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LECTURES ON TRANSFORMATION GROUPS: GEOMETRY AND DYNAMICS

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Table of Contents

0. Introduction

In these lectures we study groups of diffeomorphisms of smooth manifolds such that the action of the group, say G acting on V, preserves some geometric structure φ given on V. For example, φ may be a Riemannian or pseudo-Riemannian metric on V (i.e., a nonsingular quadratic differential form), and then φ-preserving diffeomorphisms are called *isometries* of (V, φ). In the sequel, we shall use the same "isometric" terminology for general structures φ on V.

0.1. Basic notation and conventions. Throughout these lectures, we will consider C^∞-manifolds. The geometric objects which we are interested in are assumed to be C^∞-smooth unless otherwise stated.

We denote the action of an element $g \in G$ on V by $g: V \to V$ for all $g \in G$, the action of g on each $v \in V$ by $gv \in V$, the tangent bundle of V by $T(V)$, and by $\mathrm{Is}(V, \varphi)$ the group of (global) isometries of (V, φ), i.e., of those diffeomorphisms of V which preserve φ.

We use $\mathrm{Is}^{\mathrm{loc}}(V, \varphi)$ to denote the pseudogroup of local isometries of (V, φ). (These are isometries $(U_1, \varphi|U_1) \to (U_2, \varphi|U_2)$ for all open subsets U_1 and U_2 in V.)

The local isotropy subgroup consisting of the germs of diffeomorphisms at $v \in V$, which fix φ and v, is denoted by $\mathrm{Is}^{\mathrm{loc}}(v) = \mathrm{Is}^{\mathrm{loc}}(V, v, \varphi)$.

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We shall often speak of an action of G on (V, φ) meaning an action on V preserving φ. For the most part our G-actions are faithful and thus correspond to subgroups $G \subset \mathrm{Is}(V, \varphi)$.

0.2. We begin this subsection by indicating the basic problems we are concerned with.

0.2.A. *The existence problem for invariant structures.* Assume we are given V and an action of G on V, and we want to find an invariant structure φ on V of prescribed type. For example, we may look for an invariant pseudo-Riemannian metric on V of prescribed signature (p, q) for $p + q = \dim V$.

If the group G is *compact*, then the existence of such a φ is a purely topological problem. For example, if G is finite and the action is free, then G-invariant structures 1-1 correspond to structures on the quotient space V/G. In the general case (G is compact, infinite, the action is nonfree) the situation is more complicated, but still one may relate G-invariant structures on V with appropriate structures on V/G. The key fact is the existence of the quotient V/G which is a *Hausdorff* topological space for compact G.

The properties of compact actions are very beautiful and useful for a geometric and topological study. However, in these lectures we are interested in the actions with nontrivial dynamics, where the word "dynamics" refers to the asymptotic behavior of orbits gv, $v \in V$, $g \in G$, for $g \to \infty$. Of course, this definition leaves no room for any nontrivial dynamics of compact group actions.

0.2.B. There is no simple recipe to decide whether there exists some kind of invariant φ, but there are dynamical constructions which are interesting in this regard.

As an example we point out the following construction which often provides an invariant subbundle T^+ of the tangent bundle $T(V)$ (compare 2.1). Let $G = \mathbf{Z}$ or $G = \mathbf{R}$. Define $T^+ \subset T(V)$ by

$$T^+ = \left\{ \tau \in T(V) \,\middle|\, \lim_{g \to +\infty} \|Dg(\tau)\| = 0 \right\},$$

where $\| \ \|$ denotes some Riemannian metric given on V. Notice that this T^+ is not in general a subbundle of $T(V)$ since the dimension of the "fiber" over v, $T_v^+ \subset T^+$, may be nonconstant in $v \in V$. Clearly, T^+ does not depend on the choice of the Riemannian metric in case V is compact. This shows T^+ is indeed *invariant* in the compact case.

A geometrically interesting case where T^+ *is* a subbundle occurs, when V is the unit tangent bundle $UT(W)$ of a complete Riemannian manifold W of negative sectional curvature, and $G = \mathbf{R}$ acts by the geodesic flow.

One knows in this case (see [3]) that $\dim T^+ = n-1$ for $n = \dim V$, and moreover there exists a foliation of V into $(n-1)$-dimensional submanifolds (called the *stable* manifolds for this flow (see [3] and also see 2.2(b)) such that T^+ is the tangent bundle of this foliation. If W is simply connected, the projections of stable manifolds to V, under the projection $V = UT(W) \to W$, are *horospheres* (i.e., spheres of infinite radius) in W. If $n = 2$ and horospheres are 1-dimensional, then they are called horocycles.

We shall see in 2.6 that the above geodesic flow admits an invariant pseudo-Riemannian metric which is continuous but in general not smooth. In fact, such nonsmoothness is typical for the invariant structures obtained in differential dynamics by limit arguments. For example, an action may easily admit an invariant measure without having any smooth invariant measure (see, e.g., [13, p. 43]).

0.2.C. *The isometry group problem.* Now, let us *start* with a geometric structure φ on V and ask what is the isometry group G of (V, φ). Note that if we want to have a sufficiently large (or even just nontrivial) isometry group G, then we must start with a very special structure φ, as for most "sufficiently rigid" φ the full isometry group, called $\mathrm{Is}(V, \varphi)$, is trivial. For example, everybody knows that $\mathrm{Is}(V, \varphi) = \mathrm{Id}$ for *generic* pseudo-Riemannian metrics φ on V, for $\dim V \geq 2$. This is also true for generic subbundles $T \subset T(V)$ (viewed as geometric structures on V) such that $2 \leq \dim T \leq \dim T(V) - 2$, where "dim" refers to the dimension of the fibers of T. In fact, the same applies to general *rigid structures* which generalize pseudo-Riemannian and connection type structures (see 0.4 and §5 for the definition of a rigid structure).

0.3. In view of the above discussion, one does not expect rigid geometry to be accompanied by rich dynamics. In fact, a cohabitation of a big enough G with a rigid φ makes both G and φ extremely special. However, these special situations are very often encountered in mathematical practice. In fact, by looking at available examples (especially at those which arise in homogeneous surroundings (see 0.5 below)) one may come to the conclusion that genericity is exceptional while nongenericity is predominant.

0.4. Special or nonspecial, we want to study an action of G on (V, φ) where G is a *noncompact* Lie group, and (V, φ) is a compact manifold with a *rigid* geometric structure φ. The precise definition of the term rigid is given in 5.10 and 5.11. Here the reader may restrict to φ being one of the following:

(a) A pseudo-Riemannian metric.

(b) The conformal structure associated to a pseudo-Riemannian metric.

Notice that the conformal structure is rigid for dimension $n \geq 3$ but is not in our sense for $n = 1$ and $n = 2$. In fact, the local isometry pseudogroup (i.e., the group of local conformal transformations) is infinite dimensional for $n = 1, 2$, which is incompatible with rigidity according to 5.16.E.

(c) A subbundle or a system of subbundles in $T(V)$ with a certain nondegeneracy condition (see 2.6) which should rule out, for example, a single *integrable* subbundle $T \in T(V)$. (The nonrigidity of an integrable φ is manifested by the fact that the local isometry group of (V, φ) is infinite dimensional.) Since we have not yet given the definition of rigidity, we shall make two comments. First, one can think of rigidity of a geometric structure as, essentially, finite dimensionality of the local isometry pseudogroup of (V, φ), called $\mathrm{Is}^{\mathrm{loc}}(V, \varphi)$.

Secondly, we should notice that if one starts with a rigid structure φ and then add another structure φ' which does not have to be rigid, then the structure represented by the pair (φ, φ') is rigid.

(d) An affine connection φ on V is a rigid structure in our sense. In fact, every rigid structure (see 5.11) can be viewed as a kind of higher order connection (see 5.16.C).

0.5. There are two radically different aspects in the study of G and (V, φ).

0.5.A. *Dynamical aspect.* Here one wants to understand the dynamical properties of an action by taking into account the fact that the action preserves some geometric structure. A special (and probably most interesting) case is that of the action of a Lie group G (or of a subgroup $H \subset G$) on some homogeneous space G/Γ, where $\Gamma \subset G$ is a discrete subgroup (see 6.7). Such "homogeneous" actions often come along with natural invariant structures. For example, the standard conformal structure φ_0 on S^2 underlies the theory of *Kleinian groups* which are discrete subgroups in $\mathrm{PO}(3, 1) = \mathrm{Is}(S^2, \varphi_0)$. (Some properties of these are briefly discussed in 1.7.)

The most striking example of an interaction of homogeneous local geometry and ergodic theory is the recent theorem of Margulis concerning the action of $H = \mathrm{O}(2, 1) \subset G = \mathrm{SL}(3, \mathbf{R})$ on G/Γ for $\Gamma = \mathrm{SL}(3, \mathbf{Z})$. Namely, Margulis has proven that every compact minimal H-invariant subset is a smooth submanifold in G/Γ and hence consists of a single compact $\mathrm{O}(2, 1)$-orbit (see [51] for the proof and spectacular applications to the arithmetic of quadratic forms. See [61 $\frac{1}{2}$] for further developments.)

We want to emphasize once again that the dynamical depth and the beauty of the above examples is related to the presence of invariant structures and is unparalleled by what one sees in the systems of generic type, which preserve no smooth rigid structure.

0.5.B. *Geometric aspect.* Here we ask ourselves what is the *geometry* of (V, φ) provided that the isometry group $G = \mathrm{Is}(V, \varphi)$ is "sufficiently large". The most general "largeness" condition is noncompactness of G. One may impose stronger conditions by requiring, for example, that G has sufficiently fast rate of growth, or by insisting that G contains a given group G (i.e., a free group on two generators or such group as $\mathrm{SL}(2, \mathbf{R})$). Besides conditions imposed on G one may also require that the *action* of G on V is dynamically speaking "large" or, better to say, "ample". Two such ampleness conditions which are especially useful are *ergodicity* and *topological transitivity*. First, let us recall the definitions.

(i) *Ergodicity.* An action of a group G on a space V with a measure μ is called *ergodic* if $A \subset V$ is G-invariant implies

$$\mu(A) = 0 \quad \text{or} \quad \mu(V \backslash A) = 0.$$

Any transitive action is clearly ergodic. More generally, any essentially transitive action (i.e., transitive on the complement of a null set) is ergodic.

(ii) *Topological transitivity.* We say that an action of G on V is *topologically transitive* if there exists a dense orbit $G(v) \subset V$, $v \in V$. The following example shows how such a condition may effect an invariant structure.

0.6. Example. Let V be a compact connected surface, and φ a C^2-smooth pseudo-Riemannian metric on V. Notice that such a V of type $(1, 1)$ is homeomorphic to the torus or to the Klein bottle. Indeed, the existence of a Lorentz metric on V gives a vector field on a double covering of V, and so the Euler-Poincaré characteristic $\chi(V) = 0$ if V is a closed manifold (surface in our case).

Assume the action of G on V is topologically transitive. Then the Gauss curvature K_φ (being invariant as φ is invariant) is constant on each orbit $G(v) \subset V$, $v \in V$, and by continuity it is constant on all of V, as V equals the closure of a dense orbit. Since $\chi(V) = 0$, the Gauss-Bonnet theorem shows that the constant is zero, $K_\varphi = 0$ on V, and so V is locally flat. (The notion of Gauss curvature and the Gauss-Bonnet theorem automatically extend to the case of an indefinite metric. In fact, the reader who remembers the proof of the Gauss-Bonnet theorem will see that the positivity of the metric is never used there. See [5] for a proof of

Gauss-Bonnet formula for Lorentzian manifolds).

Now, it is not difficult to show (see, e.g., [5]) that $V = \mathbf{R}^{1,1}/\mathbf{Z}^2$, and G is a subgroup of affine transformations acting on the torus T^2 (compare 6.6.B(ii)). Thus we have got a good picture of both V and G in this case, and from our (geometric) point of view this is the end of the story. (But, of course, one may insist on further study of the dynamics of G.)

0.7. Remarks. The key step in the above argument is the passage from topological transitivity to local homeogeneity by means of the Gauss curvature which displays very well how the tensorial nature of the structure influences the dynamics. We shall see in §5 that the same idea can be applied to all geometric structures which have (essentially) tensorial nature. Unfortunately, this requires a somewhat unpleasant but unavoidable formal language of higher order jets and their infinitesimal transformations (see 5.2), and the conclusion is weaker. Namely, we have the following theorem (see 5.14.C).

0.7.A. *If the isometry group $G = \mathrm{Is}(V, \varphi)$ is topologically transitive on V, then there exists an open dense subset $U \subset V$ such that the structure φ is locally homogeneous on U, i.e., every two points in U have φ-isometric neighborhoods.*

Notice that there are simple (but not very natural) examples where $U \neq V$, and it would be nice to ensure the equality $U = V$ by a reasonable condition on (V, φ).

Also notice that in the case of an invariant Riemannian metric the passage from topological transitivity to local homogeneity is trivial, since the full isometry group G of every compact Riemannian manifold is compact, and therefore a dense orbit is necessarily equal to all of V.

0.8. As we have already remarked, generic manifolds have no isometries at all, therefore the presence of an isometry on V makes the manifold V quite special. Now, if we insist on the assumption that the isometry group $G = \mathrm{Is}(V, \varphi)$ is noncompact, then this makes the manifold V even more special. These considerations seem to indicate that there are good reasons to conjecture that it should be possible to classify all compact rigid manifolds having noncompact isometry groups. More precisely, we have the following.

Vague general conjecture. All triples (G, V, φ), where V is compact (or has finite volume) and G is "sufficiently large" (e.g., G is noncompact), are almost classifiable.

We are still far from proving (or even starting) this conjecture, but there are many concrete results which confirm it (see, e.g., 0.9.A below). On the other hand, we shall give in §6 a list of known (G, V, φ), which gives the idea of what kind of classification one may expect.

0.9. The following is a theorem supporting the conjecture.

0.9.A. **Theorem** (Obata [59], Lelong-Ferrand [44]). *Let V be a compact connected Riemannian manifold of dimension n. If the group of conformal transformations of V is noncompact, then V is conformally equivalent to the Euclidean sphere S^n. (Note that the group of conformal transformations of S^n equals* $\mathrm{PO}(n+1, 1) = \mathrm{O}(n+1, 1)/\{\pm 1\}$.

This statement confirms the fact that the existence of an action of a noncompact group on a manifold which preserves some geometric structure is a rather unusual phenomenon, and this completely agrees with the philosophy on which the conjecture was based.

0.10. It should be noted that the noncompactness of the group of conformal transformations of S^n is a nontrivial phenomenon which contradicts everybody's geometric intuition. It is not clear at all why there exists a single conformal transformation of S^n, which is not a rigid rotation. Similarly, one cannot see by a plain eye not equipped with the mathematical machinery any nontrivial conformal transformation of $\mathbf{R}^n$ (which, as we know, maps round spheres to round spheres) where "trivial" refers to the similarity transformation.

Even geometrically minded artists, designers of symmetric patterns, could not overcome this limitation of human imagination. If we look at the incredible variety of ornaments designed through the centuries all over the world, we see all kinds of translational and rotational symmetries but never a conformal symmetry. Yet, in recent times conformal symmetries were displayed in many beautiful drawings by Escher. However, the idea of those was communicated to the artist by a mathematician, namely Coxeter.

The most important transformation group in the world is the Lorentz group $\mathrm{O}(3, 1)$ of the special relativity. The group is noncompact; this appears to be one of the major obstacles for intuitively understanding the special relativity. Notice that the special relativity has replaced one infinity by another, namely it has banished the infinite (or unbounded) speed of motion but introduced arbitrarily large Lorentz transformations. But it is not easy (at least for a mathematician) to reconcile any kind of infinity with the intuitive vision of the physical universe.

0.10.A. **Remark.** The group of conformal transformations of S^n may be hard to see intuitively, but it can be easily introduced from the linear algebraic point of view.

In fact, this group is identified with the projectivized group $\mathrm{O}(n+1, 1)$ of linear transformations of $\mathbf{R}^{n+2}$ which preserve the quadratic form

$$q(x_0, x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1}(x_i)^2 - (x_0)^2$$

(see 6.5.D). The sphere S^n appears here as the set of those lines in $\mathbf{R}^{n+1}$ passing through the origin on which $q = 0$. (The reader can see 1.6 and 6.3 for a general discussion on algebraic actions.)

Notice that algebraically the group $\mathrm{O}(n+1, 1)$ is very close to the group $\mathrm{O}(n+2)$ of the linear transformations of $\mathbf{R}^{n+2}$, which preserve the *definite* quadratic form $\sum_{i=1}^{n+2}(x_i)^2$. Yet geometrically $\mathrm{O}(n+1, 1)$ and $\mathrm{O}(n+2)$ are radically different as one is compact and the other noncompact.

0.11. **Remark.** Now, we want to exhibit a rigid group action of non-algebraic nature which is infinitely more complicated than what we have seen earlier.

0.11.A. *Very important example.* Our group here is $\mathrm{SL}(2, \mathbf{R})$ which acts on the homogeneous space $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$. If Γ is a connected subgroup, this action is of the same level of complexity as the action of $\mathrm{O}(n+1, 1)$ on S^n.

But now we take a *discrete* subgroup $\Gamma \subset \mathrm{SL}(2, \mathbf{R})$ such that $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ is compact. (The existence of such Γ is not at all a trivial matter: see 1.9.D(ii).) This action is far from anything algebraic, and one cannot gain much intuition here by appealing to algebra. Yet, the geometry remains useful here as this action preserves a rigid geometric structure (see 5.11, 5.12 for the definition).

In fact, there exists on this V an invariant pseudo-Riemannian metric of type $(1, 2)$ coming from the Killing form on the Lie Algebra $\mathrm{sl}(2, \mathbf{R})$. (For more details on this $\mathrm{SL}(2, \mathbf{R})/\Gamma$-example the reader can see 1.9.C.)

0.11.B. **Remark.** The fundamental group $\pi_1(V)$ of $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ is quite large. In fact, it is at least as large as Γ since the quotient map $\mathrm{SL}(2, \mathbf{R}) \to V$ is a covering map. ($\pi_1(\mathrm{SL}(2, \mathbf{R})/\Gamma)$ is *strictly* larger than Γ as $\pi_1(\mathrm{SL}(2, \mathbf{R})) \neq \{0\}$.)

The significance of $\pi_1(V)$ being large will be clarified later on (see 1.12).

0.12. It is hard to reconcile our intuition on isometries of compact manifolds with having such a huge noncompact group as $\mathrm{SL}(2, \mathbf{R})$ for the group of isometries. But the intuition regains some ground if we are willing to sacrify the fundamental group of V. Namely, we can prove the following.

0.12.A. **Theorem** [14]. *Let (V, φ) be a compact simply connected real analytic Lorentz manifold (i.e., φ is a pseudo-Riemannian metric of type*

$(n-1, 1))$. *Then the isometry group* $\mathrm{Is}(V, \varphi)$ *is compact.*

0.12.B. **Remarks.** (i) If (V, φ) is a compact Riemannian manifold, i.e., φ is a *definite* quadratic differential form in V, then the group $\mathrm{Is}(V, \varphi)$ is compact without any extra assumption on V.

(ii) Our Theorem 0.12.A is vaguely similar to the Obata–Lelong-Ferrand theorem (see 0.9.A) on conformal transformations.

(iii) Our compactness theorem does not directly generalize to pseudo-Riemannian manifolds (V, φ), where φ has type (p, q) for $\min(p, q) \geq 2$ (see §4 in [14]). However, one has the following weaker compactness result which can be used to prove the compactness of $\mathrm{Is}(V, \varphi)$ in the Lorentz case (see [14]).

0.12.C. **Theorem** [29] (Compare 3.2.B.(i)). *If* V *is a compact simply connected real analytic pseudo-Riemannian manifold then the orbits of the full isometry group* $\mathrm{Is}(V)$ *are compact.*

This result is a specialized (to the case of a pseudo-Riemannian metric φ) version of a more general statement in [29] which, under the same assumption on V as in 0.12.B, ensures the compactness of the $\mathrm{Is}(V, \varphi)$-orbits when φ is a C^{an} smooth rigid structure of algebraic type (see 5.5, 5.11, 5.12, for the definitions) provided that $\mathrm{Is}(V, \varphi)$ preserves a smooth volume element on V.

Notice that by imposing certain conditions on V and φ one necessarily gets that compactness of orbits (see 3.7.A in [29] and also 1.11.B in these lectures) strongly restricts the range of all possible (G, V, φ). In fact, rigid actions with compact orbits should be regarded as classifiable in our sense (compare with the twisted rotation example in 1.11.D).

0.12.D. *Warning.* Amazingly, there exists examples of actions with compact but not uniformly compact orbits (see [64]), but these do not appear in our framework.

0.13. While the above Theorems 0.9.A and 0.12.A indicate that noncompactness of $\mathrm{Is}(V, \varphi)$ makes V rather special, the following Theorem 0.13.A states that if $\mathrm{Is}(V, \varphi)$ is a *very large* noncompact group, then V is very special.

0.13.A. **Splitting theorem** (See 0.8.B and 5.4. in [29]). *Let* (V, φ) *be a connected Lorentz manifold of finite volume (e.g., compact) such that the isometry group* $\mathrm{Is}(V, \varphi)$ *contains* $\mathrm{SL}(2, \mathbf{R})$ *as a subgroup. Then the action of* $\mathrm{SL}(2, \mathbf{R})$ *on* V *is everywhere locally free (i.e., the isotropy subgroup is discrete at all* $v \in V$*). The metric* φ *is nonsingular on the (3-dimensional) orbits and the normal subbundle to the orbits is integrable with totally geodesic leaves. Furthermore, some infinite covering* $\tilde{V}$ *of* V

is split by the lifts to $\tilde{V}$ *of the two* (*into the orbits and the normal leaves*) *foliations* (*see* 4.9 *for a more detailed discussion*).

0.13.B_1. **Remark.** Our theorem refines an earlier result of Zimmer who proved that the above action of $SL(2, \mathbf{R})$ on V is almost everywhere locally free and that the full isometry group $Is(V)$ containing $SL(2, \mathbf{R})$ differs from $SL(2, \mathbf{R})$ roughly by a compact group (see [72], [69] for the precise statements and more general results).

0.13.B_2. **Historical remark and references.** Until recently, the transformation groups of (V, φ) were pursued from a geometric point of view, as can be seen by looking in the monographs by Lichnerowicz [45], Kobayashi [41], Koszul [42].

A dynamical approach was developed in a series of papers by Zimmer (see, e.g., [68]–[76]) and an intermediate approach was attempted in [29].

This paper grew out of the lectures given by the second author in Cagliari in October 1988 and made into notes by the first author. The problem section (§7) was written later by the second author following the suggestion by Professor S.-T. Yau. The second author wants to thank the Department of Mathematics at the University of Cagliari for organizing the lectures, while the first author acknowledges the hospitality of IHES where the preparation of the manuscript was essentially concluded.

1. Dynamics of A-actions

1.1. As we mentioned earlier the structure of an action of a *compact* group G on a smooth manifold V is quite easy and transparent from the dynamical point of view. Namely, each orbit of G is closed, the quotient space V/G is Hausdorff, and each orbit has a "nice" invariant neighborhood by the slice theorem (see [60]) if the action is smooth. In fact, the whole subject of compact (in particular finite) group actions belongs to the algebraic and geometric topology rather than to the dynamics.

1.2. Now, if we turn to the really interesting case where G is noncompact, we shall see a more complicated and tantalizing picture. For example, a noncompact isometry group G may easily have a *dense* orbit $G(v) \subset V$, such that $\dim G(v) < \dim V$. Certainly, such an orbit cannot be closed in V! The standard example of that is an *irrational rotation* of the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ which is a flow (i.e. an action of $\mathbf{R}$) on T^2 induced by parallel translations of $\mathbf{R}^2$. These are defined with a fixed real number α, by

$$(x_1, x_2) \to (x_1 + t, x_2 + \alpha t),$$

where $x_1, x_2 \in \mathbf{R}^2$, and $t \in \mathbf{R}$ is the flow parameter.

If α is a *rational* number, then all orbits of the induced action of $\mathbf{R}$ on T^2 are compact. In fact, such a rational action of $\mathbf{R}$ factors through that of the circle $\mathbf{R}/\mathbf{Z}$ on T^2, where $\mathbf{Z}$ is the intersection of the line $\mathbf{R} = \{t, \alpha t\} \subset \mathbf{R}^2$ with the integral lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$. On the contrary, if α is irrational, each orbit $\mathbf{R}(v)$ in T^2 is noncompact. Moreover, by the famous theorem of Kronecker each orbit is dense (moreover, uniformly distributed) in T^2.

1.3. Another important feature of noncompact group actions is the *recurrency* phenomenon.

1.3.A. **Definition.** A point $v \in V$ is called *recurrent* for an action of G on V if $g(v)$ comes back infinitely often to an arbitrary small neighborhood $U \subset V$ of v. That is, the set $\{g \in G | g(v) \in U\} \subset G$ is noncompact for every neighborhood U of v.

Notice that recurrency does not exclude closed orbits. In fact if the orbit $G(v)$ is compact, then v is obviously recurrent, insofar as G is noncompact. But the real case of interest is that where some orbit $G(v)$ is noncompact while v is recurrent. In this case the closure of $G(v)$ in V is significantly bigger than $G(v)$ itself. For example, each point $v \in V = T^2$ is recurrent for the above action of $\mathbf{R}$ on T^2, in the rational as well as in the irrational case. This fact can easily be derived from the famous

1.3.B. **Poincaré recurrence theorem** (see [71], [13]). *If an action of G preserves a finite measure on V, then almost all points $v \in V$ are recurrent. ("Almost all" here signifies, according to the usual convention, that nonrecurrent points constitute a set of measure zero.)*

1.3.B$_1$. **Remark.** The Poincaré theorem ensures a somewhat stronger recurrency than that given by Definition 1.3.A. Namely, the theorem guarantees a certain kind of "relative density" of the subset $\{g \in G | g(v) \subset U\}$ in G.

1.4. Instead of dealing with individual orbits one may look at the quotient space V/G and see many features of the orbit structure of the action reflected in the topology of V/G. The simplest actions from our point of view are those where the orbit space V/G is Hausdorff.

An especially simple class of group actions which have this property is described in the following.

1.4.A. **Definition.** An action of G on a manifold V is called *proper* if for each pair x, y of points in V there exist neighborhoods U_x of x and U_y of y in V such that the subset $\{g \in G | gU_x \cap U_y \neq \varnothing\}$ is relatively compact in G.

1.4.B. **Examples.** (i) A well-known and easy theorem from the theory of Lie transformation groups states that the standard (transitive) action of G on G/H, where H is a Lie subgroup of the group G, is proper iff H is compact (see, e.g., [66]).

(ii) It follows from (i) above that every closed (e.g., discrete) subgroup H of a connected Lie group G admits a smooth proper action on some Euclidean space. The proof of this fact follows from a well-known result about Lie groups, which states that if G is a connected Lie group, there is a (maximal) compact subgroup H (unique up to conjugation) such that G/H is diffeomorphic to a Euclidean space (see, e.g., [35]).

A typical example of this situation is the hyperbolic space H^n (which is topologically $\mathbf{R}^n$) viewed as the homogeneous space $\mathrm{SO}(n, 1)/\mathrm{SO}(n-1)$.

1.5. After proper actions come stratified actions which can be defined as follows:

1.5.A. **Definition.** We call an action of G on V *stratified* if V can be decomposed into the union of *locally closed* subsets called strata, $V = V_0 \cup V_1 \cup \cdots \cup V_n$, where V_0 is open, V_1 is open in $V_1 \cup V_2 \cup \cdots \cup V_n$, V_2 is open in $V_2 \cup V_3 \cup V_4 \cup \cdots$, and so on, such that V_i/G is a Hausdorff space for all i. (We remind the reader that a subset in (a topological space) V is called locally closed if it is contained and closed in some open subset $U \supset S$ in V.)

1.5.B. **Example.** Let G be a Lie group, and $H \subset G$ a closed subgroup. Consider the one-point compactification $(G/H)^* = G/H \cup \{\infty\}$. Then the action of G on $(G/H)^*$ is stratified with two strata, $\{\infty\}$ and G/H. For example, if $G = \mathbf{R}^n$ and $H = \{0\}$, then $(G/H)^* = S^n$, and a remarkable additional property of the resulting action of $\mathbf{R}^n$ on S^n is the existence of an invariant structure. Namely, the conformal structure of $\mathbf{R}^n$ extends to a smooth conformal structure on $S^n \supset \mathbf{R}^n$ (via the stereographic projection).

1.5.C. **Exercise.** Take $G/H = \mathbf{R}^n$ with an action of G equal to $\mathrm{GL}(n, \mathbf{R})$, $\mathrm{SL}(n, \mathbf{R})$, or the group of all orthogonal isometries. Then the corresponding action on $S^n = \mathbf{R}^n \cup \{\infty\}$ is obviously stratified. The question for the reader is to decide in which case there is an invariant A-structure (see 1.8, 5.5) on S^n.

1.5.D. **Remarks.** The dynamical complexity of a stratified action is comparable to that of transitive actions. For example, one can easily prove the following:

(i) If a point $v \in G$ is recurrent for a stratified action of G on V, then v is recurrent for the (transitive) action of G on the orbit $G(v) = G/G_v$.

(ii) Another important (and easy to prove) property states that *every finite invariant measure* μ *on* V *decomposes into finite invariant measures on the orbits.* In fact, the existence of μ on V implies that the orbit G/G_v admits a finite invariant measure for almost all $v \in V$.

1.6. The importance of stratified actions stems from the fact that every *algebraic* action is *stratified* (see, e.g., [71] and §2 in [29]), where *algebraic* means that the manifold V is given a structure of *real algebraic manifold* such that the group G acts on V as an *algebraic group.* We do not need and shall not give the detailed abstract definition in these lectures. On the other hand, we want to point out that all basic features of algebraic actions can be seen in the following example (compare discussion in 6.3, 6.4, 6.5).

1.6.A. *Linear actions.* A subgroup $G \subset \mathrm{GL}(N, \mathbf{R})$ is called *algebraic* if it equals to zero set of a polynomial map $f\colon \mathrm{GL}(N, \mathbf{R}) \to \mathbf{R}^k$. Notice that $\mathrm{GL}(N, \mathbf{R})$ is an open subset in the Euclidean space $\mathbf{R}^{N^2}$ of $(N \times N)$-matrices, and polynomials on $\mathrm{GL}(N, \mathbf{R})$ by definition are the functions which are polynomial in the Euclidean coordinates. Also notice that by taking $\|f\|^2$ one may restrict oneself to a single polynomial $\mathrm{GL}(N, \mathbf{R}) \to \mathbf{R}$.

