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# FLIPS, FLOPS, MINIMAL MODELS, ETC.

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One of the most interesting and profound developments of algebraic geometry in the past ten years is the "minimal model program," also called "Mori's program." The aim of the program, originating in [78], can be summarized as follows:

Let $X$ be a smooth projective algebraic variety. One would like to construct another algebraic variety $X'$ such that:

(i) $X'$ is obtained from $X$ by a series of simple "surgery type" operations, and

(ii) the global structure of $X'$ is simple.

The usefulness of the program depends on how well we understand the "surgery type" operations and how simple the structure of $X'$ is.

In dimension one the program does not exist; every smooth compact curve is as simple as possible. In dimension two one recovers the construction of minimal models of smooth surfaces, which has already been done by the Italian geometers around the turn of the century.

For a long time it was believed that a similar program is impossible in higher dimensions. The main reason behind this belief was that $X'$ cannot be chosen to be smooth. Only beginning with the works of Reid [96] and Mori [78] did it become clear that by allowing certain singularities the local structure becomes only a little more complicated while the global structure becomes much simpler.

After this conceptual obstacle was removed, the hardest part of the program turned out to be to show the existence of certain special "surgery type" operations. This was finally completed in dimension three by Mori [82] and is still unknown in higher dimensions.

There have been several survey articles recently about the program. [54] is aimed at a very general readership. [120] assumes some familiarity with algebraic geometry while [49] is aimed at those who wish to become experts. The booklet [13] grew out of a seminar aimed at advanced graduate

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students; Chapters 8–13 present the simplest known proof of (1.4.2–7). [57] is intended as an introduction to [82].

In this article I try to be self-contained without reproducing earlier surveys. §1 discusses the structure of Mori's program mostly without proofs. The most difficult "surgery type" operations are discussed in §2. These are called flip and flop. After a general discussion of the codimension two surgery problem in §2.1 a detailed description of flops is presented in §2.2. The description is very explicit, and this is useful for applications. §2.3 contains some examples and results concerning flips. See [57] for a more detailed introduction.

The applications of the program are presented in §3 together with several open problems and conjectures.

The remaining sections are more independent of the main body of the program and of each other. Each one presents old and new results, conjectures, and speculations centered around a question which originated in the minimal model program.

By [78] every step in the construction of $X'$ is related to some simple geometric configuration inside $X$. However, it is not clear when a similar-looking configuration corresponds to a step of the program. This leads to some very interesting questions and examples which are discussed in §4.

It turns out that flips and flops also play a crucial role in understanding proper but nonprojective varieties. This approach leads to simplifications of several results and to numerous new problems. These can be found in §5.

§6 is devoted to conjectures about deformations of rational surface singularities that grew out of studying minimal models of threefolds. These conjectures are quite interesting themselves, and a conceptual proof of them may lead to a better understanding of flips.

The groundfield is the field $\mathbb{C}$ of complex numbers, unless the contrary is stated at the beginning of a section.

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## 0. Notation and terminology

1. A line bundle will always be locally free in the Zariski topology. This is important for spaces like $\mathbb{C}^2 - 0$ since $\operatorname{Pic}_{\text{Zariski}}(\mathbb{C}^2 - 0) = 0$ but $\operatorname{Pic}_{\text{Euclidean}}(\mathbb{C}^2 - 0)$ is infinite dimensional.

2. A line bundle $L$ on an $n$-dimensional proper irreducible variety or complex space $X$ is called *big* if $h^0(X, L^{\otimes m}) > \text{const}\cdot m^n$ for $m \gg 0$.

3. A line bundle $L$ on a scheme $X$ is called *nef* if for every compact curve $C \subset X$ we have $\deg(L|C) \geq 0$.

4. A normal variety $X$ has $\mathbb{Q}$-*factorial* singularities if for every codimension-one subvariety $V \subset X$ there is an integer $m > 0$ such that $mV$ is locally definable by one equation. The main consequence of this is that every codimension-one subvariety will have a cohomology class in $H^2(X, \mathbb{Q})$.

5. *Terminal* and *canonical* singularities are defined in (1.3.2). *Du Val* singularities are defined in (1.3.3.2).

6. *Minimal models* and *relative Fano models* are defined in (1.4.8).

7. *Curve neighborhoods* are defined in (2.1.1) and the *opposite* in (2.1.5).

8. *Flops* are defined in (2.2.1) and *flips* in (2.3.1).

9. If $I \subset \mathscr{O}_X$ is an ideal sheaf, then $B_I X$ denotes the blow-up of $I$. If $Z \subset X$ is a closed subvariety, then $B_Z X$ denotes the blow-up of the ideal sheaf of $Z$.

10. The word *morphism* will be reserved for maps that are everywhere defined. In diagrams they will be denoted by a solid arrow: $\rightarrow$. Maps that need not be everywhere defined will be indicated by a broken arrow: $\dashrightarrow$.

11. Let $X$ be a smooth proper variety or complex manifold. Let $K_X$ denote the canonical line bundle. For every $m > 0$ the global sections of $K_X^{\otimes m}$ define a meromorphic map

$$X \dashrightarrow I_m(X) \subset \mathbb{P}(H^0(X, K_X^{\otimes m})).$$

If $m$ is sufficiently large and divisible, then $\dim I_m(X)$ is independent of $m$. This number is called the *Kodaira dimension* of $X$ and is denoted by $\kappa(X)$. We set $\kappa(X) = -\infty$ if $I_m(X) = \emptyset$ for every $m > 0$.

12. If $(x_1, \cdots, x_n)$ is a coordinate system on $\mathbb{C}^n$, then the symbol $\mathbb{C}^n/\mathbb{Z}_m(a_1, \cdots, a_n)$ denotes the quotient of $\mathbb{C}^n$ by the group action $(x_1, \cdots, x_n) \mapsto (\zeta^{a_1}x_1, \cdots, \zeta^{a_n}x_n)$, where $\zeta$ is a primitive $m$th root of unity.

## 1. Construction of minimal models

**1.1. Introductory remarks.** Let $C$ be a smooth proper algebraic curve over $\mathbb{C}$ (equivalently, a compact Riemann surface). $C$ can be endowed with a metric of constant curvature, and one has the following classification according to the sign of the curvature ($\mathbb{H}$ denotes the upper half plane):

| curvature | structure |
|---|---|
| positive | $\mathbb{CP}^1$ |
| zero | $\mathbb{C}/\mathbb{Z}^2$ |
| negative | $\mathbb{H}/\pi_1(C)$ |

In higher complex dimensions there are several possible curvatures to consider. The holomorphic bisectional curvature is a very strong invariant, and there are only few varieties where it can be semidefinite.

**1.1.1. Theorem** [77], [107]. *Let $X$ be a compact complex Kähler manifold. Assume that $T_X$ admits an Hermitian metric whose holomorphic bisectional curvature is everywhere positive. Then $X \cong \mathbb{P}^n$.*

Much less is known about manifolds that admit an Hermitian metric whose holomorphic bisectional curvature is everywhere negative. It was conjectured in [122, Problem 35] that their universal cover is always Stein. The following result indicates that this may not be true:

**1.1.2. Theorem** [68], [76]. *There are simply connected algebraic surfaces $X$ such that $\Omega_X^1$ is ample.*

For such a surface $X$ some symmetric power of $T_X$ admits an Hermitian metric whose holomorphic bisectional curvature is everywhere negative. However it is not clear whether $T_X$ itself admits such a metric.

One can take the trace of holomorphic bisectional curvature to get the Ricci curvature of the tangent bundle. I prefer to think of it as the curvature of the determinant of the tangent bundle. These two approaches are not completely equivalent since a metric on $\det T_X$ may not always be liftable to a metric on $T_X$.

In algebraic geometry it is customary to consider the canonical bundle:

$$K_X \stackrel{\text{def}}{=} (\det T_X)^*$$

The sheaf of local sections of the canonical bundle is called the dualizing sheaf and is denoted by $\omega_X$. I will try to be systematic, and say canonical bundle when I mean a line bundle and dualizing sheaf when I mean a sheaf.

Dualizing changes the sign of the curvature, creating the possibility of confusion.

Even in dimension two it is not true that every variety has a semidefinite canonical bundle, but the exceptions are easy to enumerate:

**1.1.3. Theorem.** *Let $X$ be a smooth proper algebraic surface. If $K_X$ does not admit a metric $h$ whose curvature $\Theta = \bar{\partial}\partial \log h$ is semipositive, then there is a morphism $f: X \to Y$ which is one of the following types:*

(1.1.3.1) *$Y$ is a smooth surface, and $X$ is obtained from $Y$ by blowing up a point.*

(1.1.3.2) *$Y$ is a smooth curve, and $X$ is a $\mathbb{P}^1$ bundle over $Y$.*

(1.1.3.3) *$Y$ is a point and $X \cong \mathbb{P}^2$.*

*In all cases there is an embedded copy of $\mathbb{P}^1 \cong C \subset X$ such that $(\sqrt{-1}/2\pi)\int_C \Theta < 0$.*

These cases are very different in nature. (1.1.3.2) and (1.1.3.3) are very precise global structural statements. One can hardly wish for more. (1.1.3.1) merely identifies (and removes) a small part of $X$ and gives no global information. The corresponding morphism $f\colon X \to Y$ is called the contraction of a $(-1)$-curve. This is the simplest example of an algebraic surgery operation.

The contraction (1.1.3.1) introduces a new surface $Y$ which is simpler than $X$ since $b_2(Y) = b_2(X) - 1$. We can apply (1.1.3) to $Y$ and continue if possible. This gives the following:

**1.1.4. Theorem.** *Let $X$ be a smooth proper algebraic surface. Then there is a sequence of contractions $X \to X_1 \to \cdots \to X_n = X'$ such that $X'$ satisfies exactly one of the following conditions*:

(1.1.4.1) *$K_{X'}$ admits a metric whose curvature is semipositive.*

(1.1.4.2) *$X'$ is a $\mathbb{P}^1$-bundle over a curve $C$.*

(1.1.4.3) *$X' \cong \mathbb{P}^2$.*

Based on this presentation and using good hindsight we can formulate the aim of the minimal model program:

**1.1.5. Hope.** Let $X$ be a smooth, projective algebraic variety. Then there are certain "elementary surgery operations" such that repeated application of them produces a variety $X'$ and

either: $K_{X'}$ admits a metric whose curvature is semipositive,

or: there is a structure theorem for $X'$.

This has been achieved only in dimension three, and there are many surprises along the way.

**1.2. Extremal rays on smooth varieties.** Curvature assumptions are very difficult to handle in algebraic geometry. The following observation leads to a slightly different notion, which is easier to deal with.

Let $L$ be a line bundle on a complex manifold $M$ with metric $h$ and curvature $\Theta$, and let $C \subset M$ be any proper curve. Then

$$c_1(L) \cap C = \frac{\sqrt{-1}}{2\pi} \int_C \Theta .$$

We will denote this number by $C \cdot L$. In particular, if $\Theta$ is semipositive, then $C \cdot L \geq 0$ for every $C$.

**1.2.1. Definition.** A line bundle $L$ on a variety $X$ is called nef if $C \cdot L \geq 0$ for every compact curve $C \subset X$. (This replaces the earlier confusing terminology "numerically effective".)

It is conjectured that for the canonical line bundle being nef is equivalent to admitting a metric with semipositive curvature. In general however these two notions are slightly different.

The intersection product $C \cdot L$ depends only on the homology class of $C$ but not on $C$ itself. With this in mind we introduce:

**1.2.2. Definition.**

(1.2.2.1) Let $X$ be a smooth projective variety over $\mathbb{C}$. The cone of curves of $X$—denoted by $NE(X)$—is the convex cone generated by the homology classes of effective curves in $H_2(X, \mathbb{R})$, where $N$ stands for "Numerical equivalence", and $E$ for "Effective".

The closed cone of curves or the Kleiman-Mori cone of $X$—denoted by $\overline{NE}(X)$—is the closure of $NE(X)$ in $H_2(X, \mathbb{R})$.

The definitions of course make sense for any complex manifold $X$.

(1.2.2.2) If $X$ is a singular variety, then instead of $H_2(X, \mathbb{R})$ one can use

$$N_1(X) \stackrel{\text{def}}{=} \frac{\{1-\text{cycles with real coefficients}\}}{\{\text{numerical equivalence}\}} \cong (\operatorname{Pic} X \otimes \mathbb{R})^* .$$

For smooth varieties (or for varieties with rational singularities) over $\mathbb{C}$ there is a natural linear embedding $N_1(X) \hookrightarrow H_2(X, \mathbb{R})$, and under this identification the corresponding cones are the same.

(1.2.2.3) If $L$ is a line bundle or a Cartier divisor, then taking cap product determines a linear map

$$L^{\cap} \colon N_1(X) \to \mathbb{R} .$$

Reformulating (1.2.1) we obtain that $L$ is nef iff $L^{\cap}|NE(X)$ is semipositive.

**1.2.3. Definition.**

(1.2.3.1) Let $V \subset \mathbb{R}^m$ be a convex cone. A subcone $W \subset V$ is called extremal if

$$u, v \in V, u+v \in W \Rightarrow u, v \in W .$$

Informally: $W$ is a face of $V$.

(1.2.3.2) A one-dimensional subcone is called a ray. A ray is of the form $\mathbb{R}^+ v$ for some $v \in V$.

(1.2.3.3) Let $X$ be a smooth proper algebraic variety. A ray $\mathbb{R}^+[Z] \subset \overline{NE}(X)$ is called a $K_X$-negative extremal ray if it is extremal and $K_X^{\cap} \colon \mathbb{R}^+[Z] \to \mathbb{R}$ is strictly negative on $\mathbb{R}^+[Z] - \{0\}$. If no confusion is likely, we will use "extremal ray" to mean a $K_X$-negative extremal ray.

The first step toward (1.1.5) is to understand the part of $\overline{NE}(X)$ where $K_X^\cap$ is negative. It turns out to have a relatively simple structure:

**1.2.4. Theorem** [78, 1.4]. *Let $X$ be a smooth projective variety (any dimension) over an algebraically closed field. Then the extremal rays of the closed cone of curves $\overline{NE}(X)$ are discrete in the open halfspace $\{z \in N_1(X) | z \cdot K_X < 0\}$. If $R \subset \overline{NE}(X)$ is an extremal ray, then there is a rational curve $C \subset X$ such that $[C] \in R$.*

Once an extremal ray is identified as the source of the trouble, one would like to use it to construct a map as in (1.1.3). In dimension three a complete description is known:

**1.2.5. Theorem** [78, 3.1–5]. *Let $X$ be a smooth projective threefold over $\mathbb{C}$ (or an algebraically closed field of characteristic zero). Let $R$ be an extremal ray of the closed cone of curves. Then the following holds:*

(1.2.5.1) *There is a normal projective variety $Y$ and a surjective morphism $f: X \to Y$ such that an irreducible curve $C \subset X$ is mapped to a point by $f$ iff $[C] \in R$; one can always assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$, and then $Y$ and $f$ are unique up to isomorphism.*

*The following is a list of all the possibilities for $f$ and $Y$.*

(1.2.5.2) *First case: $f$ is birational.*

*Let $E \subset X$ be the exceptional set of $f$. One has the following possibilities for $E$, $Y$, and $f$:*

(1.2.5.2.1) *$E$ is a smooth minimal ruled surface with typical fiber $C$, and $C \cdot E = -1$; $Y$ is smooth, and $f$ is the inverse of the blowing up of a smooth curve in $Y$.*

(1.2.5.2.2) *$E \cong \mathbb{P}^2$, and its normal bundle is $\mathcal{O}(-1)$. $Y$ is smooth, and $f$ is the inverse of the blowing up of a point in $Y$.*

*In the remaining cases $Y$ has exactly one singular point $P$, and $f$ is the inverse of the blowing up of $P$ in $Y$. Let $\widehat{\mathcal{O}}_{P,Y}$ be the completion of the local ring of $P \in Y$. Then the following hold:*

(1.2.5.2.3) *$E \cong \mathbb{P}^2$, and its normal bundle is $\mathcal{O}(-2)$. $\widehat{\mathcal{O}}_{P,Y} \cong \mathbb{C}[[x, y, z]]^{\mathbb{Z}_2}$, where $\mathbb{Z}^2$ denotes the ring of invariants under the group action $(x, y, z) \mapsto (-x, -y, -z)$.*

(1.2.5.2.4) *$E \cong Q$, where $Q$ is a quadric cone in $\mathbb{P}^3$, and its normal bundle is $\mathcal{O}_{\mathbb{P}^3}(-1)|Q$. $\widehat{\mathcal{O}}_{P,Y} \cong \mathbb{C}[[x, y, z, t]]/(xy - z^2 - t^3)$.*

(1.2.5.2.5) *$E \cong Q$, where $Q$ is a smooth quadric surface in $\mathbb{P}^3$, the two families of lines on $Q$ are numerically equivalent in $X$, and its normal bundle in $\mathcal{O}_{\mathbb{P}^3}(-1)|Q$. $\widehat{\mathcal{O}}_{P,Y} \cong \mathbb{C}[[x, y, z, t]]/(xy - zt)$.*

(1.2.5.3) *Second case: $f$ is not birational.*

*Then we have one of the following cases*:

(1.2.5.3.1) $\dim Y = 2$; $Y$ *is smooth and* $f$ *is a flat conic bundle* (*i.e., every fiber is isomorphic to a conic in* $\mathbb{P}^2$).

(1.2.5.3.2) $\dim Y = 1$; $Y$ *is a smooth curve, and every fiber of* $f$ *is irreducible. The generic fiber is a smooth surface* $F$ *such that* $-K_F$ *is ample.*

(1.2.5.3.3) $\dim Y = 0$ *and* $X$ *is a Fano variety* (*i.e.,* $-K_X$ *is ample*) *and* $b_2(X) = 1$.

**1.2.6. Comments.**

(1.2.6.1) The second case should be considered a fairly complete structural description. Conic bundles are well understood. If $\dim Y = 1$, then there is a complete list of the possible fibers. Fano varieties have been completely classified by Fano-Iskovskikh [36], [37].

(1.2.6.2) The cases listed under (1.2.5.2) should constitute the desired "elementary surgery operations." The first two cases are as expected, but the last three are unexpected and create a serious problem since $Y$ is singular. Thus, we cannot continue as in (1.1.4).

The realization that singularities must appear in three-dimensional minimal models was apparently made by Ueno [116, Chapter 16]. At that time this was interpreted as a sign that there are no minimal models in dimension three. The crucial conceptual step of allowing singularities was taken by Reid [96] and Mori [78]. Choosing the right class of singularities is a technical but very important part of the program.

**1.3. Terminal and canonical singularities.**

**1.3.1. Guiding principles.** We want to investigate varieties $X$ for which $K_X$ is not nef. In order to do this, $K_X$ should exist and being nef should make sense.

The usual definition of $K_X$ works over the smooth locus of $X$. If $X$ is normal (a harmless assumption), then $\operatorname{codim}(\operatorname{Sing} X) \geq 2$, hence $K_X$ has a well-defined homology class in $H_{2\dim X-2}(X, \mathbb{Z})$. However, because of the singularities there is no product between $H_{2\dim X-2}$ and $H_2$. Thus, the symbol $C \cdot K_X$ makes no sense in general.

If $K_{X-\operatorname{Sing} X}$ extends to a line bundle over $X$, then its first Chern class is in $H^2(X, \mathbb{Z})$ and we can take the cap product with $[C] \in H_2(X, \mathbb{Z})$. For the singularity given in (1.2.5.2.3), this condition is not satisfied because of the group action. However, $K_{X-\operatorname{Sing} X}^{\otimes 2}$ will extend to a line bundle over $X$. Thus, we can still define a first Chern class $c_1(K_X) \in H^2(X, \mathbb{Q})$, and this is also satisfactory.

For smooth varieties the plurigenera

$$P_m(X) = \dim \Gamma(X, K_X^{\otimes m}) \qquad (m \geq 0) \tag{1.3.1.1}$$

are birational invariants, and they are the most important discrete birational invariants that we know. The birational invariance is implied by the following more local result:

Let $f\colon X \to Y$ be a proper birational morphism between smooth varieties, and let $E_i \subset X$ be the $f$-exceptional divisors. Then

$$(1.3.1.2) \qquad K_X = f^*K_Y \otimes \mathscr{O}_X\left(\sum a_i E_i\right), \qquad a_i > 0 \text{ for every } i.$$

This can be reformulated as

$$(1.3.1.3) \qquad f_*(\omega_X^{\otimes m}) = \omega_Y^{\otimes m} \quad \text{for every } m > 0.$$

Observe that $a_i \geq 0$ would be sufficient to conclude the birational invariance of plurigenera.

**1.3.2. Definition.** An algebraic variety $X$ is said to have canonical (resp. terminal) singularities if the following three conditions are satisfied:

(1.3.2.1) $X$ is normal.

(1.3.2.2) $K_{X-\operatorname{Sing} X}^{\otimes m}$ extends to a line bundle over $X$ for some $m > 0$. This unique extension will be denoted by $K_X^{[m]}$. The smallest such $m$ is called the index of $K_X$.

(1.3.2.3) Let $f\colon X' \to X$ be a resolution of singularities, and let $E_i \subset X$ be the $f$-exceptional divisors. Let

$$K_{X'}^{\otimes m} = f^*(K_X^{[m]}) \otimes \mathscr{O}_{X'}\left(\sum a_i E_i\right) \qquad (m = \operatorname{index}(K_X)).$$

Then

$$X' \text{ has canonical singularities} \Leftrightarrow a_i \geq 0 \quad \text{for every } i,$$
$$X' \text{ has terminal singularities} \Leftrightarrow a_i > 0 \quad \text{for every } i.$$

(1.3.2.4) For arbitrary $i$ one can define

$$\omega_X^{[i]} \overset{\text{def}}{=} \text{double dual of } \omega_X^{\otimes i}.$$

This is a torsion free sheaf, locally free iff $\operatorname{index} X | i$. If $X$ has canonical singularities, and $f\colon X' \to X$ is a resolution of singularities, then

$$f_*(\omega_{X'}^{\otimes i}) = \omega_X^{[i]} \quad \text{for } i \geq 0.$$

(1.3.2.5) Another useful consequence of the definition is the following. Let $U \subset X$ be any open set (in the Euclidean topology). Let

$$f(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m} \in \Gamma(U - \operatorname{Sing} U, K_{U-\operatorname{Sing} U}^{\otimes m}).$$

Then for any compact $K \subset U$

$$\int_K |f|^{2/m} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n < \infty.$$

In dimension two it is easy to get a complete list by looking at the minimal resolution:

**1.3.3. Proposition.**

(1.3.3.1) *A two-dimensional terminal singularity is smooth.*

(1.3.3.2) *Two-dimensional canonical singularities are exactly the DuVal singularities (also called rational double points), and are given by equations:*

$$(1.3.3.3)\qquad \begin{aligned} &A_n: xy + z^{n+1} = 0;\\ &D_n: x^2 + y^2 z + z^{n-1} = 0;\\ &E_6: x^2 + y^3 + z^4 = 0;\\ &E_7: x^2 + y^3 + yz^3 = 0;\\ &E_8: x^2 + y^3 + z^5 = 0. \end{aligned}$$

In dimension three there is a complete list of terminal singularities. See [100] for a very nice survey.

**1.3.4. Theorem** [98], [17] , [85], [79], [61], [109].

(1.3.4.1) *Three-dimensional terminal singularities are isolated.*

(1.3.4.2) *A three-dimensional hypersurface singularity is terminal iff it is isolated and can be given by an equation*

$$g(x, y, z) + th(x, y, z, t) = 0,$$

*where $g$ is one of the equations from* (1.3.3.3).

(1.3.4.3) *Every other three-dimensional terminal singularity is the quotient of a hypersurface terminal singularity (called the index-one cover) by a cyclic group. The typical case is*

$$(xy + f(z^n, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_n(1, -1, a, 0), \quad \text{where } (a, n) = 1.$$

*The exceptional cases can be written as*

$$(x^2 + f(y, z, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_n(a, b, c, d) \quad \text{for some } n \le 4.$$

*There is a complete list of the possibilities.*

(1.3.4.4) *Every terminal singularity can be deformed into a collection of terminal cyclic quotient singularities $\mathbb{C}^3/\mathbb{Z}_n(1, -1, a)$, where $(a, n) = 1$.*

**1.4. Extremal rays on singular varieties.** The original deformation theoretic arguments of Mori [77], [78] do not seem to work for singular varieties. Substantially new ideas are required to extend the results. The new approach relies very heavily on vanishing theorems, and therefore it works in characteristic zero only. On the other hand, applied even in the smooth case, it gives results not accessible by the previous method; namely, it proves that extremal rays can always be contracted in any dimension. Since the proofs are reviewed in [13, Lectures 8–13], we restrict ourselves to stating the theorems and to some comments.

**1.4.1. Definition.**

(1.4.1.1) Let $x \in \mathbb{R}$. We define $\ulcorner x \urcorner$ to be the smallest integer $\geq x$.

(1.4.1.2) Let $X$ be an algebraic space. Let $D = \sum a_i D_i$ be a formal sum of distinct irreducible divisors on $X$ with rational coefficients. Let

$$\ulcorner D \urcorner = \ulcorner \sum a_i D_i \urcorner \overset{\text{def}}{=} \sum \ulcorner a_i \urcorner D_i .$$

(1.4.1.3) The fractional part of $D$ is the collection of those $D_i$ such that $a_i$ is not an integer.

**1.4.2. Vanishing Theorem** [42], [117]. *Let $X$ be a smooth, proper algebraic space. Let $D = \sum a_i D_i$ be a nef and big $\mathbb{Q}$-divisor. Assume that the fractional part of $D$ has only normal crossing singularities. Then*

$$H^i(X, K_X \otimes \mathscr{O}_X(\ulcorner D \urcorner)) = 0 \quad \textit{for } i > 0 .$$

**Comments.** If the $a_i$ are integers, and $D$ is ample, then this is the Kodaira vanishing theorem. Thus one can express the result as follows: if a divisor is close to being ample, then Kodaira vanishing still holds.

The above technical formulation seems artificial, but divisors of the form $\ulcorner D \urcorner$ appear very frequently. We will see an example in (5.3.8).

**1.4.3. Nonvanishing Theorem** [105]. *Let $X$ be a nonsingular projective variety. Let $D$ be a nef Cartier divisor and let $G$ be a $\mathbb{Q}$-divisor such that $\ulcorner G \urcorner$ is effective. Suppose that $aD + G - K_X$ is ample for some $a > 0$, and that the fractional part of $G$ has only simple normal crossings. Then, for all $m \gg 0$,*

$$H^0(X, mD + \ulcorner G \urcorner) \neq 0 .$$

**Comments.** The divisor $G$ is here for the purpose of certain applications. The proof does not simplify if we assume that $G = 0$. By (1.4.2) the higher cohomologies vanish, thus

$$H^0(X, mD + \ulcorner G \urcorner) = \chi(X, mD + \ulcorner G \urcorner) .$$

Therefore, the claim is that a certain expression involving Chern classes is not zero. In dimension three one can understand the precise form of this expression and prove the result in several cases. In higher dimensions however this approach seems to fail.

**1.4.4. Basepoint-Free Theorem** [6], [44], [45], [97]. *Let $X$ be a proper algebraic space with only canonical singularities. Let $D$ be a nef Cartier divisor such that $aD - K_X$ is nef and big for some $a > 0$. Then $|mD|$ has no basepoints for all $m \gg 0$.*

**Comments.** In the literature this result is stated for projective varieties only. The proof, however, analyzes only various proper modifications, and these can always be chosen to be projective.

