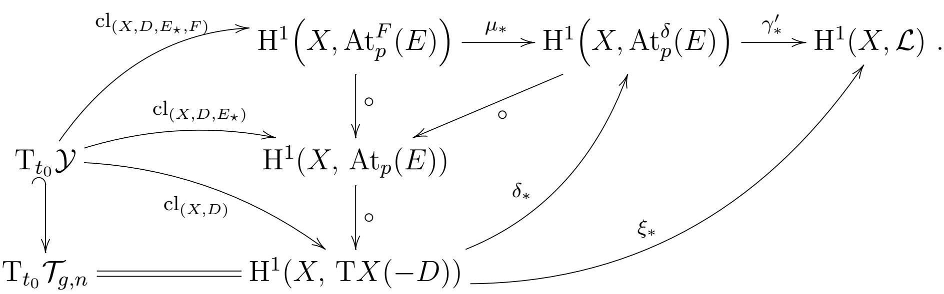
Proof. Assume that the subbundle $F \subset E$ extends to the first order infinitesimal neighborhood of $t_0 \in \mathcal{Y} \subset \mathcal{T}_{g,n}$. Consequently, we have a classifying homomorphism

$$\mathrm{cl}_{(X,D,E_\star,F)} \,:\, \mathrm{T}_{t_0}\mathcal{Y} \longrightarrow \mathrm{H}^1\Big(X,\, \mathrm{At}_p^F(E)\Big)$$

to the space of infinitesimal deformations of

$$(X,\, D,\, E_\star,\, F)$$

(that is of quadruples given by curve, punctures, parabolic bundle and subbundle in the isomonodromic deformation). Denoting forgetful morphisms simply by "$\circ$", and also adopting a similar notation for the other classifying maps, by Lemma 3.2 and Lemma 4.3, the following diagram of homomorphisms is commutative:



(in the above diagram "$\circ$" denotes the homomorphisms of cohomologies induced by the natural inclusions of coherent sheaves). The result now simply follows from the fact that the top row is exact according to (5.5). $\square$

Theorem 5.2. *Let X be a compact Riemann surface of genus $g \geq 2$, and let D be a divisor on X. Let δ be an irreducible logarithmic connection, singular* *over D, on a holomorphic vector bundle $E \longrightarrow X$. Consider the family of parabolic vector bundles*

$$\mathcal{E}_\star \longrightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}_{g,n}$$

underlying the isomonodromic deformation of (E, δ) as in Section 4.3, and denote, for any $t \in \mathcal{T}_{g,n}$, by $\mathcal{E}^t_\star$ the corresponding parabolic vector bundle over $\mathcal{X}_t = p^{-1}(t)$ with parabolic structure over the divisor $(s_1(t), \cdots, s_n(t))$. Denote

$$\begin{aligned} \mathcal{Y} &:= \{t \in \mathcal{T}_{g,n} \mid \mathcal{E}^t_\star \textit{ is not parabolically semistable.}\} \\ \mathcal{Y}' &:= \{t \in \mathcal{T}_{g,n} \mid \mathcal{E}^t_\star \textit{ is not parabolically stable.}\} \end{aligned}$$

Then $\mathcal{Y}$ and $\mathcal{Y}'$ are closed analytic subsets of $\mathcal{T}_{g,n}$, whose codimensions are bounded as follows:

$$\operatorname{codim}(\mathcal{Y}) \geq g\,; \quad \operatorname{codim}(\mathcal{Y}') \geq g-1\,.$$