1.6.B. *Basic example.* Let φ be a *tensor* on $\mathbf{R}^N$, e.g., a multilinear (say quadratic) form. Then the subgroup G of $\mathrm{GL}(N, \mathbf{R})$ consisting of transformations of $\mathbf{R}^N$ preserving φ is *algebraic* as a simple (and well-known) argument shows. Let us indicate an important construction leading to many concrete examples. Start with the standard action of $\mathrm{GL}(n, \mathbf{R})$ on $\mathbf{R}^n$, and consider the induced action of $\mathrm{GL}(n, \mathbf{R})$ on $\mathbf{R}^M$ for $M = n^k$ on the kth tensor power $\mathbf{R}^M = \bigotimes_k \mathbf{R}^n$. This defines an embedding $\mathrm{GL}(n, \mathbf{R}) \to \mathrm{GL}(M, \mathbf{R})$ whose image is an *algebraic* subgroup in $\mathrm{GL}(M, \mathbf{R})$. Moreover, for any algebraic subgroup $G \subset \mathrm{GL}(n, \mathbf{R})$ (notice that the case $G = \mathrm{GL}(n, \mathbf{R})$ is already interesting) and for every G-invariant subspace $\mathbf{R}^N \subset \mathbf{R}^M$ (e.g., the subspaces of symmetric or antisymmetric tensors) the homomorphism $G \to \mathrm{GL}(N, \mathbf{R})$ corresponding to the restriction to $\mathbf{R}^N$ has algebraic image in $\mathrm{GL}(N, \mathbf{R})$.

1.6.C. **Remarks.** (i) Note that the geometry (as opposed to the dynamics) of an algebraic action can be quite intricate. To see an example of this, we suggest that the readers first look at the action of $(\mathbf{R}^{\times})^2$ on the projective space $P^{n-1} = P(\mathbf{R}^n)$, given by

$$(t_1, t_2)(x_1, \cdots, x_n) = (t_1^{k_1} t_2^{m_1} x_1, t_1^{k_2} t_2^{m_2} x_2, \cdots, t_1^{k_n} t_2^{m_n} x_n)$$

for given integers $k_1, m_1, k_2, m_2, \cdots, k_n, m_n$, and then try to understand the geometry of the closure of a given orbit in P^{n-1}. (Compare with the discussion in 6.5.A.)

(ii) Notice that the dynamical simplicity of algebraic actions does not extend to *subalgebraic* actions. This term refers to an action of a subgroup $G_0 \subset G$, where G acts algebraically on V. If G_0 is not algebraic, the action of G_0 may have nontrivial dynamics. One gets especially interesting actions by looking at actions of discrete subgroups $G_0 \subset G$. The classical example is that of a discrete subgroup $G_0 \subset \mathrm{PGL}(2, \mathbf{C}) = \mathrm{PSO}(3, 1)$ acting on $\mathbf{C}P^1 \cong S^2$. These are Kleinian groups already mentioned in 0.5.A.

1.7. Little digression. Kleinian groups are very beautiful, and there is a wide variety of productive ways to think about them. The basic feature of Kleinian groups is the *existence* of the *limit set* $\Omega \subset S^2$ with the following properties:

(i) Ω is a closed G_0-invariant subset in S^2 such that the closure of the orbit $G_0(\omega)$ equals Ω for every $\omega \in \Omega$.

(ii) The action of G_0 on $S^2 \backslash \Omega$ is proper (see 1.4.A).

(iii) For every point $s \in S^2$ the set of the accumulation points of the orbit $G_0(s) \subset S^2$ equals Ω. It is well known (and easy to prove) that the limit set exists and is unique for all *nonelementary* Kleinian groups, where G_0 is called elementary if some orbit of G_0 is finite. What is much harder and really exciting is understanding the geometry of limit sets. One knows, for instance, that the limit set of a nonelementary Kleinian group is either the whole sphere S^2 or a round circle or an amazingly complicated fractal subset.

Nowadays one can see fractal limit sets displayed on beautiful multicoloured posters (see, e.g., [48]).

1.8. Now we turn to a class of actions which generalize algebraic and subalgebraic actions and play the central role in these lectures.

1.8.A. *Basic nondefinition.* We say that an action of a Lie group G on V is *A-rigid* if the action preserves some rigid structure φ of algebraic type given on V or, for brevity, an *A-structure.* The precise definitions of the terms "rigid" and "algebraic type" are given in §5. Here we only recall that most structures encountered in differential geometry are of algebraic type and that the rigidity for these structures is essentially equivalent to the finite-dimensionality of the pseudogroup $\mathrm{Is}^{\mathrm{loc}}(V, \varphi)$ of local isometries of the structure φ. For example, every tensor field on V is an A-structure, and most of tensorial structures are rigid (see 5.11).

In the language of §0 (see 0.2), A-actions correspond to subgroups $G \subset \mathrm{Is}(V, \varphi)$ for a rigid A-structure φ. Thus, for example, A-actions include isometries of pseudo-Riemannian manifolds, groups of conformal transformations, and connection preserving transformations.

1.9. One of the reasons for a nontrivial dynamics of an A-rigid action may be the fact that the group G in question is *strictly smaller* than the full isometry group $\mathrm{Is} = \mathrm{Is}(V, \varphi) \supset G$. For example, let us look again at the Kleinian group acting on S^2 (see 1.7). The relevant *rigid* structure φ here is the *flat complex projective structure* for which $\mathrm{Is}(S^2, \varphi) = \mathrm{PSO}(3, 1)$. (The conformal structure on S^2 does not formally fit into our discussion since it is nonrigid).

As we have mentioned before, the action of a Kleinian group $G \subset \mathrm{Is}$ on S^2 may have a remarkably rich dynamics. This is due to two properties:

(a) Is is noncompact;

(b) G is a proper subgroup in Is.

Moreover G, being discrete infinite, is not an algebraic subgroup in $\mathrm{Is} = \mathrm{PSO}(3, 1)$.

1.9.A. **Remark.** If we start with a structured manifold (V, φ) with compact isometry group $\mathrm{Is}(V, \varphi)$, then a subgroup $G \subset \mathrm{Is}(V, \varphi)$ may have a dynamically interesting action on V only if G is *nonclosed* in Is. What happens in this case can be seen in the example of the irrational translation of the torus (see 1.2).

1.9.B. Now we want to look at A-rigid actions where $G = \mathrm{Is}(V, \varphi)$, but the dynamics is at least as rich as that of any subalgebraic (e.g., Kleinian) action. We start by looking at our "very important example" of 0.11.A.

1.9.C. The group G in this example is $\mathrm{SL}(2, \mathbf{R})$, and V is a compact G-homogeneous space. Namely, $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$, where Γ is a discrete cocompact subgroup ("cocompact" signifies the compactness of $\mathrm{SL}(2, \mathbf{R})/\Gamma$).

We already mentioned in 0.11.A that the action of $\mathrm{SL}(2, \mathbf{R})$ on this V is A-rigid as there exists an invariant pseudo-Riemannian metric of type $(1, 2)$ defined on V. To see this, let us start with the Killing form φ_0 on the Lie algebra $\mathrm{sl}(2, \mathbf{R})$ identified with the tangent space $T_e(\mathrm{SL}(2, \mathbf{R}))$ of $\mathrm{SL}(2, \mathbf{R})$ at the identity. We get a pseudo-Riemannian metric $\tilde{\varphi}$ on $\mathrm{SL}(2, \mathbf{R})$ by left translations of $T_e(\mathrm{SL}(2, \mathbf{R}))$ to the tangent spaces $T_g(\mathrm{SL}(2, \mathbf{R}))$, $g \in \mathrm{SL}(2, \mathbf{R})$.

Since φ_0 is Adj-invariant, the form $\tilde{\varphi}$ is invariant under the right translations as well as under left translations. It follows that $\tilde{\varphi}$ descends to an *invariant* metric φ on the quotient $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$, and thus $\mathrm{Is}(V, \varphi) \supset \mathrm{SL}(2, \mathbf{R})$. Furthermore, one can show that the full isometry group $\mathrm{Is}(V, \varphi)$ equals $\mathrm{SL}(2, \mathbf{R})$ or $\mathrm{SL}(2, \mathbf{R})/\{\pm 1\}$.

1.9.D. *Digression.* (i) We have considered so far *cocompact* discrete subgroups $\Gamma \subset G = \mathrm{SL}(2, \mathbf{R})$, but noncocompact subgroups may also be interesting from our point of view. For example, one may distinguish (discrete) *lattices* $\Gamma \subset G$, where a subgroup Γ is called a lattice if G/Γ admits a finite G-invariant measure. Notice that every discrete cocompact subgroup Γ in a unimodular Lie group G (e.g., in $\mathrm{SL}(2, \mathbf{R})$) is necessarily a lattice as the Haar measure on G (being bi-invariant on the unimodular group) descends to an invariant measure on G/Γ.

(ii) The existence of (cocompact and noncocompact) discrete lattices in a Lie group is a highly nontrivial matter. Of course some cases are quite easy. For example, if $G = \mathbf{R}^n$, one can immediately see a discrete lattice in there, namely $\mathbf{Z}^n \subset \mathbf{R}^n$, and then one can prove that any lattice is obtainable from $\mathbf{Z}^n$ by an automorphism of $\mathbf{R}^n$. The situation is by far more subtle and complicated for semisimple Lie groups G, such as $\mathrm{SL}(2, \mathbf{R})$. The simplest example here is $\Gamma = \mathrm{SL}(n, \mathbf{Z})$, the group of matrices with integral entries and determinant one.

There is a nontrivial (though not very difficult) theorem saying that $\mathrm{SL}(n, \mathbf{R})/\mathrm{SL}(n, \mathbf{Z})$ has *finite measure*, and so $\mathrm{SL}(n, \mathbf{Z})$ is indeed a *lattice* in $\mathrm{SL}(n, \mathbf{R})$. This lattice is not cocompact, and getting cocompact lattices requires more effort. In general, there exist three methods of constructing discrete lattices in semisimple Lie groups G.

1. *Arithmetic method.* This amounts to getting an appropriate monomorphism $G \to \mathrm{SL}(n, \mathbf{R})$ for large n and obtaining $\Gamma \subset G$ as the intersection $G \cap \mathrm{SL}(n, \mathbf{Z})$. By this method Borel [9] has proven that every semisimple G contains a cocompact as well as a noncocompact lattice.

2. *Geometric method.* This is especially clear for $\mathrm{SL}(2, \mathbf{R})$ viewed as the isometry group on the hyperbolic plane $\mathbf{H}^2$. One gets our Γ by taking, for example, a regular n-gon, $n > 5$, in $\mathbf{H}^2$ with $90°$-angles and then generating Γ by the n-reflection of $\mathbf{H}^2$ in the sides of this n-gon.

3. *Differential equations.* Monodromy groups of some totally integrable system of differential equations are lattices in their Zariski closures. (See Mostow [56] for an extensive discussion of all these matters.)

1.10. To appreciate the action of $\mathrm{SL}(2, \mathbf{R})$ on $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ it is useful to look on how one-parameter subgroups of $\mathrm{SL}(2, \mathbf{R})$ act on this V. A somewhat unexpected fact is that every such subgroup $G_0 \subset \mathrm{SL}(2, \mathbf{R})$ equals the *full* isometry group of some rigid A-structure on V. In fact let X_0 denote the vector field on V generating G_0, and consider the "sum" of X_0, viewed as a geometric structure, with the Killing metric φ on V. What we get (by the definition of the "sum") is the pair $\varphi' = (\varphi, X_0)$ and the isometry group of (V, φ') is the intersection $\mathrm{Is}(V, \varphi) \cap \mathrm{Is}(V, X_0)$.

In other words, $\mathrm{Is}(V, \varphi')$ consists of those isometries of (V, φ) which also preserve X_0. Since X_0 is a vector field, to "preserve" X_0 means to commute with X_0. Therefore, $\mathrm{Is}(V, \varphi')$ consists of the centralizer C_0 of G_0 in $\mathrm{SL}(2, \mathbf{R})$. As it is well known (and obvious), every subgroup C in $\mathrm{SL}(2, \mathbf{R})$ with 1-dimensional center is 1-dimensional, and so C_0 is at most a finite extension of G_0.

The action of noncompact one-parameter subgroups G_0 on $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ displays a wide spectrum of beautiful dynamical properties such as topological transitivity, ergodicity, etc. Notice that there are only two essentially different actions corresponding to two conjugacy classes of noncompact one-dimensional subgroups in $\mathrm{SL}(2, \mathbf{R})$. Namely, we have the following.

(a) The action of the one-parameter subgroup

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{on } V = \mathrm{SL}(2, \mathbf{R})/\Gamma.$$

If the group Γ in question has no torsion, then this action can be identified with the geodesic flow on a compact surface W of constant negative curvature. Namely, one notices first that the (left) quotient $\mathrm{SL}(2, \mathbf{R})/S^1$ admits a left invariant metric of constant curvature -1 and equals the hyperbolic plane $\mathbf{H}^2$. Then the unit tangent bundle $UT(\mathbf{H}^2)$ can be identified with $\mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\{\pm 1\}$, such that the action of g_t corresponds to the geodesic flow on $UT(\mathbf{H}^2)$. Finally, we take the quotient $\mathbf{H}^2/\Gamma$ for our W which is a *smooth* surface as Γ has no torsion and so acts *freely* on $\mathbf{H}^2$.

(b) The second action is that of the (unipotent) subgroup $\mathrm{O}_t = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ on V. This corresponds to the horocycle flow on the unit tangent bundle of the above W. (This flow moves each tangent vector along the oriented horocycle normal to and normally oriented by this vector; see [4].) Notice that, in the above $\mathrm{SL}(2, \mathbf{R})/\Gamma$-example, rich dynamics goes along with a large fundamental group $\Pi = \pi_1(V)$. In fact, Π is the central extension

$$0 \to \mathbf{Z} \to \Pi \to \pi \to 0$$

where $\mathbf{Z}$ equals the Galois group of the universal covering $\widetilde{\mathrm{SL}(2, \mathbf{R})}$ of $\mathrm{SL}(2, \mathbf{R})$.

The following Theorem 1.11.A shows, in contrast, that for simply connected manifolds the dynamics of the full isometry group is almost as simple as that of an algebraic group.

1.11.A. **Theorem** [29]. *Let (V, φ) be a compact manifold with a rigid real analytic structure φ of algebraic type (i.e., a pseudo-Riemannian metric* *or a conformal structure). If* V *is simply connected, then the action of* $G = \mathrm{Is}(V, \varphi)$ *is stratified. Moreover, the isotropy subgroup* $G_v \subset G$ *of each point* $v \in V$ *has at most finitely many connected components.*

An easy corollary of 1.11.A is the following:

1.11.B. (Compare 0.12.C.). *Let* V *be as in* 1.11.A *and assume, in addition,* $G = \mathrm{Is}(V, \varphi)$ *preserves a smooth measure on* V. *Then all* G*-orbits are compact.*

1.11.C. *Unbounded volume property* (see 6.4.B$_5$). The above $\mathrm{Is}(V, \varphi)$ actions for $\pi_1(V) = 0$ may look rather nonalgebraic in spite of the above stratification Theorem 1.11.A.

For example, the volumes of the graphs $\Gamma_g \subset V \times V$ of the transformation $g: V \to V$ may be unbounded as $g \in G$ goes to infinity (compare 6.4.B$_5$).

1.11.D. *Twisted rotation example.* Let us indicate a specific A-rigid action on $V = S^3$ where $\mathrm{Vol}(\Gamma_g) \to \infty$ for $g \to \infty$. Let S^3 be fibered over S^2 in the usual way, $p: S^3 \to S^2$, and let $S^1 \times \mathbf{R}$ act on S^3 in the following way. S^1 acts by the usual rotations having the Hopf fibers for the orbits, while $\mathbf{R}$ rotates the fibers with variable speed. That is, for a given function, say, a on S^1, let $Y = aX$ be the vector field on S^3 where X is the generating field for the S^1 action. Then Y integrates to an action of $\mathbf{R}$ which rotates the circle $p^{-1}(s) \subset S^3$, $s \in S^2$, with the speed $a(s)$. If a is nonconstant, then by an easy argument $\mathrm{Vol}(\Gamma_g)$ is unbounded for $g \to \infty$. On the other hand, this action is A-rigid analytic for all real analytic a (see 6.6).

1.11.E. **Remark.** Another way to see that the above twisted action is not algebraic is by looking at the corresponding diagonal action of our $G = S^1 \times \mathbf{R}$ on the products $S^3 \times S^3 \times \cdots \times S^3$ and by observing that there exist no nonempty open invariant subsets where this action is proper (compare 6.4.B$_4$).

1.12. The role of the simply connectedness condition in Theorem 1.11.A is based on the fact that every local isometry of a *rigid simply connected real analytic* manifold V extends to a global isometry of V. If the manifold V is not simply connected, then local isometries do not always extend to global isometries. For example, take $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ where the group $\mathrm{SL}(2, \mathbf{R})$ has the Lorentz metric $\tilde{\varphi}$ determined by the Killing form on $\mathrm{sl}(2, \mathbf{R})$. Then we know (see 1.9.C) that $\mathrm{Is}(V, \varphi) = \mathrm{SL}(2, \mathbf{R})$.

Yet, there are many local isometries of V which are not extendible to all of V. Namely, every isometry of $(\tilde{V}, \tilde{\varphi}) = (\mathrm{SL}(2, \mathbf{R}), \tilde{\varphi})$ which covers (V, φ) defines a local isometry of (V, φ). Now, as $\tilde{\varphi}$ is bi-invariant, both the left and the right translations are $\tilde{\varphi}$-isometric. Thus $\mathrm{Is}(\mathrm{SL}(2, \mathbf{R}), \tilde{\varphi}) \supset \mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ and so $(\mathrm{SL}(2, \mathbf{R})/\Gamma, \varphi)$ has (at least) twice as much local isometries compared to global isometries.

Nonextendible local isometries are the major source of dynamical complexity of the action of $G = \mathrm{Is}(V, \varphi)$ on non-simply-connected V. To see this, let us show how the recurrence of a point necessarily brings along local isometries near this point which may not be exendible to global isometries.

Let $v \in V$ be a recurrent point, and let $g_i \to \infty$ be local isometries such that $v_i = g_i(v) \to v$. Thus we have isometries g_i^{-1} moving v to v_i, and as v_i are close to v we may expect these isometries form part of a *connected* local isometry group G_0 acting near the point v. If the structure φ in question is a Riemannian metric, then all g_i are *uniformly bounded*, and we construct G_0 by taking convergent subsequences of g_i. This does not work for general A-structures, but a certain modification of this argument does apply (see [72] and 4.4, 4.6.A in [29]). The conclusion one obtains roughly says that the local isotropy group $\mathrm{Is}^{\mathrm{loc}}(v)$, $v \in V$, is "essentially" as "big" as the full isometry group $\mathrm{Is}(V, \varphi)$ provided V is compact or (and) admits a finite smooth G-invariant measure. An instance of such a result is Theorem 4.6.A for semisimple groups. The following is a specific example.

1.12.A. **Example.** Consider again our "very important" $\mathrm{SL}(2, \mathbf{R})/\Gamma$-example (see 0.11.A, 1.9.C). Then $\mathrm{Is}(V) = \mathrm{SL}(2, \mathbf{R})$ and also the local isotropy group $\mathrm{Is}^{\mathrm{loc}}(v) = \mathrm{SL}(2, \mathbf{R})$ acting by conjugation, for all $v \in V$.

To see another example, we state the following.

1.12.B. **Proposition** (Compare 4.4 and 4.6 in [29]). *Let (V, φ) be a compact manifold endowed with a rigid C^{an} structure φ of algebraic type (see 5.5). If $G = \mathrm{Is}(V, \varphi)$ is noncompact, then there exists a point $v \in V$ such that the local isotropy subgroup G_v is noncompact. If, moreover, there is a smooth finite G-invariant measure on V, then G_v is not compact for all points $v \in V$.*

2. Geometric structures associated with Anosov actions

2.1. We describe in this subsection certain situations where an a priori complicated dynamics may preserve a continuous (and in rare cases smooth) rigid structure.

First, we recall the definition of a *hyperbolic* action (as introduced by Anosov in 1966 [2]) of the groups $G = \mathbf{Z}$ and $G = \mathbf{R}$ on a compact manifold V. We assume the action is locally free (which is automatic for

Z) and denote by $I \subset T = T(V)$ the subbundle of the vectors tangent to the orbits. Notice that the fiber dimension of I is

$$\dim I = \dim G = \begin{cases} 0 & \text{for } G = \mathbf{Z}, \\ 1 & \text{for } G = \mathbf{R}. \end{cases}$$

We start by recalling the contracting and expanding subbundles T^+ and T^- defined as follows (compare 0.2.A):

$$T^+ = \left\{\tau \in T(V) | \lim_{g \to +\infty} \|Dg(\tau)\| = 0\right\},$$
$$T^- = \left\{\tau \in T(V) | \lim_{t \to -\infty} \|Dg(\tau)\| = 0\right\},$$

where Dg denotes the differential of the action $g\colon V \to V$, $g \in G$, and $\| \ \|$ refers to a fixed Riemannian metric on V.

Recall (see 0.2.A) that T^+ and T^- are not, in general, subbundles (see examples in 2.3 following the definition of Anosov action below) as the dimension of the fiber may not be constant on V. Now, we have the first Anosov axiom:

A_1. T^+ and T^- are continuous subbundles in $T(V)$, and their fiber dimensions satisfy

$$\dim T^+ + \dim T^- = \dim V - \dim G.$$

Moreover, the subbundles T^+, T^-, and I Whitney split the tangent bundle $T(V) = T^+ \oplus T^- \oplus I$. The second axiom states that the convergence to 0 in the definition of T^+ and T^- is exponential.

A_2. There exist constants $C > 0$ and $\lambda > 1$ such that

$$\|Dg(\tau)\| \leq C\lambda^{-|g|}\|\tau\|$$

for $g \geq 0$ and $\tau \in T^+$ as well as for $g \leq 0$ and $\tau \in \Gamma^-$.

2.2. **Remarks.** (a) Notice that the definition of an Anosov action does not depend on the choice of the Riemann metric since V is compact.

(b) There exist two invariant foliations S^+ and S^- on V, whose tangent bundles are T^+ and T^- respectively. These are defined by the following construction: points v_1 and v_2 in V lie in a leaf of S^+ iff $\operatorname{dist}(g(v_1), g(v_2)) \to 0$ for $g \to +\infty$, and S^- is similarly defined for $g \to -\infty$. The leaves of S^+ are called *stable* manifolds and those of S^- *unstable*.

2.3. **Examples.** (i) Let $V = S^n$, where S^n is identified with $\mathbf{R}^n \cup \{\infty\}$ via the stereographic projection. If the action of $G = \mathbf{R}$ on S^n is given by the scaling

$$x \mapsto e^g x \quad \text{on } \mathbf{R}^n,$$

then $T^+ \subset T(S^n)$ consists of all vectors tangent to S^n outside the south pole (which corresponds to $0 \in \mathbf{R}^n$). Similarly, T^- misses the north pole.

(ii) Another example is that of the action on S^n corresponding to a parallel translation on $\mathbf{R}^n$. Here both T^+ and T^- consists of the tangent vectors outside the south pole. The proof is an easy exercise for the reader.

2.4. The simplest example of an Anosov action. Let V be the 2-dimensional torus, i.e., $V = \mathbf{R}^2/\mathbf{Z}^2$, and let f be a diffeomorphism of V whose lift to $\mathbf{R}^2$ is the linear map f_0 given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then f generates an Anosov system. In fact, the eigendirections of the corresponding automorphism of the plane $\mathbf{R}^2$ define invariant line fields on the torus: the contracting and the expanding ones. Notice that the invariant line fields are smooth in this example.

Another remark is that there exists a quadratic form on $\mathbf{R}^2$ of signature $(1, 1)$ invariant under f_0. Such a form gives rise to an f-invariant Lorentz metric on T^2, which is also invariant under the translations of T^2 (see 3.4.1(i)). We shall show later in this section (see 2.6) that the existence of such a metric is rather typical for Anosov systems. Also, notice that the action of f on T^2 is topologically transitive and even ergodic (see 0.5.B) for the Haar measure on T^2.

2.4.A. **Remark.** The Anosov property is stable under C^1-small perturbation of the action. For example, a torus diffeomorphism C^1-close to the automorphism $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ of the above example is always Anosov although its contracting and expanding direction fields may be less than C^2-smooth even in the case when the diffeomorphism is analytic (see [3]).

2.5. Example. *The stable horospherical foliation.* Let us describe the stable and unstable foliations for the geodesic flow in the unit tangent bundle $V = UT(W)$ of a complete manifold W of negative sectional curvature K.

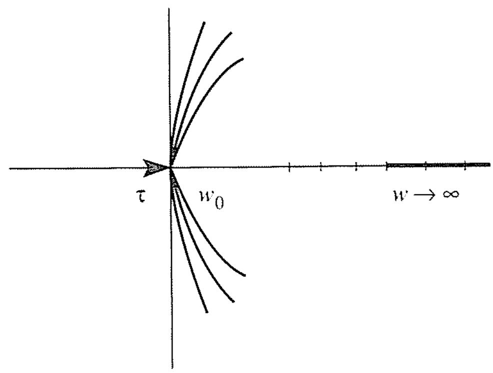
We start with the case where W is simply connected. We take a tangent vector $\tau \in UT(W)$ at $w_0 \in W$, and let $R = \mathbf{R}_+ \subset W$ be the geodesic ray issuing from w_0 and directed by τ. The conditions $K \leq 0$ and $\pi_1(W) = 0$ imply that (see, e.g., [40])

$$(*) \qquad \operatorname{dist}_W(t_1, t_2) = t_2 - t_1$$

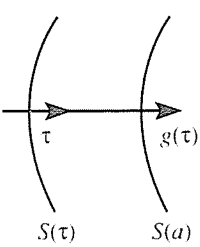
for all segments $[t_2 - t_1] \subset R = \mathbf{R}_+$.

Now for each $w \in R \subset W$ we take the ball $B_w(r)$ of radius $r = r(w) = \operatorname{dist}(w, w_0)$ and observe with $(*)$ that $B_w(r') \supset B_w(r)$ for $r' \geq r$. The increasing union $B_\infty = \bigcup_{w \in R} B_w(r)$ is called the *horoball* defined by τ, and can be viewed as the ball of infinite radius with the center at infinity.

The topological boundary of B_∞ is called the *horosphere* $S = S(\tau)$ normal to τ. By using $K \le 0$ it is easy to see that S is a smooth hypersurface in W passing through w_0 and normal to τ.

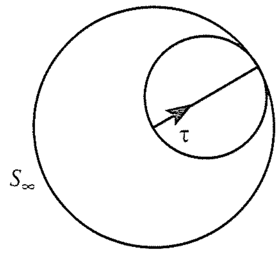


Next we consider the *normal lift* $\tilde{S}(\tau) \subset V = UT(w)$ of $S(\tau)$ where the point $\tilde{s}$ over each $s \in S(\tau)$ is represented by the unit tangent vector $\tau(s)$ normal to S and directed in the same way as τ. Since the lifted horospheres corresponding to different vectors, say $\tilde{S}(\tau_1)$ and $\tilde{S}(\tau_2)$, either coincide or, otherwise, are disjoint in V, they foliate the unit tangent bundle $V = UT(W)$. This foliation is invariant under the geodesic flow. In fact if we apply the geodesic flow g to $\tilde{S}(\tau)$, we obtain the lift of another horosphere, namely $\tilde{S}(g(\tau))$. The following diagram depicts what is happening downstairs in W.



Now let W be compact and let us apply the above construction to the universal covering of W. Then we get a foliation of the universal covering which clearly is *invariant* under the deck transformation group and so defines a foliation of our compact $V = UT(W)$. It is not hard to see (for $K < 0$) that this is exactly the stable foliation for the geodesic flow. One gets the nonstable one by taking the horospheres $S(-\tau)$ for $\tau \in UT(W)$.

2.5.A. **Remark.** If W has constant negative curvature, then the horospheres in the Poincaré model are represented by the spheres tangent to the boundary S_∞ of the ball giving the Poincaré model of $W = \mathbf{H}^n$.



2.6. Quadratic forms on split spaces. The pair of subbundles $\pi = (T^+, T^-)$ associated to an Anosov $\mathbf{Z}$-action (see (2.1)) is not a rigid structure as defined in 5.10 and 5.11. In fact, since T^+ and T^- are integrable, the local isometry group of π is infinite dimensional; it consists of the diffeomorphisms preserving the corresponding foliations S^+ and S^- and thus (locally) isomorphic to $\operatorname{Diff}\mathbf{R}^k \times \operatorname{Diff}\mathbf{R}^l$, where k and l are the dimensions of S^+ and S^-.

However, if an Anosov system preserves an additional structure φ, such as a symplectic (see 2.6.B below) or contact structure, then the "sum" (π, φ) may very well be rigid. In fact, by coupling π and φ we can obtain our old friend pseudo-Riemannian metric.

To do this, we start with a simple algebraic observation.

Let L be a linear space split into the direct sum $L = L' \oplus L''$, and let ω be a bilinear form defined on L. Then one can canonically construct a quadratic form, say ϖ, on L as follows:

$$\varpi(l, l) = \omega(l', l'') \quad \text{for all } l \in L,$$

where l' and l'' are projections of l to L' and L'', respectively.

2.6.A. **Lemma.** *Let ω be a nonsingular antisymmetric form on L, and let L', L'' be isotopic (often called Lagrangian) subspaces. Then the form ω is a nonsingular quadratic form of signature (n, n) for $2n = \dim L$, for which L' and L'' are isotropic. (We recall that a subspace in L' is called ω-isotropic if $\omega|L' \equiv 0$.)*

Proof. Take bases $\{l_i'\}$ and $\{l_j''\}$ of L' and L'' respectively in such a way that $\omega(l_i', l_j'') = \delta_{ij}$. Then, in the basis $\{l_i', l_j''\}$ of L, the form ω also has $\varpi(l_i', l_j'') = \delta_{ij}$.

2.6.B. Now, let an Anosov action of $\mathbf{Z}$ on V preserve a *symplectic* structure, that is, a closed nonsingular exterior 2-form ω. (The closeness of ω is immaterial at this point.) Then the expanding and contracting bundles T^+ and T^- are clearly ω-isotropic, and so the above lemma provides a continuous invariant pseudo-Riemannian metric φ on V, of signature (n, n) for $2n = \dim W$.

2.6.C. **Remark.** The above metric has not yet been successfully applied by anybody to study general Anosov systems. However, if one assumes that the subbundles T^+ and T^- are smooth, then the metric is also smooth, and so our theory of rigid invariant structures fully applies. For example, if one applies Theorem 0.7.D to this case, one obtains the following.

