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Evolution and monotonicity of a geometric constant under the Ricci flow*

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Abstract: Let $(M, g(t))$ be a compact Riemannian manifold and the metric $g(t)$ evolve by the Ricci flow. In the paper we derive the evolution equation for a geometric constant λ under the Ricci flow and the normalized Ricci flow, such that there exist positive solutions to the nonlinear equation

$$-\Delta_\phi f + af \ln f + bRf = \lambda f,$$

where Δ_ϕ is the Witten-Laplacian operator, $\phi \in C^\infty(M)$, a and b are both real constants, and R is the scalar curvature with respect to the metric $g(t)$. As an application, we obtain the monotonicity of the geometric constant along the Ricci flow coupled to a heat equation for manifold M with some Ricci curvature condition when $b > \frac{1}{4}$.

Keywords: Eigenvalue, Perelman's μ-entropy, Witten-Laplacian operator, Ricci flow.

1. Introduction

Let M be an n-dimensional compact Riemannian manifold with a time-dependent Riemannian metric $g(t)$, which is a smooth solution to the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t), \tag{1}$$

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where $R_{ij}(t)$ is the Ricci curvature of metric $g(t)$. The Ricci flow was first introduced by Hamilton [9] to research the geometry of positive Ricci curvature on three dimensional manifolds. More precisely, he proved that the solution to Ricci flow converges to a constant curvature metric on a compact three dimensional manifold with positive Ricci curvature. Later, the Ricci flow was first treated as a gradient flow by Perelman. In his seminal preprint [16], Perelman introduced the so-called $\mathcal{F}$-entropy functional and showed that in a certain sense the Ricci flow is a gradient flow of the functional $\mathcal{F}$. Moreover, he also proved that along the Ricci flow coupled to a backward heat-type equation the functional $\mathcal{F}$ is nondecreasing, which implies the monotonicity of the first eigenvalue of $-4\Delta+R$ along the Ricci flow. As an application of the monotonicity, Perelman was able to rule out nontrivial steady or expanding breathers on compact manifolds.

Since then there has been increasing attentions on the eigenvalue problems under various geometric flows, especially the Ricci flow. In [15] Ma gave a monotonicity formula of the first eigenvalue of the Laplacian operator on a domain with Dirichlet boundary condition under the Ricci flow. Cao [1] studied the eigenvalues of $-\Delta + \frac{R}{2}$ and showed that they are nondecreasing along the Ricci flow for manifolds with nonnegative curvature operator. Li [11] obtained the monotonicity of eigenvalues of the operator $-4\Delta + kR$ and ruled out compact steady Ricci breathers by using their monotonicity. Later, Cao [2] also improved his own previous results and proved that the first eigenvalues of $-\Delta + cR(c \geq \frac{1}{4})$ are nondecreasing under the Ricci flow on the manifolds without curvature assumption. Ling considered the first nonzero eigenvalue under the normalized Ricci flow, gave a Faber-Krahn type of comparison theorem and a sharp bound [13], and constructed a class of monotonic quantities on closed n-dimensional manifolds [14]. Moreover, Zhao got the evolution equation for the first eigenvalue of the Laplacian operator along the Yamabe flow, gave some monotonic quantities under the Yamabe flow [18], and proved that the first eigenvalue of the p-Laplace operator is increasing and differentiable almost everywhere along the unnormalized powers of the mth mean curvature flow [19] and the unnormalized H^k-flow [20]. Guo and his collaborators [8] derived an explicit formula for the evolution of the lowest eigenvalue of the Laplace-Beltrami operator with potential in abstract geometric flows. Recently, the first author and his collaborators proved that the eigenvalues of some geometric operators related to the Witten-Laplacian are nondecreasing under the Ricci flow in [4], [6] and [7], and derived some monotonicity formulas of the eigenvalues for some geometric operators along the Yamabe flow in [5].