Proof. The mechanics of the proof of this theorem are identical to the proofs of Proposition 5.3 of [4, p. 138] (concerning $\mathcal{Y}$) and Proposition 5.4 of [4, p. 139] (concerning $\mathcal{Y}'$) up to some minor modifications. We will therefore be brief. The fact that $\mathcal{Y}$ and $\mathcal{Y}'$ defined as in the statement are closed analytic subsets of $\mathcal{T}_{g,n}$ follows from [17]. Indeed, one can write $\mathcal{Y}'$ as a union of strata corresponding to types k of nontrivial Harder-Narasimhan filtrations, and the results of [17] tell us that the union of strata corresponding to types greater or equal to a fixed k forms a closed subset. On the other hand, within the moduli space of semi-stable objects, stable ones form an open subset. Let $0 \neq F \subset E$ be a destabilizing subbundle, i.e.,

$$\frac{\text{par-deg}\,(F_\star)}{\text{rank}\,(F_\star)} > \frac{\text{par-deg}\,(E_\star)}{\text{rank}\,(E_\star)}\,,$$
$$\text{(respectively, } \frac{\text{par-deg}\,(F_\star)}{\text{rank}\,(F_\star)} \geq \frac{\text{par-deg}\,(E_\star)}{\text{rank}\,(E_\star)}).$$

Then, as is Section 5.1, we have a short exact sequence of sheaves on X

$$0 \longrightarrow \mathrm{T}X(-D) \xrightarrow{\xi} \mathcal{L} \longrightarrow T^\delta \longrightarrow 0\,, \tag{5.7}$$

where T^δ is a torsion sheaf because $\xi \neq 0$ by irreducibility of δ.

We will show that

$$\text{degree}(\mathcal{L}) < 0\,, \quad (\text{respectively,} \quad \text{degree}(\mathcal{L}) \leq 0) \tag{5.8}$$

in the stable (respectively, semistable) case.

For this first consider the Harder-Narasimhan filtration of the parabolic endomorphism bundle $\text{End}(E_\star) = E_\star \otimes E_\star^*$. Let $W_\star \subset \text{End}(E_\star)$ is the part of this filtration for nonnegative parabolic weights. Then all the successive quotients of the Harder-Narasimhan filtration of the quotient parabolic bundle $\text{End}(E_\star)/W_\star$ have negative parabolic degree. On the other hand, when $E_\star$ is parabolic semistable, for the socle filtration of $\text{End}(E_\star)$, all the successive quotients of the filtration have parabolic degree to be zero. In the stable case, $\mathcal{L}$ is a subsheaf of the quotient parabolic bundle $\text{End}(E_\star)/W_\star$, and hence the parabolic degree of $\mathcal{L}$ with the induced parabolic structure is negative. This implies that the degree of $\mathcal{L}$ is negative. In the semistable case, $\mathcal{L}$ is a subsheaf of the quotient of the socle filtration, so the parabolic degree of $\mathcal{L}$ with the induced parabolic structure is nonpositive. Hence the degree of $\mathcal{L}$ is nonpositive in this case. Also form the result on p. 705 of [1] it follows that that the degree of $\mathcal{L}$ must be negative, in the stable case, and negative or zero, in the semi-stable case.

From the long exact sequence associated to the short exact sequence (5.7), one then deduces

$$\begin{gathered} \dim\Big(\xi_*\mathrm{H}^1(X,\, \mathrm{T}X(-D))\Big) \,\geq\, g \\ (\text{respectively,}\ \ \dim\Big(\xi_*\mathrm{H}^1(X,\, \mathrm{T}X(-D))\Big) \,\geq\, g-1)\,. \end{gathered} \tag{5.9}$$

Up to replacing t_0 by a generic element of $\mathcal{Y}$ respectively, $\mathcal{Y}'$, we may assume that in the infinitesimal neighborhood of t_0 in $\mathcal{Y}$ respectively, $\mathcal{Y}'$, the destabilizing subbundle F, which we take to be maximal, in the Harder-Narasimhan sense, extends; this follows from the picture of $\mathcal{Y}$ respectively, $\mathcal{Y}'$ as a union of strata. Then Proposition 5.1, in combination with (5.9), yields the desired estimate for the codimension. □

6. Infinitesimal deformations of parabolic Higgs bundles

This section is dedicated to prove our second main result: in the rank two case, if the initial connection is irreducible, the vector bundle corresponding to a generic fiber of the parameter space in the (universal) isomonodromic deformation is parabolically very stable. We shall proceed in a way similar to what lead to the first main result. Namely, after recalling the basic definitions, we will establish the deformation theory of parabolic Higgs bundles over varying base curves, as well as the obstruction space of deformations of non-zero nilpotent Higgs fields. These results will then be applied to the isomonodromic deformation.

