Rationality of adjoint orbits

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To Corrado De Concini

Abstract: We prove that every orbit of the adjoint representation of any connected reductive algebraic group G is a rational algebraic variety. For complex simply connected semisimple G, this implies rationality of homogeneous affine Hamiltonian G-varieties (which we classify).

Keywords: Orbit of linear algebraic group, rational algebraic variety, Hamiltonian variety.

1. Introduction

Let G be a connected affine algebraic group and let H be its closed subgroup. Whether the algebraic variety G/H is rational is a well-known old problem closely related to the rationality problem of invariant fields of linear representations of algebraic groups (see [12, 1.5], [14, Thm. 1, Cor. 2]).

As is shown in [13, p. 298], [14, Thm. 2], [12, Rem. 1.5.9], for some G and *finite H*, the variety G/H is nonrational (and even not stably rational). However, the existence of nonrational varieties G/H with a *connected* group H is still an intriguing open problem.

At the same time, for many pairs (G, H) with connected group H either rationality or stable rationality of the variety G/H is proved; for instance, G/H is rational whenever $\dim(G/H) \leqslant 10$ (see [5]).

The conference talk [1] served for me as an impetus to explore rationality of orbits of the adjoint representations of connected reductive groups. Searching for some special rational coordinates on the adjoint orbits of $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$, the author of [1] proved, as a byproduct, rationality of the majority of these orbits. He uses the "method of the canonical orbit parameterization" that dates back to the work of I. M. Gelfand and M. I. Naimark on unitary representations of classical groups (1950). The parameterization of

(co)adjoint orbits was of interest to many researchers because of its connection with the problems of the theory of integrable systems (see introduction and related references in [2]).

In the present paper, is proved the following theorem announced in [15] that yields infinitely many new examples of rational varieties of the form G/H, including those for which H is a disconnected group of positive dimension.

Theorem 1. *Let G be a connected reductive algebraic group. Every G-orbit of the adjoint representation of G is a rational algebraic variety.*

In view of [16, Cor. 1], Theorem 1 provides new examples of algebraic varieties X that are locally flattenable in the sense of [16, Def. 1] (i.e., such that for every point $x \in X$ there is an open subset of X containing x and isomorphic to an open subset of an affine space):

Corollary 1. *Let G be a connected reductive algebraic group. Every G-orbit of the adjoint representation of G is a locally flattenable algebraic variety.*

The G-orbits in Theorem 1 are exactly the varieties G/H, where H is the G-centralizer $C_G(x)$ of an element x of the Lie algebra

$$\mathfrak{g} := \mathrm{Lie}(G).$$

The groups $C_G(x)$ have been thoroughly studied (see [6, 9, 20]). Among them there are both connected and disconnected groups. Their dimensions are not less than $r := \mathrm{rk}(G)$.

Note that Theorem 1 establishes a specific property of the adjoint representation: by Remark 2 below, there exist representations of some G not all of whose orbits are rational algebraic varieties.

Theorem 1 is applied in the proof of the following Theorem 2 about the classification and properties of some Hamiltonian G-spaces in the sense of Kostant (see definition given in the first paragraph of [11, Sect. 5.1]). In terms of the currently used definition (see [3, Def. 7]), homogeneous Hamiltonian G-manifolds X are meant. We consider the cases, in which X is an algebraic variety and G acts on X algebraically. In Theorem 2, the following notation is used:

> $\mathscr{H}$ is the set of isomorphism classes of homogeneous affine
> Hamiltonian G-varieties,
> $\mathscr{S}$ is the set of G-orbits of nonzero semisimple elements of $\mathfrak{g}$.