In fact, Perelman [16] also present the other important functional $\mathcal{W}$ in order to study the shrinking breather. The $\mathcal{W}$ functional is defined by

$$\mathcal{W}(g,f,\tau)=\int_M\left[\tau(R+|\nabla f|^2)+f-n\right]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}d\nu,$$

where f is a smooth function on M, $d\nu$ is the Riemannian volume measure on (M,g), and τ is a positive scale parameter. Like the functional $\mathcal{F}$, the $\mathcal{W}$ functional is also nondecreasing along the Ricci flow coupled to a backward heat-type equation. The associated μ-entropy is given by the infimum of the $\mathcal{W}$ functional

$$\mu(g,\tau)=\inf\left\{\mathcal{W}(g,f,\tau)|f\in C^\infty(M),\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_M e^{-f}d\nu=1\right\}.$$

Thus the monotonicity of μ-entropy is same with the $\mathcal{W}$ functional under the Ricci flow. Now if we let $u=e^{-\frac{f}{2}}$, it is obvious that the μ-entropy corresponds to the best logarithmic Sobolev constant. More importantly, one can show that the μ-entropy is achieved by some positive smooth function u (cf. Cao and Zhu [3] and references therein) which satisfies the Euler-Lagrange equation

$$\tau(-4\Delta u+Ru)-2u\ln u-nu=\mu(g,\tau)u.$$

Recently, the geometric constant μ-entropy has been generalized by Huang and Li [10]. Under the Ricci flow they obtained the monotonicity of the lowest constant such that there exist positive solutions to the following nonlinear equation

$$-\Delta u+au\ln u+bRu=\lambda_a^b u$$

with a normalized condition $\int_M u^2 d\nu=1$, where a and b are both real constants. As can be seen, these geometric constants are very similar to the eigenvalues of geometric operators, which have become a powerful tool in the study of geometry and topology of manifolds.

In this paper, we consider an n-dimensional compact Riemannian manifold M with a time-dependent Riemannian metric $g(t)$, which evolves by the (normalized) Ricci flow. Inspired by Perelman's μ-entropy and Huang and Li [10], we study a lowest geometric constant λ which satisfies the following nonlinear equation

$$-\Delta_\phi f+af\ln f+bRf=\lambda f, \tag{2}$$

where f is a positive smooth solution with the normalization $\int_M f^2 d\mu = 1$, $d\mu = e^{-\phi(x)}d\nu$ is the weighted volume measure on M, a and b are the same with above, $\phi \in C^\infty(M)$, and Δ_ϕ is the Witten-Laplacian (also called symmetric diffusion operator), i.e.

$$\Delta_\phi = \Delta - \nabla\phi\nabla.$$

When ϕ is a constant function, the Witten-Laplacian operator is just the Laplace-Beltrami operator. In particular, the Witten-Laplacian is also a symmetric operator on $L^2(M)$ analogous to the Laplace-Beltrami operator, and satisfies the following integration by parts formula

$$\int_M (\nabla u, \nabla v)d\mu = -\int_M \Delta_\phi uvd\mu = -\int_M \Delta_\phi vud\mu, \forall u, v \in C^\infty(M). \tag{3}$$

Therefore, many classical results of the Laplace-Beltrami operator can be extended to the Witten-Laplacian operator. For example, we can see these papers ([6], [12] and [17]). The main purpose of this paper is to investigate the monotonicity of the geometric constant λ along the Ricci flow coupled to a heat equation on compact Riemannian manifolds under some curvature assumption for the case $b > \frac{1}{4}$.

The following theorem is our main result.

Theorem 1.1. *Let $g(t), t \in [0, T)$, be a solution to the Ricci flow (1) on an n-dimensional compact Riemannian manifold M, and $\lambda(t)$ be the lowest constant of the nonlinear equation (2). Suppose that the Ricci curvature satisfies*

$$|Rc| \geq \frac{1}{2\sqrt{b} - 1}|\nabla\nabla\phi|, \forall t \in [0, T),$$

where $b > \frac{1}{4}$ and $\phi(\cdot, t) \in C^\infty(M)$ satisfies the heat equation $\frac{\partial\phi}{\partial t} = \Delta\phi$. Then $\lambda(t) + \frac{na^2}{8}t$ is nondecreasing.

