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Immaculate line bundles on toric varieties

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Abstract: We call a sheaf on an algebraic variety immaculate if it lacks any cohomology including the zero-th one, that is, if the derived version of the global section functor vanishes. Such sheaves are the basic tools when building exceptional sequences, investigating the diagonal property, or the toric Frobenius morphism.

In the present paper we focus on line bundles on toric varieties. First, we present a possibility of understanding their cohomology in terms of their (generalised) momentum polytopes. Then we present a method to exhibit the entire locus of immaculate divisors within the class group. This will be applied to the cases of smooth toric varieties of Picard rank three and to those being given by splitting fans.

The locus of immaculate line bundles contains several linear strata of varying dimensions. We introduce a notion of relative immaculacy with respect to certain contraction morphisms. This notion will be stronger than plain immaculacy and provides an explanation of some of these linear strata.

Keywords: Toric variety, immaculate line bundle, splitting fan, toric varieties of Picard rank 3, primitive collections.

1. Introduction

We work over an algebraically closed field $\Bbbk$ of any characteristic.

1.1. Exceptional sequences ask for immaculacy

A major tool for the process of understanding derived categories $\mathcal{D}(X)$ on an algebraic variety X is full exceptional sequences $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ of sheaves or complexes. That is, its members are supposed to generate $\mathcal{D}(X)$ and, up to $\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}_i, \mathcal{F}_i) = \Bbbk$, one asks for $\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}_i, \mathcal{F}_j[p]) = 0$ for all shifts $p \in \mathbb{Z}$

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and pairs $i \geq j$. These conditions call to mind the shape of unitary upper triangular matrices. If full exceptional sequences exist, then they provide a semi-orthogonal decomposition of $\mathcal{D}(X)$ into the simplest summands possible.

Whenever the $\mathcal{F}_i$ are sheaves, then $\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}_i, \mathcal{F}_j[p])$ can alternatively be written as the classical group $\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{F}_j)$. If, moreover, $\mathcal{F}_i$ are locally free, e.g. invertible sheaves, then this equals $\mathrm{H}^p(X, \mathcal{F}_i^{-1} \otimes \mathcal{F}_j)$. Thus, we require certain sheaves $\mathcal{G} = \mathcal{F}_i^{-1} \otimes \mathcal{F}_j$ to lack any cohomology, including the seemingly innocent 0-th one:

$$\mathbb{R}\Gamma(X, \mathcal{G}) = 0.$$

We will call this property of a sheaf $\mathcal{G}$ immaculate, see Definition 4.1 in Subsection 4.1.

We are going to focus on invertible sheaves on smooth, projective varieties X with $\mathbb{R}\Gamma(\mathcal{O}_X) = \mathbb{k}$. So, when looking for exceptional sequences of line bundles, the case $i = j$ yielding $\mathcal{G} = \mathcal{O}_X$ is already taken care of. That is, whenever we have sufficiently good knowledge of the locus of immaculate sheaves within the Picard or class group $\mathrm{Cl}(X)$, then we can freely use its elements $\mathcal{G}_\nu = \mathcal{O}_X(D_\nu)$ as building blocks to mount exceptional sequences via $\mathcal{F}_i := \mathcal{O}_X(\sum_{\nu=2}^{i} D_\nu)$. The defining property of the vanishing Ext groups can then be understood as asking consecutive sums of the D_ν to be immaculate, too.

The comparison of the shape of several full exceptional sequences can shed light on several features of the given variety X. Thus, the shape of the tool box of immaculate line bundles should serve as a rich invariant. In addition, immaculate line bundles appear in different contexts. In [Ach15] they are exploited to show a characterisation of toric varieties in terms of Frobenius splitting property. In [PSP08] they are used to study the diagonal property of smooth projective varieties (see for instance [PSP08, Thm 4]). For a surface of general type, the property of immaculacy of line bundles is relevant to the spectral theory [KZ17].

1.2. The situation on toric varieties

Suppose that X is a smooth, projective toric variety. The main result in this context is Kawamata's proof of the existence of full exceptional sequences of complexes of sheaves on smooth, projective toric Deligne-Mumford stacks, see [Kaw06, Kaw13]. The original claim made in [Kaw06] that plain sheaves suffice was corrected in [Kaw13, Remark 7]. An earlier conjecture of King about the existence of full, strongly exceptional ($\mathrm{Ext}^{\geq 1}(\mathcal{F}_i, \mathcal{F}_j) = 0$ for all i, j) sequences of line bundles was disproved in [HP06], [Mic11]. Even when abstaining from the additional property "strong", Efimov has shown in [Efi14] that for smooth toric stacks one cannot hope for the existence of full exceptional sequences of line bundles in general. Moreover, we are far away from an understanding of which equivariant divisors represented by which abstract polyhedra might form those sequences. The only rather general, positive result is that of [CM04, Theorem 4.12] where the existence of those sequences was established for splitting fans, see Subsection 7. From a different viewpoint, this was reproven for a special case in [Cra11].

Another remarkable result can be found in [HP11]. There, the authors start with an arbitrary, that is, not necessarily toric, smooth projective rational surface and show that full exceptional sequences of line bundles do always exist. A second interesting point is that these sequences can easily be transformed into a cycle of divisors imitating the toric situation, that is, to each full exceptional sequences one can associate a toric surface materialising this sequence.

1.3. Visualising the cohomology of toric line bundles

In the present paper, we keep the notion of exceptional sequences in the background. Instead, for a given projective (often smooth) toric variety we are just interested in the immaculacy property of divisor classes. Classically, the cohomology of a reflexive rank one sheaf, that is, of a Weil divisor on a toric variety X can be expressed in terms of special polyhedral complexes whose vertices are some rays of the fan Σ of X. In particular, the complexes live in $N_{\mathbb{R}} = N \otimes \mathbb{R}$, where N is the lattice of one parameter subgroups of the torus acting on X.

We propose a different point of view on the cohomology of toric $\mathbb{Q}$-Cartier Weil divisors. We will make it literally visible in terms of polytopes in the dual space $M_{\mathbb{R}}$. As usual, one writes $M_{\mathbb{R}} = M \otimes \mathbb{R}$ with $M = \mathrm{Hom}(N, \mathbb{Z})$ being the monomial lattice of the acting torus T. Since each $\mathbb{Q}$-Cartier Weil divisor can be decomposed into a difference $D = D^+ - D^-$ of nef (or even $\mathbb{Q}$-ample) ones, this means that the T-invariant ones among them can be encoded by a pair of polytopes (Δ^+, Δ^-), see Subsection 3.3 for more details and the more general situation of semi-projective varieties.

Polytopes form a cancellative semigroup under Minkowski addition. In this context, the pair (Δ^+, Δ^-) represents the formal difference

$$D = \Delta^+ - \Delta^-$$

within the Grothendieck group of generalised polytopes. On the other hand, each T-invariant Weil divisor D leads to a (possibly empty) polytope of sections $\Delta(D) \subseteq M_{\mathbb{R}}$. Its lattice points parameterise the monomial basis of

$\Gamma(X,\mathcal{O}_X(D))$. If D is nef, then the pair consisting of $\Delta^+ := \Delta(D)$ and $\Delta^- := 0$ can be used to represent D. For general D being represented by some (Δ^+,Δ^-), one can still recover the polytope of sections as

$$\Delta(D) = \{r \in M_{\mathbb{R}} \mid \Delta^- + r \subseteq \Delta^+\},$$

cf. Remark 3.9. This can be visualised as a kind of a materialised shadow of the abstract difference $\Delta^+ - \Delta^-$.

So it is quite a surprising fact that, after using the formal difference $\Delta^+ - \Delta^-$ and its shadow $\Delta(D)$, the cohomology of $D = \Delta^+ - \Delta^-$ can be understood by a third flavour, namely by the naive and original meaning of the set theoretic differences of these polytopes.

Theorem 1.1. *On a projective toric variety X the cohomology groups $\mathrm{H}^i(X,\mathcal{O}_X(D))$ are M-graded, and for each $m \in M$, the homogeneous component of degree m equals the reduced cohomology group $\widetilde{\mathrm{H}}^{i-1}(\Delta^- \setminus (\Delta^+ - m), \Bbbk)$. Here $\Delta^+ - m$ means the shift by m of Δ^+ in $M \otimes \mathbb{R}$.*

See Example 3.13 for an illustration of this claim. The theorem is stated more generally as Theorem 3.6 in the context of semi-projective toric varieties. It implies that the immaculacy of $D = (\Delta^+,\Delta^-)$ can be measured by the fact whether the topological space $\Delta^- \setminus (\Delta^+ - m)$ is $\Bbbk$-acyclic for all shifts $m \in M$. See Subsection 4.1 for a discussion of the notion of being $\Bbbk$-acyclic.

Besides its elementary geometric nature, the description of sheaf cohomology via the defining polyhedra in the vector space $M_{\mathbb{R}}$ also has another advantage. It allows one to think about a generalisation to the more general setup of Okounkov bodies, as introduced in [LM09]: after fixing a complete flag of subspaces in an arbitrary (not necessarily toric) smooth projective variety X, convex polytopes of sections $\Delta(\mathcal{L})$ are assigned to each invertible sheaf $\mathcal{L}$. Thus, a description of Cartier divisors D via pairs of polytopes (Δ^+,Δ^-) is possible, and one can ask for the relation between $\mathrm{H}^i(X,\mathcal{O}_X(D))$, and the cohomology of the set theoretic differences $\Delta^- \setminus (\Delta^+ - m)$. Since in especially nice situations the Okounkov bodies induce a toric degeneration of X, see [And13], semi-continuity suggests that the latter might serve as an upper bound for the former.

1.4. Immaculate loci for toric varieties

The ultimate goal of this project is to understand the structure of the set of all immaculate line bundles on a fixed toric variety $X = \mathbb{TV}(\Sigma)$ as a subset of the class group of X. Although some of our statements are more general, throughout this introduction we will assume X is in addition smooth and projective.

We show that in sufficiently nice situations the immaculacy is preserved under pullback, see Proposition 4.5 and Corollary 4.6. Moreover, in Definition 4.8 we introduce a relative version of immaculacy, and we show how this stronger version is responsible for the presence of certain linear strata within the immaculacy locus, see Theorems 4.11 and 4.13. However, the example of the flag variety $\mathbb{F}(1,2,3)$ depicted in Figure 5 shows that not all of them (here they are affine lines) can be explained by this notion. The diagonal immaculate line is not induced from any map giving rise to relative immaculacy. Some features of Corollary 4.6 and Theorem 4.13 are summarised as the following statement.

Theorem 1.2. *Suppose X and Y are projective toric varieties and $p\colon X \to Y$ is a surjective toric morphism with connected fibres. Let $\mathcal{L}$ be a line bundle on Y, and let D^- be a nef line bundle on X.*

(i) $\mathcal{L}$ is immaculate if and only if $p^*\mathcal{L}$ is immaculate.
(ii) If $\mathcal{L}$ is ample on Y, then the following conditions are equivalent:

- *for infinitely many integers a the divisor $a\cdot p^*\mathcal{L}-D^-$ is immaculate,*
- *$p^*\mathcal{L}' - D^-$ is immaculate for any line bundle $\mathcal{L}'$ on Y,*
- *the image of the polytope Δ^- (of sections of D^-) under the quotient map $M_X \to M_X/M_Y$ has no internal lattice points.*

In Section 5 we demonstrate our principal approach to obtain the immaculacy locus. It uses the natural map $\pi : \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X)$ assigning to each T-invariant divisor its class. All non-immaculate classes, that is, those carrying some cohomology, must be contained in some of the so-called $\mathcal{R}$-maculate images

$$\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) = \pi(\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}})$$

for certain "tempting" subsets $\mathcal{R} \subseteq \Sigma(1)$. The notion of temptation is introduced in Definition 5.1; it selects those subsets such that the induced subcomplexes of Σ in $N_{\mathbb{R}}$ have some cohomology after being intersected with the unit sphere. Hence, our basic approach is similar to the treatment of acyclic line bundles in [BH09, Sect. 4] and [Efi14, Ch. 4]. However, while the main focus of [BH09] and [Efi14] is to detect full exceptional sequences (of line bundles), we are going to display the entire structure of the immaculate locus. Note also that being "acyclic" in [BH09] does not involve the vanishing of the 0-th cohomology of a sheaf.

To recognise the immaculacy locus in the Picard group involves two different problems. First, one has to find an efficient method to identify the tempting subsets $\mathcal{R} \subseteq \Sigma(1)$. In Subsection 5.2 we have collected some standard situations implying or avoiding immediate temptation. In small examples they already suffice to check the status of most subsets of $\Sigma(1)$. The second problem is to keep control over the interrelation of the different maculate sets or of their convex counterparts, the so-called maculate regions. While a divisor class cannot be immaculate if it is touched by one single maculate set, one has to consider all of these regions for checking the opposite. This behaviour is much better around the vertices of the maculate regions – and this is the content of Theorem 5.24.

1.5. Special situations

After these general investigations, we turn to very concrete situations. In Section 7 we look at the situation of splitting fans, that is, of those fans where all primitive collections (see Subsection 5.2.3 for a definition) are mutually disjoint. While we have already remarked in Subsection 1.2 that the existence of full exceptional sequences is known for this class, we understand this situation from a different viewpoint – namely by describing the entire locus of immaculate line bundles. The main result is contained in Theorem 7.12. A special case of this class is the smooth, projective toric varieties of Picard rank 2. We have, nevertheless, decided to treat these varieties in a separate section. On the one hand, the result can be described in a very clear manner – we do this in Theorem 6.2 – and serves as a concrete example to prepare and illustrate the more general situation of Section 7. On the other, it is a good starting point for the much tougher situation of Picard rank 3 coming in Section 8.

Without going into details of the notation, the highlights of the results in Sections 6–8 can be summarised in the following theorem:

Theorem 1.3. *Suppose X is a smooth projective toric variety.*

- *If the Picard rank of X is 2 and X is not a product of projective spaces, then the set of immaculate line bundles in the Picard group forms a union of finitely many parallel (infinite) lines* (arising as in Theorem 1.2(ii) from a projection $p\colon X \to \mathbb{P}^{\ell_1-1}$) *and two bounded triangles.*
- *If the fan of X is a splitting fan, in particular $X = X_k = \mathbb{P}(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{\ell_k})$ for line bundles $\mathcal{L}_i$ on a smaller splitting fan variety X_{k-1}, then set of immaculate line bundles contains the pullbacks of immaculate*

line bundles from X_{k-1}, their Serre duals, and a family of $\ell_k - 1$ hyperplanes arising as in Theorem 1.2(ii) from the projection $p\colon X_k \to X_{k-1}$. Moreover, for sufficiently "general" choices of $\mathcal{L}_i$, these are all immaculate line bundles on X (see Theorem 7.12 for the exact phrasing of the sufficiently "general" condition).

- *If the Picard rank of X is 3 and X does not have a splitting fan, then the set of immaculate line bundles contains a collection of parallel lines (parameterised by lattice points in the union of two parallelograms), and a finite collection of bounded line segments. For sufficiently general* (see Proposition 8.7) *choices of such X, these are all immaculate line bundles.*

The article concludes with Section 9, which briefly treats the computational aspects of the approach.

Throughout the paper the theory will be illustrated by one running example. We call it the hexagon example since Σ equals the normal fan of a lattice hexagon in $\mathbb{R}^2$. The associated toric variety is the del Pezzo surface of degree 6, which equals the blowing up of $\mathbb{P}^2$ in three points. In particular, it has Picard rank 4 which makes it possible to demonstrate many possible features explicitly. The example is spread under the names Example 3.2, 3.13, 4.7, 4.17, 5.2, 5.10, 5.14, 5.17, 5.20, and 5.27. In addition, its immaculate locus and exceptional sequences can be completely recovered from computer calculations, which are summarised in Section 9.

2. Differences of polytopes

In Section 3.3 we will encode invertible sheaves on projective toric varieties by pairs of polytopes. Then, the cohomology of these sheaves will be expressed by the differences of shifts of the polytopes. Hence, we will start with gathering some general remarks about this construction and we will provide homotopy equivalences between several of its variants.

The arguments here are rather standard in algebraic topology and the results are not very surprising. Similar methods are also used in the context of toric geometry in (for instance) [Ful93, Exercises in Section 4.4], or [Mus05, Chapt. 7, Lem. 4], and those claims can also be deduced from the results in this section.

2.1. Removing open subsets

Fix a real vector space, e.g. $\mathbb{R}^d$ with the Euclidean topology. In this subsection we will show that certain subsets of $\mathbb{R}^d$ are homotopy equivalent. In fact, in most statements below, for $A \subset B \subset \mathbb{R}^d$ we will show that A is a *strong deformation retract* of B. Recall, that a *retract* is a continuous map $r\colon B \to A$, such that $r|_A = \mathrm{id}_A$, and a *strong deformation retract* is a retract which is homotopic to the identity id_B in a way that preserves A, that is there exists continuous $H\colon B \times [0,1] \to B$, such that $H|_{A\times[0,1]}(a,t) = a$, $H(\cdot,0) = \mathrm{id}_B$, and $H(\cdot,1)\colon B \to A$ is the retract of B to A. We will mostly use the *standard* "strong deformation", that is, once we have defined r, the standard definition of H is $H(b,t) = tb + (1-t)r(b)$. Note that this requires that the interval between b and $r(b)$ is contained in B, which will often be guaranteed by some sort of convexity. This standard way of defining H will allow us to glue together several such homotopies.

For a convex subset $P \subset \mathbb{R}^d$, by its *span* we mean the smallest affine subspace containing P. The *relative interior* P° of P is its interior as a subset of its span. Analogously, the *relative boundary* ∂P is the boundary of P within $\operatorname{span} P$. Note that every convex subset of $\mathbb{R}^d$ contains an open subset of its span, so the relative interior of non-empty P is never empty either.

Lemma 2.1. *Let $P \subset \mathbb{R}^d$ be a compact convex subset and let $Q \subset \mathbb{R}^d$ be an open convex subset. If $P \cap Q \neq \emptyset$, then $(\partial P) \setminus Q$ is a strong deformation retract of $P \setminus Q$.*

Proof. Since Q is open and $P \cap Q \neq \emptyset$, there exists a point $p_0 \in P^\circ \cap Q$. Define the retract $r\colon P \setminus \{p_0\} \to \partial P$ by $r(p)$ to be the unique point on the boundary ∂P that is contained in the semiline originating at p_0 and passing through p. Since P and Q are convex, the standard strong deformation map H is well defined, showing the claim. $\square$

We adapt the convention that polyhedra are intersections of finitely many closed halfspaces, polytopes denote bounded hence compact polyhedra, that the empty set is a (-1)-dimensional face of every convex polytope, and that each P is a face of itself. In particular, polytopes and polyhedra are always convex. A proper face is any face that is not $\emptyset$ or P. By a *(finite) polytopal complex* we mean a finite collection Ξ of compact convex polytopes in $\mathbb{R}^d$ satisfying the usual conditions:

- if $P \in \Xi$, then every face of P is in Ξ, and
- if $P_1, P_2 \in \Xi$, then $P_1 \cap P_2$ is a face of both P_1 and P_2.

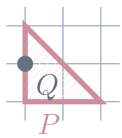
Note that the *support* of a polytopal complex Ξ, $\operatorname{supp}\Xi := \bigcup\{P : P \in \Xi\} \subset \mathbb{R}^d$ is compact. A convex polytope P gives rise to a natural polytopal complex $\{F : F \text{ is a face of } P\}$, whose support is P.

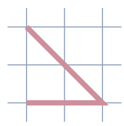
For a polytopal complex $\Xi \subset \mathbb{R}^d$, and a convex subset $Q \subset \mathbb{R}^d$ we denote by $\mathcal{C}(\Xi, Q)$ the polytopal complex

$$\mathcal{C}(\Xi, Q) = \{F \in \Xi \mid F \cap Q = \emptyset\}\,.$$

If $P \subset \mathbb{R}^d$ is a convex polytope, then this gives rise to the special case

$$\mathcal{C}(P, Q) = \{F \mid F \text{ is a face of } P \text{ with } F \cap Q = \emptyset\}\,.$$

For example, for 

we get  as $\mathcal{C}(P, Q)$.

This leads to an analogue of Lemma 2.1 for P replaced with a polytopal complex.

Proposition 2.2. *Let Ξ be a polytopal complex and Q an open convex set. Then* $\operatorname{supp} \mathcal{C}(\Xi, Q)$ *is a strong deformation retract of* $(\operatorname{supp} \Xi) \setminus Q$*.*

Proof. We argue by induction on the number of elements (faces) of Ξ. If $Q \cap \operatorname{supp} \Xi = \emptyset$, or equivalently, $\mathcal{C}(\Xi, Q) = \Xi$, then there is nothing to prove. So suppose $P \in \Xi$ is such that $P \cap Q \neq \emptyset$ and assume that P has maximal possible dimension among such faces. Then there is no other face $F \in \Xi$ that intersects the relative interior P°. In particular, $\Xi' := \Xi \setminus \{P\}$ is a polytopal complex, such that $\operatorname{supp} \Xi' \cap P = \partial P$. By the inductive assumption, $\operatorname{supp} \mathcal{C}(\Xi, Q) = \operatorname{supp} \mathcal{C}(\Xi', Q)$ is a strong deformation retract of $(\operatorname{supp} \Xi') \setminus Q$.

It remains to show, that $(\operatorname{supp} \Xi') \setminus Q$ is a strong deformation retract of $(\operatorname{supp} \Xi) \setminus Q$. But this follows directly by applying Lemma 2.1. $\square$

2.2. Compact approximation of open semialgebraic sets

A subset $X \subset \mathbb{R}^d$ is *semialgebraic* if it can be expressed as a finite union of sets given by polynomial equalities and inequalities. In particular, the support of a polyhedral complex $\operatorname{supp} \Xi$, and an open subset $\{F(x) < 0\} \subset \mathbb{R}^d$ (where $F \in \mathbb{R}[x_1, \ldots, x_d]$) are both semialgebraic subsets, and so is their intersection.

By *piecewise polynomial function* we mean a continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$, such that there exists a finite covering $\mathbb{R}^d = U_1 \cup \cdots \cup U_k$ by semialgebraic subsets U_e (for $e \in 1, \ldots, k$) and polynomials $\phi_i \in \mathbb{R}[x_1, \ldots, x_d]$, and $\phi|_{U_i} = (\phi_i)|_{U_i}$.

We discuss a way of replacing a semialgebraic set in $\mathbb{R}^d$ with a homotopy equivalent subset that is additionally closed in $\mathbb{R}^d$.

Proposition 2.3. *Suppose $X \subset \mathbb{R}^d$ is a compact semialgebraic subset of $\mathbb{R}^d$. Let $\phi\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous, piecewise polynomial function. Denote by $\phi_{>0} := \phi^{-1}((0,\infty))$ the set of points that are mapped to the positive axis, and for $\epsilon \in \mathbb{R}$ define $\phi_{\geq\epsilon} := \phi^{-1}([\epsilon,\infty))$. Then there exists a real number $c > 0$ such that for all $0 < \epsilon \leq c$ the intersection $X \cap \phi_{\geq\epsilon}$ is a strong deformation retract of $X \cap \phi_{>0}$.*

Proof. We may and will assume that X is contained in $\phi_{\geq 0}$. We use the Whitney stratification of X, see for example [Tho69] or [Kal05]. We argue by restricting to one stratum of X at a time. When c is sufficiently small, then the strata whose closures do not intersect $\phi_0 := \phi^{-1}(0)$ are contained in $X \cap \phi_{\geq\epsilon}$. Hence the homotopy does not move these strata. The strata that are contained in ϕ_0 are neither existent in $X \cap \phi_{\geq\epsilon}$ nor $X \cap \phi_{>0}$. Hence it is enough to consider the strata whose closures intersect ϕ_0, but are not contained in ϕ_0. Let M be such a stratum, and suppose that M has a maximal dimension among all such strata.

Define $M_{<\epsilon} \subset M$ to be the intersection $M \cap \phi^{-1}(0,\epsilon)$. Similar to the proof of Proposition 2.2, we can find a strong deformation retract of $\overline{M} \cap \phi_{>0}$ onto $(\partial M \cap \phi_{>0}) \cup (M \cap \phi_{\geq\epsilon}) = \overline{M} \setminus M_{<\epsilon}$. Then we replace X with $X' = \overline{X \setminus (M_{<\epsilon} \cup \phi_0)}$, and we can argue inductively to show the claim. ☐

Suppose $Q \subset \mathbb{R}^d$ is a (compact) polytope defined by affine inequalities $\phi_i(v) \geq 0$ for $i \in \{1,\ldots,k\}$. Let $\epsilon > 0$ be a positive real number. Then the *ϵ-widening of Q* (with respect to the collection of inequalities $\{\phi_i(v) \geq 0 \mid i \in \{1,\ldots,k\}\}$) is the set:

$$Q_{>-\epsilon} := \left\{ v \in \mathbb{R}^d \mid \forall_i\ \phi_i(v) > -\epsilon \right\}.$$

Note that $Q_{>-\epsilon}$ is open and contains Q. The shape of $Q_{>-\epsilon}$ may depend on the choice of the inequalities defining Q, but we will ignore this dependence in our notation, as it will be irrelevant to our statements.