Let (X, D) be as before a compact Riemann surface of genus $g \geq 2$ endowed with n ordered marked points. Let $E_\star$ be a vector bundle $E \longrightarrow X$ endowed with a parabolic structure over D as before. However, from now on we will always assume that

$$\mathrm{rank}(E) = 2 .$$

For each $x_j \in D$, the parabolic filtration of E_{x_j} in (2.10) then is of length $n_j \leq 2$.

6.1. Very stable parabolic Higgs bundles

Let us recall the notion of $E_\star$ being parabolically very stable.

Consider the vector bundle $\mathrm{End}_p(E_\star)$ in (3.1) and define

$$\mathrm{End}_p^0(E_\star) \subset \mathrm{End}_p(E_\star) \tag{6.1}$$

to be the coherent subsheaf defined by the endomorphisms that are nilpotent with respect to the quasiparabolic filtration over every point of D, *i.e.,* for a section s of $\mathrm{End}_p^0(E_\star)$, we have

$$s\left(E_j^i\right) \subseteq E_j^{i+1}$$

for all $x_j \in D$ in the domain of definition of s and all $1 \leq i \leq n_j$ (as in (2.10)).

Remark 6.1. Since $\mathrm{End}(E)^\vee = \mathrm{End}(E)$ with the isomorphism given by the bilinear pairing defined by $A \otimes B \longmapsto \mathrm{trace}(AB)$, we have a fiberwise non-degenerate pairing

$$(\mathrm{End}(E) \otimes \mathcal{O}_X(D)) \otimes (\mathrm{End}(E) \otimes \mathcal{O}_X(D)) \longrightarrow \mathcal{O}_X(2D)$$

given by trace. For this pairing, the image of $\mathrm{End}_p(E_\star) \otimes (\mathrm{End}_p^0(E_\star) \otimes \mathcal{O}_X(D))$ is evidently contained in $\mathcal{O}_X \subset \mathcal{O}_X(2D)$. It is now straightforward to check that this restricted pairing produces an isomorphism

$$\mathrm{End}_p(E_\star)^\vee = \mathrm{End}_p^0(E_\star) \otimes \mathcal{O}_X(D) . \tag{6.2}$$

A *Higgs field* on a parabolic vector bundle $E_\star$ is a holomorphic section of $\text{End}^0_p(E_\star)\otimes \text{K}_X\otimes\mathcal{O}_X(D)$, where $\text{End}^0_p(E_\star)$ is the vector bundle constructed in (6.1). A *Higgs bundle* is a pair $(E_\star,\, \theta)$, where $E_\star$ is a parabolic vector bundle and θ is a Higgs field on $E_\star$. The Higgs field

$$\theta \,\in\, \text{H}^0\Big(X,\, \text{End}^0_p(E_\star)\otimes \text{K}_X\otimes\mathcal{O}_X(D)\Big)$$

is called *nilpotent* if $\theta^2 \,=\, 0$. A parabolic Higgs bundle $(E_\star,\, \theta)$ is called nilpotent if θ is nilpotent.

A parabolic vector bundle $E_\star$ is called *parabolically very stable* if it does not admit any nonzero nilpotent Higgs field. It can be proved that a parabolically very stable vector bundle $E_\star$ is automatically parabolically stable. To prove this, assume that $E_\star$ is not stable. then there is a line subbundle $L\,\subset\, E$ such that

$$\text{par-deg}\,(L_\star)\,\geq\,\frac{\text{par-deg}\,(E_\star)}{2}\,, \tag{6.3}$$

where $L_\star$ is the parabolic line bundle given by the parabolic structure on L induced by the parabolic structure on $E_\star$. Denote $D' \,:=\, \{x_j\in D \mid E^2_j\neq \{0\}\}$ and

$$D_L \,:=\, \{x_j \,\in\, D' \,\mid\, L_{x_j} \,=\, E^2_j\}\,. \tag{6.4}$$

From (6.3) it follows that

$$\begin{gathered}\text{degree}\,(\text{Hom}\,(E/L\,,\, L)\otimes\mathcal{O}_X(D_L))\\ \geq\, \text{degree}(D_L)+\sum_{D'-D_L}(\alpha^2_j-\alpha^1_j)-\sum_{D_L}(\alpha^2_j-\alpha^1_j)\geq 0\,.\end{gathered}$$