Remark 1.1. *In fact, when ϕ is a constant, our result reduces to Theorem 1.1 of Huang-Li in [10].*

The rest of this paper is organized as follows. In Section 2, we will derive the evolution equation of the geometric constant under the Ricci flow. In Section 3, we consider the system of Ricci flow coupled to a heat equation. We will first calculate the evolution equation of the geometric constant along the Ricci flow coupled to the heat equation, and then prove Theorem 1.1 by using it. In Section 4, we will deduce the evolution equation and monotonicity of the geometric constant under the normalized Ricci flow.

2. Evolution equation of the geometric constant

In this section, we establish the evolution equation of the geometric constant λ in the nonlinear equation (2) under the Ricci flow. For simplicity, the smooth function ϕ on M is assumed to be independent of t in the section.

Let $(M, g(t))$ be a compact Riemannian manifold, and $(M, g(t)), t \in [0, T)$ be a smooth solution to the Ricci flow equation (1). Let λ be the lowest constant of the nonlinear equation (2) at time t where $0 \leq t < T$, and f be the corresponding positive solution with the normalization

$$\int_M f^2 d\mu = 1.$$

We assume that $f(x,t)$ is a C^1-family of smooth functions on M, and satisfies the following condition

$$\frac{d}{dt}\left[\int_M f^2 d\mu\right] = 0.$$

Hence, we have

$$\int_M f\left[f_t d\mu + (f d\mu)_t\right] = 0, \tag{4}$$

where $f_t = \frac{\partial f}{\partial t}$.

We also need to define a functional

$$\begin{aligned}\lambda(f,t) &= \int_M \left(-f\Delta_\phi f + af^2 \ln f + bRf^2\right) d\mu \\ &= \int_M \left(-\Delta_\phi f + af \ln f + bRf\right) f d\mu,\end{aligned}$$

where f satisfies the equality (4). At time t, if f is the positive solution to the nonlinear equation (2) corresponding to λ, then

$$\lambda(f,t) = \lambda(t).$$

Let us first derive the evolution equation of the above functional under the general geometric flow.

Lemma 2.1. *Suppose that λ is the lowest constant of the equation (2), f is the corresponding positive solution of λ at time t_0, and the metric $g(t)$ evolves by*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij},$$

where v_{ij} is a symmetric two-tensor. Then we have

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j - \frac{a}{4} V f + b\frac{\partial R}{\partial t} f \right) f d\mu + \int_M \left(v_{ij,i} - \frac{V_j}{2} \right) f_j f d\mu, \tag{5}$$

where $V = Tr(v)$.

Proof. The proof is only a direct computation. Notice that

$$\frac{\partial}{\partial t}\Delta_\phi = \Delta_\phi \frac{\partial}{\partial t} - v_{ij}\nabla_i\nabla_j - \frac{1}{2} g^{kl}(2(\mathbf{div} v)_k - \nabla_k V)\nabla_l + v_{ij}\nabla_i\phi\nabla_j.$$

Hence we have

$$\begin{aligned}
\frac{d}{dt}\lambda(f,t) =& \frac{d}{dt}\int_M (-\Delta_\phi f + af\ln f + bRf)\, f d\mu \\
=& \int_M \left(v_{ij} f_{ij} + \frac{1}{2} g^{kl}(2v_{ki,i} - V_k) f_l - v_{ij}\phi_i f_j + a f_t + b\frac{\partial R}{\partial t} f \right) f d\mu \\
&+ \int_M (-\Delta_\phi f_t + a f_t \ln f + bR f_t)\, f d\mu \\
&+ \int_M (-\Delta_\phi f + af\ln f + bRf)\,(f d\mu)_t \\
=& \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j + a f_t + b\frac{\partial R}{\partial t} f \right) f d\mu \\
&+ \int_M \left(v_{ij,i} - \frac{1}{2} V_j \right) f_j f d\mu \\
&+ \int_M (-\Delta_\phi f + af\ln f + bRf)\left[f_t d\mu + (f d\mu)_t \right],
\end{aligned}$$

where we used (3) in the last equality. At time t_0, f is the corresponding positive solution of λ, i.e., the equation (2) holds. Combining (2) with (4), the last term in the above evolution equation vanishes. Moreover, it follows from (4) that

$$\int_M f f_t d\mu = -\frac{1}{2}\int_M f^2 (d\mu)_t = -\frac{1}{4}\int_M V f^2 d\mu.$$