Lemma 2.4. *Suppose $P, Q \subset \mathbb{R}^d$ are two polytopes. Then there exists a positive constant $c > 0$, such that for all $0 < \epsilon \leq c$, the difference $P \setminus Q_{>-\epsilon}$ is a strong deformation retract of $P \setminus Q$. Similarly, if Ξ is a polytopal complex, then $\operatorname{supp}\Xi \setminus Q_{>-\epsilon}$ is a strong deformation retract of $\operatorname{supp}\Xi \setminus Q$ for sufficiently small ϵ.*

Proof. Suppose $Q = \{\phi_i(v) \geq 0 \mid i \in \{1,\ldots,k\}\}$. If the intersection $Q \cap P$ is empty, then the statement is easy, just choose c such that $P \cap Q_{>-c} = \emptyset$. So assume otherwise $Q \cap P \neq \emptyset$ and fix a point $v \in Q \cap P$. For any $x \in P \setminus Q$ consider the unique line ℓ_x passing through x and v. Let

$$c_x = -\frac{1}{\min\{\phi_i(y) \mid i \in \{1,\dots,k\}, y \in \ell_x \cap P\}}.$$

Note that $c_x > 0$ and the set $\{c_x \mid x \in P \setminus Q\}$ is closed, as its values are equal to those on $\operatorname{supp} \mathcal{C}(P,Q)$, which is compact. So let $c = \min\{c_x \mid x \in P \setminus Q\}$ and choose $0 < \epsilon \leq c$. Then for every $x \in P \setminus Q$ the line ℓ_x has non-empty intersection with the compact set $P \setminus Q_{>-\epsilon}$. Define the retract as $x \mapsto r(x) = v + \lambda_x(x-v)$ where $\lambda_x = \min\{\lambda : \lambda \geq 0, v + \lambda_x(x-v) \in P \setminus Q_{>-\epsilon}\}$. The standard homotopy $H(x,t) = tx + (1-t)r(x)$ gives the desired strong deformation.

Note that in the above arguments, r and H preserve faces of P, in the sense, that if F is a (closed) face of P, and r_F and H_F are the retract and its deformation as above, but defined for F, then $r_F = r_P|_F$ and $H_F = H_P|_{F \times [0,1]}$. Thus, they glue well to define the appropriate retract and its strong deformation of $\operatorname{supp} \Xi \setminus Q_{>-\epsilon}$ onto $\operatorname{supp} \Xi \setminus Q$. ☐

As a corollary we have an analogue of Proposition 2.2 and Lemma 2.1 for polytopes Q:

Lemma 2.5. *Let $\Xi \subset \mathbb{R}^d$ be a polytopal complex, and let $Q \subset \mathbb{R}^d$ be a polytope. Then* $\operatorname{supp} \mathcal{C}(\Xi, Q)$ *is a strong deformation retract of* $\operatorname{supp} \Xi \setminus Q$. *In particular, if $P \subset \mathbb{R}^d$ is a polytope, then* $\operatorname{supp} \mathcal{C}(P,Q)$ *is a strong deformation retract of $P \setminus Q$.*

Proof. This is a combination of Lemma 2.4 and Proposition 2.2, together with an observation that $\mathcal{C}(\Xi, Q) = \mathcal{C}(\Xi, Q_{>-\epsilon})$ for sufficiently small $\epsilon > 0$. ☐

Corollary 2.6. *Let $P, Q \subset \mathbb{R}^d$ be two polytopes and assume their intersection is non-empty. Then $\partial P \setminus Q$ is homotopy equivalent to $P \setminus Q$.*

Proof. The complex of P consists of all faces of ∂P and in addition P. Since $Q \cap P \neq \emptyset$, the complexes $\mathcal{C}(P,Q)$ and $\mathcal{C}(\partial P, Q)$ are equal. Therefore, by Lemma 2.5 both $P \setminus Q$ and $\partial P \setminus Q$ are homotopy equivalent to $\operatorname{supp} \mathcal{C}(P,Q) = \operatorname{supp} \mathcal{C}(\partial P, Q)$. ☐

2.3. Allowing common tail cones

Finally, we conclude this section with an argument that reduces considerations of homotopy types of differences of (closed) polyhedra to the case of (compact) polytopes. A simplifying assumption is that the polyhedra have the same tail cone. Recall that $\operatorname{tail}(P) := \{v \in \mathbb{R}^d \mid P + v \subseteq P\}$ is the polyhedral cone indicating the unbounded directions of a polyhedron P. A cone is *pointed* if 0 is its vertex, or equivalently, if the dual cone has full dimension.

Proposition 2.7. *Suppose $P \subset \mathbb{R}^d$ is a polyhedron with a pointed tail cone and $Q \subset \mathbb{R}^d$ is a polyhedron or the interior of a polyhedron with the same tail cone* $\operatorname{tail} Q = \operatorname{tail} P$. *Then there exists a sequence of linear forms $H_1, \ldots, H_k$ and sufficiently large numbers $t_1, \ldots, t_k \in \mathbb{R}$, such that the truncated difference*

$$\operatorname{Trunc}(P \setminus Q) := (P \setminus Q) \cap \bigcap_i \{H_i \leq t_i\}$$

is compact and a strong deformation retract of $P \setminus Q$.

Proof. Proceeding inductively on the number of rays of the common tail cone, we may assume that there are polyhedra P' and Q' with $\operatorname{tail} P' = \operatorname{tail} Q'$ not containing a certain ray ρ such that $P = P' + \rho$ and $Q = Q' + \rho$. We choose a linear form H with $H(\rho) > 0$ that is non-positive on $\operatorname{tail} P' = \operatorname{tail} Q'$. Then there exists a real number t such that both P' and Q' are contained in the halfplane $\{H < t\}$. It follows that $(P \setminus Q) \cap \{H \leq t\}$ is a strong deformation retract of $P \setminus Q$. Indeed, the map $r_\rho \colon \mathbb{R}^d \to \{H \leq t\}$ projecting along ρ does the job. $\square$

As a conclusion, we remark that the homotopy equivalences, such as that in Lemma 2.5, are valid also for polyhedra with common tail cones.

2.4. Smale theorem

The following statement is well known in algebraic topology, and it captures our motivation for replacing all the sets by compact sets.

Proposition 2.8. *Suppose $f \colon X \to Y$ is a continuous surjective proper map of metrisable connected CW-complexes X and Y and that each fibre is contractible and locally contractible. Then f is a homotopy equivalence.*

Proof. By Whitehead's Theorem ([Hat02, Thm 4.5] or [Spa66, Cor. 7.6.24]) it is enough to show that the map induces an isomorphism on all homotopy groups $\pi_i(X) \to \pi_i(Y)$. This claim follows from [Sma57], which is a version of a previously known theorem of Vietoris (with similar assumptions, it claims that induced maps on homology groups are isomorphisms).

Note that by [Hat02, Prop. A.4] both X and Y are locally contractible (in particular LC^n for any n by [Lef42, IV.1.4.1]). Also the same statement implies that X and Y are pathwise connected (that is, 0-connected, in the language of [Sma57]). $\square$

Corollary 2.9. *Suppose X and Y are connected semialgebraic subsets of real affine spaces. If $f\colon X \to Y$ is a proper continuous surjective map with every fibre convex and non-empty, then f is a homotopy equivalence.*

Proof. X and Y are CW-complexes by the Whitney stratification theorem (see [Tho69], [Kal05]), and they are clearly metrisable as subsets of real affine spaces. Since the fibres are convex, they are all contractible and locally contractible. Thus Proposition 2.8 implies the claim. □

3. Toric geometry

The main subject of our paper is to investigate a toric variety X and its immaculacy locus within $\mathrm{Cl}(X)$. For this we will make use of the classical method of calculating the cohomology of equivariant line bundles from the fan in $N_{\mathbb{R}}$. However, after introducing the usual toric notation in Subsection 3.1, we will provide an alternative method using the momentum polyhedra in $M_{\mathbb{R}}$ in Subsection 3.3. It is appropriate to make the cohomology of equivariant line bundles or its absence visible.

3.1. Basic toric notation

All our toric varieties are normal. Our main references for dealing with toric varieties are [CLS11, Ful93, KKMSD73]. We denote by N the lattice of one-parameter subgroups of the torus acting on the toric variety, and by M the character lattice. Throughout Σ denotes a fan in N and $X = \mathbb{TV}(\Sigma)$ the corresponding toric variety. Occasionally, if there is more than one toric variety involved, we may add a subscript N_X, M_X, $\Sigma_X, \ldots$. For a cone σ in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ or $M_{\mathbb{R}} = M \otimes \mathbb{R}$ we denote the dual cone in $M_{\mathbb{R}}$ or $N_{\mathbb{R}}$, respectively, by $\sigma^{\vee}$.

The set of all cones of dimension k of a fan Σ is denoted $\Sigma(k)$. Similarly, for a cone σ, by $\sigma(k)$ we mean the set of all faces of dimension k. In general, every cone σ generates a unique fan consisting of all faces of σ, and the fan will be denoted by the same letter σ. In order to reduce the notation, we will follow the standard convention to denote rays (one dimensional strictly convex lattice cones) and their primitive lattice generators by the same letter, usually ρ.

We will frequently assume that our toric variety X is *semiprojective*, that is projective over an affine (toric) variety. This means that the fan of X has a convex support $\operatorname{supp}\Sigma \subseteq N_{\mathbb{R}}$. Another assumption simplifying the notation in the proofs is that X has no torus factors. In particular, (with both these assumptions) the fan Σ is generated by cones of dimension equal to $\dim X$.

Every Weil divisor on X is linearly equivalent to a torus invariant divisor $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho \cdot D_\rho$ with $D_\rho := \overline{\mathrm{orb}(\rho)}$. If in addition D is $\mathbb{Q}$-Cartier, then there exists a continuous function $u\colon \operatorname{supp}\Sigma \to \mathbb{R}$, which is linear on the cones of Σ, and such that $u(\rho) = -\lambda_\rho$ for every $\rho \in \Sigma(1)$. In particular, for every maximal cone $\sigma \in \Sigma$ there is a unique $u_\sigma \in M_{\mathbb{Q}}$, such that $u|_\sigma = \langle \cdot, u_\sigma\rangle$. We call u the *support function* of D. The divisor D is Cartier if and only if each u_σ is contained in the lattice M.

The *polyhedron of sections* $\Delta = \Delta(D) \subset M_{\mathbb{R}}$ of an equivariant Weil divisor D is defined by its inequalities:

$$\Delta = \{r \in M_{\mathbb{R}} \mid \langle \rho, r\rangle \geq -\lambda_\rho \text{ for all } \rho \in \Sigma(1)\}\,.$$

The name was derived from the fact that $\Delta \cap M$ provides a (monomial) basis of the global sections of $\mathcal{O}_X(D)$. If D is in addition $\mathbb{Q}$-Cartier, then we can describe it also as an intersection of shifted cones that depend on the support function u:

$$\Delta = \bigcap_{\sigma\in\Sigma \text{ maximal}} (u_\sigma + \sigma^\vee)\,.$$

A $\mathbb{Q}$-Cartier Weil divisor D on a semiprojective toric variety X of dimension d is nef if and only if its support function u is concave, that is, for all $a, b \in \operatorname{supp}\Sigma$ and for all $0 \leq t \leq 1$, we have $u(ta+(1-t)b) \geq tu(a)+(1-t)u(b)$. Equivalently, if $a \in \sigma$ for some $\sigma \in \Sigma(d)$, then for every $\sigma' \in \Sigma(d)$ we have: $\langle a, u_\sigma\rangle \leq \langle a, u_{\sigma'}\rangle$. Another way to understand nefness is that all u_σ are contained in Δ; in fact, the set of its vertices equals the set $\{u_\sigma\}$. Observe that this property fails for many quasiprojective varieties which are not semiprojective. As the simplest example, consider $\mathbb{P}^2 \setminus \{pt\}$, and $D = \mathcal{O}(1)$. Then its polytope of sections is a triangle, while there are only two cones of maximal dimension, thus only two u_σ. Note that some of the u_σ might coincide. Moreover, in contrast to the projective case treated, e.g., in [CLS11, (4.2)], for semiprojective X, the polyhedron Δ is no longer compact but has $(\operatorname{supp}\Sigma)^\vee \subseteq M_{\mathbb{R}}$ as its tail cone. Nevertheless, one may still recover the support function of a nef divisor from its polyhedron Δ by

$$u(a) = \min\langle a, \Delta\rangle := \min\{\langle a, r\rangle \mid r \in \Delta\}.$$

Note that the minimum is well-defined for $a \in \operatorname{supp}\Sigma = (\operatorname{tail}\Delta)^\vee$.

A fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^d$ gives rise to a map $\rho : \mathbb{Z}^{\Sigma(1)} \to N$, which takes the basis element indexed by a ray of Σ to the corresponding primitive element on that ray in N. If the underlying toric variety $X = \mathbb{TV}(\Sigma)$ has no torus factors, then the cokernel of ρ is finite. For simplicity, we always assume that this is the case. If, moreover, X is smooth, then ρ is surjective. We denote the kernel by K, and we obtain an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{\rho} N.$$

It is well known that the so-called Gale dual of this sequence yields

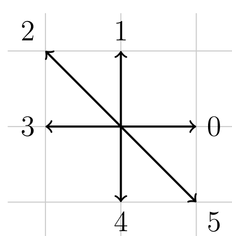
$$0 \longleftarrow \mathrm{Cl}(X) \xleftarrow{\pi} \mathrm{Div}_T(X) \xleftarrow{\rho^*} M \longleftarrow 0, \tag{3.1}$$

where $\mathrm{Div}_T(X) = \left(\mathbb{Z}^{\Sigma(1)}\right)^*$ denotes the group of torus invariant Weil divisors on X. Note that $\mathrm{Cl}(X)$ may have torsion, which corresponds to the torsion of the cokernel of ρ. The anticanonical class of X is $-K_X = \pi(\underline{1})$. The set of effective classes is $\mathrm{Eff}_{\mathbb{Z}}(X) = \pi\left(\mathbb{Z}^{\Sigma(1)}_{\geq 0}\right)$, although often we really consider the effective cone $\mathrm{Eff}_{\mathbb{R}}(X) = \pi\left(\mathbb{R}^{\Sigma(1)}_{\geq 0}\right)$, where π is now considered as the map $\mathbb{R}^{\Sigma(1)} \to \mathrm{Cl}(X) \otimes \mathbb{R}$.

Example 3.2. Throughout the text we will regularly come back to the example of the del Pezzo surface of degree 6, which is the blow up of $\mathbb{P}^2$ in three points, also referred to as a *hexagon* due to the shapes of its fan and the polytopes of sections of ample divisors. This is also a smooth projective toric variety of Picard rank 4, which illustrates that our methods go beyond the main results presented in this article (splitting fans and Picard rank 3 cases). The exact sequence (3.1) in this example is given by the matrices

$$\rho^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The rows of ρ^* form the rays of our fan Σ, meaning we work with the following 2-dimensional fan:



With this choice of ρ^* and π the Nef cone is generated by the following 5 rays, where we write its polytope of sections Δ next to it:

$\operatorname{Pic}(X)$ coordinates	$\Delta \subset M_{\mathbb{R}}$
1 1 0 0	
1 0 1 1	
0 1 1 0	
1 1 1 0	
1 1 1 1	

3.2. Toric cohomology

Let us review the classical method of calculating the cohomology groups of toric divisors. Afterwards, in Subsection 3.3, we dualise it to obtain another method that exploits the polyhedra of sections of nef divisors.

If $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho \cdot D_\rho$ is a Weil divisor on a toric variety $X = \mathbb{TV}(\Sigma)$, then for every $m \in M$ we define

$$V_{D,m} := \bigcup_{\sigma\in\Sigma} \operatorname{conv}\{\rho \mid \rho \in \sigma(1), \langle \rho, m\rangle < -\lambda_\rho\} \subseteq N_{\mathbb{R}}. \tag{3.3}$$

It is a classical result [CLS11, Thm. 9.1.3] (see also [Ach15, Thm. 2.2 and Rem. 2.3] for the characteristic free proof), that one obtains the m-th homogeneous piece of the sheaf cohomology of $\mathcal{O}_X(D)$ as

$$\mathrm{H}^i\left(\mathbb{TV}(\Sigma), \mathcal{O}_X(D)\right)_m = \widetilde{\mathrm{H}}^{i-1}(V_{D,m}, \Bbbk) \quad \text{for all} \quad i \geq 0.$$

Recall that the reduced cohomology of a topological space S is defined via the cochain double complex of S mapping to a point. In particular, there arises a (-1)-st reduced cohomology, and

$$\widetilde{\mathrm{H}}^i(S,\Bbbk) = 0 \text{ for } i < -1, \qquad \widetilde{\mathrm{H}}^{-1}(S,\Bbbk) = \begin{cases} \Bbbk & \text{if } S = \emptyset \\ 0 & \text{if } S \neq \emptyset, \end{cases}$$

$$\mathrm{H}^0(S,\Bbbk) = \widetilde{\mathrm{H}}^0(S,\Bbbk) \oplus \Bbbk, \text{ and} \qquad \widetilde{\mathrm{H}}^i(S,\Bbbk) = \mathrm{H}^i(S,\Bbbk) \text{ for } i > 0.$$

Here $\mathrm{H}^i(S,\Bbbk)$ are the classical singular cohomology groups of the topological space S (with coefficients $\Bbbk$). See [Spa66, §4.3, §5.4] and [Hat02, §2.1, §3.1] for more details about singular and reduced (co)homology groups. See also a brief but relevant summary at the end of [CLS11, §9.0].

Note that, since $0 \notin V_{D,m}$, one might retract $V_{D,m}$ onto a subset of the sphere $S^{d-1} \subseteq N_{\mathbb{R}}$ (where d is the dimension of X, and hence also of $N_{\mathbb{R}}$) without changing their cohomology. Alternatively, we can replace $V_{D,m}$ with $V^{>}_{D,m} := \mathbb{R}_{>0} \cdot V_{D,m}$. If Σ is simplicial, then we can also consider the "full" or "induced" subcomplexes $V^{\geq}_{D,m}$ of Σ, defined as $V^{\geq}_{D,m} := \left\{\sigma \in \Sigma \mid \sigma \setminus \{0\} \subset V^{>}_{D,m}\right\}$. Then $V^{>}_{D,m}$ is (up to $0 \notin V^{>}_{D,m}$) the support of $V^{\geq}_{D,m}$.

If $D = \sum_{\rho\in\Sigma(1)} \lambda_\rho \cdot D_\rho$ is at least a $\mathbb{Q}$-Cartier divisor on X, then one can alternatively use its support function u to calculate the cohomology of $\mathcal{O}_X(D)$. The subset

$$V^{\mathrm{supp}}_{D,m} = \{a \in \operatorname{supp}\Sigma \mid \langle a, m\rangle < u(a)\} \subseteq \operatorname{supp}\Sigma \tag{3.4}$$

contains $V^{>}_{D,m}$ as a strong deformation retract. One can easily prove this using homotopies as in Subsection 2.1. See [CLS11, Thm. 9.1.3], [Ful93, Sect. 4.4] or [Mus05, Chapt. 4, Lem. 7] for slightly weaker claims. Actually, in the original [KKMSD73, p.42], it was exactly the sets $V^{\mathrm{supp}}_{D,m}$ which were used to describe $\mathrm{H}^i\left(\mathbb{TV}(\Sigma), \mathcal{O}_X(D)\right)_m$.

3.3. Cohomology using polyhedra

From now on we assume X to be a semiprojective toric variety (as defined in Section 3.1), in particular it is quasiprojective, but the converse does not hold. In Example 3.12 we illustrate that for the main results of this section to work it is not enough to assume that X is quasiprojective and the semiprojective assumption is strictly needed. We also indicate in the proofs were we exploit this assumption. In Remark 3.14 we also explain what can be deduced if X is not quasiprojective.

Let Y be a projective toric variety containing X as an open torus invariant subset. Fix a torus invariant ample Cartier divisor L on Y such that $L + K_Y$ is effective, where $K_Y = -\sum_{\rho\in\Sigma_Y(1)} D_\rho$ is the canonical divisor of Y. Then the piecewise linear function $\|\cdot\| := -u$ corresponding to L is a norm on the vector space $N_{\mathbb{R}}$. The closed balls centred at 0 with respect to this norm are convex polytopes, whose vertices are on rays of Σ_Y. This is an important property of the norm that we will use in the proof of Lemma 3.5. We need X to be quasiprojective (which is implied by semiprojective) for such norm to exist. Otherwise, the balls are not necessarily convex, or not all rays of X are among the vertices of such a ball.

Since X is quasiprojective, every $\mathbb{Q}$-Cartier Weil divisor is a difference of nef divisors: $D = D^+ - D^-$, with both D^+ and D^- nef $\mathbb{Q}$-Cartier Weil [CLS11, Thm 6.3.22(a)]. Again, we use quasiprojective here. If not, the nef cone needs not to span the Picard lattice. Thus every such Cartier divisor on $X = \mathbb{TV}(\Sigma)$ is (non-uniquely) represented by a pair of polyhedra (Δ^+, Δ^-) sharing the same tail cone $|\Sigma|^\vee \subseteq M_\mathbb{R}$. Polyhedra form a semigroup under Minkowski addition. Restricting to polyhedra with a fixed tail cone, one ensures that this semigroup is cancellative. In this context, the pair (Δ^+, Δ^-) represents the formal difference $D = \Delta^+ - \Delta^-$ within the Grothendieck group of generalised polyhedra.

The goal of this section is to reinterpret the toric cohomology in terms of this pair of polyhedra.

Lemma 3.5. *Let X be a semiprojective toric variety with no torus factors and $D = D^+ - D^-$ be a $\mathbb{Q}$-Cartier Weil divisor on X with D^+ and D^- nef. Assume that Δ^+ and Δ^- are the associated polyhedra of D^+ and D^- and denote by u the support function of D. Then the sets*

$$V^{\mathrm{supp}}_{D,0} = \{a \in \operatorname{supp}\Sigma \mid u(a) > 0\} \subseteq \operatorname{supp}\Sigma$$

and $\Delta^- \setminus \Delta^+$ are homotopy equivalent.

Proof. Let $u^\pm$ be the support functions of the nef divisors $D^\pm$. For each full-dimensional $\sigma \in \Sigma$ we denote by $u^+_\sigma \in \Delta^+$ and $u^-_\sigma \in \Delta^-$ the unique vertices minimising $\langle a, \bullet\rangle$ on the respective polytopes for $a \in \operatorname{int}\sigma$, hence for all $a \in \sigma$. Note that to get the uniqueness of $u^\pm_\sigma$ we exploit that σ is full-dimensional, and implicitly, that the fan of X is generated by the cones of full-dimension, so that the collection of $u^\pm_\sigma$ determines $u^\pm$ and thus $D^\pm$. This property of X is guaranteed by the semiprojectivity. Thus, for $a \in \sigma$, we have $\min\langle a, \Delta^\pm\rangle = \langle a, u^\pm_\sigma\rangle = u^\pm(a)$. Moreover, we can write

$$V^{\mathrm{supp}}_{D,0} = \{a \in \operatorname{supp}\Sigma \mid u^-(a) < u^+(a)\}.$$

Since $V^{\mathrm{supp}}_{D,0} \subseteq N_\mathbb{R}$ and $\Delta^- \setminus \Delta^+ \subseteq M_\mathbb{R}$ are contained in mutually dual spaces, we are going to compare these two sets via the following incidence set:

$$W := \{(a, r) \in V^{\mathrm{supp}}_{D,0} \times (\Delta^- \setminus \Delta^+) \mid \langle a, r\rangle < u^+(a)\}.$$

It comes with two natural, surjective projections

$$\begin{tikzcd} & W \arrow[dl, "p_V"'] \arrow[dr, "p_\Delta"] & \\ N_\mathbb{R} \supseteq V^{\mathrm{supp}}_{D,0} & & \Delta^- \setminus \Delta^+ \subseteq M_\mathbb{R}, \end{tikzcd}$$

with contractible fibres: let us start with checking the map p_V. If $a \in V_{D,0}^{\mathrm{supp}}$, then there is a cone $\sigma \in \Sigma(d)$ containing a (we use here that Σ is generated by the cones of maximal dimension, as mentioned above). We obtain

$$p_V^{-1}(a) \cong \{r \in \Delta^- \setminus \Delta^+ \mid \langle a, r\rangle < \langle a, u_\sigma^+\rangle\} = \{r \in \Delta^- \mid \langle a, r\rangle < \langle a, u_\sigma^+\rangle\}.$$

Obviously, the latter is a convex set. However, it is non-empty, too, due to the fact that $a \in V_{D,0}^{\mathrm{supp}}$ (together with $a \in \sigma$) implies that $\min\langle a, \Delta^-\rangle < \langle a, u_\sigma^+\rangle$. We turn to the second map p_Δ. Fixing an element $r \in \Delta^- \setminus \Delta^+ \subseteq \Delta^-$ we have

$$p_\Delta^{-1}(r) \cong \{a \in V_{D,0}^{\mathrm{supp}} \mid \langle a, r\rangle < u^+(a)\} = \{a \in \operatorname{supp}\Sigma \mid \langle a, r\rangle < \min\langle a, \Delta^+\rangle\}.$$

Again, the latter is a convex set, because $\operatorname{supp}\Sigma$ is convex, which is guaranteed by semiprojectivity of X. Moreover, because the same latter set is non-empty because $r \notin \Delta^+$.