**1.4.5. Rationality Theorem** [45], [53]. *Let $X$ be a projective variety with only canonical singularities. Let $H$ be an ample Cartier divisor, and let*

$$r = \max\{t \in \mathbb{R} | H + tK_X \text{ is nef}\}.$$

*Assume that $K_X$ is not nef (i.e., $r < \infty$). Then $r$ is a rational number of the form $u/v$ where $0 < v \leq (\operatorname{index} X)(\dim X + 1)$.*

**1.4.6. Cone Theorem** [45], [97]. *Let $X$ be a projective variety with only canonical singularities. Then the extremal rays of the closed cone of curves $\overline{NE}(X)$ are discrete in the open halfspace $\{z \in N_1(X) | z \cdot K_X < 0\}$.*

**1.4.7. Contraction Theorem** [44], [97]. *Let $X$ be a projective variety with only $\mathbb{Q}$-factorial terminal (resp. canonical) singularities. Then the following hold:*

(1.4.7.1) *For every extremal ray $R \subset \overline{NE}(X)$ there is a contraction map $f: X \to Y$ such that an irreducible curve $C \subset X$ is mapped to a point by $f$ iff $[C] \in R$. One can always assume that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and then $f$ and $Y$ are unique.*

(1.4.7.2) *We have the following possibilities for $f$ and $Y$:*

(1.4.7.2.1) *$f$ is birational, and the exceptional set is an irreducible divisor. Then $Y$ again has $\mathbb{Q}$-factorial terminal (resp. canonical) singularities. Such a contraction is called divisorial.*

(1.4.7.2.2) *$f$ is birational, and the exceptional set has codimension at least two in $X$. In this case $K_{Y-\operatorname{Sing} Y}^{\otimes m}$ never extends to a line bundle over $Y$ for $m > 0$. Such an $f$ is called a small extremal contraction.*

(1.4.7.2.3) $\dim Y < \dim Y$. *The general fiber $F$ has negative canonical class. Such a contraction is called a Fano contraction.*

*Proof.* We will show that in case (1.4.7.2.2) $K_{Y-\operatorname{Sing} Y}^{\otimes m}$ never extends to a line bundle over $Y$. Assume the contrary. Then $K_X^{[m]}$ and $f^*K_Y^{[m]}$ are two line bundles on $X$, and are isomorphic outside the exceptional set. Since the exceptional set has codimension at least two, these line bundles are isomorphic. On the other hand, if $[C] \in R$, then,

$$\deg(K_X^{[m]} | C) < 0 = \deg(f^*K_Y^{[m]} | C).$$

This is a contradiction.

**Comments.** (1.4.7.2.3) should be considered a structure theorem. It describes $X$ in terms of the lower-dimensional varieties $F$ and $Y$. Of course it may happen that $\dim Y = 0$ which means that $K_X$ is ample. Even in dimension three these singular Fano varieties are not fully understood.

(1.4.7.2.1) is very satisfactory. The new variety $Y$ has the same properties as $X$, but the rank of $H^2$ is one less. We can continue the above process with $Y$ as in (1.1.4). So far there is no complete description of divisorial contractions in dimension three, but it seems to be attainable with a finite amount of work.

(1.4.7.2.2) is the bad news, since $Y$ does not have canonical singularities. In this case something new must be done. This new operation is called flip. Currently its existence is known in dimension three only . We will discuss it in detail in the next section. For now assume that we can do something, and let us formulate the main theorem.

**1.4.8. Definition.** Let $Z$ be a projective variety with $\mathbb{Q}$-factorial terminal singularities.

(1.4.8.1) $Z$ is called a minimal model if $K_Z$ is nef.

(1.4.8.2) $Z$ is called a relative Fano model if there is an extremal ray $R$ such that the corresponding contraction $f\colon Z \to Y$ maps onto a lower-dimensional variety. Thus if $F$ is a general fiber, then $-K_F$ is ample.

**1.4.9. Minimal Model Theorem for Threefolds** [82]. *Let $X$ be a smooth projective three-dimensional algebraic variety over $\mathbb{C}$. Then a succession of divisorial contractions and flips transforms $X$ into a projective variety $X'$ which has the following properties*:

(1.4.9.1) *$X'$ and $X$ are birationally equivalent.*

(1.4.9.2) *$X'$ has only $\mathbb{Q}$-factorial terminal singularities.*

(1.4.9.3) *Either $X'$ is a minimal model or $X'$ is a relative Fano model.*

*This $X'$ is not unique, but only one of the alternatives in* (1.4.9.3) *can occur.*

*Proof.* Starting with a smooth threefold $X$ we define inductively a series of threefolds as follows. Let $X_0 = X$. If $X_i$ is already defined, we consider $K_{X_i}$. If $K_{X_i}$ is nef, then let $X' = X_i$. If $K_{X_i}$ is not nef, then we contract an extremal ray. If we obtain a Fano contraction, then again we set $X' = X_i$. If the contraction $f_i\colon X_i \to Y_i$ is divisorial, then we set $X_{i+1} = Y_i$. If the contraction is small, then we set $X_{i+1} =$ the flip of $f_i$. All that remains is to prove that the process will terminate.

A divisorial contraction decreases $\dim H^2$ by one, so we can have only finitely many of these. A flip leaves $\dim H^2$ unchanged. Shokurov [105] proves that a flip "improves" the singularities, and this easily implies that any sequence of flips is finite.

## 2. Flip and Flop

Studying threefolds one frequently encounters the situation where the "bad set" is a curve and one wants to change the threefold in codimension two only. There are at least three such examples:

1. In (1.4.7.2.2) we saw that in trying to construct the minimal model of a threefold one may encounter a contraction $f: X \to Y$ which contracts only finitely many curves $C_i \subset X$. These curves have negative intersection with $K_X$, and therefore one would like to get rid of them. Blowing up introduces a whole family of curves that have negative intersection with the canonical class. Thus we need to search for some other operation that changes $X$ in codimension two only. This operation will be the "flip."

2. A threefold $X$ may have several different minimal models $X_i$. It turns out that the induced bimeromorphic map $X_i \dashrightarrow X_j$ is an isomorphism in codimension one. Thus the difference between $X_i$ and $X_j$ is in finitely many curves only. One would like to understand such maps by factoring them into a sequence of "elementary" maps. "Elementary" may mean for instance that only one irreducible curve is changed. These are the so-called "flops."

3. A threefold $X$ may have a divisor $D$ which is ample except on finitely many curves. One can get rid of these curves by blowing up. However, the pullback of $D$ will not be ample, only the proper transform. Thus, for instance, $h^0(mD)$ will change in this process. One could try to make $D$ ample by changing $X$ in codimension two only. This will not affect $h^0(mD)$. This situation arises very naturally for nonprojective threefolds and will be discussed in detail in §5.

A large part of the difficulty in three-dimensional geometry comes from these codimension-two surgery problems. It is not too hard to find examples that show that even under quite reasonable conditions codimension-two modifications do not exist. There are some theorems that assert the existence of codimension-two modifications under very strict restrictions. These theorems are very hard, and have important consequences in three-dimensional geometry.

**2.1. Curve surgery on threefolds.** The aim of this subsection is to discuss general facts about algebraic curve surgery on threefolds. The two most important special cases, flips and flops, will be discussed in detail in subsequent subsections.

**2.1.1. Definition.**

(2.1.1.1) A three-dimensional curve neighborhood is a pair $C \subset X$, where $C$ is a proper connected curve, and $X$ is the germ of a normal threefold along $C$. We will frequently think of $X$ as an analytic representative of the germ. For some purposes, especially if one wants to consider positive characteristic as well, one can think of $X$ as a formal scheme along $C$. This would require changing some definitions.

(2.1.1.2) A three-dimensional curve neighborhood $C \subset X$ is called contractible if there is a morphism $f: (C \subset X) \to (P \in Y)$ satisfying the following properties:

(i) $Y$ is the germ of a normal singularity around the point $P$;
(ii) $f(C) = P$;
(iii) $f: X - C \to Y - P$ is an isomorphism.

$f$ and $Y$ are uniquely determined by $C \subset X$. $f$ is called the contraction morphism of $C \subset X$.

(2.1.1.3) $C \subset X$ is called a three-dimensional irreducible curve neighborhood if $C$ is irreducible.

(2.1.1.4) Two three-dimensional curve neighborhoods $C_i \subset X_i$ are called bimeromorphic if there is an isomorphism $(X_1 - C_1) \cong (X_2 - C_2)$. If both neighborhoods are contractible to $P_i \in Y_i$, then $(Y_1 - P_1) \cong (Y_2 - P_2)$, thus in fact $Y_1 \cong Y_2$. Therefore, two contractible curve neighborhoods $f_i: X_i \to Y_i$ are bimeromorphic iff $Y_1 \cong Y_2$.

**2.1.2. Proposition.** *Let $Z$ be a normal threefold, and let $C \subset Z$ be a connected proper curve, $C \subset X$ be the germ of $Z$ along $C$, and $C' \subset X'$ be a three-dimensional curve neighborhood bimeromorphic to $C \subset X$. One can patch $Z - C$ and $X'$ along $X - C \cong X' - C'$ to get a new threefold $C' \subset Z'$. Then*

(2.1.2.1) *$Z'$ is proper iff $Z$ is, and*

(2.1.2.2) *the composite map $\phi: Z \dashrightarrow Z - C \cong Z' - C' \hookrightarrow Z'$ is bimeromorphic.*

*Proof.* $\phi$ is meromorphic in codimension one, and is therefore meromorphic by the Levi extension theorem. The properness of $Z'$ is clear.

**2.1.3. Definition.** We say that the above $Z'$ is obtained from $Z$ by the curve surgery $(C \subset X) \dashrightarrow (C' \subset X')$.

**2.1.4. Examples.**

(2.1.4.1) Let $V$ be the total space of the line bundle $\mathscr{O}(-1, -1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. Both of the projections $\pi_i: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ can be extended to morphisms

$$p_i: (\mathbb{P}^1 \times \mathbb{P}^1 \subset V) \to (C_i \cong \mathbb{P}^1 \subset X_i).$$

It is easy to see that $X_i$ is smooth, and the normal bundle of $C_i \subset X_i$ is $\mathscr{O}(-1) + \mathscr{O}(-1)$. $p_2 \circ p_1^{-1}: X_1 \dashrightarrow X_2$ is bimeromorphic but is not an isomorphism.

This example was first used in a systematic way by Kulikov [62] to study the birational transformations of threefolds that have a basepoint-free pencil of $K3$ surfaces.

(2.1.4.2) Let $V$ be the total space of the line bundle $\mathscr{O}(-1, -1)$ over $E \times E$ for some elliptic curve $E$. Any of the projections $\pi_{ij}: E \times E \to E$ given by $\pi_{ij}: (z_1, z_2) \mapsto iz_1 + jz_2$, $i, j \in \mathbb{Z}$, can be extended to a morphism

$$p_{ij}: (E \times E \subset V) \to (C_{ij} \cong E \subset X_{ij}).$$

The $X_{ij}$ are singular along $C_{ij}$, and normal iff $(i, j) = 1$.

Thus there are infinitely many nonisomorphic curve neighborhoods bimeromorphic to each other. (One can produce such examples with rational singularities too.)

We will see several problems about threefolds where the main difficulty turns out be able to understand certain curve surgeries. Frequently the main problem is to show that there are nontrivial curve surgeries. The following definition singles out the curve surgery that "changes $X$ the most":

**2.1.5. Definition.**

(2.1.5.1) Let $C_1 \subset X_1$ be a three-dimensional curve neighborhood. Let $H_1 \in H^2(X_1, \mathbb{Q})$ be a cohomology class such that $H_1 \cap [C_1^j] < 0$ for every irreducible component $C_1^j$ of $C_1$. (Choosing $H_1$ to be negative is just a matter of preference. This choice conforms to the most important special case, the flip, where $H_1 = c_1(K_{X_1})$ is negative.) A three-dimensional curve neighborhood $C_2 \subset X_2$ bimeromorphic to $C_1 \subset X_1$ is called the opposite of $C_1 \subset X_1$ with respect to $H_1$ if there is a cohomology class $H_2 \in H^2(X_2, \mathbb{Q})$ such that $H_2 \cap [C_2^j] > 0$ for every irreducible component $C_2^j$ of $C_2$ and

$$(X_2 - C_2, H_2|X_2 - C_2) \cong (X_1 - C_1, H_1|X_1 - C_1).$$

In general, the opposite may not exist and depends on the choice of $H_1$.

(2.1.5.2) If $C_1$ is irreducible, then $H_1$ is unique up to a multiplicative constant since $H^2(X_1, \mathbb{Q}) \cong H^2(C_1, \mathbb{Q}) \cong \mathbb{Q}$. As we will see, there is at most one opposite.

(2.1.5.3) If $C \subset X$ is given, and the choice of $H$ is understood, then the opposite (if it exists) will be denoted by $C^+ \subset X^+$.

The uniqueness of the opposite follows from:

**2.1.6. Proposition.** *Let $C_i \subset X_i$ $(i = 1, 2)$ be three-dimensional curve neighborhoods. Let $H_i \in H^2(X_i, \mathbb{Q})$ be cohomology classes such that $H_1 \cap [C_1^j] \leq 0$ for every irreducible component $C_1^j$ of $C_1$, and $H_2 \cap [C_2^j] < 0$*

*for every irreducible component* $C_2^j$ *of* $C_2$. *Assume that*

$$(X_1 - C_1, H_1|X_1 - C_1) \cong (X_2 - C_2, H_2|X_2 - C_2).$$

*Then the above isomorphism extends to a morphism*

$$g: X_1 \to X_2.$$

*In particular, if* $H_1 \cap [C_1^j] < 0$ *for every irreducible component* $C_1^j$ *of* $C_1$ *then* $X_1 \cong X_2$.

*Proof.* Let $Z$ be a desingularization of the graph $\Gamma$ of the bimeromorphic map $X_1 \dashrightarrow X_2$, and let $q_i: Z \to X_i$ be the projections. Let $E_j \subset Z$ be the exceptional divisors (note that every $q_1$ exceptional divisor is also $q_2$ exceptional, and vice versa). By assumption

$$q_1^* H_1 = q_2^* H_2 + \sum d_j [E_j]$$

for some rational numbers $d_j$. Use the cohomological version of (5.2.5.3) with $L = -H_1$, $M = -q_2^* H_2$, $G = 0$ to conclude that $d_j \leq 0$. Interchanging $X_1$ and $X_2$ gives that $d_j \geq 0$, thus

$$q_1^* H_1 = q_2^* H_2.$$

If $\Gamma \to X_1$ is not an isomorphism then there is a proper curve $B \subset Z$ such that $q_1(B) =$ point but $q_2(B) \subset C_2$ is one dimensional. Thus

$$0 = q_1^* H_1 \cap [B] = q_2^* H_2 \cap [B] < 0.$$

This is a contradiction. q.e.d.

To get some further results we have to impose some restrictions on $C \subset X$. A weak but useful requirement is the rationality of the singularities of $X$. The following result collects the basic topological consequences of rationality.

**2.1.7. Proposition.** *Let* $X$ *be a normal variety, and let* $Z \subset X$ *be a closed subvariety. Then the following hold*:

(2.1.7.1) *The local cohomology sheaves* $\mathscr{H}_Z^0(X, \mathbb{Z})$ *and* $\mathscr{H}_Z^1(X, \mathbb{Z})$ *are zero.*

(2.1.7.2) $H^1(X, \mathbb{Z}) \to H^1(X - Z, \mathbb{Z})$ *is injective.*

(2.1.7.3) *If* $X$ *has rational singularities and* $\operatorname{codim}(Z, X) \geq 2$, *then* $\mathscr{H}_Z^2(X, \mathbb{Q}) = 0$.

(2.1.7.4) *If* $X$ *has rational singularities and* $\operatorname{codim}(Z, X) \geq 2$, *then* $H^2(X, \mathbb{Q}) \to H^2(X - Z, \mathbb{Q})$ *is injective.*

(2.1.7.5) *If* $x \in X$ *is local, then* $\operatorname{Pic}(X - x) \to H^2(X - x, \mathbb{Z})$ *is an isomorphism.*

(2.1.7.6) *If* $x \in X$ *is local, and* $\dim X \leq 3$, *then* $X$ *is a rational homology manifold iff* $X$ *is* $\mathbb{Q}$*-factorial.*

*Proof.* (2.1.7.1) and (2.1.7.2) are obvious since normal implies topologically unibranch. To see (2.1.7.3) we may assume that $X$ is local, and $Z = \{0\}$ is the closed point. Using the exact sequence

$$H^1(X - 0, \mathbb{Q}) \to H_0^2(X, \mathbb{Q}) \to H^2(X, \mathbb{Q}) = 0,$$

it is sufficient to see that $H^1(X - 0, \mathbb{Q}) = 0$. Let $f: X' \to X$ be a resolution of singularities and let $E = f^{-1}(0)$. Since $X$ has rational singularities, $R^1 f_* \mathbb{Q} = 0$ (see [108, 2.14]; [60, 12.1.3]). Therefore, $H^1(X - 0, \mathbb{Q}) \cong H^1(X' - E, \mathbb{Q})$ and $H^1(X', \mathbb{Q}) = 0$. By [29, §3] the restriction $0 = H^1(X', \mathbb{Q}) \to H^1(X' - E, \mathbb{Q})$ is surjective. This shows (2.1.7.3) which in turn implies (2.1.7.4). (2.1.7.5) is [21, 6.1], and this implies the last claim.

**2.1.8. Lemma.** *Let* $X$ *be a variety with rational singularities, and* $Z \subset X$ *be a closed subvariety such that* $\operatorname{codim}(Z, X) \geq 2$. *If* $L$ *is a line bundle on* $X - Z$ *such that its first Chern class is liftable to* $H^2(X, \mathbb{Q})$, *then* $L^k$ *extends to a line bundle over* $X$ *for some* $k > 0$.

*Proof.* This will be done by induction on $\dim Z$. The $\dim Z = 0$ case follows from (2.1.7.5). Taking a general hyperplane section and using induction we see that $L^{k_1}$ extends across an open dense subset $U \subset Z$ for some $k_1 > 0$. Again by induction $L^{k_2 k_2}$ extends across $Z - U$ for some $k_2 > 0$.

**2.1.9. Proposition.** *Let* $C \subset X$ *be a three-dimensional curve neighborhood. Assume that* $X$ *has only rational singularities. Let* $H \in H^2(X, \mathbb{Q})$ *be a cohomology class such that* $H \cap [C_j] < 0$ *for every irreducible component* $C_j$ *of* $C$. *If the opposite* $C^+ \subset X^+$ *exists, then* $C$ *is contractible.*

*Proof.* Let $C_i^+$ be the irreducible components of $C^+$. For every $i$ let $D_i^+$ and $E_i^+$ be two-dimensional germs intersecting $C_i^+$ in a single point and disjoint otherwise. Let $\sum a_i[D_i^+] = \sum a_i[E_i^+] = mH^+$, where $a_i, m \in \mathbb{Z}$ are positive. Let $D_i$ (resp. $E_i$) be the proper transforms of $D_i^+$ (resp. $E_i^+$) in $X$. Then $D = k\sum a_i D_i$ and $E = k\sum a_i E_i$ extend to Cartier divisors on $X$ by (2.1.8) for some $k > 0$. The ideal sheaf $(\mathscr{O}_X(-D), \mathscr{O}_X(-E)) \subset \mathscr{O}_X$ defines a one-dimensional subscheme $\overline{C}$ of $X$ whose normal bundle $\mathscr{O}_X(D) + \mathscr{O}_X(E)|C$ is negative. Thus, $C$ is contractible by [3, 6.2]; [7, 6.1].

**2.1.10. Proposition.** *Let* $C_i \subset X_i$ *be three-dimensional curve neighborhoods. Let*

$$g: X_1 \dashrightarrow X_2$$

*be a bimeromorphic map. Let* $C_1^0 \subset C_1$ *be an irreducible component. Assume that* $X_2$ *has rational singularities and that* $g$ *is not holomorphic at the generic point of* $C_1^0$. *Then* $C_1^0$ *is a rational curve.*

*In particular, if* $X_2$ *is the opposite of* $X_1$, *then every component of* $C_1$ *is rational.*

*Proof.* Let $\Gamma$ be the closure of the graph of $g$ with projections $p_i : \Gamma \to X_i$. By assumption, $p_1^{-1}(C_1^0) \subset \Gamma$ contains a surface $E$. $p_2(E)$ is positive dimensional. Let $D \subset X_2$ be a general local divisor intersecting $p_2(E)$. Since $X_2$ has rational singularities, so does $D$. Therefore, every exceptional curve of $p_2^{-1}(D) \to D$ is rational. One of these exceptional curves maps surjectively onto $C_1^0$.

**2.1.11. Remark.** Even if both $X_i$ are smooth, the above $C_1^0$ can be singular [55, 4.8].

**2.1.12. Proposition.** *Let* $C_i \subset X_i$ *be bimeromorphic three-dimensional curve neighborhoods. Assume that both have rational* $\mathbb{Q}$*-factorial singularities. Then*

$$\begin{aligned}\#\{\textit{irreducible components of } C_1\} &= \operatorname{rank} H^2(X_1, \mathbb{Q}) \\ &= \operatorname{rank} H^2(X_1 - C_1, \mathbb{Q}) \\ &= \operatorname{rank} H^2(X_2, \mathbb{Q}) \\ &= \#\{\textit{irreducible components of } C_2\}.\end{aligned}$$

The following result is needed in factoring curve surgeries as a succession of "elementary" curve surgeries.

**2.1.13. Proposition.** *Let* $C_i \subset X_i$ *be three-dimensional curve neighborhoods. Let*

$$g : X_1 \dashrightarrow X_2$$

*be a bimeromorphic map. Assume that* $X_1$ *has* $\mathbb{Q}$*-factorial singularities and that* $g$ *is not an isomorphism. Then there is an irreducible component* $C_1^0 \subset C_1$ *which is contractible.*

*Proof.* As in the proof of (2.1.9) we choose $D_i^+$ and $E_i^+$ on $X_2$. We obtain $D$ and $E$ on $X_1$. By (2.1.6) there is an irreducible component $C_1^0 \subset C_1$ such that $C_1^0 \cdot D = C_1^0 \cdot E < 0$. Thus, $C_1^0$ is contractible (cf. [55, 4.10]).

**2.1.14. Problem.** Prove (2.1.13) without assuming that $X_1$ has $\mathbb{Q}$-factorial singularities (maybe assuming that it has rational singularities).

In general very little is known about the existence of the opposite of a neighborhood $C \subset X$. A positive answer is known only in the following cases.

**2.1.15. Theorem.** *Let $C \subset X$ be a three-dimensional contractible curve neighborhood. Then the opposite $C^+ \subset X^+$ exists in any of the following cases:*

(2.1.15.1) *$X$ has terminal singularities, and $K_X|C$ is numerically trivial ($H$ arbitrary)* [98].

(2.1.15.2) *$X$ has canonical singularities, and $K_X|C$ is numerically trivial ($H$ arbitrary)* [47].

(2.1.15.3) *$X$ has canonical singularities, and $-K_X|C$ is ample, $H = K_X$* [82].

(2.1.15.4) *$X$ is toric* [99].

**Comments.** We will give a proof of (2.1.15.1), which provides a very explicit description of $C^+ \subset X^+$ in the next subsection. See [55] for a shorter proof of (2.1.15.2).

Some important special cases of (2.1.15.3) were done independently by using different methods by Tsunoda [114]; Shokurov [106]; Mori [80]; Kawamata [47]. The proof of the general case is very long and complicated. See [58] for an introduction.

**2.2. Flops.** The aim of this section is to develop a detailed description of flops in dimension three.

**2.2.1. Definition.** Let $f: C \subset X \to P \in Y$ be a three-dimensional contractible curve neighborhood. Assume that $X$ has terminal or canonical singularities and that $K_X|C = 0$. Let $H$ be a line bundle on $X$ such that $H^{-1}|C$ is ample.

The opposite of $C \subset X$ will be called the flop of $C \subset X$. It will turn out that the flop is independent of the choice of $H$ if $X$ has terminal singularities.

In the terminal singularity case the existence of flops was proved by Reid [98]. The following explicit description is based on an idea of Mori (cf. [55, §2]).

**2.2.2. Theorem.** *Let $f: C \subset X \to P \in Y$ be as above. Assume that $X$ has terminal (resp. canonical) singularities. Then $Y$ has a terminal (resp. canonical) singularity at $P$. If $X$ has terminal singularities and $C$ is irreducible, then $P \in Y$ and the flop can be described in one of the following ways:*

$$(P \in Y) \cong (0 \in (x^2 + F(y, z, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_n(a, b, c, d)), \quad (n \geq 1). \tag{2.2.2.1}$$

*If $\tau: Y \to Y$ is the involution $(x, y, z, t) \mapsto (-x, y, z, t)$, then*

$$(C^+ \subset X^+) \cong (C \subset C) \quad \textit{and} \quad f^+ = \tau \circ f,$$

*or*

(2.2.2.2)
$$\begin{aligned}(P \in Y) &\cong (0 \in (xy - g(z^n, t)h(z^n, t) = 0)\\ &\subset \mathbb{C}^4/\mathbb{Z}_n(1, -1, a, 0)) \qquad (n \geq 3).\end{aligned}$$

*$C \subset X$ can be written explicitly as follows. In*

$$(\mathbb{C}^4 \times \mathbb{P}^1)/\mathbb{Z}_n(1, -1, a, 0, -1, 0)$$

*with coordinates $(x, y, z, t) \times (p : q)$ the equations of $X$ are*

$$xy - gh = 0 \quad \text{and} \quad px - qg = 0.$$

*In the affine chart $U = (p \neq 0)$ we let $x_1 = q/p$ and $y_1 = y$. Then*

$$\begin{aligned}(C \subset X) \cap U &\cong (x_1\text{-axis} \subset (x_1 y_1 - h = 0))\\ &\subset \mathbb{C}^4(x_1, y_1, z, t)/\mathbb{Z}_n(1, -1, a, 0).\end{aligned}$$

*In the affine chart $V = (q \neq 0)$ we let $y^1 = p/q$ and $x^1 = x$. Then*

$$\begin{aligned}(C \subset X) \cap V &\cong (y^1\text{-axis} \subset (x^1 y^1 - g = 0))\\ &\subset \mathbb{C}^4(x^1, y^1, z, t)/\mathbb{Z}_n(1, -1, a, 0).\end{aligned}$$

*To obtain the description of $C^+ \subset X^+$ one only has to interchange $g$ and $h$ in the above formulas.*

**2.2.3. Remark.** If $C$ is reducible, then the statement about the singularities of $Y$ still holds. In case (2.2.2.1) the flop is described the same way. In case (2.2.2.2) the flop will be described in (2.2.8).

**2.2.4. Corollary** [55, 4.11]. *Notation as above. If $X$ has terminal singularities, then $X$ and $X^+$ have the same (analytic) singularities.*

*Proof.* If we have (2.2.2.1), then $X$ and $X^+$ are even isomorphic. In the second case the singularities are at the origins of the charts $U$ and $V$, and the isomorphism is clear from the explicit description. The local isomorphism does not take $C$ to $C^+$, therefore it will not extend to a global isomorphism.

**2.2.5. Proof of** (2.2.2). Choose $m > 0$ such that $K_X^{[m]}$ is a line bundle. By a suitable analytic version of the Basepoint-Free Theorem (1.4.4) [84, 5.5] we obtain that there is a line bundle $L$ on $Y$ such that $f^*L \cong K_X^{[m]}$. Thus $L \cong K_Y^{[m]}$. A resolution $g: Z \to X$ of $X$ is also a resolution $f \circ g: Z \to Y$ of $Y$, and $g^*K_X^{[m]} \cong (f \circ g)^* K_Y^{[m]}$. This shows that $P \in Y$ is terminal (resp. canonical). By (1.3.4) we obtain that in the terminal case $P \in Y$ can be described as in (2.2.2.1) or (2.2.2.2).

(2.2.5.1) Let $P \in Y$ be as described in (2.2.2.1). Since the flop of $C \subset X$ is unique (2.1.5.2), it is sufficient to check that $f^+: C^+ \subset X^+ \to Y$

satisfies the conditions of (2.1.5.1). Using the identification $(C^+ \subset X^+) \cong (C \subset X)$ set $H^+ = H^{-1}$. The only requirement that needs verification is that

$$c_1(H^{-1}|X - C) \cong c_1(\tau^*(H|X - C)).$$

Let $M$ be any line bundle on $Y - P \cong X - C$. Observe that $(Y - P)/\tau \cong (\mathbb{C}^3 - 0)/\mathbb{Z}_n$. Therefore, $M \otimes \tau^* M$ is the pullback of a line bundle on $(\mathbb{C}^3 - 0)/\mathbb{Z}_n$, and is torsion. Hence, $c_1(\tau^* M) = c_1(M^{-1})$ as required.