Finally, at time t_0 we get

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j - \frac{a}{4} V f + b\frac{\partial R}{\partial t} f \right) f d\mu$$

$$+\int_M \left(v_{ij,i} - \frac{1}{2}V_j\right) f_j f d\mu.$$

□

Remark 2.1. *In fact, Lemma 2.1 also tell us the evolution of the geometric constant λ. From the above proof it is easy to see that the evolution equation (5) does not depend on the evolution equation of f, as long as f satisfies (4). Hence we have*

$$\frac{d}{dt}\lambda(t) = \frac{d}{dt}\lambda(f,t) \tag{6}$$

for any time t, when f is the corresponding positive function with λ at time t.

Now we can calculate the evolution equation of the geometric constant under the Ricci flow. In Lemma 2.1, if we choose the symmetric two-tensor $v_{ij} = -2R_{ij}$, the following result holds.

Theorem 2.1. *Let $g(t), t \in [0,T)$, be a solution to the Ricci flow (1) on a compact manifold M^n. Assume that there is a C^1-family of smooth functions $f(x,t) > 0$, which satisfy*

$$-\Delta_\phi f(x,t) + af\ln f + bRf(x,t) = \lambda(t)f(x,t),$$

and the normalization

$$\int_M f(x,t)^2 d\mu = 1.$$

Then the lowest geometric constant $\lambda(t)$ satisfies

$$\begin{aligned} \frac{d}{dt}\lambda(t) =& \frac{1}{2}\int_M |R_{ij} + \psi_{ij}|^2 e^{-\psi} d\mu + \frac{4b-1}{2}\int_M |Rc|^2 e^{-\psi} d\mu \\ &+ \frac{a}{2}\int_M Re^{-\psi} d\mu + \frac{a}{2}\int_M \psi\Delta e^{-\psi-\phi} d\nu \\ &+ \int_M \left(\psi_{ij}\phi_{ij} + \frac{1}{2}\psi_i(\Delta\phi)_i\right) e^{-\psi} d\mu, \end{aligned} \tag{7}$$

where ψ satisfies $e^{-\psi} = f^2$.

Proof. The proof also follows from a direct computation. Note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2,$$

and

$$\mathbf{div} Rc = \frac{1}{2}\nabla R.$$

Using (6) and substituting $v_{ij} = -2R_{ij}$ into the equality (5), we have

$$\begin{aligned}(8)\qquad \frac{d}{dt}\lambda(t) = &\int_M \left(-2R_{ij}f_{ij}f + 2R_{ij}\phi_i f_j f + \frac{a}{2}Rf^2\right) d\mu \\ &+ \int_M \left(b\Delta Rf^2 + 2b|Rc|^2 f^2\right) d\mu.\end{aligned}$$

Using integration by parts and $\frac{1}{2}\Delta R = \mathbf{div}(\mathbf{div} Rc)$, we get

$$\begin{aligned}(9)\qquad \frac{1}{2}\int_M \Delta Rf^2 d\mu = &\int_M (2R_{ij}f_i f_j + 2R_{ij}f_{ij}f - 4R_{ij}\phi_i f_j f)\, d\mu \\ &+ \int_M \left(R_{ij}\phi_i\phi_j f^2 - R_{ij}\phi_{ij}f^2\right) d\mu.\end{aligned}$$