2.6.D. *Local homogeneity property.* If an Anosov $\mathbf{Z}$-action preserves a smooth symplectic structure ω on V, and the bundles T^+ and T^- are C^∞ smooth, then there is an open dense invariant subset $V_0 \subset V$ admitting a structure of locally homogeneous space, and the implied (local) Lie group G acting on V_0 preserves ω, T^+, T^-, and the metric φ.

2.6.E. **Remark.** The idea of the proof of 2.6.D can be seen in the discussion in 0.6, where we have already encountered this phenomenon in the case of a 2-dimensional manifold V and a C^2-diffeomorphism. In 0.6 we only needed C^2-smoothness, as we used the curvature of the manifold to obtain the homogeneity of all V (namely, the open dense invariant subset $V_0 \subset V$ admitting a structure of a locally homogeneous space turned out to be all of V). Thus, returning to our situation in the 2-dimensional case, we come to the following well-known fact (see [5]).

2.6.E$_1$. *If f is a C^2-smooth Anosov diffeomorphism of a smooth, compact connected and orientable surface preserving a smooth measure and having C^2-smooth stable and unstable foliations, then f is smoothly conjugate to a linear automorphism of the torus T^2.*

Recall that each automorphism of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ is given by an integral linear transformation of $\mathbf{R}^2$, that is,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with $a, b, c, d \in \mathbf{Z}$ and $ad - bc = \pm 1$.

2.6.E$_2$. One may hope that there is a similar classification in the higher-dimensional case (see the references at the end of 2.11).

2.7. R-actions and geodesic flows. We want to extend the previous discussion to the case of Anosov actions of the group $G = \mathbf{R}$. In fact, in this case we stand a much better chance of having an invariant rigid structure since the Anosov splitting $T(V) = I \oplus T^+ \oplus T^-$ together with a vector field X generating the one-dimensional subbundle I form, generically, a *rigid* structure (as opposed to the case $G = \mathbf{Z}$ where $\dim I = 0$). Here X denotes the field generating the action, and the notion of genericity applies to the triples (X, T^+, T^-) where the subbundles T^+ and T^- are integrable.

To see that genericity implies rigidity we consider the (one-codimensional) subbundle $T' = T^+ \oplus T^- \subset T(V)$, and let ξ be the 1-form defined by the following two conditions:

$$\ker \xi = T', \quad \xi(X) = 1.$$

We take the differential $\omega = d\xi$, and assume that the 2-form ω is nonsingular on T'. Then, as earlier, we obtain a quadratic form φ' on T'.

2.7.A. *Warning.* The notions of genericity and rigidity in the above discussion are by no means rigorous. First of all, both notions apply, strictly speaking, to smooth objects, while T^+ and T^- are in general only continuous. (One also needs C^1-smoothness to take the differential of ξ, but this is a less serious matter.)

Secondly, a 2-form on T' can be nonsingular only if $\dim T'$ is even.

Granted this, the nonsingularity can be generically ensured locally, near a fixed point, but not over all of V. Finally, one cannot honestly apply the notion of genericity to T^+ and T^-, as these come as a result of a specific (infinite) construction and cannot be changed with a sufficient freedom underlying the idea of genericity.

Yet, for all these shortcomings the above discussion put forth strong evidence in favor of (generic) rigidity of (X, T^+, T^-).

2.8. Symplectic systems. Now, let us suppose that an $\mathbf{R}$-action on a smooth manifold preserves a symplectic form ω. Notice that such an action cannot be Anosov.

In fact, since ω and the generating field X are invariant, the condition $\lim_{g\to\infty} \|Dg(\tau)\| = 0$, $g = \mathbf{R}$, for some tangent vector field τ implies that $\omega(\tau, X) = 0$. It follows that the subbundles T^+ and T^- are contained in the (one-codimensional) subbundle $\operatorname{Ker} X^*_\omega$, where X^*_ω denotes the 1-form $\tau \mapsto \omega(\tau, X)$. Since ω is antisymmetric, the field X is also contained in $\operatorname{Ker} X^*_\omega$ and so T^+, T^-, and X do not span $T(X)$ as required by the Anosov condition.

Next we recall that the Lie derivative L_X acting on the exterior forms can be expressed in terms of the exterior differential d and the interior product with X, called i_X, by the following (obvious and well-known) formula

$$(*) \qquad L_X = i_X \circ d + d \circ i_X.$$

Then, by the definition of i_X, we have $i_X \circ \omega = X^*_\omega$ and observe that $L_X\omega = 0$ as ω is invariant under the flow generated by X. Since also $d\omega = 0$, from $(*)$ it follows that

$$0 = L_X\omega = dX^*_\omega,$$

which implies that X^*_ω is closed, and so the subbundle $\operatorname{Ker} X^*_\omega$ is integrable. Since $X \subset \operatorname{Ker} X^*_\omega$, the leaves of the resulting foliation are invariant under our $\mathbf{R}$-action and we fix our attention on one such leaf, denoted by V. Now the bundles I, T^+, and T^- restricted to V are contained in $T(V)$ and may very well provide the Anosov splitting of the $\mathbf{R}$-action restricted to V. If indeed the action is Anosov on V, then the restriction of ω on the 1-codimensional subbundle $T^+ \oplus T^- \subset T(V)$ is nonsingular, because ω is nonsingular, and $T^+ \oplus T^-$ is transversal and ω-orthogonal to X at the same time. With a nonsingular ω on $T^+ \oplus T^-$ we obtain as earlier a nonsingular quadratic form of signature $(n-1, n-1)$ on $T^+ \oplus T^-$ for $2n-1 = \dim V$. This form orthogonally adds up with the form on I defined by being equal to 1 on X, and thus we get a pseudo-Riemannian metric of type $(n, n-1)$ on V.

2.9. Geodesic flows. Let us recall the symplectic description of the geodesic flow on $V = UT(W)$ for a manifold W with a Riemannian metric h. First we define the *canonical symplectic form* ω on the cotangent bundle $T^*(W)$ by $\omega = d\eta$, where η is the tautological 1-form on $T^*(W)$. Namely, this η is uniquely characterized by the following identity which must hold for all smooth 1-forms α on W, which are also viewed as sections $\alpha\colon W \to T^*(W)$,

$$(+) \qquad \alpha^*(\eta) = \alpha,$$

where α^* denotes the form on W induced from η by the map $\alpha\colon W \to T^*(W)$, and α on the right-hand side of $(+)$ is thought of as a 1-form on W.

In the case $W = \mathbf{R}^n$ one can see that

$$\eta = \sum_{i=1}^{n} y_i\, dx_i,$$

where x_i are the coordinates of $\mathbf{R}^n$, and y_i are the (impulse) coordinates in the cotangent space $T^*_0(\mathbf{R}^n)$ $(= \mathbf{R}^n)$, where we use the splitting

$$T^*(\mathbf{R}^n) = \mathbf{R}^n \times T^*_0(\mathbf{R}^n).$$

The above η can be thought of as a (universal) form on $\mathbf{R}^n$ with undetermined coefficients y_i. Then every α-section is given by n functions

$$y_i = y_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n,$$

and the (induced) α-form on $\mathbf{R}^n$ becomes

$$\sum_{i=1}^{n} y_i(x_j)\, dx_i ,$$

which agrees with the tautological definition of η. The advantage of the above local formula for η is the following expression for ω,

$$\omega = d\eta = \sum_{i=1}^{n} dy_i \wedge dx_i ,$$

which clearly shows ω is nonsingular.

To go further we invoke the metric h (which has not been used so far) and define the following (Hamilton) function on $V' = T^*(W)$,

$$H(Q) = \|Q\|_h^2 ,$$

for all covectors Q in $T^*(W)$. Then we take the ω-gradient $Y = \operatorname{grad}_\omega H$, defined by the equality,

$$i_Y \omega = dH ,$$

which means $\omega(Y, \tau') = dH(\tau')$ for all tangent vectors τ' on V'. Then the above $(*)$ for the Lie derivative tells us that

$$L_Y \omega = ddH + i_Y\, d\omega = 0 ,$$

which makes ω invariant under the flow generated by Y.

Next, the (already used) relation

$$Y_\omega^* \overset{\text{def}}{=\!=} i_Y \omega = dH$$

shows that the levels of the function H are invariant under this flow. In particular, the flow preserves the unit cotangent bundle $UT^*W = \{H = 1\} \subset T^*(W)$.

Finally, we use the metric h to identify vectors with covectors. Thus we obtain a diffeomorphism between $T(W)$ and $T^*(W)$, which brings ω and the flow from $T^*(W)$ to $T(W)$, while the level $\{H = 1\} \subset T^*(W)$ goes to the unit tangent bundle $V = UT(W) \subset T(W)$. This is invariant under the flow as well as the level $\{H = 1\}$ in $T^*(W)$.

Now, comes the punchline: *The flow in $V = UT(W)$ which we have just constructed, is identical with the geodesic flow.*

To prove this one should write down the defining vector field X for the geodesic flow and identify it with Y transported to $T(W)$. This is quite easy (see, e.g., [40]). Yet, if one wants to avoid explicit formulas for the geodesic flow, one may proceed in a less formal way which is conveniently divided into three steps.

Step 1. Show that the two flows are identical in the (simplest) case o.f.: $W = \mathbf{R}$ with the standard metric. This is immediate.

Step 2. Do the same for $W = \mathbf{R}^n$ by the splitting $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$, where the pertinent $\mathbf{R}$ is the line in $\mathbf{R}^n$ defined by the vector $\tau \in T(\mathbf{R}^n)$ at which we want to check the equality of the two fields.

Step 3. Recall that the vector fields on $T(W)$ defining the geodesic flow is algebraically expressed in terms of the first jet of h. In fact, one uses the Γ^h_{ij}-coefficients, but we do not care about the specific shape of the formula. Since every Riemannian metric h on W is Euclidean up to the second order at every point $w \in W$, the conclusion of Step 2 extends to all (W, h).

This argument still does not quite reveal the geometric reason for the existence of an invariant symplectic (or any other) structure for the geodesic flow. Here is an alternative geometric description of this structure, which makes the invariance clear.

First we recall that every normally oriented hypersurface $S \subset W$ lifts to $UT(W)$ by the unit normal field to S defined by the coorientation (compare 2.5). The lifted hypersurfaces give us a distinguished class of $(n-1)$-dimensional submanifolds in $UT(W)$ called *base* submanifolds. Distinguishing such a class is, in fact, equivalent to giving a geometric structure to $UT(W)$. In the present case one can define this structure by taking all base submanifolds in $V = UT(W)$ passing through a given point $v \in V$ and taking the linear subspace $K_v \subset T_v(W)$ generated by all these submanifolds. This K_v happens to be of codimension one for all $v \in V$ and the resulting codimension-one subbundle $K \subset T(V)$ is called the *canonical contact structure* on $UT(W)$. In fact, one can easily see that K is identical with $T^+ \oplus T^-$ for the geodesic flow in the negative curvature case.

Now we apply the geodesic flow g_t to some base manifold $\tilde{S}$ which is the lift of a smooth hypersurface $S \subset W$. Then, at least for small t, $g_t(\tilde{S})$ equals the lift $\tilde{S}_t$ for the *equidistant* hypersurface $S_t \subset W$ having

$$\operatorname{dist}(s, S) = t$$

for all $s \in S_t$ (compare 2.5). Thus the structure defined by the base manifolds (whatever its name) is invariant under the geodesic flow.

We invite the reader to follow this discussion to the end and to relate the above geometric picture with the symplectic formalism. We also suggest one look at what happens to the round spheres $S^{n-1} \subset \mathbf{R}^n$ lifted to

$UT(\mathbf{R}^n)$ when the equidistant hypersurface reduces to a single point in the center of S^{n-1}.

2.10. Hyperbolic geodesic flows. Once we obtain an invariant symplectic structure ω for the geodesic flow (recall that this ω comes from the canonical form on $T^*(W)$) we can construct an invariant pseudo-Riemannian metric on $V = UT(W)$ in the Anosov case.

2.10.A. **Theorem** (Cartan-Kanai). *Let W be a compact manifold of negative curvature $K < 0$. Then the geodesic flow on $V = UT(W)$ preserves a continuous pseudo-Riemannian metric φ on V.*

Furthermore, if the horospherical foliations in $UT(W)$ are C^k-smooth, then φ is also C^k.

2.10.B. **Remarks.** (i) In general, the horospherical foliations are only continuous (in fact they are Hölder continuous), and so is φ.

(ii) One knows (see [34] that for $\frac{1}{4}$-pinched manifolds, $-1 < K < -\frac{1}{4}$, the foliations are C^1-smooth, and so is φ.

(iii) If we assume the foliation C^∞, then we can use the rigidity theory to obtain, in particular, an open dense locally homogeneous subset in $UT(W)$ as in the case of $\mathbf{Z}$-actions. In fact, one expects W is a locally symmetric manifold in this case.

(iv) Many properties of geodesic flows for $K < 0$ can be seen by looking at the *ideal boundary* $\partial_\infty = \partial_\infty \tilde{W}$ of the universal covering $\tilde{W}$ of W (see [55], [15], [6]). This ∂_∞ is a topological space homeomorphic to S^{m-1}, $m = \dim W$, and the deck transformation group $\Gamma = \pi_1(W)$ acts on ∂_∞ by homeomorphisms.

The smoothness of the horospherical foliations implies the existence of a Γ-*invariant smooth structure* (i.e., the structure of a smooth manifold) on ∂_∞, and one wants to know how this structure on ∂_∞ influences the (dynamics of the) action of Γ on ∂_∞. More precisely, one has the following.

2.10.C. **Questions.** When does ∂_∞ admit a Γ-invariant C^α-structure? Here α may be a real number $\alpha = k + \varepsilon$, $0 \leq \varepsilon \leq 1$, where ε refer to the Hölder continuity of the k th derivatives. For example, does the existence of a Γ-invariant C^α-structure for $\alpha \geq 2$ imply that W is homotopy equivalent to a locally symmetric manifold? Another question (motivated by the Mostow rigidity theorem) concerns the uniqueness of Γ-invariant structures. A natural approach to these problems would be to pass from a smooth structure to a Γ-invariant rigid A-structure. For example, one may seek a (generalized) conformal structure on S_∞ and (or) a symplectic structure on $S_\infty \times S_\infty$.

2.10.D. **Remark.** If one goes back from a structure on ∂_∞ to $V = UT(W)$, one does not obtain a structure invariant under the geodesic flow but rather some *transversally invariant* structure for the 1-dimensional foliation into the orbits of this flow.

2.11. A very brief overview of some results and problems in Anosov systems. The most important and quite amazing property of Anosov systems is their *topological stability.* That is, if A' is a small C^1-perturbation of our Anosov action A on V, then there exists a homeomorphism (which is not, in general, C^1-smooth) of V close to the identity which sends each orbit of A' to an orbit of A. Notice that for $G = \mathbf{Z}$ such a homeomorphism necessarily commutes with the action, but this is not so for $G = \mathbf{R}$. This result in the complete generality is due to Anosov, but in many important cases this goes back to Birkhoff and Morse. In fact, Morse [54] has essentially proven a deeper global version of the stability theorem for compact manifolds W_1 and W_2 of negative curvature: *If* $\pi_1(W_1) = \pi_1(W_2)$, *then there exists a homeomorphism* $UT(W_1) \to UT(W_2)$ *which sends each orbit of the geodesic flow of* W_1 *to that of* W_2.

A similar global result is known (see [19], [48]) for Anosov $\mathbf{Z}$-systems on *infra-nil-manifolds* V. These systems generalize linear automorphisms of tori and exhaust all *known* Anosov $\mathbf{Z}$-actions up to topological equivalence. One may think that the infra-nil-systems give a complete list of Anosov actions. Yet one cannot even prove that *no simply connected* manifold supports an Anosov action.

There are by far more known Anosov actions for $G = \mathbf{R}$ than for $\mathbf{Z}$. This is almost entirely due to the abundance of manifolds of negative curvature. But one does not know, for example, if the existence of an Anosov $\mathbf{R}$-action on V implies that the fundamental group of V has exponential growth.

Notice that the above mentioned problems have motivated the study of the relations between $\pi_1(V)$ and $\mathrm{Is}(V)$ for V with a rigid structure. Unfortunately, there is no feedback so far except for Anosov systems with *smooth* foliations S^+ and S^- which is an extremely restrictive assumption.

The second beautiful feature of Anosov actions is the density of the periodic points (i.e., the points with compact orbits) provided the recurrent points are dense. This is due to Hedlund, Hopf, and Busemann for certain classes of hyperbolic geodesic flows, and to Anosov in the general case. One does not know if the recurrency condition is always satisfied for $G = \mathbf{Z}$.

The third basic property is the *ergodicity* of the actions preserving a smooth invariant measure. This general result is due to Sinai and Anosov, while some special cases of hyperbolic geodesic flows go back to Hedlund [33] and Hopf [37]. Nowadays one knows completely the measure theoretic structure of an Anosov system A. This structure is determined by a single invariant, the *measure entropy* of A, unless $G = \mathbf{R}$ and the system is obtained as the mapping cylinder of a $\mathbf{Z}$-action. (See [2], [3], [13] for a more complete discussion.)

Geometric structures invariant under hyperbolic (and not only hyperbolic) actions have been studied since the earlier days of classical mechanics and differential geometry. For example, the symplectic structure for the geodesic flow was revealed by Poincaré (following Lagrange, Hamilton, Liouville, etc.). The Anosov structure for manifolds of negative curvature can be traced back to Lobachevski and Hadamard, and the invariant pseudo-Riemannian metric to E. Cartan. Interest in these structures has recently been revived in an attempt to classify (in the smooth category) Anosov's systems with the smooth stable and unstable foliations (see [7], [18], [17], [39]). A typical result in this direction is the following theorem of Hurder-Katok and Ghys: *If a compact negatively curved surface W has C^2-smooth horospherical foliations, then W has constant curvature.*

For higher-dimensional manifolds W with $K(W) < 0$ the results are less complete. For example, one knows (see [16]) that *if the horospherical foliations are C^∞, then the geodesic flow is C^∞-isomorphic to that of some manifold W' with constant curvature $K' < 0$, provided that either* $\dim W$ *is odd or* $-4 < K(W) < -1$. (Of course one expects that $K(W)$ itself is constant.)

Another result in this direction concerns arbitrary (not necessarily geodesic) Anosov flows whose *Lyapunov subbundles* are C^∞-smooth. Namely every such flow on a compact manifold is C^∞*-isomorphic* to the geodesic flow on a locally symmetric manifolds of $\mathbf{R}$-rank $= 1$ (see [7]). (A priori, a Lyapunov subbundle $\tau \subset T(V)$ is only measurable. Thus the theorem applies to those flows where *every* invariant measurable subbundle is smooth.) Recently, this result was extended in [7] to all contact flows with smooth stable and unstable foliations (this more general result makes the above Lyapunov bundles irrelevant). Furthermore, the C^∞-condition was reduced to some C^k, $k < \infty$, in [32].

3. Isometries of simply connected real analytic manifolds

3.1. Now we return to a smooth A-rigid (see 1.8.A and also §5) manifold (V, φ). In this section we want to justify our earlier claim that the assumption $\pi_1(V)=0$ roots out any nontrivial dynamics of $G=\mathrm{Is}(V,\varphi)$-actions on V.

Unfortunately, all results which we state in this section need the real analyticity of φ, and we do not know what happens in the C^∞-case. We start by recalling some basic properties of $\mathrm{Is}(V,g)$ for C^{an} manifolds (the reader can consult §3 in [29] for more details).

3.2. Let φ be a C^{an} smooth rigid A-structure (see 0.4, 1.8, and also §5 for the definitions) on a compact manifold V. Then we have the following Propositions 3.2.A and 3.2.B.

3.2.A. *If V is simply connected, then the following hold:*

(i) *The group* $\mathrm{Is}(V,\varphi)$ *has at most finitely many connected components.*

(ii) *The isotropy subgroup* $\mathrm{Is}(V,\varphi,v)\subset\mathrm{Is}(V,\varphi)$ *also has finitely many connected components for all* $v\in V$.

(iii) *The topology induced from V on the* $\mathrm{Is}(V,\varphi)$*-orbit of each point* $v\in V$ *equals the quotient topology on* $\mathrm{Is}(V,\varphi)/\mathrm{Is}(V,\varphi,v)$. *In fact the orbits are semianalytic subsets (see* 3.5.1.B) *in V, and there exists at least one orbit which is a closed C^{an}-submanifold in V.*

3.2.B. *Assume as earlier* $\pi_1(V)=0$ *and let* $\mathrm{Is}(V,\varphi)$ *preserve a smooth volume element on V. Then* (i) *all orbits of* $\mathrm{Is}(V,\varphi)$ *are compact*; (ii) *the group* $\mathrm{Is}(V,\varphi)$ *contains a closed connected normal abelian subgroup* $A_0\subset\mathrm{Is}(V,\varphi)$ *such that* $\mathrm{Is}(V,\varphi)/A_0$ *is compact.*

The complete proof of these statements (which can be found, together with an extensive discussion of related theorems, in [29, §3.5 and 3.7]) is too technical and involved to be presented in detail here. However, in order to give some indications of why 3.2.A and 3.2.B are true, we shall briefly describe in 3.5 below a proof of 3.2.A(i) and 3.2.B(ii) valid for the special case when the structure φ in quesiton is a pseudo-Riemannian metric on V.

3.3. Remark. The reader might notice that the result 3.2.B(i) ensuring compactness of $\mathrm{Is}(V,\varphi)$-orbits under certain conditions on V and φ has already been quoted twice in these notes (compare 0.12.C where this was stated for a pseudo-Riemannian metric φ, and see also 1.11.B).

3.4. Isometries of Lorentz manifolds. Assume that (V,φ) is an n-dimensional C^{an} compact and simply connected Lorentz manifold. Then we know from Theorem 0.12.A in the Introduction that the isometry group $\mathrm{Is}(V,\varphi)$ is compact.

In fact, a key ingredient of the proof of Theorem 0.12.A (see [14]) is the compactness of $\mathrm{Is}(V,\varphi)$-orbits. This follows from Proposition 3.2.B(i) applied to the pseudo-Riemannian metric (which is an A-rigid structure) where the needed invariant measure comes from the pseudo-Riemannian volume element on V (which is obviously invariant under $\mathrm{Is}(V, \varphi)$, and finite for compact V).

The following remarks apply, strictly speaking, to Theorem 0.12.A concerning pseudo-Riemannian manifolds, but may also be used in the general discussion.

3.4.1. **Remarks.** (a) In Theorem 0.12.A real analyticity is only used for the following: each point $v \in V$ admits a neighborhood $U_v \subset V$ such that for every smaller connected $U' \subset U_v$, every Killing field on U' extends to U_v (compare with 5.15). This property for Riemannian C^{an} manifolds was proven by Nomizu [58], and his argument immediately generalizes to pseudo-Riemannian manifolds. The simple connectedness in 0.12.A is needed for the extension of local Killing fields to all of V (see 1.12).

(b) It is unknown whether Theorem 0.12.A remains true for C^∞ metrics. The major difficulty in the C^∞ case is that there may exist nonextendible local Killing fields. (For more on this point the reader can see §1.7 in [29], and also 1.12 and 5.15 in these lectures.)

(c) The Lorentz $(n-1, 1)$ condition on the signature of the metric tensor φ in 0.12.A is essential. In fact, the manifold $V = S^3 \times S^3 \times S^3$ admits an analytic metric of type $(7, 2)$ whose isometry group is $T^3 \times \mathbf{R}$ (see §5 in [14] and 1.11.D in these lectures).

(d) Note that if one allows $\pi_1(V) \neq \{0\}$, one may have a noncompact isometry group starting from dimension 2.

The simplest example is as follows (compare 2.4):

(i) Take $V = T^2 = \mathbf{R}^2/\mathbf{Z}^2$ and an automorphism $A\colon T^2 \to T^2$ which lifts to the linear map $\tilde{A}\colon \mathbf{R}^2 \to \mathbf{R}^2$ which has real eigenvalues λ_1 and λ_2, $\lambda_1 \neq \lambda_2$. For example, $\tilde{A}$ is given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and then $\lambda_1 = (3+\sqrt{5})/2$, $\lambda_2 = (3-\sqrt{5}/2)$.

Let x_1 and x_2 be the corresponding eigenvectors, and define a quadratic form $\tilde{g}$ on $\mathbf{R}^2$ by $\tilde{g}(\tilde{x}_1, \tilde{x}_1) = \tilde{g}(\tilde{x}_2, \tilde{x}_2) = 0$ and $\tilde{g}(\tilde{x}_1, \tilde{x}_2) = 1$. This $\tilde{g}$ is clearly $\tilde{A}$-invariant, and so induces an A-invariant metric g on T^2. It follows that the isometry group of (T^2, g) is noncompact as it contains $\mathbf{Z}$ (generated by A) as a closed subgroup, unless $A^2 = \mathrm{Id}$.

(ii) Another very important example is that of $V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ discussed in 0.11.A (see also 1.9.C).

Finally, observe that in example (i) above the fundamental group $\pi_1(T^2/\mathbf{Z}^2)$ is $\mathbf{Z}^2$, which is not too much. But in the case of

$V = \mathrm{SL}(2, \mathbf{R})/\Gamma$ the fundamental group is at least as large as Γ. One may wonder if the size of $\pi_1(V)$ influences that of the isometry group $\mathrm{Is}(V)$; some answers in this direction are presented in [29].

3.5. Outline of the proof of 3.2.A(i), (ii) and of 3.2.B for the case of a pseudo-Riemannian structure. We describe here a somewhat simplified version of the argument in [29] for the case when (V, φ) is a pseudo-Riemannian manifold of type (p, q), $p + q = \dim V$. In the original proof (see §§3.5 and 3.7 in [29]), 3.2.A and 3.2.B are derived from an appropriate form of the Frobenius theorem combined with a result of Rosenlicht (see 6.4.A and see also 5.14) on the orbit structure of algebraic actions. Here, since we work in the C^{an} category, we do not make explicit use of the Frobenius theorem.

3.5.1. Let (V, φ) be an n-dimensional connected pseudo-Riemannian manifold. Denote by $E \to V \times V$ the fibration where the fiber E_{v_1, v_2} equals the set of linear isometries between the tangent spaces $T_{v_1} \to T_{v_2}$. By using exponential coordinates at v_1 and v_2 we assign to each $e: T_{v_1} \to T_{v_2}$ a germ of a diffeomorphism of V sending $v_1 \to v_2$ and call it $\tilde{e}$. Then we define $E^r \subset E$ as the subset consisting of maps $e: T_{v_1} \to T_{v_2}$ for all v_1, v_2 such that the metric φ and the induced metric $\varphi' = \tilde{e}^*(\varphi)$ have equal Taylor coefficients of order $\leq r$ at v_1.

Next we compactify E and E^r as follows. Define a pseudo-Riemannian metric $\overline{\varphi}$ on $V \times V$ by $\overline{\varphi} = \varphi \oplus -\varphi$, and let $\overline{E} \to V \times V$ be the space of n-dimensional $\overline{\varphi}$-isotropic subspaces in $T(V \times V)$. The space E naturally embeds into $\overline{E}$ as an open dense subset. That is, each linear isometry $e: T_{v_1} \to T_{v_2}$ goes to its graph $\Gamma_e \subset \Gamma_{v_1} \times \Gamma_{v_2} = T_{(v_1, v_2)}$.

Then, we use the exponential map in the product manifold $V \times V$, and denote by $\hat{e}$ the germ at the origin of the exponential image of $\overline{e} \in \overline{E}$. This is a local n-dimensional submanifold in V. If $\overline{e} \in E \subset \overline{E}$, then the corresponding $\hat{e}$ is the graph of the germ $\tilde{e}$. Let $\overline{E}^r \subset \overline{E}$ be the set of those $\overline{e}$ where the metric $\overline{\varphi}|\hat{e}$ vanishes with order r at the origin. Clearly, $\overline{E}^r \supset E^r \subset E \subset \overline{E}$. Notice that if V is compact, then $\overline{E}$ and $\overline{E}^r$ are compact for all r.

We are now in a position to prove the following statement:

3.5.1.A. *If V is C^{an}, then $\overline{E}^r \subset E$ is a compact analytic subset.*

Proof. For every $\overline{e}$ at a point $w \in W \times W$ we denote by $J^r_{\overline{e}}$ the space of r-jets of quadratic differential forms at $w \in \hat{e} \subset V \times V$ on the submanifold $\hat{e} \subset V$. Clearly, this is a finite-dimensional vector space, and the union $J^r = \bigcup_{\overline{e}} J^r_{\overline{e}}$ has a natural structure of a real analytic vector bundle $J^r \to \overline{E}$.

This bundle J^r comes along with a section $j: \overline{E} \to J^r$, where to each $\overline{e}$ we assign the jet of the form $\varphi \oplus -\varphi$ on $V \times V$ restricted to $\hat{e}$. Clearly, j is real analytic, and the above condition "equal Taylor coefficients of order r" is equivalent to the vanishing of j. Thus $\overline{E}^r$ is represented as the zero set of an analytic section. q.e.d.

Now, note that the complement $\overline{\Sigma}^r = \overline{E}^r \backslash E^r$ consists of those $\overline{e} \subset T(V \times V)$ whose projection to $T(V)$ is *not* injective. It follows that $\overline{\Sigma}^r$ is a compact analytic subset in E^r. Since $\overline{E}^{r+1} \subset E^r$, we conclude, in the compact analytic case (see 3.5.1.B) that, $\overline{E}^{r+1} = \overline{E}^r$ for large r and similarly $\overline{\Sigma}^{r+1} = \overline{E}^r$. Hence, $E^{r+1} = E^r$ for large r (see 3.5.1.B$_2$).

Let $E^\infty = E^r$ for large r. Then using analyticity of φ one can see that $e \in E^\infty$ if and only if the germ $\hat{e}$ is a local isometry. Next, fix a point $v_0 \in V$, and let $E(V_0) \subset E$ consist of linear isometries $T_{v_0} \to T_v$ for all $v \in V$. Notice that $E(v_0)$ is a real analytic set.

Let $E^\infty(v_0) = E^\infty \cap E(v_0)$, and observe that the isometry group $\mathrm{Is}(V, \varphi)$ embeds into $E^\infty(v_0)$ by $f \to D_{v_0} f$, where $D_{v_0} f$ stands for the differential of the isometry f at the point v_0. If V is a simply connected C^{an}-manifold, then it is not difficult to show (see [29]) that the image of $\mathrm{Is}(V, \varphi)$ in $E^\infty(v_0)$ is the union of some connected components of $E^\infty(v_0)$. In fact, this is equivalent to the following property of Killing fields proven by Nomizu in [58]: If V is a simply connected real analytic manifold, then every germ of a Killing field at v_0 extends to a Killing field on all of V. Nomizu's original result refers to the case of a C^{an} Riemannnian manifold, but this argument immediately extends to pseudo-Riemannian manifolds (compare 3.4.1(a)).