(2.2.5.2) Assume that $P \in Y$ is as described in (2.2.2.2). In (2.2.8) we will give a complete description of all curve neighborhoods $C \subset X$ which contract to $P \in Y$. (2.2.2.2) will be a special case of this more general result.

**2.2.6. Proposition.** *Assume that* $Y = (xy - F(z, t) = 0) \subset \mathbb{C}^4$ *defines an isolated singularity. Let* $F = f_1 \cdots f_k$ *be the irreducible factors. Set*

$$D_i = (x - f_1 \cdots f_i = f_{i+1} \cdots f_k - y = 0) \subset Y \qquad (i = 1, \cdots, k - 1).$$

*Then* $\operatorname{Pic}(Y - 0)$ *is freely generated by the* $[D_i]$.

We start with the following special:

**2.2.6.1. Lemma** [27, 1.2]. *Let* $Y = (xy - F(z, t) = 0)$. *Assume that* $F(z, t)$ *is irreducible. Then* $\operatorname{Pic}(Y - 0) \cong 0$. *In fact,* $Y$ *is a topological manifold at the origin.*

*Proof.* Let $C = (F = 0)$. $C - 0$ is a punctured disc since $F$ is irreducible. Thus by [67, 8.5], if $\Delta_C(t)$ denotes the monodromy on the Milnor fiber, then $\Delta_C(1) = \pm 1$. Also, $\Delta_Y(t) = \Delta_C(t)$ hence $\Delta_Y(1) = \pm 1$ and again by [67, 8.5] we obtain that $Y$ is a topological manifold. Thus by (2.1.7.5) $\operatorname{Pic}(Y - 0) \cong H^2(Y - 0, \mathbb{Z}) \cong 0$.

**2.2.6.2. Computations.** Let $Y_1$ be the blow up of $Y$ at $(x = f_1 \cdots f_j = 0)$. In (2.2.2.2) set $g = f_1 \cdots f_j$, $h = f_{j+1} \cdots f_k$, and $C = C_1$.

The proper transform $D_i'$ of $D_i$ on $Y_1$ is given as follows:

(i) If $i < j$, then $D_i'$ intersects $C_1$ at the origin of the $V$-chart, and there it is given by equations

$$x_1 - f_1 \cdots f_i = f_{i+1} \cdots f_j - y^1 = 0.$$

(ii) If $i = j$ then $D_i'$ intersects $C_1$ transversally at a single point which is not the origin of either charts and is given by equations

$$x_1 - 1 = f_{j+1} \cdots f_k - y_1 = 0 \quad (\text{resp. } x^1 - f_1 \cdots f_j = 1 - y^1 = 0).$$

(iii) If $i > j$, then $D_i'$ intersects $C_1$ at the origin of the $U$-chart, and is given there by equations

$$x_1 - f_{j+1} \cdots f_i = f_{i+1} \cdots f_k - y_1 = 0.$$

*Proof of* (2.2.6). First blow-up $(x = f_1 = 0)$. Then on the $U$-chart blow-up $(x_1 = f_2 = 0)$ and continue. After $k-1$ steps we obtain a threefold $p\colon Z \to Y$ with $k-1$ curves $C_1, \cdots, C_{k-1}$ which are $p$-exceptional. The proper transform $\overline{D}_j$ of $D_j$ intersects $\bigcup C_i$ at a single point of $C_j$ transversally. The singularities of $Z$ are of the form $uv - f_i = 0$. Thus by (2.2.6.1)

$$\operatorname{Pic}(Y-0) = \operatorname{Pic}\left(Z - \bigcup C_i\right) \cong \operatorname{Pic}(Z) \cong H^2(Z, \mathbb{Z}) \cong H^2\left(\bigcup C_i, \mathbb{Z}\right).$$

Now it is clear that the $\overline{D}_i$ freely generate $\operatorname{Pic} Z$.

(2.2.6.3) For some purposes another basis of $\operatorname{Pic}(Y-0)$ is useful. If we write $F = gh$, then

$$\begin{aligned}(xy - gh, x - g) &= (x - g, y(x-g) - (xy - gh)) \\ &= (x-g, g(h-y)) = (x-g, h-y) \cap (x, g).\end{aligned}$$

Therefore, $[x - g = h - y = 0] = -[x = g = 0] = [x = h = 0]$. Hence,

$$D_i = \sum_{s=i+1}^{k} [x = f_s = 0].$$

Thus, by (2.2.6) $[x = f_i = 0]$, $i = 1, \cdots, k$, generate $\operatorname{Pic}(Y-0)$ and satisfy a single relation

$$\sum [x = f_i = 0] = 0.$$

**2.2.7. Proposition** [98]; **Mori, Shepherd-Barron, Ue** (unpublished). *Assume that* $Y = (xy - F(z, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_n(1, -1, a, 0)$ *defines an isolated singularity. Let* $F = f_1 \cdots f_k$, *where* $f_i(0, 0) = 0$, *the* $f_i$ *being* $\mathbb{Z}_n$*-invariants and irreducible among such power series. (Note that there cannot be multiple factors since the singularity is isolated.) Then*

$$\operatorname{Pic}(Y-0) \cong \mathbb{Z}/n\mathbb{Z} \cdot [K_{Y-0}] + \frac{\mathbb{Z}[x = f_1 = 0] + \cdots + \mathbb{Z}[x = f_k = 0]}{([x = f_1 = 0] + \cdots + [x = f_k = 0])}.$$

*Thus, as an abstract group*

$$\operatorname{Pic}(Y-0) \cong \mathbb{Z}/n\mathbb{Z} + \mathbb{Z}^{k-1}.$$

*Proof.* Let $Y' = (xy - F(z, t) = 0)$ and let $q\colon Y' \to Y$ be the quotient map. Each $f_i$ decomposes as $f_i = \prod f_{ij}$, where the $f_{ij}$ are irreducible but not necessarily $\mathbb{Z}_n$-invariant.

The kernel of the pullback map

$$q^*\colon \operatorname{Pic}(Y-0) \to \operatorname{Pic}(Y'-0)^{\mathbb{Z}_n}$$

is $n$-torsion. The torsion in $\mathrm{Pic}(Y-0) \cong H^2(Y-0, \mathbb{Z})$ is dual to the torsion in $H_1(Y-0, \mathbb{Z})$. By [30, X.3.4], $Y'-0$ is simply connected, thus $H_1(Y-0, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. One can easily compute that $[K_{Y-0}]$ has order $n$ in $\mathrm{Pic}(Y-0)$, so that it generates the torsion.

For fixed $i$ the $\mathbb{Z}_n$-action is transitive on the $f_{ij}$ since $f_i$ is irreducible among $\mathbb{Z}_n$-invariant power series. By (2.2.6.3) the classes $[x = f_i = 0]$ generate $\mathrm{Pic}(Y'-0)^{\mathbb{Z}_n}$ and satisfy a single relation

$$\sum [x = f_i = 0] = 0.$$

The divisors $[x = f_i = 0]$ descend to $Y = 0$, thus $q^*$ is surjective.

**2.2.8. Theorem.** *Assume that*

$$Y = (xy - F(z, t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_n(1, -1, a, 0)$$

*defines an isolated singularity. Let $C \subset X$ be a curve neighborhood with contraction map*

$$f\colon C \subset X \to 0 \in Y.$$

(2.2.8.1) *Every such $X$ can be constructed as follows:*

*First write $F = f_1 \cdots f_m$, where the $f_i$ are $\mathbb{Z}_n$-invariant but not necessarily irreducible. Let $Y_0' = (xy - F = 0)$. Next blow up $(x = f_1 = 0) \subset Y_0'$. The resulting $Y_1'$ has a singularity of the form $x_1y_1 - f_2 \cdots f_m = 0$. Blow up $(x_1 = f_2 = 0) \subset Y_1'$ to obtain $Y_2'$ and continue. After $m-1$ blow-ups we obtain $Y_{m-1}'$. Finally let $X = Y_{m-1}'/\mathbb{Z}_n$.*

(2.2.8.2) *$C^+ \subset X^+$ is obtained in the same way from the reverse order product $F = f_m \cdots f_1$.*

*Proof.* Let $E$ be an $f$-ample line bundle on $X$. Its restriction to $\mathrm{Pic}(X-C) = \mathrm{Pic}(Y-0)$ will be denoted again by $E$. Let $F = g_1 \cdots g_k$, where $g_i(0, 0) = 0$, the $g_i$ being $\mathbb{Z}_n$-invariants and irreducible among such power series. By (2.2.7) we can write

$$[E] = \sum a_i [x = g_i = 0] \quad \text{(modulo torsion)}.$$

By adding a suitable multiple of $0 = \sum [x = g_i = 0]$ and rearranging the $g_i$ we can assume that $0 = a_1 \le \cdots \le a_k$. Set $b_i = a_{i+1} - a_i$, $i = 1, \cdots, k-1$. By (2.2.6.3)

$$[E] = \sum b_i [D_i] \quad \text{(modulo torsion)}.$$

Let $i_1, i_2, \ldots$ be those indices such that $b_i \ne 0$. Let

$$f_j = \prod_{s=i_{j-1}+1}^{i_j} g_s \quad (\text{set } i_{-1} = -1).$$

By the computations (2.2.6.2) if we perform the series of blow-ups in the statement of the theorem, then we obtain a proper modification $\overline{f}\colon Y'_{m-1} \to Y'_0$ such that $E|Y'_0-0$ extends to an ample line bundle on $Y'_{m-1}$. By (2.1.6) this implies that $X \cong Y'_{m-1}/\mathbb{Z}_n$.

The opposite is obtained in a similar way by using $-[E]$;

$$-[E] = (a_k - a_{k-1})[x = g_{k-1} = 0] + \cdots + (a_k - a_1)[x = g_1 = 0],$$

and this implies the second part.

**2.2.9. Corollary.** *Notation as in* (2.2.2). *Assume that* $X$ *has terminal singularities and* $C$ *is irreducible. Then there is an isomorphism* $\phi\colon H^2(X, \mathbb{Z}) \to H^2(X^+, \mathbb{Z})$ *which makes the following diagram commutative*:

$$\begin{array}{ccc} H^2(X, \mathbb{Z}) & \longrightarrow & H^2(Y - 0, \mathbb{Z}) \\ \phi\downarrow & & \| \\ H^2(X^+, \mathbb{Z}) & \longrightarrow & H^2(Y - 0, \mathbb{Z}) \end{array}$$

(*The horizontal maps are the natural restrictions.*)

*Proof.* In case (2.2.2.1) $X \cong X^+$ and let $\phi = -\mathrm{id}$. In case (2.2.2.2), $H^2(X, \mathbb{Z})$ is generated by $n[x = g = 0]$, and $H^2(X^+, \mathbb{Z})$ is generated by $n[x = h = 0] = -n[x = g = 0]$. (The $n$ comes in because of the group action.) Let $\phi(n[x = g = 0]) = -n[x = h = 0]$.

**2.3. Flips.**

**2.3.1. Definition.** Let $f\colon C \subset X \to P \in Y$ be a three-dimensional contractible curve neighborhood. Assume that $X$ has terminal or canonical singularities and that $-K_X|C$ is ample. $C \subset X$ will also be called an extremal neighborhood (extremal nbd in the terminology of [60]).

The opposite $C \subset X$ with respect to $K_X$ will be called the flip $C \subset X$.

**2.3.2 Examples of flips.** Examples of flips are not easy to get because $X$ cannot be smooth, in fact $X$ cannot have only hypersurface singularities either. The best hope is to find an example which is globally the quotient of a flop. This is indeed possible.

(2.3.2.1) First we consider the simplest example of flops. $Y = (xy - uv = 0) \subset \mathbb{C}^4$ has an isolated singularity at the origin. Let

$$X = B_{(x, v)}Y \quad \text{and} \quad X^+ = B_{(x, u)}Y.$$

Let $C \subset X$ (resp. $C^+ \subset X^+$) be the exceptional curves of $X \to Y$ (resp. $X^+ \to Y$). Thus, we have the following varieties and maps:

$$(C^+ \subset X^+) \to (0 \in Y) \leftarrow (C \subset X).$$

(2.3.2.2) Consider the action of the cyclic group $\mathbb{Z}_n$: $(x, y, u, v) \mapsto (\zeta x, y, \zeta u, v)$, where $\zeta$ is a primitive $n$th root of unity. This defines an action on all of the above varieties. The corresponding quotients are denoted by a subscript $n$.

The fixed point set of the action of $Y$ is the 2-plane $(x = u = 0)$. On the projective quadric $(xy - uv = 0) \subset \mathbb{P}^3$ the action has two fixed lines: $(x = u = 0)$ corresponding to the above fixed 2-plane and $(y = v = 0)$ corresponding to the $\zeta$-eigenspace. On $X$ therefore the fixed point set has two components: the proper transform of the $(x = u = 0)$ plane and the image of the $(y = v = 0)$ line. The latter is an isolated fixed point. $(x, v' = vx^{-1}, u)$ give local coordinates at the isolated fixed point. The group action is $(x, v', u) \mapsto (\zeta x, \zeta^{-1}v', \zeta u)$. In particular, the quotient is a terminal singularity (1.3.4).

On $X^+$ the fixed point set will have only one component and contains the exceptional curve $C^+$. Thus $X_n^+$ is smooth.

It is not too hard to compute the intersection numbers of the canonical classes with the exceptional curves. We obtain that

$$C_n \cdot K_{X_n} = -\frac{n-1}{n} \quad \text{and} \quad C_n^+ \cdot K_{X_n^+} = n - 1.$$

Thus, $X_n^+ \to Y_n$ is the flip of $X_n \to Y_n$ for $n \geq 2$.

(2.3.2.3) Before going further let us note two special properties of this example. At the isolated fixed point on $X$ we have coordinates $(x, v', u)$, and the curve $C$ is the $v'$-axis. A typical local $\mathbb{Z}_n$-invariant section of $K_X^{-1}$ is given by $\sigma = (v'^{n-1} - x)(dx \wedge dv' \wedge du)^{-1}$, which has intersection number $(n-1)$ with $C$. Since this section is invariant, it descends to a local section $\sigma_n$ of $K_{X^n}^{-1}$. Let $D_n = (\sigma_n = 0)$. By construction, $D_n \cong \{(v', u) - \text{plane}\}/\mathbb{Z}_n$, which is a DuVal singularity (1.3.3) of type $A_{n-1}$. Since $C_n \cdot D_n = C_n \cdot K_{X_n}^{-1}$, one can easily see that even globally $D_n$ is a member of $|K_{X_n}^{-1}|$.

Another simple way of getting a surface singularity out of the above construction is to consider the general hyperplane section $H_n$ of $Y_n$. This is given as the quotient of an invariant section of $Y$. $v - u^n = 0$ is such a section whose zero set on $Y$ is isomorphic to the singularity $(xy - u^{n+1} = 0)$. This itself is a quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}_{n+1}$. Using this, $H_n$ can be written as a quotient of $\mathbb{C}^2$, and we easily get that $H_n$ is isomorphic to the singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}(1, 1)$.

The first observation made in (2.3.2.3) together with the examination of many other examples leads to the following conjecture:

**2.3.3. Reid's Conjecture on General Elephants** [100], [60]. The contraction map provides a one-to-one correspondence between the following two sets.

Extremal neighborhoods:

$$EN := \left\{ \begin{array}{l} \text{three-dimensional contractible curve neighborhoods } C \subset X \\ \text{such that } X \text{ has canonical singularities and } -K_X|C \text{ is ample}, \end{array} \right\}$$

and flipping singularities:

$$FS := \left\{ \begin{array}{l} \text{three-dimensional normal singularities } P \in Y \text{ such that } K_Y \text{ is} \\ \text{not } \mathbb{Q}\text{-Cartier, and the general } D \in |-K_Y| \text{ has a DuVal} \\ \text{singularity at } P. \end{array} \right\}$$

Reid's original hope was that this equivalence can be used to obtain a proof of the existence of flips. To do this one needs to produce a "nice" member of $|-K_Y|$ and then to use this member to construct $X^+$. It is still to be seen whether either of these steps can be done in the spirit envisaged by Reid.

One direction of the conjecture is proved, and the other is also known with some restrictions:

**2.3.4. Theorem** [60, 3.1]. *For a singularity $P \in Y$ as in (2.3.3.$FS$) there is a curve neighborhood $C \subset X$ as in (2.3.3.$EN$) such that $P \in Y$ is the contraction of $C \subset X$.*

The proof of this result uses the full force of the existence of flips. Thus at the moment it cannot be used to show their existence.

**2.3.5. Theorem** [82] [60, 1.7]. *Let $f: C \subset X \to P \in Y$ be a three-dimensional contractible curve neighborhood. Assume that $X$ has only terminal singularities, $C \cdot K_X < 0$, and $C$ is irreducible. Then the general member of $|-K_X|$ and the general member of $|-K_Y|$ have only DuVal singularities.*

From the point of view of (2.3.3) the proof is again unsatisfactory. This result appears very close to the end of a nearly complete classification of extremal neighborhoods. At the moment it is easier to prove that flips exist than to show (2.3.5).

The second observation of (2.3.2.3) again gives a general feature of extremal neighborhoods. In this case though there are some exceptions:

**2.3.6. Theorem** [60, 1.8]. *Let $f: C \subset X \to P \in Y$ be a three-dimensional contractible curve neighborhood. Assume that $X$ has only terminal singularities. $C \cdot K_X < 0$, and $C$ is irreducible. Let $P \in H \subset Y$ be a general hypersurface section of $Y$ through $P$. Then $H$ is either a cyclic quotient singularity or one of six exceptional singularities listed in* [60, 1.8];

[58, 7.2]; *the exceptional ones are all rational and have multiplicity at most five.*

**Comments.** We except that the list of exceptions is not much longer if $C$ is allowed to be reducible. However, if $X$ is allowed to have canonical singularities, infinitely many new cases seem to appear. We do not know what to except.

The exceptional hyperplane sections do not correspond in any simple way to extremal neighborhoods $C \subset X$. Several of the cases come from infinitely many different families of extremal neighborhoods. The pull-back $f^*H$ frequently has much worse singularities than $H$.

The advantage of (2.3.6) is that one can view $Y$ as a family of fairly simple surface singularities. $X$ and $X^+$ appear as a family of modifications of these surface singularities. This makes it possible to have a reasonably explicit construction of $X^+$ in terms of $H$. This description is crucial in understanding more delicate properties of flips. One of the most important applications is to consider flips in families of three-dimensional contractible curve neighborhoods. From this point of view flipping in families is not harder than flipping the individual neighborhoods. This yields:

**2.3.7. Theorem** [60, 11.7]. *Let $f\colon C_t \subset X_t \to Y_t\colon t \in \Delta$ be a flat family of three-dimensional contractible curve neighborhoods. Assume that $X_0$ has only terminal singularities, $C_0 \cdot K_{X_0} < 0$, and $C_0$ is irreducible. Then the flips $X_t^+$ fit into a flat family over $\Delta$.*

**Comments.**

(2.3.7.1) It happens frequently that $C_t\colon t \neq 0$ is reducible.

(2.3.7.2) The opposites of a flat family of curve neighborhoods usually do not fit into a flat family. Consider for instance the family of vector bundles

$$0 \to \mathscr{O}_{\mathbb{P}^1}(-3) \to E_t \to \mathscr{O}_{\mathbb{P}^1}(-1) \to 0, \qquad t \in \operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}(-3)) \cong \mathbb{C}.$$

Then

$$E_t \cong \begin{cases} \mathscr{O}_{\mathbb{P}^1}(-2) + \mathscr{O}_{\mathbb{P}^1}(-2) & \text{if } t \neq 0, \\ \mathscr{O}_{\mathbb{P}^1}(-3) + \mathscr{O}_{\mathbb{P}^1}(-1) & \text{if } t = 0. \end{cases}$$

Let $C_t \cong \mathbb{P}^1 \subset X_t$ be the total space of $E_t$. Every $X_t$ is toric, and it is not hard to see that

$$X_t^+ \cong \begin{cases} \mathbb{P}^1 \times (x^2 + y^2 + z^2 = 0 \subset \mathbb{C}^2) & \text{if } t \neq 0, \\ \text{a curve neighborhood with an isolated} & \\ \quad \text{singularity of index 3} \quad \text{if } t = 0. & \end{cases}$$

Therefore the $X_t^+$ do not fit into a flat family. (This example also shows that the so-called "log-flip" is not continuous in families.)

(2.3.7.3) By using (2.3.7) the construction of flips leads to very interesting conjectures concerning deformations of rational singularities. These will be discussed in §6.

## 3. Applications of minimal models

**3.1. Further study of relative Fano models.** Let $X$ be an algebraic variety with $\mathbb{Q}$-factorial terminal singularities, which is a relative Fano model (1.4.8.2). Let $F \subset X$ be a general fiber. Then $F$ is an algebraic variety with terminal singularities such that $-K_F$ is ample. Thus, $F$ is a Fano variety, possibly with large Picard number.

In low dimensions the typical examples of Fano varieties are $\mathbb{P}^n$ and the smooth quadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$. Thus the basic problem about Fano varieties is:

**3.1.1. Problem.** How similar are Fano varieties to $\mathbb{P}^n$ ?

This is a very general question and of not much use without further clarification. Here are some—successively weaker—technical versions of being "similar to $\mathbb{P}^n$."

**3.1.2. Definition.** Let $X$ be an $n$-dimensional variety.

(3.1.2.1) $X$ is said to be rational if there is a generically one-to-one map $g: \mathbb{P}^n \dashrightarrow X$.

(3.1.2.2) $X$ is said to be unirational if there is a generically finite map $g: \mathbb{P}^n \dashrightarrow X$.

(3.1.2.3) $X$ is said to be rationally connected if through any two general points $x, y \in X$ there is an irreducible rational curve $C_{x,y} \subset X$.

(3.1.2.4) $X$ is said to be uniruled if there is an $(n-1)$-dimensional variety $Y^{n-1}$ and a generically finite map $g: \mathbb{P}^1 \times Y^{n-1} \dashrightarrow X$.

**3.1.3. Theorem.**

(3.1.3.1) [12]. *A smooth cubic threefold* $X_3 \subset \mathbb{P}^4$ *is not rational.*

(3.1.3.2) [41]. *A smooth quartic threefold* $X_4 \subset \mathbb{P}^4$ *is not rational.*

*Methods of proof.* The cubic case relies on the observation that the intermediate Jacobian is a birational invariant up to direct factors which are Jacobians of curves. Therefore, one needs to compute the intermediate Jacobian. This method is applicable to several other Fano threefolds and to conic bundles, but it seems to work only in dimension three.

Iskovskikh and Manin prove (along the lines indicated by Fano) that the birational automorphism group of a smooth quartic is finite. Such computations are fairly hard. This method again can be applied more generally, even in higher dimensions [102], [103], [94].

**3.1.4. Problems.** Is every Fano variety unirational? Is every Fano variety rationally connected?

Every smooth Fano threefold is rationally connected (cf. [38, 4.1]). Miyaoka [72] shows that if the Picard number is one, then there is a connected chain of rational curves through any two points. Unirationality is not known already for smooth quartic threefolds $X_4 \subset \mathbb{P}^4$. The answer is probably negative.

**3.1.5. Theorem** [73]. *Every Fano variety is uniruled, in fact it is covered by rational curves* $C$ *such that* $-C \cdot K_X \leq 2 \dim X$.

For smooth varieties the method of Mori [77] works; this was observed in [52]. In the singular case a more refined version is needed.

**3.1.6. Corollary** [73]. *Every relative Fano model is uniruled.*

**3.1.7. Problems.** (3.1.7.1) Are there only finitely many deformation types of Fano varieties with terminal singularities of a given dimension?

(3.1.7.2) Let $X$ be a Fano variety. Is the self-intersection of $-K_X$ bounded by a function depending only on the dimension? Is the index (1.3.2.2) of $X$ bounded by a function depending only on the dimension?

(3.1.7.3) Let $X$ be a Fano variety, and let $x, y \in X$ be sufficiently general points. Is there an irreducible rational curve $C_{xy}$ containing $x$ and $y$ such that

$$C_{xy} \cdot (-K_X) \leq (\text{some function of } \dim X)?$$

**Comments.** In dimension three there are 104 deformation types of smooth Fano varieties [36], [37], [83], [84].

Kawamata [48] proves (3.1.7.1) in dimension three under the additional assumption that the rank of the Picard group is one.

By [59], [66], (3.1.7.1) is equivalent to (3.1.7.2).

The argument of Iskovskikh [38, 4.1] shows that (3.1.7.3) also implies (3.1.7.1).

**3.1.8.** In dimension three the most interesting open question about relative Fano models is their birational classification. Given two relative Fano models how do we decide when they are birational? A somewhat simpler question is to decide which ones are rational. This problem is settled for most smooth Fano varieties [5] [38], [39], and a lot of work has been done about conic bundles [5], [102], [103]. See [40] for the conjectured rationality criterion.

**3.1.9. Problem.** (3.1.9.1) Find "standard" birational models for families of Del Pezzo surfaces over curves. By this we mean that given

$f: X \to C$ whose general fiber is a Del Pezzo surface, find a "standard form" $f': X' \to C$ and a birational map $g$ which fits into a commutative diagram:

$$\begin{array}{ccc} X & \stackrel{g}{\dashrightarrow} & X' \\ \downarrow f & & \downarrow f' \\ C & = & C \end{array}$$

For example, if the general fiber of $f$ is $\mathbb{P}^2$, then we can choose $X' = C \times \mathbb{P}^2$. By [14] $f$ always has a section. Maybe one can arrange that the section does not pass through any singular points, and every fiber is irreducible and reduced.

(3.1.9.2) Study the birational types of the "standard" birational models. Some results are due to Alekseev [1].

**3.2. Further study of minimal models.** Minimal models of a given threefold $X$ are not unique. In some cases there can be infinitely many nonisomorphic minimal models. Fortunately different minimal models are closely related:

**3.2.1. Theorem** [47], [55]. *Let $X_1$ and $X_2$ be three-dimensional minimal models with $\mathbb{Q}$-factorial singularities. Then any birational map $f: X_1 \dashrightarrow X_2$ can be obtained as a composite of flops.*

*Proof.* First one needs to prove that $f$ is an isomorphism in codimension one and that $K_{X_1}$ is trivial along the locus $\bigcup C_i$, where $f$ is not defined [55, 4.3]. If $f$ is not an isomorphism, then by (2.1.12) one of the $C_i$ can be flopped. By [55, 2.4] after finitely many flops we get $X_2$. q.e.d.

Since minimal models are not unique, we need to find properties that are invariant under flops. There are surprisingly many such:

**3.2.2. Theorem.** *Let $X$ be a three-dimensional $\mathbb{Q}$-factorial minimal model. Then the following objects are unchanged under flops, and hence do not depend on the $\mathbb{Q}$-factorial minimal model chosen:*

(3.2.2.1) *The intersection homology groups $IH^i(X, \mathbb{C})$ together with their Hodge structures* [44, 4.12].

(3.2.2.2) *The collection of analytic singularities of $X$* [55, 4.11].

(3.2.2.3) *The miniversal deformation space* $\operatorname{Def} X$ [60, 12.6].

(3.2.2.4) *The integral cohomology groups $H^i(X, \mathbb{Z})$.*

(3.2.2.5) $\operatorname{Pic} X \subset \operatorname{Weil} X$.

(3.2.2.6) $h^0(X, \mathcal{O}(D))$ *for every Weil divisor $D$.*

(3.2.2.7) $h^i(X, \mathcal{O}(mK_X))$ *for every $i$ and $m$.*

*Proof.* (3.2.2.2) follows from (2.2.4).

If $C \subset X \dashrightarrow C^+ \subset X^+$ is a flop, then the cohomology of $X$ and of $X^+$ can be computed from a Mayer-Vietoris sequence involving $X - C$, a neighborhood $U$ of $C$, and $U - C$. $U$ retracts to $C \cong \mathbb{P}^1$, thus the only nontrivial cohomologies of $U$ are $H^0$ and $H^2$. Using (2.2.9) we get that the sequences for $X$ and $X^+$ are isomorphic. (The product structure is not an invariant).