Now, the idea is to apply Corollary 2.9. For this purpose, we will replace the three objects in the above diagram with homotopy equivalent gadgets which are all compact. Recall the notions of sufficiently large truncation $\operatorname{Trunc}(\Delta^-)$ of Δ^- as in Proposition 2.7 and the ϵ-widening $(\Delta^+)_{>-\epsilon}$ as in Subsection 2.2. For any $R > 0$ and any sufficiently small $\epsilon > 0$ we consider the following three compact sets:

$$\begin{aligned}
V_{D,0}^{\mathrm{supp}}(R,\epsilon) &:= \{a \in \operatorname{supp}\Sigma \mid u(a) \geq \epsilon \text{ and } \|a\| \leq R\},\\
W(R,\epsilon) &:= \{(a,r) \in \operatorname{supp}\Sigma \times \operatorname{Trunc}(\Delta^-) \mid \\
&\qquad\qquad \langle a, r\rangle \leq u^+(a) - \epsilon,\ u(a) \geq \epsilon,\ \text{and } \|a\| \leq R\},\ \text{and}\\
&\operatorname{Trunc}(\Delta^-) \setminus (\Delta^+)_{>-\epsilon}.
\end{aligned}$$

By the results from Section 2 these are homotopy equivalent to $V_{D,0}$, W, and $\Delta^- \setminus \Delta^+$, respectively (see Lemma 2.4, Propositions 2.3 and 2.7). We need to carefully choose the inequalities used in the ϵ-widening so that the projection $W(R,\epsilon) \to \operatorname{Trunc}(\Delta^-) \setminus (\Delta^+)_{>-\epsilon}$ is well defined and surjective. This is possible due to the choice of the norm with nice properties as the introductory paragraphs of this subsection. Then with the same arguments as above we show that the fibres of projections $W(R,\epsilon) \to V_{D,0}^{\mathrm{supp}}(R,\epsilon)$ and $W(R,\epsilon) \to \operatorname{Trunc}(\Delta^-) \setminus (\Delta^+)_{>-\epsilon}$ are non-empty convex polytopes. Thus by the criterion of Smale (Corollary 2.9), the projection maps are homotopy equivalences, and consequently, $V_{D,0}$ is homotopy equivalent to $\Delta^- \setminus \Delta^+$. ☐

Theorem 3.6. *Let X be a semiprojective toric variety and $D = D^+ - D^-$ be a $\mathbb{Q}$-Cartier Weil divisor on X with D^+ and D^- nef. Denote by Δ^+*

and Δ^- *the polyhedra of* D^+ *and* D^-*, respectively. Then* $\mathrm{H}^i(X, \mathcal{O}(D)) = \bigoplus_{m\in M} \widetilde{\mathrm{H}}^{i-1}(\Delta^- \setminus (\Delta^+ - m), \Bbbk)$.

Proof. We will show that $\mathrm{H}^i(X, \mathcal{O}(D))_m = \widetilde{\mathrm{H}}^{i-1}(\Delta^- \setminus (\Delta^+ - m), \Bbbk)$. From Subsection 3.2 together with Lemma 3.5 we obtain this claim for $m = 0$.

For general $m \in M$ we define $D(m) := D + \mathrm{div}(x^m) = D + \sum_{a\in\Sigma(1)} \langle a, m\rangle \cdot D_a$. Compared with D, its associated sheaf is twisted by

$$\mathcal{O}_X(m) := \mathcal{O}_X\big(\mathrm{div}(x^m)\big) = x^{-m} \cdot \mathcal{O}_X.$$

Since the polyhedra $\Delta^\pm$ encode, for each affine chart, the minimal generators of the sheaves $\mathcal{O}_X(D^\pm)$, this means that the divisor $D(m)$ is represented by the pair $(\Delta^+ - m, \Delta^-)$ or, equivalently, by $(\Delta^+, \Delta^- + m)$. In particular, for the support functions we have $u^{D(m)} = u^D - m$. Thus, $V_{D(m),0} = V_{D,m}$. □

Remark 3.7. Note that the presentation of a toric divisor $D = D^+ - D^-$ as a difference of nef divisors is by far not unique. Thus, one of the consequences of Theorem 3.6 is that the reduced cohomology of the difference of polyhedra is independent of the choice of this presentation. In particular, choosing a suitable semiprojective toric variety X, for any three rational polyhedra $\Delta_0, \Delta_1, \Delta_2$ in $M_\mathbb{R}$ with the same tail cone, the differences $\Delta_1 \setminus \Delta_2$ and $(\Delta_1 + \Delta_0) \setminus (\Delta_2 + \Delta_0)$ are homotopy equivalent.

Remark 3.8. Suppose X is a projective toric variety and D is a $\mathbb{Q}$-Cartier Weil divisor on X. Observe that despite that there are at most finitely many degrees m for which $\mathrm{H}^i(\mathcal{O}(D))_m \neq 0$, in the definitions of $V_{D,m}$ and $V^{\mathrm{supp}}_{D,m}$ it is not immediately clear, which $m \in M$ can potentially lead to non-zero cohomology. Instead, the description in Theorem 3.6 provides such a criterion. If Δ^- and $\Delta^+ + m$ are disjoint, then the difference is contractible. We will elaborate more on this criterion in a follow up article about related computational issues.

Remark 3.9. Let the $\mathbb{Q}$-Cartier Weil divisor D be encoded by the pair of polyhedra (Δ^+, Δ^-). Then, the polyhedron of sections $\Delta(D)$ mentioned in Subsection 3.1 can be recovered as

$$\Delta(D) = \bigcap_{r\in\Delta^-} (\Delta^+ - r) = \{r \in M_\mathbb{R} \mid \Delta^- + r \subseteq \Delta^+\}.$$

Example 3.10. If D is a nef divisor on a projective toric variety X, one can choose $\Delta^- = \{0\}$, and then the formula from Remark 3.9 implies $\Delta = \Delta(D) = \Delta^+$. Thus for $r \in \Delta$ the set $\Delta^- \setminus (\Delta^+ - r)$ is empty (thus only has 1-

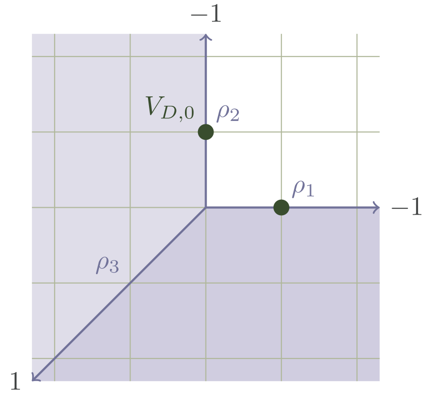
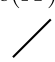
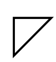


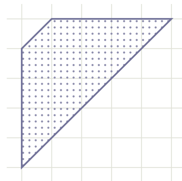
Figure 1: The fan of $X = \mathbb{P}^2 \setminus [0,0,1]$, the divisor $D = -D_{\rho_1} - D_{\rho_2} + D_{\rho_3} \simeq \mathcal{O}_X(-1)$, and the complex $V_{D,0}$ consisting of 2 points responsible for $\mathrm{H}^1(\mathcal{O}_X(D))_0 \neq 0$.

dimensional (-1)st cohomology), or for $r \notin \Delta$ the set $\Delta^- \setminus (\Delta^+ - r)$ is a single point, hence it has no reduced cohomology at all. Therefore, $h^0(X, \mathcal{O}(D)) = \#(\Delta \cap M)$ and $\mathrm{H}^i(X, \mathcal{O}(D)) = 0$ for all $i > 0$.

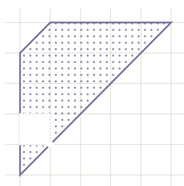
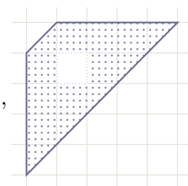
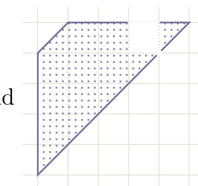
Example 3.11. If on the other hand $-D$ is a nef divisor, then $\Delta^+ = \{0\}$, and Δ^- is the polytope of $-D$. Let $i = \dim \Delta^-$. Thus for $r \in -\operatorname{relint} \Delta^-$ the set $\Delta^- \setminus (\Delta^+ - r)$ is homotopic to a sphere of dimension $i-1$, while for $r \notin -\operatorname{relint} \Delta^-$ the set $\Delta^- \setminus (\Delta^+ - r)$ is contractible. Therefore, $h^i(X, \mathcal{O}(D)) = \#(\operatorname{relint} \Delta^- \cap M)$ and $\mathrm{H}^j(X, \mathcal{O}(D)) = 0$ for all $j \neq i$.

Example 3.12. The claim of Theorem 3.6 does not hold for quasiprojective toric varieties, that are not semiprojective. To see this, consider $X = \mathbb{P}^2 \setminus \{[0,0,1]\}$ and let $D \simeq \mathcal{O}_X(-1)$ be the negative of the hyperplane divisor. Then $D^+ \simeq 0$ and $D^- \simeq \mathcal{O}_X(1)$, the polytopes are a point and a basic triangle, respectively. Thus $\Delta^- \setminus (\Delta^+ - m)$ is always non-empty and contractible, hence the difference never has any reduced cohomologies. But $\mathrm{H}^1(\mathcal{O}_X(-1)) \neq 0$ as shown on Figure 1.

Example 3.13. In the notation and coordinates of the "hexagon" example (Example 3.2), consider the divisor $D = (-4,-4,-2,1) \in \operatorname{Pic}(X)$. We have $D = D^+ - D^-$ for instance as $D^+ =$ $+$ and $D^- =$  $+\,4$ , where we represent the nef divisors with their polytopes. Thus we obtain

$\Delta^- =$  and $\Delta^+ =$  .

The only non-contractible differences of Δ^- and a shift $(\Delta^+ + m)$ are:

 ,  , and  .

Therefore $h^0(D) = 0$, $h^1(D) = 2$ and $h^2(D) = 1$.

Remark 3.14. If X is not quasiprojective then there might be no non-trivial nef line bundles at all. But if nevertheless D is a Cartier divisor which is a linear combination of nef divisors, then we can still express $D = D^+ - D^-$ with $D^\pm$ nef, and consider the polyhedra $\Delta^\pm$. If the fan of X has convex support and it is generated by cones of maximal dimension (we could call such X *semicomplete*, by analogy to semiprojective), then an analogous claim to Theorem 3.6 is true. To see that, one uses a surjective torus invariant morphism $X \to Y$, where Y is a semiprojective toric variety determined by the polytope $\Delta^+ + \Delta^-$. Then D is a pullback of a line bundle on Y, and argue similarly to the proofs of Proposition 4.5 and Corollary 4.6 to show that cohomology of D can be calculated on Y. We skip the details since non-quasiprojective toric varieties are not among our primary interest.

4. The immaculacy locus in Pic(X)

In this section we introduce the notion of an immaculate sheaf, concentrating on the case of line bundles. We also study a relative version of this notion, and how the immaculacy interacts with morphisms.

4.1. Immaculate line bundles

Recall, that a sheaf is called *acyclic*, if it has all higher cohomology groups equal to zero. We will also say that for a field $\Bbbk$, a topological space V is

$\Bbbk$-acyclic, if it is non-empty, arcwise connected, and its singular cohomologies $\mathrm{H}^i(V, \Bbbk) = 0$ vanish for all $i > 0$. Note that in such case $\mathrm{H}^0(V, \Bbbk) = \Bbbk$. For example, all non-empty contractible spaces are $\Bbbk$-acyclic (for any $\Bbbk$), and the real projective plane $\mathbb{RP}^2$ is $\Bbbk$-acyclic (except if $\operatorname{char} \Bbbk = 2$). Spheres S^k are never $\Bbbk$-acyclic.

Definition 4.1. We call a sheaf $\mathcal{F}$ on a variety X *immaculate* if all cohomology groups $\mathrm{H}^p(X, \mathcal{F})$ ($p \in \mathbb{Z}$) vanish. The difference from the usual notion of acyclic sheaves is that we ask for the vanishing of H^0, too.

Remark 4.2. In [PSP08] immaculate line bundles are called *"cohomologically trivial"*. Although descriptive, we find this terminology rather confusing. It could mean a line bundle, whose class in the cohomology is 0, such as the line bundle corresponding to a difference of two points on an algebraic curve of genus $g \geq 1$, analogously to the notion of "homologically trivial" cycle (or submanifold) in the context of Chow groups [Hul94, p. 132], or combinatorics [GOT18, Chapter 23], It could also mean a line bundle, whose cohomologies are isomorphic to the cohomologies of the trivial sheaf, $H^i(L) = H^i(\mathcal{O}_X)$ for all i, analogously to the notion of a "(co)homologically trivial" topological space (see for instance [ES52, Def. I.9.1 and I.9.1c]) or supermanifold (see for instance [BBHR91, Prop. V.3.1]). Also other interpretations are possible, and the discussions with our colleagues confirm that the phrase "cohomologically trivial" always requires a clarification. Moreover it is difficult to build further terminology on this notion, as we do in Section 5.

In particular, a toric sheaf of a $\mathbb{Q}$-Cartier Weil divisor $\mathcal{O}_X(D)$ is immaculate if and only if all sets $V_{D,m}$ are $\Bbbk$-acyclic. Equivalently, as an immediate consequence of Theorem 3.6, we can identify the immaculate line bundles in terms of properties of polyhedra Δ^+ and Δ^-.

Proposition 4.3. *Let $D = D^+ - D^-$ be a $\mathbb{Q}$-Cartier Weil divisor on a semiprojective toric variety X with D^+ and D^- nef. Denote the polyhedra of D^+ and D^- by Δ^+ and Δ^-, respectively. Then $\mathcal{O}_X(D)$ is immaculate if and only if for all $m \in M$ the set $\Delta^- \setminus (\Delta^+ - m) \subseteq M_{\mathbb{R}}$ is $\Bbbk$-acyclic.*

Example 4.4. With $D = D^+ - D^-$ and Δ^+ and Δ^- as in Proposition 4.3, if Δ^+ is a (lattice) point, say $\Delta^+ = \{0\}$, that is $D^+ = 0$, then $D = -D^-$ is immaculate if and only if Δ^- does not contain (relatively) interior lattice points at all. Note that this is supposed to exclude the case $\dim \Delta^- = 0$, too. See Theorem 4.13 for a generalisation.

Next, we discuss the behaviour of immaculacy with respect to some special morphisms. For a variety X we denote by $\mathbb{R}\Gamma_X$ the derived global sections

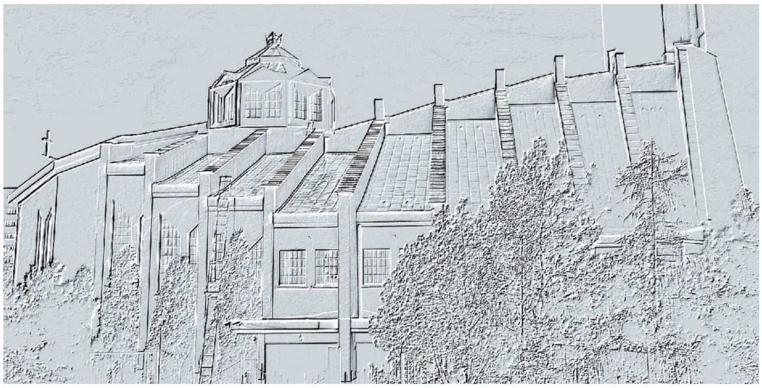


Figure 2: The church of the Immaculate Conception of Blessed Virgin Mary in Warsaw. The shape of the roof resembles an illustration of a line bundle.

functor for coherent sheaves on X. Thus, a sheaf $\mathcal{F}$ is immaculate if and only if $\mathbb{R}\Gamma_X(\mathcal{F}) = 0$ in the derived category $\mathcal{D}(X)$, that is, if $\mathbb{R}\Gamma_X(\mathcal{F})$ is exact. Similarly, for a morphism $p\colon X \to Y$ we denote by $\mathbb{R}p_*$ the derived push forward functor.

Proposition 4.5. *Suppose X and Y are algebraic varieties over $\Bbbk$ with Y normal, and $p\colon X \to Y$ is a surjective proper morphism with connected fibres such that $R^i p_* \mathcal{O}_X = 0$ for $i > 0$, that is, $\mathbb{R}p_*\mathcal{O}_X = \mathcal{O}_Y$. Assume $\mathcal{E}$ is a locally free sheaf on Y. Then $\mathcal{E}$ is immaculate if and only if $p^*\mathcal{E}$ is immaculate on X.*

Proof. This follows from $\mathbb{R}\Gamma_Y\mathcal{E} = \mathbb{R}\Gamma_Y(\mathbb{R}p_*p^*\mathcal{E}) = \mathbb{R}\Gamma_X(p^*\mathcal{E})$. □

The assumptions of Proposition 4.5 are satisfied in the typical setting of the morphisms arising from the Minimal Model Program.

Corollary 4.6. *Suppose X and Y are toric varieties, and $p\colon X \to Y$ is a surjective toric projective morphism with connected fibres, that is a toric projective morphism corresponding to a surjective map of one-parameter subgroups lattices $N_X \to N_Y$. Assume $\mathcal{E}$ is a locally free sheaf on Y. Then $\mathcal{E}$ is immaculate if and only if $p^*\mathcal{E}$ is immaculate.*

Proof. To apply Proposition 4.5 we must ensure that $R^i p_*\mathcal{O}_X = 0$ for $i > 0$. For this, we may assume that Y is affine and have to check that $\mathrm{H}^i(\mathcal{O}_X) = 0$. Since p is projective, the support of the fan Σ_X of X is a convex cone. Thus for $m \in M_X$ the m-th grading of $\mathrm{H}^i(\mathcal{O}_X)$ is calculated by $V_{0,m}^{\mathrm{supp}} = \{a \in \operatorname{supp}\Sigma_X \mid \langle a, m\rangle < 0\}$ which is convex, hence either contractible or empty. □

Example 4.7. In the notation of Example 3.2 (“hexagon”), consider the divisor

$$D = (-2, -2, -2, -2) = -2$$  .

It is a pullback of the immaculate line bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ under the blow-down map to $\mathbb{P}^2$ (contracting three disjoint exceptional divisors), thus D is also immaculate.

4.2. Relative immaculacy and affine spaces of immaculate line bundles

The main goal of this subsection is to explain the occurrence of some infinite families of immaculate line bundles. For this we present a more restrictive notion than plain immaculacy, which leads to a construction of such families.

Definition 4.8. Suppose $p\colon X \to Y$ is a morphism of algebraic varieties. We say that a sheaf $\mathcal{F}$ on X is *p-immaculate* if the direct image sheaves $R^i p_* \mathcal{F}$ vanish in all cohomological degrees $i \in \mathbb{Z}$, that is, if $\mathbb{R}p_*\mathcal{F} = 0$.

Clearly, a sheaf on X is immaculate if and only if it is p-immaculate for the map $p\colon X \to \{*\}$. Moreover, for any map $p\colon X \to Y$, the equality $\mathbb{R}\Gamma_X = \mathbb{R}\Gamma_Y \circ \mathbb{R}p_*$ implies that each p-immaculate sheaf is automatically immaculate. And, finally, it is a consequence of cohomology and base change that for a flat morphism the relative immaculacy of locally free sheaves can be checked fiberwise:

Proposition 4.9. *Suppose that $p\colon X \to Y$ is a flat proper morphism of algebraic varieties. Let $\mathcal{E}$ be a locally free sheaf on X and for $y \in Y$ denote by $X_y := f^{-1}(y)$ the fibre of y. Then $\mathcal{E}$ is p-immaculate if and only if $\mathcal{E}_y := \mathcal{E}|_{X_y}$ is immaculate for every closed point y.*

Proof. If $\mathcal{E}|_{X_y}$ is immaculate for every closed point y, then the functions $y \mapsto \dim \mathrm{H}^i(X_y, \mathcal{E}|_{X_y})$ are constantly equal to 0, on closed points. Hence, by semicontinuity [Mum08, Cor. 1 in Sect. 5, p. 50], they are also zero on non-closed points. Thus by the implication (i) $\Longrightarrow$ (ii) in [Mum08, Cor. 2 in Sect. 5, pp. 50–51], the sheaf $R^i p_*(\mathcal{E})$ is locally free and zero at every point of Y, that is $R^i p_*(\mathcal{E}) = 0$.

If $\mathcal{E}$ is p-immaculate, then for sufficiently large i the equivalent conditions of [Mum08, Cor. 2 in Sect. 5, pp. 50–51] are satisfied, hence by the last paragraph of that corollary the map $R^{i-1}p_*(\mathcal{E}) \otimes \kappa(y) \to \mathrm{H}^{i-1}(X_y, \mathcal{E}|_{X_y})$ is an isomorphism. Moreover, $R^{i-1}p_*(\mathcal{E}) = 0$, hence the condition (ii) is satisfied for a smaller value of i and hence also condition (i) is satisfied. Going down with i, we eventually get the claim. $\square$

Proposition 4.10. *Suppose that X and Y are varieties and $p\colon X \to Y$ is a morphism. Assume that $\mathcal{F}$ is a p-immaculate coherent sheaf on X. Then, for any locally free sheaf $\mathcal{E}$ on Y, the sheaf $\mathcal{F} \otimes p^*\mathcal{E}$ is p-immaculate, hence immaculate.*

Proof. The projection formula implies $R^i p_*(\mathcal{F} \otimes p^*\mathcal{E}) = R^i p_*\mathcal{F} \otimes \mathcal{E}$ and the latter is zero by the definition of a p-immaculate sheaf. $\square$

While the previous claims followed from rather standard arguments, it is quite nice that, in the projective setting, also the converse of the above statement holds true:

Theorem 4.11. *Suppose X and Y are varieties, $p\colon X \to Y$ is a morphism, and Y is projective. Assume $\mathcal{F}$ is a coherent sheaf on X, and L is an ample line bundle on Y. Then the following conditions are equivalent.*

(a) $\mathcal{F}$ is p-immaculate,
(b) for any Cartier divisor D on Y the sheaf $\mathcal{F} \otimes \mathcal{O}_X(p^*D)$ is immaculate,
(c) for infinitely many integers $k > 0$ the sheaf $\mathcal{F} \otimes p^*L^{\otimes k}$ is immaculate.
(d) for any locally free sheaf $\mathcal{E}$ on Y the sheaf $\mathcal{F} \otimes p^*\mathcal{E}$ is immaculate,

Proof. The implication (a) $\Longrightarrow$ (d) is shown in Proposition 4.10. The implications (d) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are clear. Thus we only have to show (c) $\Longrightarrow$ (a).

By the derived projection formula $(\mathbb{R}p_*\mathcal{F}) \otimes L^{\otimes k} \simeq \mathbb{R}p_*(\mathcal{F} \otimes p^*L^{\otimes k})$ in $\mathcal{D}(Y)$. Applying the derived global sections functor $\mathbb{R}\Gamma_Y$ we obtain that

$$\mathbb{R}\Gamma_Y(\mathbb{R}p_*\mathcal{F} \otimes L^{\otimes k})) \overset{\text{q.is.}}{\simeq} (\mathbb{R}\Gamma_Y \circ \mathbb{R}p_*)(\mathcal{F} \otimes p^*L^{\otimes k}) = \mathbb{R}\Gamma_X(\mathcal{F} \otimes L^{\otimes k}) = 0$$

by our assumption in (c). The entries in the second table, that is, in the E_2 layer of the spectral sequence for $\mathbb{R}\Gamma_Y(\mathbb{R}p_*\mathcal{F} \otimes L^{\otimes k}))$ are $\mathrm{H}^i(Y, R^j p_*\mathcal{F} \otimes L^{\otimes k})$ for varying i, j.

By Serre vanishing and our assumptions, for infinitely many sufficiently large k, we have $\mathrm{H}^i(R^j p_*\mathcal{F} \otimes L^{\otimes k}) = 0$ for all $i > 0$ and all j. Hence for such k the spectral sequence stabilises immediately and thus (since it converges to 0) also the H^0 row is identically zero. That is $\mathrm{H}^0(R^j p_*\mathcal{F} \otimes L^{\otimes k}) = 0$ for infinitely many sufficiently large k. Again by Theorem of Serre [Har77, Thm 5.17] the coherent sheaves $R^j p_*\mathcal{F}$ are identically zero, which is the content of (a). $\square$

We now switch our attention back to toric varieties. Our goal is to reinterpret p-immaculacy and apply Theorem 4.11 in terms of toric geometry. The following statement captures our main reason to study the cohomology of divisors on semiprojective varieties, despite that we are principally interested in projective varieties. For a projective toric morphism $X \to Y$, we can restrict to an open affine subset of Y, and our theory still works, despite that we no longer live in the projective world. Technically, the following characterisation of p-immaculacy differs from the characterisation of plain immaculacy in Proposition 4.3 just by enlarging the tail cones.

Proposition 4.12. *Suppose $p\colon X \to Y$ is a toric map of semiprojective toric varieties, and let $p^*\colon M_Y \to M_X$ be the corresponding map of monomial lattices. Let D be a $\mathbb{Q}$-Cartier Weil divisor on X and write $D = D^+ - D^-$ as a difference of nef divisors, as usual. Then $\mathcal{O}_X(D)$ is p-immaculate if and only if for all maximal cones σ in the fan of Y and for all $m \in M_X$ the difference $(\Delta^- + p^*(\sigma^\vee)) \setminus (\Delta^+ + p^*(\sigma^\vee) - m)$ is $\Bbbk$-acyclic.*

Proof. Let Σ_Y be the fan of Y and for a maximal cone $\sigma \in \Sigma(\dim Y)$ denote by U_σ the open affine subset of Y corresponding to σ. By [Har77, Prop. III.8.5] the sheaf $\mathcal{O}_X(D)$ is p-immaculate if and only if the cohomology groups $\mathrm{H}^i(\mathcal{O}_{p^{-1}(U_\sigma)}(D)) = 0$ for all i and for all $\sigma \in \Sigma_Y(\dim Y)$. Equivalently, for all σ the restriction of D to $p^{-1}(U_\sigma)$ is immaculate. The restriction of D^+ to $p^{-1}(U_\sigma)$ is still nef and the polyhedron of the restriction is equal to $\Delta^+ + p^*(\sigma^\vee)$. Analogous statements hold for D^- and Δ^-. Therefore, the claim follows from Proposition 4.3 applied to each $p^{-1}(U_\sigma)$ separately. $\square$

A sublattice $M' \subset M$ is *saturated* if $M \cap M'_{\mathbb{R}} = M'$ (the intersection is taken in $M_{\mathbb{R}}$). For a non-empty rational polyhedron $\Delta \subset M_{\mathbb{R}}$ define its *linear sublattice span* to be the smallest saturated sublattice $M' \subset M$ such that $M'_{\mathbb{R}}$ contains a translate of Δ. Equivalently, choosing a translate of Δ that contains 0, the vector space $M'_{\mathbb{R}}$ coincides with the $\mathbb{R}$-linear span of such translate, and $M' = M'_{\mathbb{R}} \cap M$. In particular, $\Delta \subset m + M'_{\mathbb{R}}$ for any $m \in \Delta$, and $\dim \Delta = \dim M'$. Note that (if Δ has no integral points) there could be no integral translate of Δ containing 0.