Since $E^\infty(v_0) = \overline{E}^\infty \cap \overline{E}(v_0) \backslash \overline{\Sigma}^\infty$ is a difference of two compact analytic sets, it has at most finitely many connected components (see 3.5.1.B$_3$).

Similarly, one can see that the isotropy subgroup $\mathrm{Is}_v(V, \varphi)$ has finitely many connected components. In fact, Is_v equals the intersection $E_{v,v} \cap E^\infty$ and hence is an analytic (even algebraic) subset in $E_{v,v} = \mathrm{Is}(T_v(V)) \equiv \mathrm{O}(p, q))$. (Recall that here φ is a pseudo-Riemannian metric of type (p, q).)

3.5.1.B. *Semianalytic sets.* Let us state here the properties of semianalytic sets used in the above argument. We start with

3.5.1.B$_1$. **Definition.** A subset A in a compact real analytic manifold V is called *analytic* if A equals the zero set of a system of real analytic functions on V.

It is clear that the pullback of an analytic set under an analytic map is analytic, and that the zero set of an analytic section of an analytic vector bundle is real analytic.

It is equally clear that the finite union and the finite intersection of real analytic sets are real analytic. Less obvious is the following classical result.

3.5.1.B$_2$. **Nöther property.** *Every decreasing sequence of real analytic sets*

$$A_0 \supset A_1 \supset A_2 \cdots$$

stabilizes. That is, there exists an r, such that

$$A_\infty \stackrel{\text{def}}{=} \bigcap_i A_i = A_r.$$

Next, we call a subset $A \subset V$ An-*constructive* if it is the difference of two analytic sets, i.e., if $A = A_1 - A_2$. For us an important property of An-constructive sets reads:

3.5.1.B$_3$. *Every* An-*constructive set has at most finitely many connected components.* In fact one knows that, more generally, every *semianalytic* subset in V has at most finitely many connected components when one defines semianalytic sets as the sets of solutions of finite systems of analytic inequalities.

For a more extensive treatment of the basic geometric facts in the theory of semianalytic sets we refer the reader to the work of Lojasiewicz (see [46], [47]).

3.5.2. **Useful remark.** Observe that, besides the group $\mathrm{Is}(V, \varphi)$ itself, one can find other subgroups, say, $A \subset \mathrm{Is}(V, \varphi)$, whose action on V enjoys properties 3.2.A(i), (ii) discussed in 3.5.1 above.

For example, if A is the centralizer of a system of connected subgroups in $\mathrm{Is}(V, \varphi)$, then A is the full isometry group of the structure, call it φ', in V obtained by augmenting φ with a system of Killing fields generating the subgroup in question (compare with 1.10). Since the metric φ is rigid, the structure φ' is also rigid (see 0.4(c)). Moreover, the previous discussion also applies to the structure φ' so that 3.2.A(i), (ii) hold for $\mathrm{Is}(V, \varphi')$ and its isotropy subgroup $\mathrm{Is}_v(V, \varphi')$ for all $v \in V$.

A useful application of this is the following 3.5.2.A which allows us to reduce a large part of the proof of Theorem 0.12.A to the case of an *Abelian* isometry group. (For more on this, see 3.5.5 at the end of this section.)

3.5.2.A. *Abelianization trick.* Let $(X_1, \cdots, X_m)$ be a *maximal* system of commuting Killing fields on V, and let $A \subset \mathrm{Is}(V, \varphi)$ be the connected Abelian subgroup generated by $(X_1, \cdots, X_m)$. Then the subgroup $\mathrm{Is}(V, \varphi') \subset \mathrm{Is}(V, \varphi)$ for $\varphi' = (\varphi, X_1, \cdots, X_m)$ equals the centralizer of A in $\mathrm{Is}(V, \varphi)$.

Since A is maximal, the connected identity component $\mathrm{Is}^0(V, \varphi) \subset \mathrm{Is}(V, \varphi)$ equals A.

3.5.3. We explain now why the orbits of $\mathrm{Is}(V, \varphi)$ are compact. We begin by considering the space $\mathbf{R}^{p,q}$ with the standard form

$$h_0 = \sum_{i=1}^{p} dx_i^2 - \sum_{j=p+1}^{n} dx_j^2,$$

where (p, q) is the type of φ. Consider pseudo-Riemannian metrics h on $\mathbf{R}^{p,q}$ such that $h - h_0$ together with its first derivatives vanishes at the origin, and let H^r be the space of the first r Taylor coefficients of all such h. Note, that H^r is a linear space of dimension

$$\frac{n(n+1)}{2}\left(1 + n + \cdots + \frac{(n+r-1)!}{(n-1)!r!}\right) - \frac{n(n+1)}{2} - \frac{n^2(n+1)}{2},$$

and that the group $\mathrm{O}(p, q)$ naturally (and linearly) acts on H^r.

Next, we take an orthonormal frame F at some point $v \in V$, and let $D^r(F, v, \varphi) \subset H^r$ denote the string of the first r Taylor coefficient of φ in the exponential coordinates in V corresponding to F. Changes of the frame F correspond to the above action of $\mathrm{O}(p, q)$ on H^r, and so we get a map of V into $H^r/\mathrm{O}(p, q)$, say $\mathscr{D}_\varphi^r \colon V \to H^r/\mathrm{O}(p, q)$. The quotient space $H^r/\mathrm{O}(p, q)$ is not a Hausdorff space. However, by the algebraic quotient theorem (see, for example, [71], [62], [57], and also 6.4.A, 5.14.B), there exists an $\mathrm{O}(p, q)$-invariant real algebraic stratification of H^r, say $H^r = H_0^r \cup H_1^r \cup \cdots \cup H_{i_r}^r$, such that each quotient space $H_i/\mathrm{O}(p, q)$, $0 \leq i \leq i_r$, is a manifold, and the quotient map $H_i \to H_i/\mathrm{O}(p, q)$ is a smooth fibration (compare with 5.14.B). Then one can easily show (see §3 in [29] for details) that for each r there exists a stratum H_i^r, for some $0 \leq i \leq i_r$, such that the pullback $V_i^r = (\mathscr{D}_\varphi^r)^{-1}(H_i^r/\mathrm{O}(p, q))$ is an open dense subset in V, and the map $\mathscr{D}_\varphi^r$ is continuous (in fact, real analytic, if φ is C^{an}) on V_i^r. Note that V_i^r is invariant under $\mathrm{Is}(V, \varphi)$.

If V is compact, real analytic, and r is sufficiently large, then by the previous discussion each "fiber" $(\mathscr{D}_\varphi^r)^{-1}(h)$ for $h \in H_i^r/\mathrm{O}(p, q)$ is a union of finitely many orbits of $\mathrm{Is} = \mathrm{Is}(V, \varphi)$ since the action of Is preserves the pseudo-Riemannian measure on V and this measure is finite for compact V; the orbit $\mathrm{Is}(v) \subset V$ for almost all $v \in V_i^r$ also admits a finite $\mathrm{Is}(V, \varphi)$-invariant measure by the classical measure decomposition theorem (see 1.5.D(ii)).

It follows that almost all orbits of $\mathrm{Is}(V, \varphi)$ in V_i^r are compact (see Remarks 3.5.4 below and 3.7.A in [29] for an explanation). Finally, one obtains by continuity that *all* orbits of $\mathrm{Is}(V, \varphi)$ are compact.

3.5.4. **Remark.** The above compactness of almost all orbits of $\mathrm{Is}(V, \varphi)$ in V_i^r is a consequence of a well-known result by Montgomery (see [53]). For the reader's sake we recall the statement:

3.5.4.A. *Consider a homogeneous space $X = G/H$ with a finite G-invariant measure, where G is a connected Lie group. If the isotropy subgroup $G_x \subset G$, $x \in X$, is connected, then X is compact, and therefore each maximal compact subgroup $K \subset G$ is transitive on X.*

By applying the above 3.5.4.A to our $\mathrm{Is}(V, \varphi)$-orbits we do not only obtain the compactness of orbits but also infer the following useful proposition.

3.5.4.B. **Proposition.** *Let $K \subset \mathrm{Is}^0(V, \varphi)$ be a maximal compact subgroup in the connected identity component $\mathrm{Is}^0(V, \varphi) \subset \mathrm{Is}(V, \varphi)$. Then the orbits of $\mathrm{Is}^0(V, \varphi)$ equal those of K.*

(See 3.7.A and 3.7.A$_3$ in [29] for a proof.)

Note that in the case when $\mathrm{Is}(V, \varphi)$ is connected Abelian, 3.5.4.A is immediate without connectedness of the isotropy subgroup. Also, in this case the orbits of $\mathrm{Is}(V, \varphi)$ equal those of the maximal torus $T \subset \mathrm{Is}^0(V, \varphi)$.

3.5.5. Finally, we observe that the discussion in 3.5.1 and 3.5.3 equally applies to the structure $\varphi' = (\varphi, X_1, \cdots, X_m)$ introduced in 3.5.2.A. To do this, one only needs to modify the definition of the space H^r by adding the Taylor coefficients of the fields $X_1, \cdots, X_m$. If the fields X_i constitute a *maximal* system of commuting Killing fields, then the connected identity component of $\mathrm{Is}(V, \varphi')$ is Abelian and thus (by a trivial case of 3.5.4.A) yields the compactness of orbits of $\mathrm{Is}(V, \varphi')$ as $\mathrm{Is}(V, \varphi')$ has only finitely many connected components. This is exactly what is used in [14] for the proof of Theorem 0.12.A. In fact, the compactness of the isometry group $\mathrm{Is}(V, \varphi)$ for Lorentz manifolds is achieved by showing that every maximal connected Abelian subgroup $A \subset \mathrm{Is}(V, \varphi)$ is compact. (See §7 in [14] for an elementary proof of the fact that if all maximal connected Abelian subgroups in a connected Lie group G are compact, then G is compact.)

4. Actions of semisimple Lie groups

4.1. When a semisimple Lie group G acts on a smooth manifold V, this action often looks as if it preserves some rigid geometric structure. In fact, any Lie group G acting on V preserves certain geometric structures related to the corresponding action of the Lie algebra $L = L(G)$ on V.

The action of L on V can be seen as a system of vector fields $X_1, \cdots, X_k$ on V corresponding to a basis $l_1, \cdots, l_k$ in L such that $[X_i, X_j]_V = [l_i, l_j]_L$ for all $i, j = 1, \cdots, k$. Note that the fields X_i in general are not preserved by the action of G. More precisely, if some $g \in G$ sends the point v to v', then the frame at v given by $\{X_1, \cdots, X_k\}$ is sent to the frame at v' given by $\{\mathrm{ad}_g X_1, \cdots, \mathrm{ad}_g X_k\}$ via the adjoint action of G on L. (We use the word "frame" even though the vector fields $X_1, \cdots, X_k$ may be dependent.) For example, if the group G is abelian, then the action of G preserves the frame. In general, one can take some adjoint-invariant polynomial function or tensor on the Lie algebra of G and this will give a geometric structure on V invariant under G. (More general structures are obtained by considering invariant functions on the spaces of jets of vector fields.) A similar kind of invariant structure arises when the linear maps $L \to T_v(V)$ given by $(l_1, \cdots, l_k) \to (X_1, \cdots, X_k)_v$ have ranks r independent of $v \in V$, for $0 \le r \le k = \dim L$.

Notice that the map $L \to T_v(V)$ is essentially the differential of our action at $v \in V$, and that its kernel, say $K_v \subset L$, invariantly depends on the point $v \in V$. That is, the map $V \to \mathrm{Gr}_r(L)$ defined by $v \mapsto K_v \in \mathrm{Gr}_r(L)$ is G-equivariant for the action of G on the Grassmann manifold $\mathrm{Gr}_r(L)$ induced by the adjoint action of G on L.

The map $V \to \mathrm{Gr}_r(L)$, which is a special case of the generalized Gauss map in the sense of [29], was introduced by Zimmer in [71] where he proves the following among other things:

4.1.A. **Theorem.** *If a noncompact simple Lie group G acts on a compact manifold V preserving a finite measure μ on V such that the above maps $L \to T_v(V)$, $v \in V$, have rank r independent of v, then either $r = 0$ or $r = k = \dim L$. In both cases the Gauss-Zimmer map $\alpha\colon V \to \mathrm{Gr}_r(L)$ is constant as the Grassmann manifold $\mathrm{Gr}_r(L)$ reduces to a single point.*

Proof. Consider the push-forward measure $\alpha_*(\mu)$ on $\mathrm{Gr}_r(L)$, and observe that $\alpha_*(\mu)$ is invariant under the ad_G-action. Since the later action is algebraic, we can use the following important lemma.

4.1.B. **Furstenberg-Tits lemma** (see §3.2. in [71] and 6.4.B$_4$ in these lectures). *If an algebraic action preserves a finite measure ν, then this action when restricted to the support of ν factors through an action of a compact group. In particular, if the algebraic group in question has no nontrivial compact factor group, then the action fixes* $\mathrm{supp}(\nu)$.

Now, return to the above measure $\alpha_*(\mu)$ on $\mathrm{Gr}_r(L)$ where L is simple, and note that the action of ad_g, $g \in G$, on $\mathrm{Gr}_r(L)$ has no fixed point for $1 \le r \le \dim L - 1$. Indeed, a fixed point would be an ad_g-invariant r-dimensional subspace in L, namely an ideal of L, and this is ruled out since we assumed L to be simple. This concludes the proof of 4.1.A.

4.2. Remark. The above Theorem 4.1.A is a modified version of the original Zimmer's result. In fact, Zimmer does not assume $r = \text{const}$ and proves the following by using the above argument.

4.3. Full Zimmer theorem (see [72], [70]). *Let G be a noncompact simple Lie group acting on (V, μ). Then for almost all (with respect to μ) points $v \in V$ the isotropy subgroup $G_v \subset G$ is either discrete or equals all of G.*

4.4. A natural question that arises in light of Theorem 4.3 is the following.

4.4.A. *Open problem.* Let a noncompact simple Lie group G smoothly and faithfully act on a connected manifold V, such that the action preserves a smooth finite measure. Is then the isotropy group G_v discrete for *all* $v \in V$?

4.4.B. **Example** (A. Connes). Let $\Gamma \subset G$ be a noncocompact lattice. Then G/Γ admits no smooth G-invariant compactification. (We invite the reader to find a proof on his own.)

4.4.C. **Remark.** If the action of G is faithful on each connected component of V, then the fixed point set $V_0 \subset V$ where $G_v = G$ is nowhere dense. In fact we have the following.

4.4.D. **Thurston's stability theorem** [65]. *If some group G of diffeomorphisms of V fixes a point $v_0 \in V$, and the action of G on $T_{v_0}(V)$ is trivial, then G admits a nontrivial homomorphism into* $\mathbf{R}$.

As simple groups admit no such homomorphisms, the action of our G is nontrivial on $T_{v_0}(V)$ for all v_0 in the fixed point set $V_0 \subset V$, and so V_0 is nowhere dense.

4.5. Remark. A nontrivial action of a simple Lie group G on $T_{v_0}(V)$ cannot fix a hyperplane in $T_{v_0}(V)$, since the group $\mathrm{LinAut}(\mathbf{R}^n, \mathbf{R}^{n-1})$ is solvable and thus contains no simple Lie group, where $\mathrm{LinAut}(\mathbf{R}^n, \mathbf{R}^{n-1})$ denotes the subgroup in $\mathrm{GL}(n, \mathbf{R})$ fixing $\mathbf{R}^{n-1} \subset \mathbf{R}^n$.

It follows that the codimension of V_0 is at least 2, for an appropriate notion of C^1-codimension. This can be seen even better in the case when the maximal compact subgroup $K \subset G$ is nontrivial. (Recall that the only simple Lie group where the maximal subgroup $K = \{\mathrm{Id}\}$ is the universal cover $\widetilde{\mathrm{SL}(2, R)}$ of $\mathrm{SL}(2, \mathbf{R})$). Here obviously, we have

$$V_0 = \mathrm{Fix}(G) \subset \mathrm{Fix}(K) \subset V,$$

where the set $\mathrm{Fix}(K)$ of fixed points of K is a smooth submanifold of codimension ≥ 2 in V.

It seems that $V_0 = \mathrm{Fix}(G)$ is a C^r-submanifold for C^r-actions and the action is linearizable near V_0. This is probably known to experts but the only published result (see [43]) known to the authors is more special.

4.6. Now, let us look at the action for our simple Lie group G near a *recurrent* point $v \in (V, \mu)$ (see 1.3.A) such that $g_i(v) \to v$ for some divergent sequence $g_i \in G$. These g_i act on the frame $\{X_1, \cdots, X_k\}$ by ad_{g_i}, where $k = \dim G$, and one can show that this action comes from some kind of infinitesimal action of ad_G at $v \in V$ (compare the recurrency discussion in 1.3). This was done by Zimmer in a general dynamical framework. Here, following [29] we state (without proof) a geometric version of Zimmer's theorem for G-actions preserving a rigid A-structure φ on V, when G is as earlier assumed to be noncompact simple and the action of G is faithful.

4.6.A. **Theorem.** *Let (V, φ) be a C^∞-smooth rigid A-manifold. If $G \subset \mathrm{Is}(V, \varphi)$, and the action of G preserves a finite smooth measure μ on V, then for almost all $v \in V$ the local isotropy group $\mathrm{Is}_v(V, \varphi)$ contains an isomorphic copy of the group $\mathrm{ad}\, G \subset \mathrm{Aut}\, L(G)$ (see 5.2 in [29]).*

The above result shows that locally the structure φ is highly symmetric, and this may be used often for a detailed study of (V, φ). In fact, such a study may be carried over in the special case of Lorentz manifolds as we shall see presently.

4.7. Consider a compact Lorentz manifold (V, φ) and assume that the isometry group $G = \mathrm{Is}(V, \varphi)$ contains $\mathrm{SL}(2, \mathbf{R})$ as a subgroup (see 0.13.A). We want to complement Theorem 0.13.A by giving a *complete* geometric description of such (G, V, φ). To do this, we begin with the following construction which specializes the one given in 6.9.

4.7.1. Let $\tilde{\varphi}_0$ be a bi-invariant metric on a Lie group G_0, $(\tilde{V}_1, \tilde{\varphi}_1)$ be a Riemannian manifold, and $G_1 = \mathrm{Is}(\tilde{V}_1, \tilde{\varphi}_1)$ be its isometry group. Set $\tilde{V} = G_0 \times \tilde{V}_1$, and let Γ be a discrete subgroup in the product group $G = G_0 \times G_1$. The left- and right-translations of G_0 on itself together with the (only one) action of G_1 on $\tilde{V}_1$ give rise to two actions of each of the groups G_0, $G = G_0 \times G$ and $\Gamma \subset G$ on $\tilde{V}$, which are called left and right action respectively.

Consider the left action of G_0 on $\tilde{V}$ and the right action of Γ on $\tilde{V}$. Since these actions commute, we obtain an action of the group G_0 on the quotient manifold $V = \tilde{V}/\Gamma$. Note that this action of G_0 preserves the metric φ on V which descends from the metric $\tilde{\varphi} = \tilde{\varphi}_0 + \tilde{\varphi}_1$ on $\tilde{V}$ and that the type of φ is $(m_+ + m, m_-)$, where (m_+, m_-) is the type of $\tilde{\varphi}_0$ and $m = \dim \tilde{V}_1$.

4.7.2. One obtains an invariant metric of more general kind by *warping* $\tilde{\varphi}$ on $\tilde{V}$ with an arbitrary strictly positive G_1-invariant function $\tilde{f}$ on $\tilde{V}_1$. Namely, one takes

$$\tilde{\varphi}_{\tilde{f}} = \tilde{f}\tilde{\varphi}_0 + \tilde{\varphi}_1$$

or, more pedantically,

$$\tilde{\varphi}_{\tilde{f}}(g_0, v_1) = \tilde{f}(v_1)\tilde{\varphi}_0(g_0) + \tilde{\varphi}_1(v_1)$$

(if one wants to keep track of the arguments $g_0 \in G$ and $v_1 \in \tilde{V}_1$).

Clearly $\tilde{\varphi}_{\tilde{f}}$ is Γ-invariant and thus defines a metric $\tilde{\varphi}_{\tilde{f}}$ on V, which is G_0-invariant as well as φ. Finally, we notice that the manifold V is compact if $\tilde{V}_1/G_1$ is compact and $\Gamma \subset G$ is cocompact in $G = G_0 \times G_1$.

4.8. **Remark.** One could consider a more general situation where $(\tilde{V}_1, \tilde{\varphi}_1)$ is pseudo-Riemannian. In this case one should insist on properness of the action of the isometry group G_1 on $\tilde{V}_1$ (which is automatic in the Riemannian case even if the isometry group is not compact) in order to obtain a Hausdorff quotient space $V = \tilde{V}/\Gamma$.

4.9. Splitting theorem (second version, compare 0.13.A). *Let* (V, φ) *be a compact Lorentz manifold such that* $\mathrm{Is}(V, \varphi) \supset G_0 = \mathrm{SL}(2, \mathbf{R})$. *Then* V *is obtained by the above construction for a form* $\tilde{\varphi}_0$ *on* G_0 *proportional to the Killing form and for some* $(\tilde{V}_1, \tilde{\varphi}_1)$, $\tilde{f}$ *on* $\tilde{V}_1$, *and* $\Gamma \subset G_0 \times \mathrm{Is}(\tilde{V}_1, \tilde{\varphi}_1)$.

(The reader can consult §5.4 in [29] for a discussion related to this theorem.)

Notice that the present version of the splitting theorem is equivalent to 0.13.A. Namely, the following properties (i)–(iii) claimed by 0.13.A are direct corollaries of 4.9.

(i) The isotropy subgroups of the action of G_0 are discrete for all $v \in V$ (compare 4.10(b) below).

(ii) The metric φ is nonsingular on the 3-dimensional G_0-orbits.

(iii) The normal subbundle to the orbits is integrable and the leaves are totally geodesic.

4.10. Conversely, Theorem 4.9 can be easily deduced from the above properties (i)–(iii).

4.10.1. **Remark.** (a) Property (i) above confirms (in a very special situation) our conjecture 4.4.A claiming local freedom of measure preserving actions of simple groups.

(b) The above Theorem 4.9 generalizes to pseudo-Riemannian manifolds of arbitrary type (p, q) if the isometry group contains a semisimple subgroup G_0 large enough with respect to $\min(p, q)$.

(c) There exist pseudo-Riemannian manifolds V with "insufficiently large" subgroups of isometries $G_0 \subset \mathrm{Is}(V)$ where $\tilde{V}$ do not split in the above sense. One can see this by looking at the action of a subgroup $G_0 \subset G$ on G/Γ where G/Γ is given a pseudo-Riemannian metric coming from a bi-invariant metric on G (see 6.7).

4.11. In order to get some additional insight into (semi)simple group actions we shall now consider some examples which are more complicated than the algebraic actions and those on G/Γ considered in §1.9.

4.11.1. *Induced actions* (see 6.9). Let G be a semisimple Lie group acting on G/Γ, where $\Gamma \subset G$ is a discrete subgroup, and consider an action of Γ on some connected manifold F. Then we take the quotient $V = (G \times F)/\Gamma$ for the diagonal Γ-action and observe that G naturally acts on this V. Furthermore if G/Γ and F are compact, then also V is compact. Similarly, if $\mathrm{Vol}(V/\Gamma) < \infty$ and the action of Γ on F preserves some measure, then the same is true for the action of G on V. Now, to make a practical use of the above construction one needs interesting examples of Γ-actions on F, which do not come from an action of $G \supset \Gamma$.

But the work by Zimmer suggests (see [73]) that for $\mathrm{rank}_{\mathbf{R}} G \geq 2$ (recall that G is assumed to be noncompact simple and that $\mathrm{rank}_{\mathbf{R}}$ stands for the rank of the symmetric space G/K, where $K \subset G$ is the maximal compact subgroup in G) an action of Γ extends to that of some Lie group G' with finitely many connected components if we assume, for example, that V is compact and (or) that the action of Γ preserves a smooth finite measure and $\mathrm{Vol}(G/\Gamma)$ is finite. (If the action of Γ on F admits no invariant Riemannian metric, then the above G' should be equal to G.)

On the other hand, one can construct different kinds of actions for many cases of $\mathrm{rank}_{\mathbf{R}} = 1$. For example if there is a nontrivial homomorphism $\Gamma \to \mathbf{Z}$, i.e., $H^1(\Gamma, \mathbf{Z}) \neq 0$, then every action of $\mathbf{Z}$, i.e., a single diffeomorphism of F, gives rise to a Γ-action on F.

4.11.2. Here is a potential generalization of the above "induced construction" in the case $G = \mathrm{PO}(n, 1) = \mathrm{IS}(\mathbf{H}^n)$, where $\mathbf{H}^n$ is the hyperbolic space. Let W be a compact manifold foliated into n-dimensional leaves, and assume that these leaves carry metrics of constant curvature -1. Then the above G naturally acts on the manifold V of orthonormal n-frames tangent to the leaves. In fact, V equals the space of those maps $f \colon \mathbf{H}^n \to W$ which locally isometrically send V onto a leaf. Then the action of G on $\mathbf{H}^n$ induces an action of G on V. To make this construction work, one needs foliations of curvature -1, which in general are hard to come by.

However, if $n = 2$, one can start with a 2-dimensional foliation $\mathscr{F}$ with an arbitrary metric and then use the Riemann mapping theorem to produce a new metric of constant curvature. (According to D. Sullivan this idea is due to Winkelnkemper.) To be precise, we assume that V is compact and that the foliation is hyperbolic, i.e., there exists no nonconstant conformal map of $\mathbf{R}^2$ into a leaf of the foliation. Then we consider the space $\mathscr{H}$ of *conformal* maps of $\mathbf{H}^2$ into the leaves of $\mathscr{F}$, and observe that $\mathrm{PSL}(2, \mathbf{R}) = \mathrm{IS}(\mathbf{H}^2)$ naturally acts on $\mathscr{H}$.

The space $\mathscr{H}$ is infinite dimensional but contains an invariant submanifold $V \subset \mathscr{H}$ consisting of the maps which are *covering maps* of $\mathbf{H}^2$ onto the leaves. The manifold V is diffeomorphic to a S^1-bundle over W, namely to the bundle of unit vectors which are tangent to the leaves.

The action of $\mathrm{PSL}(2, \mathbf{R})$ on V obtained in this way is continuous but not, in general, smooth. In fact smooth actions of this type seem quite rare especially if one wants a smooth invariant measure. (Note that one does have such a measure if the foliation admits a transversal measure.)

4.12. The above discussion raises the following question: Does there exist a compact *simply connected* manifold V which admits a faithful smooth (C^∞ or C^{an}) action of a simple Lie group preserving a smooth finite measure μ on V?

We know that the answer is "no," if the action is C^{an} and preserves a rigid structure (see [29] and §3 in these lectures).

4.13. We end this section with the following:

4.13.A. **Remark.** The above construction can be generalized by taking for $\mathscr{H}$ another space of maps $\mathbf{H}^2 \to W$ satisfying some system of partial differential equations.

For example, one can use harmonic maps into Riemannian manifolds W (preferably of negative curvature) and holomorphic maps into (Kobayashi hyperbolic) complex manifolds W. Of course $\mathscr{H}$ itself is too large for us, and we are interested in G-invariant submanifolds and (or) finite G-invariant measures in $\mathscr{H}$.

There is no systematic theory here, but at least there is one beautiful example. Namely, let W be the Riemann moduli space (or rather a finite nonsingular covering of the moduli space), and let V consist of those (extremal) holomorphic maps $\mathbf{H}^2 \to W$ whose lifts to the Teichmüller space $\tilde{W}$, which is the universal covering of W, are isometric embeddings with respect to the Teichmüller metric in $\tilde{W}$ (this metric equals the Kobayashi metric by Royden's theorem). The reader can see references [52] and [68] for the study of the $\mathbf{R}$-part of this action.

5. Infinitesimal geometric structures

5.1. Classically, a geometric structure φ on V is expressed in local coordinates by finitely many, say by s, functions, and these transform by certain rules depending on a particular *type* of φ when one changes the coordinates. In other words, φ is an $\mathbf{R}^s$-valued function on the space $\mathscr{U} = \mathscr{U}(V)$ of coordinate charts in V where a point in $\mathscr{U}$ is a pair (v, u), $v \in V$ and $u = (u_1, \cdots, u_n)$, for $n = \dim V$, is a local coordinate system around v, i.e., u is a locally diffeomorphic map of a small neighborhood of $v \in V$ into $\mathbf{R}^n$ sending $v \mapsto 0$. When we pass to another coordinate system u' around v, the value $\varphi(v, u')$ is expressed by certain algebraic formulas depending on the type of φ in terms of $\varphi(v, u)$ and the partial derivatives of certain orders $\leq r$ (also depending on the type of φ) at v of the coordinates u'_i viewed as functions of u_i. In particular, if u_i and u'_i only differ in order $> r$ (i.e., $u_i - u'_i$ vanishes at v along with all derivatives of order $\leq r$), then $\varphi(v, u) = \varphi(v, u')$. That is, $\varphi(v, u)$ depends only on the rth order jet (or rth order differential) of u at v.

Let us express this property with the following.

5.2. Jet language. By definitions the jet $J^r_v f$ of a smooth function f on V defined in a neighborhood of a point $v \in V$ is the equivalence class of f under the following relation: two smooth maps f and f' defined in a small neighborhood of $v \in V$ are called *r-equivalent* at v if the partial derivatives of these maps of orders $0, 1, \cdots, r$ are equal at v, where the partial derivatives are taken in a fixed coordinate system around v.

This notion of r-equivalence (and hence, of an r-jet) does not depend on the coordinate system. In fact, the chain rule shows that the partial derivatives of order $\leq r$ in one coordinate system can be expressed in terms of those in another system.

5.2.A. **Remarks.** (i) The definitions of the r-equivalence and jets immediately extend to maps $f: V \to \mathbf{R}^s$. Moreover, these notions can be applied to maps into an arbitrary smooth manifold X. This is done by embedding X into some $\mathbf{R}^N$ and then by noticing that the r-equivalence for maps $V \to X \hookrightarrow \mathbf{R}^N$ does not depend on the embedding.

(ii) According to our definition, the only meaningful expression involving jets which we can write so far is $J^r_v f = J^r_v f'$; this is just another way to say that f and f' are r-equivalent at v. But, if we fix local coordinates around v, we see more structure as a jet of any map is the same thing as the totality of partial derivatives of orders $\leq r$ evaluated at $v \in V$.