The group of Weil divisors modulo linear equivalence is clearly invariant under flops. The invariance of $\operatorname{Pic} X$ follows from (2.2.9). Any two minimal models are isomorphic in codimension one; this implies (3.2.2.6).

If $X \to Y \leftarrow X^+$ is a flop, then

$$h^i(X, \mathscr{O}(mK_X)) = h^i(Y, \mathscr{O}(mK_Y)) = h^i(X^+, \mathscr{O}(mK_{X^+})).$$

This essentially follows from the fact that $K_X$ is the pullback of $K_Y$ and that $Y$ has rational singularities.

Concerning (3.2.2.1) it is worth pointing out that if $X'$ is any smooth projective variety birational to $X$, then $IH^i(X, \mathbb{C})$ is naturally a direct summand of $H^i(X', \mathbb{C})$.

**3.2.3.** The whole point of constructing minimal models was to simplify the global structure of the canonical bundle. The following immediate consequence of (1.4.4) is the first important general result exploiting this:

**3.2.4. Theorem** [6], [44], [45], [97], [105]. *Let $X$ be a proper algebraic variety with only canonical singularities. Assume that $K_X$ is nef and big. Then $|mK_X|$ has no basepoints for all $m \gg 0$, $\operatorname{index} X | m$.*

**3.2.5. Problem.** Let $X$ be a proper algebraic variety with only canonical singularities. Assume that $K_X$ is nef and big. Is there a constant $N$ depending only on the dimension such that $|mK_X|$ determines a birational map for $m \geq N$?

**Comments.** It is certainly not true that one can even get a morphism. This is prevented by the presence of high index singular points.

There has been some positive results in dimension three. If one assumes that $\chi(\mathscr{O}_N) \leq 1$, then one can take $N = 269$ [22]. In general, if $\chi(\mathscr{O}_X) \leq k$, then there is a bound depending on $k$. $\chi(\mathscr{O}_X)$ can be arbitrarily large for threefolds of general type if the minimal model is sufficiently singular. Compare (3.2.6.2) and (3.2.7).

In dimension three the above problem reduces to finding an $N_0$ such that the $m$th-plurigenus is at least 2 for $m \geq N_0$. For this and other purposes it is very useful to have a plurigenus formula. As usual, we can compute only $\chi(\mathscr{O}_X(mK_X))$. Singular Riemann-Roch gives a formula, but an explicit computation of the occurring terms is not easy.

**3.2.6. Plurigenus Formula.** Let $X$ be a proper threefold with terminal singularities. As was mentioned in (1.3.4.4), and arbitrary terminal singularity $(P_i \in X)$ deforms to a collection $S_i$ of cyclic quotient singularities of the form $\mathbb{C}^3/\mathbb{Z}_r(1, -1, a)$. Let $S = S(X)$ be the collection of all the $S_i$ (with multiplicities counted).

For a pair of integers $r, n$ define $\overline{n}$ by $0 \le \overline{n} < r$ and $n \equiv \overline{n} \pmod r$.

The following is the Barlow-Fletcher-Reid plurigenus formula [100, 10.3]:

$$\begin{aligned}\chi(\mathcal{O}_X(mK_X)) = {} & \frac{m(m-1)(2m-1)}{12}K_X^{(3)} + (1-2m)\chi(\mathcal{O}_X) \\ & + \sum_S \left\{ (m-\overline{m})\frac{r^2-1}{12r} + \sum_{j=0}^{\overline{m}-1} \frac{\overline{aj}(r-\overline{aj})}{2r} \right\}\end{aligned} \tag{3.2.6.1}$$

and

$$\chi(\mathcal{O}_X) = -\frac{1}{24}K_X \cdot c_2(X) + \sum_S \frac{r^2-1}{24r}. \tag{3.2.6.2}$$

One can rewrite (3.2.6.1) as

$$\begin{aligned}\chi(\mathcal{O}_X(mK_X)) = {} & \frac{m(m-1)(2m-1)}{12}K_X^{(3)} + \frac{m}{12}K_X \cdot c_2 + \chi(\mathcal{O}_X) \\ & + \sum_S \left\{ -\overline{m}\frac{r^2-1}{12r} + \sum_{j=0}^{\overline{m}-1} \frac{\overline{aj}(r-\overline{aj})}{2r} \right\}.\end{aligned} \tag{3.2.6.3}$$

**Comments.** Since $X$ has only isolated singularities, $c_2$ makes sense. Alternatively, one can take $c_2$ of any resolution of singularities and push it down to $X$.

The sum in (3.2.6.3) is a periodic function whose period divides $\operatorname{index}(X)$. Also, if $\operatorname{index}(X)|m$, then the sum is zero, hence we have only the polynomial part.

It is conceivable that (3.2.5) can be answered by understanding the combinatorics of (3.2.6.3). It is not at all clear what the nature of the periodic part is.

The following result helps us understand $K_X \cdot c_2$:

**3.2.7. Theorem.** *Let $X$ be a projective threefold. Assume that $K_X$ is nef.*

(3.2.7.1) [121], [112] *If $X$ is smooth, then*

$$K_X^{(3)} \le \tfrac{8}{3}K_X \cdot c_2 = -64\chi(\mathcal{O}_X).$$

(3.2.7.2) [64] *If $X$ has isolated singularities (e.g., if $X$ is a minimal model), then*

$$K_X^{(3)} \le 3K_X \cdot c_2.$$

If $K_X$ is not big, then we do not have a vanishing for the higher cohomologies of $\mathscr{O}_X(mK_X)$ on a minimal model. Still the above results can be used to gain some information about the plurigenera:

**3.2.8. Theorem** [69], [70], [71]. *Let $X$ be a three-dimensional minimal model. Then $h^0(X, \mathscr{O}_X(mK_X)) \geq 1$ for some $m > 0$.*

The above two results of Miyaoka are some of the least understood theorems in threefold theory. I think that it would be worthwhile to investigate them again for simplification or clarification.

The following is one of the basic open problems in the theory of minimal models:

**3.2.9. Abundance Conjecture** [98, 4.6]. Let $X$ be an $n$-dimensional minimal model. Then $\mathscr{O}_X(mK_X)$ is generated by global sections for some $m > 0$.

There are some results of Kawamata [46] and Miyaoka [70] about special cases. It seems that even in dimension three new ideas are needed for the proof.

Finally we pose a problem about generalizing the canonical bundle formula of elliptic surfaces to higher dimensional minimal models.

**3.2.10. Conjecture** (canonical Bundle Formula). Let $X$ be an $n$-dimensional minimal model. Assume that $\mathscr{O}_X(mK_X)$ is generated by global sections for some $m > 0$. Let $g: X \to I(X)$ be the Stein factorization of the corresponding morphism, and $\Delta \subset I(X)$ be the locus of the singular fibers. Then there are

(i) a nef line bundle $L$ on $I(X)$,
(ii) an effective divisor $\sum a_i D_i$ on $I(X)$ such that $\bigcup D_i \subset \Delta$;
(iii) An integer $m > \max a_i$,

such that

$$\mathscr{O}_X(mK_X) \cong g^*\left(L \otimes \mathscr{O}_{I(X)}\left(\sum a_i D_i\right) \otimes \mathscr{O}_{I(X)}(mK_{I(X)})\right).$$

To be more precise one would like to show that $a_i$ depends only on the degeneration of certain Hodge structures along $D_i$. See [115], [25], [81, 5.13] for closely related results.

**3.3. Applications to threefolds.** While the current point of view in three-dimensional algebraic geometry is that minimal models are the basic objects, it is important to note that the theory can be successfully applied to several old problems that seemed completely intractable before. The first such examples are about various birationally invariant properties of threefolds. It is natural to expect that these are easier to study on a birational model whose global structure is comparatively simple.

**3.3.1. Theorem** [73], [70] ($\kappa = -\infty$ characterization). *Let $X$ be a smooth projective threefold. Then the following three statements are equivalent*:

(3.3.1.1) *There is a rational curve through every point of $X$.*

(3.3.1.2) *$X$ is uniruled.*

(3.3.1.3) $H^0(X, \omega_X^{\otimes i}) = 0$ *for every* $i > 0$.

*Proof.* Consider all components of the Hilbert scheme of rational curves on $X$. There are only countably many such. If there is a rational curve through every point, then one of these components must give a dominant map. If necessary, we can take a lower-dimensional subvariety to get a generically finite dominant map.

If $X$ is uniruled, then there is a rational curve through a general point of $X$. If we specialize to any $x \in X$, this rational curve will specialize to a collection of rational curves, one of which will pass through $x$. If $g: \mathbb{P}^1 \times Y \dashrightarrow X$ is generically finite, then

$$h^0(X, \omega_X^{\otimes i}) \leq h^0(\mathbb{P}^1 \times Y, \omega_{\mathbb{P}^1 \times Y}^{\otimes i}) = h^0(Y, \omega_Y^{\otimes i}) \cdot h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes i}) = 0.$$

Assume (3.3.1.3). Let $X'$ be the model whose existence is guaranteed by (1.4.9). By (3.2.8) it cannot be a minimal model. Therefore, $x'$ is a relative Fano model, thus $X'$ and hence $X$ are uniruled by (3.1.6).

**3.3.2. Theorem** [44], [6], [25] (Finite generation of the canonical ring). *Let $X$ be a smooth projective threefold. Then the canonical ring*

$$R(X) = \sum_{i=0}^{\infty} H^0(X, \omega_X^{\otimes i})$$

*is finitely generated.*

*Proof.* If $X$ is uniruled, then $R(X) \cong \mathbb{C}$. The canonical ring is a birational invariant, therefore it is sufficient to consider the case when $X$ is a minimal model. If $K_X$ is big, then (3.2.4) and [123] imply the result. If $\kappa(X) \leq 1$, then the claim is fairly easy [23]. The remaining case is when $X$ is an elliptic fiber space over a surface $S$. Here one uses a version of (3.2.10) to reduce the problem to $S$ [25]. The corresponding finite generation problem for surfaces was solved by Fujita [24]. q.e.d.

Minimal models also help in trying to understand the group of birational self-maps: $\operatorname{Bir} X$. This direction was started by Hanamura [31] who later proved several of his results without assuming the existence of minimal models [32]. Here we mention a related result:

**3.3.3. Proposition.** *Let $X$ be a three-dimensional minimal model. Then there is a natural representation*

$$\operatorname{Bir} X \to \operatorname{Pic} X / \operatorname{Pic}^{\tau} X,$$

*whose kernel is a compact group.*

*Proof.* Let $p: X \dashrightarrow X$ be a birational map. For any Cartier divisor $D$ the divisor $p^*D$ is again Cartier by (3.2.2.5). This defines the representation. If $H$ is an ample divisor and $p^*H$ is also ample, then $p$ is an automorphism by (2.1.6) q.e.d.

The existence of flips in families (2.3.7) implies that suitably chosen minimal models of a flat family of smooth threefolds again form a flat family. This way one can prove some results about families of threefolds.

**3.3.4. Theorem** [60, 1.3] (Deformation invariance of plurigenera). *Let $\{X_t : t \in T\}$ be a flat family of smooth projective threefolds. Assume that $T$ is connected, and that for some $0 \in T$ and some $m > 0$ we have $h^0(X_0, \omega_{X_0}^{\otimes m}) \geq 2$. Then $h^0(X_t, \omega_{X_t}^{\otimes n})$ is independent of $t \in T$ for every $n$.*

The assumption $h^0(X_0, \omega_{X_0}^{\otimes m}) \geq 2$ is probably not needed.

**3.3.5. Theorem** [60, 1.4] (Moduli space for threefolds of general type). *Let $\mathcal{M}$ be the functor "families of threefolds of general type modulo birational equivalence," i.e.,*

$$\mathcal{M}(S) = \left\{ \begin{array}{l} \textit{Smooth projective families } X/S \textit{ such that every fiber is a} \\ \textit{threefold of general type. Two families } X^1/S \textit{ and } X^2/S \\ \textit{are equivalent if there is a rational map } f: X^1/S \dashrightarrow X^2/S \\ \textit{which induces a birational equivalence on each fiber.} \end{array} \right.$$

*Then there is a separated algebraic space* $\mathbf{M}$ *which coarsely represents* $\mathcal{M}$. *Every connected component of* $\mathbf{M}$ *is of finite type.*

**3.4. Applications to deformations of surface singularities.** Let $S_t : t \in \Delta$ be a one-parameter family of surfaces. Then the total space $X = \bigcup S_t$ is a threefold. Studying birational models of $X$ may lead to various results on families of surfaces. A special case of this problem, when $S_t : t \in \Delta$ is a degeneration of $K3$ surfaces, was the first instance that flops appeared in the literature as a tool [62].

**3.4.1. Degenerations of surfaces.** Let $S_t : t \in \Delta$ be a one-parameter family of surfaces. We assume that $S_t : t \neq 0$ is smooth but $S_0$ may be even reducible. By the semistable reduction theorem [50, Chapter 2] after a suitable base change we can take a new degeneration $S'_t : t \in \Delta$ such that $S'_0$ is reduced with normal crossings only as singularities. Let $X' = \bigcup S'_t$ be the total space of the family.

Due to the special structure of $X'$ the minimal model program for $X'$ is easier than that for arbitrary threefolds. This special case of (1.4.9) was settled earlier by [114], [106], [79], [80], [47] (the latter is the only complete published proof). The end result is the following:

**3.4.2. Theorem.** *Assume that* $S_t: t \neq 0$ *are not ruled. Then, the relative minimal model of* $X'/\Delta$ *is a threefold* $\overline{X}/\Delta$ *satisfying the following properties*:

(3.4.2.1) $\overline{X}$ *has only* $\mathbb{Q}$*-factorial terminal singularities, and* $K_{\overline{X}}$ *is nef on every fiber* $\overline{S}_t$.

(3.4.2.2) $\overline{S}_t$ *for* $t \neq 0$ *is a smooth minimal surface.*

(3.4.2.3) $\overline{S}_0$ *is reduced with only the following types of singularities*:

(i) $(xy = 0) \subset \mathbb{C}^3$ *or* $(xyz = 0) \subset \mathbb{C}^3$;

(ii) $(t = 0)/\mathbb{Z}_n \subset (xy + f(z^n, t) = 0)/\mathbb{Z} \subset \mathbb{C}^4/\mathbb{Z}_n(1, -1, a, 0)$, *where* $(a, n) = 1$.

**Comments.** If one restricts the birational type of the general fiber $S_t$, then frequently one can restrict the class of singularities even more. These questions have been worked out in detail for the case where $S_t$ is trivial canonical class.

The above result can serve as a guiding principle to determine which singular surfaces should appear at the boundary of the moduli of surfaces. The best choice seems to be the one given in [61, Chapter 5].

The following question is the only missing ingredient in the construction of a compactification of the moduli of surfaces of general type:

**3.4.3. Problem.** Find a bound on the order of the group occurring in (3.4.2.3ii) in terms of the general fiber $S_t: t \neq 0$ alone. This is not known even when $S_t$ is a quintic in $\mathbb{P}^3$.

Let $f: X \to Y$ be a flat projective family of surfaces. In general the minimal resolutions of the fibers do not form a flat family. The following result gives a necessary and sufficient condition.

**3.4.4. Theorem** ([63] for Gorenstein singularities; [61, 2.10] in general). *Let* $f: X \to Y$ *be flat family of projective surfaces with isolated singularities. Assume that* $Y$ *is connected. Then the following are equivalent*:

(3.4.4.1) *The self-intersection* $K_{\overline{X}_y} \cdot K_{\overline{X}_y}$ *of the canonical class of the minimal resolution* $\overline{X}_y$ *of the fiber* $X_y$ *is independent of* $y \in Y$.

(3.4.4.2) *There is a smooth family of projective surfaces* $\overline{f}: \overline{X} \to \overline{Y}$ *and a finite and surjective morphism* $p: \overline{Y} \to Y$ *such that* $(\overline{X})_y \cong \overline{X}_{p(y)}$ *for every* $y \in \overline{Y}$.

**3.4.5. Definition.**

(3.4.5.1) For a sequence of natural numbers $a_1, \cdots, a_n$ we define the continued fraction $[a_1, \cdots, a_n]$ recursively by

$$[a_n] = a_n, \qquad [a_i, \cdots, a_n] = a_i - \frac{1}{[a_{i+1}, \cdots, a_n]}.$$

(3.4.5.2) Let $K_{e-2}$ be the set of sequences $k_2, \cdots, k_{e-1}$ such that $[k_2, \cdots, k_{e-1}] = 0$. (The indices are chosen to work well for the next definition.) It is known (see, e.g., [110]) that

$$\# K_{e-2} = \frac{1}{e-2}\binom{2e-6}{e-3}.$$

(3.4.5.3) Given a cyclic quotient singularity $\mathbb{C}^2/\mathbb{Z}_n(1, q)$ let

$$\frac{n}{n-q} = [a_2, \cdots, a_{e-1}].$$

($e$ is the multiplicity of the quotient singularity.)

(3.4.5.4) For a cyclic quotient singularity $\mathbb{C}^2/\mathbb{Z}_n(1, q)$ let

$$K(\mathbb{C}^2/\mathbb{Z}_n(1, q)) = \{[k_2, \cdots, k_{e-1}] \in K_{e-2} | k_i \le a_i \ \forall i\}.$$

**3.4.6. Theorem** [61], [11], [110]. *The number of irreducible components of the versal deformation space of* $\mathbb{C}^2/\mathbb{Z}_n(1, q)$ *is exactly* $K(\mathbb{C}^2/\mathbb{Z}_n(1, q))$.

**Comments.** [61, Chapter 3] establishes a one-to-one correspondence between the components of the versal deformation space and certain partial resolutions of the singularity. The above formula was found by Christophersen by studying explicit equations of the versal deformation space. Some technical details of his proof are still unfinished. Based on the observation of Christophersen, Stevens showed that there is a one-to-one correspondence between the partial resolutions studied in [61] and the above continued fractions.

In§6 we will consider the possibility of extending some of these results to deformations of arbitrary rational surface singularities.

**3.5. Applications to the resolution of singularities.** Two-dimensional singularities have unique minimal resolutions, and this is very useful in their study. In dimension three there is no unique minimal resolution, and this is one of the reasons why we know much less about three-dimensional singularities.

As a consequence of the Minimal Model Program we obtain two candidates as substitutes for the minimal resolution. Let us formulate their existence as a separate theorem:

**3.5.1. Theorem** [82]. *Let $X$ be a three-dimensional algebraic variety (not necessarily proper). Then there are two birational modifications*

$$X^{\text{term}} \xrightarrow{t} X^{\text{can}} \xrightarrow{c} X$$

*with the following properties*:

(3.5.1.1) $X^{\text{term}}$ *has only $\mathbb{Q}$-factorial terminal singularities, and* $X^{\text{can}}$ *has only canonical singularities.*

(3.5.1.2) $K_{X^{\text{can}}}$ *is c-ample*; $K_{X^{\text{term}}}$ *is* $(t \circ c)$*-nef.*

(3.5.1.3) $X^{\text{can}}$ *is unique*; $X^{\text{term}}$ *is unique up to flops.*

(3.5.1.4) $t$ *is crepant, i.e.*, $t^* K_{X^{\text{can}}} \equiv K_{X^{\text{term}}}$.

Since we understand three-dimensional canonical and terminal singularities reasonably well, these objects can be viewed as suitable intermediaries between $X$ (which has arbitrary singularities) and a resolution (which is locally nice but globally uncontrollable).

Both (3.4.4) and (3.4.6) can be viewed as applications of this principle. We hope that in the future (3.5.1) will be used even more.

(3.5.2) Here we want to explain how the point of view given by minimal models helps in the resolution process of singularities. This answers an "old dream of many specialists" [28, p. 60]. (To be fair, this is probably not the kind of answer the specialists dreamed about.)

The Jungian method [28] starts with a threefold $X$ and a finite morphism onto a smooth threefold $f: X \to Y$ (e.g., $Y = \mathbb{P}^3$). By embedded resolution of surfaces we can assume that the branch locus $B \subset Y$ is a divisor with normal crossings. This implies that the singularities of $X$ are toroidal, hence it is much easier to resolve them. In dimension three it is however not clear how the local toroidal resolutions can be patched together (especially if we want to preserve projectivity). This is the point where minimal models come in.

Since $X^{\text{can}}$ is unique and locally definable, it is sufficient to construct it locally. The local existence of $X^{\text{can}}$ follows from the general result, but in the toroidal case it is much easier and was established by [99]. Thus we obtain $X^{\text{can}}$.

In dimension three the structure of canonical singularities is sufficiently understood to proceed with the resolution.

Reid [96], [98] investigates partial resolutions of canonical singularities. He shows how to construct $t: X^{\text{term}} \to X^{\text{can}}$ as a sequence of explicit blow-ups. Most of the time we need to blow up only closed points and curves, but in certain cases some other ideal sheaves are blown up. If the higher-dimensional minimal model program works, then this step should also be possible in higher dimensions.

$X^{\text{term}}$ has only isolated, terminal and toroidal singularities. For most applications it is better not to resolve these. Also, the resolution process is now completely local: there are no compatibility questions involved if the local resolution process is via blowing up closed subvarieties that dominate the singular points.

## 4. How to find extremal rays?

In this section the characteristic of the base field is arbitrary.

The definition of extremal rays is very simple, and in some cases it is easy to exhibit all extremal rays on a threefold. However, by the nature of the definition this requires an overview of all curves on a threefold. This is sometimes very hard. Therefore it is of interest to find other methods that are more local in nature.

Let us review the situation for surfaces. There are three kinds of extremal rays.

If $C \subset S$ is a $(-1)$-curve, then this fact is shown by the first order infinitesimal neighborhood of $C$ in $S$. Therefore no global information is required to assert that $C$ is a $(-1)$-curve, and hence generates an extremal ray.

If $C \subset S$ is a fiber of a $\mathbb{P}^1$-bundle or a line $\mathbb{P}^2$, then local information is not sufficient. Indeed, we can always destroy the extremality of $C$ by blowing up any point away from $C$. Thus we need some global information. Let $C \subset S$ be a smooth rational curve of self-intersection 0 or 1. Then $C$ generates an extremal ray iff every deformation of $C$ stays irreducible. Thus again a fairly limited amount of global information allows us to decide whether $C$ generates an extremal ray or not.

The aim of this section is to consider the analogous problem for threefolds. Even for smooth threefolds I am unable to give a complete answer at this moment.

**4.1. Seemingly extremal rays.**

**4.1.1. Definition.** Let $X$ be an algebraic space.

(4.1.1.1) Let $Z_k^+X$ be the free abelian semigroup generated by closed $k$-dimensional irreducible and reduced subspaces of $X$. Thus an element of $Z_k^+X$ is of the form

$$\sum a_i[V_i],$$

where the $V_i$ are $k$-dimensional closed, irreducible and reduced subspaces, $a_i > 0$ are natural numbers, and the sum is finite.

(4.1.1.2) We define effective algebraic equivalence for elements of $Z_1$, $Z_2 \in Z_k^+X$ as follows:

(i) If $W$ is a normal (possibly reducible) pure dimensional variety, and $p: W \to C$ is a flat morphism onto a smooth connected curve, then any fibers are effectively algebraically equivalent.

(ii) If $g: W \to X$ is a proper morphism, and $Z_1, Z_2 \in Z_k^+W$ are effectively algebraically equivalent, then $g_*[Z_1], g_*[Z_2] \in Z_k^+X$ are also effectively algebraically equivalent, where $g_*$ is the push-forward of cycles [26, 1.4].

(iii) Finally, take the transitive hull of the relation given by (i) and (ii);

i.e., this definition is like the definition of algebraic equivalence but we require all intermediate cycles to be effective.

Effective algebraic equivalence of $Z_1$, $Z_2 \in Z_k^+X$ will be denoted by

$$Z_1 \overset{\text{eff}}{\approx} Z_2 .$$

(4.1.1.3) Given a cycle $Z$ the symbol $\mathbb{Q}^+_{\text{eff}}[Z]$ will denote all effective cycles $Z_i$ such that a positive multiple of $Z$ is effectively algebraically equivalent to a positive multiple of $Z_i$.

One reason for this definition is that effective algebraic equivalence has the following property:

**4.1.2. Proposition.** *Let $f: X \to Y$ be a proper morphism between algebraic spaces. Let $\sum a_j[V_j]$ and $\sum b_j[U_j]$ be two effective k-cycles which are effectively algebraically equivalent. Assume that every $V_j$ is contained in a fiber of $f$. Then every $U_j$ is also contained in a fiber of $f$.*

*Proof.* Choose morphisms $p_i: W_i \to C_i$, fibers $A_i, B_i \subset W_i$ of $p_i$, and proper morphisms $g_i: W_i \to X$ which show the effective algebraic equivalence between $\sum a_j[V_j]$ and $\sum b_j[U_j]$; i.e.,

$$\sum a_j[V_j] = g_{1*}[A_1], \qquad g_{i*}[B_i] = g_{i+1*}[A_{i+1}] \quad \text{for } i = 1, \cdots, n-1,$$
$$g_{n*}[B_n] = \sum b_j[U_j].$$

By induction on $i$ we get that $f \circ g_i: W_i \to Y$ maps $A_i$ to a finite point set. Therefore by [13, 1.5] $f \circ g_i$ factors through the Stein factorization of $p_i$, hence $f \circ g_i(B_i) = f \circ g_{i+1}(A_{i+1})$ is again a finite point set. q.e.d.

Algebraic equivalence does not have the above property if $Y$ is non-projective:

**4.1.3. Example.** Let $Z \subset \mathbb{P}^4$ be a cubic hypersurface with a single ordinary node locally given by the equation $xy - uv = 0$. Let $p: X \to Z$ be the blow-up of this node. The exceptional divisor of $p$ is $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $L_1, L_2 \subset E$ be two intersecting lines. The normal bundle of $E$ is

$\mathscr{O}_E(-1, -1)$. Therefore $E \subset X$ can be contracted in any of the fiber directions to get a pair of smooth nonprojective threefolds $f_i \colon X \to X_i$. $f_i(L_i) =$ point but $f_i(L_{3-i})$ is one dimensional. This shows that the $L_i \notin \mathbb{Q}^+_{\mathrm{eff}}[L_{3-i}]$.

However, $L_1$ and $L_2$ are algebraically equivalent. This can be seen as follows. The projection of $Z$ from the node exhibits $X$ as the blow-up of $\mathbb{P}^3$ along a curve of type (3,3) on a smooth quadric surface $Q$. Let $F \subset X$ be the exceptional set of the projection $q \colon X \to \mathbb{P}^3$. The restriction $q \colon E \to Q$ is an isomorphism. If a general line $l \subset \mathbb{P}^3$ degenerates to a line on $Q$, then in the corresponding limit on $X$ we obtain a line on $E$ and three lines on $F$. Thus, $L_1$ and $L_2$ are algebraically equivalent on $X$.

It is worthwhile to note that if $Z \subset \mathbb{P}^4$ is a generic hypersurface of degree at least five with a single ordinary node, then the corresponding lines $L_i$ are not algebraically equivalent [10].

**4.1.4. Definition.** Let $X$ be a proper algebraic space (smooth or possibly with canonical singularities).

(4.1.4.1) A curve $C \subset X$ is said to generate a seemingly extremal ray $R = \mathbb{Q}^+_{\mathrm{eff}}[C]$ if the following conditions hold:

(i) $C \cdot K_X < 0$;

(ii) if $Z_1, Z_2 \in Z_1^+ X$ are two 1-cycles such that

$$Z_1 + Z_2 \in \mathbb{Q}^+_{\mathrm{eff}}[C],$$

then $Z_1$ is numerically equivalent to a multiple of $C$.

(4.1.4.2) A morphism $f \colon X \to Y$ is called the contraction associated with $R$ if the following conditions hold:

(i) $Y$ is normal and $f_* \mathscr{O}_X = \mathscr{O}_Y$;

(ii) an irreducible curve $D \subset X$ is mapped to a point by $f$ iff there is a cycle $Z_2$ such that $D + Z_2 \in \mathbb{Q}^+_{\mathrm{eff}}[C]$. $R$ is called contractible if $f$ exists.