The following theorem can be interpreted as a relative version of Example 4.4.

Theorem 4.13. *Assume X is a projective toric variety, D^- is a nef $\mathbb{Q}$-Cartier Weil divisor, and D' is a nef Cartier divisor on X. Suppose Δ^- and Δ' are their respective polytopes, and let $M' \subset M$ be the linear sublattice span of Δ'. Let Y be the projective toric variety corresponding to Δ' and $p\colon X \to Y$ be the natural map of toric varieties. Then the following conditions are equivalent.*

(1) The divisor $-D^-$ is p-immaculate.
(2) For all integers a the divisors $aD' - D^-$ are immaculate on X.
(3) For infinitely many integers a the divisors $aD' - D^-$ are immaculate.

(4) The image of Δ^- under the projection

$$\varphi\colon M_{\mathbb{R}} \to {M_{\mathbb{R}}}\big/{M'_{\mathbb{R}}} = \left({M}\big/{M'}\right) \otimes \mathbb{R}$$

has no lattice points in the relative interior.

Items (1), (2), and (3) should be seen as parallel to, respectively, (a), (b) and (c) of Theorem 4.11. Note that in (c) we only allow twists by positive powers of the ample line bundle L, whereas in the toric setting as in (3) it is also possible to use anti-ample twists.

Proof. The implication (2) $\Longrightarrow$ (3) is clear.

To show (3) $\Longrightarrow$ (4) we consider two cases, *positive* or *negative*. That is, among the integers a such that $D_a := aD' - D^-$ is immaculate, there exists a subsequence either of positive a_i converging to $+\infty$ or of negative a_i converging to $-\infty$.

In the positive case, suppose by contradiction, that there exist an interior lattice point of $\varphi(\Delta^-)$. Replacing Δ^- with its translate (and D^- with a linearly equivalent divisor) if necessary, we may assume that, say, $0 \in \operatorname{relint} \varphi(\Delta^-)$. Choosing a subsequence if necessary, assume that every $|a_i|\Delta'$ has a lattice point $m_i \in M$ in the relative interior such that the distance (with respect to any fixed norm on $M_{\mathbb{R}}$) of m_i to the boundary $\partial(|a_i|\Delta')$ converges to $+\infty$. A nef decomposition of $D_{a_i} = a_iD' - D^-$ is exactly $D^+_{a_i} = a_iD'$ and $D^-_{a_i} = D^-$. By Proposition 4.3 for any i the difference $\Delta^- \setminus (a_i\Delta' - m_i)$ is $\Bbbk$-acyclic. Since Δ^- is compact, taking a_i very large we have $\Delta^- \setminus (a_i\Delta' - m_i) = \Delta^- \setminus M'_{\mathbb{R}}$. By Corollary 2.9, the restricted projection map $\varphi\colon \Delta^- \setminus M'_{\mathbb{R}} \to \varphi(\Delta') \setminus \{0\}$ is a homotopy equivalence, a contradiction, since the first one $\Delta^- \setminus M'_{\mathbb{R}}$ is $\Bbbk$-acyclic, and the latter one $\varphi(\Delta^-) \setminus \{0\}$ is either homeomorphic to a sphere (if $\dim \varphi(\Delta^-) > 0$) or empty (if $\varphi(\Delta^-) = \{0\}$).

In the negative case, the nef decomposition of D_{a_i} is $D^+_{a_i} = 0$ and $D^-_{a_i} = D^- - a_iD'$. By Example 4.4 for any a_i the Minkowski sum $\Delta^- + |a_i|\Delta'$ has no lattice points in the relative interior. Taking $|a_i|$ very large, we see that there are no lattice points in the relative interior of $\Delta^- + M'_{\mathbb{R}}$. Equivalently, there is no (relative) interior lattice point in $\varphi(\Delta^-)$. This concludes the proof of (3) $\Longrightarrow$ (4).

Next we prove (4) $\Longrightarrow$ (2). Assume (by shifting Δ' if necessary) that $\Delta' \subseteq M'_{\mathbb{R}}$, that is, that $\varphi(\Delta')$ equals $0 \in {M}\big/{M'}$. Assume a is a non-negative integer. We must show that

- the Minkowski sum $\Delta^- + a\Delta'$ has no interior lattice points (hence $-D^- - aD'$ is immaculate), and

- $\Delta^- \setminus (a\Delta' - m)$ is contractible and non-empty for all $m \in M$ (hence $-D^- + aD'$ has no cohomology in degree m).

The first claim is straightforward: Such an interior lattice point would be mapped to an interior lattice point of $\varphi(\Delta^-)$, which is impossible by the assumptions of (4). Also the second claim is easy. Let $P := (a\Delta' - m) \cap \Delta^-$, which is a convex set contained in Δ^- such that $\Delta^- \setminus (a\Delta' - m) = \Delta^- \setminus P$. Since Δ' is contained in $M'_{\mathbb{R}}$, also $a\Delta' \subset M'_{\mathbb{R}}$, and consequently $P \subset M'_{\mathbb{R}} - m$. If $\varphi(-m) = \varphi(P) \notin \varphi(\Delta^-)$, then P is disjoint with Δ^- and $\Delta^- \setminus P = \Delta^-$, which is contractible and non-empty as claimed. Since $\varphi(\Delta^-)$ has no interior lattice points, it remains to consider $\varphi(-m) \in \partial(\varphi(\Delta^-))$ and, consequently, $P \subset \partial\Delta^-$. So the difference is non-empty and by Corollary 2.6 it is homotopic to $\partial\Delta^- \setminus P$, which is contractible (a sphere with a convex disc taken out).

To show (2) $\iff$ (1) note that $D' = p^*L$ for an ample line bundle L on Y. We apply the implications (c) $\implies$ (a) $\implies$ (b) of Theorem 4.11. $\square$

The following examples obey the notation of Theorem 4.13.

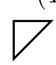
Example 4.14. If Δ' is full dimensional, that is if D' is big, then there is no antinef divisor which is p-immaculate, as in this case, M' is the whole lattice M, and $\varphi(\Delta^-)$ is a point, thus having an interior lattice point by definition.

Example 4.15. If Δ' is just a point, then $M' = 0$ and the question becomes whether Δ^- contains any interior lattice points, as already discussed in Example 4.4.

Example 4.16. If Δ' has codimension one, then M' is a hyperplane. The divisor $-D^-$ is p-immaculate if Δ^- cannot be divided by integral shifts of M'. In case D^- is in addition Cartier, this is equivalent to

$$\max\langle \Delta^-, M' \rangle - \min\langle \Delta^-, M' \rangle \leq 1,$$

where we think of the hyperplane $M' \subset M$ as a primitive element of N dual to the hyperplane.

Example 4.17. In the hexagon case (Example 3.2), let $D' = (1, 1, 0, 0)$ so that $\Delta' =$ , and let $D^- = (1, 1, 1, 0)$ so that $\Delta^- =$ . Then all combinations $aD' - D^- = (a-1, a-1, -1, 0)$ are immaculate. Other lines of immaculate divisors on this surface are listed in Table 2 in Section 9.

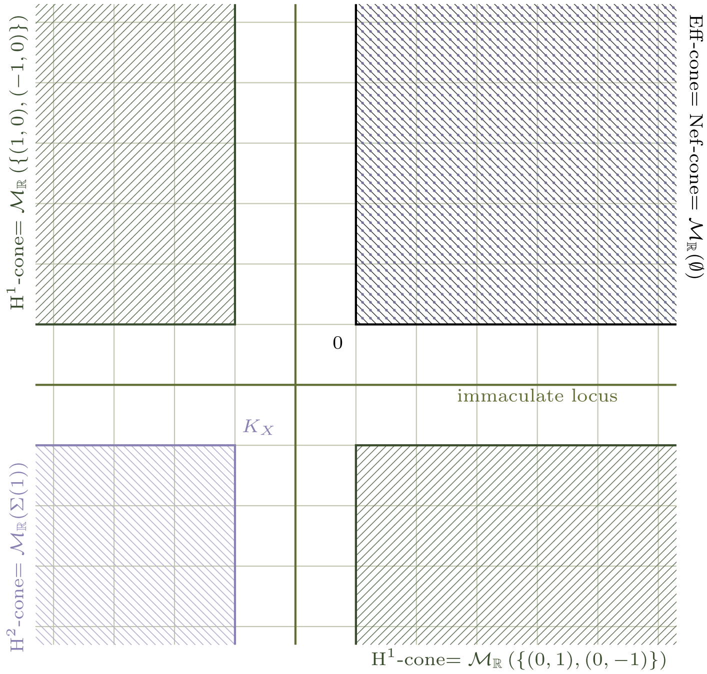


Figure 3: The Picard lattice of the surface $\mathbb{P}^1 \times \mathbb{P}^1$. The effective cone Eff is the cone of divisors with non-zero H^0 and it coincides with the Nef-cone. There are two cones of divisors with non-zero H^1, and one cone with non-zero H^2. The remaining line bundles are immaculate, and the immaculate locus consist of two lines parallel to the common facets of the Nef- and Eff-cones. These two lines correspond to the two projections to $\mathbb{P}^1$. The notation $\mathcal{M}_{\mathbb{R}}(\bullet)$ is explained in Section 5.1.

5. Immaculacy by avoiding temptations

Let us compare three examples of smooth projective varieties with Picard rank 2: the product projective space $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_1$ and the flag variety $\mathbb{F}(1,2;3) := \{(p,\ell) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \mid p \in L\}$. Note that the first one is simultaneously a toric variety and a homogeneous space (for the semisimple group $\mathrm{SL}_2 \times \mathrm{SL}_2$), the second is a toric variety, while the third one is a homogeneous space for the simple group SL_3. Figures 3, 4, 5 illustrate the Picard lattices of these examples, indicating the regions of line bundles with non-trivial cohomologies.

For homogeneous spaces, the regions for various H^i are disjoint, that is, for every line bundle L there is at most one value of i, such that $\mathrm{H}^i(L) \neq 0$,

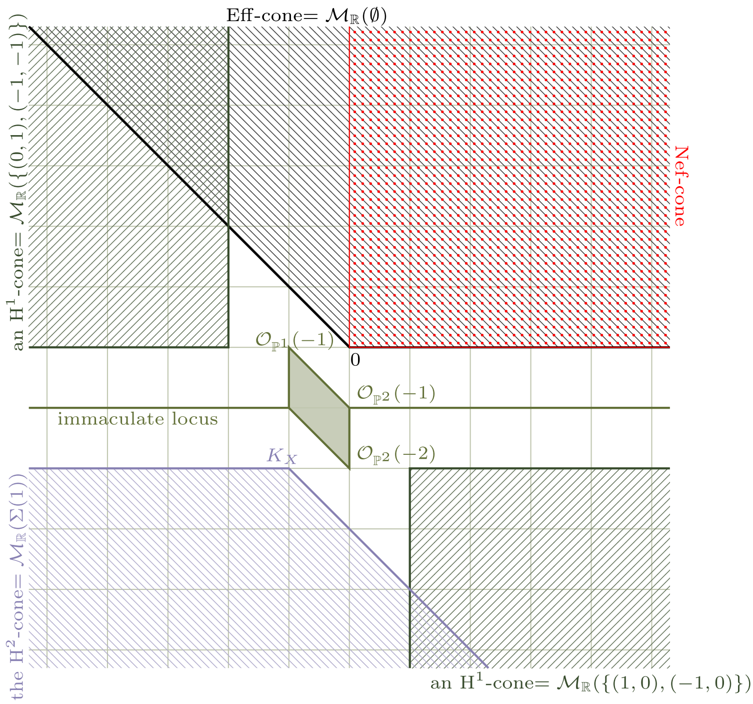


Figure 4: The Picard lattice of the Hirzebruch surface $\mathbb{F}_1 = \mathbb{TV}(\Sigma)$, where Σ has rays $\Sigma(1) = \{(0,1),(-1,-1),(1,0),(-1,0)\}$. The effective cone Eff is the cone of divisors with non-zero H^0. There are two cones of divisors with non-zero H^1, and one cone with non-zero H^2. In addition the Nef-cone is marked; it is a proper subset of the Eff-cone. The remaining line bundles are immaculate, and the immaculate locus consist of a bounded polytope and a line parallel to the unique common facet of the Nef- and Eff-cones. This common facet $\mathbb{R}_{\geq 0}\cdot(1,0)$ corresponds to the fibration $p\colon \mathbb{F}_1 \to \mathbb{P}^1$. The other face $\mathbb{R}_{\geq 0}\cdot(0,1)$ of the Nef-cone corresponds to the blow-down map $\mathrm{bl}\colon \mathbb{F}_1 \to \mathbb{P}^2$. The line bundle $\mathrm{bl}^*\,\mathcal{O}_{\mathbb{P}^2}(-1)$ is p-immaculate, hence Theorem 4.13 explains the line of immaculate divisors. The line bundles $\mathrm{bl}^*\,\mathcal{O}_{\mathbb{P}^2}(-2)$ and $p^*\mathcal{O}_{\mathbb{P}^1}(-1)$ are immaculate by Corollary 4.6. For clarity the figure omits p^* and bl^* in the names of line bundles.

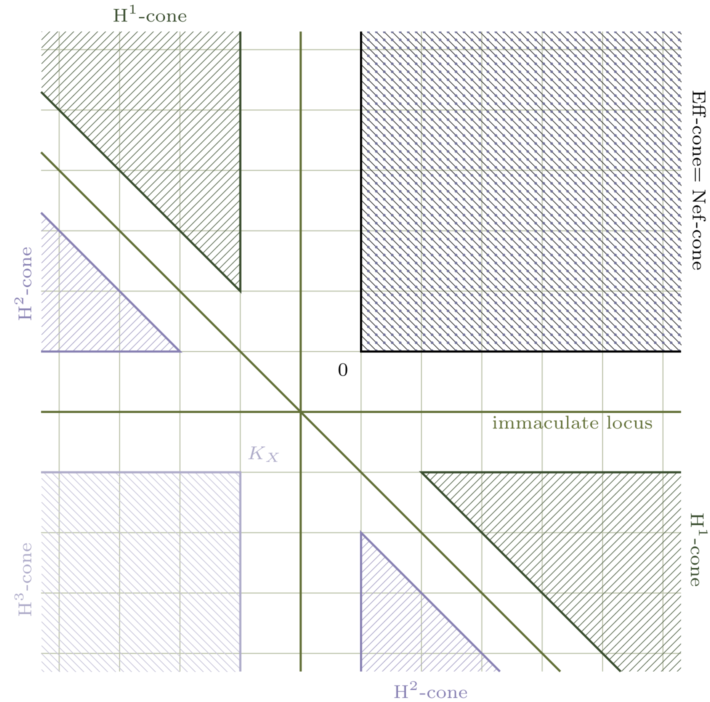


Figure 5: The Picard lattice of the threefold flag variety $\mathbb{F}(1,2;3)$. The effective cone Eff is the cone of divisors with non-zero H^0 and it coincides with Nef-cone. There are two cones of divisors with non-zero H^1, two cones with non-zero H^2, and one cone with non-zero H^3. The remaining line bundles are immaculate, and the immaculate locus consists of three lines. The horizontal and vertical lines are parallel to the faces of Nef- and Eff-cones, and these two lines arise using Theorem 4.11(b) for the two Mori fibrations $\mathbb{F}(1,2;3) \to \mathbb{P}^2$ and $\mathbb{F}(1,2;3) \to (\mathbb{P}^2)^*$ of the flag variety. Informally, since there are no more interesting contractions of $\mathbb{F}(1,2;3)$, the third line ℓ, the diagonal, does not arise with this method. More strictly, suppose by contradiction that ℓ consists of line bundles of type $\mathcal{O}_{\mathbb{F}(1,2;3)}(E) \otimes p^*\mathcal{O}_Y(D)$ for some map $p\colon \mathbb{F}(1,2;3) \to Y$ and a p-immaculate line bundle E as in Theorem 4.11(b). Then $p^* \operatorname{Pic} Y$ is the line through 0 parallel to ℓ, but it does not contain non-zero effective divisors, a contradiction.

see for instance [Kos61, Thm 5.14]. For toric varieties this is not necessarily the case. As illustrated by the $\mathbb{F}_1$ example, the regions may intersect. The goal of this section is to show how to obtain these regions of line bundles with various cohomologies for any toric variety.

5.1. Temptations

Let $X = \mathbb{TV}(\Sigma)$ be a toric variety with no torus factors. The lack of torus factors is intended to simplify statements and proofs, as claims about cohomologies of line bundles for general case (with torus factors) can be easily deduced from the simpler case: first any line bundle $\tilde{L}$ on $X \times (\mathbb{G}_{\mathfrak{m}})^t$ is isomorphic to a pullback of a line bundle L on X, and secondly, $H^i(\tilde{L}) = H^i(L) \otimes \Bbbk[(\mathbb{G}_{\mathfrak{m}})^t]$ by the Künneth formula [Kem93, Prop. 9.2.4]. However, since the map ρ^* of (3.1) is not injective for toric varieties with torus factors, the statements and proofs become more complicated and less clear.

For any subset $\mathcal{R} \subseteq \Sigma(1)$ we define $V^{>}(\mathcal{R}) \subset N_{\mathbb{R}}$, similar to $V^{>}_{D,0}$ as in Section 3.2:

$$V^{>}(\mathcal{R}) := \mathbb{R}_{>0} \cdot \left(\bigcup_{\sigma \in \Sigma} \operatorname{conv}(\mathcal{R} \cap \sigma(1)) \right).$$

Moreover define $V^{\geq}(\mathcal{R})$ as the complex of cones $\{\operatorname{cone}(\mathcal{R} \cap \sigma(1)) \mid \sigma \in \Sigma\}$ in $N_{\mathbb{R}}$, so that

$$\operatorname{supp} V^{\geq}(\mathcal{R}) = V^{>}(\mathcal{R}) \cup \{0\}.$$

In fact, $V^{>}(\mathcal{R}) = V^{>}_{-\sum_{\rho \in \mathcal{R}} D_\rho, 0}$ and analogously for $V^{\geq}$. Thus, as in Section 3.2, if Σ is in addition simplicial, then $V^{\geq}(\mathcal{R})$ is the full ("induced") subcomplex of Σ generated by $\mathcal{R}$. This notion has an analogous function as that of "$\operatorname{supp}(\mathbf{r})$" in [BH09, Sect. 4].

Definition 5.1. We call $\mathcal{R} \subseteq \Sigma(1)$ *tempting* if the geometric realisation $V^{>}(\mathcal{R})$ of $V^{\geq}(\mathcal{R}) \setminus \{0\}$ admits some reduced cohomology, that is if it is not $\Bbbk$-acyclic.

Example 5.2. Following up with our "hexagon" example (see notation in Example 3.2), the fan Σ of this surface has the following 34 tempting subsets $\mathcal{R} \subseteq \Sigma(1)$:

$$\begin{gathered}\emptyset, \{0,2\}, \{0,3\}, \{0,4\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}, \{0,1,3\},\\ \{0,1,4\}, \{0,2,3\}, \{0,2,4\}, \{0,2,5\}, \{0,3,4\}, \{0,3,5\}, \{1,2,4\}, \{1,2,5\},\\ \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{0,1,2,4\}, \{0,1,3,4\},\\ \{0,1,3,5\}, \{0,2,3,4\}, \{0,2,3,5\}, \{0,2,4,5\}, \{1,2,3,5\}, \{1,2,4,5\},\\ \{1,3,4,5\}, \{0,1,2,3,4,5\}.\end{gathered}$$

As in Section 3.1 we denote both natural maps $\mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X)$ and $\mathbb{R}^{\Sigma(1)} \to \mathrm{Cl}(X) \otimes \mathbb{R}$ by π.

Definition 5.3. Let $\mathcal{R} \subseteq \Sigma(1)$ be a subset. Then, we denote the images

$$\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) := \pi\left(\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}\right),$$
$$\mathcal{M}_{\mathbb{R}}(\mathcal{R}) := \pi\left(\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq -1}^{\mathcal{R}}\right).$$

If $\mathcal{R}$ is tempting as defined above, then $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ is called the $\mathcal{R}$-*maculate set* of $\mathrm{Cl}(X)$, respectively, $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ is the $\mathcal{R}$-*maculate region* of $\mathrm{Cl}(X) \otimes \mathbb{R}$.

Remark 5.4. Suppose that the fan Σ is complete. The empty set $\mathcal{R} = \emptyset$ yields $\mathcal{M}_{\mathbb{R}}(\emptyset) = \mathrm{Eff}(X)$. Moreover, Alexander duality implies that switching between $\mathcal{R}$ and $\Sigma(1)\setminus\mathcal{R}$ does not change the temptation status. After applying $\mathcal{M}$, the relation between the subsets $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ and $\mathcal{M}_{\mathbb{Z}}(\Sigma(1) \setminus \mathcal{R})$ of $\mathrm{Cl}(X)$ becomes Serre duality in $X = \mathbb{TV}(\Sigma)$.

Proof. Denote by $\sim$ the homotopy equivalence of topological spaces. Define the set $V(\mathcal{R}) = \bigcup_{\sigma\in\Sigma} \mathrm{conv}(\mathcal{R} \cap \sigma(1))$, which is a support of a polytopal complex (in particular, a compact CW-complex) and $V(\mathcal{R}) \sim V^{>}(\mathcal{R})$. Moreover, $V(\Sigma(1))$ is homeomorphic to a sphere S^{d-1} and $V(\mathcal{R}) \subset V(\Sigma(1))$. We show inductively that $V(\Sigma(1)) \setminus V(\mathcal{R}) \sim V(\Sigma(1) \setminus \mathcal{R})$ by removing one cone at a time.

Pick two subsets $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \Sigma(1)$ and a cone $\sigma \in \Sigma$. Let $\sigma_1 = \mathrm{cone}(\sigma(1) \cap \mathcal{R}_1)$. We claim $V(\mathcal{R}_2)\setminus V(\mathcal{R}_1) \sim V(\mathcal{R}_2\setminus\sigma_1(1))\setminus V(\mathcal{R}_1\setminus\sigma_1(1))$. After repeating this for all cones σ we eventually obtain $V(\mathcal{R}_2)\setminus V(\mathcal{R}_1) \sim V(\mathcal{R}_2\setminus\mathcal{R}_1)\setminus V(\emptyset) = V(\mathcal{R}_2 \setminus \mathcal{R}_1)$.

To show the claim we use the strong deformation retracts leading to $V(\mathcal{R}_i) \setminus \sigma_1 \sim V(\mathcal{R}_i \setminus \sigma_1(1))$ from Lemma 2.5. The standardness of the retracts discussed in Section 2.1 implies that they restrict well to a (strong deformation) retract of $V(\mathcal{R}_2) \setminus V(\mathcal{R}_1)$ onto $V(\mathcal{R}_2 \setminus \sigma_1(1)) \setminus V(\mathcal{R}_1 \setminus \sigma_1(1))$.

The Alexander duality [Hat02, Cor. 3.45] shows that

$$\widetilde{\mathrm{H}}^i(V(\mathcal{R}), \mathbb{Z}) = \widetilde{\mathrm{H}}_{d-i-2}(V(\Sigma(1)) \setminus V(\mathcal{R}), \mathbb{Z}) = \widetilde{\mathrm{H}}_{d-i-2}(V(\Sigma(1) \setminus \mathcal{R}), \mathbb{Z})$$

(the latter equality follows from the homotopy equivalence shown above). The universal coefficient theorems for homologies (see for instance [Spa66, Thm 5.2.8 and Lem. 5.2.5] or [Hat02, Thm 3A.3]) and cohomologies (see for instance [Spa66, Thm 5.5.3 and Cor. 5.5.4] or [Hat02, Thm 3.2]) together with Tor- and Ext-freeness of modules over $\Bbbk$ (that is, vector spaces) show that:

$$\widetilde{\mathrm{H}}^i\left(V(\mathcal{R}),\Bbbk\right)=\widetilde{\mathrm{H}}_i\left(V(\mathcal{R}),\Bbbk\right)^*=\left(\widetilde{\mathrm{H}}_i(V(\mathcal{R}),\mathbb{Z})\otimes\Bbbk\right)^*=\widetilde{\mathrm{H}}^i\left(V(\mathcal{R}),\mathbb{Z}\right)\otimes\Bbbk$$

and analogously for $V(\Sigma(1)\setminus\mathcal{R})$. Summarising:

$$\widetilde{\mathrm{H}}^i\left(V(\mathcal{R}),\Bbbk\right)=\widetilde{\mathrm{H}}_{d-i-2}\left(V(\Sigma(1)\setminus\mathcal{R}),\Bbbk\right)=\widetilde{\mathrm{H}}^{d-i-2}\left(V(\Sigma(1)\setminus\mathcal{R}),\Bbbk\right)^*,$$

hence $\mathcal{R}$ is tempting if and only if $\Sigma(1)\setminus\mathcal{R}$ is tempting.

The reinterpretation in terms of Serre duality in the projective case is the following: the canonical divisor on X is $K_X=-\sum_\rho D_\rho$, and the Serre dual divisor to $D:=\sum_\rho a_\rho D_\rho$ is $K_X-D=\sum_\rho(-1-a_\rho)D_\rho$. Thus the duality swaps the sets $\mathcal{M}_\mathbb{Z}(\mathcal{R})$ and $\mathcal{M}_\mathbb{Z}(\Sigma(1)\setminus\mathcal{R})$ and their cohomologies are dual one to the other. □

Remark 5.5. In [Efi14, Prop. 4.1] *forbidden sets* K_I are defined for tempting subsets $I\subset\Sigma(1)$. They correspond to the $\mathcal{R}$-maculate sets $\mathcal{M}_\mathbb{Z}(\mathcal{R})$.