It follows that the space J^r_v of all r-jets, say for maps $f: (V, v) \to \mathbf{R}^m$, can be identified (the identification depends on a choice of local coordinates) with the Euclidean space $\mathbf{R}^{mN_0}$ for $N_0 = 1 + n + n(n+1)/2 + \cdots + (n+r-1)!/(n-1)!r!$. Furthermore, the space

$$J^r(V, \mathbf{R}^m) = \bigcup_{v \in V} J^r_v$$

has a natural structure of a smooth manifold fibered over V with the fibers J^r_v. This fibration is trivial over each coordinate chart $(U, u_1, \cdots, u_n)$ in V, as the coordinates $u_1, \cdots, u_n$ (obviously) define a splitting $J^r(U) = J^r_v \times U$, for each $v \in U$.

5.3. The fibration $\mathscr{D}^r(V) \to V$. If we apply the r-equivalence relation to all local coordinate systems u in $(v, u) \in \mathscr{U}$, we obtain a quotient of $\mathscr{U}$, call it $\mathscr{D}^r(V)$, whose points are pairs (v, δ), where $v \in V$, and δ is an element in the space of r-jets of locally diffeomorphic maps of V (or rather of a small neighborhood of $v \in V$) into $\mathbf{R}^n$ sending $v \mapsto 0$. We denote this space by $\mathscr{D}^r_v$, and then see that

$$\mathscr{D}^r(V) = \bigcup_{v \in V} \mathscr{D}^r_v.$$

Each *fiber* $\mathscr{D}^r_v$ of $\mathscr{D}^r_v(V)$ is naturally embedded into the jet space J^r_v of all jets of maps $V \to \mathbf{R}^n$ and the subset $\mathscr{D}^r_v$ in $J^r_v = \mathbf{R}^N$, $N = n(1 + n + n(n+1)/2 + \cdots + (n+r-1)!/(n-1)!r!)$ is distinguished by two conditions. The first amounts to vanishing of the first n coordinates, which correspond to the requirement $u(v) = 0$, and the second is given by nonvanishing of the Jacobian matrix at v, which reflects the locally diffeomorphic nature of u. To be more precise, we look at the tautological projections between the jet spaces $J^r_v \to J^{r'}_v$, for all $r' \leq r$ (to go from J^r to $J^{r'}$ we just forget derivatives above order r'), and we also note that the space of 1-jets of maps $(V, v) \to \mathbf{R}^n$ sending $v \mapsto 0$ is nothing else but the space of linear maps of the tangent space $T_v(V) \to \mathbf{R}^m$. Now, we first invoke the condition $u(v) = 0$ thus restricting to the subspace $\mathbf{R}^{N-n} \subset \mathbf{R}^N = J^r_v$ containing $\mathscr{D}^r_v$, and then use the above projection on the 1-jets, say

$$p_1 : \mathbf{R}^{N-n} \to \mathrm{Hom}(T_v(V), \mathbf{R}^n).$$

Now $\mathscr{D}^r_v \subset \mathbf{R}^{N-n}$ is given by the condition $\mathscr{D}^r_v = \{\delta \in \mathbf{R}^{N-n} | \operatorname{rank} p_1(\delta) = n\}$.

Notice that $\mathscr{D}^r_v$ is an open subset in $\mathbf{R}^{N-n}$ and thus has a natural structure of a smooth manifold. Furthermore, the space $\mathscr{D}^r(V)$ also has a natural structure of a smooth manifold such that the projection $\mathscr{D}^r(V) \to V$ becomes a smooth locally trivial fibration with fiber $\mathscr{D}^r_v(V)$. In fact, a coordinate system $u_1, \cdots, u_n$ in a neighborhood U of a point $v_0 \in V$ defines in an obvious way a splitting $\mathscr{D}^r(U) = \mathscr{D}^r_{v_0} \times U$.

5.4. Geometric structures. A geometric structure of order r can now be defined as a map φ of $\mathscr{U} = \mathscr{U}(V)$ into a smooth manifold Φ such that $\varphi(v, u)$ only depends on $J^r_v(u)$. Thus φ factors through the projection $\mathscr{U} \to \mathscr{D}^r(V)$ and so defines a map of $\mathscr{D}^r(V)$ to Φ, which we also denote by $\varphi: \mathscr{D}^r(V) \to \Phi$. In fact, we usually refer to a structure as to a map defined on $\mathscr{D}^r(V)$ rather than on $\mathscr{U}$, and we assume it is C^∞-smooth.

5.4.A. **Remark on $\Phi = \mathbf{R}^s$.** The study of general structures can be reduced to those where the target space Φ is the euclidean space $\mathbf{R}^s$ as our smooth manifold Φ can be embedded into some euclidean space.

5.5. A-structures. First we recall that $\mathscr{D}^r_v(V)$ is an *open* subset in $\mathbf{R}^{N-n}$, and thus we can speak of polynomials, rational and algebraic functions on $\mathscr{D}^r(V)$ defined as restrictions to $\mathscr{D}^r(V)$ of corresponding functions on $\mathbf{R}^{N-n}$.

Now, a structure $\varphi: \mathscr{D}^r(V) \to \mathbf{R}^s$ is called of *algebraic type* or an *A-structure* if the restriction of φ to each fiber $\mathscr{D}^r_v(V)$ is algebraic. Similarly, we define the notion of algebraic type for structures $\varphi: \mathscr{D}^r(V) \to \Phi$, where Φ is an arbitrary real algebraic manifold.

5.6. Our definition attaches no transformation law to a geometric structure φ. In fact, we do not need this for our applications. On the other hand, all natural structures come along with some rule of transformation, under the (jets of) coordinate changes.

Let us explain the meaning of this in our language. We start with describing the corresponding transformation group.

5.7. The group $\mathscr{D}^r$. Let $(V, v) = (\mathbf{R}^n, 0)$ and $\mathscr{D}^r = \mathscr{D}^r_0(\mathbf{R}^n)$. Since we can compose (locally diffeomorphic) maps $\mathbf{R}^n \to \mathbf{R}^n$ sending $0 \mapsto 0$, this induces a composition law in the jet space and the space $\mathscr{D}^r$ acquires a structure of a group. Recall that $\mathscr{D}^r$ is realized as an open subset in $\mathbf{R}^N$ and $N = n(n + n(n+1)/2 + \cdots + (n+r-1)!/(n-1)!r!)$ distinguished by the nondegeneracy of the matrix of the first $n \times n$ coordinates. The composition law $\mathscr{D}^r \times \mathscr{D}^r \to \mathscr{D}^r$ is given by a polynomial map on $\mathscr{D}^r \times \mathscr{D}^r \subset \mathbf{R}^N \times \mathbf{R}^N$ which corresponds to the chain rule (for higher derivatives). The inverse map $\delta \to \delta^{-1}$ is a *rational* map on $\mathbf{R}^N \supset \mathscr{D}^r$ with poles on $\mathbf{R}^N \backslash \mathscr{D}^r$.

The group $\mathscr{D}^r$ naturally acts on each $\mathscr{D}^r_v$, as the r-jet of the composed map $u' = \delta \circ u: (V, v) \xrightarrow{u} \mathbf{R}^n \xrightarrow{\delta} \mathbf{R}^n$ only depends on those of u and δ, and so we have an action of $\mathscr{D}^r$ on $\mathscr{D}^r(V)$. This action is free and so every orbit $\mathscr{D}_v^r$ is identical to $\mathscr{D}^r$. It is also clear that this action is exactly what remains of coordinate changes when we pass to r-jets.

5.8. $\mathscr{D}$-invariant structures. A structure $\varphi: \mathscr{D}^r(V) \to \Phi$ is called *$\mathscr{D}$-invariant* or a *$\mathscr{D}^r$-structure* if the manifold Φ is endowed with an action of the group $\mathscr{D}^r$, and φ is $\mathscr{D}^r$-equivariant.

We say in this case that the structure is of *type* Φ, where "the type" refers to the transformation law of φ encompassed by the action of $\mathscr{D}^r$ on Φ.

5.7.A. **Remarks.** (i) The type Φ of φ is not uniquely defined in terms of φ. In fact, we always can enlarge (and sometimes diminish) Φ by embedding it in a larger $\mathscr{D}^r$-space. However, the type is clear from the context. For example, if φ is a Riemannian metric on V (cf. 5.9.A(a)), then the type Φ is the space of all positive quadratic forms on $\mathbf{R}^n$ with the standard action of $\mathscr{D}^1 = \mathrm{GL}(n, \mathbf{R})$. But, if we do not care much about the positivity of φ, we look at the larger space of *all* quadratic forms as the pertinent type.

(ii) The $\mathscr{D}$-condition is not as restrictive as one may a priori think. For example, every structure φ of *algebraic type* defines in a canonical way another A-structure, say φ', which does have the $\mathscr{D}$-invariance property. To see the idea, let us consider a smooth map $p: T(V) \to \mathbf{R}$ whose restriction to each tangent space $T_v(V) = \mathbf{R}^n$ is a polynomial of degree $\leq d$ for a fixed d. Such polynomials on $\mathbf{R}^n$ form a linear space, say Φ of finite dimension $N = 1 + n + n(n+1)/2 + \cdots + (n+d-1)!/(n-1)!d!$, and $\mathscr{D}^1 = \mathrm{GL}(n, \mathbf{R})$ naturally acts on Φ. Now, each frame $\delta \in \mathscr{D}_v^1(V)$, $v \in V$, gives us an identification of $T_v(V)$ with $\mathbf{R}^n$, and so the map p on $T_v(V)$ gives us a vector, say $\varphi_p(\delta) \in \Phi$. The resulting map $\delta \mapsto \varphi_p(\delta)$ clearly is $\mathscr{D}^1$-equivariant as well as fiberwise polynomial and thus provides us with an A-structure on V in the narrow equivariant sense.

5.9. Structures as sections. Notice that the action of $\mathscr{D}^r$ on $\mathscr{D}^r(V)$ is free and proper with the orbits $\mathscr{D}_v^r(V)$, and so $\mathscr{D}^r(V)$ is the principal $\mathscr{D}^r$-bundle over V. The equivariant maps $\mathscr{D}^r(V) \to \Phi$ can be identified with the sections of the associated bundle, denoted by $\Phi(V) \to V$.

In fact, most geometric structures naturally appear as sections of such associated bundles. For example, *tensors* φ on V are the sections of (tensor) bundles which are associated to the principal $\mathrm{GL}(n, \mathbf{R})$-bundle $\mathscr{D}^1(V)$, that is, the frame bundle on V (see below for further examples).

5.9.A. **Examples.** (a) Riemannian metrics are geometric structures of the first order which have both properties A and $\mathscr{D}^1$. One sees this immediately by recalling that a Riemannian metric g is represented in local coordinates $u = u_k$, $k = 1, \cdots, n$, around $v \in V$ by $s = \frac{1}{2}n(n+1)$ smooth functions

$$g_{lk}(v) = g\left(\frac{\partial}{\partial u_l}, \frac{\partial}{\partial u_k}\right), \qquad 1 \le l \le k \le n,$$

i.e., by the components of the metric tensor at $v \in V$, and, if g'_{ij} represent g with respect to another coordinate system u', then

$$g'_{ij} = \sum_{k,l} \frac{\partial u_k}{\partial u'_i} \frac{\partial u_l}{\partial u'_j} g_{kl}.$$

(b) Affine connections on V are locally defined by Γ^k_{ij} coefficients which transform according to a certain rule under coordinate changes. The transformation rule involves first and second derivatives of the coordinates. Thus affine connections have second order. Note that if φ is an affine connection on V, then its type Φ (see 5.8) is the Euclidean space of Γ^k_{ij} coefficients with the action of the group $\mathscr{D}^2$ defined by the coordinate changes. Thus affine connections are $\mathscr{D}^2$-invariant. It is clear that they are of algebraic type.

(c) A field of k-planes on V is an A-structure of the first order. In fact, φ is a section $V \to \mathrm{Gr}_k(V)$, where $\mathrm{Gr}_k(V)$ is the Grassmann manifold of V, namely the set of all k-dimensional vector spaces of the vector spaces $T_v(V)$, $v \in V$. The full linear group $\mathscr{D}^1 = \mathrm{GL}(n, \mathbf{R})$, for $n = \dim V$, naturally acts on the fiber $\mathrm{Gr}_k(T_v(V)) = \mathrm{Gr}_k(\mathbf{R}^n)$ of the fibration $\mathrm{Gr}_k(V) \to V$. Thus, if φ is a field of k-planes on V, its type Φ is the Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$ of the k-dimensional subspaces of $\mathbf{R}^n$ with the standard action of the group $\mathscr{D}^1$ on $\mathbf{R}^n$.

5.10. Isometries and the idea of rigidity. A diffeomorphism between smooth manifolds $f\colon V_1 \to V_2$ induces, by passing to jets of f, a diffeomorphism

$$\mathscr{D}^r(V_1) \to \mathscr{D}^r(V_2),$$

and so every structure $\varphi_2\colon V_2 \to \Phi$ induces a structure of the same type Φ on V_1. Now, given two structured manifolds (V_1, φ_1) and (V_2, φ_2) of the same type Φ one says that $f\colon V_1 \to V_2$ is an *isometry* if the induced structure on V_1 equals φ_1. The following definition may serve as a motivation to our (more elaborate) definition of rigidity in 5.11.

5.10.A. *Iso-rigidity.* A structure manifold (V, φ) is called *Iso*^{k}*-rigid at* $v \in V$ if for every germ at v of an isometry $f\colon V \to V$ fixing v, the equality of the jets

$$(*) \qquad J^k_f(v) = J^k_{\mathrm{Id}}(v)$$

implies that

$$f = \mathrm{Id}, \tag{$**$}$$

where Id denotes the identity map $V \to V$, and the equalities $(*)$ and $(**)$ (according to the "germ" terminology) are meant in a small neighborhood of $v \in V$.

Next, (V, φ) is called *Iso^k-rigid* if it is rigid at all $v \in V$.

The following simple proposition allows us to pass from local to global isometries.

5.10.B. **Proposition.** *Let (V, φ) be a connected rigid manifold and f_1 and f_2 be isometries $(V, \varphi) \to (V', \varphi')$, where (V', φ') is another manifold of the same type as (V, φ). If $J^k_{f_1}(v_0) = J^k_{f_2}(v_0)$ at some point $v_0 \in V$, then $f_1 = f_2$.*

Proof. By applying the Iso-definitions to $f_2^{-1} \circ f_1$ at the points $v \in V$, where $f_1(v) = f_2(v)$, one concludes that the set of those $v \in V$, where $J^k_{f_1}(v) = J^k_{f_2}(v)$ is *open* in V. On the other hand, this set is obviously closed in V as we (tacitly) assume f is C^r-smooth.

5.10.C. **Examples.** (i) One knows that Riemannian and pseudo-Riemannian metrics φ are Iso^1-rigid, as every isometry can be recaptured from the differential at a single point by using the exponential map. In fact these φ satisfy the stronger rigidity property defined in 5.11.A.

(ii) If (V, φ) has no local isometries at all, then it is trivially Iso-rigid though not necessarily rigid in the sense of 5.11.A.

5.11. Infinitesimal isometries and k-rigidity. We recall the natural action of $\mathscr{D}\mathrm{iff}(V)$ on $\mathscr{D}^r(V)$ which induces an action of $\mathscr{D}\mathrm{iff}(V)$ on i-jets of maps $\varphi: V \to \Phi$. In fact, an action of a diffeomorphism f on the ith jet of φ at a given point $\delta \in \mathscr{D}^r(V)$ only depends on the jet J^{r+i}_f at the point $v \in V$ under δ. Now we consider the group $\mathscr{D}^{r+i}(V)$ of $(r+i)$-jets of diffeomorphisms fixing v (this group is isomorphic to $\mathscr{D}^{r+i} = \mathscr{D}^{r+i}(\mathbf{R}^n, 0)$), and then define the *infinitesimal isotropy subgroup of the isometries* of a given structure $\varphi: \mathscr{D}^r(V) \to \Phi$ as the subgroup

$$\mathrm{Is}^{r+i}(v) = \mathrm{Is}^{r+i}(V, \varphi, v) \subset \mathscr{D}^{r+i}(v)$$

consisting of (jets of) diffeomorphisms fixing J^i_φ on the fiber $\mathscr{D}^r_v(V) \subset \mathscr{D}^r(V)$. That is,

$$J^i_{\varphi \circ \tilde{f}} | \mathscr{D}^r_v = J^i_\varphi | \mathscr{D}^r_v, \tag{$*$}$$

where $\tilde{f} = J^r_f$. (If φ is a $\mathscr{D}^r$-structure represented by a section $V \to \Phi(V)$, then $(*)$ reduces to the corresponding jet equality at v.)

Let us observe natural maps

$$\mathrm{Is}(v) \to \mathrm{Is}^{\mathrm{loc}}(v) \to \cdots \to \mathrm{Is}^{r+i}(v) \xrightarrow{p_{r+i-1}} \mathrm{Is}^{r+i-1}(v) \to \cdots \to \mathrm{Is}^{r}(v),$$

where $\mathrm{Is}(v)$ denotes the isotropy subgroup of the isometry group $\mathrm{Is}(V, \varphi)$ at v, and $\mathrm{Is}^{\mathrm{loc}}(v)$ consists of the germs of the isometries fixing v. Also notice that one cannot define Is^{r-i} for $i > 0$ for the structures of order r, but we extend the notation p_k to $k = r$ for the projection

$$p_{r-1}\colon \mathrm{Is}^{r}(v) \to \mathscr{D}^{r-1}(v).$$

5.11.A. **Definition.** A structure φ is called *k-rigid at v* for $k \geq r-1$ if the map p_k is injective. The structure φ is *k-rigid* if it is rigid at all points $v \in V$.

5.11.A$_1$. **Remark on rigidity and Iso-rigidity.** Notice that Iso^k-rigidity states in this language that the map $\mathrm{Is}^{\mathrm{loc}} \to \mathrm{Is}^{k}$ is injective. It will become clear later on (see 5.13) that rigid manifolds are Iso-rigid but there are nonrigid Iso-rigid manifolds (see 5.11.B(v) below). The rigidity in the following examples will be clarified in §5.16.

5.11.B. **Examples.** (i) Let φ be a full frame field on V, i.e., a system of $n = \dim V$ independent vector fields. Then φ is a 0-rigid structure. (The Iso-rigidity is apparent in this case.)

(ii) Affine connections as well as pseudo-Riemannian metrics are 1-rigid. This can be seen either by looking at the 1-jets of exponential maps (compare 5.10.c(i)) or using (i) above (compare 5.16.C).

(iii) Conformal pseudo-Riemannian structures on V are 2-rigid for $\dim V \geq 3$. This is a classical result, probably due to E. Cartan (see 5.16.F(iv)).

(iv) Conformal structures on surfaces are *not* rigid. In fact these are not even Iso-rigid. The same applies to complex analytic structures in all dimensions and to symplectic (see 2.6.B) and contact structures.

(v) Take a generic Riemannian metric φ_0 on V and a C^∞-function ψ vanishing at a single point with infinite order. Then the structure $\varphi = \psi\varphi_0$ is Iso-rigid, as there exist no nontrivial isometries for this φ. But it is nonrigid as $\mathrm{Is}^{r}(V, \varphi, v_0) = \mathscr{D}^{r}_{v_0}(V)$, and so none of the projections $\mathrm{Is}^{k} \to \mathrm{Is}^{k-1}$ is injective.

This example also shows that a small perturbation of an Iso-rigid structure does not have to be Iso-rigid. In fact a small perturbation of the above φ may be identically zero in a neighborhood of v_0, which would make $\mathrm{Is}^{\mathrm{loc}}(v_0) = \mathscr{D}\mathrm{iff}^{\mathrm{loc}}(v_0)$. On the other hand, we shall see in 5.18 that the rigidity is stable under small perturbations of φ.

5.12. Isometries of rigid structures. It is immediate from the definition of rigidity that the action of the isometry group $\mathrm{Is} = \mathrm{Is}(V, \varphi)$ on $\mathscr{D}^s$ is free insofar as φ is k-rigid and $s \geq k$.

It follows that $\dim \mathrm{Is} \leq \dim \mathscr{D}^k$.

In fact one classically knows (since S. Lie) that Is has a natural structure of a Lie group such that the action of Is on V is smooth. Another important classical property is the following.

5.12.A. *Properness of the action of* Is *on* $\mathscr{D}^s(V)$ *The action of* Is *on* $\mathscr{D}^s(V)$ *is proper as well as free* (see [29] for the proof adapted to our language).

Let us indicate a converse to the above property 5.12.A.

5.12.B. *Let a Lie group G act on V. If the induced action on $\mathscr{D}^k(V)$ is free and proper, then there exists a G-invariant rigid structure φ on V which, moreover, is $\mathscr{D}$-invariant.*

Proof. Observe that if a Lie group G smoothly acts on a manifold V such that the corresponding action on $\mathscr{D}^k(V)$ for some $k = 1, 2, \ldots$ is free and proper, then the quotient space $\Phi_k = \mathscr{D}^k(V)/G$ is a smooth Hausdorff manifold. The action of $\mathscr{D}^k$ on $\mathscr{D}^k(V)$ induces a smooth action on Φ_k, and the structure $\varphi_k \colon V \to \Phi_k(V)$ corresponding to the quotient map $\mathscr{D}^k(V) \to \Phi_k$ is G-invariant.

This structure is not necessarily rigid. But one can pass to φ_{k+1} for $\Phi_{k+1} = \mathscr{D}^{k+1}(V)/G$, and this φ_{k+1} is rigid since the action of G on $\mathscr{D}^k(V)$ is free.

5.12.B$_1$. **Remark.** The structure which we have constructed in the proof of 5.12.B is by no means an A-structure. The following is a typical example.

5.12.C. **Example.** Let $G = \mathbf{Z}$, and consider an Anosov action of $\mathbf{Z}$ on a compact manifold V (see 2.1). Then the corresponding action of $\mathbf{Z}$ on $\mathscr{D}^1(V)$ is free and proper (the proof is not difficult), and the corresponding structure φ_1 is rigid as the action of $\mathscr{D}^1 = \mathrm{GL}(n, \mathbf{R})$ on $\mathscr{D}^1(V)/\mathbf{Z}$ is locally free.

5.12.D. **Remark.** Let us give another construction of an invariant structure by appealing to the following trivial fact.

If an action of G on W is proper, then there exists an invariant Riemannian metric g on W. (In fact, the existence of g is characteristic for proper actions; see [60].) Now, such a metric g on $W = \mathscr{D}^k(V)$ can easily be interpreted as a geometric structure φ on V. The rigidity of g implies that of φ. Notice that this structure is not $\mathscr{D}$-invariant.

5.13. Local integrability of infinitesimal isometries. Denote by $\mathrm{Dif}^{r+i}(v, v')$ the space of $(r+i)$-jets (of germs) of diffeomorphisms $V \to V$ mapping $v \mapsto v'$ and then, for a given structure φ of order r on V, consider the subset $\mathrm{Is}^{r+i}(v, v') \subset \mathrm{Dif}^{r+i}(v, v')$ of the jets which send $J^i_\varphi|\mathscr{D}^r_v(V) \to J^i_\varphi|\mathscr{D}^r_{v'}(V)$. The jets preserving J^i_φ are called *infinitesimal isometries of order* i.

Notice that $\mathrm{Is}^{r+i}(v, v)$ coincide with $\mathrm{Is}^{r+i}(v)$ defined in 5.11. Also observe that the group $\mathrm{Is}^{r+i}(v)$ naturally acts on $\mathrm{Is}^{r+i}(v, v')$, and this action is free and transitive. It follows that the k-rigidity of φ as defined in 5.11.A implies that the natural projection $\mathrm{Is}^{k+1}(v, v') \to \mathrm{Is}^k(v, v')$ is injective for all v and v' in V. This injectivity property explicitly states that rigidity amounts to *uniqueness* of an extension of the infinitesimal isometries of (V, φ) of order k to those of order $k+1$.

Now, an infinitesimal isometry $d \in \mathrm{Is}^i(v, v')$ is called *locally integrable* if there is an isometry $f\colon U \to V$ for some neighborhood $U \subset V$ of v, such that $d = J^{r+i}_f(v)$.

Global integrability means, by definition, the existence of a global isometry $f\colon V \to V$ such that $J^{r+i}_f(v) = d$.

Infinitesimal isometries in general are not locally integrable, and local integrability does not imply the global one.

5.13.A. **Example.** Consider a C^∞ metric φ on V, which is flat on a given open subset $U \subset V$ and has variable curvature outside U. Then for each $v \in \partial U$ the infinitesimal isometry groups satisfy

$$\mathrm{O}(n) = \mathrm{Is}^1(v) = \mathrm{Is}^2(v) = \cdots = \mathrm{Is}^k(v) = \cdots ,$$

while $\mathrm{Is}^{\mathrm{loc}}(v)$ is generically trivial. Furthermore $\mathrm{Is}^{\mathrm{loc}}(v') = \mathrm{O}(n)$ for all $v' \in U$. Yet the full isometry group of V may be trivial.

The following proposition, which probably goes back to S. Lie, shows that local integrability fails only on some nowhere dense subset in V.

5.13.B. *If the structure φ is $(r+i)$-rigid, then there exist an integer i_0 (which depends only on $r+i$ and* $\dim V$*) and an open dense subset $U \subset V$ depending on φ such that every infinitesimal isometry in $\mathrm{Is}^{r+i_0}(u, v)$ is locally integrable for all $u \in U$ and $v \in V$.*

A well-known corollary for Riemannian metrics g is due to Singer [63].

5.13.C. **Corollary.** *If (V, φ) is infinitesimally homogeneous (namely, if $\mathrm{Is}^{r+j}(v, v')$ is nonempty for all $j = 1, 2, \ldots$ and all v and v' in V), then V is locally homogeneous (i.e., the pseudogroup of local isometries is transitive).*

The proof is based on the Frobenius theorem for totally integrable systems (see 5.17.B and also §1.6 in [29]).

To construct a local isometry sending v to v' we take some point u in the above U. Then Proposition 5.13.B ensures local isometries sending $v \to u$ and $u \to v'$. The composition of these is what we need.

5.13.D. **Remark.** One does not know how big $r+j$ is required to be for a given type of structure to insure the local homogeneity. But one knows that in the Riemannian case $r+j$ must be of order $n = \dim V$ (see [67]; also see [25, p. 165] for a related discussion).

5.14. Partition into Is^{r+j} and $\mathrm{Is}^{\mathrm{loc}}$-orbits. Now, let V be a manifold with a C^∞ (C^{an}) smooth (possibly nonrigid) structure φ of order r. Then, for each $j = 1, 2, \dots$ we have the partition of V into the orbits under infinitesimal isometries of order j, called Is^{r+j}-orbits: two points v_1 and v_2 are in the same orbit if the set $\mathrm{Is}^{r+j}(v_1, v_2)$ (see 5.13) is nonempty. The quotient space for this Is^{r+j}-partition is denoted by V/Is^{r+j}. Similarly, we define $\mathrm{Is}^{\mathrm{loc}}$-orbits referring to the local isometry pseudogroup of V.

For general structures of *nonalgebraic* type the Is^{r+j}- and $\mathrm{Is}^{\mathrm{loc}}$-partitions may be arbitrarily complicated. For example, the partition into the orbits by an Anosov diffeomorphism is an Is^{r+j}-partition for the invariant structure constructed in 5.12.C.

On the other hand, if the structure is A, then the Is^{r+j}-partition is *regular* (at least on an open dense set in V) in the following sense.

5.14.A. **Definition.** A partition of V is called *regular* if it equals the partition into the level sets of a smooth submersion of V into some smooth manifold W.

Now, we can state the regularity property of the Is-partition.

5.14.B. **Regular partition theorem.** *Let φ be a C^∞-smooth A-structure of order r. Then for every $j \geq 0$ there exists an open dense subset $V_j \subset V$ such that the restriction of the Is^{r+j}-partition to V_j is regular. Furthermore if φ is rigid, there exists an open dense subset, say $V_\infty \subset V$, such that all Is^{r+j}-partitions, $j = 0, 1, \cdots$, are regular on V_∞, and $\mathrm{Is}^{\mathrm{loc}}$ is regular on V_∞ as well. In fact, the $\mathrm{Is}^{\mathrm{loc}}$-partition equals the Is^{r+j}-partition on V_∞ for all sufficiently large j. Moreover, this V_∞ is invariant under local (and hence global) isometries of V. (It is even invariant under infinitesimal isometries.)*

Idea of the proof. The Is^{r+j}-partition can be thought of as the partition into the levels of the map which assigns to each v the infinitesimal isometry class of φ at v of order $r+j$. If the structure φ is A, one can actually produce such a map (called the generalized Gauss map in [29]) which ranges in the quotient $\Phi'/\mathscr{D}^{r+j}$, where Φ' is some algebraic manifold acted upon the group $\mathscr{D}^{r+j}$. Then the regularity theorem for Is^{r+j}-partition follows from the algebraic stratification Theorem 6.4.A which ensures an open dense invariant subset $\Phi'' \subset \Phi'$ such that the partition into $\mathscr{D}^{r+j}$-orbits is regular in Φ''. The passage from the infinitesimal partition to the $\mathrm{Is}^{\mathrm{loc}}$ partition is then achieved by using the Frobenius theorem (see 5.17.B; also see §1 in [29]).

As an immediate corollary we get the following proposition for topologically transitive actions claimed in 0.7.A.

5.14.C. **Locally homogeneity theorem.** *If the isometry group* $\mathrm{Is}(V, \varphi)$ *is topologically transitive on* V *(i.e., if there exists a dense orbit), then there exists an open dense locally homogeneous subset in* V.

5.15. Killing fields and globalization. The essential results which we have stated so far (see 5.13.B and 5.14.B) concern local rather than global isometries where the step from "local" to "global" is obstructed by nonextendibility of isometries (compare Example 5.13.A). Now, we want to impose a *regularity condition* on the *Killing fields* of (V, φ) in order to remove this obstruction. Here, a tangent field X on some open subset $U' \subset V$ is called *Killing* if it integrates to isometries $X_t\colon U' \to U$ for the open subsets $U' \subset U$ which are relatively compact in U and where $t \in [0, \varepsilon]$ for some $\varepsilon = \varepsilon(U', X) > 0$. We call φ *regular* if the sheaf of Killing fields is locally constant (i.e., if the *dimension* of the space of Killing fields on a small connected subset in V is independent of this subset). If φ is rigid, this is equivalent to the following.

5.15.A. *Extension property.* Each point $v \in V$ admits a neighborhood $U \subset V$ such that every Killing field on every connected subset $U' \subset U$ extends to U.