**4.1.5. Comments.**

(4.1.5.1) If $C$ generates an extremal ray, then $C$ also generates a seemingly extremal ray. This is clear from the definition.

(4.1.5.2) It may be more natural to require in (4.1.4.1) that $Z_1 \in \mathbb{Q}^+_{\mathrm{eff}}[C]$. This is a stronger restriction. I do not know if the two versions are equivalent. At the moment I see some advantages of both.

(4.1.5.3) The definition corresponds to what we observed in the surface case: The only information used is about deformations of multiples of $C$.

Allowing multiples of $C$ and deformations over several irreducible curves is necessary for threefolds.

(4.1.5.4) My hope is that on a projective threefold a seemingly extremal ray looks very much like an extremal ray and that for nonprojective threefolds this is the correct generalization of the notion of extremal rays. There are several results supporting the first hope. Unfortunately, I know very little about the nonprojective case that supports the second hope.

(4.1.5.5) It seems that in positive characteristic there are more deformations of curves than over $\mathbb{C}$. Therefore it is possible that the above definition needs to be modified over $\mathbb{C}$. In the theorems this is reflected by the fact that over $\mathbb{C}$ we need to assume the existence of certain rational curves whose existence can be proved in positive characteristic.

**4.1.6. Proposition.** *Let $X$ be a smooth projective surface and let $Z_1$, $Z_2 \in Z_1^+X$ be 1-cycles. Then the following hold:*

(4.1.6.1) *If* $\dim \operatorname{Alb} X = 0$ *or if $Z_1$ is irreducible and $Z_1 \cdot Z_1 > 0$, then*

$$Z_1 \equiv Z_2 \quad \textit{iff} \quad \exists m > 0 \textit{ such that } mZ_1 \overset{\text{eff}}{\approx} mZ_2 .$$

(4.1.6.2) *An irreducible curve $C \subset X$ generates an extremal ray iff it generates a seemingly extremal ray. If $D \in Z_1^+X$, then*

$$[D] \in \mathbb{R}^+[C] \Leftrightarrow D \in \mathbb{Q}^+_{\text{eff}}[C].$$

*Every seemingly extremal ray is contractible.*

(4.1.6.3) *The usual cone theorem holds for $\overset{\text{eff}}{\approx}$ in the following form: If $H$ is ample on $X$ and $\varepsilon > 0$ are fixed, then for any $Z \in Z_1^+X$ there is an effective cycle*

$$B + \sum a_i E_i \in \mathbb{Q}^+_{\text{eff}}[Z]$$

*such that the $E_i$ are extremal rational curves and $\varepsilon B \cdot H \geq B \cdot (-K_X)$.*

*Proof.* $Z_1 \equiv Z_2$ if $\exists m > 0$ such that $[mZ_1] = [mZ_2]$ as elements in $NS(X)$. Thus if $\dim \operatorname{Alb} X = 0$, then $mZ_1$ and $mZ_2$ are linearly equivalent. If $Z_1$ is irreducible, $Z_1 \cdot Z_1 > 0$, and $m$ is sufficiently large, then every linear system numerically equivalent to $mZ_1$ is nonempty, thus we can deform $mZ_1$ into $mZ_2$.

If $C \subset X$ generates an extremal ray, then by definition it also generates a seemingly extremal ray. Conversely, assume that $C$ generates a seemingly extremal ray. We can write

$$C \equiv B + \sum a_i E_i , \tag{4.1.6.4}$$

where $\varepsilon B \cdot H \geq B \cdot (-K_X)$, and the $E_i$ are extremal rational curves. If $C \cdot C > 0$, then (4.1.6.4) converts into an effective algebraic equivalence by (4.1.6.1).Thus $\mathbb{Q}^+_{\text{eff}}[C] = \mathbb{Q}^+_{\text{eff}}[E_1]$. If $C \cdot C \leq 0$, then either $C$ is a $(-1)$-curve, and we are done, or $C$ is a fiber of a not necessarily minimal ruled fibration and for such curves the claim is clear. This shows (4.1.6.2).

It is sufficient to show (4.1.6.3) for irreducible curves. If $C \cdot C > 0$, then the usual cone theorem implies (4.1.6.3). If $C$ is irreducible and $C \cdot C \leq 0$, then the previous argument works.

**4.1.7. Example.** Even for surfaces $\overset{\text{eff}}{\approx}$ may behave unusually. For instance let $E$ be an elliptic curve, and let $L$ be a nontorsion degree zero line bundle. Let $X = \operatorname{Proj}_E(\mathcal{O} + L)$. Then, $X$ has two sections $S_1$ and $S_2$ with self-intersection zero. These sections are algebraically equivalent but $S_i \notin \mathbb{Q}^+_{\text{eff}}[S_{3-i}]$.

For threefolds there are only partial results.

**4.1.8. Definition.** Assume that $C \subset X$ generates a seemingly extremal ray $R = \mathbb{Q}^+_{\text{eff}}[C]$.

(4.1.8.1) We say that $R$ covers $X$ if for every $x \in X$ there is a 1-cycle $Z_x \in \mathbb{Q}^+_{\text{eff}}[C]$ such that $x \in \operatorname{supp} Z_x$.

(4.1.8.2) We say that $R$ rationally covers $X$ if in addition there is a rational component of $Z_x$ containing $x$. If $x$ is smooth, then in positive characteristic these notions are equivalent (cf. [13, 1.8]).

**4.1.9. Theorem.** [36, §4] *Let $X$ be a normal threefold with $\mathbb{Q}$-factorial singularities. Assume that $C \subset X$ generates a seemingly extremal ray $R$ which rationally covers $X$. Then $R$ is contractible, and we have one of the following cases:*

(4.1.9.1) $\dim N_1(X) = 1$ ; *in particular, $X$ is Fano.*

(4.1.9.2) $\dim N_1(X) = 2$ *and the contraction $g: X \to Y$ is onto a smooth curve $Y$. The fibers of $g$ are all irreducible.*

(4.1.9.3) *The contraction $g: X \to Y$ is onto a normal projective surface $Y$. If the characteristic is different from two, then $g$ is generically a $\mathbb{P}^1$ bundle (in the étale topology). In characteristic two the generic fiber can also be a planar double line.*

*Proof.* Conditions (4.1) and (4.3) of [56] are clearly satisfied, hence the results follow from [ibid. 4.5-6].

**4.1.10. Theorem** [56, 1.4]. *Let $X$ be a smooth projective threefold whose Kodaira dimension is nonnegative, i.e., $h^0(X, K_X^{\otimes m}) > 0$ for some $m > 0$. Assume that $C \subset X$ generates a seemingly extremal ray $R$. In characteristic zero assume in addition that $C$ is rational. Then $R$ is contractible. The collection of all curves in $R$ covers a surface $E$, and we have one of the following situations:*

(4.1.10.1) *$E$ is a smooth minimal ruled surface with typical fiber $C$ and $C \cdot E = -1$. Only the fibers of $E$ are in $R$.*

(4.1.10.2) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathscr{O}(-1)$.

(4.1.10.3) $E \cong \mathbb{P}^2$ *and its normal bundle is* $\mathscr{O}(-2)$.

(4.1.10.4) $E \cong Q$, *where* $Q$ *is a quadric cone in* $\mathbb{P}^3$, *and its normal bundle is* $\mathscr{O}_{\mathbb{P}^3}(-1)|Q$.

The cone theorem for $\overset{\text{eff}}{\approx}$ is not known for smooth threefolds. The proof of [78, 1.4.1] gives the following:

**4.1.11. Theorem** [78, 1.4.1]. *Let* $X$ *be a smooth projective variety over a field of positive characteristic. If* $H$ *is ample on* $X$ *and* $\varepsilon > 0$ *are fixed, then for any* $Z \in Z_1^+X$ *there are effective cycles.*

$$A + \sum a_i E_i \in \mathbb{Q}^+_{\text{eff}}[Z] \quad \textit{and} \quad A + B + \sum (a_i + b_i) E_i \in \mathbb{Q}^+_{\text{eff}}[Z]$$

*such that the* $E_i$ *are rational curves satisfying* $0 < E_i \cdot (-K_X) \leq 1 + \dim X$, *and* $\varepsilon B \cdot H \geq B \cdot (-K_X)$.

**4.1.12. Remarks.** (4.1.12.1) This formulation is stronger than [78, 1.4.1]. Of course $mZ \approx B + \sum b_i E$, but again we restrict the kind of algebraic equivalence that we allow.

(4.1.12.2) Essentially nothing is known about characteristic zero versions of this result.

**4.2. How to recognize extremal rays?** We consider the following question: Given a curve $C$ which generates a seemingly extremal ray, how do we decide whether $C$ generates an extremal ray? We will always assume that the seemingly extremal ray is one of those described in (4.1.9–10).

The above question has a finer version. Namely, if $C$ does generate an extremal ray we may want to know whether

$$[D] \in \mathbb{R}^+[C] \Leftrightarrow D \in \mathbb{Q}^+_{\text{eff}}[C].$$

If this holds, then we will say the seemingly extremal ray generated by $C$ equals the extremal ray generated by $C$.

**4.2.1. Proposition.** *Let* $X$ *be a projective variety. Assume that* $C \subset X$ *generates a contractible seemingly extremal ray* $R$. *Let* $f: X \to Y$ *be the associated contraction. If* $Y$ *is projective, then* $C$ *generates an extremal ray. The converse is also true in characteristic zero if* $X$ *has canonical singularities.*

*Proof.* Assume that $Y$ is projective, and let $H$ be ample on $Y$. Let $[D] \in \mathbb{R}^+[C]$. Then $f(D) \cdot H = \text{const} \cdot f(C) \cdot H = 0$, hence $f(D)$ is a point.

Conversely, assume that $C$ also generates an extremal ray of $NE(X)$. The contraction of $\mathbb{R}^+[C]$ (1.4.7) has to coincide with $f$. In positive characteristic the missing ingredient is the vanishing $R^1 f_* \mathscr{O}_X = 0$.

**4.2.2. Corollary.** *Let $X$ be a normal projective threefold with isolated $\mathbb{Q}$-factorial singularities. Assume that $C \subset X$ generates a seemingly extremal ray $R$ which rationally covers $X$. Then $C \subset X$ generates an extremal ray.*

*Proof.* Let $g: X \to Y$ be the morphism constructed in (4.1.9) ($Y =$ point for (4.1.9.1)). If $Y$ is projective, then $C \subset X$ generates an extremal ray. The projectivity of $Y$ is clear except when $\dim Y = 2$. $Y$ is a normal surface, and every point of $Y$ is finitely dominated by a smooth point. Therefore $Y$ has only $\mathbb{Q}$-factorial singularities, hence $Y$ is projective (4.3.4). q.e.d.

If $\dim N(X) > 1$, then it is probably always true that the seemingly extremal ray generated by $C$ equals the extremal ray generated by $C$. The analogous question for Fano threefolds raises very interesting questions whose answers are known in special cases only.

**4.2.3. Problems.** Let $X$ be a smooth Fano variety (i.e., $-K_X$ is ample).

(4.2.3.1) Is the vector space $AE_1(X) = \{\text{1-cycles}\}/\{\text{algebraic equivalence}\}$ finitely generated? The answer is yes in positive characteristic by [78, 1.2].

(4.2.3.2) Assume that $\dim X = 3$ and $\operatorname{Pic} X \cong \mathbb{Z}$. Let $C, D \in X$ be two curves. Is $D \in \mathbb{Q}^+_{\text{eff}}[C]$?

The situation turns out to be more complicated for the seemingly extremal rays described in (4.1.10).

**4.2.4. Proposition.** *Let $D \subset X$ be a Cartier divisor on a smooth projective variety. Assume that $-K_X|D$ is ample and $-D|D$ is nef and numerically nontrivial. Then there is an extremal ray $R \subset \overline{NE}(X)$ such that if $C \subset X$ is an irreducible curve and $[C] \in R$, then $C \subset D$.*

*Proof.* Observe that the assumptions imply that $D$ is a (possibly singular) Fano variety. Thus the conditions are very restrictive.

Let $H \subset X$ be an ample divisor. Let

$$t = \max\{s | H + sD \text{ in nef}\},$$

and let

$$F = \{z \in \overline{NE}(X) | z \cdot (H + tD) = 0\}.$$

By construction $F$ is an extremal subset of $\overline{NE}(X)$ such that $K_X|(F - 0)$ is negative. Therefore $F$ contains the class of an effective curve, in particular $t$ is rational. Thus $F$ is an extremal face and contains at least one extremal ray.

**4.2.5. Corollary.** *Let $X$ be a smooth projective threefold, and let $C \subset X$ generate a seemingly extremal ray $R$ which is one of those described in*

(4.1.10). *If* $E \cong \mathbb{P}^2$ *or* $E \cong Q$ *(the quadric cone in* $\mathbb{P}^3$*), then the seemingly extremal ray generated by* $C$ *equals the extremal ray generated by* $C$.

In the next subsection we will see an example of a seemingly extremal ray of type (4.1.10.1) which is not an extremal ray. In the positive direction we have the following useful result:

**4.2.6. Proposition** (Pinkham, unpublished). *Let* $X$ *be a smooth projective threefold, and let* $E \subset X$ *be a smooth minimal ruled surface with typical fiber* $C$ *such that* $C \cdot E = -1$. *In* $N_{E|X}^{-1}$ *is not ample, then the seemingly extremal ray generated by* $C$ *equals the extremal ray generated by* $C$.

*Proof.* It is clearly sufficient to show that $C$ generates an extremal ray. By (1.2.4) we can write

$$C \equiv [\lim D_i] + \sum a_i E_i ,$$

where the $E_i$ generate extremal rays, and $D_i \cdot K_X \geq 0$. $\sum a_i E_i \neq 0$ since $C \cdot K_X = -1$. $\overline{NE}(E)$ is generated by $C$ and by another element $S \in \overline{NE}(E)$ which may not be represented by a cycle with rational coefficients. $N_{E|X}^{-1}$ is not ample iff $S \cdot E \geq 0$.

Split $D_i$ into two parts $D_i = A_i + B_i$, where $A_i$ is made up of those irreducible components that are contained in $E$, and $B_i$ is made up of those irreducible components that are not contained in $E$. Thus we obtain

$$C \equiv aC + bS + [\lim B_i] + \sum a_i E_i ,$$

where $a, b \geq 0$. If $a \geq 1$, then

$$0 \equiv (a-1)C + bS + [\lim B_i] + \sum a_i E_i ,$$

which is impossible since $X$ is projective. Therefore taking intersection with $E$ gives

$$0 > (1-a)C \cdot E = bS \cdot E + [\lim B_i] \cdot E + \sum a_i (E_i \cdot E) .$$

On the right-hand side, $S \cdot E \geq 0$ since $N_{E|X}^{-1}$ is not ample and $[\lim B_i] \cdot E \geq 0$ by construction. Thus there is a curve $E_1$ such that $E_1 \cdot E < 0$, hence $E_1 \subset E$. Of course $E_1$ has to generate an extremal ray in $NE(E)$ too. $S \notin \mathbb{R}^+[E_1]$ since $S \cdot E \geq 0$ and $E_1 \cdot E < 0$. Thus $[C] \in \mathbb{R}^+[E_1]$.

**4.2.7. Corollary.** *Notation as in* (4.2.6). *If* $C$ *does not generate an extremal ray in* $X$, *then* $E$ *is contractible to a point.*

**4.2.8. Corollary.** *Let* $X$ *be a proper smooth nonprojective threefold. Let* $C \subset X$ *be a smooth rational curve. Assume that* $B_C X$ *is projective.*

*Then*

*either* $C \equiv 0$, $N_{C|X} \cong \mathcal{O}(-1) + \mathcal{O}(-1)$, *C is contractible* $f\colon (C \subset X) \to (0 \in Y)$ *and* $Y$ *is projective,*

*or* $C \not\equiv 0$, *the opposite* $X \dashrightarrow X^-$ *exists and* $X^-$ *is projective.*

*Proof.* By (4.2.6) $C$ has negative bundle. Therefore $C$ can be contracted, and the inverse flip $X \dashrightarrow X^-$ exists by (4.3.5). Let $H'$ be a very ample divisor on $B_C X$, and let $H \subset X$ be the image of $H'$.

If $C \cdot H > 0$, then $X$ is projective by (4.1.1). If $C \equiv 0$, then $C \cdot K_X = 0$, and the adjunction formula gives that $N_{C|X} \cong \mathcal{O}(-1) + \mathcal{O}(-1)$. Applying the Basepoint-Free Theorem (1.4.4) to $H$ gives the morphism $f\colon X \to Y$ onto a projective variety $Y$ with a single ordinary node.

Otherwise $C \cdot M < 0$ for some divisor $M$ and $C \cdot H \leq 0$. Hence, for the inverse flip $C^- \subset X^-$ we have

$$C^- \cdot M^- > 0 \quad \text{and} \quad C^- \cdot H^- \geq 0.$$

Therefore, $mH^- + M^-$ is ample on $X^-$ for $1 \ll m$ (5.1.1.).

**4.3. Examples of seemingly extremal rays.** (4.1.3) shows that there are seemingly extremal rays which are not equal to an extremal ray. It is not too hard to get similar examples of seemingly extremal rays which do not generate an extremal ray. However, in all known constructions the exceptional surface was rational. Moishezon asked whether this was necessarily the case. The following example answers this question.

**4.3.1. Example.** There is a smooth proper scheme $X$ of dimension 3 and a smooth curve $C \subset X$ such that $X$ is not projective but $B_C X$ is. $C$ can have arbitrarily high genus.

The construction will be done in two steps with auxiliary lemmas collected at the end.

**4.3.2. Step 1.** Assume that there are projective varieties $U$ and $V$ of dimension three and a morphism $f\colon U \to V$ satisfying the following properties:

(4.3.2.1) $U$ is smooth.

(4.3.2.2) There are two points $P, Q \in V$ such that $f^{-1}$ is an isomorphism over $V - \{P, Q\}$.

(4.3.2.3) $f^{-1}(P) = C$ is a smooth curve of genus $g$, and $f^{-1}(Q) = L$ is a smooth rational curve.

(4.3.2.4) The normal bundle of $C$ is negative.

(4.3.2.5) The normal bundle of $L$ is the direct sum of two line bundles $\mathcal{O}(a)$ and $\mathcal{O}(b)$ such that $a < 0$, $b < 0$, and $(a, b) = 1$.

(4.3.2.6) $\mathbb{R}^+[C] = \mathbb{R}^+[L] \subset NE(U)$.

Then one can construct an example as in (4.3.1).

*Proof.* Let $(L^- \subset U^-)$ be the opposite of $L \subset U$ given by (4.3.5), and let $f^- : U^- \to V$ be the corresponding morphism. $U^- - C$ and $U^- - L^-$ are both quasiprojective, but $U^-$ is not projective since $[C]$ and $[L^-]$ generate a line in $NE(U^-)$. We claim that $B_C U^-$ is projective. To see this it is sufficient to show that $B_C U^-/V$ is relatively projective.

Let $H^- \subset U^-$ be a divisor such that $H^- \cdot L^- > 0$. Let $E \subset B_C U^-$ be the exceptional divisor of the blow-up $\pi: B_C U^- \to U^-$. By assumption (4.3.2.4) the normal bundle of $E$ is negative. If $m$ is sufficiently large, then the divisor

$$-mE + \pi^* H^-$$

is $B_C U^-/V$-ample.

Of course $U^-$ has singularities along $L^-$. By (4.3.5) these singularities are isolated; let $R \subset U^-$ be the singular set. By (4.3.4), $(U^- - R)$ is not quasiprojective. Let $X \to U^-$ be a relatively projective resolution of these singularities such that $(U^- - R) \subset X$. Then $X$ is not projective but $B_C X$ is.

**4.3.3. Step 2.** Construction of $f: U \to V$ as in (4.3.2).

**Construction.** Let us start with a surface $S \subset \mathbb{P}^3$ of degree $d_2$. Assume that $S$ contains a line $L$ and a smooth planar curve $C$ of degree $d_1$. Assume that $L \cap C = \emptyset$. The normal bundles are

$$N_{L|S} \cong \mathcal{O}(2 - d_2)|L \quad \text{and} \quad N_{C|S} \cong \mathcal{O}(1 + d_1 - d_2)|C.$$

We can apply (4.3.6) to conclude that there is a morphism $\pi: S \to \overline{S}$ onto a projective surface $\overline{S}$ such that $L$ and $C$ are contracted to points, and $\pi$ is an isomorphism on $S - (L \cup C)$. Let $\overline{G} \subset \overline{S}$ be a smooth ample curve disjoint from $\pi(L \cup C)$, and let $G = \pi^{-1}\overline{G}$.

Embed $S \subset \mathbb{P}^3 \subset \mathbb{P}^4$ and let $V \in \mathbb{P}^4$ be a sufficiently general point. Let $F \subset \mathbb{P}^4$ be the cone over $G$ with vertex $V$. Thus $S \cap F$ consists of the curve $G$ and a finite point set $P(S, F)$. Finally, let $H \subset \mathbb{P}^4$ be a sufficiently general hypersurface of degree $d_3$ containing $S$ and $F$.

**4.3.3.1. Lemma.** *Assume that $d_3$ is sufficiently large. Then $H$ has the following properties:*

(Sing $H$) *the singularities of $H$ are the following:*

- (i) *An isolated singularity at $V$.*
- (ii) *Ordinary double points at $P(S, F)$.*
- (iii) *Ordinary double points at some other points of $S$. We may assume that the set of these—$P(S)$—is disjoint from $L$ and $C$. In general there will be singular points along $G$.*
- (iv) *Ordinary double points at some other points of $F$. We may assume that the set of these—$P(F)$—is disjoint from $G$.*

(Div $H$) *The group of Weil divisors modulo algebraic equivalence is generated by* $S$, $F$, *and* $[\mathscr{O}(1)]$ *over* $\mathbb{Q}$.

*Proof.* The statements about the singularities are easy.

In order to see (Div $H$) let us take a general hyperplane $W \subset \mathbb{P}^4$. Then $S \cap W$ and $F \cap W$ are two smooth curves intersecting transversally. By the Noether-Lefschetz theorem the Picard group of a general surface of large degree containing these curves has rank three. On the other hand (cf. [18]),

$$\mathrm{Weil}(H) = \mathrm{Pic}(H - \mathrm{Sing}\, H) \hookrightarrow \mathrm{Pic}(W \cap H) \cong \mathbb{Z}^3 .$$

This shows (Div $H$). q.e.d.

We will resolve the singularities of $H$ in three steps. First we blow up the sheaf $\mathscr{O}(S)$ to obtain $p_1 \colon H_1 \to H$, i.e.,

$$H_1 = \mathrm{Proj}_H \sum_{i=0}^{\infty} \mathscr{O}(iS) .$$

$\mathscr{O}(S)$ is Cartier outside $P(S) \cup P(S, F)$, thus $p_1^{-1}$ is an isomorphism outside $P(S) \cup P(S, F)$. Let $S_1 \subset H_1$ be the proper transform of $S$. By construction $\mathscr{O}(S_1)$ is $p_1$-ample. Also, $S_1 \cong S$ and

$$N_{S_1|H_1} \cong \mathscr{O}(1 + d_2 - d_3)|S .$$

Next we resolve the singularity at $V$. This gives $p_2 \colon H_2 \to H_1$. The exceptional set is disjoint from $S_1$. Let $\mathscr{O}(-E)$ be $p_2$-ample where $E$ is a suitable $p_2$-exceptional divisor. Let $F_2$ be the proper transform of $F$.

Finally we blow up $\mathscr{O}(F_2)$. This gives $p_3 \colon H_3 \to H_2$ which is an isomorphism outside $P(F)$. Let $F_3$ be the proper transform of $F_2$. Let $U = H_3$.

Observe that by construction

$$M = \mathscr{O}(aF_3) \otimes p_3^*\mathscr{O}(-bE) \otimes p_3^* p_2^* \mathscr{O}(cS_1) \otimes p_3^* p_2^* p_1^* \mathscr{O}(d)$$

is very ample on $U$ for suitable $a \ll b \ll c \ll d$. We may also assume that $H^1(U, M) = 0$ and that $d = e(d_3 - d_2 - 1)$ for some $e$.

If $1 \le j \le e - c - 1$ and $d_2 \ll d_3$, then

$$H^1(S_1, M \otimes \mathscr{O}(jS_1)) = H^1(S, \mathscr{O}_S((e - c - j)(d_3 - d_2 - 1)) \otimes \mathscr{O}_S(aG)) = 0,$$

and

$$M \otimes \mathscr{O}((e - c)S_1)|S_1 \cong \mathscr{O}_S(aG) \quad \text{is generated by global sections.}$$

Therefore the conditions of (4.3.6) are satisfied, and we obtain a contraction map $f \colon U \to V$ which is an isomorphism outside $S_1$. On $S_1$ it induces the map given by $\mathscr{O}(aF)$. Thus $f$ contracts the two curves $L$ and $C$ and is an isomorphism elsewhere.

The normal bundles of $L$ and $C$ in $U$ can be computed from the sequences

$$0 \to N_{L|S} \to N_{L|U} \to N_{S_1|U}|L \to 0,$$
$$0 \to N_{C|S} \to N_{C|U} \to N_{S_1|U}|C \to 0.$$

In particular they are both negative and $\deg N_{L|U} = 3 - d_3$. We can choose $d_3$ in such a way that $d_3 - 3$ is a prime number, therefore condition (4.3.2.5) can also be satisfied.

The Picard group of $U$ is generated (over $\mathbb{Q}$) by the classes $\mathscr{O}(S_1)$, $\mathscr{O}(F_3)$, $p_3^*p_2^*p_1^*\mathscr{O}(d)$ and by $p_2$-exceptional divisors. The $p_2$-exceptional divisors are all disjoint from $L$ and $C$ and so is $F_3$.

$$\mathscr{O}(S_1)|S_1 \cong p_3^*p_2^*p_1^*\mathscr{O}(1 + d_2 - d_3)|S_1,$$

thus $\mathbb{R}^+[C] = \mathbb{R}^+[L] \subset NE(U)$. Therefore all the conditions of (4.3.2) are satisfied for a suitable choice of $S$ and $H$.

**4.3.4. Lemma** [51, p. 328, Corollary 3]. *Let $W$ be a normal, proper algebraic space with $\mathbb{Q}$-factorial singularities. Let $T \subset W$ be a finite set. If $W - T$ is quasiprojective, then $W$ is projective.*

**4.3.5. Lemma.** *Let $L \subset U$ be a smooth rational curve in a smooth threefold. Assume that the normal bundle of $L$ is the direct sum of two line bundles $\mathscr{O}(a)$ and $\mathscr{O}(b)$ such that $a < 0$ and $b < 0$. Then the opposite $(L \subset U) \dashrightarrow (L^- \subset U^-)$ exists in the category of algebraic spaces. If $(a, b) = 1$, then $U^-$ has isolated singularities.*

*Proof.* The problem is local around $L$. As in [78, 3.33] we obtain that a suitable neighborhood of $L$ in $U$ is analytically equivalent to a neighborhood of $L$ in the total space of the vector bundle $\mathscr{O}_L(-a) + \mathscr{O}_L(-b)$. The latter admits a torus action, and the existence of the opposite becomes an exercise in toric geometry. See [99] for similar computations.

**4.3.6. Theorem** (Castelnuovo's contractibility criterion). *Let $W$ be a proper algebraic space and let $S \subset W$ be a Cartier divisor. Let $M$ be a line bundle on $W$ generated by global sections. Let $\phi: W \to W'$ be the Stein factorization of the corresponding morphism. Assume the following:*

(4.3.6.1) $H^1(W, M) = 0$,

(4.3.6.2) $H^1(S, \mathscr{O}_S \otimes M(jS)) = 0$ *for* $1 \le j \le k - 1$, *and*

(4.3.6.3) $\mathscr{O}_S \otimes M(kS)$ *is generated by global sections. Let* $\mathrm{cont}: S \to \overline{S}$ *be the Stein factorization of the corresponding morphism.*

*Then $M(kS)$ is generated by global sections, and the Stein factorization* $\mathrm{Cont}: W \to \overline{W}$ *of the corresponding morphism has the following properties:*

(4.3.6.4) $\mathrm{Cont}\,|W - S = \phi|W - S$,
(4.3.6.5) $\mathrm{Cont}\,|S = \mathrm{cont}$, *and*
(4.3.6.6) $\overline{W}$ *is projective if* $W$ *is.*

*Proof.* One can easily check that only the above conditions are used in the proof given in [34, V.5.7] for the classical case.

