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# Symplectic Geometry and the Verlinde formulas

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ABSTRACT. The purpose of this paper is to give a proof of the Verlinde formulas by applying the Riemann-Roch-Kawasaki theorem to the moduli space of flat $G$-bundles on a Riemann surface $\Sigma$ with marked points, when $G$ is a connected simply connected compact Lie group $G$. Conditions are given for the moduli space to be an orbifold, and the strata are described as moduli spaces for semisimple centralizers in $G$. The contribution of the strata are evaluated using the formulas of Witten for the symplectic volume, methods of symplectic geometry, including formulas of Witten-Jeffrey-Kirwan, and residue formulas. Our paper extends prior work by Szenes on SU(3) and Jeffrey-Kirwan for SU($n$) to general groups $G$.

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## Introduction

The Verlinde formula [62], [3] computes the dimension of spaces of holomorphic sections of canonical line bundles over the moduli space $\mathcal{M}$ of semistable $G_{\mathbf{C}}$-bundles on a Riemann surface $\Sigma$ with marked points, with $G$ a connected and simply connected compact Lie group. For a given "level" $p$, the Verlinde formula is a sum over the finite collection of weights parametrizing the representations of the central extension of the loop group $LG$ at level $p$. This formula, discovered by Verlinde in the context of quantum field theory, has received a number of rigorous proofs first for $G = \mathrm{SU}(2)$ by Thaddeus [57], Bertram and Szenes [7] and Szenes [52] (see also Donaldson [17] and Jeffrey-Weitsman [31] for related questions), and for more general groups by Tsuchiya-Ueno-Yamada [59], Beauville-Laszlo [4] (for $G = \mathrm{SU}(n)$), Faltings [20], Kumar, Narasimhan, Ramanathan [38] for general groups $G$. A common feature of many of these proofs is that establishing the fusion rules for the Verlinde numbers is an essential step in the proof, a second step being the description of the fusion algebra.

The purpose of this paper is to give a proof of the Verlinde formula for connected simply connected compact Lie groups, by methods of symplectic geometry. This program has been already carried out by Szenes [53] for SU(3), and Jeffrey-Kirwan [30] in the case of $\mathrm{SU}(n)$. The main point of the paper is to extend their approach a to general groups. More precisely, we will obtain the Verlinde formula by an application of Riemann-Roch.

The theorem of Narasimhan-Sheshadri [46] asserts that $\mathcal{M}$ can be identified with the set of representations of $\pi_1(\Sigma)$ with values in $G$, with given conjugacy classes of holonomies at the marked points. For generic choices of holonomies, this last space is a symplectic orbifold $(M/G, \omega)$, which carries an orbifold Hermitian line bundle with connection $(\lambda^p, \nabla^{\lambda^p})$, such that $c_1(\lambda^p, \nabla^{\lambda^p}) = p\omega$. In particular the orbifold $M/G$ is complex. This orbifold carries a canonical Dirac operator $D_{p,+}$, unique up to homotopy. Its index $\mathrm{Ind}(D_{p,+})$ is the Euler characteristic of $\lambda^p$. We will compute the index $\mathrm{Ind}(D_{p,+})$ using the formula of Riemann-Roch-Kawasaki [32, 33]. We show that, for $p$ large, it is given by the Verlinde formula. The fact that $p$ has to be large may be related to the fact that a priori, higher cohomology may well not vanish for small $p$. In fact, the Verlinde formula computes $\dim H^0(\mathcal{M}, \lambda^p)$, while $\mathrm{Ind}(D_{p,+})$ is the corresponding Euler characteristic (we will come back to this point at the end of the introduction in connection with results by Teleman [55, 56]). For non generic holonomies, we also show that small perturbations of the holonomies still produce an orbifold moduli space. For suitable perturbations, we show that for any $p$, the index of the corresponding Dirac operator $D_{p,+}$ is still given by the Verlinde formula. The typical case where such a perturbation is needed is when $\Sigma$ does not have marked points.

Our proof contains various interrelated steps.

- A first step is the description of the strata of the orbifold moduli space $M/G$. These strata are in fact moduli spaces for the semisimple centralizers in $G$. Up to conjugacy, there is only a finite family of such centralizers. In general, they are non simply connected. The strata split into a union of substrata indexed by the fundamental group of the centralizers. The description of the geometry of $M/G$ involves results contained in Sections 1, 4, 5, 6. Observe that if $G = \mathrm{SU}(n)$, there are no non trivial semisimple centralizers, which explains the smoothness of the moduli space $M/G$ (this case which was already considered by Jeffrey-Kirwan [**30**]).