Then the line bundle $\text{Hom}\,(E/L\,,\, L)\otimes \text{K}_X\otimes\mathcal{O}_X(D')$ has a non-zero holomorphic section by Riemann–Roch theorem. A nonzero holomorphic section ζ of $\text{Hom}\,(E/L\,,\, L)\otimes \text{K}_X\otimes\mathcal{O}_X(D')$ defines a nonzero nilpotent Higgs field on $E_\star$ using the composition

$$E\,\longrightarrow\, E/L\,\stackrel{\zeta}{\longrightarrow}\, L\otimes\text{K}_X\otimes\mathcal{O}_X(D')\,\longrightarrow\, E\otimes\text{K}_X\otimes\mathcal{O}_X(D')\,,$$

where $\longrightarrow\, E/L$ is the quotient map; the other homomorphism

$$L\otimes\text{K}_X\otimes\mathcal{O}_X(D')\,\longrightarrow\, E\otimes\text{K}_X\otimes\mathcal{O}_X(D')$$

is the tensor product of the inclusion $L\,\hookrightarrow\, E$ with the identity map of $\text{K}_X\otimes\mathcal{O}_X(D')$. Therefore, $E_\star$ is not parabolically very stable.

Note that the kernel of the above composition homomorphism is precisely L.

6.2. Infinitesimal deformations of a parabolic Higgs bundle on a fixed curve

Let $(E_\star, \theta)$ be a parabolic Higgs bundle of rank 2 over a fixed pointed curve (X, D). As recalled in Section 3.1, the infinitesimal deformations of $E_\star$ are parametrized by $\mathrm{H}^1(X, \mathrm{End}_p(E_\star))$. These of course need to be reflected in the infinitesimal deformations of the pair $(E_\star, \theta)$. Using Serre duality, and (6.2), the dual of the space of infinitesimal deformations of $E_\star$ is $\mathrm{H}^0\Big(X, \mathrm{End}_p^0(E_\star) \otimes \mathcal{O}_X(D) \otimes \mathrm{K}_X\Big)$, where K_X is the holomorphic cotangent bundle of X. As shown in [7], this dual space corresponds to the infinitesimal deformations of Higgs fields θ on a fixed parabolic bundle $E_\star$. Let us recall how these two infinitesimal deformation spaces fit together to construct the infinitesimal deformation space of pairs $(E_\star, \theta)$.

Let

$$f_\theta \,:\, \mathrm{End}_p(E_\star) \longrightarrow \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \tag{6.5}$$

be the homomorphism defined by $A \longmapsto \theta \circ A - A \circ \theta$. Now we have a two-term complex $\mathcal{C}_\bullet^{(E_\star, \theta)}$ of sheaves on X

$$\mathcal{C}_0^{(E_\star,\theta)} \,:=\, \mathrm{End}_p(E_\star) \xrightarrow{f_\theta} \mathcal{C}_1^{(E_\star,\theta)} \,:=\, \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\,.$$

The infinitesimal deformations of

$$(E_\star,\, \theta)\ ,$$

keeping $(X,\, D)$ fixed, are parametrized by the hypercohomology $\mathbb{H}^1\Big(\mathcal{C}_\bullet^{(E_\star,\theta)}\Big)$ [7]. Consider the following short exact sequence of complexes.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \\
\downarrow & & \| \\
\mathrm{End}_p(E_\star) & \xrightarrow{f_\theta} & \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \\
\| & & \downarrow \\
\mathrm{End}_p(E_\star) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