The integral sets $\mathcal{M}_\mathbb{Z}(\mathcal{R})\subseteq\mathrm{Cl}(X)$ reflect more precisely the properties we need, but the real regions $\mathcal{M}_\mathbb{R}(\mathcal{R})$ are easier to control and they already contain a lot of information. Note that under the natural map $\kappa\colon\mathrm{Cl}(X)\to\mathrm{Cl}(X)\otimes\mathbb{R}$, $[D]\mapsto[D]\otimes 1$, the $\mathcal{R}$-*maculate* set is mapped into the $\mathcal{R}$-*maculate* region, that is $\kappa\colon\mathcal{M}_\mathbb{Z}(\mathcal{R})\to\mathcal{M}_\mathbb{R}(\mathcal{R})$. In other words, the preimage $\kappa^{-1}\mathcal{M}_\mathbb{R}(\mathcal{R})$ in $\mathrm{Cl}(X)$ contains $\mathcal{M}_\mathbb{Z}(\mathcal{R})$, or, slightly incorrect, $\mathcal{M}_\mathbb{Z}(\mathcal{R})\subseteq\mathcal{M}_\mathbb{R}(\mathcal{R})\cap\mathrm{Cl}(X)$. We will encounter several situations when $\kappa^{-1}\mathcal{M}_\mathbb{R}(\mathcal{R})$ and $\mathcal{M}_\mathbb{Z}(\mathcal{R})$ are either equal or not equal, depending on the saturation of respective cones.

Proposition 5.6. *Suppose $X=\mathbb{TV}(\Sigma)$ is a toric variety with no torus factors.*

(i) Let $\mathcal{R}\subseteq\Sigma(1)$ be a subset, and suppose $[D]\in\mathrm{Cl}(X)$ is a class of a Weil divisor D on X. Then $[D]$ belongs to $\mathcal{M}_\mathbb{Z}(\mathcal{R})$ if and only if D is linearly equivalent to some $\sum_{\rho\in\Sigma(1)}\lambda_\rho\cdot D_\rho$ with $\lambda_\rho\in\mathbb{Z}$ and $\mathcal{R}=\{\rho\in\Sigma(1)\mid\lambda_\rho<0\}$.

(ii) Again, let $\mathcal{R}\subseteq\Sigma(1)$, and suppose $[D]\in\mathrm{Cl}(X)$ is a class of a Weil divisor D on X. Then $[D]_\mathbb{R}\in\mathrm{Cl}(X)\otimes\mathbb{R}$ belongs to $\mathcal{M}_\mathbb{R}(\mathcal{R})$, if and only if D is $\mathbb{Q}$-linearly equivalent to $\sum_{\rho\in\Sigma(1)}\lambda_\rho\cdot D_\rho$ (for rational λ_ρ) with $\mathcal{R}=\{\rho\in\Sigma(1)\mid\lambda_\rho<0\}$.

(iii) If $\mathcal{R}\subseteq\Sigma(1)$ is tempting, then for any i such that $\widetilde{\mathrm{H}}^{i-1}(V^>(\mathcal{R}),\Bbbk)\neq 0$ and any Weil divisor $[D]\in\mathcal{M}_\mathbb{Z}(\mathcal{R})$, we have $\mathrm{H}^i(\mathcal{O}_X(D))\neq 0$.

(iv) A rank one reflexive sheaf $\mathcal{O}_X(D)$ *for* $[D] \in \mathrm{Cl}(X)$ *is immaculate if and only if* $D \notin \bigcup_{\mathcal{R}=tempting} \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$.

(v) A rank one reflexive sheaf $\mathcal{O}_X(D)$ *such that* $[D]_{\mathbb{R}} \notin \bigcup_{\mathcal{R}=tempting} \mathcal{M}_{\mathbb{R}}(\mathcal{R})$ *is immaculate.*

This statement is comparable with [BH09, Prop. 4.3 and 4.5] and [Efi14, Prop. 4.2].

Proof. The divisor D of (i) or (ii) belongs to $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ or $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ if and only if it is an image under π of $\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}$ or $\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq -1}^{\mathcal{R}}$, respectively. The kernel of π is the set of principal torus invariant divisors, hence the claim holds.

To see (iii), take $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$, and a linearly equivalent

$$D' = \sum_{\rho\in\Sigma(1)} \lambda_\rho \cdot D_\rho = D + \mathrm{div}(x^m)$$

as in (i). Then by [CLS11, Thm 9.1.3] the appropriate cohomology group is $\mathrm{H}^i(\mathcal{O}_X(D))_m = \mathrm{H}^i(\mathcal{O}_X(D'))_0 \neq 0$.

If D is immaculate, then it is not contained in $\bigcup_{\mathcal{R}=\text{tempting}} \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ by (iii). Conversely, if D is not immaculate, then pick a linearly equivalent divisor $\sum_{\rho\in\Sigma(1)} \lambda_\rho \cdot D_\rho$ which has non-trivial cohomologies in degree $0 \in M$. By [CLS11, Thm 9.1.3] the set $\mathcal{R} = \{\rho \in \Sigma(1) \mid \lambda_\rho < 0\}$ is tempting and $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$, concluding the proof of (iv).

Finally, (v) follows from (iv), since $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ implies $[D]_{\mathbb{R}} \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$. □

It is not always true, that $[D]_{\mathbb{R}} \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$ implies $[D] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ as the following example shows.

Example 5.7. Let $X = \mathbb{TV}(\Sigma) = \mathbb{P}(2,3,5)$, the weighted projective plane with weights 2, 3, 5. Consider the $\mathbb{Q}$-Cartier Weil divisor $D \simeq \mathcal{O}_X(1)$ which can be written as the difference $D_{\rho_2} - D_{\rho_1}$. Then D is immaculate, but $[D]_{\mathbb{R}} \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$ for $\mathcal{R} = \emptyset$ (corresponding to the $\mathrm{Eff}_{\mathbb{R}}$-cone).

This leads to the following definition:

Definition 5.8. A divisor D is $\mathbb{R}$-*immaculate*, if

$$[D]_{\mathbb{R}} \in \mathrm{Cl}(X) \otimes \mathbb{R} \setminus \bigcup_{\mathcal{R}=\text{tempting}} \mathcal{M}_{\mathbb{R}}(\mathcal{R}).$$

Thus Example 5.7 shows a simple case of an immaculate Weil divisor that is not $\mathbb{R}$-immaculate. In Example 7.8 we construct a line bundle on a smooth toric projective variety with the same property. Up to the zero-th cohomology group, the concept of $\mathbb{R}$-immaculate divisor here is an analogue of the *strongly acyclic* line bundle in [BH09, Def. 4.4].

Definition 5.9. The *immaculate loci* of X are

$$\operatorname{Imm}_{\mathbb{Z}}(X) = \operatorname{Cl}(X) \setminus \bigcup_{\mathcal{R}\subset\Sigma(1),\ \mathcal{R}\text{ is tempting}} \mathcal{M}_{\mathbb{Z}}(\mathcal{R}), \text{ and}$$

$$\operatorname{Imm}_{\mathbb{R}}(X) = \kappa^{-1}\left((\operatorname{Cl}(X)\otimes\mathbb{R}) \setminus \bigcup_{\mathcal{R}\subset\Sigma(1),\ \mathcal{R}\text{ is tempting}} \mathcal{M}_{\mathbb{R}}(\mathcal{R})\right) \subset \operatorname{Cl}(X),$$

where $\kappa\colon \operatorname{Cl}(X) \to \operatorname{Cl}(X)\otimes\mathbb{R}$ is the natural map $[D] \mapsto [D]\otimes 1 = [D]_{\mathbb{R}}$.

Thus $\operatorname{Imm}_{\mathbb{Z}}(X)$ is the collection of all immaculate divisors. By Proposition 5.6(v) all the divisors in $\operatorname{Imm}_{\mathbb{R}}(X)$ are immaculate, that is $\operatorname{Imm}_{\mathbb{R}}(X) \subset \operatorname{Imm}_{\mathbb{Z}}(X)$. More precisely, $\operatorname{Imm}_{\mathbb{R}}(X)$ is the set of all $\mathbb{R}$-immaculate divisors as in Definition 5.8.

Example 5.10. In contrast to Examples 5.7 and 7.8, we can see that in the case of the hexagon (Example 3.2), all immaculate line bundles are $\mathbb{R}$-immaculate. This follows since the matrix π defining the map $(\mathbb{Z}^{\Sigma(1)})^* \to \operatorname{Pic}(X)$ is totally unimodular.

Example 5.11. We illustrate Proposition 5.6 with the example of the Hirzebruch surface $\mathbb{F}_a = \mathbb{TV}(\Sigma_a)$. The special cases $a = 0$ and $a = 1$ are presented in the Figures 3 and 4, respectively. More general cases are explained in Subsection 6.2 — our surface case corresponds to $\ell_1 = \ell_2 = 2$ there.

The Gale transform, that is the map π, is given by the matrix

$$\pi = \left(\begin{array}{cc|cc} 1 & 1 & 0 & -a \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The associated rays of the fan Σ_a are given by the matrix

$$\rho = \left(\begin{array}{cc|cc} 0 & -a & 1 & -1 \\ 1 & -1 & 0 & 0 \end{array}\right).$$

If we denote the four columns, that is the rays, by $\rho_1, \ldots, \rho_4$, then the tempting subsets of $\Sigma_a(1)$ are just $\emptyset$, $\Sigma_a(1)$, $\mathcal{R}_1 = \{\rho_1, \rho_2\}$, and $\mathcal{R}_2 = \{\rho_3, \rho_4\}$. The corresponding maculate regions are

$$\begin{aligned}\mathcal{M}_{\mathbb{R}}(\emptyset) &= \operatorname{cone}\big\langle(1,0),(0,1),(-a,1)\big\rangle = \operatorname{cone}\big\langle(1,0),\,(-a,1)\big\rangle,\\ \mathcal{M}_{\mathbb{R}}(\Sigma_a(1)) &= (a-2,-2)+\operatorname{cone}\big\langle(-1,0),(a,-1)\big\rangle,\\ \mathcal{M}_{\mathbb{R}}(\mathcal{R}_1) &= (-2,0)+\operatorname{cone}\big\langle(-1,0),(0,1),(-a,1)\big\rangle\\ &= (-2,0)+\operatorname{cone}\big\langle(-1,0),(0,1)\big\rangle,\\ \mathcal{M}_{\mathbb{R}}(\mathcal{R}_2) &= (a,-2)+\operatorname{cone}\big\langle(1,0),(0,-1)\big\rangle.\end{aligned}$$

The lattice points within the complement of the union of these four regions consist of the line $(*,-1)$ and, if $a \geq 1$, the two isolated points $(-1,0)$ and $(a-1,-2)$. In the degenerate case of $a=0$, there is an additional line $(-1,*)$, see Figure 3. Here, all immaculate divisors are $\mathbb{R}$-immaculate.

5.2. Conditions on presence or absence of temptations

In this section we describe straightforward criteria that imply that a given subset of rays is tempting or it is non-tempting. The upshot is that, for all sets $\mathcal{R} \subseteq \Sigma(1)$ covered by one of these claims, one does not need to look at the topology of $V^{>}(\mathcal{R}) = \operatorname{supp} V^{\geq}(\mathcal{R}) \setminus \{0\}$.

5.2.1. Monomials do not lead into temptation The first criterion is similar to the boundedness condition in [HKP06, Prop. 2].

Proposition 5.12. *Suppose $X = \mathbb{TV}(\Sigma)$ is a complete toric variety and $\mathcal{R} \subset \Sigma(1)$ is a tempting subset. Denote by $\rho^*\colon M_{\mathbb{R}} \to \mathbb{R}^{\Sigma(1)}$ the natural embedding of the principal torus invariant divisors into all torus invariant divisors. Then*

$$\rho^*(M_{\mathbb{R}}) \cap \left(\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq 0}^{\mathcal{R}}\right) = \{0\}\,.$$

Proof. Suppose on the contrary, that $(\rho^*)^{-1}\left(\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq 0}^{\mathcal{R}}\right)$ is a positive dimensional cone $\tau \subset M_{\mathbb{R}}$. Consider the divisor $D = \sum_{\varrho\in\mathcal{R}} -D_{\varrho}$. Since $\mathcal{R}$ is tempting, the divisor has non-zero cohomologies in degree $-m$ for all $m \in \tau\cap M$. Thus, the cohomology groups $\bigoplus_{i=0}^{\dim X} \mathrm{H}^i(D)$ are infinitely dimensional, a contradiction to the completeness of X. □

Example 5.13. Consider the Hirzebruch surface $\mathbb{F}_a$ as in Example 5.11, and suppose $a > 0$. Then out of 16 subsets of $\{\rho_1,\rho_2,\rho_3,\rho_4\}$, only six survive the test provided by Proposition 5.12. Namely, these are the four tempting subsets as listed in Example 5.11, and $\{\rho_4\}$ and its complement $\{\rho_1,\rho_2,\rho_3\}$ having the property of the associated cone intersecting M in just $\{0\}$.

Example 5.14. In the "hexagon" case (see Examples 3.2 and 5.2), Proposition 5.12 shows that the following 18 out of $64 = 2^6$ subsets of $\Sigma(1)$ are non-tempting:

$$\{0,1\}\,,\{0,5\}\,,\{1,2\}\,,\{2,3\}\,,\{3,4\}\,,\{4,5\}\,,\{0,1,2\}\,,\{0,1,5\}\,,\{0,4,5\}\,,$$
$$\{1,2,3\}\,,\{2,3,4\}\,,\{3,4,5\}\,,\{0,1,2,3\}\,,\{0,1,2,5\}\,,\{0,1,4,5\}\,,\{0,3,4,5\}\,,$$
$$\{1,2,3,4\}\,,\{2,3,4,5\}\,.$$

5.2.2. Faces are not tempting

Proposition 5.15. *Suppose $X = \mathbb{TV}(\Sigma)$ is a complete toric variety and $\sigma \in \Sigma$ is any cone (or a proper subfan with strictly convex support). Then the subsets $\mathcal{R} = \sigma(1) \subset \Sigma(1)$ and $\Sigma(1) \setminus \mathcal{R}$ are not tempting.*

Proof. The complex $V^{>}(\mathcal{R})$ is equal to the convex set $\sigma \setminus \{0\}$, hence it is contractible. By Alexander duality (see Remark 5.4) the complement is also not tempting. □

Example 5.16. For the Hirzebruch surface $\mathbb{F}_a$, only the four tempting subsets fail this test. All the other subsets are either faces or complements of faces.

Example 5.17. According to Proposition 5.15, in the "hexagon" case (see Examples 3.2, 5.2), the following 24 subsets of $\Sigma(1)$ are non-tempting: all single element subsets $\{i\}$, all consecutive two elements subsets $\{i, i+1\}$, and their complements (which have either four or five elements), which are all faces or their complements. Moreover, considering also three consecutive elements $\{i, i+1, i+2\}$ (which are rays of a subfan with a strictly convex support), we obtain 30 subsets, which are all the non-tempting subsets of $\Sigma(1)$. Alternatively, the three element subsets can be understood from Example 5.14.

5.2.3. Primitive collections delude A *primitive collection* of a simplicial fan Σ is a "minimal non-face", that is, a subset of rays $\mathcal{R} \subset \Sigma(1)$, such that the cone spanned by $\mathcal{R}$ is not in Σ, but the cone spanned by $\mathcal{R} \setminus \{\rho\}$ is in Σ for every $\rho \in \mathcal{R}$. More generally, a subset $\mathcal{R} \subset \Sigma(1)$ of any fan is a primitive collection, if $\mathcal{R}$ is not contained in any single cone of Σ, but every proper subset is. See [Bat91], [CvR09] for more details and explanations why this notion is important and relevant to projective toric varieties, see also Section 7.1.

Proposition 5.18. *Suppose $X = \mathbb{TV}(\Sigma)$ is a complete simplicial toric variety with no torus factors. Let $\mathcal{R} \subset \Sigma(1)$ be either empty or a primitive collection. Then $\mathcal{R}$ and its complement are tempting.*

Proof. If $\mathcal{R} = \emptyset$ then the claim is clear, so suppose $\mathcal{R}$ is a primitive collection, that is, a subset which is does not generate a cone of Σ, but all its proper subsets do generate such cones. By Alexander duality it is enough to prove that $\mathcal{R} = \{\rho_1, \ldots, \rho_k\}$ is tempting. Since every ray belongs to Σ, we have $k \geq 2$. We distinguish between two cases: either $\mathcal{R}$ is linearly independent or not.

If $\mathcal{R}$ is linearly independent, then $\mathcal{V} := \operatorname{span}_{\mathbb{R}} \mathcal{R}$ is k-dimensional, and $\mathcal{R}^+ := \sum_{j=1}^k \mathbb{R}_{\geq 0} \cdot \rho_j$ is a k-dimensional simplicial cone in $\mathcal{V}$ which does not belong to Σ. On the other hand, its boundary $\partial \mathcal{R}^+$ is a subcomplex of Σ; it is exactly the complex $V^{\geq}(\mathcal{R})$ as in Section 5.1. Thus, $|V^{\geq}(\mathcal{R})| \setminus \{0\} = |\partial \mathcal{R}^+| \setminus \{0\}$ is homotopy equivalent to a sphere S^{k-2}. In particular, it is not $\Bbbk$-acyclic.

On the other hand, suppose $\mathcal{R}$ is linearly dependent. Since $\mathcal{R}$ is a primitive collection, all the cones generated by $\mathcal{R} \setminus \{\rho_j\}$ are necessarily simplicial. In particular, $\mathcal{V} := \operatorname{span}_{\mathbb{R}} \mathcal{R}$ is $(k-1)$-dimensional, and each $\mathcal{R} \setminus \{\rho_j\}$ spans a full-dimensional cone in $\mathcal{V}$ that belongs to Σ. Thus, these cones generate $V^{\geq}(\mathcal{R})$, and this is a complete fan in $\mathcal{V}$ which (up to $\mathbb{R}$-linear change of coordinates) looks like the $\mathbb{P}^{k-1}$-fan in $\mathbb{R}^{k-1}$. Again, $V^{>}(\mathcal{R}) = |V^{\geq}(\mathcal{R})| \setminus \{0\}$ is homotopy equivalent to S^{k-2}, hence it is not $\Bbbk$-acyclic. $\square$

Example 5.19. For the Hirzebruch surface $\mathbb{F}_a$, all tempting subsets are predicted by Proposition 5.18. That is all four of them are either empty, or $\Sigma(1)$, or a primitive collection.

Example 5.20. Proposition 5.18 applied to the hexagon example (see Examples 3.2, 5.2), implies that the following 20 subsets are tempting:

$$\emptyset, \{0,2\}, \{0,3\}, \{0,4\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\},$$
$$\{0,1,2,4\}, \{0,1,3,4\}, \{0,1,3,5\}, \{0,2,3,4\}, \{0,2,3,5\}, \{0,2,4,5\},$$
$$\{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \Sigma(1).$$

Remark 5.21. In [Efi14, Prop. 4.4] it is shown that if $\mathcal{R}$ is tempting, then it is a union of primitive collections. The converse of this statement does not hold.

5.3. The cube

Throughout this subsection we will assume $X = \mathbb{TV}(\Sigma)$ is a complete and simplicial toric variety.

Let $\mathcal{R} \subseteq \Sigma(1)$ be an arbitrary, not necessarily tempting subset. This gives rise to a vertex $v(\mathcal{R}) := -(0^{\Sigma(1)\setminus\mathcal{R}}, 1^{\mathcal{R}})$ of the cube W spanned by all points of $\mathbb{R}^{\Sigma(1)}$ with $0/-1$ coordinates. It is the only vertex of the polyhedral cone $\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq -1}^{\mathcal{R}}$, that is, the class of the corresponding divisor $D_{\mathcal{R}} := -\sum_{\rho\in\mathcal{R}} D_\rho$ is the most prominent element of the $\mathcal{R}$-maculate region $\mathcal{M}_{\mathbb{R}}(\mathcal{R}) := \pi\big(\mathbb{R}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{R}_{\leq -1}^{\mathcal{R}}\big)$ introduced in Definition 5.3.

We have discussed in Proposition 5.6 that the temptation of $\mathcal{R}$ implies the maculacy of $D_{\mathcal{R}}$. In the following we will show in Theorem 5.24 that for $[D_{\mathcal{R}}] \in \mathrm{Cl}(X)$ this is the only source of disgrace.

Lemma 5.22. *Suppose $X = \mathbb{TV}(\Sigma)$ is a complete simplicial toric variety, and $\mathcal{R} \subseteq \Sigma(1)$ is an arbitrary subset. Let $m \in M \setminus \{0\}$. Then the complex $V^{>}_{D_{\mathcal{R}},m}$ (or $V_{D_{\mathcal{R}},m}$) from (3.3) in Subsection 3.2 is contractible and non-empty.*

Proof. We prove a slightly more general claim. Let Σ be a simplicial, complete fan, let $m \in M \setminus \{0\}$ and $S \subseteq \Sigma(1)$ such that

$$\{\rho \in \Sigma(1) \mid \langle \rho, m\rangle < 0\} \subseteq S \subseteq \{\rho \in \Sigma(1) \mid \langle \rho, m\rangle \leq 0\},$$

then the punctured full subcomplex $\langle S\rangle \setminus \{0\} \subset \Sigma$ is contractible and non-empty: First, since the cohomology of $\langle S\rangle \setminus \{0\}$ is stable under small perturbations of the position of the rays from $\Sigma(1)$, we may move those from $\Sigma(1)\cap m^\perp$ into the open halfspaces $(m < 0)$ or $(m > 0)$ depending on whether they belong to S or if they do not, respectively. Hence, we may assume that $\Sigma(1)\cap m^\perp = \emptyset$, that is, that $S = [m< 0]\cap\Sigma(1) = [m\leq 0]\cap\Sigma(1)$. But now, the proof follows from the fact that $\langle S\rangle$ is a deformation retract of the $(m \leq 0)$ part of the geometric realisation of Σ, that is, of $(\operatorname{supp}\Sigma) \cap (m \leq 0)$. This is a closed half space which stays cohomologically trivial even after removing the origin. $\square$

Remark 5.23. Note that full subcomplex generated by $S = [m< 0] \cap \Sigma(1)$ equals $V^{\geq}_{0,m}$ for the zero divisor. Its contractibility for $m \neq 0$ is reflected by the fact that the structure sheaf is almost immaculate.

As an immediate corollary of Lemma 5.22 we obtain

Theorem 5.24. *Suppose $X = \mathbb{TV}(\Sigma)$ is a complete simplicial toric variety, and $\mathcal{R} \subseteq \Sigma(1)$ is a subset. It gives rise to the divisor $D_{\mathcal{R}} := -\sum_{\rho\in\mathcal{R}} D_\rho$.*

(i) *Let* $D = \sum_{\rho \in \Sigma(1)} \lambda_\rho D_\rho$ *be a Weil divisor linearly equivalent to* $D_{\mathcal{R}}$*. Then either* $D = D_{\mathcal{R}}$*, or* $\mathcal{R}_D := \{\rho \in \Sigma(1) \mid \lambda_\rho < 0\}$ *is not tempting.*

(ii) *If* $D_{\mathcal{R}}$ *is not immaculate, then* $\mathcal{R}$ *is tempting,* $D_{\mathcal{R}}$ *is the only divisor from* $\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}$ *that maps to* $[D_{\mathcal{R}}]$ *under* π*, and* $\mathcal{R}$ *is the only tempting set containing* $[D_{\mathcal{R}}] \in \mathrm{Cl}(X)$ *in its maculate region* $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$*.*

(iii) *Non-tempting* $\mathcal{R}$ *lead to immaculate* $[D_{\mathcal{R}}]$*, and the class map* π *is injective on the maculate vertices of the cube* W*.*

Proof. (i) If $D \neq D_{\mathcal{R}}$, then there is an $m \in M \setminus \{0\}$ such that $D_{\mathcal{R}} = D - \mathrm{div}(x^m) = D(-m)$. Thus, as in the proof of Theorem 3.6, we obtain $V_{D,0} = V_{D_{\mathcal{R}}(m),\,0} = V_{D_{\mathcal{R}},m}$. The latter is cohomologically trivial by Lemma 5.22, and the first is generated by $\mathcal{R}_D$.

(ii) Since $[D_{\mathcal{R}}]$ is maculate, there must be a tempting $\mathcal{R}'$ such that $[D_{\mathcal{R}}] \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R}')$, that is there is a divisor D with $[D] = [D_{\mathcal{R}}]$ and $\mathcal{R}_D = \mathcal{R}'$. By (i) this implies $D = D_{\mathcal{R}}$, that is $\mathcal{R}_D = \mathcal{R}$.

(iii) This is just a reformulation of (ii). □

Remark 5.25. Immaculate divisors on X do always exist: There are always non-tempting subsets induced from cones by Proposition 5.15. Other possibilities to produce such subsets arise from Lemma 5.22 – just take $\mathcal{R} := \Sigma(1) \cap [m < 0]$ for some $m \in M \setminus \{0\}$. In any case, the corresponding vertex of the cube is immaculate by Theorem 5.24(iii).

Example 5.26. We work with the Hirzebruch surface $X = \mathbb{F}_a$, and we follow the coordinates and notation of Example 5.11. The cube W is 4-dimensional, so that it has 16 vertices. They correspond to subsets $I \subseteq \{\rho_1, \rho_2, \rho_3, \rho_4\}$, and only four of them are tempting. These tempting ones lead to the four different images

$$\begin{aligned} (0,0) &= -\pi(0000), & (0,-2) &= -\pi(1100), \\ (a,-2) &= -\pi(0011), & (-2+a,-2) &= -\pi(1111) \end{aligned}$$

in $\mathbb{Z}^2 = \mathrm{Cl}(\mathbb{F}_a) = \mathrm{Cl}(\mathbb{F}_a)$. The remaining twelve are

$$\begin{gathered} (-1,0) = -\pi(1000) = -\pi(0100), \\ (-1+a,-2) = -\pi(1011) = -\pi(0111), \\ (-1,-1) = -\pi(1010) = -\pi(1001), \\ (-1+a,-1) = -\pi(1001) = -\pi(0101) \end{gathered}$$

(appearing twice) and, with single occurrence,

$$\begin{aligned} (0,-1) &= -\pi(0010), & (a,-1) &= -\pi(0001), \\ (-2,-1) &= -\pi(1110), & (-2+a,-1) &= -\pi(1101). \end{aligned}$$

Thus, the two isolated points of the immaculate locus plus two points from the immaculate line could have been guessed from the non-injectivity of π alone. More precisely, it is caused by $(1,-1,0,0) \in \ker \pi$, which corresponds to the second row of the matrix ρ describing the fan.