**4.3.7. Remark.** The above example is also interesting from the point of view of projectivization by blowing up. Let $X$ be a proper Moishezon threefold. By [75] there is a sequence of smooth blow-ups such that the end result is projective. By (4.3.4) the last necessary blow-up is the blow-up of a curve. We see that this curve need not be rational.

However, the example also shows that there are rational curves around, namely the proper transform of $L^-$, and we could also blow up the proper transform of $L^-$ to achieve projectivity. Some considerations suggest that this may always be possible.

**4.3.8. Conjecture.** Let $X$ be a proper Moishezon threefold. Then there is a sequence of blow-ups centered at points or smooth rational curves such that the resulting threefold is projective.

## 5. Nonprojective threefolds

The aim of this section is to review and simplify some results about nonprojective threefolds and to ask some questions suggested by flips and flops. We will consider only threefolds that are close to being algebraic, namely compact Moishezon threefolds (possibly with some mild singularities). These are the same as proper algebraic spaces. We start by discussing some ampleness criteria.

**5.1. Ampleness and projectivity criteria.** All results of this subsection are valid over an arbitrary ground field.

The basic ampleness criteria of Nakai-Moishezon, Seshadri and of Kleiman can be formulated for algebraic spaces too. The proofs given in [51], [33] are for schemes, but they apply to algebraic spaces without modification once the conditions are suitably changed.

**5.1.1. Theorem** (Nakai-Moishezon criterion [86], [74], [51, III.1]). *Let* $Z$ *be a proper algebraic space and let* $H$ *be a line bundle on* $Z$. *Then* $H$ *is ample on* $Z$ *iff for every irreducible closed subspace* $X \subset Z$ *the* $\dim X$*-fold self-intersection of* $H|X$ *is positive.*

**5.1.2. Theorem** (Seshadri criterion [33, 1.7]). *Let* $Z$ *be a proper algebraic space, and let* $H$ *be a line bundle on* $Z$. *Then* $H$ *is ample on* $Z$ *iff there is a positive constant* $\varepsilon$ *such that for every irreducible curve* $C \subset Z$

*we have*

$$H \cdot C \geq \varepsilon \max_{x \in C}\{\operatorname{mult}_x C\}.$$

**5.1.3. Theorem** (Kleiman criterion [5, IV.2]). *Let $Z$ be a proper algebraic threefold, and let $H$ be a line bundle on $Z$. Assume that $Z$ has only $\mathbb{Q}$-factorial singularities. Then $H$ is ample on $Z$ iff $H$ induces a strictly positive linear function on*

$$\overline{NE}(Z) - \{0\},$$

*and no irreducible curve $C \subset Z$ is numerically equivalent to $0$.*

*In particular, $Z$ is nonprojective iff either $\overline{NE}(Z)$ contains a line (through the origin) or there is an irreducible curve $C \subset Z$ which is numerically trivial.*

*Proof.* If $Z$ is a proper algebraic space of dimension $n$ with $\mathbb{Q}$-factorial singularities, then the quasidivisoriality condition of [51, IV.2] is satisfied by subspaces of dimensions $n$ and $n-1$. In dimension three this is sufficient to make the proof work. I do not know what happens in higher dimensions.

Peternell noted (unpublished) that for smooth Moishezon $n$-folds in characteristic zero one can prove Kleiman's criterion using projectivization with a sequence of smooth blowups. q.e.d.

For smooth threefolds (5.1.3) was strengthened by [92] using analytic techniques. We will give a simple algebraic proof of his result which also works in the presence of certain singularities.

**5.1.4. Theorem** (cf. [92]). *Let $Z$ be a proper algebraic threefold, and assume that $Z$ has only normal $\mathbb{Q}$-factorial singularities. Then $Z$ is nonprojective iff there is an irreducible curve $C \subset Z$ such that*

$$-[C] \in \overline{NE}(Z).$$

*Proof.* The conditions are sufficient by (5.1.3). To get the converse let $C \subset Z$ be an irreducible curve. If $C$ is numerically trivial, then

$$-[C] = [C] \in \overline{NE}(Z),$$

and we are done. So we may assume that there is no such $C$, thus by (5.1.3) $\overline{NE}(Z)$ contains a line. This means that there are sequences of effective 1-cycles $C_i$ and $D_i$ such that

$$\lim[C_i] + \lim[D_i] = 0 \quad \text{and} \quad \lim[C_i] \neq 0.$$

By the Chow lemma we can find a projective threefold $Z'$ and a proper birational morphism $f: Z' \to Z$. There are only finitely many curves $B_j \subset Z$ such that $f^{-1}$ is not an isomorphism generically along $B_j$. Let

$B_j'' \subset Z'$ be any irreducible curve such that $f\colon B'' \to B_j$ is finite, and let $B_j'$ be a rational multiple of $B_j''$ such that $f_*[B_j'] = [B_j]$. For any other curve $B \subset Z$ let $B' \subset Z'$ be the proper transform. Thus for any 1-cycle $A$ in $Z$ we have defined a 1-cycle $A'$ in $Z'$ (with possibly rational coefficients) such that if $L$ is a line bundle on $Z$, then

$$L \cdot A = f^* L \cdot A'.$$

Take an ample divisor $H'$ on $Z'$ and let $H = f(H')$. Since $Z$ is $\mathbb{Q}$-factorial, some multiple of $H$ is Cartier; we may assume that in fact $H$ is Cartier. Then $f^* H = H' + E$, where $E \subset Z'$ is an effective divisor and $f(E) \subset \bigcup B_j$.

$$0 = H \cdot (\lim[C_i] + \lim[D_i]) = H' \cdot (\lim[C_i'] + \lim[D_i']) + E \cdot (\lim[C_i'] + \lim[D_i']).$$

Also, $\lim[C_i] \neq 0$ hence $\lim[C_i'] \neq 0$ and therefore

$$H' \cdot (\lim[C_i'] + \lim[D_i']) \geq H' \cdot (\lim[C_i']) > 0.$$

Hence

$$E \cdot (\lim[C_i'] + \lim[D_i']) < 0.$$

$E$ is an effective divisor, and the $B_j'$ are the only transforms contained in $E$. Therefore there is at least one curve—call it $B_1$—and a positive constant $\varepsilon$ such that if $a_i$ is the coefficient of $B_1$ in $C_i + D_i$, then $a_i > \varepsilon$ holds for infinitely many values of $i$. Now take $C = B_1$ and then

$$-[C] = \varepsilon^{-1} \lim[C_i + D_i - \varepsilon B_1] \in \overline{NE}(Z). \quad \text{q.e.d.}$$

The following result is interesting because it provides a characterization of projectivity without giving a criterion of ampleness.

**5.1.5. Corollary.** *Let $Z$ be a proper algebraic threefold, and assume that $Z$ has only normal $\mathbb{Q}$-factorial singularities. Then $Z$ is projective iff there is a line bundle $L$ on $Z$ such that $L \cdot C > 0$ for every irreducible curve $C \subset Z$. ($L$ need not be ample.)*

The following is a very interesting open problem:

**5.1.6. Problem.** Let $X$ be a smooth proper Moishezon threefold. Assume that $X$ is not projective. Can one find an effective 1-cycle $C \subset X$ such that $C$ is numerically trivial?

**5.2. Projectivization with flip or flop.**

5.1.2 Let us return to a construction used in the proof of (5.1.4). Let $Z$ be a proper normal $\mathbb{Q}$-factorial algebraic space of dimension three. Let $f\colon Z' \to Z$ be a birational projectivization. Let $H'$ be a very ample divisor on $Z'$ and let $H = f(H')$. We may assume that $H$ is Cartier. Let

$B = \bigcup B_j \subset Z$ be the fundamental set of $f^{-1}$. Then $H$ is very ample on $Z - B$ and has positive intersection with every curve not contained in $B$. If $H \cdot B_j > 0$, for every $j$, then $Z$ is projective by (5.1.5). (In fact it is easy to see that $H$ is ample.) If $H \cdot B_j < 0$, then $B_j$ can be contracted (cf. [55, 4.10]), and we can hope that the opposite (2.1.5) exists. If we are lucky, then after finitely many such curve surgeries we obtain a threefold $X^+$ and a line bundle $H^+$ such that $H^+$ is nef and big. In general $X^+$ may have fairly complicated singularities. We may hope that some multiple of $H^+$ is basepoint free, thus $X^+$ dominates a projective variety. I do not know any example where this cannot be done with suitable choice of $H$. On the other hand, the procedure should involve inverses of flips, which sometimes do not exist. Therefore I do not want to make any conjecture. The following form of the problem is very interesting:

**5.2.2. Problem.** Let $X$ be a proper algebraic threefold (smooth or with mild singularities). Can one find a (possibly very singular) projective variety $X^+$ such that $X$ and $X^+$ are isomorphic in codimension one? This means that there are subsets $B \subset X$ and $B^+ \subset X^+$ and an isomorphism $X - B \cong X^+ - B^+$ such that $\dim B \le 1$, $\dim B^+ \le 1$.

This question is especially interesting for threefolds with $K_X$ nef, since extremal ray theory does not give anything for them. For these threefolds the answer is very satisfactory.

**5.2.3. Theorem.** *Let $X$ be a proper algebraic threefold with $\mathbb{Q}$-factorial canonical singularities. Assume that $K_X$ is nef. Then after finitely many flops one obtains a proper algebraic threefold $X^+$ with $\mathbb{Q}$-factorial canonical singularities which is a small modification of a projective variety $Y$. $K_Y$ is nef and $Y$ has only canonical singularities. Furthermore,*

$$IH^i(X, \mathbb{Q}) \cong IH^i(Y, \mathbb{Q}).$$

*The isomorphism preserves Hodge structures but it does not preserve products.*

*If $X$ is smooth, then $X^+$ is also smooth, and $Y$ has only terminal hypersurface singularities* (1.3.4.2).

*Proof.* Let $H$ be as in (5.2.1) and let $L = H + mK_X$ for a sufficiently large $m$. Let $C \subset X$ be an irreducible curve such that $L \cdot C < 0$. By [55, 4.10] $C$ is contractible. Since $m$ is sufficiently large, $K_X \cdot C = 0$. Therefore the flop of $C$ exists [47, 6.10] and any sequence of $L$-flops terminates [55, 6.2]. Hence after finitely many flops we obtain $X^+$ and $L^+$ such that $X^+$ has $\mathbb{Q}$-factorial canonical singularities, $K_{X^+}$ is nef and $L^+$ is nef and big. By the Basepoint-Free Theorem (1.4.4) some multiple of $L^+ + K_{X^+}$ is basepoint-free. This gives

$$X \overset{q}{\dashrightarrow} X^+ \xrightarrow{p} Y,$$

where $q$ is a composition of flops, and $p$ is the morphism induced by a large multiple of $L^+ + K_{X^+} = H^+ + (m+1)K_{X^+}$. There are only finitely many curves $C^+ \subset X^+$ such that $H^+ \cdot C^+ \leq 0$, thus the same holds for $L^+ + K_{X^+}$. In particular, $p$ contracts only finitely many curves. The statement about the intersection homology groups follows easily from general results [55, 4.12].

**5.2.4. Corollary.** *Let $X$ be a proper algebraic threefold with $\mathbb{Q}$-factorial canonical singularities. Assume that $K_X$ is nef. If $X$ is not projective, then it contains a smooth contractible rational curve.*

One can weaken (5.2.2) by allowing $B \subset X$ or $B^+ \subset X^+$ to have dimension larger than one. If we require only that $\dim B \leq 1$, then any proper birational morphism $f\colon X^+ \to X$ provides an example. It is more difficult to find $X^+$ if we require that $\dim B^+ \leq 1$.

**5.2.5. Proposition.** *Let $X$ be a proper algebraic threefold with terminal singularities. Assume that $X$ is not uniruled. Then there is a projective variety $X^+$ and closed subsets $B \subset X$ and $B^+ \subset X^+$ such that* $\dim B^+ \leq 1$ *and $X - B$ is isomorphic to $X^+ - B^+$.*

*Proof.* By (1.4.9) there is a projective threefold with terminal singularities $X^+$ such that $K_{X^+}$ is nef, and $X^+$ is birational to $X$. We claim that this is the required example. Let $Y$ be a resolution of the graph of a birational equivalence between $X$ and $X^+$:

$$X \xleftarrow{p} Y \xrightarrow{q} X^+.$$

We can write

$$\begin{aligned} K_Y &= p^*K_X + E_1 + F_1, \\ K_Y &= q^*K_{X^+} + E_2 + G_2, \end{aligned} \tag{5.2.5.1}$$

where the $E_i$, $F_1$, $G_2$ are positive linear combinations of divisors satisfying the following conditions: For an irreducible divisor $B \subset Y$,

if $B$ is a component of $E_i$, then $\dim p(B) \leq 1$ and $\dim q(B) \leq 1$;

if $B$ is a component of $F_1$, then $\dim p(B) \leq 1$ and $\dim q(B) = 2$;

if $B$ is a component of $G_2$, then $\dim p(B) = 2$ and $\dim q(B) \leq 1$.

These conditions determine the divisors $E_i$, $F_1$, $G_2$ uniquely. Moreover, since $X$ has terminal singularities, the support of $E_1 + F_1$ contains all $p$-exceptional divisors. By (5.2.5.1) we have

$$p^*K_X = q^*K_{X^+} + G_2 + (E_2 - E_1 - F_1). \tag{5.2.5.2}$$

We need the following easy lemma:

**5.2.5.3. Lemma.** *Let $g: U \to V$ be a proper birational morphism between algebraic spaces. Assume that $U$ is smooth and projective. Let $D_i \subset U$ be the $g$-exceptional divisors, $L$ be a line bundle on $V$, $M$ be a $g$-nef line bundle on $U$, and $G \subset U$ be an effective divisor such that none of the $D_i$ is a component of $G$. Assume that*

$$g^*L \equiv M + G + \sum d_i D_i .$$

*Then $d_i \geq 0$ for every $i$.*

*Proof.* Taking general hyperplane sections of $U$ the problem can be reduced to the case when $U$ is a surface. Then $G$ becomes $g$-nef and hence can be absorbed into $M$. Let us write

$$g^*L \equiv M + D^+ - D^- ,$$

where the $D^+$ (resp. $D^-$) are nonnegative linear combinations of $g$-exceptional curves without common components. The intersection matrix of the exceptional curves is negative definite, thus if $D^- \neq 0$, then there is an exceptional curve $C \subset \operatorname{supp} D^-$ such that $C \cdot D^- < 0$. Thus

$$0 = C \cdot f^*L = C \cdot M + C \cdot D^+ - C \cdot D^- > 0 .$$

This is a contradiction. q.e.d.

Applying the lemma to (5.2.5.2) we conclude that $F_1 = \emptyset$. Therefore $B = p(E_1 + E_2 + G_2)$ and $B^+ = q(E_1 + E_2)$ satisfy the requirements. q.e.d.

My main interest in this proposition is that in some cases it answers the following question:

**5.2.6. Conjecture.** Let $X$ be a proper Moishezon space with terminal singularities, and let $S \subset X$ be a proper subset of codimension at least two. Then there is a proper and irreducible curve $C \subset X$ which is disjoint from $S$. More generally, there should exist such a curve $C$ through any sufficiently general point of $X$.

To put this into perspective observe that in Zariski's example of a non-projective surface (blow up 12 general points of a plane cubic and contract the proper transform of the cubic) every curve of the surface passes through the unique singular point.

**5.2.7. Proposition.** *Let $X$ be a Moishezon threefold with terminal singularities. Assume that $X$ is not uniruled. Let $S \subset X$ be a subset of dimension 1. Then through any sufficiently general point $x \in X$ there is a smooth irreducible curve $C \subset X$ which is disjoint from $S$.*

*Proof.* We use the notation of (5.2.5). Let $x \in X - (S \cup B)$. Since $X^+$ is projective, two general hypersurface sections of $X^+$ through $x$

give a smooth curve $C^+ \subset X^+$ which is disjoint from $B^+$, and from the proper transform of $S$, $p \circ q^{-1}$ is an isomorphism along $C^+$, hence $C = p \circ q^{-1}(C^+)$ is a curve with the required properties.

For uniruled manifolds a different argument, communicated to me by J.-M. Hwang, settles the conjecture:

**5.2.8. Proposition** (Hwang, unpublished). *Let $X$ be a smooth, proper uniruled Moishezon manifold of dimension $n$, and let $S \subset X$ be a subset of codimension at least 2. Then through any sufficiently general point $x \in X$ there is a rational curve $C \subset X$ disjoint from $S$.*

*Proof.* Since $X$ is uniruled, there is a (nonproper) variety $Y$ of dimension $n-1$ and a dominant morphism $g\colon Y \times \mathbb{P}^1 \to X$. For general $y \in Y$ the map

$$\mathcal{O}_{\mathbb{P}^1}(2) + \underbrace{\mathcal{O}_{\mathbb{P}^1} + \cdots + \mathcal{O}_{\mathbb{P}^1}}_{(n-1)-\text{times}} \cong T_{Y\times\mathbb{P}^1}|\{y\} \times \mathbb{P}^1 \to g^* T_X|\{y\} \times \mathbb{P}^1$$

is generically injective. In particular, $g^*T_X|\{y\} \times \mathbb{P}^1$ is generated by global sections. Let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the space parametrizing morphisms of $\mathbb{P}^1$ to $X$. Let $U \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be an open neighborhood of $g|\{y\} \times \mathbb{P}^1$ such that for every $f \in U$ the pullback $f^*T_X$ is generated by global sections. As in [77, §1] we obtain that $U$ is smooth of dimension $h^0(\mathbb{P}^1, f^*T_X)$ at $f$.

For $s \in X$ and $p \in \mathbb{P}^1$ let $U_{s,p} \subset U$ be those morphisms $f$ such that $f(p) = s$. Since $f^*T_X$ is generated by global sections, by [ibid, Proposition 3] we obtain that $U_{s,p}$ is smooth of dimension

$$h^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-p)) = h^0(\mathbb{P}^1, f^*T_X) - n .$$

Let $U_S \subset U$ be those morphisms whose image intersects $S$. Then

$$U_S = \bigcup_{p\in\mathbb{P}^1} \bigcup_{s\in S} U_{s,p} .$$

Thus, $\dim U_S \le \dim U - 1$. Therefore, if $f \in U$ is sufficiently general, then the image of $f$ is disjoint from $S$.

**5.3. Moishezon threefolds with $b_2 = 1$.**

**5.3.1. Notation.** For the rest of this section $X$ will be a smooth Moishezon threefold such that the rank of the Néron-Severi group is one. In particular, this condition is satisfied if $b_2(X) = 1$. We fix a generator $\mathcal{O}(1)$ of the Néron-Severi group such that $\mathcal{O}(k)$ is effective for some $k > 0$. The construction of (5.2.1) gives an effective divisor $H$ which has to be numerically equivalent to a positive multiple of $\mathscr{O}(1)$. Therefore, there are only finitely many irreducible curves $C \subset X$ such that $C \cdot \mathscr{O}(1) \leq 0$.

Note that it is possible that the self-intersection $\mathscr{O}(1)^{(3)}$ of $\mathscr{O}(1)$ is negative.

Up to numerical equivalence we can write

$$K_X \equiv \mathscr{O}(m), \qquad m = m_X \in \mathbb{Z}.$$

We try to classify $X$ according to the sign of $m$.

**5.3.2. Theorem.** *Notation as above. Assume that* $m > 0$. *Then there is a morphism*

$$f \colon X \to \overline{X}$$

*onto a projective variety* $\overline{X}$ *with at most terminal hypersurface singularities* (1.3.4.2), *and there is an ample line bundle* $\mathscr{O}_{\overline{X}}(1)$ *on* $\overline{X}$ *such that*

$$\mathscr{O}_X(1) \cong f^*(\mathscr{O}_{\overline{X}}(1)).$$

$f$ *is small and the self-intersection of* $\mathscr{O}_X(1)$ *is positive. Furthermore, the group of Weil divisors of* $\overline{X}$ *modulo algebraic equivalence has rank one.*

*Conversely, if* $\overline{X}$ *is a projective variety with at most terminal hypersurface singularities such that the group of Weil divisors modulo algebraic equivalence has rank one, and* $f \colon X \to \overline{X}$ *is a small resolution, then* $X$ *is a Moishezon threefold with* $\operatorname{rk} NS(X) = 1$ *which is nonprojective if* $f$ *is not an isomorphism.*

*Proof.* By (5.3.1) there are only finitely many curves $B_j \subset X$ such that $\mathscr{O}(1) \cdot B_j < 0$. Since $m > 0$, we conclude that there are only finitely many curves $B_j \subset X$ such that $K_X \cdot B_j < 0$. If $K_X \cdot B_j < 0$ then by [19, Lemma 5; 58, 5.1] one can deform $B_j$ inside $X$. This is impossible, thus $\mathscr{O}(1)$ and $K_X$ are nef. By the Basepoint-Free Theorem (1.4.4), some multiple of $\mathscr{O}(1)$ gives the required morphism onto $\overline{X}$.

**5.3.2.1. Remark.** If $X$ is allowed to have $\mathbb{Q}$-factorial terminal or canonical singularities, then there can be curves which have negative intersection with $K_X$. These can be flipped, and after finitely many flips some multiple of $H^+$ becomes basepoint free.

**5.3.2.2. Example.** Let $\overline{X}$ be a general hypersurface of degree $k \geq 3$ with an ordinary node given locally by the equation $xy - zt = 0$. Let $X$ be obtained by blowing up (locally in the Euclidean topology) the ideal $(x, z)$. Then $\operatorname{Pic} X \cong \mathbb{Z}$, and $X$ is not projective. $K_X = f^*\mathscr{O}(k-5)$.

**5.3.2.3. Corollary.** *Let* $g \colon X \to T$ *be a smooth, proper, holomorphic morphism between complex spaces. Assume that* $T$ *is connected and further that for some* $0 \in T$ *the fiber* $X_0$ *is a smooth projective threefold with*

$b_2(X_0) = 1$ *such that* $m = m_{X_0} > 0$. *Then every fiber of* $g$ *is projective, in fact* $g$ *is projective.*

*Proof.* We may assume that $T$ is irreducible. Let us consider the line bundle $\omega_{X/T}$. By assumption $\omega_{X/T}^k$ is very ample on $X_0$ for $k \gg 0$, hence there is a Zariski open set $0 \in U_k \subset Z$ such that $\omega_{X_t}^k$ is very ample on $X_t$ for $t \in U_k$. By upper semicontinuity of $h^0$ we conclude that $h^0(X_t, \omega_{X_t}^k) < \text{const} \cdot k^3$ for $k \gg 0$ for every $t \in T$. Thus every fiber of $g$ is Moishezon.

From (5.3.2) and (1.4.2) we see that $h^1(X_t, \mathcal{O}_{X_t}(m+s)) = 0$ for $s > 0$. Thus

$$h^0(X_t, \mathcal{O}_{X_t}(m+s))$$

is independent of $t \in T$ for $s > 0$. As in (5.3.2) there are morphisms $\overline{g}: \overline{X} \to T$ and $f: X \to \overline{X}$ such that

$$\mathcal{O}_X(1) \cong f^*\mathcal{O}_{\overline{X}}(1).$$

We claim that $f$ is an isomorphism. If $f$ is not an isomorphism over a point $\tau \in T$, then we take a general disc $h: \Delta \to T$ through $\tau$ and consider the family over $\Delta$ induced by base change. Thus we have

$$f_\Delta : X_\Delta \to \overline{X}_\Delta.$$

$f_\Delta$ is an isomorphism except over finitely many points of $\overline{X}_\Delta$. The central fiber of $\overline{X}_\Delta$ has only isolated hypersurface singularities, hence $\overline{X}_\Delta$ itself has only isolated hypersurface singularities. The claim follows from the next easy result:

**5.3.2.4. Lemma.** *Let* $0 \in U$ *be a four-dimensional isolated hypersurface singularity. Let* $f: V \to U$ *be a prop morphism. Assume that* $f$ *is an isomorphism over* $U - 0$ *and that* $f^{-1}(0)$ *is at most one-dimensional. Then* $f$ *is an isomorphism.*

*Proof.* Let $D \subset V$ be a small three-dimensional disc intersecting $f^{-1}(0)$ at a single point. Then $f(D) \subset U$ is a divisor near 0, Cartier on $U - 0$. By [30, XI.3.1.4] this implies that $f(D)$ is Cartier at 0. Therefore $f$ is an isomorphism (cf. (6.1.2)). q.e.d.

If $m = 0$, then (5.2.3) gives the following:

**5.3.3. Theorem.** *Let* $X$ *be a smooth Moishezon threefold such that* $K_X$ *is trivial. Then after finitely many flops one obtains a smooth Moishezon threefold* $X^+$ *which is a small resolution of a projective variety* $Y$. $K_Y$ *is trivial, and* $Y$ *has only terminal hypersurface singularities. Furthermore*

$$H^i(X, \mathbb{Q}) \cong IH^i(Y, \mathbb{Q}).$$

*The isomorphism preserves Hodge structures, but it does not preserve products.*

The $m < 0$ case seems the hardest, and there are only partial results.

**5.3.4. Theorem** [87], [88]. *Notation as above. Assume that* $m < 0$. *Then*

(5.3.4.1) $m \geq -4$;

(5.3.4.2) $m = -4$ *iff* $X \cong \mathbb{P}^3$;

(5.3.4.3) $m = -3$ *iff* $X \cong \mathbb{Q}^3$ (*the smooth quadric in* $\mathbb{P}^4$).

**5.3.5. Corollary.** *Let* $X$ *be a Moishezon threefold which is homeomorphic to* $\mathbb{P}^3$ (*resp.* $\mathbb{Q}^3$). *Then* $X$ *is isomorphic to* $\mathbb{P}^3$ (*resp.* $\mathbb{Q}^3$).

**5.3.6. Remark.** (5.3.4) and (5.3.5) are claimed [91]. Unfortunately there is a gap in the proof of [ibid, Lemma 3.5]. Therefore (5.3.4) should be attributed to Nakamura, and (5.3.5) seems new. In a letter (March 1990) Peternell informed me that he was preparing an article containing a proof of (5.3.5). Nakamura [87], [88] already observed that (5.3.4) implies (5.3.5) provided $\kappa(X) < 3$.

*Proof of* (5.3.5). Let $L$ be the generator of $H^2(X, \mathbb{Z})$ with positive self-intersection. As in the case when $X$ is assumed projective [35], [8] one can easily compute that $K_X = -4L$ (resp. $K_X = -3L$). With the notation of (5.3.1) $L = \mathscr{O}(\pm 1)$. Assume that $L = \mathscr{O}(-1)$. Then $K_X = \mathscr{O}(4)$ (resp. $K_X = \mathscr{O}(3)$) hence by (5.3.2) we get that

$$0 < \mathscr{O}(1)^{(3)} = -L^{(3)} < 0,$$

which is impossible. Therefore $K_X = \mathscr{O}(-4)$ (resp. $K_X = \mathscr{O}(-3)$), and the result follows from (5.3.4).

**5.3.7. Remarks.** (5.3.7.1) The proof presented for (5.3.4) gives useful information about the $m = -2$ case too. I know very little about the case $m = -1$. Recently Nakamura extended his method to the case $m = -2$. It seems that he will be able to go further than (5.3.12).

(5.3.7.2) The idea of the proof is to study the linear system $|\mathscr{O}(1)|$. We have to show that it is very ample. Nakamura [87], [88] studies the base locus very carefully. Here we study the image of the rational map given by $|\mathscr{O}(1)|$. Then only new ingredients are (5.3.8) and (5.3.11). The other lemmas can all be found in [87], [88].