It produces an exact sequence of hypercohomologies

$$\mathrm{H}^0\Big(X,\, \mathrm{End}^0_p(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\Big) \\ \xrightarrow{\ a\ } \mathbb{H}^1\Big(\mathcal{C}_\bullet^{(E_\star,\theta)}\Big) \xrightarrow{\ b\ } \mathrm{H}^1(X,\, \mathrm{End}_p(E_\star))\ .$$

The above homomorphism a corresponds to changing the Higgs field keeping $E_\star$ fixed, and b corresponds to the forgetful map that sends an infinitesimal deformation of $(E_\star,\, \theta)$ to the corresponding infinitesimal deformation of $E_\star$ by simply forgetting θ.

6.3. Infinitesimal deformations of a parabolic Higgs bundle on moving curve

In Section 6.2, we recalled the infinitesimal deformation space of parabolic Higgs fields with fixed pointed base curve. On the other hand, in Section 3.2, we stated that the infinitesimal deformation space of the triple $(X, D, E_\star)$ is given by $\mathrm{H}^1(X, \mathrm{At}_p(E))$. We shall now explain how these two spaces fit together to form the infinitesimal deformation space of the quadruple $(X, D, E_\star, \theta)$.

There is a natural homomorphism

$$\eta\ :\ \mathrm{At}_p(E) \\ \longrightarrow \mathrm{Diff}^1_X\left(\mathrm{End}^0_p(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D),\, \mathrm{End}^0_p(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\right)\ , \tag{6.6}$$

where $\mathrm{At}_p(E)$ is constructed in (3.4). To construct η, consider the homomorphism

$$\mathrm{At}_D(E) \longrightarrow \mathrm{Diff}^1_X\left(\mathrm{End}(E) \otimes \mathrm{K}_X,\, \mathrm{End}(E) \otimes \mathrm{K}_X\right)$$

constructed in [5, p. 635, (4.1)], where $\mathrm{At}_D(E)$ is constructed in (2.3); in essence, one combines the action on sections of $E, \mathit{End}(E)$ with a Lie derivative on K (but see [5]). It is straight-forward to check that this homomorphism produces a homomorphism as in (6.6) (see Section 4.1 of [5]). We have the homomorphism

$$\eta_\theta\ :\ \mathrm{At}_p(E) \longrightarrow \mathrm{End}^0_p(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\,, \quad s \longmapsto \eta\,(s)\,(\theta)\ . \tag{6.7}$$

Denote the quadruple $(X,\, D,\, E_\star,\, \theta)$ by $\underline{z}$. Let $\mathcal{A}_\bullet^{\underline{z}}$ be the following two-term complex of sheaves on X:

$$\mathcal{A}_0^{\underline{z}} := \mathrm{At}_p(E) \xrightarrow{\ \eta_\theta\ } \mathcal{A}_1^{\underline{z}} := \mathrm{End}^0_p(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\,,$$

where η_θ is the homomorphism in (6.7). The infinitesimal deformations of

$$\underline{z} = (X,\, D,\, E_\star,\, \theta)$$

are parametrized by the hypercohomology $\mathbb{H}^1(\mathcal{A}_\bullet^{\underline{z}})$. Consider the following short exact sequence of complexes.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \\
\downarrow & & \| \\
\mathrm{At}_p(E) & \xrightarrow{\eta_\theta} & \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \\
\| & & \downarrow \\
\mathrm{At}_p(E) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

It produces an exact sequence of hypercohomologies

$$\mathrm{H}^0\Big(X,\, \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\Big) \xrightarrow{a'} \mathbb{H}^1(\mathcal{A}_\bullet^{\underline{z}}) \xrightarrow{b'} \mathrm{H}^1(X,\, \mathrm{At}_p(E)) \,.$$

The above homomorphism a' corresponds to changing the Higgs field keeping the triple $(X,\, D,\, E_\star)$ fixed, and b' corresponds to the forgetful map that sends an infinitesimal deformation of $(X,\, D,\, E_\star,\, \theta)$ to the corresponding infinitesimal deformation of $(X,\, D,\, E_\star)$ by simply forgetting θ; recall from Lemma 3.1 that $\mathrm{H}^1(X,\, \mathrm{At}_p(E))$ parametrizes the infinitesimal deformations of $(X,\, D,\, E_\star)$.