Theorem 2. *If G is a simply connected complex semisimple algebraic group and X is a homogeneous affine Hamiltonian G-variety, then the following holds.*

(a) *X is isomorphic to a unique G-orbit $\mathcal{O}_X \in \mathscr{S}$ endowed with the standard structure of a Hamiltonian G-variety (see* [11], *Sects. 5.2, 5.3, Thm. 5.3.1]).*

(b) *The map $X \mapsto \mathcal{O}_X$ yields a bijection*

$$\tag{1} \mathscr{H} \to \mathscr{S}.$$

(c) *X is a simply connected rational variety.*

(d) *The G-stabilizer of any point of X is a Levi subgroup of a proper parabolic subgroup of G.*

(e) *Every Levi subgroup of every proper parabolic subgroup of G is the G-stabilizer of a point of some homogeneous affine Hamiltonian G-variety.*

Recall that for the adjoint action of G on $\mathfrak{g}$, the categorical quotient $\mathfrak{g}/\!\!/ G$ is isomorphic to the affine space $\mathbb{A}^r$, and every fiber of the quotient morphism $G \to \mathfrak{g}/\!\!/ G$ contains a unique G-orbit from $\mathscr{S}$ (see [10], [17, 8.5]). Combined with the existence of bijection (1), this yields a parametrization of $\mathscr{H}$ by $\mathbb{A}^r \setminus \{0\}$.

The proofs of Theorems 1 and 2 are given in Section 3.

Conventions and notation. Our basic reference for algebraic groups and algebraic geometry is [4] and we follow the conventions therein. Unless otherwise stated, all algebraic groups and algebraic varieties are taken over an algebraically closed field k whose characteristic is not a bad prime for reductive G (see [20, Chap. I, Def. 4.1]).

We use the following notation:

$C_G(M)$ is the G-centralizer of a subset M of $\mathfrak{g}$ or G.

$\mathrm{Rad}_u(Q)$ is the unipotent radical of an affine algebraic group Q.

$\mathbb{A}^n$ is the n-dimensional affine space.

2. Birational complements

Definition 1. Let G and H be as in Section 1. A sequence $S_1, \ldots, S_m$ of locally closed subsets of G is called a *birational complement to H in G* if the morphism

$$\tag{2} \lambda \colon S_1 \times \cdots \times S_m \times H \to G, \quad (s_1, \ldots, s_m, h) \mapsto s_1 \cdots s_m h$$

is an open embedding.

Remark 1. One can show that this notion is order-sensitive, i.e., reshuffling the terms of a sequence that is a birational complement to H in G, one obtains a sequence that, generally speaking, is not a birational complement to H in G.

If $S_1, \ldots, S_m$ is a birational complement to H in G, then Definition 1 implies, in view of the connectedness of G, that H is connected as well. It also implies that open embedding (2) is H-equivariant with respect to the action of H on G by right translations and on $S_1 \times \cdots \times S_m \times H$ by right translations of the last factor. Thus, the image of open embedding (2) is an H-invariant open subset of G that is H-equivariantly isomorphic to $S_1 \times \cdots \times S_m \times H$.

Example 1. Let G be a semidirect product $A \ltimes B$ of closed subgroups A and B. Then the one-term sequence A (resp. B) is a birational complement to B (resp. A) in G. For instance, one can take A and B to be respectively a Levi subgroup of G and the unipotent radical $\mathrm{Rad}_u(G)$.

Example 2. Let G be a reductive algebraic group and let P be a parabolic subgroup of G. Let P^- be the parabolic subgroup of G opposite to P. Then the one-term sequence $\mathrm{Rad}_u(P^-)$ is a birational complement in G to P (see [4, IV, Prop. 14.21(iii)]).

Lemma 1. *Let G and H as in Section 1 and let Q be a closed subgroup of H. Let $S_1, \ldots, S_m$ be a birational complement to H in G.*

(a) *If $Z_1, \ldots, Z_n$ is a birational complement to Q in H, then*

$$S_1, \ldots, S_m, Z_1, \ldots, Z_n$$

is a birational complement to Q in G.