Example 5.27. In the notation of Examples 3.2 and 5.2 (the hexagon) the image of $[-1,0]$-cube in $\mathrm{Pic}(X) \otimes \mathbb{R}$ is a lattice polytope P with 46 vertices and 54 lattice points. 34 of the vertices come from the 34 tempting vertices of the cube. The remaining 12 vertices of P are images of 12 non-tempting vertices of the cube. The 8 lattice points in P that are not vertices (including 2 interior points), are images of the remaining 18 non-tempting vertices of the cube, each with a repetition (the two interior lattice points appear three times each as images of the vertices of the cube, the other points appear twice each). In particular, the cube produces 20 immaculate line bundles on X.

6. Toric manifolds with Picard rank 2

We commence this section with recalling a well-known fact about smoothness of toric varieties in terms of Gale duality. Then we study our first family of examples, that is smooth complete toric varieties of Picard rank 2. Such varieties are described in [Kle88], and we can classify all the immaculate line bundles on them. While the case of Picard rank 2 is a special case of Section 7, it will be helpful to spend some time on this. Here we are in a situation where all loci can be completely described and depicted, and this will be very helpful for understanding the general situation.

6.1. Spotting smoothness via Gale duality

When working with fans having only few generators, the Gale transform becomes the essential tool to investigate their combinatorial structure. We recall an argument showing that this instrument, considered for abelian groups instead of vector spaces, can spot smoothness, too. Let

$$0 \longrightarrow K \xrightarrow{\iota} \mathbb{Z}^n \xrightarrow{\rho} N \longrightarrow 0$$

be an exact sequence of free abelian groups with $d := \mathrm{rk}\, N$. This situation gives rise to the Gale transform being just the dual sequence

$$0 \longleftarrow K^* \xleftarrow{\iota^*} \mathbb{Z}^{n*} \longleftarrow N^* \longleftarrow 0.$$

Denote by $\mathbb{Z}^d \subseteq \mathbb{Z}^n$ and $\mathbb{Z}^{(n-d)*} \subseteq \mathbb{Z}^{n*}$ the orthogonal subgroups being generated by $\{e_1, \ldots, e_d\}$ and $\{e^{d+1}, \ldots, e^n\}$, respectively.

Proposition 6.1. *The determinant of* $\{\rho(e_1),\ldots,\rho(e_d)\}$ *equals, maybe up to sign, the determinant of* $\{\iota^*(e^{d+1}),\ldots,\iota^*(e^n)\}$.

Proof. Assuming that the restriction $\rho|_{\mathbb{Z}^d}:\mathbb{Z}^d\to N$ has a finite cokernel C (which is equivalent to $\rho|_{\mathbb{Z}^d}$ being injective or to $\mathbb{Q}^d\xrightarrow{\rho}N\otimes\mathbb{Q}$ being an isomorphism), we obtain

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbb{Z}^d \arrow[r, equal] \arrow[d] & \mathbb{Z}^d \arrow[d, "\rho"] & \\
0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, equal] & \mathbb{Z}^n \arrow[r, "\rho"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "\bar\iota"] & \mathbb{Z}^{n-d} \arrow[r, "\rho"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Dualising the bottom row yields $\operatorname{coker}\left(\mathbb{Z}^{(n-d)*}\xrightarrow{\iota^*}K^*\right)=\operatorname{Ext}^1_{\mathbb{Z}}(C,\mathbb{Z})$. That is, the cokernels of $\mathbb{Z}^d$ in N and of $\mathbb{Z}^{(n-d)*}$ in K^* have the same order. ☐

6.2. Immaculate locus for Picard rank two

After illustrating the general method for classifying immaculate line bundles in Section 5, we describe now the immaculate loci in the specific case of smooth (complete) toric varieties of Picard rank 2.

Investigating Gale duals leads to the well-known classification of the combinatorial type of d-dimensional, simple, convex polytopes with $d+2$ vertices – they are $(\triangle^{\ell_1-1}\times\triangle^{d-\ell_1+1})^\vee$ for some $\ell_1=2,\ldots,d$, where $\triangle^r$ means the r-dimensional simplex and $(\ldots)^\vee$ denotes the dual of a polytope. This is a special case of the situation we will meet in Subsection 7.2.

Explicitly, in [Kle88, Thm 1], this classification was refined to find all complete smooth d-dimensional fans with $d+2$ rays, that is, all smooth complete toric varieties with Picard rank two. They are parameterised by the following data:

(i) a decomposition $d+2=\ell_1+\ell_2$ with $\ell_1,\ell_2\geq 2$ and
(ii) a choice of non-positive integers $0=c^1\geq\ldots\geq c^{\ell_2}$ which are jointly denoted by $c\in\mathbb{Z}^{\ell_2}_{\leq 0}$.

These data provide the $2\times(\ell_1+\ell_2)$-matrix

$$\left(\begin{array}{ccc|cccc} 1 & \ldots & 1 & 0 & c^2 & \ldots & c^{\ell_2} \\ 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \end{array}\right)$$

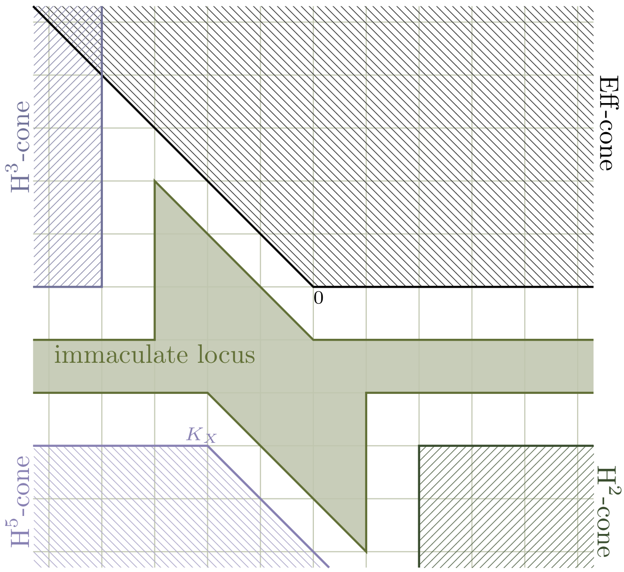


Figure 6: The Picard lattice and immaculate locus of a smooth projective toric 5-fold X with $\operatorname{Cl} X = \mathbb{Z}^2$ and the matrix $\pi = \left(\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$, that is, $c = (0,\ -1,\ -1)$.

encoding $\pi\colon (\mathbb{Z}^{(d+2)})^* \twoheadrightarrow \mathbb{Z}^2 = \operatorname{Cl} X$ (compare with Example 5.11, where we had $a = -c^2$). That is, the rays of the associated fan Σ_c are $u_i = \rho(e_i)$ and $v_j = \rho(f_j)$ in

$$N := \mathbb{Z}^{\ell_1+\ell_2} / \big(\mathbb{Z} \cdot (\underline{1}, c) + \mathbb{Z} \cdot (\underline{0}, \underline{1})\big) \cong \mathbb{Z}^d$$

where $\{e_1, \ldots, e_{\ell_1}, f_1, \ldots, f_{\ell_2}\}$ denotes the canonical basis in $\mathbb{Z}^{d+2} = \mathbb{Z}^{\ell_1+\ell_2}$ and $\rho : \mathbb{Z}^{d+2} \to N$ is the canonical projection. The fan structure is easy – the d-dimensional cones are σ_{ij} which are generated by $\Sigma_c(1) \setminus \{u_i, v_j\}$ ($i = 1, \ldots, \ell_1,\ j = 1, \ldots, \ell_2$). That is, $\#\Sigma_c(d) = \ell_1 \ell_2$. Comparing with (6.1), one sees that the corresponding cross-frontier (2×2)-minors of the above matrix are $\det \begin{pmatrix} 1 c^i \\ 0 1 \end{pmatrix} = 1$, that is, by Proposition 6.1, the cones σ_{ij} are indeed smooth. We will denote $\overline{c} := \sum_{\nu=1}^{\ell_2} c^\nu$. Note that in [Kle88, Thm 2] it is shown that X_c is Fano if and only if $-\overline{c} \le \ell_1 - 1$.

Theorem 6.2. *Suppose $X = \mathbb{TV}(\Sigma_c)$ is a smooth complete toric variety of Picard rank 2. Then $\operatorname{Imm}_{\mathbb{Z}}(X) = \operatorname{Imm}_{\mathbb{R}}(X)$. Moreover, the line bundle* *represented by* $(x, y) \in \mathbb{Z}^2 = \mathrm{Cl}\, X$ *is immaculate if and only if one of the following holds:*

- $-\ell_2 < y < 0$ *or*
- $y \geq 0$ *and* $-\ell_1 < x < c^{\ell_2} y$ *or*
- $y \leq -\ell_2$ *and* $0 > x + \overline{c} > c^{\ell_2}(y + \ell_2) - \ell_1$.

Note that the second and the third case in the theorem are Serre dual to one another, while the first item is self-dual. The first item is a bunch of (horizontal) affine lines of the same type as in Theorem 4.13, corresponding to p-immaculate line bundles $(0, -1), \ldots, (0, -\ell_2 + 1)$, where $p\colon X \to \mathbb{P}^{\ell_1 - 1}$ is the natural projection. If $c = \underline{0}$, that is, if $X \simeq \mathbb{P}^{\ell_1-1} \times \mathbb{P}^{\ell_2-1}$, then the divisors appearing in the second and third item form parts of the lines corresponding to the other projection $X \to \mathbb{P}^{\ell_2 - 1}$. Otherwise, $c^{\ell_2} < 0$, and there are only finitely many line bundles in the second and third items (the inequalities define triangles). The special case of $\ell_1 = \ell_2 = 2$ is illustrated on Figure 4, and another case of a 5-fold is on Figure 6. Points of the form $(-1, 0), \ldots, (-\ell_1 + 1, 0)$ are always contained in the second item (independently of c). Later, in the more general setup of Section 7, these points, together with the lines from the first item, will form the "generating seeds" in the sense of Definition 7.9.

Proof of Theorem 6.2. The only tempting subsets of $\Sigma_c(1)$ are $\emptyset$, $U = \{u_1, \ldots, u_{\ell_1}\}$, $V = \{v_1, \ldots, v_{\ell_2}\}$ and $\Sigma_c(1) = U \sqcup V$. We proceed along the lines of Example 5.11 and calculate the maculate loci:

$$
\begin{aligned}
\mathcal{M}_{\mathbb{R}}(\emptyset) &= \operatorname{cone}\big\langle (1,0),\, (c^{\ell_2}, 1)\big\rangle,\\
\mathcal{M}_{\mathbb{R}}(\Sigma_c(1)) &= (-\overline{c} - \ell_1, -\ell_2) + \operatorname{cone}\big\langle (-1,0),\, (-c^{\ell_2}, -1)\big\rangle,\\
\mathcal{M}_{\mathbb{R}}(U) &= (-\ell_1, 0) + \operatorname{cone}\big\langle (-1,0),\, (0,1)\big\rangle,\\
\mathcal{M}_{\mathbb{R}}(V) &= (-\overline{c}, -\ell_2) + \operatorname{cone}\big\langle (1,0),\, (0,-1)\big\rangle.
\end{aligned}
$$

Note that for every maculate $\mathcal{R} \subset \Sigma_c(1)$, the tail cone in the above locus is smooth and the primitive generators of rays are all in the image of the set $\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}$. Thus, the map $\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}} \to \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \mathrm{Cl}(X)$ is surjective, i.e. $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) = \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \mathrm{Cl}(X)$. It follows that $\mathrm{Imm}_{\mathbb{Z}}(X) = \mathrm{Imm}_{\mathbb{R}}(X)$ and the explicit description of the immaculate locus follows by an explicit calculation of the inequalities of the cones above, and by taking the complement in $\mathrm{Cl}(X)$. $\square$

Proposition 6.3. *Suppose as above that* $X = \mathbb{TV}(\Sigma_c)$ *is a smooth complete toric variety of Picard rank* 2*. If* L *is a line bundle on* X *such that* $\mathrm{H}^i(X, L) \neq 0$*, then* $i \in \{0, \ell_1 - 1, \ell_2 - 1, \dim X\}$.

Proof. As in the previous proof, the only tempting subsets are $\emptyset$, U, V and $U \sqcup V$. Each of them leads to line bundles with non-trivial cohomology in one of the degrees: 0, $\ell_1 - 1$, $\ell_2 - 1$ or $\dim X$, respectively. Thus no other H^i can be non-zero. □

7. The immaculate locus for splitting fans

In this section we apply the theory of Section 5 to the case of splitting fans and calculate the essential part of the immaculate locus of line bundles in this setup. Let $X = \mathbb{TV}(\Sigma)$ be a smooth complete toric variety. Recall from Subsection 5.2.3 that a primitive collection of a (smooth, hence simplicial) fan Σ is another word for a "minimal non-face". We say Σ is a *splitting fan*, if the primitive collections of Σ are pairwise disjoint. This is equivalent to an existence of a chain $\Sigma = \Sigma_k, \ldots, \Sigma_1$ of fans such that $\mathbb{TV}(\Sigma_1) = \mathbb{P}^n$ and $\mathbb{TV}(\Sigma_{i+1}) \to \mathbb{TV}(\Sigma_i)$ is a toric split bundle, that is a projectivisation of a direct sum of toric line bundles (see [Bat91, Cor. 4.4]). In particular, all such X are projective. Note that every smooth complete toric variety with Picard rank two satisfies this property with $k = 2$, see Subsection 6.2.

7.1. Primitive relations

In this subsection we recall the notion of the primitive relation associated to a primitive collection and express all such relations for a splitting fan. We also give a lower bound on the number of primitive collections and characterise the splitting fans as those smooth complete fans that have the least possible number of primitive collections, with respect to that lower bound. Having in mind the application to splitting fans, which are smooth by definition, we restrict our presentation of primitive relations to the smooth case, following [Bat91]. See [CvR09, §1.3] for a more general treatment.

Let Σ be a fan of a smooth complete toric variety X. For every primitive collection $\mathcal{P} \subseteq \Sigma(1)$ we denote $e_{\mathcal{P}} := \sum_{\rho \in \mathcal{P}} e_\rho$, where the $e_\rho \in \mathbb{Z}^{\Sigma(1)}$ is the basis element corresponding to ρ, which under the natural map $\mathbb{Z}^{\Sigma(1)} \to N$ is mapped to the primitive generator of the corresponding ray. We denote by $\sigma(\mathcal{P})$ (called the "focus of $\mathcal{P}$") the unique cone $\sigma \in \Sigma$ such that the image of $e_{\mathcal{P}}$ in N is contained in $\operatorname{int} \sigma \subset N_{\mathbb{R}}$. It leads to a unique element $f(\mathcal{P}) \in \mathbb{Z}_{\geq 1}^{\sigma(\mathcal{P})(1)}$ with $e_{\mathcal{P}} - f(\mathcal{P}) \in \ker(\mathbb{Z}^{\Sigma(1)} \to N)$. (Here, by convention, $\mathbb{Z}_{\geq 1}^{\emptyset} = \{0\}$.) The expression $e_{\mathcal{P}} - f(\mathcal{P})$ is called the *primitive relation* associated to $\mathcal{P}$. As an element of $\mathrm{Cl}(X)^*$, it represents a class of 1-cycles.

In [Bat91, Prop. 3.1] it is shown that $\mathcal{P} \cap \sigma(\mathcal{P}) = \emptyset$, that is the elements of $\mathcal{P}$ are not among the generators of $\sigma(\mathcal{P})$. Moreover, if Σ is projective, then there exists a primitive collection $\mathcal{P}$ with $\sigma(\mathcal{P}) = 0$, see [Bat91, Prop. 3.2 and Thm 4.3].

Proposition 7.1. *Let $X = \mathbb{TV}(\Sigma)$ be a complete toric variety. Then every ray of Σ is contained in some primitive collection. If X is in addition $\mathbb{Q}$-factorial, then the number of primitive collections is at least the rank of $\mathrm{Cl}(X)$. Moreover, if X is smooth, then equality holds if and only if Σ is a splitting fan.*

Proof. Let $\{\mathcal{P}_1, \ldots, \ldots \mathcal{P}_k\}$ be the primitive collections of Σ. For each $\rho \in \Sigma(1)$ there exists a cone $\tau \in \Sigma$ such that $\tau(1) \cup \{\rho\}$ is not contained in any cone of Σ (otherwise, Σ would not be complete). Thus ρ is also contained in a minimal set of this type. This proves $\bigcup_{i=1}^k \mathcal{P}_i = \Sigma(1)$ as claimed.

Let X be $\mathbb{Q}$-factorial, so its fan is simplicial. In particular, $\#\Sigma(1) = \operatorname{rk}\mathrm{Cl}(X) + \dim X$. For each $i = 1, \ldots, k$ we choose a $\rho_i \in \mathcal{P}_i$. Then, $\Sigma(1) \setminus \{\rho_1, \ldots, \rho_k\}$ does not contain any of the primitive collections, hence it generates a cone in Σ. Thus,

$$(\dim X + \operatorname{rk}\mathrm{Cl}(X)) - k = \#\Sigma(1) - k \leq \#\left(\Sigma(1) \setminus \{\rho_1, \ldots, \rho_k\}\right) \leq \dim X.$$

If there is some overlap, say $\mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset$, then we might choose $\rho_i = \rho_j$, thus the above inequalities even yield $\operatorname{rk}\mathrm{Cl}(X) \leq k - 1$. On the other hand, if all $\mathcal{P}_i$ are pairwise disjoint, then we know that the facets of Σ look like $\sqcup_{i=1}^k (\mathcal{P}_i \setminus \{\rho_i\})$, in particular, $\#\Sigma(1) - k = \dim X$. ☐

7.2. Temptation for splitting fans

Here we assume X is a smooth toric projective variety of dimension d whose fan Σ is a splitting fan. We will first identify all of the tempting subsets $\mathcal{R} \subseteq \Sigma(1)$. Later in Section 7.3 we will investigate the associated π-images $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ or $\mathcal{M}_{\mathbb{Z}}(\mathcal{R})$ as introduced in Definition 5.3.

Let $\Sigma(1) = \mathcal{P}_1 \sqcup \ldots \sqcup \mathcal{P}_k$ be the decomposition into primitive collections of lengths $\ell_1, \ldots, \ell_k \geq 2$, respectively. Thus, maximal cones $\sigma \in \Sigma(d)$ correspond to maximal subsets of $\Sigma(1)$ not containing any entire set $\mathcal{P}_i$ $(i = 1, \ldots, k)$. This establishes a bijection

$$\mathcal{P}_1 \times \ldots \times \mathcal{P}_k \xrightarrow{\sim} \Sigma(d), \quad (p_1, \ldots, p_k) \mapsto \Sigma(1) \setminus \{p_1, \ldots, p_k\}.$$

In particular, Σ is combinatorially equivalent to the normal fan of

$$\square = \square(\{1, \ldots, k\}) := \triangle^{\ell_1 - 1} \times \ldots \times \triangle^{\ell_k - 1}$$

where $\triangle^{\ell-1}$ denotes the $(\ell-1)$-dimensional simplex with ℓ vertices. In particular, we know that $\#\Sigma(1) = \sum_{i=1}^{k} \ell_i,\ d = \sum_{i=1}^{k}(\ell_i - 1),\ \#\Sigma(d) = \prod_{i=1}^{k} \ell_i$, and, in compliance with Proposition 7.1, $\operatorname{rk}(\operatorname{Cl} X) = \#\Sigma(1) - d = k$.

Now, the essential point is that the temptation of a subset $\mathcal{R} \subseteq \Sigma(1)$ depends only on the combinatorial structure of Σ. The finer structure, the true shape of the fan reflected by the maps $\rho : \mathbb{Z}^{\Sigma(1)} \to N$ or $\pi\colon \mathbb{Z}^{\Sigma(1)} \to \operatorname{Cl}(X)$, does matter only for the second step of turning the tempting sets $\mathcal{R}$ into the maculate regions $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$.

Lemma 7.2. *If Σ is a splitting fan with the decomposition $\Sigma(1) = \mathcal{P}_1 \sqcup \ldots \sqcup \mathcal{P}_k$ into primitive collections $\mathcal{P}_i$, then the tempting subsets of $\Sigma(1)$ are $\mathcal{R}(J) := \bigcup_{j\in J} \mathcal{P}_j$ with $J \subseteq \{1, \ldots, k\}$.*

Proof. Instead of the complex $(\operatorname{supp} V^{\geq}(\mathcal{R})) \setminus \{0\} \subseteq \operatorname{supp}\Sigma \setminus \{0\} \sim S^{d-1}$ we consider its dual version $G(\mathcal{R})$ built as the union of all (closed) facets $G(\rho) < \square$ dual to $\rho \in \mathcal{R}$. Clearly, $\operatorname{supp} V^{\geq}(\mathcal{R}) \setminus \{0\}$ is homotopy equivalent to $G(\mathcal{R})$, thus one is $\Bbbk$-acyclic if and only if the other is. A subset $J \subseteq \{1, \ldots, k\}$ defines a splitting $\square = \square(J) \times \square(\{1, \ldots, k\} \setminus J)$ and accordingly we have $G(\mathcal{R}(J)) = \partial\square(J) \times \square(\{1, \ldots, k\} \setminus J)$, which is not $\Bbbk$-acyclic. Thus every set $\mathcal{R}(J)$ is tempting.

On the other hand, suppose that, for some j, the set $\mathcal{R} \subset \Sigma(1)$ does not contain the whole P_j, but at least one element of P_j. We claim $G(\mathcal{R})$ is contractible. Indeed, if, without loss of generality, $j = 1$, then we split off $\square = \triangle^{\ell_1-1} \times \square(2, \ldots, k)$. Let $f < \triangle^{\ell_1-1}$ be the (non-empty) face corresponding to the subset $\mathcal{R} \cap P_1 \subset P_1$ and pick a standard strong deformation retract $\triangle^{\ell_1-1} \to f$. Then $G(\mathcal{R})$ can be retracted to the contractible $f \times \square(2, \ldots, k)$ by gluing together the retractions of the contributing faces: each of the faces is either of the form $\triangle^{\ell_1-1} \times F$ for some face $F < \square(2, \ldots, k)$ or of the form $F' \times \square(2, \ldots, k)$ for a face $F' < \triangle^{\ell_1-1}$ containing f. Note that the image of any face of the latter type is just all of $f \times \square(2, \ldots, k)$, hence $G(\mathcal{R})$ is contractible. $\square$

Remark 7.3. The 2^k different sets $J \subseteq \{1, \ldots, k\}$ yield 2^k tempting sets $\mathcal{R}(J)$, hence 2^k maculate regions $\mathcal{M}_{\mathbb{R}}(\mathcal{R}(J))$ within the k-dimensional space $\operatorname{Cl}(X) \otimes \mathbb{R} \cong \mathbb{R}^k$. This looks a little like the structure of 2^k octants in this space, but we will see that typically the octants are "leaning", and they may intersect as illustrated on Figures 4 and 6.

Proposition 7.4. *Let $X = \mathbb{TV}(\Sigma)$ with Σ a splitting fan with k primitive collections, and L be a line bundle on X such that $\mathrm{H}^i(X, L) \neq 0$, then $i \in \left\{\sum_{j\in J}(\ell_j - 1)\right\}_{J\subset\{1,\ldots,k\}}$.*

Proof. In the previous proof we have seen that $\mathcal{R}(J)$ leads to the non-$\Bbbk$-acyclic $G(\mathcal{R}(J)) = \partial\square(J) \times \square(\{1,\ldots,k\} \setminus J)$. For cohomological considerations we can focus on the first factor $\partial\square(J) = \partial\left(\prod_{j\in J} \triangle^{\ell_j-1}\right)$. Thus we have the boundary of a polytope of dimension $\sum_{j\in J}(\ell_j - 1)$, so $\mathcal{R}(J)$ is homotopy equivalent to a $(\sum_{j\in J}(\ell_j - 1) - 1)$-dimensional sphere. $\square$

7.3. The refined structure of the fan and the class map

We have treated the combinatorial structure of the splitting fan Σ in the previous section. Here we concentrate on the more refined information, specifically, we focus on the detailed structure of the class map $\pi\colon \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X)$, where $X = \mathbb{TV}(\Sigma)$.

Write $\Sigma(1) = \bigsqcup_{i=1}^k \mathcal{P}_i$ the decomposition of the rays into the disjoint sets of primitive collections. In [Bat91, Corollary 4.4], Batyrev has proved that X can be obtained via a sequence of projectivisations of decomposable bundles. Within the fan language this means that we can assume that there is a sequence of fans $\Sigma = \Sigma_k, \ldots, \Sigma_1, \Sigma_0 = 0$ in abelian groups $N = N_k \to\!\!\!\to \ldots \twoheadrightarrow N_1 \twoheadrightarrow N_0 = 0$ such that the focus $\sigma(\mathcal{P}_j) = 0$ in N_j and $N_{j-1} = N_j/\operatorname{span}\mathcal{P}_j$. The fans Σ_j in N_j are splitting with $\Sigma_j(1) = \sqcup_{i=1}^j \mathcal{P}_i$, and they admit subfans $\widetilde{\Sigma}_{j-1} \subset \Sigma_j$ such that $\psi_j : N_j \twoheadrightarrow N_{j-1}$ induces an isomorphism $\widetilde{\Sigma}_{j-1} \xrightarrow{\sim} \Sigma_{j-1}$ (piecewise linear on the geometric realisations) and Σ_j consist of the sums of cones from $\widetilde{\Sigma}_{j-1}$ and proper subsets of $\mathcal{P}_j$.