6.4. Infinitesimal deformations of a nilpotent parabolic Higgs bundle

We shall now construct the obstruction space, *i.e.* when the infinitesimal deformation of a nonzero nilpotent parabolic Higgs field remains nilpotent.

Let $(E_\star,\, \theta)$ be a parabolic Higgs bundle of rank 2 over a fixed pointed curve (X, D) as before. Now assume that the Higgs field θ on $E_\star$ is nonzero nilpotent. Let

$$L := \mathrm{kernel}\,(\theta) \subset E$$

be the corresponding holomorphic line subbundle and denote $Q := E/L$ the quotient bundle. From the exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0$$

and its dual sequence, we obtain an exact sequence

$$\begin{gathered} 0 \longrightarrow \mathrm{End}_n^L(E) := Q^\vee \otimes L \longrightarrow E^\vee \otimes L \oplus Q^\vee \otimes E \\ \longrightarrow \mathrm{End}(E) \longrightarrow L^\vee \otimes Q \longrightarrow 0\,, \end{gathered}$$

factoring through

$$\mathrm{End}^L(E) := \{s \in \mathrm{End}(E) \mid s(L) \subset L\}$$

such that we have the following two short exact sequences:

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & \mathrm{End}_n^L(E) \\ & & \downarrow \\ 0 & & E^\vee \otimes L \oplus Q^\vee \otimes E \\ \downarrow & & \downarrow \\ \mathrm{End}^L(E) & = & \mathrm{End}^L(E) \\ \downarrow & & \downarrow \\ \mathrm{End}(E) & & 0 \\ \downarrow & & \\ L^\vee \otimes Q & & \\ \downarrow & & \\ 0 & & \end{array}$$

We note that $\mathrm{rank}\left(\mathrm{End}^L(E)\right) = 3$, and

$$\mathrm{rank}\left(textEnd_n^L(E)\right) = 1\,.$$

The line bundle $\mathrm{End}_n^L(E) = \mathrm{Hom}(Q, L)$ defined above corresponds to those endomorphisms of E which respect the filtration $0 \subset L \subset E$ and which are moreover nilpotent; it is also the kernel of the natural projection

$$\mathrm{End}^L(E) \longrightarrow \mathrm{End}(L) \oplus \mathrm{End}(Q)\,.$$

Now define

$$\mathrm{End}_p^L(E_\star) := \mathrm{End}^L(E) \cap \mathrm{End}_p(E_\star) \subset \mathrm{End}(E)$$

as in Section 3.3 and set

$$\mathrm{End}_n^L(E_\star) := \mathrm{End}_n^L(E) \cap \mathrm{End}_p(E_\star) \subset \mathrm{End}(E)\,.$$

We have the following two term complex $\mathcal{D}_\bullet^{(E_\star,\theta)}$ of sheaves on X:

$$\mathcal{D}_0^{(E_\star,\theta)} = \mathrm{End}_p^L(E_\star) \xrightarrow{f'_\theta} \mathcal{D}_1^{(E_\star,\theta)} = \mathrm{End}_n^L(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\,,$$

where f'_θ is the restriction of the homomorphism f_θ in (6.5). The infinitesimal deformations of $(E_\star,\, \theta)$ in the moduli of nilpotent parabolic Higgs bundles (keeping $(X,\, D)$ fixed) are parametrized by $\mathbb{H}^1\Big(\mathcal{D}_\bullet^{(E_\star,\theta)}\Big)$ [7].