Let ψ be a smooth function satisfying $e^{-\psi} = f^2$ and plug it and (9) into (8), we have

$$\begin{aligned}(10)\qquad \frac{d}{dt}\lambda(t) =&(1-2b)\int_M R_{ij}\psi_{ij}e^{-\psi}d\mu + (2b-\frac{1}{2})\int_M R_{ij}\psi_i\psi_j e^{-\psi}d\mu \\ &+ 2b\int_M |Rc|^2 e^{-\psi}d\mu - (1-4b)\int_M R_{ij}\phi_i\psi_j e^{-\psi}d\mu \\ &+ 2b\int_M R_{ij}(\phi_i\phi_j - \phi_{ij})e^{-\psi}d\mu + \frac{a}{2}\int_M Re^{-\psi}d\mu.\end{aligned}$$

By the contracted second Bianchi identity $\nabla_i R_{ij} = \frac{1}{2}\nabla_j R$ and integration by parts, it follows that

$$\begin{aligned}(11)\qquad \int_M R_{ij}(\psi_{ij} - \psi_i\psi_j)e^{-\psi}d\mu = &\int_M R_{ij}\phi_i\psi_j e^{-\psi}d\mu - \frac{1}{2}\int_M R\Delta e^{-\psi-\phi}d\nu \\ &+ \frac{1}{2}\int_M R\Delta e^{-\phi}e^{-\psi}d\nu + \frac{1}{2}\int_M R\psi_i\phi_i e^{-\psi}d\mu,\end{aligned}$$

and

$$\begin{aligned}\int_M |\psi_{ij}|^2 e^{-\psi}d\mu =&\frac{1}{2}\int_M |\nabla\psi|^2\Delta e^{-\psi-\phi}d\nu - \int_M (\Delta\psi)_i\psi_i e^{-\psi}d\mu \\ &- \int_M R_{ij}\psi_i\psi_j e^{-\psi}d\mu \\ =&\frac{1}{2}\int_M |\nabla\psi|^2\Delta e^{-\psi-\phi}d\nu - \int_M \Delta\psi\Delta e^{-\psi-\phi}d\nu\end{aligned}$$

$$\begin{aligned}
&+\int_M \Delta\psi\Delta e^{-\phi}e^{-\psi}d\nu+\int_M \Delta\psi\phi_i\psi_i e^{-\psi}d\mu\\
&-\int_M R_{ij}\psi_i\psi_j e^{-\psi}d\mu\\
=&2b\int_M R\Delta e^{-\psi-\phi}d\nu-\int_M \psi_i\phi_i\Delta e^{-\psi-\phi}d\nu\\
&-a\int_M \psi\Delta e^{-\psi-\phi}d\nu+\int_M \Delta\psi\Delta e^{-\phi}e^{-\psi}d\nu\\
&+\int_M \Delta\psi\phi_i\psi_i e^{-\psi}d\mu-\int_M R_{ij}\psi_i\psi_j e^{-\psi}d\mu.
\end{aligned} \tag{12}$$

In the above formula the last equality holds because of (2) and the relation between f and ψ, i.e.

$$2\lambda(t)=\Delta_\phi\psi-\frac{1}{2}|\nabla\psi|^2-a\psi+2bR.$$

Similar to (11), we also have

$$\begin{aligned}
\int_M R_{ij}(\phi_{ij}-\phi_i\phi_j)e^{-\psi}d\mu=&\int_M R_{ij}\phi_i\psi_j e^{-\psi}d\mu-\frac{1}{2}\int_M R\Delta e^{-\psi-\phi}d\nu\\
&+\frac{1}{2}\int_M R\Delta e^{-\psi}e^{-\phi}d\nu+\frac{1}{2}\int_M R\psi_i\phi_i e^{-\psi}d\mu.
\end{aligned} \tag{13}$$