(b) *The variety G/Q contains an open subset isomorphic to*

$$(3) \qquad S_1 \times \cdots \times S_m \times (H/Q).$$

Proof. (a) Let $X := S_1 \times \cdots \times S_m$ and $Y := Z_1 \times \cdots \times Z_n$. By Definition 1,

$$\mu \colon Y \times Q = Z_1 \times \cdots \times Z_n \times Q \to H, \quad (z_1, \ldots, z_n, q) \mapsto z_1 \cdots z_n q$$

is an open embedding, therefore,

$$\nu := \mathrm{id}_X \times \mu \colon X \times (Y \times Q) \to X \times H$$

is an open embedding. Since λ (see (2)) is also an open embedding, this implies that the morphism

$$\lambda \circ \nu \colon X \times (Y \times Q) = S_1 \times \cdots \times S_m \times Z_1 \times \cdots \times Z_n \to G,$$

$$(s_1, \ldots, s_m, z_1, \ldots, z_n, q) \mapsto s_1 \cdots s_m z_1 \cdots z_n q$$

is an open embedding as well. This proves (a).

(b) As noted above, G contains an H-invariant open subset U that is H-equivariantly isomorphic to $S_1 \times \cdots \times S_m \times H$. Therefore, by [4, II, Thm. 6.8 and Cor. 6.6], a geometric quotient U/Q exists and is isomorphic to variety (3). On the other hand, U/Q is isomorphic to an open subset of G/Q because the canonical morphism $G \to G/Q$ is open (see [4, II, 6.1]). This proves (b). $\square$

Example 3. Taking $H = Q$ in Lemma 1(b) yields that G/H contains an H-invariant open subset isomorphic to $S_1 \times \cdots \times S_m$. For instance, G/P in Example 2 contains an open subset isomorphic to the underlying variety of $\mathrm{Rad}_u(P^-)$, i.e., to an affine space (see [4, IV, 14.4, Rem.]); whence, G/P is rational.

Example 4. Let G be a reductive group, let P be a parabolic subgroup of G, and let L be a Levi subgroup of P. Then by Lemma 1 and Examples 1, 2, the two-term sequence $\mathrm{Rad}_u(P^-)$, $\mathrm{Rad}_u(P)$ is a birational complement to L in G and G/L contains an open subset isomorphic to the underlying variety of $\mathrm{Rad}_u(P^-) \times \mathrm{Rad}_u(P)$, i.e., to an affine space; whence G/L is a rational variety.

Example 5. Let B be a Borel subgroup of G and let Q be a closed subgroup of B. In view of Lemma 1 and Example 2, the variety G/Q contains an open subset isomorphic to $\mathbb{A}^d \times (B/Q)$ for some d. By [19, Thm. 5], the variety B/Q is isomorphic to $\mathbb{A}^s \times (\mathbb{A}^1 \setminus \{0\})^t$ for some s, t. Whence G/Q is a rational variety (this statement is Theorem 2.9 of [5]). Since every connected solvable subgroup of G lies in a Borel subgroup of G, this implies that the variety G/H is rational if H is connected solvable.

3. Proofs of Theorems 1 and 2

In the proof of Theorems 1 and 2, we shall use the following

Theorem 3.

- *Let G be a connected reductive algebraic group and let x be an element of $\mathfrak{g}$. The following properties are equivalent:*

(a) *x is semisimple;*
(b) *the G-orbit $\mathcal{O}$ of x is an affine variety;*
(c) *$C_G(x)$ is reductive;*
(d) *$\mathcal{O}$ is a closed subset of $\mathfrak{g}$.*

• *Let G be a simply connected semisimple algebraic group. Then properties* (a)–(d) *are equivalent to the following property*

(e) $C_G(x)$ *is a Levi subgroup of a parabolic subgroup of G.*

For any parabolic subgroup P of G and any Levi subgroup L of P, there is a semisimple element $s \in \mathfrak{g}$ such that $C_G(s) = L$.

Proof.

(a)⇔(d) is well-known (see [10, Rem. 11], [17, 8.5]).

(d)⇒(b) is clear.

(b)⇔(c) follows from Matsushima's criterion (see [17, Thm. 4.17], [18]).