Let $\mathrm{At}_p^L(E) \subset \mathrm{At}_p(E)$ be as in Section 3.3 (with $F = L$). The homomorphism η_θ in (6.7) maps $\mathrm{At}_p^L(E)$ to $\mathrm{End}_L^n(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)$. As before, denote the quadruple $(X,\, D,\, E_\star,\, \theta)$ by $\underline{z}$. We have the following two term complex $\mathcal{B}_\bullet^{\underline{z}}$ of sheaves on X:

$$\mathcal{B}_0^{\underline{z}} = \mathrm{At}_p^L(E) \xrightarrow{\eta'_\theta} \mathcal{B}_1^{\underline{z}} = \mathrm{End}_n^L(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D)\,,$$

where η'_θ is the restriction of the homomorphism η_θ in (6.7).

The infinitesimal deformations of $\underline{z} = (X,\, D,\, E_\star,\, \theta)$ in the moduli of nilpotent parabolic Higgs bundles are parametrized by $\mathbb{H}^1(\mathcal{B}_\bullet^{\underline{z}})$. The morphism $\mathbb{H}^1(\mathcal{B}_\bullet^{\underline{z}}) \longrightarrow \mathbb{H}^1(\mathcal{A}_\bullet^{\underline{z}})$ forgetting that the Higgs field remains nilpotent along the infinitesimal deformation is obtained from the morphism of complexes

$$\begin{array}{ccc} \mathcal{B}_0^{\underline{z}} = \mathrm{At}_p^L(E) & \xrightarrow{\eta'_\theta} & \mathcal{B}_1^{\underline{z}} = \mathrm{End}_n^L(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \\ \downarrow & & \downarrow \\ \mathcal{A}_0^{\underline{z}} = \mathrm{At}_p(E) & \xrightarrow{\eta'_\theta} & \mathcal{A}_1^{\underline{z}} = \mathrm{End}_p^0(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \end{array}$$

induced by the identity. The homomorphism

$$\mathbb{H}^1(\mathcal{B}_\bullet^{\underline{z}}) \longrightarrow \mathrm{H}^1\Big(X, \mathrm{At}_p^L(E)\Big)$$

however, which to a infinitesimal deformation of

$$(X,\, D,\, E_\star,\, \theta)$$

with nilpotent Higgs field associates the underlying infinitesimal deformation of $(X,\, D,\, E_\star,\, L)$ with $L = \text{kernel}(\theta)$ is obtained from the natural morphism of complexes

$$\begin{array}{ccc} \mathcal{B}_0^z = \mathrm{At}_p^L(E) & \xrightarrow{\eta'_\theta} & \mathcal{B}_1^z = \mathrm{End}_n^L(E_\star) \otimes \mathrm{K}_X \otimes \mathcal{O}_X(D) \\ \downarrow & & \downarrow \\ \mathrm{At}_p^L(E) & \longrightarrow & 0\,; \end{array}$$

note that the first hypercohomology space of the complex below coincides with $\mathrm{H}^1\Big(X, \mathrm{At}_p^L(E)\Big)$.

6.5. The isomonodromic deformation contains very stable parabolic bundles

We have now established the necessary ingredients of our second main result:

Theorem 6.2. *Let X be a Riemann surface of genus $g \geq 2$ and let D be a divisor on X. Let δ be an irreducible logarithmic connection, singular over D, on a rank 2 vector bundle $E \longrightarrow X$. Consider the family of parabolic bundles*

$$\mathcal{E}_\star \longrightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}_{g,n}$$

underlying the universal isomonodromic deformation of (E, δ) as in Section 4.3 and denote, for any $t \in \mathcal{T}_{g,n}$, by $\mathcal{E}_\star^t$ the corresponding parabolic vector bundle over $\mathcal{X}_t = p^{-1}(t)$ with parabolic structure over $(s_1(t), \cdots, s_n(t))$. Denote

$$\mathcal{Y}'' \quad := \quad \{t \in \mathcal{T}_{g,n} \mid \mathcal{E}_\star^t \ \ \textit{is not parabolically very stable.}\}$$

Then $\mathcal{Y}''$ is a proper closed analytic subset of $\mathcal{T}_{g,n}$.