Plugging (11), (12) and (13) into (10), we arrive at

$$\begin{aligned}
\frac{d}{dt}\lambda(t)=&\int_M R_{ij}\psi_{ij}e^{-\psi}d\mu-\frac{1}{2}\int_M R_{ij}\psi_i\psi_j e^{-\psi}d\mu+2b\int_M |Rc|^2e^{-\psi}d\mu\\
&-\int_M R_{ij}\phi_i\psi_j e^{-\psi}d\mu+b\int_M R\Delta e^{-\psi-\phi}d\nu+\frac{a}{2}\int_M Re^{-\psi}d\mu\\
=&\int_M R_{ij}\psi_{ij}e^{-\psi}d\mu+2b\int_M |Rc|^2e^{-\psi}d\mu-\int_M R_{ij}\phi_i\psi_j e^{-\psi}d\mu\\
&+\frac{1}{2}\int_M |\psi_{ij}|^2e^{-\psi}d\mu+\frac{1}{2}\int_M \psi_i\phi_i\Delta e^{-\psi-\phi}d\nu-\frac{1}{2}\int_M \Delta\psi\Delta e^{-\phi}e^{-\psi}d\nu\\
&-\frac{1}{2}\int_M \Delta\psi\phi_i\psi_i e^{-\psi}d\mu+\frac{a}{2}\int_M \psi\Delta e^{-\psi-\phi}d\nu+\frac{a}{2}\int_M Re^{-\psi}d\mu\\
=&\frac{1}{2}\int_M |R_{ij}+\psi_{ij}|^2e^{-\psi}d\mu+\frac{4b-1}{2}\int_M |Rc|^2e^{-\psi}d\mu+\frac{a}{2}\int_M Re^{-\psi}d\mu\\
&-\int_M R_{ij}\phi_i\psi_j e^{-\psi}d\mu+\frac{1}{2}\int_M \psi_i\phi_i\Delta e^{-\psi-\phi}d\nu-\frac{1}{2}\int_M \Delta\psi\Delta e^{-\phi}e^{-\psi}d\nu\\
&-\frac{1}{2}\int_M \Delta\psi\phi_i\psi_i e^{-\psi}d\mu+\frac{a}{2}\int_M \psi\Delta e^{-\psi-\phi}d\nu.
\end{aligned} \tag{14}$$

Integrating by parts again, one has the following identity (cf. (2.17) in [4]).

$$\int_M \psi_i\phi_i\Delta e^{-\psi-\phi}d\nu - 2\int_M R_{ij}\psi_i\phi_j e^{-\psi}d\mu - \int_M \Delta\psi(\Delta e^{-\phi} + \phi_i\psi_i e^{-\phi})e^{-\psi}d\nu$$
$$= 2\int_M \psi_{ij}\phi_{ij}e^{-\psi}d\mu + \int_M \psi_i(\Delta\phi)_i e^{-\psi}d\mu. \tag{15}$$

Finally, the desired result (7) is achieved from the above two formulas (14) and (15). □

Remark 2.2. *In particular, if ϕ is a constant, our theorem had been proved by Huang-Li in [10].*

Remark 2.3. *Moreover, when $a = 0$, the above theorem had been obtained by Fang-Xu-Zhu in [4]. Furthermore, if we also let ϕ be a constant function on M, our theorem coincides with Cao's Theorem 1.5 in [2].*

3. Monotonicity of the geometric constant

In this section, we consider the system of Ricci flow coupled to a heat equation. We will derive the evolution equation of the lowest geometric constant along the Ricci flow coupled to the heat equation, and prove the monotonicity of the geometric constant under the system.

In the above section ϕ does not depend on the time t. Now, we let it evolve by the following heat equation

$$\frac{\partial\phi}{\partial t} = \Delta\phi. \tag{16}$$

By the same calculation as Section 2, we can easily get the evolution equation of the lowest geometric constant λ under the system of Ricci flow coupled to the above heat equation.