The first step in the proof of (5.3.4) is the following lemma.

**5.3.8. Lemma.** *Let* $X$ *be a normal proper* $n$*-dimensional algebraic space. Let* $M$ *be a Cartier divisor on* $X$ *which is ample in codimension one* (*i.e., there is a codimension two subset* $Z \subset X$ *such that* $M|X - Z$ *is ample*). *Then*

$$h^{n-1}(X, \mathscr{O}(K_X + M)) = 0.$$

*Proof.* If $r$ is sufficiently large, then the linear system $|rM|$ is very ample when restricted to $X - Z$. Let $f\colon X' \to X$ be a proper, smooth, birational projectivization such that

$$f^*|rM| = |H'| + \sum a_i E_i,$$

where $|H'|$ is basepoint free and big, and $\sum E_i$ is a divisor with normal crossings only. Then

$$f^* M \equiv \frac{1}{r} H' + \sum \frac{a_i}{r} E_i.$$

Let [ ] denote the integral part of a real number. Then by (1.4.2) and [89, 3.6]

$$h^j\left(X', \mathscr{O}\left(K_{X'} + f^*M - \sum \left[\tfrac{a_i}{r}\right] E_i\right)\right) = 0 \quad \text{for } j > 0,$$

and

$$R^j f_* \mathscr{O}\left(K_{X'} + f^*M - \sum \left[\tfrac{a_i}{r}\right] E_i\right) = 0 \quad \text{for } j > 0.$$

Let

$$F = f_* \mathscr{O}\left(K_{X'} + f^*M - \sum \left[\tfrac{a_i}{r}\right] E_i\right).$$

Then $H^j(X, F) = 0$ for $j > 0$, and we have an injection $i\colon F \to \mathscr{O}(K_X + M)$ which is an isomorphism in codimension one. Thus

$$i\colon 0 = h^{n-1}(X, F) \to h^{n-1}(X, \mathscr{O}(K_X + M))$$

is surjective.

**5.3.9. Corollary.** *Notation as in* (5.3.1). *Assume that* $m < 0$. *Then*

(5.3.9.1) $\operatorname{Pic} X \cong \mathbb{Z}$;

(5.3.9.2.0) $h^0(X, \mathscr{O}(k)) = 0$ *if* $k <$;

(5.3.9.2.1) $h^1(X, \mathscr{O}(k)) = 0$ *if* $k \leq 0$;

(5.3.9.2.2) $h^2(X, \mathscr{O}(k)) = 0$ *if* $k \geq m$;

(5.3.9.2.3) $h^3(X, \mathscr{O}(k)) = 0$ *if* $k > m$;

(5.3.9.3) $\chi(\mathscr{O}_X) = 1$.

*Proof.* (5.3.9.2.0) and (5.3.9.2.3) are clear. (5.3.9.2.1) and (5.3.9.2.2) are dual. Since $\mathscr{O}(1)$ is ample in codimension one, (5.3.8) implies (5.3.9.2.2) for $k > m$.

The only remaining vanishing is $h^1(X, \mathscr{O}) = 0$. This will be done by studying the Albanese map. Assume that we have a nontrivial albanese map $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$. $\mathrm{Alb}(X)$ is Moishezon and hence projective. Let $L$ be ample on $\mathrm{Alb}(X)$. Then $\mathrm{alb}^* L$ is a nontrivial line bundle on $X$

which is trivial on the fibers of alb. Since $\operatorname{rk} NS(X) = 1$, this implies that alb is generically finite. Therefore $h^0(X, \mathscr{O}(m)) = h^0(X, \mathscr{O}(K_X)) > 0$ [116, 10.1], which is impossible.

These results imply that $\chi(\mathscr{O}_X) = 1$ and that $\operatorname{Pic} X$ is discrete. If $\operatorname{Pic} X$ contains torsion, then we can construct an étale cover $\widetilde{X} \to X$. (5.3.8) gives that $h^2(\mathscr{O}_{\widetilde{X}}) = h^3(\mathscr{O}_{\widetilde{X}}) = 0$. On the other hand,

$$1 - h^1(\mathscr{O}_{\widetilde{X}}) = \chi(\mathscr{O}_{\widetilde{X}}) = \deg(\widetilde{X}/X)\chi(\mathscr{O}_X) = \deg(\widetilde{X}/X).$$

Therefore $\operatorname{Pic} X$ is torsion free and has rank one.

**5.3.10 Corollary.** *Notation as above. Then the following hold*:

(5.3.10.1) $m \geq -4$;

(5.3.10.2) *If* $m = -4$, *then*

$$\chi(X, \mathscr{O}(k)) = \tfrac{1}{6}(k+1)(k+2)(k+3) \quad \textit{and} \quad \mathscr{O}(1)^{(3)} = 1.$$

(5.3.10.3) *If* $m = -3$, *then*

$$\chi(X, \mathscr{O}(k)) = \tfrac{1}{6}(k+1)(k+2)(2k+3) \quad \textit{and} \quad \mathscr{O}(1)^{(3)} = 2.$$

(5.3.10.4) *If* $m = -2$, *then*

$$\chi(X, \mathscr{O}(k)) = \mathscr{O}(1)^{(3)}\tfrac{1}{6}k(k+1)(k+2) + k + 1.$$

*Proof.* (5.3.9) computes $\chi(X, \mathscr{O}(k))$ for $m \leq k \leq 0$. Since $\chi(X, \mathscr{O}(k))$ is a polynomial of degree at most three, knowing it at four places specifies it exactly. The leading coefficient is obtained from Riemann-Roch. q.e.d.

The following is the main step in the proof:

**5.3.11 Lemma.** *Notation as above. Let* $s = h^0(X, \mathscr{O}(1))$ *and let*

$$h: X \dashrightarrow Y \subset \mathbb{P}^{s-1}$$

*be the induced map. If* $m \leq -3$ *or if* $m = -2$ *and* $s \geq 4$, *then* $h$ *is generically finite.*

*Proof. Step* 0. We will need several times the Del Pezzo-Bertini classification of varieties of minimal degree in $\mathbb{P}^n$ (cf. [20]). The following statements will be needed:

If $Y \subset \mathbb{P}^n$ is a nondegenerate $k$-fold of degree $d$, then $d \geq n+1-k$. If equality holds and there is a nonempty open subset $U \subset Y$ such that every hyperplane section of $U$ is irreducible, then either $Y = \mathbb{P}^n$ or $Y$ is a quadric hypersurface of rank at least 4.

Since every member of $|\mathscr{O}_X(1)|$ is irreducible, this implies that if $\deg Y = s - \dim Y$, then $Y$ is either $\mathbb{P}^3$ or $\mathbb{Q}^3$ ($\mathbb{P}^1$ and $\mathbb{P}^2$ are excluded since $s \geq 4$).

*Step* 1. By (5.3.9–10) we get that $s \geq 4$ if $m = -4$ and $s \geq 5$ if $m = -3$. Thus in all cases $s \geq 4$. $\mathscr{O}(1)$ is the effective generator of $\operatorname{Pic} X$, therefore every member of $|\mathscr{O}(1)|$ is irreducible. Since $\dim |\mathscr{O}(1)| \geq 2$, this implies that $|\mathscr{O}(1)|$ is not composed of a pencil. Thus, $\dim Y \geq 2$.

*Step* 2. Let $H_1, H_2 \in |\mathscr{O}(1)|$ be two different members, and let $C = H_1 \cap H_2$ be the intersection curve (with scheme structure). Then

(i) $H^0(X, \mathscr{O}(1)) \to H^0(C, \mathscr{O}(1)|C)$ is surjective.

(ii) $h^0(\mathscr{O}_C) = 1$.

(iii)
$$h^1(\mathscr{O}_C) = \begin{cases} 0 & \text{if } m \leq -3, \\ 1 & \text{if } m = -2. \end{cases}$$

(iv)
$$h^1(C, \mathscr{O}(-1)|C) = \begin{cases} 0 & \text{if } m = -4, \\ 1 & \text{if } m = -3, \\ s-2 & \text{if } m = -2. \end{cases}$$

All these statements can be obtained from (5.3.9) by standard diagram chasing.

*Step* 3. Assume that $\dim Y = 2$. Then $Y \subset \mathbb{P}^{s-1}$ is a nondegenerate surface, and hence has degree $d \geq s-2$. By Step 0 we have in fact $d > s-2$. $f$ cannot be a morphism since $A_i \cdot \mathscr{O}(1) = 0$ holds for only finitely many curves $A_i$.

Every section $s \in H^0(X, \mathscr{O}(1))$ gives a map
$$s(-1)\colon \mathscr{O}_X(-1) \to \mathscr{O}_X.$$

Let $I_B \subset \mathscr{O}_X$ be the ideal generated by the images for all $s \in H^0(X, \mathscr{O}(1))$. Then $B = \operatorname{Spec}(\mathscr{O}_X/I_B) \subset X$ is the scheme theoretic base locus of $|\mathscr{O}(1)|$.

For general $H_1$ and $H_2$ the intersection curve decomposes as
$$C = F \cup D_1 \cup \cdots \cup D_d,$$
where $D_i$ are the moving components, and $\operatorname{supp} F = \operatorname{supp} B$. The reduced structure of $F$ is independent of the $H_i$ (for general $H_i$), but the scheme structure along $F$ may depend continuously on the choice of $H_i$. The moving curves are parametrized by the points of $Y$, at least generically. For generic $y \in Y$ the arithmetic genus of the curve $D_y$ and the intersection number $\mathscr{O}(1) \cdot D_y$ are independent of $y$.

Since
$$1 \geq h^1(\mathscr{O}_C) \geq d \cdot h^1(\mathscr{O}_{D_1}),$$

we conclude that $h^1(\mathscr{O}_{D_i}) = 0$. Thus $D_i \cong \mathbb{P}^1$. Also,

$$s-2 \geq h^1(C, \mathscr{O}_X(-1)|C) \geq d \cdot h^1(D_1, \mathscr{O}_X(-1)|D_1).$$

Since $d > s-2$, this implies that $h^1(C, \mathscr{O}_X(-1)|D_1) = 0$, so that

$$\deg \mathscr{O}_X(1)|D_i \leq 1.$$

Let $H \in |\mathscr{O}_X(1)|$ be a divisor not containing $D_i$. $H$ contains $B$, and $D_i$ intersects $B$ since $C$ is connected by (ii) of Step 2. Therefore, $H$ and $D_i$ have at least one intersection point along $F$. Thus, their intersection number is 1, and they intersect transversally at a smooth point of $H$.

Assume that the intersection points of $D_y$ and $B$ do not depend on the choice of $y \in Y$ for generic $y$. Then $D_1 \cup \cdots \cup D_d$ is connected, hence

$$\begin{aligned} s-2 &\geq h^1(\mathscr{O}_X(-1)|C) \geq h^1(\mathscr{O}_X(-1)|D_1 \cup \cdots \cup D_d) \\ &= d - \chi(D_1 \cup \cdots \cup D_d) \geq d-1. \end{aligned}$$

Therefore, $d = s-1$ and $\chi(D_1 \cup \cdots \cup D_d) = 1$. This implies that the $D_i$ intersect at a single point $p$, and the embedding dimension of $D_1 \cup \cdots \cup D_d$ at $p$ is $d \geq s-1$. On the other hand, the general $H \in |\mathscr{O}_X(1)|$ is smooth at $p$, thus $s-1 \leq d \leq 2$. This contradicts $s \geq 4$. Therefore, the intersection points of $D_y$ and $F$ move with $y$. Thus, $\dim F = 1$, the $D_i$ are disjoint and they intersect the same irreducible component $E \subset F$.

By the above results, the general $H \in |\mathscr{O}_X(1)|$ is generically smooth along $E$, thus at the generic point of $E$ the scheme $B$ is contained in a smooth surface. In particular, $I_B$ is a local complete intersection at the generic point of $E$. Therefore, if $s_1, s_2 \in H^0(X, \mathscr{O}(1))$ are sufficiently general, then the inclusion

$$(s_1(-1), s_2(-1)) \subset I_B$$

is an equality at the generic point of $E$. This means that $I_F$ and $I_B$ agree generically along $E$.

A general section $h$ of $\mathscr{O}_X(1)$ gives a map $h: \mathscr{O}_C \to \mathscr{O}_C(1)$. Let

$$G = \operatorname{im}[\mathscr{O}_C \xrightarrow{h} \mathscr{O}_C(1)].$$

$G$ is a subsheaf of $\mathscr{O}_C(1)$ which is generated by a global section. Its image is contained in $I_B \otimes \mathscr{O}_C(1)$, hence it is generically zero along $E$. On each of the curves $D_i$ it is nontrivial with at least one section. Therefore

$$\begin{aligned} s = h^0(X, \mathscr{O}(1)) &= 2 + h^0(C, \mathscr{O}(1)) \geq 2 + h^0(C, G) \\ &\geq 2 + d \geq 2 + s - 1 = s + 1. \end{aligned}$$

This contradiction shows that $Y$ cannot be a surface.

*Proof of* (5.3.4). The linear system $|\mathscr{O}_X(1)|$ gives a map

$$f\colon X \dashrightarrow Y_d^3 \subset \mathbb{P}^{s-1},$$

where $Y$ is a threefold of degree $d$. Two general members $H_1, H_2 \in |\mathscr{O}_X(1)|$ intersect in a curve $C = F \cup D$, where $F$ is the fixed part, and $D$ is the irreducible moving curve.

$$\deg \mathscr{O}_X(1)|D = \deg f \cdot \deg Y + b,$$

where $b = 0$ iff $f$ is a morphism. Since $m \leq -3$, $D \cong \mathbb{P}^1$ by (iii) of Step 2. By (iv) of Step 2,

$$\begin{aligned}\deg f \cdot \deg Y + b - 1 = h^1(D, \mathscr{O}_X(-1)|D) &\leq h^1(C, \mathscr{O}_X(-1)|C) \\ &\leq \begin{cases} 0 & \text{if } m = -4, \\ 1 & \text{if } m = -3. \end{cases}\end{aligned}$$

For $m = -4$ this implies that $b = 0$, $\deg f = \deg Y = 1$. Thus $Y \cong \mathbb{P}^3$, and $f\colon X \to \mathbb{P}^3$ is a birational morphism. Since $\mathbb{P}^3$ is smooth, the exceptional set is a divisor. Since $\operatorname{Pic} X \cong \mathbb{Z}$, there cannot be any exceptional divisor. Thus, $f$ is an isomorphism.

For $m = -3$ we obtained $s \geq 5$, thus $\deg Y \geq 2$. Therefore, again $b = 0$, and $f$ is a birational morphism onto a quadric in $\mathbb{P}^4$. A singular quadric has reducible hyperplane sections, and these give reducible divisors in $|\mathscr{O}_X(1)|$, a contradiction. Thus $Y$ is smooth, and as before $f\colon X \to \mathbb{Q}^4 \subset \mathbb{P}^4$ is an isomorphism.

**5.3.12. Theorem.** *Notation as above. Assume that* $m = -2$. *Let* $s = h^0(X, \mathscr{O}(1))$ *and let* $f\colon X \dashrightarrow Y$ *be the map given by* $|\mathscr{O}_X(1)|$. *Then the following hold*:

(5.3.12.1) $s \leq 7$.

(5.3.12.2) *If* $s \geq 4$, *then either*

(5.3.12.2.1) $f$ *is a morphism with Stein factorization* $X \to \overline{Y} \to Y$, *and* $\overline{Y}$ *is a Fano variety of index two possibly with terminal hypersurface singularities*; *or*

(5.3.12.2.2) $s = 4$, *and* $f\colon X \dashrightarrow \mathbb{P}^3$ *is birational*; *or*

(5.3.12.2.3) $s = 5$, *and* $f\colon X \dashrightarrow \mathbb{Q}^3$ *is birational.*

**Remark.** The corresponding Fano varieties are the following:

(i) $s = 4$, $\overline{Y}$ is a double cover of $\mathbb{P}^3$ ramified along a quartic.

(ii) $s = 5$, $\overline{Y} = Y$ is a cubic hypersurface in $\mathbb{P}^4$.

(iii) $s = 6$, $\overline{Y} = Y$ is a complete intersection of quadric hypersurfaces in $\mathbb{P}^5$.

(iv) $s = 7$, $\overline{Y} = Y$ is a 3-fold hyperplane section of the Grassmannian $\mathrm{Gr}(1, 4) \subset \mathbb{P}^9$.

A Fano variety of this latter type is called $V_5$ (see, e.g. [38, II.1]).

**5.3.13. Corollary.** *Let $X$ be a Moishezon threefold which is homeomorphic to the Fano variety $V_5$. Then $X$ is isomorphic to the Fano variety $V_5$.*

*Proof.* The Betti numbers of $V_5$ are $b_1 = b_3 = 0$ and $b_2 = 1$ [38, IV.3.5]. Thus, any Moishezon threefold homeomorphic to $V_5$ has $\chi(\mathscr{O}) = 1$. Let $L$ be the generator of $H^2(X, \mathbb{Z})$ which satisfies $L^{(3)} = 5$. Let $K_X = -xL$. From Riemann-Roch it follows that

$$5x^3 + 4x = 48\chi(\mathscr{O}) = 48.$$

This has $x = 2$ as the only integral solution. As in the proof of (5.3.5), $L$ is the effective generator, thus $K_X = \mathscr{O}_X(-2)$. By (5.3.10.4)

$$s = h^0(X, \mathscr{O}_X(1)) \geq \chi(X, \mathscr{O}_X(1)) = 7.$$

Therefore, by (5.3.12), $|\mathscr{O}_X(1)|$ is base free and maps birationally onto a Fano variety $Y$ of index two and degree 5 in $\mathbb{P}^6$, possibly with terminal hypersurface singularities. If $Y$ indeed has a singular point $y$, then projecting from $y$ we obtain a threefold $Z \subset \mathbb{P}^5$ of degree 3. Thus $Z$ has minimal degree, and therefore we obtain a contradiction as in Step 0 of (5.3.11). Thus $Y$ is smooth, and $f: X \to Y$ has to be an isomorphism.

*Proof of* (5.3.12). We use the notation of the proof of (5.3.11). If $|\mathscr{O}_X(1)|$ is basepoint free, then $\overline{Y}$ is a Fano variety of index two. Therefore we need to consider the case when there are base points, i.e., $b > 0$. By (ii) of Step 2, $1 = h^1(\mathscr{O}_C) \geq h^1(\mathscr{O}_D)$.

Assume first that $h^1(\mathscr{O}_D) = 1$. Then

$$\begin{aligned} \deg f \cdot \deg Y + 1 \leq \deg f \cdot \deg Y + b &= h^1(D, \mathscr{O}_X(-1)|D) \\ &\leq h^1(C, \mathscr{O}_X(-1)|C) = s - 2. \end{aligned}$$

Since $\deg Y \geq s - 3$, this implies that $\deg f = 1$ and $\deg Y = s - 3$. If $s \geq 6$, then $Y$ contains reducible hyperplane sections, a contradiction. Thus $Y = \mathbb{P}^3$ or $Y = \mathbb{Q}^3$. As before, the singular quadric threefold is excluded.

Now assume that $h^1(\mathscr{O}_D) = 0$. If $b \geq 2$, then the above argument works. If $b = 1$, then as in the proof of (5.3.11) we obtain that $\mathscr{O}_C(1)$ has a subsheaf isomorphic to a line bundle of degree $\deg f \cdot \deg Y$ on $D$.

Thus

$$s = 2 + h^0(D, \mathscr{O}(1)|D) \geq 2 + h^0(\mathbb{P}^1, \mathscr{O}(\deg f \cdot \deg Y))$$
$$= \deg f \cdot \deg Y + 3.$$

This is the same inequality as before, and leads to the same numerical possibilities.

**5.3.14. Examples.** There are infinitely many examples of nonprojective threefolds which behave as in (5.3.12.2.2–3). These were independently discovered by Hironaka, Fujiki, and perhaps others.

In $\mathbb{P}^4$ take a smooth quadric $Q$ and blow up a smooth curve of type $(3, n)$ on $Q$. The proper transform of the quadric has normal bundle $(-1, 2-n)$, thus it can be contracted in one direction to get a smooth Moishezon threefold. For every $n \geq 4$ this gives an example for (5.3.12.2.2).

Similarly, taking a smooth quadric $Q$ in $\mathbb{Q}^3$ and blowing up a curve of type $(2, n)$ produce examples for (5.3.12.2.3).

In all of these cases one can contract the proper transform of $Q$ in the other direction. This gives a singular projective Fano variety satisfying the requirements of (5.2.2).

## 6. Deformations of rational surface singularities

In this section we want to explore some questions raised again by codimension two modifications, namely various aspects of small resolutions. Applied to deformations of rational surface singularities we obtain several interesting conjectures and results.

**6.1. Small modifications of threefold singularities.**

**6.1.1. Definition.** Let $0 \in X$ be a germ of a three-dimensional normal singularity. A small modification of $X$ is a three-dimensional contractible curve neighborhood $C \subset Y$ together with the contraction map $f: C \subset Y \to X$.

The following easy proposition connects small modifications with the divisor class group $\mathrm{Pic}(X - 0)$ [47, 3.1]

**6.1.2. Proposition.** (6.1.2.1) *Let $f: Y \to X$ be a small modification as above, and let $D \subset Y$ be an $f$-ample Cartier divisor. Then the following hold*:

(6.1.2.1.1) *$mf(D) \subset X$ is not Cartier if $m > 0$,*

(6.1.2.1.2) *$f_*\mathscr{O}_Y(mD) = \mathscr{O}_X(mf(D))$ for $m \geq 0$, and*

(6.1.2.1.3) *$\sum_{m=0}^{\infty} \mathscr{O}_X(mf(D))$ is a finitely generated $\mathscr{O}_X$ algebra.*

(6.1.2.2) *Conversely, let* $D' \subset X$ *be a divisor such that*

(6.1.2.2.1) *no multiple of* $D'$ *is Cartier, and*

(6.1.2.2.2) $\sum_{m=0}^{\infty} \mathscr{O}_X(mD')$ *is finitely generated* $\mathscr{O}_X$ *algebra.*

*Then* $Y = \operatorname{Proj}_X \sum_{m=0}^{\infty} \mathscr{O}_X(mD')$ *and the projection map* $f: Y \to X$ *give a small modification of* $X$.

One can always find an ideal $I \subset \mathscr{O}_X$ which is isomorphic to $(D')$ (as a module), and then the $m$th symbolic power of $I$ is $I^{(m)} \cong \mathscr{O}(mD')$. For this reason the algebra $\sum_{m=0}^{\infty} \mathscr{O}_X(mD')$ is called the symbolic power algebra of $D'$.

This gives a purely algebraic approach to finding small modifications: we have to compute the local divisor class group and then decide when is the symbolic power algebra finitely generated. There are several results concerning the first problem.

(6.1.3) Let $0 \in X$ be a three-dimensional isolated singularity. Let $S$ be a small sphere around 0, and let $Z = X \cap S$ be the link of $X$. This is a compact 5-dimensional manifold, independent of $S$ (up to diffeomorphism). Then the first Chern class

$$c_1: \operatorname{Pic}(X - 0) \to H^2(Z, \mathbb{Z})$$

is an injection, and an isomorphism if $0 \in X$ is rational [21, 6.1]. This result is very useful, but it is frequently very hard to compute $Z$ and its cohomologies. The simplest case should be isolated hypersurface singularities, but even here there are many unsolved problems.

Let $0 \in X \subset \mathbb{C}^4$ be an isolated hypersurface singularity. By [30, X.3.4] its local Picard group is torsion free, and hence a finitely generated free abelian group. There are some easy ways of recognizing nontrivial elements of $\operatorname{Pic}(X - 0)$. Assume for instance that the equation of $X$ can be written as

$$u(\underline{x})v(\underline{x}) - s(\underline{x})t(\underline{x}) = 0.$$

Then

$$D = (u(\underline{x}) = s(\underline{x}) = 0)$$

is a nontrivial element of $\operatorname{Pic}(X - 0)$. Every time a divisor $D \subset X$ is a complete intersection in $\mathbb{C}^4$, we can write the equation of $X$ in the above form.

**6.1.4. Example.** If $X$ is defined by a homogeneous cubic form $C(\underline{x}) = 0$, then $\operatorname{Pic}(X - 0)$ has rank 6 and is generated by the (cones over the) lines on the corresponding cubic surface in $\mathbb{P}^3$. In analogy with the case of curves on surfaces in $\mathbb{P}^3$ one can expect that $\operatorname{Pic}(X - 0)$ is generated by simple divisors if $X$ itself is simple. See (2.2.7) for another example.

(6.1.5) Let $X$ be a canonical singularity, and let $p\colon Y \to X$ be a resolution of singularities such that the exceptional divisor $E = E_1 \cup \cdots \cup E_k$ has normal crossings only. Then every $E_i$ is rational or ruled [96, 2.14], hence the groups

$$H^2(E_i, \mathbb{Z}) \cong \operatorname{Pic}(E_i)/\operatorname{Pic}^0(E_i)$$

are readily computable. Thus we can also compute $H^2(Y, \mathbb{Z})$. Since

$$\operatorname{rank}(\operatorname{Pic}(X-0)) = \operatorname{rank}(H^2(Y, \mathbb{Z})) - k, \tag{6.1.5.1}$$

we can compute the rank of $\operatorname{Pic}(X-0)$. However the computations get very messy even in simple examples, unless some shortcuts are used.

Let $X = (f=0) \subset \mathbb{C}^4$ be an isolated singularity. Let us assign weights to the variables by $w\colon x_i \mapsto w_i \in \mathbb{Z}$. Let $f = f_d + f_{d+1} + \cdots$ be the $w$-homogeneous decomposition of $f$. The coordinate ring of the $w$-tangent cone

$$T(X) := \mathbb{C}[x_1, x_2, x_3, x_4]/(f_d)$$

is $w$-homogeneous. Let $a_i$ be the dimension of its $w$-degree $i$ piece.

**6.1.6. Theorem** [21, 7.5]. *Notation as above. Assume that $f_d = 0$ defines an isolated singularity (i.e., $f$ is semiquasihomogeneous in the sense of* [2, 12.1]). *Let $N = d - \sum w_i$. Then*

$$\operatorname{rank}(\operatorname{Pic}(X-0)) \le a_{N+d} - \sum a_{N+w_i},$$

*and equality holds if $X$ is rational.*

*Proof.* Let $p\colon Y \to X$ be the $w$-weighted blow up. The fiber over $0$ is the weighted hypersurface $E = (f_d = 0) \subset \mathbb{P}(w_i)$. By assumption $Y$ has only quotient singularities, hence the local divisor class group of every point is torsion. By Hodge theory (see, e.g., [21, §8])

$$\operatorname{rank}(H^2(E, \mathbb{C})) = 1 + a_{N+d} - \sum a_{N+w_i}.$$

Now use (6.1.5.1). q.e.d.

At least for terminal singularities this approach should give a complete description of the local divisor class group.

Very little is known about the finite generation of symbolic power algebras of rank-one sheaves: $\sum_{m=0}^{\infty} \mathscr{O}_X(mD')$. There are some old examples due to [95] that show that in general the above symbolic power algebra is not finitely generated. Recently more examples were found by [16]. In the positive direction the best result so far is the following:

**6.1.7. Theorem** [98, 2.12], [47, 6.1]. *Let $x \in X$ be a three-dimensional canonical singularity. Then the symbolic power algebra of any divisor is finitely generated.*

**Comments.** In general, different divisors $D' \subset X$ may give rise to the same small modification. This defines an equivalence relation on the divisors, or even on the elements of the vector space $\operatorname{Pic}(X-0)\otimes\mathbb{Q}$. This equivalence relation gives a polyhedral decomposition of the vector space $\operatorname{Pic}(X-0)\otimes\mathbb{Q}$ [98, §7]; [47, §6]. For terminal singularities one can relate this decomposition to the Weyl chambers of the corresponding Dynkin diagram, but in the canonical case there is no known relation to reflection groups.

It would be very interesting to extend the above results to more general singularities. A reasonable question seems to be the following.