Proof. The proof of this theorem is identical to the proof of Theorem 5.2 of [5, p. 639] after some minor modifications. We will therefore be brief. Let θ be a nonzero nilpotent Higgs bundle on the parabolic vector bundle $E_\star$ corresponding to the initial parameter of the isomonodromic deformation. Denote $L := \text{kernel}(\theta)$ as before and let D_L be as in equation (6.4). Recall the commutative diagram (5.1) with exact rows and columns:

(6.8)
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{End}_p^L(E_\star) & \longrightarrow & \mathrm{At}_p^L(E) & \longrightarrow & \mathrm{T}X(-D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mu_1} & & \| & & \\
0 & \longrightarrow & \mathrm{End}_p(E_\star) & \longrightarrow & \mathrm{At}_p(E) & \xrightarrow{\sigma'} & \mathrm{T}X(-D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \gamma_1} & \swarrow{\scriptstyle q} & & & \\
 & & \mathcal{Q} & = & \mathcal{Q} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0\,. & & & &
\end{array}
$$

Here

$$\mathcal{Q} := L^\vee \otimes Q \otimes \mathcal{O}_X(-D_L)$$

with the notation of (6.4), and $q := \gamma_1 \circ \delta$. Since δ is irreducible, we have $q \neq 0$. Since $\mathcal{Q}$ is a line bundle, we obtain an exact sequence

$$0 \longrightarrow \mathrm{T}X(-D) \xrightarrow{\ q\ } \mathcal{Q} \longrightarrow \mathbb{T} \longrightarrow 0\,,$$

where $\mathbb{T}$ is a torsion sheaf. From the corresponding long exact sequence, we have that the induced morphism

$$q_* \,:\, \mathrm{H}^1(X, \mathrm{T}X(-D)) \longrightarrow \mathrm{H}^1(X, \mathcal{Q})$$

of cohomology spaces is surjective. Since θ is nonzero nilpotent with kernel L, it induces a non-zero section of

$$\mathcal{Q}^\vee \otimes \mathrm{K}_X = \mathrm{Hom}(Q, L) \otimes \mathcal{O}_X(D_L) \otimes \mathrm{K}_X \subset \mathrm{Hom}(Q, L) \otimes \mathcal{O}_X(D') \otimes \mathrm{K}_X\,.$$

In particular, using Serre duality, we have $\mathrm{H}^1(X, \mathcal{Q}) \neq \{0\}$. So q_* is nonzero and surjective.

Consider the closed complex analytic subset of the universal moduli of Higgs bundles over $\mathcal{T}_{g,n}$ given by the kernel of the map $(F, \psi) \longmapsto (\mathrm{trace}(\psi), \mathrm{trace}(\psi^2))$ to the universal moduli of forms of degree 1 and 2. The $\mathcal{Y}''$ defined as in the statement of the theorem is the intersection of this closed subset with leaf of the isomonodromic deformation. Hence $\mathcal{Y}''$ is a closed complex analytic subset of $\mathcal{T}_{g,n}$. We may assume that in a neighborhood of $t_0 \in \mathcal{Y}''$, the non-zero nilpotent Higgs field θ on $E_\star$ extends to a non-zero nilpotent Higgs field in this neighborhood. Similarly to the proof of Proposition 5.1, the composition

$$T_{t_0}\mathcal{Y}'' \hookrightarrow \mathrm{H}^1(X, \mathrm{T}X(-D)) \xrightarrow{q_*} \mathrm{H}^1(X, \mathcal{Q})$$

vanishes identically because $\gamma_1 \circ \mu_1 = 0$. Therefore $\mathcal{Y}'' \neq \mathcal{T}_{g,n}$. □

Remark 6.3. In the higher rank case, not only does the deformation theory of nilpotent Higgs bundles get much more complicated, but the main argument in the proof of Theorem 6.2 breaks down: in arbitrary rank the quotient $\mathcal{Q}$ is not necessarily a line bundle and we would need additional information to ensure that q_* is surjective.

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