(b)⇒(d) Let $\mathcal{O}$ be affine and let $\overline{\mathcal{O}}$ be the closure of $\mathcal{O}$ in $\mathfrak{g}$. The set $X := \overline{\mathcal{O}} \setminus \mathcal{O}$ is closed in $\mathfrak{g}$ (see [17, 1.3]). Arguing to the contrary, assume that $X \neq \varnothing$. Since $\mathcal{O}$ is affine, this yields $\mathrm{codim}_{\overline{\mathcal{O}}}(X) = 1$ (see [8]). On the other hand, $\mathrm{codim}_{\overline{\mathcal{O}}}(X) \geqslant 2$ by [10, Cor. 1 of Thm. 3]. This contradiction shows that $\overline{\mathcal{O}} = \mathcal{O}$.

(a)⇒(e) Let x be a semisimple element. By [21, Cor. 3.8], there is a torus S in G such that $x \in \mathfrak{s} := \mathrm{Lie}(S)$ and

$$(4) \qquad\qquad C_G(x) = C_G(\mathfrak{s}).$$

By [7, Prop. 3.4.7], there is a parabolic subgroup P of G such that $C_G(S)$ is a Levi subgroup of P. Since

$$(5) \qquad\qquad C_G(S) = C_G(\mathfrak{s})$$

(see [22, Thm. 24.4.8(ii)]), from (4) and (5) we infer that $C_G(x)$ is a Levi subgroup of P.

(e)⇒(c) is clear.

To prove the last statement of Theorem 3, note that since L is a Levi subgroup of a parabolic subgroup of G, there is a torus S in G such that

$$(6) \qquad\qquad L = C_G(S)$$

(see [7, Prop. 3.4.6]). We identify S with $(k^*)^d$ by means of an isomorphism between them. Let $\mathfrak{s} = k^d$ be the Lie algebra of S. If $z = (z_i) \in \mathfrak{s}$, let $R_z := \{(m_i) \in \mathbb{Z}^d \mid \sum_i m_i z_i = 0\}$. Then the minimal algebraic subalgebra $\mathfrak{a}(z)$ of $\mathfrak{s}$ containing z is $\{(s_i) \in \mathfrak{s} \mid \sum_i m_i s_i = 0 \text{ for all } (m_i) \in R_z\}$ (see [4, II, 7.3(2)]). Since the degree of k over its prime subfield is infinite, this implies the existence of $x \in \mathfrak{s}$ such that $\mathfrak{s} = \mathfrak{a}(x)$. By [21, Cor. 3.8], we then have $C_G(x) = C(\mathfrak{s})$. In view of (5) and (6), this yields $L = C_G(x)$. □

Proof of Theorem 1. Since the center of G lies in the kernel of the adjoint representation of G, without changing the G-orbits in $\mathfrak{g}$, we may (and shall) assume that G is a simply connected semisimple group.

Let x be an element of $\mathfrak{g}$. Our goal is to prove that $G/C_G(x)$ is a rational variety. We may (and shall) assume that $x \neq 0$. We shall consider separately three cases:

   (i) x is nilpotent,
  (ii) x is semisimple,
 (iii) x is neither nilpotent, nor semisimple.

Case (i). Let x be nilpotent. Then by the Jacobson–Morozov theorem, there are elements $h, y \in \mathfrak{g}$ such that $\{x, h, y\}$ is an $\mathfrak{sl}_2$-triple, i.e., $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$. For every $i \in \mathbb{Z}$, put

$$\mathfrak{g}(i) := \{z \in \mathfrak{g} \mid [h, z] = iz\}.$$

Then we have the decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, which is a structure of a $\mathbb{Z}$-graded Lie algebra on $\mathfrak{g}$.

The subspace $\mathfrak{p} := \bigoplus_{i \geqslant 0} \mathfrak{g}(i)$ is a parabolic subalgebra of $\mathfrak{g}$. Let P be the parabolic subgroup of G such that $\mathrm{Lie}(P) = \mathfrak{p}$. Then the P-stable subspace $\mathfrak{u} := \bigoplus_{i > 0} \mathfrak{g}(i)$ is $\mathrm{Lie}(\mathrm{Rad}_u(P))$.

We have $x \in \mathfrak{g}(2) \subseteq \mathfrak{u}$. By [20, Chap. III, Sect. 4.20(i)], the P-orbit of x is open in $\mathfrak{u}$, therefore,

(7) $P/C_P(x)$ is isomorphic to an open subset of an affine space.