- A second step consists in reproving Witten's formula [**63**] for the symplectic volume of the moduli spaces.
- Another step is the detailed construction of the orbifold line bundle $\lambda^p$. Also we have to compute the action of the finite stabilizers of elements of $M$ on $\lambda^p$. This is done in Sections 4 and 5.
- In Section 5, we show that the formalism of the moment map can be applied to each stratum of the moduli space. In Section 6, we use a formula of Witten [**64**] , Jeffrey-Kirwan [**28**] (see Vergne [**61**]) and Liu [**39, 40**] to express the contribution of each stratum as the action of a differential operator on a locally polynomial function on a maximal torus $T$. This locally polynomial function is just the symplectic volume of a deformation of the moduli space $M/G$. Our treatment of these formulas is very close in spirit to Liu [**39, 40**].
- Witten's formula [**63**] for the deformed symplectic volume of each stratum is a Fourier series on $T$. In order to calculate the contribution of each stratum explicitly, it is of critical importance to express Witten's formula using residues techniques, which will make obvious the fact that the given Fourier series is indeed a local polynomial on $T$. These residue techniques are developed in Section 2.
- In Section 7, we give a residue formula for the index of the Dirac operator on $M/G$, by putting together the contribution of all strata.
- Another step is to express the Verlinde sums as residues. This step, which is carried out of Section 7, has many formal similarities with what is done in Section 2 for the Fourier series on $T$.
- In Section 8, a comparison of the results of Sections 6 and 7 leads us to our main result.

We now review our techniques in more detail.

## 1. The residue techniques

Residue techniques play an important role in the whole paper, in order to convert the Witten Fourier series [**63**] for the symplectic volumes into expressions which make them local polynomials in an "obvious" way, so that differential operators can be applied to these polynomials. Similar residue techniques are also applied to the Verlinde sums.

Szenes [**53, 54**] initiated the use of residue techniques to treat Verlinde formulas. In [**53**], Szenes applied such residue techniques to the case of SU(3) and obtained the corresponding Verlinde formulas. In [**54**], Szenes developed a cohomological approach in terms of arrangement of hyperplanes to treat the Witten sums for any group $G$.

In [**30**], Jeffrey and Kirwan gave a proof of the Verlinde formulas for SU($n$) using residue techniques from a point of view which is very different from ours. In fact they use non abelian localization formulas [**64**], [**28**] applied to an extended moduli space. In the spirit of the formula of Duistermaat-Heckman [**19**], they use residue techniques to evaluate the Fourier transform of the contribution of the fixed points. In particular they reobtain Witten's formulas [**63**] for the symplectic volume of the moduli space of $G = \mathrm{SU}(n)$.

The strategy used in the present paper goes in some sense in an opposite direction. First, as in Liu [**39, 40**], our computations are local, and only use the action of differential operators on the symplectic volumes of the moduli spaces. Then, we express the symplectic volumes as residues in order to evaluate explicitly the action of certain differential operators on the symplectic volumes .

In the present paper, multidimensional residues are used in a rather "naive" way. The Witten sums are expressed as sums over a lattice identified to $\mathbf{Z}^r$. We compute the given sums by summing in succession in the variables $k_1, \dots, k_r \in \mathbf{Z}$, and by applying standard residue techniques to these one dimensional sums. Handling the recursion requires the development of a trivial, but heavy linear algebra. We believe that Szenes's techniques [**54**] can put put to fruitful use to give a more conceptual approach to this part of our work.

**2. A combinatorial description of the moduli spaces**

Let $\mathcal{O}_1, \dots, \mathcal{O}_s$ be $s$ adjoint orbits in $G$. Put $X = G^{2g} \times \prod_{j=1}^s \mathcal{O}_j$. Let $\phi : X \to G$ be given by

$$\phi(u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_s) = \prod_{i=1}^g [u_i, v_i] \prod_{j=1}^s w_j. \tag{0.1}$$

Put $M = \phi^{-1}(1)$. Under a genericity assumption on the $\mathcal{O}_j, 1 \leq j \leq s$ , the condition $(A)$ of Definition 5.17, which requires that $s \geq 1$, in Theorem 5.18, we show that $M$ is a smooth manifold on which $G$ acts locally freely. The moduli space is the orbifold $M/G$.

To prove Witten's formula [**63**], we show in Theorem 5.45 that the image by $\phi$ of the Haar measure on $X$ has a density with respect to the Haar measure on $G$, which is essentially given by the symplectic volume of the quotient fibres of $\phi$, which are themselves moduli spaces with an extra marked point. This approach was initiated by Liu [**39, 40**], who showed in particular that the differential of $\phi$ can be expressed in terms of the combinatorial complexes which compute the cohomology of the flat adjoint vector bundle $E$. Liu then obtains the intersection numbers of the moduli spaces by applying certain differential operators to the symplectic volumes. Inspired by Witten [**63, 64**], Liu gave a special role to the heat kernel on the group $G$ to establish Witten's formula, while in our approach, we do not use any heat kernel. Needless to say, the heat kernel on $G$ remains crucial in understanding connections with 2-dimensional Yang-Mills theory, and also with Witten's non Abelian localization [**64**].