Theorem 3.1. *Let $g(t), t \in [0, T)$, be a solution to the Ricci flow (1) on a compact manifold M^n. Assume that there is a C^1-family of smooth functions $f(x,t) > 0$, which satisfy*

$$-\Delta_\phi f(x,t) + af\ln f + bRf(x,t) = \lambda(t)f(x,t),$$

and the normalization

$$\int_M f(x,t)^2 d\mu = 1,$$

where $\phi(\cdot,t)\in C^\infty(M)$ is a solution to the heat equation (16). Then the lowest geometric constant $\lambda(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}\lambda(t)=&\frac{1}{2}\int_M |R_{ij}+\psi_{ij}|^2e^{-\psi}d\mu+\frac{4b-1}{2}\int_M |Rc|^2e^{-\psi}d\mu\\&+\int_M \psi_{ij}\phi_{ij}e^{-\psi}d\mu+\frac{a}{2}\int_M Re^{-\psi}d\mu\\&+\frac{a}{2}\int_M \psi\Delta e^{-\psi-\phi}d\nu+\frac{a}{2}\int_M \Delta\phi e^{-\psi}d\mu,\end{aligned}\tag{17}$$

where ψ satisfies $e^{-\psi}=f^2$.

Proof. When the function ϕ satisfies the heat equation (16), one can find that the evolution equation of the geometric constant will have two additional terms

$$\frac{a}{2}\int_M \phi_t f^2d\mu+\int_M f_i(\phi_t)_i f d\mu=\frac{a}{2}\int_M \Delta\phi e^{-\psi}d\mu-\frac{1}{2}\int_M \psi_i(\Delta\phi)_ie^{-\psi}d\mu.$$

Therefore, it is obvious that (17) holds by same arguments with Theorem 2.1. □

Now let us complete the proof of Theorem 1.1.

Proof. (Proof of Theorem 1.1) Let $\lambda(t)$ be the lowest constant of the nonlinear equation (2), and $f(x,t)$ its corresponding solution with the normalization at time t. From Theorem 3.1 we have

$$\begin{aligned}\frac{d}{dt}\lambda(t)=&\frac{1}{2}\int_M |R_{ij}+\psi_{ij}|^2e^{-\psi}d\mu+\frac{4b-1}{2}\int_M |Rc|^2e^{-\psi}d\mu+\int_M \psi_{ij}\phi_{ij}e^{-\psi}d\mu\\&+\frac{a}{2}\int_M \psi\Delta e^{-\psi-\phi}d\nu+\frac{a}{2}\int_M Re^{-\psi}d\mu+\frac{a}{2}\int_M \Delta\phi e^{-\psi}d\mu\\=&\frac{1}{2}\int_M |R_{ij}+\psi_{ij}+\frac{a}{2}g_{ij}|^2e^{-\psi}d\mu+\frac{4b-1}{2}\int_M |Rc|^2e^{-\psi}d\mu\\&+\int_M \psi_{ij}\phi_{ij}e^{-\psi}d\mu-\frac{na^2}{8}+\frac{a}{2}\int_M \Delta\phi e^{-\psi}d\mu\\=&\frac{1}{2}\int_M |R_{ij}+\psi_{ij}+\phi_{ij}+\frac{a}{2}g_{ij}|^2e^{-\psi}d\mu+\frac{4b-1}{2}\int_M |Rc|^2e^{-\psi}d\mu\\&-\frac{1}{2}\int_M |\phi_{ij}|^2e^{-\psi}d\mu-\int_M R_{ij}\phi_{ij}e^{-\psi}d\mu-\frac{na^2}{8}\\\geq&\frac{1}{2}\int_M |R_{ij}+\psi_{ij}+\phi_{ij}+\frac{a}{2}g_{ij}|^2e^{-\psi}d\mu-\frac{na^2}{8}\end{aligned}$$

$$+ (2b - \sqrt{b}) \int_M \left(|Rc|^2 - \frac{1}{(2\sqrt{b}-1)^2} |\phi_{ij}|^2 \right) e^{-\psi} d\mu.$$

Hence it follows from the assumption of Ricci curvature that $\lambda(t) + \frac{na^2}{8}t$ is nondecreasing. □

When M is a two-dimensional surface, the Ricci curvature $R_{ij} = \frac{1}{2}Rg_{ij}$. Hence we have the following result in dimension two.