By [20, Chap. III, Sect. 4.16], we have $C_G(x) \subset P$; whence

(8) $C_P(x) = C_G(x).$

In view of (8), we have the following tower of algebraic groups:

(9) $G \supset P \supset C_G(x),$

By Example 2 and Lemma 1(b) applied to (9), we infer that $G/C_G(x)$ contains an open subset isomorphic to $(\mathrm{Rad}_u(P^-)) \times (P/C_G(x))$. Since the underlying variety of $\mathrm{Rad}_u(P^-)$ is isomorphic to an affine space, we infer from (7) that $G/C_G(x)$ contains an open subset isomorphic to an open set of an affine space. Therefore, $G/C_G(x)$ is a rational variety.

Case (ii). Let x be semisimple. Then $C_G(x)$ is a Levi subgroup of a parabolic subgroup of G in view of Theorem 3. Hence the variety $G/C_G(x)$ is rational by Example 4.

Case (iii). Let x be neither nilpotent, nor semisimple. Let $x = x_s + x_n$ be the Jordan decomposition of x. By Theorem 3, the group $C_G(x_s)$ is a Levi subgroup of a parabolic subgroup of G. We have $x_n \in \mathrm{Lie}(C_G(x_s))$ and it follows from the uniqueness of the Jordan decomposition that

$$(10) \qquad C_G(x) = C_{C_G(x_s)}(x_n).$$

By Example 4, there is a two-term birational complement S_1, S_2 to $C_G(x_s)$ in G such that

$$(11) \qquad S_i \text{ is isomorphic to an affine space for every } i.$$

Applying Lemma 1 to $H = C_G(x_s)$, $Q = C_{C_G(x_s)}(x_n)$, we obtain from (10) that $G/C_G(x)$ contains an open set isomorphic to

$$S_1 \times S_2 \times (C_G(x_s)/C_{C_G(x_s)}(x_n)).$$

By (i), the variety $C_G(x_s)/C_{C_G(x_s)}(x_n)$ is rational. This and (11) imply that $G/C_G(x)$ is rational.   □

Remark 2. Theorem 1 establishes a specific property of the adjoint representation: for some connected reductive groups G, there are finite-dimensional algebraic representations not all of whose G-orbits are rational algebraic varieties.

Indeed, in [13, Example 1.22], [14, Thm. 2], [12, Rem. 1.5.9] are constructed connected reductive algebraic groups G with a finite subgroup H such that the algebraic variety G/H is nonrational. Being finite, H is reductive; hence, by Matsushima's criterion, the variety G/H is affine. Therefore, by the embedding theorem (see [17, Thm. 1.5]), there exists a G-equivariant (with respect to the natural action of G on G/H) closed embedding of G/H into some finite-dimensional algebraic G-module.

Proof of Theorem 2.

(a) By [11, Thm. 5.4.1, Prop. 5.1.1], there are a unique G-orbit $\mathcal{O}_X \subset \mathfrak{g}$ and a unique morphism $\tau_X \colon X \to \mathcal{O}_X$ of Hamiltonian G-varieties. By [11, Prop. 5.1.1], τ_X is a covering, and for every $x \in X$, the identity component of the G-stabilizer G_x of x coincides with that of the G-stabilizer $G_{\tau_X(x)}$ of $\tau_X(x)$. Since G acts on X transitively and X is affine, G_x is reductive by Matsushima's criterion. Therefore, $G_{\tau_X(x)}$ is reductive as well. Then from the equivalence of properties (a), (c), and (e) in Theorem 3 we infer that $\mathcal{O}_X \in \mathscr{S}$ and $G_{\tau_X(x)}$ is connected. Since G is simply connected, the latter

implies that $\mathcal{O}_X$ is simply connected as well. This, in turn, implies that τ_X is an isomorphism since τ_X is a covering.

(b) In view of (a), this follows from the fact that each G-orbit in $\mathfrak{g}$ is endowed with the standard structure of a Hamiltonian G-variety.