With $\ell_i = \#\mathcal{P}_i$, this explicit structure of Σ can be translated into the fact that π is a triangular block matrix

$$\pi = \begin{pmatrix} \underline{1} & c_{12} & \ldots & c_{1k} \\ \underline{0} & \underline{1} & \ldots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \ldots & \underline{1} \end{pmatrix} \tag{7.5}$$

with k rows and k blocks of $1 \times \ell_j$-matrices ($j = 1, \ldots, k$). While $\underline{1}$ denotes $(1, 1, \ldots, 1)$ with ℓ_j entries, we have $c_{ij} \in \mathbb{Z}^{\ell_j}_{\le 0}$. If needed, its entries will be denoted by $c_{ij}^{\nu} \in \mathbb{Z}_{\le 0}$ ($i < j$, $\nu = 1, \ldots, \ell_j$). Every row encodes a primitive relation, hence each c_{ij} has at least one zero entry (the support of $c_{i\bullet}$ is supposed to be a face, that is to not contain any full $\mathcal{P}_j$).

Proposition 7.6. *Each matrix π in a triangular block form as in (7.5) with $c_{ij} \in \mathbb{Z}^{\ell_j}$ and $\ell_j \ge 2$ for all j gives rise to a smooth splitting fan of dimension $d := \sum_{i=1}^k (\ell_i - 1)$.*

Proof. We argue inductively on the number of rows k (and, at the same time, the number of blocks of columns). For $k = 1$ there is nothing to prove, so assume that Σ_{k-1} is a splitting fan obtained from π', a matrix with last row and last block of columns removed from π. For each $\nu \in \{1, \dots, \ell_k\}$ let $\mathcal{L}_\nu$ be the line bundle on $X_{k-1} = \mathbb{TV}(\Sigma_{k-1})$ corresponding to the point c^ν_{*k} in $\operatorname{Cl} X_{k-1}$. Then $X = \mathbb{P}(\bigoplus_{\nu=1}^{\ell_k} \mathcal{L}_\nu)$ is the desired smooth toric variety. □

Note that the smoothness of the variety $\mathbb{TV}(\Sigma)$ associated to the matrix π can also be derived directly from the method of Subsection 6.1. The co-facets of the fan Σ give rise to choosing one column of π in every block. But this yields an upper triangular matrix with only 1 as the diagonal entries. Hence, the determinant equals 1, too.

Example 7.7. A simple case to have in mind is $k = 2$. The matrix of π is

$$\left(\begin{array}{ccc|cccc} 1 & \dots & 1 & 0 & c^2 & \dots & c^{\ell_2} \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{array}\right) = \begin{pmatrix} \underline{1} & c \\ \underline{0} & \underline{1} \end{pmatrix}$$

It covers the case of Hirzebruch surfaces. In Subsection 6.2 we have discussed the immaculate locus of this matrix in detail.

Example 7.8. Consider the following smooth projective three dimensional toric variety $X = \mathbb{TV}(\Sigma) = \mathbb{P}(\mathcal{O}_Y(-2,0) \oplus \mathcal{O}_Y(0,-2))$, where $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{O}_Y(i,j) := \mathcal{O}_{\mathbb{P}^1}(i) \boxtimes \mathcal{O}_{\mathbb{P}^1}(j)$. Then the fan Σ is a splitting fan with matrix

$$\pi = \begin{pmatrix} 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The line bundle represented by

$$\pi\left((0,0,0,0,\tfrac{1}{2},\tfrac{1}{2})\right) = (-1,-1,1) \in \operatorname{Cl}(X)$$

is immaculate but not $\mathbb{R}$-immaculate (in the sense of Definition 5.8).

7.4. Generating immaculate seeds

We fix a format $\ell := (\ell_1, \dots, \ell_k)$ of splitting fans, that is a block format of the associated matrix π. We interpret c, that is the entries $c_{ij} \in \mathbb{Z}^{\ell_j}_{\le 0}$ of π, as coordinates of the "moduli space" of splitting fans $\Sigma(\ell, c)$ of this fixed format ℓ. All these fans share the same combinatorial type – that of the normal fan of $\square := \triangle^{\ell_1 - 1} \times \dots \times \triangle^{\ell_k - 1}$, see Section 7.2. Similarly, the associated toric varieties share the same Picard group. Since we use the primitive relations for the rows of π, we have even distinguished coordinates leading to a simultaneous identification $\mathrm{Cl}\,\mathbb{TV}(\Sigma(\ell,c)) = \mathbb{Z}^k$. This makes it possible to compare the immaculate loci of different $\Sigma(\ell,c)$ sharing the same ℓ.

Now, the basic idea is simple: For special c, e.g. $c=\underline{0}$, the immaculate locus is large – but it becomes smaller for growing $|c| := -c$. Roughly speaking, we will show that this shrinking of the immaculate locus becomes stationary, and we are going to calculate the limit.

There is, however, a technical obstacle. The centre of symmetry $K_X/2$ arising from Serre duality moves with c. Thus, it is not the whole immaculate locus that becomes stationary – this works only for some generating seed. That is, there is a certain subset of $\mathbb{Z}^k$ which is immaculate for all $\Sigma(\ell,c)$ and which generates (via some operations/reflections corresponding to successive Serre dualities) the full immaculate locus if $-c$ is sufficiently large.

Definition 7.9. We call $\mathrm{Seed}(\ell) := \bigcup_{j=1}^{k} \left(\mathbb{Z}^{j-1} \times \{-1,\ldots,-(\ell_j-1)\} \times \underline{0}^{k-j}\right)$ the *generating immaculate seed* for ℓ in $\mathbb{Z}^k$.

Recall the integral matrix expression

$$\pi = \pi(\ell,c) = \begin{pmatrix} \underline{1} & c_{12} & \ldots & c_{1k} \\ \underline{0} & \underline{1} & \ldots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \ldots & \underline{1} \end{pmatrix}$$

from Section 7.3 with non-positive entries within the vectors c_{ij}. For fixed $i,j \in \{1,\ldots,k\}$ we set $\overline{c}_{ij} := \sum_{\nu=1}^{\ell_j} c_{ij}^{\nu} \in \mathbb{Z}_{\leq 0}$. Moreover, denote

$$v_j := (\overline{c}_{1j},\ldots,\overline{c}_{j-1,j},\ell_j,\underline{0}) \in (\mathbb{Z}^j \times \underline{0}) \subseteq \mathbb{Z}^k.$$

Depending on c, we can now define the operator enlarging a given seed in $\mathbb{Z}^k$:

Definition 7.10. For a given subset $G \subseteq \mathbb{Z}^k$ we define its *c-hull* as the smallest set $\langle G\rangle_c \supseteq G$ satisfying the following recursive property: If $a \in \langle G\rangle_c \cap (\mathbb{Z}^j \times \underline{0}^{k-j})$ for some $j=0,\ldots,k-1$, then so is the shift $a - v_{j+1} \in \langle G\rangle_c$.

Note that $a - v_{j+1} \in \mathbb{Z}^j \times \{-\ell_{j+1}\} \times \underline{0}^{k-j-1}$. Hence, to obtain $\langle G\rangle_c$ out of G one can enlarge $\langle G\rangle_c$ successively: set $\langle G\rangle_c^{-1} := G$ and let $\langle G\rangle_c^{j+1} := \left(\langle G\rangle_c^{j} \cap (\mathbb{Z}^j \times \underline{0}^{k-j}\right) - v_{j+1}$. Then $\langle G\rangle_c = \langle G\rangle_c^{k}$.

Remark 7.11. Note that Serre duality replaces $a \in \mathbb{Z}^k$ with $-a - \sum_{i=1}^{k} v_i$. As we will see, we can also use Serre duality on the level of the smaller varieties $\mathbb{TV}(\Sigma_j)$. Thus the shift $a - v_{j+1}$ is obtained by a "double Serre duality": First we dualise in $\mathbb{TV}(\Sigma_j)$ obtaining $-a-\sum_{i=1}^{j} v_i$, and then we dualise in $\mathbb{TV}(\Sigma_{j+1})$ to get the shift $-(-a-\sum_{i=1}^{j} v_i)-\sum_{i=1}^{j+1} v_i$ from Definition 7.10. Geometrically, for simplicity of notation assume that $j = k-1$, and let $p\colon X \to \mathbb{TV}(\Sigma_{k-1})$ is the projection map. Then for a line bundle L on $\mathbb{TV}(\Sigma_{k-1})$, its Serre dual is $\omega_{\mathbb{TV}(\Sigma_{k-1})} \otimes L^*$, and Serre dual of its pullback is

$$\omega_X \otimes p^* \left(\omega_{\mathbb{TV}(\Sigma_{k-1})} \otimes L^*\right)^* = p^*L \otimes \left(\omega_X \otimes p^*\omega^*_{\mathbb{TV}(\Sigma_{k-1})}\right) = p^*L \otimes \mathcal{O}_X(-v_k).$$

By Serre's duality and Corollary 4.6 the immaculacies of L, p^*L, and $p^*L \otimes \mathcal{O}_X(-v_k)$ are equivalent.

These definitions of $\operatorname{Seed}(\ell)$ and the hull operations allow to describe the locus of immaculate line bundles for "general" c. Recall the notions of maculate regions from Definition 5.3, and immaculate loci from Definition 5.9.

Theorem 7.12. *Fix ℓ and let c be a parameter leading to a matrix $\pi = \pi(\ell, c)$ with the associated splitting fan $\Sigma = \Sigma(\ell, c)$. Then:*

- *(i)* $\operatorname{Seed}(\ell) \subseteq \operatorname{Imm}_{\mathbb{R}}(\Sigma(\ell, c))$ *for all c, that is the generating seeds are $\mathbb{R}$-immaculate.*
- *(ii) Both the loci* $\operatorname{Imm}_{\mathbb{Z}}(\Sigma)$ *and* $\operatorname{Imm}_{\mathbb{R}}(\Sigma)$ *are closed under the c-hull operation.*
- *(iii) For "general" c, the immaculate loci are both equal to the minimal set satisfying the above. That is,* $\operatorname{Imm}_{\mathbb{Z}}(\Sigma(\ell, c)) = \operatorname{Imm}_{\mathbb{R}}(\Sigma(\ell, c)) = \langle \operatorname{Seed}(\ell)\rangle_c$.

More precisely, a sufficient condition for "general" in (iii) is that for each $j = 1, \dots, k-1$ the vector $c_{j,j+1} \in \mathbb{Z}^{\ell_{j+1}}$ has at least two entries differing by more than ℓ_j.

Proof. Consider the sequence of projections $X = X_k \to X_{k-1} \to X_{k-2} \to \dots \to X_1 = \mathbb{P}^{\ell_1 - 1}$ and the corresponding fans $\Sigma = \Sigma_k, \Sigma_{k-1}, \Sigma_{k-2}, \dots, \Sigma_1$, such that $X_{j+1} = \mathbb{P}(\mathcal{E}_j)$, where $\mathcal{E}_j$ is a split vector bundle over $X_j = \mathbb{TV}(\Sigma_j)$. We argue by induction on the Picard number k. If $k = 1$, then $X \simeq \mathbb{P}^{\ell_1 - 1}$, $\operatorname{Seed}(\ell) = \operatorname{Imm}(\Sigma_1) = \{-1, -2, \dots, -(\ell_1 - 1)\} \subset \mathbb{Z} \simeq \operatorname{Cl} X$, $v_1 = -\ell_1$, and the shift in the definition of c-hull operation is irrelevant since 0 is not a generating seed. Thus there is nothing to prove in this case.

So suppose that the statement holds for Picard number at most $k-1$ and denote by p the projection $p\colon X = \mathbb{P}(\mathcal{E}_{k-1}) \to X_{k-1}$. To present a description of the maculate regions, we recognise the vectors v_j (for $j = 1, \dots, k$) defined above as the building blocks of the π-images of the "maculate vertices of the cube" (see Section 5.3). The associated tail cones are built from the polyhedral cones

$$C_j := \langle (c^\nu_{1j}, \dots, c^\nu_{j-1,j}, 1, \underline{0}) \mid \nu = 1, \dots, \ell_j \rangle \subseteq (\mathbb{R}^j \times \underline{0}) \subseteq \mathbb{R}^k = \mathrm{Cl}(X) \otimes \mathbb{R}.$$

Note that v_j is the sum of the generators of C_j. The maculate regions are now parameterised by $J \subseteq \{1, \dots, k\}$ and equal

$$\mathcal{M}_\mathbb{R}(\mathcal{R}_J) = \textstyle\sum_{j \notin J} C_j + \sum_{j \in J} (-v_j - C_j).$$

We claim that the elements $(*, \dots, *, j) \in \mathbb{Z}^{k-1} \times \{j\} \subset \mathbb{Z}^k$ with $j = -1, \dots, -(\ell_k - 1)$ are always $\mathbb{R}$-immaculate. Indeed, the vectors v_j and the cones C_j for $j = 1, \dots, k-1$ have zero as their last entries. The last entry of v_k is ℓ_k, and the last entries of the generators of C_k are always 1. Thus, any point in a maculate cone $\mathcal{M}_\mathbb{R}(\mathcal{R}_J)$ has either the last entry at least 0 or at most $-\ell_k$.

We turn our attention to the points of the form $(*, \dots, *, 0) \in \mathbb{Z}^{k-1} \times \{0\} \subset \mathbb{Z}^k$. We remark that those line bundles are exactly the pullbacks of line bundles on X_{k-1}. Thus Corollary 4.6 is relevant here, if we restrict the attention to $p^*(\mathrm{Imm}_\mathbb{Z}(X_{k-1})) \subset \mathrm{Imm}_\mathbb{Z}(X)$. Instead, our argument here is stronger, about the $\mathbb{R}$ -immaculate locus. If $k \in J$, then $\mathcal{M}_\mathbb{R}(\mathcal{R}_J) \cap (\mathbb{Z}^{k-1} \times \{0\}) = \emptyset$. Thus, those J are never a source of maculacy for points with the last coordinate 0. On the other hand, if $k \notin J$, then $\mathcal{M}_\mathbb{R}(\mathcal{R}_J) \cap (\mathbb{Z}^{k-1} \times \{0\}) = \overline{\mathcal{M}}_\mathbb{R}(\mathcal{R}_J)$ and $\overline{\mathcal{M}}_\mathbb{R}$ denotes the maculate region with respect to $\overline{\pi}$ obtained from π by deleting the last row and the last block of columns. In particular, $\overline{\pi}$ corresponds to the fan Σ_{k-1}.

Therefore, by the inductive assumption, all the generating seeds are $\mathbb{R}$-immaculate concluding the proof of (i). Moreover, the inductive assumption together with Alexander/Serre duality (see Remarks 5.4, 7.11, and Corollary 4.6) shows that both loci $\mathrm{Imm}_\mathbb{Z}(X)$ and $\mathrm{Imm}_\mathbb{R}(X)$ are closed under c-hull operation, proving (ii). The induction and the above discussion also show (iii) for points of the form $(*, \dots, *, j)$ for $-\ell_k < j \le 0$. By Alexander/Serre duality (or the shift from the definition of c-hull), the case of $j = -\ell_k$ is also proved: if $a = (*, \dots, *, -\ell_k)$, then a is ($\mathbb{R}$-)immaculate if and only if its shift $a - v_k \in \mathbb{Z}^{k-1} \times 0$ is ($\mathbb{R}$-)immaculate.

The duality also swaps the points of the form $(*, \dots, *, \ge 1) \in \mathbb{Z}^{k-1} \times \mathbb{Z}_{\ge 1} \subset \mathbb{Z}^k$ with those of the form $(*, \dots, *, \le -(\ell_k + 1))$. Therefore, to complete the proof of (iii) it remains to show that no line bundle whose class in $\mathrm{Cl}(X)$ is $a = (*, \dots, *, \ge 1)$ is immaculate. We must show this under the assumptions that (iii) holds for X_{k-1} and $c_{k-1,k}$ has two entries differing by more than ℓ_{k-1}, say, $|c^1_{k-1,k} - c^2_{k-1,k}| > \ell_{k-1}$.

As before, $\mathcal{R}_J$-maculacy can only go along with $k \notin J$. Let a^k be the last coordinate of a and consider two vectors $b_\nu = a - a^k \cdot (c^\nu_{1k}, \ldots, c^\nu_{k-1,\,k}, 1) \in \mathbb{Z}^{k-1} \times 0$ for $\nu = 1$ or 2. The difference between the $(k-1)$-st coordinates of b_1 and b_2 is at least $\ell_{k-1} + 1$. Using (iii) for X_{k-1}, since all immaculate line bundles for X_{k-1} have the last coordinate in the set $\{-\ell_{k-1}, \ldots, 0\}$, at least one of b_1 or b_2 is not immaculate. Say b_1 is not immaculate. Let $J \subset \{1, \ldots, k-1\}$ be the subset such that $b_1 \in \overline{\mathcal{M}}_{\mathbb{Z}}(\mathcal{R}_J)$. Then $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}_J)$ is the sum of $\overline{\mathcal{M}}_{\mathbb{Z}}(\mathcal{R}_J)$ and the monoid generated by $(c^\nu_{1k}, \ldots, c^\nu_{k-1,\,k}, 1)$. Therefore $a = b_1 + a^k \cdot (c^1_{1k}, \ldots, c^1_{k-1,\,k}, 1) \in \mathcal{M}_{\mathbb{Z}}(\mathcal{R}_J)$ and a cannot be immaculate. ☐

We remark that for non-general values of c the conclusion of (iii) needs not to hold, see for example Figure 6.

8. The immaculate locus for Picard rank 3

In this section we finally make everything concrete in the case of Picard rank 3. We first review the classification of Batyrev, and then describe the tempting subsets of rays. Finally, we list a lot of immaculate line bundles and prove (similarly to Theorem 7.12) that for sufficiently general parameters the listed ones are all immaculate line bundles.

8.1. Classification by Batyrev

In [Bat91] a classification of smooth, projective toric varieties of Picard rank three is given by using its primitive collections.

Proposition 8.1 ([Bat91, Thm 5.7])**.** *If Σ is a complete, regular d-dimensional fan with $d+3$ generators, then the number of primitive collections of its generators is equal to 3 or 5.*

In the case that there are exactly three primitive collections the fan Σ is a splitting fan by Proposition 7.1. Thus the associated toric variety is isomorphic to a projectivisation of a decomposable bundle over a smooth toric variety of smaller dimension and Picard rank two. In particular, Theorem 7.12 provides a full description of the immaculate loci in this case.

Therefore, for the rest of this section we are going to assume that X is a smooth variety of Picard rank 3, which has exactly five primitive collections. Following [Bat91] we give a more precise description of the fan. There is a decomposition of the rays $\Sigma(1)$ into five disjoint subsets J_α and the primitive collections are given by $J_\alpha \cup J_{\alpha+1}$ for $\alpha \in \mathbb{Z}/_{5\mathbb{Z}}$.

Proposition 8.2 ([Bat91, Thm 6.6])**.** *Let us denote* $\mathcal{J}_\alpha = J_\alpha \cup J_{\alpha+1}$*, where* $\alpha \in {}^{\mathbb{Z}}/_{5\mathbb{Z}}$*,*

$$J_0 = \{v_1, \ldots, v_{p_0}\}, J_1 = \{y_1, \ldots, y_{p_1}\}, J_2 = \{z_1, \ldots, z_{p_2}\},$$
$$J_3 = \{t_1, \ldots, t_{p_3}\}, J_4 = \{u_1, \ldots, u_{p_4}\},$$

and $p_0 + \cdots + p_4 = d + 3$*. Then any complete regular* d *-dimensional fan* Σ *with the set of generators* $\Sigma(1) = \bigcup J_\alpha$ *and five primitive collections* $\mathcal{J}_\alpha$ *can be described up to a symmetry of the pentagon by the following primitive relations with non-negative integral coefficients* $c_2, \ldots, c_{p_2}$*,* $b_1, \ldots b_{p_3}$*:*

$$\begin{aligned}
\sum_{i=1}^{p_0} v_i + \sum_{i=1}^{p_1} y_i - \sum_{i=2}^{p_2} c_i z_i - \sum_{i=1}^{p_3} (b_i + 1) t_i &= 0, \\
\sum_{i=1}^{p_1} y_i + \sum_{i=1}^{p_2} z_i - \sum_{i=1}^{p_4} u_i &= 0, \\
\sum_{i=1}^{p_2} z_i + \sum_{i=1}^{p_3} t_i &= 0, \\
\sum_{i=1}^{p_3} t_i + \sum_{i=1}^{p_4} u_i - \sum_{i=1}^{p_1} y_i &= 0, \\
\sum_{i=1}^{p_4} u_i + \sum_{i=1}^{p_0} v_i - \sum_{i=2}^{p_2} c_i z_i - \sum_{i=1}^{p_3} b_i t_i &= 0.
\end{aligned}$$

It looks less scary if we write those equations as a matrix whose rows indicate the five primitive relations. This matrix consists of five blocks of columns of sizes $p_0, \ldots, p_4$. By $\underline{0} = (0, 0, \ldots, 0)$ and $\underline{1} = (1, 1, \ldots, 1)$ we mean row vectors of the appropriate size to fit into the indicated block. Denoting $c = (0, c_2, \ldots, c_{p_2}) \in \mathbb{Z}_{\geq 0}^{p_2}$ and $b = (b_1, \ldots, b_{p_3}) \in \mathbb{Z}_{\geq 0}^{p_3}$, the primitive relation matrix looks like

$$\begin{pmatrix}
\underline{1} & \underline{1} & -c & -(b+\underline{1}) & \underline{0} \\
\underline{0} & \underline{1} & \underline{1} & \underline{0} & -\underline{1} \\
\underline{0} & \underline{0} & \underline{1} & \underline{1} & \underline{0} \\
\underline{0} & -\underline{1} & \underline{0} & \underline{1} & \underline{1} \\
\underline{1} & \underline{0} & -c & -b & \underline{1}
\end{pmatrix}.$$

8.2. Tempting subsets

As above, we suppose $X = \mathbb{TV}(\Sigma)$ is a smooth projective toric variety of dimension d and Picard rank 3, whose fan Σ has five primitive relations.

For finding the immaculate line bundles the first step is to find the tempting subsets of $\Sigma(1)$. We have seen in Proposition 5.18 that some subsets are always tempting. In our situation these are already all tempting subsets.

The following lemma is shown in [Efi14, Theorem A.1.3)]. We include a proof for clarity.

Lemma 8.3. *The only tempting subsets are primitive collections, their complements, the empty set and the full subset $\Sigma(1)$.*

Proof. Let $\mathcal{R}$ be a non-empty tempting subset, which is not equal to $\Sigma(1)$. Then $\mathcal{R}$ and $\Sigma(1) \setminus \mathcal{R}$ do not span cones in Σ by Proposition 5.15. It follows that there exist two primitive collections P, P', with $P \subseteq \mathcal{R} \subseteq \Sigma(1) \setminus P'$. In the notation as above we obtain

$$J_\alpha \cup J_{\alpha+1} \subseteq \mathcal{R} \subseteq J_{\beta+2} \cup J_{\beta+3} \cup J_{\beta+4}$$

for some $\alpha, \beta \in \mathbb{Z}/_{5\mathbb{Z}}$. This already implies that $\beta = \alpha \pm 2$:

$$\begin{aligned} &J_\alpha \cup J_{\alpha+1} \subseteq \mathcal{R} \subseteq J_{\alpha-1} \cup J_\alpha \cup J_{\alpha+1}, \text{ or} \\ &J_\alpha \cup J_{\alpha+1} \subseteq \mathcal{R} \subseteq J_\alpha \cup J_{\alpha+1} \cup J_{\alpha+2}. \end{aligned}$$

For brevity we denote by J', respectively, either $J_{\alpha-1}$ or $J_{\alpha+2}$. So the only question is, what is $\mathcal{R} \cap J'$. If we show the intersection is empty or the whole J', the proof will be completed.

Denote by $\mathcal{R}' := \mathcal{R} \cap J'$, and assume conversely that the $\mathcal{R}'$ is not equal to $\emptyset$ or J', and consider $J_\alpha \cup \mathcal{R}'$. This set does not contain any primitive collection, thus it is a face. The same holds for $J_{\alpha+1} \cup \mathcal{R}'$. Hence $\mathcal{R}$ is the union of two faces which intersect in a common face $\mathcal{R}'$. This implies that $\mathcal{R}$ is not tempting. ☐

Proposition 8.4. *Suppose as above that X is a smooth projective toric variety of Picard rank 3 with five primitive collections J_i of lengths p_i. If L is a line bundle on X such that $\mathrm{H}^i(X, L) \neq 0$, then*

$$i \in \{0,\ p_\alpha + p_{\alpha+1} - 1,\ p_{\alpha-1} + p_{\alpha-2} + p_{\alpha-3} - 2,\ \dim X\}_{\alpha \in \mathbb{Z}/5\mathbb{Z}}.$$

Proof. The tempting subset $\emptyset$ and $\Sigma(1)$ lead to line bundles with non-trivial cohomology in degrees 0 and $\dim X$ respectively.

Along the lines of the proof of Proposition 5.18 we see that the primitive collection $\mathcal{J}_\alpha$ leads to line bundles with non-trivial cohomology in degree $p_\alpha + p_{\alpha+1} - 1$, and the complement of the primitive collection to line bundles with non-vanishing cohomology in degree $p_{\alpha-1} + p_{\alpha-2} + p_{\alpha-3} - 2$. Since there are no other tempting subsets by Lemma 8.3, other degrees cannot occur. ☐

8.3. Immaculate line bundles for Picard rank 3

We can compute the immaculate line bundles as described in Proposition 5.6. For this we have to consider $\pi(\mathbb{Z}_{\geq 0}^{\Sigma(1)\setminus\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}})$ for all maculate $\mathcal{R}$ where π is given as the transpose of the map embedding the kernel of the ray map into $\mathbb{Z}^{\Sigma(1)}$. This can be realised by selecting a $\mathbb{Z}$-basis out of the rows of the matrix of primitive relations presented at the end of Subsection 8.1. Picking its first, second and fourth row, we obtain

$$\pi = \begin{pmatrix} \underline{1} & \underline{1} & -c & -(b+\underline{1}) & \underline{0} \\ \underline{0} & \underline{1} & \underline{1} & \underline{0} & -\underline{1} \\ \underline{0} & -\underline{1} & \underline{0} & \underline{1} & \underline{1} \end{pmatrix}.$$

The Mori cone of a projective, simplicial toric variety is generated by the primitive relations, being understood as classes of 1-cycles, see [CLS11, Theorem 6.4.11] or for a bit more general statement [CvR09, Proposition 1.10]. In our case the Mori cone is a three-dimensional simplicial cone, and the primitive relations we chose correspond to its rays.