**6.1.8. Problem.** Let $0 \in X$ be a three-dimensional singularity. Assume that some hyperplane section $0 \in H \subset X$ is a quotient singularity. Is the symbolic power algebra of any divisor finitely generated? What happens if $H$ is any rational surface singularity?

Among the divisors of a singularity $X$ there is a distinguished one: the canonical divisor $K_X$. Finite generation of its symbolic power algebra is needed for flipping. There is much more known about this special case. One general result is:

**6.1.9. Theorem** [61, 3.5.b]. *Let $0 \in X$ be a three-dimensional isolated singularity. Assume that some hyperplane section $0 \in H \subset X$ is a quotient singularity. Then the symbolic power algebra of the canonical divisor,*

$$\sum_{n=0}^{\infty} \mathcal{O}_X(nK_X),$$

*is a finitely generated $\mathcal{O}_X$-algebra.*

**6.2. Deformations of rational surface singularities.** (6.1.9) was used in [61, §3] to describe the components of the deformation space of a quotient singularity $H$ in terms of certain partial resolutions of $H$. It seems that a large part of this correspondence can be extended to deformations of any rational surface singularity $H$. I circulated informal notes about these problems in the past two years and received very useful comments from T. de Jong, J. Stevens, D. van Straten, and J. Wahl. Several of my original conjectures were thus transformed into theorems. The starting point is the following generalization of (6.1.9):

**6.2.1. Conjecture.** Let $0 \in H$ be a rational surface singularity, and let $0 \in X$ be the total space of a one-parameter smoothing of $0 \in H$. Then the canonical algebra

$$\sum_{n=0}^{\infty} \mathcal{O}_X(nK_X)$$

is a finitely generated $\mathcal{O}_X$-algebra.

**6.2.2.** If this is true, then the Proj of the above canonical algebra gives $g: Y \to X$, where $g$ is an isomorphism outside the origin, $g^{-1}(0)$ consists of finitely many curves, some multiple of $K_Y$ is Cartier, and $K_Y$ is $g$-ample. Let $H' = g^{-1}H$; this is a proper modification of $H$.

Unfortunately, I can prove essentially nothing about the singularities of $Y$ or $H'$. For instance, is it true that $Y$ has only rational singularities? Is it true that $Y$ is Gorenstein in codimension two? A positive answer to these questions would mean that in codimension one $H'$ has only double normal crossing points. I will see (6.3.3) that $H'$ is not always normal. This makes the situation more complicated, but it also leads to very interesting examples.

For the rest of this section I will pretend that $H'$ satisfies the following condition:

(*) Reduced, Cohen-Macaulay surface with at worst double normal crossing points in codimension one.

Substantial changes are required if $H'$ does not satisfy these conditions.

A positive answer to the above conjecture gives a possible description of the components of versal deformation spaces of rational singularities. The crucial thing to notice is that locally $H' \subset Y$ is very special.

**6.2.3. Definition.** Let $0 \in S_0$ be a reduced surface singularity such that $S_0 - 0$ is Gorenstein. Let $0 \in Z$ be the total space of a one-parameter smoothing $S_t$ of $S_0$. We say that the smoothing is $\mathbb{Q}$-Gorenstein (qG for short) if some multiple of the canonical class of $Z$ is Cartier. (This makes sense since $Z$ is normal.)

In fact, in this case the order of $K_Z$ in $\mathrm{Pic}(Z - 0)$ is the same as the order of $K_{S_0}$ in $\mathrm{Pic}(S_0 - 0)$. This follows from:

**6.2.4. Lemma.** *Let $0 \in Z$ be a three-dimensional singularity with a hyperplane section $0 \in S \subset Z$. Assume that $Z - S$ is smooth (normal would be sufficient). Let $L \in \mathrm{Pic}(Z - 0)$ have finite order. Then $L|S - 0 \in \mathrm{Pic}(S - 0)$ has the same order. Equivalently, the kernel of the restriction map $\mathrm{Pic}(Z - 0) \to \mathrm{Pic}(S - 0)$ is torsion free.*

*Proof.* Assume that $L \in \mathrm{Pic}(Z - 0)$ has order $k > 0$ but $L|(S - 0) \cong \mathscr{O}_S|(S - 0)$. Using a nowhere zero section of $L^k$ we construct a $k$-sheeted cover $p: \overline{Z} \to Z$. $p^{-1}(S)$ is connected since $p^{-1}(0)$ is a single point, but $p^{-1}(S) - p^{-1}(0)$ has $k$ connected components isomorphic to $S - 0$. Such a surface singularity is not smoothable by [101, 3.4].

**6.2.5. Proposition** (Wahl). *Let $0 \in S_0$ be a surface singularity satisfying (*). A smoothing $S_t: t \in \Delta$ is qG iff $c_1(K_{S_t}) \in H^2(S_t, \mathbb{Z})$ is torsion. If*

*$S'_t : t \in \Delta$ is another smoothing of $S'_0 \cong S_0$ in the same component of* $\operatorname{Def} S_0$, *then $S'_t$ is qG iff $S_t$ is qG.*

*Proof.* Let $0 \in X$ be the total space of the smoothing. Then by the proof of [64, 5.1] $(X - 0, S_t)$ is 2-connected. Hence $H^2(X - 0, \mathbb{Z}) \to H^2(S_t, \mathbb{Z})$ is an injection and $c_1(K_X) \in H^2(X-0, \mathbb{Z})$ is torsion iff $c_1(K_{S_t}) \in H^2(S_t, \mathbb{Z})$ is.

Since the condition depends only on the topology of $S_t$, it depends only on the component of the deformation space and not on the particular smoothing.

**6.2.6. Definition.** A smoothing component of a versal deformation space of a surface singularity $0 \in S$ satisfying $(*)$ is called a qG component if one (or every) smoothing in it is qG.

A singularity will be called a qG singularity if its deformation space has at least one qG component.

**6.2.7. Remark.** It would be much nicer to have a functorial definition of a qG-family. The reasonable definition is the following:

Let $Z/T$ be flat, Cohen-Macaulay and of relative dimension two. Assume that the locus where the fibers are not Gorenstein is finite over $T$. Then $Z/T$ is qG iff $(\omega_{Z/T}^{\otimes M})^{**}$ is locally free for some $m > 0$.

At the moment I have some technical difficulties working with this definition.

It is important to note that very few singularities have qG components. One restriction is given by:

**6.2.8. Proposition** [64, 5.6-9]. *Let $0 \in S$ be a rational singularity with minimal resolution $p: T \to S$ and let $K_{T/S}$ be the relative canonical divisor, written as a $\mathbb{Q}$-linear combination of the exceptional curves. If $S$ has a qG component, then $K_{T/S} \cdot K_{T/S} \in \mathbb{Z}$.*

*Proof.* Assume for simplicity that there is a compact surface $\overline{S}$ whose only singular point is $0 \in S$ and that $\overline{S}$ admits a qG smoothing $\overline{S}_t$. Let $\overline{T} \to \overline{S}$ be the minimal resolution. Then

$$\begin{aligned} K_{T/S} \cdot K_{T/S} &= K_{\overline{K}/\overline{S}} \cdot K_{\overline{T}/\overline{S}} = K_{\overline{T}} \cdot K_{\overline{T}} - K_{\overline{S}} \cdot K_{\overline{S}} \\ &= K_{\overline{T}} \cdot K_{\overline{T}} - K_{\overline{S}_t} \cdot K_{\overline{S}_t} \in \mathbb{Z}. \end{aligned}$$

In general one can either prove that such a compactification exists or argue as in [64, 5.7].

**6.2.9. Corollary.**

(6.2.9.1) *A quotient singularity $\mathbb{C}^2/\mathbb{Z}_n(1, q)$ has a qG component if $n|(q+1)^2$* [64, 5.9].

(6.2.9.2) *Let* $0 \in S$ *be a rational singularity such that the dual graph of its minimal resolution has a curve* $C$ *of self-intersection* $(-n)$ *which is intersected transversally by* $k$ *curves of self-intersection* $(-2)$. *If* $S$ *has a qG component, then* $(2n-k)|2(n-2)^2$.

*Proof.* The first requires careful computation. The second one is easy from (6.2.8). q.e.d.

Extracting two important properties of $H'$ in (6.2.2) we arrive at the following notion which generalizes the notion of $P$-resolution introduced in [61, 3.8] for quotient singularities.

**6.2.10. Definition.**

(6.2.10.1) Let $H$ be a rational singularity, and let $g: H' \to H$ be a proper modification. Assume that $H'$ is normal. $H'$ is called in P-modification of $H$ if

(i) $K_{H'}$ is $g$-ample, and
(ii) every singularity of $H'$ has a qG component.

(6.2.10.2) Let $\mathrm{Def}^{qG} H'$ denote the subset of $\mathrm{Def} H'$ consisting of deformations that induce a qG deformation of each singularity of $H'$. Thus, up to a smooth factor, $\mathrm{Def}^{qG} H'$ is the product of the local $\mathrm{Def}^{qG}(x, H')$ for all $x \in H'$. By (6.2.5–6), $\mathrm{Def}^{qG} H'$ is the union of some components of $\mathrm{Def} H'$.

It is less clear what the correct definition is when $H'$ is not normal.

**6.2.10$'$. Definition.** (6.2.10$'$.1) Let $H$ be a rational singularity, and let $g: H' \to H$ be a proper modification. Assume that $H'$ is Gorenstein outside finitely many points. $H'$ is called a P-modification if

(i) $K_{H'}$ is $g$-ample,
(ii) $R^1 g_* \mathscr{O}_{H'} = 0$, and
(iii) $H'$ has a smoothing which induces a qG smoothing of each singularity of $H'$.

(6.2.10$'$.2) Let $\mathrm{Def}^{qG} H'$ denote the closure of the subset of $\mathrm{Def} H'$ consisting of smoothings that induce a qG smoothing of each singularity of $H'$.

**6.2.11. Definition.** Let $H$ be a rational singularity. A P-modification $g: H' \to H$ is called weakly rigid if it has no positive-dimensional deformations among P-modifications of $H$, i.e., if $g_t: H'_t \to H$ is a flat deformation of $g: H' \to H$ over $\Delta$, where $H'_t$ is a P-modification for every $t$, then $H'_t \cong H'$ for every $t$.

**6.2.12. Proposition.** *Let* $H$ *be a normal surface singularity, and let* $g: H' \to H$ *be a normal and proper modification which is dominated by* *the minimal resolution of $H$. Then $H'$ has no positive-dimensional family of deformations among proper modifications of $H$.*

*In particular, a normal $P$-modification which is dominated by the minimal resolution is weakly rigid.*

*Proof.* Let $H'_t \to H$ be a flat deformation over $\Delta$, and $\overline{H}_t$ be the minimal resolution of $H'_t$. By [61, 2.10(i)],

$$K_{\overline{H}_t} \cdot K_{\overline{H}_t} \geq K_{\overline{H}_0} \cdot K_{\overline{H}_0}, \qquad t \in \Delta.$$

On the other hand, $\overline{H}_0$ is the minimal resolution of $H$, and $\overline{H}_t$ is obtained from the minimal resolution of $H$ by blowing up some points (possibly none). Thus

$$K_{\overline{H}_t} \cdot K_{\overline{H}_t} \leq K_{\overline{H}_0} \cdot K_{\overline{H}_0}, \qquad t \in \Delta.$$

Hence the equality holds, and $\overline{H}_t$ is the minimal resolution of $H$ for every $t$. By [61, 2.10(ii)] the family $H'_t$ can be resolved simultaneously, and the claim is clear. q.e.d.

In order to formulate a general conjecture, we need a definition.

**6.2.13. Definition.** Let $U/S$ be a flat family of reduced surfaces with rational singularities only. Define a functor $P-\mathrm{mod}(U/S)$ as follows. Given $p: S' \to S$ let

$$P\mathrm{-mod}(U/S)(S') = \begin{cases} \text{Pairs } (Z'/S', g), \text{ where } Z'/S' \text{ is a qG-family} \\ \text{(6.2.7) and } g: Z' \to U \times_S S' \text{ is a proper mor-} \\ \text{phism such that for every } s' \in S' \text{ the restriction} \\ g: Z' \times_{S'} \{s'\} \to U \times_S \{p(s')\} \text{ is a P-modification.} \end{cases}$$

**6.2.14. Conjecture.** For every $U/S$ as above, the functor $P\mathrm{-mod}(U/S)$ is represented by a separated algebraic space $\mathbf{P} - \mathrm{mod}(U/S)$ which is proper over $S$.

**Comments.** One should keep in mind that the tentative definitions of P-modification and qG-family make the conjecture somewhat vague.

The conjecture is a generalization of the results in [4]. Separatedness should be expected since $K$ is assumed to be relatively ample in (6.2.10).

I expect that the proof of [4] can be modified to prove the existence and separatedness of $\mathbf{P} - \mathrm{mod}(U/S)$. The difficult part should be the establishment of properness.

Let $\mathrm{Def}\, H$ be a versal deformation space of a rational singularity, and let $\mathscr{U}/\mathrm{Def}\, H$ be the universal family. Assume that $\mathbf{P} - \mathrm{mod}(\mathscr{U}/\mathrm{Def}\, H)$ exists and is proper. Then for every component $C \subset \mathrm{Def}\, H$ there is a unique component $\mathbf{P}-\mathrm{mod}(C)$ of $\mathbf{P}-\mathrm{mod}(\mathscr{U}/\operatorname{Def} H)$ such that $\mathbf{P}-\mathrm{mod}(C) \to C$ is surjective, proper, and birational.

One can view (6.2.14) as a refinement of a conjecture posed in [119, p. 241.]

Even the existence of $\mathbf{P}-\mathrm{mod}(U/S)$ has interesting consequences. Let $g\colon H' \to H$ be a P-modification of $H$, and $H'_t$ be a sufficiently general qG-smoothing of $H'$. As in [4] we can assume that $H'_t$ contains no compact curves for $t \neq 0$. By [118, 1.4] and [60, 11.4] the family $H'_t$ can be contracted to a flat family $H_t$ of deformations of $H$. Thus, $H'_t \to H_t$ can be induced by a morphism $\rho\colon \Delta \to \mathbf{P}-\mathrm{mod}(\mathscr{U}/\operatorname{Def} H)$. Let $C_{H'}$ be a component of $\mathbf{P}-\mathrm{mod}(\mathscr{U}/\operatorname{Def} H)$ containing $\rho(\Delta)$.

Let $W \subset \operatorname{Def} H$ be the open subset parametrizing smooth deformations of $H$. It is clear that $\mathbf{P}-\mathrm{mod}(\mathscr{U}|W/W) \cong W$. By construction $C_{H'}$ and $\mathbf{P}-\mathrm{mod}(\mathscr{U}|W/W)$ intersect nontrivially. Therefore, there is an irreducible component $C_H$ of $\operatorname{Def} H$ such that the induced morphism $C_{H'} \to C_H$ is an isomorphism over an open set. If $H'$ is weakly rigid, then by the valuation criterion of properness $C_{H'} \to C_H$ is even proper. This yields the following:

**6.2.15. Conjecture–Corollary.** *Let $H$ be a rational surface singularity. There is a natural injective correspondence*

$$\left\{\begin{array}{l}\textit{qG-components of deformations of}\\ \textit{weakly-rigid P-resolutions of } H\end{array}\right\} \to \{\textit{components of } \operatorname{Def} H\}.$$

This result gives an effective procedure to exhibit irreducible components of $\operatorname{Def} H$.

**6.2.16. Known cases.** Conjectures (6.2.1) and (6.2.14) are proved in the following cases:

(6.2.16.1) If $C$ is the Artin component, then $\mathbf{P}-\mathrm{mod}(C)$ exists [4]. The minimal Du Val resolution is the corresponding P-modification. Thus the conjecture is true if the deformation space has only one component. This is for instance the case for rational double and triple points.

(6.2.16.2) quotient singularities [61, Chapter 3];

(6.2.16.3) quotient of simple elliptic and cusp singularities [61, Chapter 5];

(6.2.16.4) quadruple points [111].

(6.2.16.5) [III, Proposition 6] implies 6.2.15 in many cases.

**6.3. Examples.** (6.2.15) can be used to exhibit examples of components of deformation spaces. It is especially simple to understand those P-modifications that are dominated by the minimal resolution.

Let $H$ be a rational surface singularity with minimal resolution $f\colon \overline{H} \to H$, and $D_i \subset \overline{H}$ be connected subsets of curves, pairwise disjoint. Contracting $D_i$ to a point for every $i$ one obtains a partial resolution

$$\overline{H} \to H' \xrightarrow{g} H.$$

The following is easy:

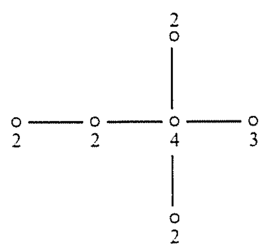
**6.3.1. Lemma.** *$K_{H'}$ is $g$-ample iff every $(-2)$-curve in $\overline{H}$ is contained in or intersects $\bigcup D_i$.*

Therefore, if every $D_i$ is a resolution of a qG singularity, then $H'$ is a P-modification. The simplest qG singularity is $\mathbb{C}^2/\mathbb{Z}_2(1, 1)$ whose resolution is a single $(4)$-curve. Thus we obtain:

**6.3.2. Example.** Let $H$ be a rational surface singularity with minimal resolution $f\colon H \to H$. Every $(-4)$-curve in $\overline{H}$ gives rise to a component of the versal deformation space of $H$.

More complicated P-modifications arise in the following examples which were developed jointly with J. Stevens.

**6.3.3. Example.** Consider any rational singularity with the following dual resolution graph:




These singularities are rational of multiplicity 5. They have at least one equisingular modulus: the cross ratio of the four curves intersecting the central $(-4)$-curve.

Two P-modifications are easy to see.

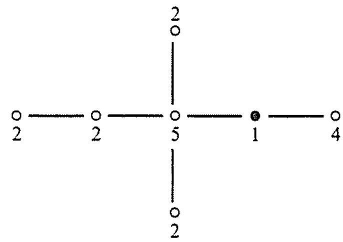
(6.3.3.1) Contract all $(-2)$-curves. This gives the Artin component.

(6.3.3.2) Contract the $(-2)$ on the left and the $(-4)$-curve.

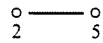
There are no other normal P-modifications dominated by the minimal resolution.

(6.3.3.3) There are at least three other normal P-modifications. To obtain these, blow up the intersection point of the $(-4)$- and the $(-3)$-curves.

Thus we have




We can contract the $(-4)$-curve on the right, any of the three




configurations, and the $(-2)$-curve on the left if it does not intersect any contracted curves. We have to check that after these contractions the canonical class is relatively ample. This is an easy computation.

(6.3.3.4) There is also a nonnormal P-modification. On the central $(-4)$-curve there are four distinguished points, corresponding to the four intersection points. We denote these intersection points by $N$, $E$, $S$, $W$ corresponding to the directions in the above diagram. There is a unique involution $\tau$ of the central $(-4)$-curve such that $\tau(N) = S$ and $\tau(E) = W$. To get the P-modification, first contract all curves except the central $(-4)$-curve $C$. This gives the normal surface $C'' \subset H''$. Then for every $x \in C''$ identify $x$ and $\tau(x)$ (cf. [3, 6.1])) to obtain a nonnormal surface germ $g\colon C' \subset H' \to H$. Along $C'$ we have generically normal crossings points. There are also two pinch points and two singularities of the form

$$(xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_2(1, -1, 1) \quad \text{and} \quad (xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_3(1, -1, 1).$$

$H'$ has only qG singularities; $xy + t = 0$ is a smoothing at the two non-Gorenstein points. Easy computation gives that

$$C' \cdot K_{H'} = \tfrac{1}{2} C'' \cdot (K_{H''} + C'') = \tfrac{1}{6},$$

hence $K_{H'}$ is $g$-ample.

We still need to check that $H'$ is qG-smoothable. This is done by using the following technical lemma whose proof we omit:

**6.3.4. Lemma.** *Let $C' \subset H'$ be the germ of a surface along a smooth curve. Assume that locally along $C'$ the surface $H'$ is one of the following:*

(6.3.4.1) *normal crossing point:* $(xy = 0) \subset \mathbb{C}^3$ *; or*

(6.3.4.2) *pinch point:* $(x^2 - y^2 z = 0) \subset \mathbb{C}^3$ *; or*

(6.3.4.3) *semi-log-terminal point:* $(xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_n(1, -1, a)$, *where* $(a, n) = 1$ (*cf.* [61, *Chapter* 4]). *For these singularities the qG deformations are exactly those that can be obtained as* $(xy + tf(z^n, t) = 0) \subset \mathbb{C}^3/\mathbb{Z}_n(1, -1, a, 0)$.

*Let the number of pinch points by* $p$, *and* $C'' \subset H''$ *be the normalization of* $H''$. *Then*

(6.3.4.4) *For* $H'$ *one can define a functor of qG deformations, in particular, the sheaf* $\mathcal{T}^1_{qG}(H') \subset \mathcal{T}^1(H')$. *These sheaves are isomorphic except over a point of type* (6.3.4.3).

(6.3.4.5) $\mathcal{T}^1_{qG}(H')$ *is concentrated along* $C'$, *it has zero dimensional torsion at the pinch points, and the quotient by this torsion is a line bundle* $L$ *on* $C'$.

(6.3.4.6) $\deg L = C'' \cdot C'' + p$ (*note that* $C'' \cdot C''$ *is automatically an integer by the choice of the singularities* (6.3.4.3)).

Applying the lemma to our situation we get that

$$\mathcal{T}^1_{qG}(H')/(\text{torsion}) \cong \mathcal{O}_{C'}(-4+2+2).$$

Therefore $H'$ has a $qG$ smoothing.

I do not know if these are all the components of $\operatorname{Def} H$ or not.

From the construction it is clear that one can produce similar examples for higher multiplicity rational singularities.

**6.3.5. Example.** Consider rational singularities of multiplicity 8 such that the minimal resolution has one $(-5)$-curve $C$ and six $(-2)$-curves intersecting $C$ transversally.

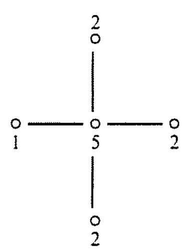
There are 22 P-modifications dominated by the minimal resolution:

(6.3.5.1) The minimal Du Val resolution; this gives the Artin component.

(6.3.5.2) We can contract any of the six subconfigurations:


This is the dual graph of the quotient singularity $\mathbb{C}^2/\mathbb{Z}_9(1, 2)$.

(6.3.5.3) We can contract any of the 15 subconfigurations:




These are dual graphs of $\mathbb{Z}_2$-quotients of simple elliptic singularities of multiplicity 6.

(6.3.5.4) There is no qG component. This follows from (6.2.9.2).

(6.3.5.5) In certain cases there are nonnormal P-modifications. Let $P_1, \cdots, P_6 \in C$ be the six intersection points. Assume that there is an involution $\tau: C \to C$ such that $\tau\{P_1, \cdots, P_6\} = \{P_1, \cdots, P_6\}$, and $\tau$ has no fixed points among $\{P_1, \cdots, P_6\}$. Under these assumptions we can repeat the construction in (6.3.3.4) to obtain a nonnormal P-modification.

If we vary the six points generically, then $\tau$ will not exist, hence there is no corresponding flat deformation of $H'$. This shows that the corresponding component does not exist for small generic deformations of the singularity.

Also note that there can be several involutions $\tau$, and these give different P-modifications. To get some examples, choose an isomorphism $C \cong \mathbb{C} \cup \{\infty\}$.

If $P_i$ are the sixth roots of unity, then there are four possible involutions:

$$\tau_0: z \mapsto -z \qquad \tau_i: z \mapsto \zeta^{2i-1}\frac{1}{z} \quad (i = 1, 2, 3),$$

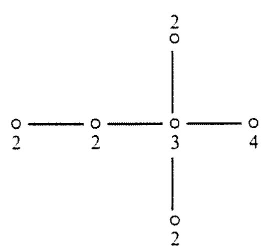
where $\zeta$ is a primitive sixth root of unity.

If $\{P_1, \cdots, P_6\} = \{0, \infty, \pm 1, \pm i\}$, then there are six involutions

$$z \mapsto \frac{\pm i}{z}, \quad z \mapsto \eta\frac{z+\eta}{z-\eta}, \quad \text{where } \eta^4 = 1.$$

Again I do not know whether these are all the components of the deformation space.

**6.3.6. Example.** Consider any rational singularity with the following dual resolution graph:

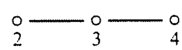



There are five P-modifications dominated by the minimal resolution:

(6.3.6.1) The minimal Du Val resolution.

(6.3.6.2) Contract all $(-2)$-curves and the $(-4)$-curve.

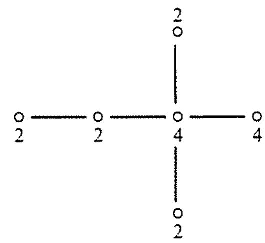
(6.3.6.3) Contract any of the three configurations




(this is the dual graph of $\mathbb{C}^2/\mathbb{Z}_{18}(1, 5)$) and the $(-2)$-curve on the left if necessary.

(6.3.6.4) There is also a one-parameter family of P-modifications. To get these, blow up any point $x$ on the $(-3)$-curve different from the intersection points. Thus we have a $(-1)$-curve and the following configuration:

(6.3.6.5)



If we contract the above configuration, then we get a proper modification $g: H_x \to H$ with one exceptional curve $C_x$ which is the image of the $(-1)$-curve. One can compute that

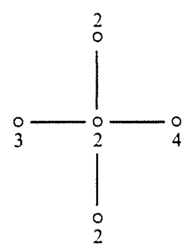
$$K_{H_x} \cdot C_x = \tfrac{1}{5}.$$

The following lemma shows that every $H_x$ is a P-modification.

**6.3.7. Lemma.** *Any singularity with dual resolution graph as in* (6.3.6.5) *has a qG component.*

*Proof.* $(x^2z+xy^2+z^5+ay^2z^2=0) \subset \mathbb{C}^3$ is an elliptic singularity with resolution:

(6.3.7.1)

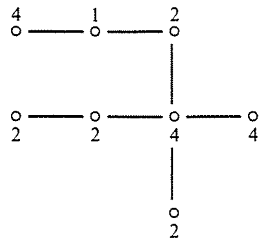


Taking the quotient by the $\mathbb{Z}_5$-action (1,2,3) we obtain a singularity as in (6.3.6.5). Conversely, for any singularity as in (6.3.6.5) one can take the 5-fold cover given by the canonical class, and the cover turns out to be one of the singularities in (6.3.7.1). The required qG smoothing is given by

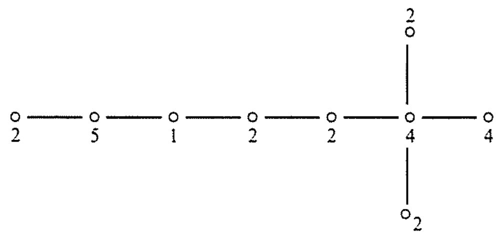
$$(x^2z + xy^2 + z^5 + ay^2z^2 + t = 0)/\mathbb{Z}_5. \quad \text{q.e.d.}$$

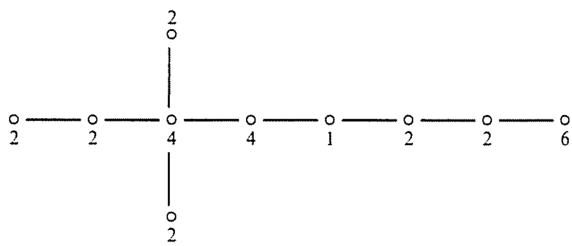
If $x$ degenerates to one of the four intersection points $N, E, S$ or $W$ on the $(-3)$-curve, the P-modification $H_x$ degenerates into a different type of P-modification. The following are the dual graphs of the minimal resolutions of these four P-modifications; in all cases everything except the

(−1)- curve is to be contracted:

$H_N$: 


$H_S$ is the same as $H_N$ upside down.

$H_W$: 


$H_E$: 


Thus we obtain a flat family of P-modifications parametrized by $\mathbb{P}^1$.

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