(c) This follows from (a) since, as was proved above, $\mathcal{O}_X$ is a simply connected and, by Theorem 1, rational variety.

(d) This follows from (a) and the equivalence of (a) and (e) in Theorem 3.

(e) This follows from the last statement of Theorem 3. $\qquad\square$

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References

[1] M. V. Babich, On birational parametrization of (co)adjoint orbits of complex classical groups. In: *Conference in Memory of A. N. Tyurin* (Steklov Math. Inst., Moscow, Russia, Oct. 26, 2015) (in Russian), http://www.mathnet.ru/php/presentation.phtml?option_lang=rus&presentid=12717

[2] M. V. Babich, *Birational Darboux coordinates on (co)adjoint orbits of* GL$(N, \mathbb{C})$, Funct. Analysis and Appl. **50** (2016), 17–30. MR3526970

[3] N. Berline, M. Vergne, *Hamiltonian manifolds and moment map*, http://www.cmls.polytechnique.fr/perso/berline/cours-Fudan.pdf

[4] A. Borel, *Linear Algebraic Groups*, 2nd enlarged ed., Graduate Texts in Mathematics, **126**, Springer-Verlag (1991). MR1102012

[5] C. Chin, D.-Q. Zhang, *Rationality of homogeneous varieties*, Trans. Amer. Math. Soc. **369** (2017), no. 4, 2651–2673. MR3592523

[6] D. H. Collingwood, W. M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, New York (1993). MR1251060

[7] F. Digne, J. Michel, *Representations of Finite Groups of Lie Type*, London Mathematical Society Student Texts, **95**, Cambridge University Press, Cambridge (2020). MR4211777

[8] J. E. Goodman, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. **89** (1969), no. 1, 160–183. MR0242843

[9] J. E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, Mathematical Surveys and Monographs, **43**, Amer. Math. Soc., Providence, Rhode Island (1995). MR1343976

[10] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404. MR0158024

[11] B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math. **170** (1970), 87–207. MR0294568

[12] V. L. Popov, Sections in invariant theory. In: *Proc. Sophus Lie Memorial Conf.* (Oslo, 1992), Scandinavian University Press, Oslo, (1994), 315–361. MR1456471

[13] V. L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, CRM Proceedings and Lecture Notes, Amer. Math. Soc. **54** (2011), 289–311. MR2768646

[14] V. L. Popov, *Rationality and the FML invariant*, J. Ramanujan Math. Soc. **28A** (2013), 409–415. MR3115201

[15] V. L. Popov, *Rationality of (co)adjoint orbits.* In: Internat. Conference (Koryazhma, Russia, August 3–9, 2016), 84–85. Abstracts of talks (in Russian), http://www.mathnet.ru/ConfLogos/805/thesis.pdf

[16] V. L. Popov, Variations on the theme of Zariski's Cancellation Problem. In: *Polynomial Rings and Affine Algebraic Geometry, PRAAG 2018* (Tokyo, Japan, February 12–16, 2018), Springer Proc. Math. Stat., **319**, Springer, Cham (2020), 233–250. MR4113950

[17] V. L. Popov, E. B. Vinberg, Invariant theory. In: *Algebraic Geometry IV*, Encyclopaedia of Mathematical Sciences, **55**, Springer-Verlag, Berlin (1994), 123–284.

[18] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), 38–41. MR0437549

[19] M. Rosenlicht, *Questions of rationality for solvable algebraic groups over nonperfect fields*, Annali di Matem. (IV) **61** (1962), 97–120. MR0158891

[20] T. A. Springer, R. Steinberg, Conjugacy classes. In: *Seminar on Algebraic Groups and Related Finite Groups* (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics **131**, Springer, Berlin (1970), 167–266. MR0268192

[21] R. STEINBERG, *Torsion in reductive groups*, Adv. in Math. **15** (1975), 63–92. MR0354892

[22] P. TAUVEL, R. W. T. YU, *Lie Algebras and Algebraic Groups*, Springer Monographs in Mathematics, Springer, Berlin (2005). MR2146652

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