5.15.B. An important consequence of the extension of Killing fields is a similar extension for the isometries in the identity components of the isometry pseudogroup (for the obvious topology in this pseudogroup). It is well known (see [58], [1]) that *rigid real analytic structures are regular.* But the regularity may easily fail for C^∞-structures. For example, every nonflat connected Riemannian manifold (V, g), where g is flat on some $U \subset V$, is *not* regular (see 5.13.A). (It is not hard to show that every Riemannian C^∞ metric g can be "regularized" by an arbitrarily small C^∞-perturbation g' such that $\mathrm{Is}(V, g) = \mathrm{Is}(V, g)$. But this is unknown for more general structures such as pseudo-Riemannian metrics.)

If V is regular and simply connected (it is enough to assume that $\pi_1(V)$ admits no nontrivial homomorphism into any finite group), then the sheaf of Killing fields is constant. If, moreover, V is compact without boundary, then global fields integrate to isometries of V. Furthermore, for every connected open set $U \subset V$, every isometry $U \to V$ in the connected component of the local isometry pseudogroup extends to an isometry of V. This allows one to globalize Proposition 5.13.B as follows.

Let $\mathrm{Dif}^{r+i}(V)$ denote the manifold of $(r+i)$-jets of germ of C^∞-diffeomorphisms $V \to V$, that is,

$$\mathrm{Dif}^{r+i}(V) = \bigcup_{v, v' \in V} \mathrm{Dif}^{r+i}(v, v'),$$

and let $\mathscr{I}^{r+i} \subset \mathrm{Dif}^{r+i}(V)$ be the corresponding union of $\mathrm{Is}^{r+i}(v, v')$. Let $\mathscr{I}_v^{r+i} = \bigcup_{v' \in V} \mathrm{Is}^{r+i}(v, v') \subset \mathscr{I}^{r+i}$, and let $\mathscr{I}_v^{r+i} = \mathscr{I}_V^{r+i}$ denote the connected component of the $(r+i)$-jet at v of the identity map $V \to V$.

Using 5.13.B one can easily show the following:

5.15.C. *If (V, φ) is regular (i.e., real analytic) compact simply connected, then for every $u \in U$ and all sufficiently large i_0 the jet map $J: f \mapsto J_f^{r+i_0}(u)$ establishes a homeomorphism of the connected component of the identity $\mathrm{Is}_0 \subset \mathrm{Is}(V, \varphi)$ onto $\mathscr{I}_u^{r+i_0}$.*

5.15.D. **Remarks.** (i) One can show (see 1.7 in [29]) that if (V, φ) is rigid *real analytic*, then infinitesimal isometries at every point $v \in V$ of sufficiently high order admit local extensions.

(ii) Let us indicate a globalization of the Regular Partition Theorem 5.14.B for regular (V, φ). (Do not confuse two notions of regularity!)

I. *f (V, φ) is regular simply connected, then there exists an open dense subset U invariant under the aciton of* $\mathrm{Is} = \mathrm{Is}(V, \varphi)$ *such that the partition of U into the orbits of the identity component* $\mathrm{Is}_0 \subset \mathrm{Is}$ *is regular.*

The proof easily follows from 5.14.B and 5.15.B.

5.16. Frame fields, rigidity and complete differential systems. We explain below how general rigid structures can be reduced to frame fields on $\mathscr{D}^r(V)$. This will provide a link of isometries with totally integrable systems needed for an application of the Frobenius theorem in the proof of 5.13.B and 5.14.B.

5.16.A. Let us look more closely at a *full frame field* φ on a manifold V, that is, a system of n independent vector fields on V for $n = \dim V$. It is clear that the infinitesimal isometry group $\mathrm{Is}^1(V, \varphi, v)$ is trivial for all $v \in V$, which amounts to 0-rigidity of φ.

Next, we observe that a structure φ of order r on $\mathscr{D}^l(V)$ naturally induces a structure on V, say φ^*, of order $r+l$. In fact, a local coordinate system $u_1, \cdots, u_n$ on V induces that on $\mathscr{D}^l(V)$, namely, the system corresponding to differentiation in these coordinates. To make sense of this, we recall that $\mathscr{D}^l(V)$ is an open subset in the space $J^l = J^l(V, \mathbf{R}^n)$ of l-jets of maps $f: V \to \mathbf{R}^n$. If we assume for example $l = 3$, then each coordinate $u = u^d_{ijk}$ in J^3, is the function on J^3 defined by the condition

$$u(J^3_f) = \frac{\partial^3 f_d}{\partial u_i \partial u_j \partial u_k},$$

where f_d denotes the d th component of f.

Now we have a natural embedding $\mathscr{D}^{r+l}(V) \to \mathscr{D}^r(\mathscr{D}^l)$ which gives the required operation $\varphi \to \varphi^*$. It is trivial that if φ is k-rigid, then φ^* is $(k+l)$-rigid.

An important example is the following.

5.16.B. *The full frame field* φ *on* $\mathscr{D}^k(V)$ *defines a* k-*rigid structure* φ' *on* V.

Our next example looks more familiar to differential geometers.

5.16.C. *Generalized connections.* A generalized connection of order r on V is a horizontal subbundle φ for the fibration $\mathscr{D}^r(V) \to V$.

That is, φ is an n-dimensional subbundle of $T(\mathscr{D}^r(V))$ transversal to the fibers. Such a connection can be viewed, by the above, as a structure of order $r+1$ on V. Next, observe that $\mathscr{D}^r(V) \to V$ is a principal fibration, and so the fiber $\mathscr{D}^r_v \subset \mathscr{D}^r(V)$ carries a full frame field (being the principal homogeneous space of the group $\mathscr{D}^r$). In fact, this field at a point $\delta \in \mathscr{D}^r_v$ corresponds to an r-jet of a local coordinate system at v. Then the first jet of this system is a frame at V, and the inverse of the differential brings this frame to φ over v. Thus φ defines in a canonical way a full frame field φ' on $\mathscr{D}^r(V)$. It is trivial to see that the rigidity of φ' implies that of φ. Thus generalized connections are r-rigid. In particular, ordinary affine connections are 1-rigid.

Now we see once again that pseudo-Riemannian metrics φ are 1-rigid, by (trivially) reducing this rigidity to that of the Levi-Civita connection φ' of φ.

5.16.D. The above implication

$$\text{rigidity of } \varphi' \Rightarrow \text{rigidity of } \varphi$$

is of quite general (and trivial) nature: suppose we have a structure φ of order r and φ' is obtained by some "canonical procedure" applied to φ. Here, "canonical procedure" means that the components of φ' are smooth functions in the partial derivatives of φ of order $\leq s$, or, more invariantly, that φ' is obtained from $\varphi: \mathscr{D}^r(V) \to \Phi$ by applying a differential operator of order s on $\mathscr{D}^r(V)$. Then it is obvious that the rigidity of φ' implies that of φ.

5.16.E. *Framed definition of rigidity.* A structure φ is *k-rigid* if there exists a full frame field on $\mathscr{D}^k(V)$ obtained from φ by a "canonical procedure."

5.16.F. **Remarks.** (i) By taking derivatives one sees that k-rigidity implies $(k+1)$-rigidity.

(ii) It is clear from the previous definition that framed rigidity implies rigidity.

The implication "rigidity" $\Rightarrow$ "framed rigidity" is not difficult, though the proof is somewhat boring.

(iii) For all natural examples of rigid structures one sees first the framed rigidity and then proceeds to rigidity. Thus the (boring) implication rigidity $\Rightarrow$ framed rigidity has little practical importance.

(iv) Frame-rigidity is very close to the notion of "structure of finite type by Cartan" where one requires a general connection of a special kind. The "finite type" terminology refers to the fact that the isometry group of such a structure is finite dimensional.

According to this terminology, structures of "infinite type," such as symplectic, foliated, etc., are those which have infinite dimensional isometry groups.

(v) The basic properties of the isometry group $\mathrm{Is}(V, \varphi)$ mentioned in 5.12 are immediate with the framed definition. Indeed, we have a frame φ' on $\mathscr{D}^r(V)$ such that isometries of (V, φ) induce those of $(\mathscr{D}^r(V), \varphi')$. These, in fact, are usual isometries for the Riemannian metric φ'' (associated to φ') with respect to which the frame φ' is orthonormal. Thus the properties of $\mathrm{Is}(V, \varphi)$ follow from the standard facts on isometries of Riemannian manifolds.

5.17. Isometries and partial differential equations. The isometry condition for a diffeomorphism $f: (V, \varphi_1) \to (V, \varphi_2)$ can be expressed by a system of partial differential equations of order r, where r is the order of the structure. In fact, the induced metric $f^*(\varphi_2)$ is expressed with J_f^r (see 5.2), and so the equality

$$f^*(\varphi_2) = \varphi_1$$

is an equation on J_f^r, i.e., a partial differential equation on f.

5.17.A. **Example.** Let the structures in question be full frame fields on V. Then the equation $f^*(\varphi_2) = \varphi_1$ has first order. At each point $v \in V$ it prescribes the differential of f, as this is uniquely defined by the condition that one given frame goes to another. Such systems of first order where the partial derivatives at each point are (smoothly) expressed in terms of the space coordinates and values of the unknown functions are called *complete.*

5.17.B. More generally, a system $\mathscr{S}$ of partial differential equations of a certain order $s+1$ is called *complete* if every partial derivative of order $s+1$ of the unknown map f can be "smoothly" expressed in terms of the derivative of order $\leq s$.

It follows that the derivatives of order $\leq s$ of every solution f of $\mathscr{S}$ satisfy, along every curve $C \subset V$, a certain system of ordinary differential equations of the first order. Therefore, if we prescribe the values of the derivatives of f of order $\leq s$ at a fixed point $v_0 \in V$ and then join v_0 with another point $v \in V$ by a curve $C \subset V$, then the solution of the partial differential equations uniquely determines $f(v) = f_C(v)$ for every solution f of $\mathscr{S}$.

If $f(v)$ does depend on the choice of C, then the system is unsolvable.

To insure solvability one imposes a *consistency* condition on $\mathscr{S}$ which amounts to a certain system of differential relations between the coefficients of the system $\mathscr{S}$.

These relations are equivalent to the (infinite) system of equations $\frac{d}{dt} f_{C_t}(v) = 0$ for all $v \in V$ and all one-parameter families C_t of curves in V between v_0 and v.

The solvability of complete integrable systems is the content of Frobenius' theorem.

5.17.C. Let us return to frame k-rigid systems (of order $r \leq k+1$), and observe that the situation here for $k > 0$ is similar to that for $k = 0$. Namely, the isometry equation

$$(*) \qquad f^*(\varphi_2) = \varphi_1$$

can be expressed by a complete system of order $k+1$. In fact, a little thought shows that the framed definition is equivalent to completeness of $(*)$.

Now, the proof of the local integrability theorem 5.13.B is reduced, by Frobenius' theorem, to verifying the integrability condition. This is of purely algebraic nature, and can be insured in our case by the constancy of the rank of a certain map (see [29]) related to equation $(*)$.

As every smooth map has constant rank on an open dense set, by applying Frobenius' theorem we obtain the desired local integrability of such a set $U \subset V$ (see 5.13.B).

5.18. Stability of rigidity. As we mentioned in 5.11.B(v) the rigidity (unlike Iso-rigidity) is stable under smooth perturbations of the structure.

In fact, a simple (but again quite boring) argument shows that rigidity is equivalent to the nonvanishing of some differential of φ. Namely, one gives the definition of rigidity as follows.

5.18.A. Denote by $\Delta^{k+r} \subset \mathscr{D}^{k+r} = \mathscr{D}_0^{k+r}(\mathbf{R}^n)$ the kernel of the natural homomorphism $\mathscr{D}^{k+r} \to \mathscr{D}^{k+r-1}$ and call a map $\psi\colon \mathscr{D}^{k+r} \to W$ (where W can be any manifold) *rigid* if for every left invariant vector field ∂ on $\mathscr{D}^{k+r}$ belonging to the Lie algebra of Δ^{k+1} the derivative is not identically zero.

Now, for every structure φ of order r (see 5.4) one can define the map $\mathscr{D}^k\varphi\colon \mathscr{D}^{k+r}(V) \to \mathbf{R}^{sN_0}$, $N_0 = 1 + n + n(n+1)/2 + \cdots$, by taking all partial derivatives of $\varphi(v, u)$ with respect to the coordinate system $u = (u_1, \cdots, u_n)$.

Then we have the following.

5.18.B. **Stable rigidity criterion.** *If the above map $\mathscr{D}^k\varphi$ is rigid, then φ is k-rigid.*

6. Examples of A-rigid actions

6.1. We present here basic examples of A-rigid actions. Most of them have already appeared throughout these lectures, but we have brought all of them together for the convenience of the reader.

6.2. Compact groups. The easiest actions from our point of view are those of compact Lie groups G on V. Every such action is rigid. In fact, starting from an arbitrary (noninvariant) metric φ one gets an invariant one by averaging our G, where the averaged metric is

$$\overline{\varphi} = \int_G (g\varphi)\, dg;$$

here dg is the Haar measure on G.

Notice that for a *generic* metric φ, $\mathrm{Is}(V, \overline{\varphi}) = G$ provided $\dim V > \dim G$ as a simple argument shows.

The basic topological property of compact group actions is the following.

6.2.A. **Compact stratification theorem.** *Let G be a compact Lie group acting on a manifold V. Then there exists a stratification (see* 1.5.A) *of V into G-invariant locally closed subsets V_i, $i = 1, \cdots, s$, such that the orbits of the action of G on V_i are mutually isomorphic (i.e., the isotropy subgroups G_v, $v \in V$, are mutually conjugate) for v running over V_i. Moreover, the quotient space V/G is a smooth manifold, and the quotient map $V \to V/G$ is a smooth fibration.*

This is well known, and the proof is not difficult (see, e.g., [11]).

6.3. Algebraic groups. After compact group actions the next remarkable class of examples is that of algebraic actions of algebraic groups on algebraic manifolds, which were introduced in 1.6.

As in 1.6, we concentrate, insofar as examples are concerned, on *algebraic* subgroups in the full matrix group $\mathrm{GL}(n, \mathbf{R})$ acting on $P^{n-1} = P(\mathbf{R}^n)$ in the usual way.

Rephrasing what was said in 1.6, we recall that a subgroup $G \subset \mathrm{GL}(n, \mathbf{R}) \subset \mathbf{R}^{n^2}$ is *algebraic* if the closure of G in $\mathbf{R}^{n^2}$ is the zero set of a system of polynomials on $\mathbf{R}^{n^2}$.

6.3.A. **Example.** Let G be a one-parameter group (i.e., G is isomorphic to $\mathbf{R}$ as a Lie group) of diagonal transformations. That is, the elements of G are matrices

$$\begin{pmatrix} t^{\alpha_1} & & & 0 \\ & t^{\alpha_2} & & \\ & & \ddots & \\ 0 & & & t^{\alpha_n} \end{pmatrix}, \qquad t \in \mathbf{R}_+^\times,$$

where $\alpha_1, \cdots, \alpha_n$ are given real numbers. Then G is algebraic if and only if α_i/α_j is *rational* for all $i, j = 1, \cdots, n$.

The proof of this is an easy exercise.

6.4. Basic facts on algebraic actions. We collect here some standard facts on algebraic groups which are used in several places in these lectures. Some of these are restatements of what has been stated earlier. For a more complete discussion the reader can see [71], [38], [10].

6.4.A. **Algebraic stratification theorem** (see [62] and compare 5.14.B). *Let G be a real algebraic group algebraically acting on a real algebraic manifold V. Then there exists a stratification $V = \bigcup_{i=1}^s V_i$, where each $V_i \subset V$ is a locally closed G-invariant algebraic submanifold, such that all orbits $G(v)$, $v \in V_i$, are closed subsets in V_i of dimension n_i. Furthermore, the quotient V_i/G has a natural structure of a smooth algebraic manifold such that the projection $V_i \to V_i/G$ is an algebraic map as well as a C^{an}-fibration.*

We state without proof another standard theorem.

6.4.B. **The orbit theorem.** *Let G be a real algebraic group algebraically acting on a real algebraic manifold V. Let $\overline{G(v)}$ denote the closure of the orbit $G(v) \subset V$ and $\partial G(V) = \overline{G(v)} - G(v)$. Then $\partial G(v)$ is a semialgebraic set of dimension* $\dim \partial G(v) < \dim G(v)$. *In particular, $\partial G(v)$ is contained in an algebraic set of dimension* $< \dim G(v)$.

6.4.B$_1$. **Corollary.** *If V is compact, then for every $v \in V$, $\overline{G(v)}$ contains a compact orbit. In fact, if $d_{\min}$ is the minimum of the dimension* *of the orbits which are contained in* $\overline{G(v)}$, *then each orbit in* $\overline{G(v)}$ *whose dimension equals* $d_{\min}$ *is compact.*

6.4.B_2. **Example.** Let $G = \mathbf{R}^n$. This G admits no compact *algebraic* homogeneous space of positive dimension. In fact, if G/H is compact and $H \neq G$, then H has infinitely many components and hence is non-algebraic. It follows that each algebraic action of $G = \mathbf{R}^n$ on a compact algebraic manifold must have a fixed point. The same is true for the group $(\mathbf{R}_+^\times)^n$ (which is isomorphic to $\mathbf{R}^n$ as a Lie group but not as an algebraic group) and also applies to complex algebraic solvable subgroups in $\mathrm{GL}(n, \mathbf{C})$. (The latter is the famous theorem of Borel-Lie.)

Next, we recall from 5.3 the principal bundle $\mathscr{D}^r(V) \to V$ whose fiber $\mathscr{D}_v^r$ over $v \in V$ consists of the r-jets (of germs) of local coordinate systems in V around v, and state the following.

6.4.B_3. **Jet properness theorem.** *Let* G *be a real algebraic group algebraically acting on a real algebraic manifold* V. *If* G *acts on* V *faithfully, then there exists an* r *such that the action of* G *on* $\mathscr{D}^r(V)$ *is free and proper. Moreover,* $\mathscr{D}^r(V)/G$ *is a smooth algebraic manifold for which the projection* $\mathscr{D}^r(V) \to \mathscr{D}^r(V)/G$ *is a smooth algebraic map.*

Notice that the map $\mathscr{D}^r(V) \to \mathscr{D}^r(V)/G$ provides a G-invariant A-structure on V. It is easy to see that this structure is rigid for large r, and so the algebraic actions are included in the class of rigid A-actions (see 1.8.A and also 5.5, 5.10, 5.11).

6.4.B_4. **Diagonal action theorem.** *If the action of* G *on* V *is faithful, then there exists an open dense subset*

$$U \subset W = \underbrace{V \times V \times \cdots \times V}_{r}$$

which is invariant for the diagonal action of G *on the* r*th Cartesian power* W *of* V *for some (sufficiently large)* r *such that the action of* G *on* U *is free and proper.*

From this one can immediately deduce the following property which we have already seen in 4.1.B.

6.4.B_4'. **Corollary.** *Let* μ *be a* G*-invariant Borel probability measure on* V, *and let* $V_\mu \subset V$ *be the Zariski closure of the support of* μ. *Then the action of* G *on* V_μ *factors through a compact action. In particular, if* $V_\mu = V$ *(e.g.,* $\operatorname{supp} \mu = V$*), then* G *is compact.*

6.4.B_5. **Finite volume property** (Compare 1.11.C.). Let $\Gamma_g \subset V \times V$ denote the graph of the action $g \colon V \to V$, $g \in G$. Then if V is compact, $\mathrm{Vol}(\Gamma_g) \leq \mathrm{const}$, where Vol denotes the n-dimensional volume ($n = \dim V$) for a fixed Riemannian metric in $V \times V$.

Idea of the proof. Algebraically embed $V \times V$ into P^N (for large N), and notice that the algebraic degree of $\Gamma_g \subset V \times V$ in P^N is bounded by a constant, say d. By the definition of degree (or Bezout theorem) the number of intersection points of Γ_g in P^N with a generic $(N-n)$-dimensional subspace P^n in P^N is at most d. Then by Crofton's formula, $\mathrm{Vol}(\Gamma_g)$ equals the average $\#(P^n \cap \Gamma_g)$ over all $P^n \subset P^N$. q.e.d.

Observe that in the above finite volume property the crux of the matter is that "const" may depend on V and on a choice of the metric in $V \times V$, but not on $g \in G$.

The above 6.4.B$_5$ is closely related to another important feature of the graphs Γ_g. Namely, we have the following.

6.4.B$_6$. **Hausdorff closure property.** If some compact subset $K \subset V \times V$ lies in the closure of the set of the graphs $\{\Gamma_g\}$, $g \in G$, then $\dim K \leq n = \dim V$, where the space of the subsets in $V \times V$ is given the Hausdorff topology corresponding to the Hausdorff metric: $\mathrm{dist}(A_1, A_2)$ for two subsets A_1, A_2 defined as the minimal ε such that the ε-neighborhoods of A_1 and A_2 satisfy:

$$U_\varepsilon(A_1) \supset A_2 \quad \text{and} \quad U_\varepsilon(A_2) \supset A_1.$$

Notice that attaching these Hausdorff limits K to the set of graphs $\{\Gamma_g\}$ provides an interesting compactification of the transformation group G, where the limit set $K \subset V \times V$ can be viewed as the "graph of an ideal transformation" of V.

6.4.C. **Remarks.** In fact, 6.4.B$_6$ above shows that the topological properties of algebraic actions are somewhat similar to those of compact group actions. Namely, in the algebraic case, where the group is noncompact, it can be naturally compactified.

A similar remark also applies to the finite volume property 6.4.B$_5$. Namely, in the compact case the sup of the pointwise norms of the differential Dg of the transformations $g: V \to V$ are uniformly bounded as g ranges over G. In the algebraic case we have bounded $\mathrm{Vol}(\Gamma_g) \subset V \times V$ which can be viewed as a kind of integral norm of $\mathscr{D}g$.

6.5. Diagonal one-parameter actions. Let us look again at the diagonal group

$$\begin{pmatrix} t^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & t^{\alpha_n} \end{pmatrix}, \qquad t \in \mathbf{R}_+^\times,$$

acting on P^{n-1}.

Here we allow arbitrary reals $\alpha_1, \cdots, \alpha_n$, so the action is not necessarily algebraic, and we assume without loss of generality that

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n.$$

Now, let us restate some elementary properties of diagonal matrices in geometric terms.

(a) The fixed points of the action are $(1, 0, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots, (0, 0, \cdots, 0, 1)$.

(b) Among these points a special role is played by the two points $(1, 0, \cdots, 0)$ and $(0, 0, \cdots, 1)$ corresponding to $\min_i \alpha_i$ and $\max_i \alpha_i$. Namely, there exists an open dense invariant set $U \subset P^{n-1}$, such that the closure of each orbit $\overline{G(v)} \subset P^n$, $v \in U$, contains these two points. In fact one may take $U = \{x_1, \cdots, x_n\}$, $x_i \ne 0$, $i = 1, \cdots, n$.

(c) Each coordinate subspace is G-invariant (coordinate subspaces are given by the equations $x_{i_j} = 0$, for given $i_1 < i_2 < \cdots < i_j < \cdots < i_n$) and these are the only closed invariant subspaces.

6.5.A. **Diagonal action of $(\mathbf{R}_+^\times)^k$ on P^{n-1}** (see 1.6.C(i)). A diagonal action of $(\mathbf{R}_+^\times)^k$ on the projective space P^{n-1} is defined by a homomorphism

$$(\mathbf{R}_+^\times)^k \to A = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \right\},$$

where A is the subgroup of the diagonal matrices in $\mathrm{GL}(n, \mathbf{R})$. More precisely, such an action is given by a matrix

$$\begin{pmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_n \end{pmatrix},$$

where each m_i, $i = 1, \cdots, n$, is a monomial:

$$m_i = t_1^{\alpha_{1,i}} t_2^{\alpha_{2,i}} \cdots t_k^{\alpha_{k,i}}, \qquad (t_1, \cdots, t_k) \in (\mathbf{R}_+^\times)^k.$$

To simplify the notation, let $k = 2$. Then our action is given by the matrices

$$\begin{pmatrix} t_1^{\alpha_1} t_2^{\beta_1} & & & 0 \\ & t_1^{\alpha_2} t_2^{\beta_2} & & \\ & & \ddots & \\ 0 & & & t_1^{\alpha_n} t_2^{\beta_n} \end{pmatrix}, \qquad (t_1, t_2) \in (\mathbf{R}_+^\times)^2.$$

6.5.A$_1$. Another example of diagonal action for $k = n-1$ is that where the action is given by

$$\begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix}, \qquad (t_1, \cdots, t_n) \in (\mathbf{R}_+^\times)^n.$$

Here $G = (\mathbf{R}_+^\times)^{n-1} = A/\Delta$, where Δ equals $\mathbf{R}_+^\times$ diagonally embedded in $A = (\mathbf{R}_+^\times)^n$, and $\Delta = \mathbf{R}_+^\times$ appears as the group of diagonal matrices

$$\begin{pmatrix} t & & & 0 \\ & t & & \\ & & \ddots & \\ 0 & & & t \end{pmatrix},$$

which trivially acts on P^{n-1}, so that one looks at the action not really of $\mathrm{GL}(n, \mathbf{R})$ but of $\mathrm{GL}(n, \mathbf{R})/\Delta$ on P^{n-1}.

Notice that this action is *universal* in the sense that every diagonal matrix is contained in the group

$$A = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \right\} \subset \mathrm{GL}(n, \mathbf{R}).$$

The geometry of the action of this $G = A/\Delta = (\mathbf{R}_+^\times)^{n-1}$ on $V = P^{n-1}$ is easy to understand. For example, the orbit $G(v)$ of a point $v = (x_1, \cdots, x_n)$ where all (homogeneous) coordinates are > 0 equals the set of *all* points with positive coordinates. This set can be identified with the open $(n-1)$-symplex. (In fact, the set $\{x_i > 0\}$ in P^{n-1} is identical with the usual symplex $\Delta^{n-1} = \{x_i > 0 | \sum_{i=1}^n x_i = 1\}$ in $\mathbf{R}^n$.) Then the closure $\overline{G(x)}$ of the orbit gives the closed symplex, and $\partial G(v) = \overline{G(v)} - G(v)$ equals the boundary of the symplex. Furthermore, the k-dimensional orbits correspond to the k-faces of our symplex in an obvious way.

Now, turn to the general diagonal action of $G = (\mathbf{R}_+^\times)^k$ on P^{n-1} and state the following classical theorem.

6.5.B. **Convexity theorem.** *The closure of each orbit $G(v) \subset P^n$ is a finite union of orbits. Moreover, there exists a convex polyhedron $H \subset \mathbf{R}^n$ and a homeomorphism $\overline{G} \to H$ such that each orbit $G(v)$ goes onto the interior of some face of H.*

Idea of proof (see [30] for details). Start with the following map M: $P^{n-1} \to \mathbf{R}^n$, $M: (x_1, \cdots, x_n) \to (x_1^2, x_2^2, \cdots, x_n^2)$ where the homogeneous coordinates are normalized by $\sum_{i=1}^n x_i^2 = 1$. Then P^{n-1} goes to the standard symplex $\Delta^{n-1} \subset \mathbf{R}^n$. If $G = (\mathbf{R}_+^\times)^{n-1} = A/\Delta$, then it is obvious that each orbit goes to some face of Δ^{n-1}.

6.5.C. Now, let us turn to the general case, but to keep the notation simple, let $k = 2$ and

$$G = \begin{pmatrix} t_1^{\alpha_1} t_2^{\beta_1} & & \\ & \ddots & \\ & & t_1^{\alpha_n} t_2^{\beta_n} \end{pmatrix}, \qquad (t_1, t_2) \in (\mathbf{R}_+^\times)^2.$$

Denote by $S \subset \mathbf{R}^2$ the set of pairs $(\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)$, and let $C = \operatorname{Conv}(S) \subset \mathbf{R}^2$ be the convex hull of S. This C is a finite polygon with at most n vertices. We may assume, by permuting the coordinates if necessary, that the first k points $(\alpha_1, \beta_1), \cdots, (\alpha_k, \beta_k)$ are the vertices while the remaining ones lie inside (or on the open edges of) C.

Notice that these k points play a special role (similar to $\max \alpha_i$ and $\min \alpha_i$ in 6.5(b)). Namely, each of the n points $(1, 0, \cdots, 0), (0, 1, \cdots, 0, 0), \cdots, (0, 0, \cdots, 1)$ in P^{n-1} is fixed under the diagonal action. But the first k, $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (\underbrace{0, \cdots, 1}_{k}, \cdots, 0)$, which correspond to the vertices of C with our notation, are *attractive* in some neighborhood in P^{n-1}. This means there exists an open dense $U \subset P^{n-1}$, such that the closure of each orbit $\overline{G(v)}$, $v \in U$, contains these k points. On the other hand, there exists no such *open* U for the remaining fixed points.

Now, let L denote the (unique) affine map $\Delta^{n-1} \to \mathbf{R}^2$ which sends the kth vertex of Δ^{n-1} to $(\alpha_k, \beta_k) \in \mathbf{R}^2$, and consider the composed map $L \circ M \colon P^{n-1} \to \mathbf{R}^2$ for the above $M \colon P^n \to \Delta^{n-1}$. One can show (see [30]) that each G-orbit in P^{n-1} homeomorphically goes under $L \circ M$ onto an open face of the convex polygon C, and thus we obtain the claimed correspondence between orbits and convex sets (see [30] for details).

6.5.D. We conclude the discussion of algebraic groups by explaining the conformal action of $\mathrm{O}(n+1, 1)$ on S^n (see 0.10.A). We start with the standard embeddings

$$S^n \subset \mathbf{R}^{n+1} \subset P^{n+1},$$

and let $G \subset \mathrm{SL}(n+2, \mathbf{R})$ be the group of projective transformations which map S^n into itself.

The action of G on S^n preserves round subspheres as these are the intersections of S^n with linear subspaces. Thus this action is conformal.

Next, we represent $S^n \subset P^{n+1}$ by a cone in $\mathbf{R}^{n+2}$. Namely, we take the cone defined by the equation

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = x_{n+2}^2,$$

and observe that the linear transformations of $\mathbf{R}^{n+1}$ preserving this cone form, up to the scalars, the orthogonal group $\mathrm{O}(n+1, 1)$ for the quadratic form

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 - x_0^2.$$

Thus we get the required conformal action of $\mathrm{O}(n+1, 1)$ on S^n.

Let us change the coordinates in order to have the form

$$x_1^2 + x_2^2 + \cdots + x_n^2 + yz.$$

Now, we have an interesting diagonal 1-parameter group, $\mathbf{R}_+^\times$, namely

$$G = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & t & \\ 0 & & & & t^{-1} \end{pmatrix}$$

acting conformally on S^n.