Corollary 3.1. *Let $g(t), t \in [0, T)$, be a solution to the Ricci flow (1) on a two-dimensional compact surface M, and $\lambda(t)$ be the lowest constant of the nonlinear equation (2). Suppose that the scalar curvature satisfies*

$$|R| \geq \frac{\sqrt{2}}{2\sqrt{b}-1} |\nabla\nabla\phi|, \forall t \in [0, T),$$

where $b > \frac{1}{4}$ and $\phi(\cdot, t) \in C^\infty(M)$ satisfies the heat equation (16). Then the geometric constant $\lambda(t) + \frac{na^2}{8}t$ are nondecreasing.

4. Geometric constant under the normalized Ricci flow

In the last section, we come to consider the normalized Ricci flow, i.e,

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2r}{n} g_{ij}, \tag{18}$$

where

$$r = \frac{\int_M R d\nu}{\int_M d\nu}$$

is the average scalar curvature. In Lemma 2.1, if we evolve the metric by the normalized Ricci flow, we can get the evolution equation of the geometric constant λ under the normalized Ricci flow.

Theorem 4.1. *Let $g(t), t \in [0, T)$, be a solution to the normalized Ricci flow (18) on a compact manifold M^n. Assume that there is a C^1-family of smooth functions $f(x, t) > 0$ which satisfy*

$$-\Delta_\phi f(x, t) + af \ln f + bRf(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\mu = 1.$$

Then the lowest geometric constant $\lambda(t)$ *satisfies*

$$\begin{aligned}\frac{d}{dt}\lambda(t) =& -\frac{2r\lambda}{n} + \frac{1}{2}\int_M |R_{ij}+\psi_{ij}|^2 e^{-\psi}d\mu + \frac{4b-1}{2}\int_M |Rc|^2 e^{-\psi}d\mu \\ &+ \frac{a}{2}\int_M Re^{-\psi}d\mu + \frac{a}{2}\int_M \psi\Delta e^{-\psi-\phi}d\nu - \frac{ar}{n}\int_M \psi e^{-\psi}d\mu \\ &+ \int_M \left(\psi_{ij}\phi_{ij} + \frac{1}{2}\psi_i(\Delta\phi)_i\right)e^{-\psi}d\mu - \frac{ar}{2},\end{aligned}$$

where ψ *satisfies* $e^{-\psi} = f^2$.

Proof. We note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 - \frac{2r}{n}R,$$

and

$$v_{ij} = -2R_{ij} + \frac{2r}{n}g_{ij}.$$

The proof can be obtained from the similar calculation with Theorem 2.1. So it is easy to get the extra term

$$-\frac{2r}{n}\int_M(-\Delta_\phi f + bRf)fd\mu - \frac{ar}{2}\int_M f^2 d\mu = -\frac{2r\lambda}{n} - \frac{ar}{n}\int_M \psi e^{-\psi}d\mu - \frac{ar}{2}.$$

□

Remark 4.1. *Here our theorem is consistent with Theorem 1.2 of Huang-Li in [10] if* ϕ *is a constant.*

When M is a two-dimensional surface, r is a constant. In fact,

$$r = 4\pi\chi(M)/A,$$

where $\chi(M)$ and A are respectively the Euler class and area of M. We can also obtain an interesting monotonicity from the above theorem.

Corollary 4.1. *Let* $g(t), t \in [0,T)$, *be a solution to the normalized Ricci flow (18) on a compact surface* M *with negative Euler characteristic class,* $\lambda(t)$ *be the lowest constant of the nonlinear equation (2), and* λ_0 *be the first eigenvalue of* $-\Delta_\phi + bR$. *Suppose that at the initial time* $\lambda_0 \geq -\frac{a}{2}$ *and the scalar curvature satisfies*

$$|R| \geq \frac{\sqrt{2}}{2\sqrt{b}-1}|\nabla\nabla\phi|, \forall t \in [0,T),$$

where $b > \frac{1}{4}$ *and* $\phi(\cdot, t) \in C^\infty(M)$ *satisfies the heat equation (16). Then* $\lambda(t) + \frac{a^2}{4}t$ *is nondecreasing.*

Remark 4.2. *In particular, when* ϕ *is a constant, our corollary reduces to Theorem 1.3 of Huang-Li in [10].*

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