Remark 8.5. (i) For all parameters b, c, the matrix π leads to a smooth fan of Picard rank 3. This means the converse of Proposition 8.2, and it follows from Subsection 6.1: The 3-minors with respect to the columns chosen from the blocks $(\alpha, \alpha+1, \alpha+3)$ for $\alpha \in \mathbb{Z}/5\mathbb{Z}$ are always 1.

(ii) It is straightforward (although tedious) to check that for all 12 tempting subsets $\mathcal{R} \subset \Sigma(1)$ the tail cone of the respective maculate region $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ is either a smooth cone or a cone with 4 rays which do also form its Hilbert basis (the latter is the case for $J_3 \cup J_4$ if $c_{p_2} < b_1 + 1$, for $J_4 \cup J_0$ if $b_1 > 0$, and for their respective complements).

(iii) From (ii) it follows that, independent of the parameters b, c, we always have that $\mathcal{M}_{\mathbb{Z}}(\mathcal{R}) = \mathcal{M}_{\mathbb{R}}(\mathcal{R}) \cap \operatorname{Pic} X$ and thus $\operatorname{Imm}_{\mathbb{Z}}(X) = \operatorname{Imm}_{\mathbb{R}}(X)$.

We will distinguish three classes (F), (A), (B) of line bundles which will become the main heroes for the immaculate locus presented in Proposition 8.7. To locate these classes in $\mathbb{Z}^3$ we will use the horizontal projection $(x, y, z) \mapsto (y, z)$ and start with some geography on the target space.

Definition 8.6. Denote by P_1 and P_2 the following two planar parallelograms P_1, P_2:

$$\begin{aligned} P_1 &= \operatorname{conv}((-p_1 - p_2 - p_3 + 2,\ p_1 - 1), (-p_1,\ p_1 - 1), \\ &\qquad (-p_2 + p_4,\ -p_3 - p_4 + 1), (p_3 + p_4 - 2,\ -p_3 - p_4 + 1)), \\ P_2 &= \operatorname{conv}((-p_1 - p_2 + 1,\ p_1 + p_2 - 2), (p_4 - 1,\ -p_4), \\ &\qquad (-p_1 - p_2 + 1,\ p_1 - p_3), (p_4 - 1,\ -p_2 - p_3 - p_4 + 2)) \end{aligned}$$

They are depicted in blue and red in Figure 7, and we will be interested in their union. Note the following two special cases:

- If $p_2 = 1$, then $P_2 \subset P_1$, see Figure 8, and the simplified vertices of P_1 are:

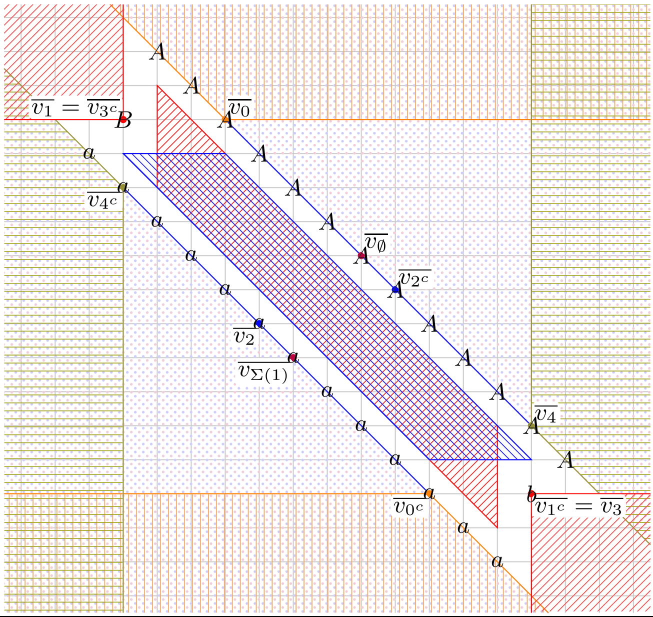
$$\begin{array}{rr} (-p_1 - p_3 + 1,\ p_1 - 1), & (-p_1,\ p_1 - 1), \\ (p_4 - 1,\ -p_3 - p_4 + 1), & (p_3 + p_4 - 2,\ -p_3 - p_4 + 1). \end{array}$$

- If $p_3 = 1$, then $P_1 \subset P_2$, see Figure 9, and the simplified vertices of P_2 are:

$$\begin{array}{rr} (-p_1 - p_2 + 1,\ p_1 + p_2 - 2), & (-p_1 - p_2 + 1,\ p_1 - 1), \\ (p_4 - 1,\ -p_4), & (p_4 - 1,\ -p_2 - p_4 + 1). \end{array}$$

Now we can describe the three classes of our immaculate candidates. They consist of entire "horizontal" lines or line segments, that are parallel to the x-axis:

- **Full horizontal lines (F).** This class consists of the union of the (infinite) lines $(*,\ y,\ z)$ with $(y,\ z) \in P_1 \cup P_2$ (including the boundary). Note that it does not depend on the values of b and c and it is self dual with respect to Serre duality: here the canonical divisor is $(-p_0 - p_1 + p_3 + \overline{c} + \overline{b},\ -p_1 - p_2 + p_4,\ p_1 - p_3 - p_4)$.
- **Line segments of Type (A).** This class consists of finite horizontal segments I_y (described below) located over the diagonal $(*,\ y,\ -y)$. Denote $D_{x,y} = (x,\ -y,\ y)$, and for any $y \in [-p_3 - p_4 + 1, p_1 + p_2 - 1]$ let
$$I_y \ := \ \{D_{x,y} \mid x_0(y) \leq x \leq x_1(y)\}$$
be the set of lattice points on the segment with x coordinate varying from $x_0(y)$ to $x_1(y)$. The values of $x_0(y), x_1(y)$ and the number of elements of I_y is in Table 1. Notice that they do not depend on b or c, as in the case of type (F).
- **Line segments of Type (B).** The segments of this type depend on p_2 and p_3 via the parallelograms P_1 and P_2 elaborated in Definition 8.6.
  - If $p_2, p_3 \geq 2$, then this type consist of just one horizontal segment whose projection to the $(y,\ z)$-plane is located left and above the intersection of the upper edges of the parallelograms P_1 and P_2, see the point marked as B on Figure 7. The line segment contains $p_0 - 1$ immaculate line bundles with coordinates
  $$([-p_0 - p_1 + \overline{c} + 1, -p_1 + \overline{c} - 1],\ -p_1 - p_2,\ p_1),$$
  where $\overline{c} := \sum c_i$.



Point	**Coordinates**	**Point**	**Coordinates**
$\overline{v_{\Sigma(1)}} = \overline{K}$	$(-p_1 - p_2 + p_4,\ p_1 - p_3 - p_4)$	$\overline{v_\emptyset}$	$(0,\ 0)$
$\overline{v_{0^c}}$	$(-p_2 + p_4,\ -p_3 - p_4)$	$\overline{v_0}$	$(-p_1,\ p_1)$
$\overline{v_{1^c}} = \overline{v_3}$	$(p_4,\ -p_3 - p_4)$	$\overline{v_1} = \overline{v_{3^c}}$	$(-p_1 - p_2,\ p_1)$
$\overline{v_{2^c}}$	$(-p_1 + p_4,\ p_1 - p_4)$	$\overline{v_2}$	$(-p_2,\ -p_3)$
$\overline{v_{4^c}}$	$(-p_1 - p_2,\ p_1 - p_3)$	$\overline{v_4}$	$(p_4,\ -p_4)$

Figure 7: $p_2, p_3 > 1$. The projected maculate regions to the $(y,\ z)$-plane for the example $(p_1,\ p_2,\ p_3,\ p_4) = (4,\ 3,\ 2,\ 5)$ and a table with the general coordinates of the projected vertices of the maculate regions, where $\overline{v_i}$ and $\overline{v_{i^c}}$ denotes the projected vertex of the maculate region $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ for $\mathcal{R} = \mathcal{J}_i$ respectively $\mathcal{R} = \mathcal{J}_i^c$. The polyhedra P_1 and P_2 from Definition 8.6 are depicted in blue and red. The letters A and B indicate where the line segments of immaculate line bundles are located in the projection, and the letters a, b denote the location of their Serre duals.

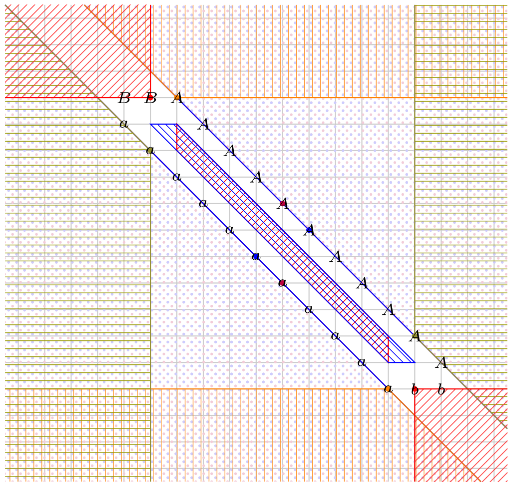


Figure 8: $p_2 = 1$. The projected maculate regions for the example $(p_1, p_2, p_3, p_4) = (4, 1, 2, 5)$. The lattice points of the white area are the lattice points of the parallelogram P_1.

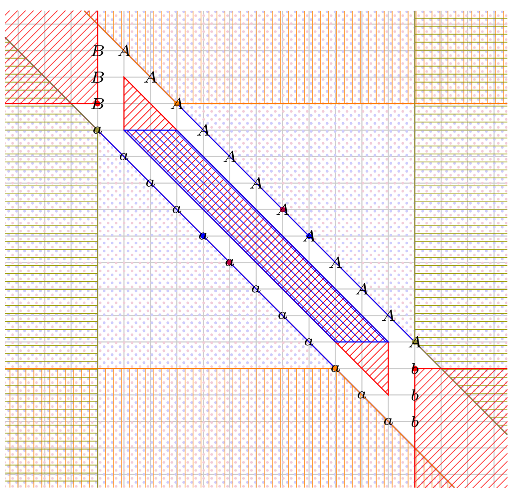


Figure 9: $p_3 = 1$. The projected maculate regions for the example $(p_1, p_2, p_3, p_4) = (4, 3, 1, 5)$. The lattice points of the white area are the lattice points of the parallelogram P_2.

Table 1: Isolated immaculate line bundles type A

$\bullet \le y$	$y \le \bullet$	$x_0(y)$	$x_1(y)$	$\#I_y$
Case $p_1 < p_4$				
$-p_3-p_4+1$	$-p_4$	$-p_0-p_4-y+1$	$-y-1$	p_0+p_4-1
$-p_4+1$	p_1-p_4	$-p_0-p_1+1$	$-y-1$	p_0+p_1-y-1
p_1-p_4+1	0	$-p_0-p_4-y+1$	$-y-1$	p_0+p_4-1
1	p_1-1	$-p_0-p_4-y+1$	-1	p_0+p_4+y-1
p_1	p_1+p_2-1	$-p_0-p_1+1$	-1	p_0+p_1-1
Case $p_1 > p_4$				
$-p_3-p_4+1$	$-p_4$	$-p_0-p_4-y+1$	$-y-1$	p_0+p_4-1
$-p_4+1$	0	$-p_0-p_1+1$	$-y-1$	p_0+p_1-y-1
1	p_1-p_4	$-p_0-p_1+1$	-1	p_0+p_1-1
p_1-p_4+1	p_1-1	$-p_0-p_4-y+1$	-1	p_0+p_4+y-1
p_1	p_1+p_2-1	$-p_0-p_1+1$	-1	p_0+p_1-1
Case $p_1 = p_4$				
$-p_3-p_4+1$	$-p_4$	$-p_0-p_4-y+1$	$-y-1$	p_0+p_4-1
$-p_4+1$	0	$-p_0-p_1+1$	$-y-1$	p_0+p_1-y-1
1	p_1-1	$-p_0-p_4-y+1$	-1	p_0+p_4+y-1
p_1	p_1+p_2-1	$-p_0-p_1+1$	-1	p_0+p_1-1

– If $p_2 = 1$, then the points of Type (B) consist of p_3 horizontal line segments, each containing $p_0 - 1$ immaculate line bundles. The coordinates are

$$([-p_0-p_1+1, -p_1-1],\ -p_1-p_2-y,\ p_1)$$

for $y \in [0, p_3-1]$. On Figure 8 their projections onto (y, z) plane are indicated by the letter B. Roughly speaking, their projections are at each lattice point directly above the upper edge of the parallelogram P_1,

– For $p_3 = 1$, there are p_2 horizontal line segments of Type (B) each containing $p_0 - 1$ immaculate line bundles. The coordinates are

$$([-p_0-p_1+\overline{c}-y(\overline{b}+1)+1, -p_1+\overline{c}-y(\overline{b}+1)-1],\ -p_1-p_2,\ p_1+y)$$

for $y \in [0, p_2-1]$. On Figure 9 their locations in the projection are indicated by the letter B, as for the previous case. In this case, the projections are located directly left of P_2.

Our result is that the types (F), (A), and (B) are always immaculate. Moreover, for sufficiently "general" parameters the listed line bundles and their Serre duals are all ($\mathbb{R}$-)immaculate line bundles. We summarise this discussion in the following proposition, but we only sketch the proof as it consists of working out the combinatorial details.

Proposition 8.7. *For X a toric projective variety of Picard number 3 with 5 primitive collections we have:*

(i) All the line bundles of type (F), (A) and (B) are immaculate.

(ii) The coordinates of the line bundles of type (F) and (A) do not depend on b and c, the coordinates of the Serre duals of the line bundles of (A), do depend on b and c.

(iii) The line bundles of type (A) are the only immaculate line bundles among the $D_{x,y} = (x,\ -y,\ y)$ with $y \in [-p_3 - p_4 + 1, p_1 + p_2 + 1]$ independent of b, c.

(iv) For b, c large enough, that is $\max(b_{p_3}, c_{p_2}) \geq p_0 + p_1 + \max(p_2, p_3) + p_4$ and if $p_2 \neq 1$ additionally $c_{p_2} \geq p_0 - 1$ and for $p_3 \neq 1$ the additional condition that $b_{p_3} - b_1 \geq p_0 - 1$, the only ($\mathbb{R}$-)immaculate line bundles are the previously mentioned and their Serre duals.

Sketch of proof. For proving all of those statements we will consider the "horizontal" projection of the Picard group and in particular of the twelve maculate regions to the y, z-plane. Figures 7, 8 and 9 illustrate the situation, for the cases $p_2, p_3 > 1$ and $p_2 = 1$, $p_3 = 1$ respectively.

- **(F) Full horizontal lines.** If the projected divisor $\overline{D}$ is not in any of the projected maculate cones $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$, then all the divisors in the line parallel to the kernel of the projection are immaculate.
- **Line segments of Type (A) and (B).** Given a divisor D we want to know whether it is immaculate. Thus we want to know whether there is an $\mathcal{R}$ such that $D \in \mathcal{M}_{\mathbb{R}}(\mathcal{R})$. We analyse the projected situation. We know that if $\overline{D} \notin \overline{\mathcal{M}_{\mathbb{R}}(\mathcal{R})}$, then $D \notin \mathcal{M}_{\mathbb{R}}(\mathcal{R})$. This eliminates a large number of candidate $\mathcal{R}$'s. Various situations that can occur are depicted in the aforementioned figures.
  The candidate $\mathcal{R}$'s for the Type (A) divisors $D_{x,y}$ are $\emptyset, \mathcal{J}_0, \mathcal{J}_2^c$ and $\mathcal{J}_4$. Studying the corresponding maculate regions, with a case analysis for the y-coordinate concludes the proof.
  For the Type (B) divisors, the candidate $\mathcal{R}$'s are $\mathcal{J}_1$ and $\mathcal{J}_3^c$. Then showing that the divisors do not lie in the corresponding maculate regions guarantees their immaculacy.
- **Statement (iv).** With the given conditions one can show that of all D with $\overline{D} \in \overline{\mathcal{M}_{\mathbb{R}}(\mathcal{J}_1)}$, the divisors of Type (B) are the only immaculate ones, by showing that all others lie in either $\mathcal{M}_{\mathbb{R}}(\mathcal{J}_1)$ or $\mathcal{M}_{\mathbb{R}}(\mathcal{J}_3^c)$. Then

for the D with $\overline{D} \in \overline{\mathcal{M}_{\mathbb{R}}(\emptyset)} \setminus \overline{\mathcal{M}_{\mathbb{R}}(\mathcal{J}_1)}$ not of Type (A), the inequalities of the parameters imply that D lies in the maculate region of either $\emptyset$ or $\mathcal{J}_2^c$. Serre duality finishes the proof. $\square$

Remark 8.8. In [Efi14, Proof of Thm. 6.2] a similar description is given implicitly. There the subclass $(p_0,\ p_1,\ p_2,\ p_3,\ p_4) = (n+2a,\ 1,\ k,\ n,\ 1)$ with $c = (0,\ldots,0,a) \in \mathbb{Z}^{p_2}_{\geq 0}$ and $b = (0,\ldots,0,a) \in \mathbb{Z}^{p_3}_{\geq 0}$ is considered. The coordinates on the Picard group are different and correspond to the fifth, second and third primitive relation (in that order). Efimov gives explicit inequalities for most of the maculate cones (forbidden sets). However there is no explicit description how the immaculate region looks like.

9. Computational aspects

In this section we want to highlight the computational advantages of immaculate line bundles and maculate regions. All of these objects and conditions give rise to nice combinatorial algorithms. Throughout the development of this paper we have implemented these as a `polymake` ([GJ00]) extension. The combinatorial nature of these algorithms makes them very fast, as opposed to many algorithms from commutative algebra. This stresses the main computational advantage of working with toric varieties. We will give a short sketch of the resulting algorithms. The `polymake` extension itself can be found at https://github.com/lkastner/immaculatePolymake. It will be further enhanced in the future.

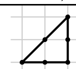
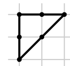
Immaculacy of a line bundle on a projective toric variety $X = \mathbb{TV}(\Sigma)$ can be checked from its representation as a difference of nef divisors. Thus we want to check all differences $\Delta^- \setminus (\Delta^+ - m)$, for any $m \in M$, for $\Bbbk$-acyclicity, via Proposition 4.3. But it is actually enough to check only finitely many m, since both Δ^- and Δ^+ are compact and thus they only intersect for finitely many shifts. Using Proposition 2.2, we just need to consider Δ^- as a polytopal complex and remove any face intersecting $\Delta^+ - m$ non-trivially, for those finitely many m. Homology computation of the resulting polytopal complex is already built in `polymake` and many other software frameworks for combinatorial software as well.

Next we want to find the tempting $\mathcal{R} \subseteq \Sigma(1)$. The easiest way is to brute force this by checking any subset of rays and then compute the homology. One can also imagine a more sophisticated approach by considering sub-diagrams of the Hasse diagram of Σ. So far this has never been a bottleneck in our examples, though in case this happens, results of Subsection 5.2 might be of use.

Table 2: Lines of immaculate line bundles for the hexagon

unbounded direction	basepoint	(Δ^+,	Δ^-)
1 1 0 0	0 0 -1 -1	$t \cdot$ ⟋	◿
	1 0 -1 0	$t \cdot$ ⟋	|
	0 0 -1 0	$t \cdot$ ⟋	◸
	-1 0 -1 -1	$t \cdot$ ⟋	—
1 0 1 1	0 -1 -1 0	$t \cdot$ —	|
	0 -1 0 0	$t \cdot$ —	◿
	-1 -1 0 0	$t \cdot$ —	⟋
	-1 -1 -1 0	$t \cdot$ —	◸
0 1 1 0	-1 0 0 0	$t \cdot$ |	◸
	-1 0 1 0	$t \cdot$ |	⟋
	-1 0 0 -1	$t \cdot$ |	◿
	-1 0 -1 -1	$t \cdot$ |	—

Table 3: Isolated immaculate line bundles for the hexagon

Pic(X) coordinates	(Δ^+,	Δ^-)
-2 -2 -2 -2	pt	
-2 -2 -2 0	pt	
0 0 0 -1	◸	◿
0 0 0 1	◿	◸

From the collection of all tempting $\mathcal{R}$ we can finally compute the immaculate locus $\mathrm{Imm}_{\mathbb{R}}(X)$, or rather the lattice points thereof. We only need to compute the intersection of all complements of the $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$. It is not difficult to see that this is a union of polyhedra. Since $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ is a rational polyhedral cone, we can write it as a finite intersection of halfspaces. Taking the complement of this cone means taking the union of the complementary halfspaces. Since we are only interested in the lattice points of $\mathrm{Imm}_{\mathbb{R}}(X)$, we just move the bounding hyperplane by one away from $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ and do not worry about openness of the complement. Now we get the polyhedra giving the lattice points of $\mathrm{Imm}_{\mathbb{R}}(X)$ by picking one complementary halfspace for every $\mathcal{R}$ and then intersecting these. Consider any possible combination and take the union of the resulting polyhedra.

We now restrict our attention to the hexagon example (see Examples 3.2 and 5.2). We immediately see that the main bottleneck of the algorithm for $\mathrm{Imm}_{\mathbb{R}}(X)$ is the amount of intersections to compute. There are 34 tempting $\mathcal{R}$'s and if every $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ was bounded by only two hyperplanes, we would

Table 4: Exceptional sequences of line bundles for the hexagon

$D^0 = [0, 0, 0, 0]$

D^1				D^2				D^3				D^4				D^5			
-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1	-2	-2	-1	0	-1	-1	-2	-1
-1	-1	-1	-1	-2	-2	-1	-1	-1	-1	-2	-1	-2	-1	-2	-2	-1	-2	-2	-1
-1	-1	-1	-1	-2	-1	-1	-1	-2	-2	-1	-1	-1	-1	-2	-1	-2	-1	-2	-2
-1	-1	-1	-1	-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1	-1	-1	-2	-1
-1	-1	-1	-1	-1	-1	-1	0	-2	-1	-1	-1	-1	-2	-1	0	-1	-1	-2	-1
-1	-1	0	0	-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1	-2	-2	-1	0
-1	-1	0	0	-1	-1	-1	-1	-2	-1	-1	-1	-1	-2	-1	0	-2	-2	-1	-1
-1	-1	0	0	-1	-1	-1	-1	-1	-1	-1	0	-2	-1	-1	-1	-1	-2	-1	0
-1	-1	0	0	-1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	-2	-1	-1	-1
-1	-1	0	0	-1	0	-1	-1	0	-1	-1	0	-1	-1	-1	-1	-1	-1	-1	0
-1	-1	0	0	0	0	-1	-1	-1	0	-1	-1	0	-1	-1	0	-1	-1	-1	-1
-1	-1	0	0	0	0	-1	-1	0	0	-1	0	-1	0	-1	-1	0	-1	-1	0
-1	0	0	-1	-1	-1	-1	-1	-2	-1	-1	-1	-2	-2	-1	-1	-2	-1	-2	-2
-1	0	0	-1	-1	-1	0	0	-1	-1	-1	-1	-2	-1	-1	-1	-2	-2	-1	-1
-1	0	0	-1	-1	-1	0	0	-1	0	-1	-1	-1	-1	-1	-1	-2	-1	-1	-1
-1	0	0	-1	-1	-1	0	0	0	0	-1	-1	-1	0	-1	-1	-1	-1	-1	-1
-1	0	0	-1	-1	0	0	0	-1	-1	0	0	-1	0	-1	-1	-2	-1	-1	-1
-1	0	0	-1	0	-1	0	0	-1	-1	0	0	-1	-1	-1	-1	-2	-2	-1	-1
-1	0	0	-1	0	-1	0	0	-1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1

have to compute 2^{34} intersections. In fact, all $\mathcal{M}_{\mathbb{R}}(\mathcal{R})$ are actually bounded by more than two hyperplanes. This issue can be overcome by building the intersections step by step and eliminating trivial intersections in between. We start by building the complementary halfspaces of $\mathcal{M}_{\mathbb{R}}(\mathcal{R}_1)$ and $\mathcal{M}_{\mathbb{R}}(\mathcal{R}_2)$, then we consider any intersection. If an intersection is empty already, we eliminate it. Furthermore, we choose the inclusion maximal intersections. Then we intersect the resulting polyhedra with the complementary halfspaces of $\mathcal{M}_{\mathbb{R}}(\mathcal{R}_3)$ and so on.

Thus we have computed the immaculate loci $\mathrm{Imm}_{\mathbb{Z}}(X) = \mathrm{Imm}_{\mathbb{R}}(X)$. They are equal to a union of three unbounded polyhedra and four isolated lattice points that are listed in Table 3. Each unbounded polyhedron consists of four parallel lines, that is lattice lines. The exact lines, together with their polytopes (Δ^+, Δ^-) are depicted in Table 2. Each pair of quadruples of lines intersects in four points.

Now it is easy to compute all exceptional sequences that are contained in the projection of the cube $\pi([-1, 0]^6) \subseteq \mathrm{Cl}(X)$. One just collects the lattice points in the projected cube and then runs a depth first search. There are 228 exceptional sequences of length six in the projected cube. Under the group action on the hexagon these 228 exceptional sequences correspond to 19 orbits of size 12. In Table 4 we list one representative from each orbit. Note that we do not need to use the four isolated points for these exceptional sequences. This is different than in the case of the splitting fans, for example the Picard rank 2 case (see [CM04]).

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