This group acts by north pole-south pole transformations (see 2.3) which can be seen geometrically if one writes $S^n = \mathbf{R}^n \cup \{\infty\}$, where the point $0 \in \mathbf{R}^n \subset S^n \subset P^{n+1}$ corresponds to the eigenspace spanned by $(0, 0, \cdots, 1)$, and ∞ corresponds to $(0, 0, \cdots, -1)$. Then G acts by scaling on $\mathbf{R}^n$, $x \mapsto t \cdot x$. This action has two fixed points on S^n. The south pole corresponds to $0 \in \mathbf{R}^n$, and the north pole to ∞. If $s \in S^n \setminus \{\text{south pole}\}$, then $ts \to$ north pole for $t \to +\infty$ and if $s \in S^n \setminus \{\text{north pole}\}$, then $ts \to$ south pole for $t \to -\infty$.

We suggest that the reader write in matrices the transformation corresponding to the parallel translation of $\mathbf{R}^n \subset S^n = \mathbf{R}^n \cup \{\infty\}$.

6.6. Twisted torus action (Compare 1.11.D). Now we want to describe a simple class of actions which look very much like algebraic actions though they are not algebraic.

Let T^n freely act on V, and let $\mathcal{G} \subset \mathrm{Diff}(V)$ be the corresponding gauge group: each $g \in \mathcal{G}$ is a fiber preserving diffeomorphism $V \hookleftarrow$, where each fiber $T_v = T^n$ goes into itself by a translation in T^n. Thus g is determined by a function, say $\varphi \colon V \to T^n$, such that $g_\varphi(v) = \varphi(v) \cdot v$ for the action $t \cdot v$ of T^n of V. Note that $\mathcal{G} \supset T^n$, and we are interested in connected finite-dimensional subgroups $G \subset \mathcal{G}$ which contain the torus

$T^n \subset \mathscr{G}$. Such a subgroup G can be given by k vector fields on V, for $k = \dim G - n$ tangent to the T^n-orbits. These fields on each orbit should correspond to some field coming from the Lie algebra $\mathbf{R}^n = L(T^n)$, and our fields are, in fact, determined by maps $f_j \colon W \to \mathbf{R}^n$, $j = 1, \cdots, k$, for $W = V/T^n$.

The following proposition shows that the action of G is rigid in most cases. For example, it is rigid if the maps f_j are real analytic.

6.6.A. **Proposition.** *The following conditions are equivalent:*

(1) *The action of G is A-rigid.*

(2) *The action is rigid.*

(3) *The induced action of G on $\mathscr{D}^r(V)$ is free for some r.*

(4) *There exists an r, such that: if a linear combination $f = \sum_{j=1}^k \lambda_j f_j$ has $\partial^I f(v_0) = 0$ for $|I| = 1, \cdots, r$, where v_0 is some point in V (and the partial derivatives are taken in some local coordinates around v_0), then f is constant on the connected component of v_0 in V.*

Sketch of the proof. Obviously, $(1) \Rightarrow (2)$ while $(2) \Rightarrow (3)$ by 5.12. Let us show $(1) \Leftrightarrow (4)$. First, observe that condition (4) is equivalent to the action of G to be locally free on $\mathscr{D}^r(V)$, and hence $(3) \Rightarrow (4)$ and $(1) \Rightarrow (4)$.

Next, to see that $(4) \Rightarrow (1)$, notice that under condition (4), the action of the isotropy subgroup G_v on $\mathscr{D}^r_v(V)$ is locally free for all $v \in V$, and that this implies that the action is in fact free. Then one can see that the embedding $G_v \subset \mathscr{D}^r_v$ is algebraic, i.e., it is injective and the image is an algebraic subgroup in $\mathscr{D}^r_v$. That is, we have a fibration $\overline{\mathscr{D}}^r \to V$ with algebraic fibers $\mathscr{D}^r_v/G_v$, and T^n naturally acts on $\overline{\mathscr{D}}^r$.

Now, observe that G-invariant functions on $\mathscr{D}^r(V)$ can be identified with T^n-invariant ones on $\overline{\mathscr{D}}^r$. These are easy to construct (for example by averaging over T^n) as T^n is compact. In fact, there exists a system of such functions algebraic on each fiber $\mathscr{D}^r_v/G_v$, $v \in V$, such that the isometry group of the corresponding structure $\mathscr{D}^r(V) \to \mathbf{R}^s$ (see 5.4) equals G. The details are left to the reader.

6.6.B. **Examples.** (i) (Compare 1.11.D.) Take $V = S^3$ and $G = S^1 \times \mathbf{R}$ acting on S^3 by twisted rotations as described in 1.1.1.D.

(ii) Here our manifold is $V = \mathbf{R}^{n-m} \times T^m$ equipped with the standard (product) affine structure, and our group G is the group of *affine* transformations of V mapping each torus $x \times T^m$ into itself by some rotation of T^m. Clearly, G is isomorphic to $T^m \times \mathbf{R}^{m(n-m)}$.

6.6.C. **Remark.** The actions described in 6.6 are *not* algebraic unless $G = T^m$.

In fact, they violate the diagonal action property (see 6.4.B$_4$) and the finite volume property (see 6.4.B$_5$) of algebraic actions as explained in 1.11.D and 1.11.E.

6.7. Homogeneous actions. Most dynamically significant rigid actions appear as we have already mentioned in 0.5.A in the homogeneous framework. Namely, V is the homogeneous space for G, and the acting group may be G itself or a subgroup H of G. If the action of G on V is faithful, then it is easy to see that the corresponding action on $\mathscr{D}^r(V)$ is free and proper for large r. Hence the action is rigid. In fact, it is not hard to show that the action is A-rigid.

These homogeneous and subhomogeneous (i.e., of subgroups $H \subset G$) actions on V include the algebraic actions as P^n is an $\mathrm{SL}(n+1, \mathbf{R})$-homogeneous space.

A twisted torus action also appears in the homogeneous framework according to 6.6.B(ii). Notice that in the algebraic and the twisted torus actions the isotropy group $G_v \subset G$ is connected and has at most finitely many components. But one obtains by far more interesting examples if the isotropy group G_v has infinitely many components. The key case here is where the isotropy group is a discrete subgroup, say $\Gamma \subset G$. Let us indicate a particularly interesting invariant structure φ on $V = G/\Gamma$ coming from a bi-invariant structure $\tilde{\varphi}$ on G as it descends to G/Γ.

6.7.A. **Example.** Let G be semisimple, and $\tilde{\varphi}$ correspond to the Killing form on G. Then φ (like $\tilde{\varphi}$) is a pseudo-Riemannian G-invariant metric on $V = G/\Gamma$.

Notice that the isotropy group G_v is conjugate to Γ for all $v \in V$ while the local isotropy group $\mathrm{Is}^{\mathrm{loc}}(V, \varphi, v)$ contains extra transformations coming from $G \times G$ acting on G by left- and right-translations. It follows, in the semisimple case, that $\mathrm{Is}^{\mathrm{loc}}$ is locally isomorphic to the group G itself. More generally, one can easily show that if G has no compact factor group, then for every G-invariant A-structure φ on V one has the Lie algebra $L(G)$ inside $L(\mathrm{Is}^{\mathrm{loc}}(v), \varphi)$. Notice that the above discussion is justified by the existence of many interesting discrete subgroups Γ in G (see 1.9.D). The first example is our famous $G = \mathrm{SL}(2, \mathbf{R})$ with the lattice $\Gamma = \mathrm{SL}(2, \mathbf{Z}) \subset G$ (see 1.9.D(ii)).

6.8. Besides pseudo-Riemannian metrics one may have other interesting invariant structures. For example, if $G = \mathrm{GL}(m, \mathbf{R})$, then the flat affine structure on G induced from $\mathbf{R}^{m^2} \subset \mathrm{GL}(m, \mathbf{R})$ is bi-invariant. Thus we get compact *affine flat manifolds* of the form $\mathrm{GL}(m, \mathbf{R})/\Gamma$ acted upon by $\mathrm{GL}(m, \mathbf{R})$ preserving the affine structure. Then passing to

$\mathrm{PSL}(m, \mathbf{R})$ one obtains similar examples with *flat projective structures*. Notice that this structure for $m \geq 3$ does not come from the Killing form (an exercise to the reader).

6.9. One can elaborate the previous examples by "twisting" them with proper actions. Namely, let another group, say G, freely and properly act on V_1, and let $\tilde{\varphi}_1$ be an invariant Riemannian metric (see 5.12.D). Then take a discrete subgroup $\Gamma \subset G \times G_1$ and observe that G acts on $(G \times V_1)/\Gamma$ preserving the (local) product structure $\varphi \times \varphi_1$ for any bi-invariant structure $\tilde{\varphi}$ on G (compare 4.7.1).

6.10. Locally homogeneous actions. Next, after homogeneous spaces come *locally homogeneous* ones. Standard examples of such manifolds are affine flat, conformally flat, projectively flat manifolds.

The isometry group $\mathrm{Is}(V)$ of a locally homogeneous space V can be arbitrarily complicated if no compactness (or finiteness of volume) assumption is imposed on V.

For example, for any countable group Γ one can take a domain $U \subset \mathbf{R}^4$ with $\pi_1(U) = \Gamma$, and then Γ isometrically acts on the universal covering V of U.

A more convincing example is as follows. Start with some Riemannian metric g_0 on a surface V, such that $\mathrm{Is}(V, g_0) = \Gamma$, and then conformally change g_0 to get a complete metric g on V with *constant* negative curvature. Thus $\mathrm{Is}(V, g) \supset \Gamma$ and it is easy to make $\mathrm{Is}(V, g) = \Gamma$ as well.

If V is compact, and (or) the action of $\mathrm{Is}(V)$ preserves a smooth volume element, then the action of the identity component $\mathrm{Is}_0(V) \subset \mathrm{Is}(V)$ on V can be understood almost as well as in the homogeneous case. But the action of discontinuous (e.g., discrete) groups on V appears more difficult to comprehend. The problem arises from the action of $\mathrm{Is}(V)$ on the fundamental group of V which makes it impossible to lift the action to the universal covering of V. On the other hand, all *known* examples of interesting discrete actions on locally homogeneous spaces are rather simple and essentially of arithmetic origin, and these can be essentially reduced to the standard action of $\mathrm{SL}(n, \mathbf{Z})$ on the torus $\mathbf{R}^n/\mathbf{Z}^n$. For example the action on a nilmanifold coming from automorphisms of the corresponding (nilpotent) Lie group is in this category.

6.11. We conclude with a particularly nice example which has homogeneous origin but is not (even locally) homogeneous.

6.11.A. **Example.** Let V be the unit tangent bundle of a complete (e.g., compact) locally symmetric space X. Consider $G = \mathbf{R}$ acting by the geodesic flow. If $\operatorname{rank} X = 1$ (e.g., X has constant sectional curvature), then V admits a natural locally homogeneous structure compatible with the flow. But for $\operatorname{rank} X \geq 2$ the manifold V is not (at least in a natural way) locally homogeneous, but is partitioned into locally homogeneous "fibers" of generic codimension $k = \operatorname{rank} X$. In fact, the quotient space of this partition can easily be identified with S^{k-1}/W, where S^{k-1} is the unit tangent sphere to some flat subspace in X, and W is the Weyl group.

6.11.A$_1$. **Remark.** The pleasant feature of this example is the existence of an invariant rigid A-structure, which can be built using the stable and the unstable foliations (as in 2.2(b)) together with the natural action of $\mathbf{R}^k$ commuting with our $G = \mathbf{R}$. It would be interesting to classify (locally homogeneous) Riemannian manifolds where the geodesic flow admits a rigid A-structure. Now, after the rows of all these rigid beauties we bring forth

6.12. An example of a nonrigid action. Take a small ball $B^n \subset V$, $n = \dim V$, and let f_1 and f_2 be generic C^∞-diffeomorphisms of V into itself (here V is any smooth manifold) sending B^n into itself and fixing the complement $V \backslash B^n$. The group generated by f_1 and f_2 obviously is free. Next take a diffeomorphism $f\colon V \to V$ such that the images $f^j(B^n)$ are disjoint for the iterates f^j of f. It is clear that the group Γ generated by f, f_1, and f_2 contains an infinite product of free groups as a subgroup. It follows (see 0.4.A and §4 in [29]) that Γ admits no rigid actions at all on any compact manifold or on a noncompact manifold with an invariant probability measure.

What we do not know, however, is the answer to the following question.

Question. Does Γ admit a faithful *real analytic* action on some compact manifold?

7. Some open problems on rigid structures and transformation groups

7.1. Rigidity problem for a geometric structure induced by a nondiffeomorphic map. Let V and W be n-dimensional manifolds with geometric structures of the same type, called φ on V and ψ on W, and let $f\colon V \to W$ be a smooth map which is an immersion (i.e., locally diffeomorphic) on an open dense subset $U \subset V$. Can the structures ϕ and ψ be A-rigid in the case where f is *not* an immersion on all of V?

7.1.A. Let, for example, ψ be a Riemannian metric on W, and φ be a quadratic form on V induced by f. Does the rigidity of φ imply that f is locally diffeomorphic?

7.1.B. Let (W, ψ) be a C^∞-smooth almost complex manifold and (V, φ) be obtained by blowing up a point $w_0 \in W$. Notice that this blown-up manifold admits a natural *continuous* almost complex structure which is our φ. Now, let φ be C^∞-smooth for some C^∞-structure on V. Does it follow that the structure φ is infinitesimally integrable (and hence nonrigid) at w_0?

7.1.B$'$. Let (W, ψ) be as above, and (V, φ) be a ramified cover of W, where the ramification locus is a smooth codimension 2 submanifold $W_0 \subset W$ whose tangent subbundle $T(W_0) \subset T(W)$ is invariant under the almost complex structure ψ (that is, an automorphism of $T(W)$ with square $-\mathrm{Id}$). Here again we ask if smoothness of (V, φ) implies infinitesimal integrability of ψ at W_0.

7.2. Extension problem for local C^{an}-isometries. Let (V, φ) and (W, ψ) be connected A-rigid n-dimensional C^{an}-manifolds of the same type, where we additionally assume V is simply connected and W is compact without boundary. Let $U \subset V$ be an open connected subset, and $f_0\colon (U, \varphi) \to (W, \psi)$ be a locally isometric C^{an}-immersion. Under which conditions does f_0 extend to a locally isometric immersion $f\colon (V, \varphi) \to (W, \psi)$? Some condition is indeed needed here as the following example shows.

7.2.A. **Example.** Let φ and ψ be flat affine connections, such that (V, φ) equals the affine space $\mathbf{R}^n$, while (W, ψ) is a compact *noncomplete* manifold. Then there is no global isometry $(V, \varphi) \to (W, \psi)$, while the local isometries are abundant.

Keeping in mind this example one may suggest that the answer to the extension question is "yes" in the case where W is *simply connected.* What is, probably, more relevant is the global extendibility of local Killing fields on (W, ψ) as is explained below.

7.2.B. Suppose φ and ψ are pseudo-Riemannian metrics of the same type, and let $I \subset \mathrm{Gr}_n(V \times W)$ denote the set of the n-planes which are tangent to the graphs of the local isometries $(U, \varphi) \to (W, \psi)$ for all $U \subset V$ (compare 3.5). If some local isometry $f_0\colon U \to W$ is not extendible to a boundary point $v_0 \in \partial U$, then the differential of f_0 blows up as $u \to v_0$. It follows that I is *not* a closed subset in Gr_n. In other words some sequence of graphs of local isometries converges to a totally geodesic isotropic n-dimensional submanifold $M \subset (V \times W, \varphi \oplus -\psi)$ which is vertical in the sense that its projection to V has everywhere rank $\leq n-1$.

Now we recall that the space of germs of n-dimensional totally geodesic submanifolds in $V \times W$ naturally *embeds* into the Grassmann bundle

$\mathrm{Gr}_n(V \times W)$, and the image is a *real analytic* subset in $\mathrm{Gr}_n(V \times W)$ (see 3.5.1). It follows that $M = M_0$ is included in a C^{an}-family of such submanifolds M_t for $t \in [0, 1]$, where each M_t for $t > 0$ is the graph of a local isometry $V \to W$. Therefore, W (as well as V) has a nonzero Killing field. Moreover, the local isometry pseudogroup of W is noncompact.

If $\pi_1(W) = 0$, then the local isometries extend to global ones on W. With these one probably can show that local isometries of V to W can be globally extended as required.

A similar argument may work for affine connections but, for example, the case of conformal pseudo-Riemannian structures looks more difficult.

Notice that the extension problem makes sense in the C^∞-case if we put it as follows.

7.2.C. Given an isometry $f_0 \colon (U, \varphi) \twoheadrightarrow (W, \psi)$ for an open subset $U \subset V$. Does it smoothly extend to the closure of U in V? This question can be viewed as a generalization of Problem 7.1.

As we have seen above the solution of the extension problem strongly depends on the structure of the closure of the set of graphs of (local) isometries $V \to W$. In the pseudo-Riemannian case every subvariety $M \subset V \times W$ in this closure (here "closure" refers to the Hausdorff topology in the space of subsets in $V \times W$) is smooth but, in general, such M may have singularities. For example, the closure of the graphs of the conformal transformations of the standard sphere S^n contains the unions $(s_1 \times S^n) \cup (S^n \times s_2)$ for all pairs of points s_1 and s_2 in S^n. This leads to the question on the possible structure of the singularities of M for general A-rigid structures. Here it may be useful to look not only at the graphs of the isometries themselves but also at the graphs (and their Hausdorff limits) of higher order jets of isometries. (Notice that all these questions apply to limits of graphs of solutions of quite general partial differential equations.)

7.2.D. **Extension to the boundary.** The seemingly easiest case of the extension problem is that where V and W are compact manifolds with boundaries and f_0 is an isometry of the *interior* U of V to that of W. The question is whether f_0 extends to the boundary ∂V. The answer is unknown even if the structures in question are pseudo-Riemannian metrics and the boundaries of V and W are totally geodesic. (If the boundary of V contains no geodesic segment, then the extension is possible by an easy limit argument.)

7.2.E. The answer to the following question needs better understanding of limits of graphs of isometries.

Let $M \subset V \times W$ be an irreducible (semi)analytic subset such that at a generic point $z \in M$ (where M is nonsingular) M is the graph of a local isometry (for given A rigid C^{an}-structures on V and W). The question is whether M is everywhere nonsingular and equals the graph of a local isometry of V to W at *every* point in M.

The earlier discussion provides the positive answer for pseudo-Riemannian structures and for the Riemannian conformal structure, but for more general A-rigid structures the answer is unknown.

7.3. Problems concerning different notions of (weak) local homogeneity of a geometric structure. We start with a list of different notions of local homogeneity of a structured manifold (V, φ) given in the order of the decreasing strength.

$(\mathrm{Ho}_1)(V, \varphi)$ admits a locally transitive action of a local Lie group.

(Ho_2) For every two points v_1 and v_2 in V there exists a local isometry of some neighborhood U_1 of v_1, say $f: U_1 \to V$, such that $f(v_1) = v_2$.

(Ho_3) For every two points v_1 and v_2 there exist arbitrarily small isometric neighborhoods U_1 of v_1 and U_2 of v_2, where the implied isometry $f: U_1 \to U_2$ does not have to send $v_1 \mapsto v_2$.

(Ho_4) There exists an open dense subset $U \subset V$ which satisfies one of the above (Ho_1), (Ho_2), or (Ho_3).

(Ho_5) There exists a dense subset $X \subset V$ such that the above (Ho_2) or (Ho_3) is satisfied for all pairs of points v_1 and v_2 in X.

(Ho_6) There exists a dense subset in $V \times V$, such that either (Ho_2) or (Ho_3) is satisfied for the pairs (v_1, v_2) in this subset.

The general problems here are as follows.

7.3.A. For a rigid (or A-rigid) structure φ of a given type and smoothness class C^k decide which of the implications $\mathrm{Ho}_{i+1} \Rightarrow \mathrm{Ho}_i$ hold true.

7.3.B. Let φ be a rigid C^k-structure of a given type. Which of the Ho_i-conditions imply that φ is real analytic?

These problems are already interesting for Riemannian and pseudo-Riemannian structures (compare 5.13.D and p. 165 in [25]). Also notice that the major difficulty appears when the smoothness class C^k is not very high and the infinitesimal techniques do not apply. In fact the symmetries of nonsmooth structures may be quite pathological, as in the case of the geodesic flows preserving continuous (and sometimes C^1-smooth) pseudo-Riemannian metrics (see 2.10). On the other hand, a slight strengthening of the smoothness condition probably makes a locally homogeneous structure (satisfying some Ho_i) real analytic. In fact, this may be already true on the infinitesimal level: if a C^2-rigid structure φ is C^k-infinitesimally homogeneous for $k \gg 2$, then one expects φ to be C^{an}. For example, if a (pseudo-)Riemannian metric φ has the same curvature tensor at all points of V, then φ should be C^{an} since the "constant curvature tensor" equation seems transversally elliptic.

7.3.C. *The* Is-*partition problem.* One may try a nonhomogeneous generalization by asking what is the possible topological structure of the partition of V into the local isometry orbits of (V, φ). As it stands, this question appears too general but one may hope for a meaningful answer in some particular cases.

7.4. Isometrices of Lorentz manifolds. Let (V, φ) be a compact simply connected Lorentz C^k-manifold. Is then the isometry group $\mathrm{Is}(V, \varphi)$ compact? By D'Ambra's theorem this is true for C^{an}-manifolds, but one expects this already for C^0.

If $\mathrm{Is}(V, \varphi)$ is noncompact, then V admits a totally geodesic foliation $\mathscr{S}$ of codimension one whose leaves are projection from $V \times V$ of totally geodesic submanifolds $M \subset V \times V$ which are limits of graphs of isometries g_i of (V, φ) for some divergent sequences $g_i \in \mathrm{Is}(V, \varphi)$.

The existence and the structure of $\mathscr{S}$ can be seen with the following definitions.

7.4.A. **Definitions.** An n-dimensional subspace $T \subset T(V \times V)$ for $n = \dim V$ is called *asymptotic* (or $\{g_i\}$-asymptotic to be precise) if it is the limit of the tangent spaces T_i to the graphs $\Gamma_i = \Gamma_{G_i} \subset V \times V$ at some points $x_i \in \Gamma_i$. That is $T_i = T_{x_i}(\Gamma_i)$ and $T = \lim_{i\to\infty} T_i$ in the Grassmann space $\mathrm{Gr}_n(V \times V)$. (Notice that this definition makes sense for arbitrary C^1-maps g_i between two manifolds.)

If the maps g_i preserve an affine connection in V (as is the case for Lorentzian isometries), then the asymptotic space $T = T_x$, $x = (v, v')$, exponentiates to a complete totally geodesic submanifold Γ in $V \times V$ which is, by definition, the limit of the graphs Γ_i with the reference points $x_i = (v_i, v_i')$ for $v_i' = g_i(v_i)$.

If g_i preserve a pseudo-Riemannian metric φ in V, then the intersection of Γ with the (vertical) fibers $v \times V$ are φ-isotropic since the graphs Γ_i are $(\varphi \oplus -\varphi)$-isotropic. In particular these intersections in the Lorentz case are at most 1-dimensional. In fact, if the sequence g_i diverges (i.e., it is nonprecompact in $\mathrm{Is}(V, \varphi)$) and V is connected, then for each $v \in V$ the intersection $\Gamma \cap (v \times V)$ is either empty or 1-dimensional. Then the projection of Γ to the first component V is a totally geodesic hypersurface denoted $\Lambda \subset V$, which is called an *asymptotic leaf* for $\{g_i\}$. If two asymptotic leaves meet, the intersection cannot be transversal as at the transversal intersection point $v \in V$ the norm of the difficulties Dg_i would be (by an easy argument) uniformly bounded for $i \to \infty$, which is impossible under our assumption $g_i \to \infty$. It follows in the Lorentz case (where the leaves have codimension one) that different leaves do not meet at all.

Now, we observe that for an arbitrary countable subset N in V there exists a subsequence g' of g_i, such that for each $v \in N$ there is a $\{g'_j\}$-asymptotic leaf Λ containing v. Then the closures of these (codimension 1) leaves form our foliation $\mathscr{S}$ in the case N is chosen *dense* in V.

7.4.B. **Remarks.** (i) The above *asymptotic foliation* $\mathscr{S}$ is by no means unique. Yet the space of these $\mathscr{S}$ form a *compact* space naturally acted upon by $\mathrm{Is}(V, \varphi)$. The compactness follows from an (obvious) uniform Lipschitz bound for the tangent subbundle $S = T(\mathscr{S}) \subset T(V)$ of an arbitrary totally geodesic foliation $\mathscr{S}$ of codimension one in V.

(ii) The above *asymptotic subbundle* S is somewhat similar to the stable subbundle T^+ discussed in 0.2.B. In fact there is a more general notion of a subbundle with a "restricted growth" which embraces S as well as T^+.

(iii) Suppose V is noncompact but the isometries g in question keep invariant some compact subset $V_0 \subset V$. Then one uses the action of (the differentials of) g on the bundle $T(V)$ restricted to V_0, and one may speak of asymptotic subbundles $S \subset T(V)|V_0$. The corresponding totally geodesic submanifolds Λ obtained by exponentiating (the fibers of) S may go outside V_0 but yet they are mutually disjoint in V (because $\mathrm{codim} = 1$).

Let us apply all this to a subgroup $G \subset \mathrm{O}(n, 1)$ acting on $V = \mathbf{R}^{n,1}$ and fixing the origin $V_0 = \{0\}$. Then we recognize the space of asymptotic subbundles as the limit set of G acting (by conformal transformations) on the sphere S^{n-1} which can be described in two equivalent ways.

(a) *Usual description.* S^{n-1} is the set of isotropic rays in $\mathbf{R}^{n,1}$.

(b) *Our description.* S^{n-1} consists of the hyperplanes $H \subset \mathbf{R}^{n,1}$ on which the form $\sum_{i=1}^n x_i^2 - x_{n+1}^2$ is singular.

7.4.C. Now we see that the space A of asymptotic foliations (or subbundles) on a (compact) Lorentz manifold generalizes the classical limit set in the theory of Kleinian groups. This leads us to the following.

Questions. How far does the Kleinian group theory extend to the action of $G = \mathrm{Is}(V, \varphi)$ (and more generally of $G = \mathrm{Is}(V, V_0, \varphi)$) on A? When is A a (smooth) manifold? When is the action minimal? Is every $g \in G$ elliptic, parabolic, or hyperbolic? etc. The answer to these questions should eventually lead to a classification of the compact Lorentzian manifolds V with noncompact isometry groups and also of noncompact manifolds admitting compact invariant subsets and/or finite invariant measures.

Another group of questions concerns a generalization of the above construction of $\mathcal{S}$ and A to (more) general manifolds V with affine connections. Here one can distinguish the case of *flat* manifolds V where asymptotic leaves tend to be parallel in V. It is also useful to consider (arbitrarily fine) partitions of V into convex (or nearly convex) subsets, and then apply $g \in \mathrm{Is}(V)$ to these partitions (and to small convex subsets in V in general) for $g \to \infty$. Thus one may obtain in the limit certain foliations which are kind of duals of $\mathcal{S}$.

Finally one may try to extend the discussion to nonaffine (e.g., conformal) structures by looking at the limits of the graphs $\Gamma_i = \Gamma_{g_i} \subset V \times V$ and their lifts to the iterated Grassmann manifolds $G' = \mathrm{Gr}_n(V \times V)$, $G'' = \mathrm{Gr}_n(G')$, etc. This evokes the compactification problem for the space of solution of totally integrable (and more general elliptic) systems of partial differential equations.

7.5. Isometric actions of $T^k \times \mathbf{R}^l$. We have seen in 3.2.B that noncompact groups acting on compact real analytic pseudo-Riemannian manifolds (V, φ) essentially reduce to the actions of $G = T^k \times \mathbf{R}^k$, such that the G-orbits equal the T^k-orbits. This leads to the following.

Questions. Let numbers p, q, k, and l is given. Does there exist a pseudo-Riemannian manifold V of type (p, q) whose isometry group G equals $T^k \times \mathbf{R}^l$, such that the orbits of G equal those of the maximal torus $T^k \subset G$? When can one find such a V with prescribed topological properties, such as $\pi_1(V) = 0$ or V being homeomorphic to S^n for $n = p + q$? (See [14] for partial results.)

An analogous but simpler question is where we are already *given* an action of the torus T^k on V and seek an invariant pseudo-Riemannian metric φ whose isometry group contains a closed subgroup isomorphic to $\mathbf{R}^l$ whose action preserves the T^k-orbits (and act on every such orbit by some "rotation" of T^k; compare the twisted torus action in 6.6).

A useful invariant of the above actions of $T^k \times \mathbf{R}^l$ is the map which assigns to each point $v \in T$ the isotropy subgroup of this point in $T^k \times \mathbf{R}^l$. The level sets of such a map are totally geodesic in V and one is led to the problem of (local and global) description of totally geodesic partitions (and of their singularities) arising from such maps and more general maps with similar singularities. (The singularities appear in the above picture if the action of T^k is *not* free.) In fact, partitions into (complete) totally geodesic submanifolds of variable dimension are quite interesting for standard manifolds such as the spheres and projective spaces.

7.6. Pseudo-Riemannian Lichnerowicz conjecture. Let (V, φ) be a compact pseudo-Riemannian manifold such that the action of the full group of conformal transformations of (V, φ) admits no smooth invariant measure. Is then (V, φ) conformally flat?

The standard conformally flat manifold is the isotropic cone V in $\mathbf{R}^{p+1,q+1}$ with the natural (p, q)-conformal structure, acted upon by $\mathrm{O}(p+1, q+1)$ (with no invariant measure). If $\min(p, q) = 1$, then there are finite coverings of this V with similar properties, and one believes these are the only examples.

7.7. Nonstable manifolds and geometric invariants. Let the diffeomorphism group $\mathscr{D}$ of a (compact) manifold V act on the space Φ of geometric structures φ on V of a given type. Then the isotropy subgroup $\mathscr{D}_\varphi \subset \mathscr{D}$ is the same thing as the isometry group $\mathrm{Is}(V, \varphi)$, and noncompactness of $\mathrm{Is}(V, \varphi)$ can be thought of as the first manifestation of *nonproperness* of the action of $\mathscr{D}$ on Φ at $\phi \in \Phi$. A phenomenon similar to nonproperness is *nonstability* (according to the terminology of algebro-geometric invariant theory) which means that the quotient space $\Phi/\mathscr{D}$ is non-Hausdorff at the point $\overline{\varphi} \in \Phi/\mathscr{D}$ under φ.

To see how badly non-Hausdorff $\Phi/\mathscr{D}$ may be one should look at the space Φ_0 of flat pseudo-Riemannian tori $\mathbf{R}^{p,q}/L$ for the lattices $L \subset \mathbf{R}^{p,q}$. Two generic (irrational) tori (T^n, φ_1) and (T^n, φ_2) are not isometric (i.e., lie on different orbits of $\mathscr{D} = \mathrm{Diff}\, T^n$) but they admit arbitrarily small (rational) perturbations φ_1' and φ_2' which are isometric (by a linear map $T^n \to T^n$ sending φ_1' to φ_2').

The properties of *nonstable* rigid manifolds (V, φ) are similar to those having $\mathrm{Is}(V, \varphi)$ noncompact. The problem is to prove specific theorems in this direction. In particular, one expects a "classification" of "sufficiently nonstable" A-rigid manifolds (V, φ).

The nonstability phenomenon makes it hard to produce geometric invariants of structured manifolds (V, φ). This is well seen if we compare Riemannian and pseudo-Riemannian manifolds. A compact Riemannian manifold (V, φ) has plenty of geometric invariants, such as $\mathrm{Vol}(V, \varphi)$, $\mathrm{Diam}(V, \varphi)$, $\mathrm{Inj\,Rad}(V, \varphi)$, spectral invariants of the Laplace operator on functions and forms, etc. Now, if φ is an indefinite quadratic form, then the only obvious invariant is $\mathrm{Vol}(V, \varphi)$. Of course, one can produce some invariants using the curvature but these are *not continuous* in the C^0-topology on the space of metrics. Probably, the nonstability makes impossible any meaningful *general* theory of rigid non-Riemannian structures. (A good example is the virtual nonexistence of pseudo-Riemannian geometry in spite of the joint efforts of mathematicians and physicists.) However one may expect such a theory for certain "stable classes" of structures.

Example. Call a pseudo-Riemannian structure φ of type (p, q) *tame* if there exist closed forms ω_+ of degree p and ω_- of degree q such that ω_+ does not vanish on the p-dimensional subspaces $T \subset T(V)$ on which φ is positive, and ω_- does not vanish where φ is negative.

A special case of a tame manifold (V, φ) is that where there exist transversal foliations S_+ and S_- on V of dimensions (p, q) admitting smooth transversal measures, such that φ is positive on the leaves of S_+ and negative on those of S_-.

Notice that tame manifolds may be unstable; yet, one expects a rich geometry for such manifolds and/or similar (?) classes of other rigid manifolds.

Remarks. (a) The above notion of tameness is borrowed from (almost) complex geometry.

(b) Stability, it seems, may have other sources than tameness. For example, the C^k-smooth conformal Riemannian structures on a compact manifold V are stable for every $k = 0, 1, \ldots$. Namely, if we take the C^k-space Φ_k of conformal C^k-structures on V, then the quotient $\Phi_k/\mathscr{D}$ is Hausdorff for every $k = 0, 1, \cdots$, as a simple argument shows. (Notice that the isotropy group $\mathscr{D}_\varphi$ may be noncompact if $V \approx S^n$, but the Hausdorff property is valid just the same.)

7.8. Problems concerning local Lie group actions and Lie algebra actions. Consider a Lie algebra L and a faithful C^∞-action of L in a (small) neighborhood of a point $v \in V$. Equivalently, one may speak of the local Lie group G corresponding to L acting on V near v. We want to have a "practical" criterion for the existence of an invariant A-rigid structure (of a certain type) defined in some neighborhood of V. A perfect answer would be in terms of the infinitesimal data at v.

7.8.A. An extreme case of nonrigidity is that where the action fixes an open subset U whose closure contains v. In this case the differential of the action is trivial at v, and by Thurston's stability theorem [65] the Lie algebra L is necessarily solvable. On the other hand every solvable algebra L seems to admit such an action which may be constructed with a smooth deformation of a trivial action to a faithful one. (For the purpose of constructing an example it is sufficient to "degenerate" a faithful *linear* action.)

7.8.B. On the other end of the spectrum one finds the action where the isotropy subalgebra $L_v \subset L$ is semisimple. Then, at least in the C^{an}-case (and probably in the C^∞-case as well) the action of L_v is linearizable by the Kushnirenko theorem (see [43]). It immediately follows that there exists an L_v-invariant submanifold transversal to the L-orbit $L(v) \subset V$ and meeting this orbit exactly at v (where the L-orbit means the orbit of the corresponding group action). Therefore, there exists an L-invariant projection of a small neighborhood u of $L(v)$ to $L(v)$. Furthermore, one can easily show that the fibration $U \to L(v)$ admits an L-invariant connection which can then be turned into an A-rigid structure on U.

7.8.C. The above discussion also shows that the action in 7.8.B can be *globalized* in the following sense. There exist a manifold V' and an action of a Lie group G with the Lie algebra L, such that the action of L near some point $v' \in V'$ is C^∞-conjugate to the original action of L in V near v. It would be interesting to give more general criteria for the existence of a globalization, where L_v is not necessarily semisimple. For example, let L be compact semisimple, and let L_v integrate to a *closed* subgroup in the simply connected Lie group corresponding to L. Is then the action globalizable? In particular, is every local action of su(2) globalizible?

7.8.D. The globalization problem becomes significantly more difficult if one requires additional properties of V' and the action of G on V', such as

(i) compactness of V' and/or topological transitivity of the action;

(ii) the existence of a smooth invariant measure μ on V' such that $\mu(V') < \infty$ and/or ergodicity of the action.

Notice that the globalization problem in cases (i) and (ii) is already interesting for *linear* actions of L on $V = \mathbf{R}^n$.

A necessary condition for (ii) for A-rigid action is given in [29]. A refinement of that to a necessary and sufficient condition faces a difficulty similar to the following more classical problem. Let $V' = G/H$ be a homogeneous space where G and H are connected Lie groups. Does there exist, for given G and H, a discrete subgroup $\Gamma \subset G$ which acts properly on V' such that V'/Γ is compact? (Instead of the compactness one may require $\mathrm{Vol}(V'/\Gamma) < \infty$ for some G-invariant measure on V'.) One can expect a comprehensive answer in the case of a nilpotent group G. On the other hand, if G is semisimple and H is noncompact, one expects the negative answer apart from a few special examples (compare [7]).

7.8.E. We conclude the discussion of the local actions of L with the following remark of a general kind. The actions of some groups (such as $\mathbf{R}$) are quite "soft," and a meaningful study usually requires some genericity condition. Other groups (e.g., compact) act in a rigid fashion, and one studies individual actions rather than representatives of generic classes. The question is how one can combine the two approaches for the study of local actions of general Lie groups and algebras.

7.9. Problems concerning the topology of G-manifolds. Here, a G-manifold means a connected manifold V with a smooth (or real analytic) faithful action of a connected Lie group G. The general question concerns the relations between the structure of the action (and G itself) and the topology of V. This has been extensively studied for compact Lie groups but almost nothing is known for noncompact G. Let us indicate specific questions.

7.9.A. Given G, what is the minimal dimension of a *compact* G-manifold V? One also asks this question in the presence of a smooth G-invariant measure on V.

7.9.B. Let $H \subset G$ be the maximal compact subgroup, and V be an H-manifold of sufficiently large dimension $n = n(G)$. Does the action of H on V extend to that of G? (To avoid local trouble one may assume H acts freely on V.)

7.9.C. Which (compact) manifolds V admit stratified (rigid) actions of G?

A great number of examples come from algebraic geometry. Namely if $G \subset \mathrm{GL}(n+1)$ is an algebraic group, one takes the Zariski closure V_0 of some G-orbit of the action on $\mathbf{R}P^n$ and then takes some equivariant non-singular resolution of V_0 for V. Thus one may produce many stratified actions of $\mathbf{R}^2$ on surfaces.

7.9.C1. Does every compact n-dimensional manifold V admit a faithful stratified C^{an}-action of $\mathbf{R}^n$ or, more generally, of a given simply connected solvable group G? When, moreover, may one have such an action with *finitely many* orbits and/or with *only one* n-dimensional orbit?

7.9.C2. Let $V_0 = G/H$ be a homogeneous space. Does there exist a compact G-manifold V containing V_0 as an open (and dense) subset? Can one take V compact with boundary such that $\operatorname{Int} V = V_0$? (Of course, the answer may be "yes" or "no" depending on G/H.)

7.9.D. It seems that G-spaces are easier to construct for groups G of subexponential growth (e.g., for nilpotent groups G) than for the exponential growth. The first "difficult" group is the (solvable) group G of the affine automorphisms of the line. The Lie algebra of this group G is 2-dimensional with X and Y satisfying $[X, Y] = 2X$. If X and Y are vector fields on V satisfying this relation, then the diffeomorphisms generated by Y, say $f_\lambda : V \to V$, send X to the multiple λX for all $\lambda > 0$. This makes the subgroup $H = \mathbf{R}$ generated by X very special (parabolic). For example, the action of H has "polynomial growth," that is, the norm of the differential of h satisfies the bound $\|\mathscr{D}_h\| \leq h^d$ for some $d > 0$ and all sufficiently large $h \in H = \mathbf{R}$.

7.9.E. If G is a *semisimple* group, the list of known compact G-spaces is rather limited, but yet there are no known topological restrictions on V (apart from those which come from the maximal compact subgroup). Compact G-manifolds look especially rigid if there is a smooth G-invariant measure on V. In this case one expects (on the basis of the known examples) that the fundamental group $\pi_1(V)$ is roughly as large as G. Moreover, one might think that the lifted action of G to the universal covering of V should be proper.

7.9.E1. If the $\mathbf{R}$-rank of G is ≥ 2, and G is simple, then one may even hope that every compact G-manifold with a smooth measure and ergodic action is isomorphic to a standard homogeneous example (see §6). On the other hand for $\mathbf{R}$-rank $= 1$ (and especially for $G = \mathrm{SL}_2\mathbf{R}$) one expects a variety of nonhomogeneous examples.

The notion of a G-manifold with semisimple G of $\mathbf{R}$-rank $= 1$ is similar to that of a foliation with a metric of negative curvature along the leaves. Here G-invariant measures should be replaced by *transversal measures*, and in the presence of these the underlying manifold must be (as one naively believes) rather special. For example, the fundamental group is expected to have exponential growth.

7.10. Problems concerning the structure of Diff. Let V be a compact smooth manifold, and let $\mathrm{Diff}^r(V)$ denote the group of C^r-diffeomorphisms of V for $r = 1, 2, \cdots, \infty$, an and also $r = \mathrm{alg}$, which means the group of real algebraic maps for a fixed (local) real algebraic structure on V. Notice that $\mathrm{Diff}^{\mathrm{alg}}$ is *not* an algebraic group. In fact $\mathrm{Diff}^{\mathrm{alg}}$ is infinite dimensional for all V of positive dimension as the algebraic degree of $f \in \mathrm{Diff}^{\mathrm{alg}}$ may be arbitrarily large.

7.10.A. Now we want to look at Diff as if it were a linear group (i.e., a subgroup in some GL_N). For example we want to define the notion of (*quasi*)-*unipotency* of an $f \in \mathrm{Diff}$. Here are some possibilities.

(U_1) The spectrum of the differential D of f acting on the linear space of vector fields on V (of given regularity class) consists of the single point $\{1\}$.

(QU_1) The spectrum of the above D lies on the unit circle.

(U_2) The *adjoint orbit* of f, that is, $\{\varphi f \varphi^{-1}\} \subset \mathrm{Diff}$ for all $\varphi \in \mathrm{Diff}$, contains the identity in its closure.

(QU_2) For every $\varepsilon > 0$ there exists a Riemannian metric g on V, such that

$$-\varepsilon g \leq g - f^*(g) \leq \varepsilon g.$$

Here it is useful to think of the space G of the Riemannian metrics on V as an infinite-dimensional manifold of *nonpositive curvature*, where f acts as an isometry. Then the above expresses the following property of the *displacement function* $\delta(g) = \mathrm{dist}(g, fg)$,

$$\inf_{g \in G} \delta(g) = 0.$$

(Of course, the specific content of this discussion depends on the choice of a particular distance function in G. Geometrically the most attractive is the one corresponding to the natural Riemannian metric on G.)

The elementary geometry of nonpositive curvature tells us that the above condition $\inf \delta(g) = 0$ is equivalent to

$$i^{-1} \mathrm{dist}(g, f^i g) \to 0 \quad \text{for every } g \in G \text{ and } i \to \infty,$$

where f^i is the ith iterate of f. Now, we observe that $\mathrm{dist}(g, f^i g)$ is bounded (for most natural definitions of dist) by const $\log \|Df^i\|$, and so the above (dist $\to 0$)-property is insured if $\|Df^i\|$ has subexponenital growth. This bound on growth is strengthened by our next definition.

(QU_3) The norm of the differential of the ith iterate of f, that is $\|Df^i\|$, grows at most polynomially in i (i.e., $\leq \mathrm{const}^i$) for $i \to \infty$. The same growth condition can be applied to the higher-order differentials of f^i, and for $f \in \mathrm{Diff}^{\mathrm{alg}}$ one may require the polynomial growth of the algebraic degree $\deg f^i$.

Notice that the specific norm $\|Df^i\|$ depends on the choice of some Riemannian metric on V, but the *equivalence class* of the sequence $\|Df^i\|$ is independent of this choice. (Here and below two positive functions (or sequences) $a(i)$ and $b(i)$ for some variable $i \in I$ are called *equivalent* if the ratio $a(i)/b(i)$ is contained in an interval $0 < \varepsilon \leq a(i)/b(i) \leq \delta < \infty$ for all $i \in I$.) Similarly, the algebraic degree is defined up to such equivalence.

What one wants to know is the relation between these definitions and their immediate variations. Also one asks for simple criteria for (quasi-)unipotency.

7.10.B. **Example.** Suppose there exists $\varphi \in \mathrm{Diff}$, such that $\varphi f \varphi^{-1} = f^2$. Then clearly f satisfies (QU_3). On the other hand, if $f \neq \mathrm{Id}$, then φ itself, probably, does not satisfy (QU_3). (This is clear if f is contained in a one-parameter group generated by a field X on V, such that φ transforms X to $2X$. In this case the spectrum of df contains 2, which is incompatible with (QU_3).)

7.10.C. Property (QU_3) leads to the following.

Growth problem. What is the possible equivalence class of the sequence $\|Df^i\|$? For example, if $\|Df^i\|$ grows very slowly (e.g., like log log log i), does it then follow that $\|Df^i\|$ is, in fact, bounded for $i \to \infty$? (A. Katok pointed out to us that the constructions in [2 $\frac{1}{2}$] provide diffeomorphisms f with a slow growth of $\|Df^i\|$. Yet such diffeomorphisms must be quite rare.)

This question may be easier for the differentials $D_r f^i$, $r \geq 2$, and for alg deg f^i. Another simplified question is whether a slow growth allows a small *periodic* perturbation f' of f. For example, let f be generated by a nonzero vector field X on V. Does the slow growth imply V admits a nontrivial circle action? In particular, does the simplicial volume of V vanish? (Compare [22].) In fact, the following strong recurrency property (which goes along with the slow growth condition) may suffice: The one-parameter subgroup generated by X is *not* closed in Diff.

7.10.D. Notice that the (strong) recurrency pattern of f^i can be measured by $\mathrm{dist}_r(f^i, \mathrm{Id})$ for the (naturally) defined C^r-distance on Diff_r. Here again one may ask what are possible equivalence classes of the sequences $\mathrm{dist}_{r_{f^i}}(f^i, \mathrm{Id})$. Of course, most diffeomorphisms f generate *discrete* (cyclic) subgroups in Diff, and so the above dist_r never approaches zero. However, if we take several diffeomorphisms $f_1, \cdots, f_k$ of V, then (the equivalence class of) the induced distance function on the free group on k-generators, $F_k \to \mathrm{Diff}$, becomes quite interesting as it approaches zero.

7.10.E. Take a (Riemannian) metric g on V, act on it by f^i, and let

$$g_j = \sum_{i=0}^{j} f^i g.$$

(Sometimes one takes $\sup_i$ rather than $\sum_i$, but the asymptotic properties of $\sup_i$ and $\sum_i$ are essentially the same.) Then we pick up some geometric invariant of the metric space (V, g_i) and obtain, by sending $i \to 0$, an invariant of f.

Examples. (a) Let N_ε^j denote the minimal number of ε-balls needed to cover (V, g_j). Then one defines the *topological entropy*

$$\operatorname{ent}_\varepsilon f = \limsup_{j\to\infty} j^{-1} \log N_\varepsilon^j ,$$

and

$$\operatorname{ent} f = \lim_{\varepsilon\to 0} \operatorname{ent}_\varepsilon f.$$

(b) The volume and the diameter of (V, g_j), say Vol_j and Diam_j, lead to the invariants closely related to the entropy, that is,

$$\limsup_{j\to\infty} j^{-1} \log \mathrm{Vol}_j ,$$

and

$$\limsup_{j\to\infty} j^{-1} \log \mathrm{Diam}_j$$

(c) There is an enumerable amount of various geometric invariants besides the volume and diameter to which the above applies. Here are some which look especially attractive.

The *k-dimensional width* of (V, g_j) for $k = 0, 1, \cdots, n = \dim V$ (see [23], [28]).

The *spherical radius* of (V, g_j), that is, the minimal radius of S^n which receives a distance decreasing map $(V, g_j) \to S^n$ of nonzero degree.

The *spectral invariants* of (V, g_j), such as the number N_λ^j of eigenvalues in the interval $[-\lambda, \lambda]$ of some operator $\mathscr{D}$ associated to g_j. For example, $\mathscr{D}$ may be the Laplace operator on functions, the operator $d+\delta$ on forms or the Dirac operator. Then the *spectral entropy* is

$$\lim_{\lambda\to\infty} \limsup_{j\to\infty} j^{-1} \log N_\lambda^j .$$

An important feature of the spectral invariants is their essential *linearity*. They can be expressed solely in terms of the linear operators $\mathscr{D}$ and f^* (induced by f) on an appropriate Hilbert space $\mathscr{H}$.

Given this variety of invariants of f one wants to know the basic relations between them. Some relations follow from pure geometry where the special origin of g_j plays no role. For example, the spherical radius gives a lower bound for N_λ for $\mathscr{D} = d + \delta$ and for the Dirac operator but not for the Laplace operator (see [24]). Furthermore, the work by Yomdin on the Shub entropy conjecture (see [26] and references therein) suggests further relations which do take into account the nature of g_j (which reflects, for example, the degree of regularity of f).

7.10.F. Let us consider the Cartesian product $W = V \times \mathbf{Z}_+$ with the *maximal* metric g_+, such that

(i) for every $v \in V$ the embedding
$$\mathbf{Z}_+ = (v \times \mathbf{Z}_+) \to V \times \mathbf{Z}_+$$
is distance nonincreasing;

(ii) for every $j \in \mathbf{Z}_+$ the embedding $(V, g_j) \to (V \times j) \subset V \times \mathbf{Z}_+$ is also distance nonincreasing.

Then the geometric invariants of $W = (V \times \mathbf{Z}_+, g_+)$ give us invariants of f. This is especially interesting for one-parameter group actions (rather than $\mathbf{Z}$ generated by f), as a continuous time reparametrization of an action does not change the *quasi-isometry* class of g_+. Then the quasi-isometry invariants of g_+ give us invariants of the (one-dimensional) orbit foliation. An example of such an invariant is the L_p-*cohomology* group $L_pH^k(W, g_+)$, where $W = V \times \mathbf{R}_+$ in the one-parameter group case. Here the L_p-cohomology is defined with ε-coverings where ε eventually goes to zero.

Example. If we apply the above to the geodesic flow of a compact manifold X of negative curvature, then the space W is quasi-isometric to the universal covering $\tilde{X}$ of X. In this case the L_p-cohomology can be equally defined with differential forms on $\tilde{X}$, and some computations of these can be found in [61]. Yet, even here our knowledge of L_pH^k is far from complete.

Remark. Forgetting the parametrization and/or passing from a group action to that of the (local) pseudogroup formed by the return maps in a neighborhood of a given point corresponds to the passage from the metrical to the *conformal* geometry. In the case of a space Y of negative curvature (as the above $\tilde{X}$) this conformal geometry lives at the ideal boundary (or the sphere at infinity) of Y.

7.10.G. The above constructions provide us with certain *invariants* of Diff that are functions Diff $\to \mathbf{R}$ which are invariant under conjugation. However, these functions usually are not continuous (though some of them are semicontinuous, as, for example, the topological entropy by the Yomdin theorem). In fact, it is hard to produce continuous invariants of Diff as the space Diff/conjugations is highly non-Hausdorff. Yet this space is Hausdorff at some points (e.g., at those corresponding to structurally stable diffeomorphisms), and one asks when two given diffeomorphisms f_1 and f_2 can be distinguished by a continuous invariant Diff $\to \mathbf{R}$. But even if f_1 and f_2 are "continuously indistinguishable," i.e., if there are arbitrarily small perturbations f_1' and f_2' which are mutually conjugate, may one still study f_1 and f_2 by means of "slightly discontinuous" invariants if the "amount" of the conjugate pairs (f_1', f_2') close to (f_1, f_2) is "not too great"?

7.10.H. *The symplectic case.* Let $V = (V, \omega)$ be a symplectic manifold with $H^1(V; \mathbf{R}) = 0$. Then every symplectic vector field ∂ on V is given by a (Hamilton) function $H: V \to \mathbf{R}$ which is unique up to an additive constant. This H provides nontrivial invariants of ∂. The simplest is the oscillation of H,

$$\operatorname{osc} H = \sup H - \inf H.$$

Further invariants are those of the push-forward measure $H^*(\mu)$ on the real line, where μ is the measure on V associated with the top exterior power of the (symplectic) form ω. Also there are more sophisticated symplectic (*width* or capacity) invariants of the subsets $H^{-1}[a, b] \subset V$ for $[a, b] \in \mathbf{R}$.

For example if $\partial \neq 0$, then also $\operatorname{osc} H \neq 0$ and so the one-parameter subgroup generated by ∂ cannot be unipotent in the sense of (U_2), in the case where V is compact. In particular, such a V admits no symplectic action of any noncompact semisimple Lie group G and also of the (solvable) group of the affine automorphisms of $\mathbf{R}$.

Hofer (see [36]) recently found a remarkable extension of the above oscillation from the Lie algebra to the group Symp Diff. His work shows, in particular, that in many cases the connected identity component of Symp Diff(V, ω) contains no (U_2)-element besides Id.

7.11. Problem concerning finitely generated subgroups in Diff. We want to find specific algebraic properties of a finitely generated group Γ which admits a faithful (or at least nontrivial) C^r-action on a compact manifold V, for a given $r = 1, 2, \cdots, \infty, \text{an}, \text{alg}$. One may ask the same question with an additional restriction on the action such as the following.

(i) $\dim V \leq k$.

(ii) The action is *rigid.*

(iii) $\Gamma \subset \operatorname{Diff}^r$ is a *discrete* subgroup.

(iv) The actions of each $\gamma \in \Gamma$ is (quasi-)unipotent in the sense of some (QU)-property discussed in 7.10.

(iv)$'$ The group $\Gamma \subset \operatorname{Diff}$ is (quasi-)unipotent as a whole in an appropriate sense. For example, one may assume that for each $\varepsilon > 0$ there exists a Riemannian metric g on V which is moved at most by ε under the action of the generators of Γ. Alternatively one may impose a bound on the size of $\|D\gamma\|$ in terms of the word length of $\gamma \in \Gamma$.

7.11.A. The properties of $\Gamma \subset \mathrm{Diff}$, one may expect by the analogy with linear groups, are as follows.

(a) residual finiteness of Γ or at least the existence of a single subgroup Γ' of finite index > 1;

(b) the solvability of the word problem;

(c) nongenericity of Γ.

Let us explain what genericity means. Take some generators $\gamma_1, \cdots, \gamma_p$, and let $r_1, r_2, \cdots, r_q, \ldots$ be an infinite sequence of relations where r_q is a "randomly chosen" word of sufficiently large length l_q. If l_q is $\geq$ const.log$q$, then the group $\Gamma = \{\gamma_1, \cdots, \gamma_p | r_1, r_2, \ldots\}$ is known to be quite large (e.g., having exponential growth, nonamenable, etc., see [27]). On the other hand, Γ has no nontrivial linear representation. Indeed, representations $\Gamma \to \mathrm{GL}_N$ given by solutions of the algebraic(!) equations imposed by $r_1 = 1, r_2 = 1, \ldots$ on matrices $M_i = 1$, $i = 1, \cdots, p$ in GL_N, and an easy argument shows that, generically, there is only one solution $M_i = 1$, $i = 1, \cdots, p$. If we fix N, then we only need finitely many relations. In fact one may think that generic groups $\Gamma = \{\gamma_1, \cdots, \gamma_p | r_1, \cdots, r_q\}$, where $p \geq 2$ and q is much greater than p, admit no monomorphisms into $\mathrm{GL}_\infty = \bigcup_N \mathrm{GL}_N$.

7.11.B. Let us see what the chances are of having (a), (b), and (c) for $\Gamma \subset \mathrm{Diff}$. The residual finiteness property seems hard to get even for $\mathcal{A}$-rigid actions.

Property (b) needs an algorithm to decide when a "given" diffeomorphism equals the identity.

Property (c), that is, nongenericity of Γ, seems the most realistic of the three. In fact, it appears quite easy for $\mathrm{Diff}^{\mathrm{alg}}$ which is, after all, a countable union of algebraic varieties. On the other hand one can probably show without much work that the p-tuples of C^r-diffeomorphisms $f_1, \cdots, f_p$, satisfying sufficiently many generic relations $r_i = \mathrm{Id}$ (thought of as difference equations), form a "rather rare" subset in the space $\underbrace{\mathrm{Diff}^r \times \cdots \times \mathrm{Diff}^r}_{p}$, provided r is sufficiently large. (Many generic groups Γ embed into $\mathrm{Diff}^0(S^4)$ and some may even embed into $\mathrm{Diff}^1(S^4)$. In fact these groups often appear as fundamental groups of singular 2-dimensional spaces of negative curvature which can be "thickened" to 5-dimensional manifolds with 4-dimensional ideal boundaries $\approx S^4$. Thus one gets $\Gamma \subset \mathrm{Diff}^0 S^4$. Furthermore, whenever one can insure the 1/4-pinching of the curvature, one obtains $\Gamma \subset \mathrm{Diff}^1 S^4$.) What looks difficult, however, is to show that this rare subset reduces to a single point, namely, $\{f_i = \mathrm{Id}, i = 1, \cdots, p\}$.

7.11.C. One can generalize the above discussion by replacing the free group on p generators by an arbitrary group Γ_0, and then consider factor groups Γ of Γ_0 obtained by adding relations $r_1 = 1, r_2 = 1, \cdots$, for some $r_1, r_2, \ldots$ in Γ_0. If Γ_0 is *hyperbolic*, then, generically, Γ also is hyperbolic and roughly of the same size as Γ_0 (see [27]). But if Γ_0 is nonfree, Γ is even less likely to lie in Diff. For example, if Γ_0 is a lattice in $\mathrm{Sp}(n, 1)$, then even the presence of a single relation may suffice. Moreover, one may expect that every noninjective homomorphism $\Gamma_0 \to \mathrm{Diff}$ has finite image. (For homomorphisms $\Gamma_0 \to \mathrm{GL}_N$ this is true by Corlette's theorem [12].)

7.11.D. Many examples of groups with the nonsolvable word problem are obtained with the use of amalgamated products and the HNN-construction. It seems that these constructions often lead to groups Γ nonrepresentable in Diff, even if the word problem is solvable.

Example. Take Γ generated by $\gamma_1, \gamma_2, \gamma_3$ with two relations $\gamma_2\gamma_1\gamma_2^{-1} = \gamma_1^2$ and $\gamma_3\gamma_2\gamma_3^{-1} = \gamma_2^2$. The first relation suggests that γ_2 is hyperbolic, while the second implies that γ_2 is parabolic (compare 7.10.B) which makes an embedding $\Gamma \subset \mathrm{Diff}$ very unlikely. (By the discussion in 7.10.B this is the case if γ_1 is contained in a one-parameter subgroup which is conjugated by γ_2 into itself.)

7.11.E. Let us indicate some properties of Diff which may be used to restrict $\Gamma \subset \mathrm{Diff}$. First of all, the algebraic topology provides a bound on finite subgroups in Diff. For example the group S_∞ of permutations of the integers with compact support embeds in no Diff. On the other hand S_∞ embeds into some *finitely generated* (even finitely presented) group Γ which then also does not embed into Diff. (This class of examples was pointed out to me by Michel Herman many years ago.)

7.11.E1. An important property of linear groups G (and Lie groups in general) is the following commutation inequality for the distance $\|g\| = \mathrm{dist}(g, \mathrm{Id})$,

$$(*) \qquad \|[g_1, g_2]\| \le \mathrm{const}\|g_1\| \cdot \|g_2\|,$$

which holds true for all g_1 and g_2 in G. This inequality implies that successive commutators of small (i.e., close to Id) elements in G converge to zero, and so *every discrete subgroup $\Gamma \subset G$ generated by small elements is nilpotent.* (This is due to Zassenhaus.)

If we try to generalize $(*)$ to the C^r-distance in Diff^r, we encounter the loss of one derivative,

$$(+) \qquad \|[g_1, g_2]\|_{r-1} \le \mathrm{const}\|g_1\|_r \cdot \|g_2\|_r,$$

which makes the above conclusion unlikely for Diff^∞. Yet some (solvable, if not nilpotent) version of that may remain valid for $\mathrm{Diff}^{\mathrm{an}}$ where the loss of derivatives is compensated by the quadratic shape of the right-hand side of $(+)$. (Notice that the sequence of functions $f_{i+1}(t) = (df_i/dt)^2$ converges to zero if it starts with a "sufficiently small" analytic function f_0.) In particular, this may be useful in the study of C^{an}-deformations of the trivial representation $\Gamma \to \mathrm{Diff}$.

7.11.E2. **Remark.** If the trivial representation of Γ to Diff admits an arbitrarily small C^1-deformation to a faithful (i.e., injective) representation $\Gamma \to \mathrm{Diff}^1$, then by Thurston's stability theorem every finitely generated subgroup $\Gamma' \subset \Gamma$ admits a nontrivial homomorphism into $\mathbf{R}$.

7.11.E2. If Γ admits a *rigid* action on V, then Γ can be *uniformly metrically embedded* (*placed* in the sense of [29]) into some Lie group. This gives a nontrivial restriction on (the "size" of) the groups Γ in $\mathrm{Diff}^{\mathrm{rigid}}$. Thus one asks if there is a similar restriction for the existence of a discrete (rather than rigid) representation $\Gamma \hookrightarrow \mathrm{Diff}$. If so, one could insure nondiscreteness of certain groups $\Gamma \subset \mathrm{Diff}$, and then try to apply the consideration of 7.11.E1 to the elements $\gamma \in \Gamma$ close to $\mathrm{Id} \in \mathrm{Diff}$.

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