**3.Moment maps and the quantization conjecture**

In Section 5.10, under genericity assumptions, we show that a $G$-invariant neighborhood $\widehat{X}$ of $M$ in $X$ can be equipped with a symplectic form, that $G$ acts on $\widehat{X}$ with a moment map, and that the standard symplectic structure on the quotients on the fibres of $\phi$ included in $\widehat{X}$ come from the symplectic structure on $\widehat{X}$. We can then use directly the formulas of Witten [**64**] and Jeffrey-Kirwan [**28**] to express the integrals of certain characteristic classes in terms of differential operators acting on the symplectic volume of the fibres.

When the genericity assumptions are not verified, we replace the given moduli space by a generic perturbation, which still carries a Dirac operator to which the

Riemann-Roch-Kawasaki theorem [**32, 33**] can be applied. We then show that the above index results still hold.

We will now put our results in perspective from the point of view of geometric quantization, especially in connection with the Guillemin-Sternberg conjecture [**23**]. By Atiyah-Bott [**2**, Section 9], we know that when there is one marked point with central holonomy, $M/G$ is a symplectic reduction of the affine space $\mathcal{A}$ of $G$-connections with respect to the action of the gauge group $\Sigma G$, which acts on $\mathcal{A}$ with a moment map $\mu$, which is the curvature, so that $M/G = \mu^{-1}(0)/\Sigma G$. Similarly the line bundle $\lambda^p$ is itself the reduction of a universal line bundle $L^p$ on $\mathcal{A}$. This theory can be extended to the case with marked points (this we do in part in Sections 4 and 5). If $\mathcal{A}$ was instead a compact manifold, and $\Sigma G$ a compact connected Lie group, the Guillemin-Sternberg conjecture [**23**] asserts that the Riemann-Roch number of $(M/G, \lambda^p)$ is equal to the multiplicity of the trivial representation in the action of $\Sigma G$ on the cohomology of $L^p$. The Guillemin-Sternberg conjecture has been proved in various stages, the most general result being given by Meinrenken [**41**],[**42**] (for a more analytic proof, see Tian and Zhang [**58**]). In Meinrenken's formalism, when the considered group does not act locally freely on $\mu^{-1}(0)$ , one replaces 0 by any regular value of $\mu$ close to 0 , and one still gets a corresponding version of Guillemin-Sternberg's conjecture.

Chang [**16**] initiated the study of Verlinde formulas in the context of geometric quantization. A new twist was introduced to the story of the proof of the Verlinde formula in work by Meinrenken and Woodward [**43**], [**44**], Alekseev, Malkin, Meinrenken [**1**], and later work by these authors. In [**43, 44**], Meinrenken and Woodward gave a symplectic proof of the fusion rules for the Verlinde numbers. In [**1**], the authors develop a theory of group actions with moment maps taking their values in the given Lie group $G$. This theory is in fact a theory of the standard moment map for an action of a central extension of the loop group $LG$. The space $X$ is the prototype of such a manifold, the moment map being just $\phi$. These authors then develop a localization formula in equivariant cohomology, which is an analogue of the formula of Duistermaat-Heckman [**19**], Berline-Vergne [**5**]. By using the theory of symplectic cuts and the previous results by Meinrenken [**41, 42**] on the Guillemin-Sternberg conjecture, they announce a proof of the Verlinde formulas.

In some way, our paper represents a direct attempt to prove Verlinde formulas directly, by a method which resembles the proof given by Jeffrey-Kirwan [**29**] of the Guillemin-Sternberg conjecture. The proof of [**29**] consists in extracting the Riemann-Roch number of $\mu^{-1}(0)/\Sigma G$ from the Lefschetz formulas.

In [**55, 56**], Teleman gave a proof of the vanishing of the higher cohomology groups of $\lambda^p$ for a small perturbation of the moduli space $M/G$. As a consequence, one should always have $\mathrm{Ind}(D_{p,+}) = \dim H^0(\mathcal{M}, \lambda^p)$. As explained before, we only prove such an equality for large $p$, and otherwise, we have to perturb the moduli space by a perturbation which can be 'large' for small $p$, so that there could be a wall-crossing discrepancy between $\mathrm{Ind}(D_{p,+})$ and $\dim H^0(\mathcal{M})$. Inspection of the proof shows that this discrepancy also vanishes for $g$ large enough, but Teleman's results indicate that the perturbation described above should never be needed to get the above equality.

Our paper is organized as follows. In Section 1, we establish basic simple facts on compact simply connected simple Lie groups and their semisimple centralizers. In Section 2, we develop our basic residue techniques in several variables. In particular, we express certain Fourier series on $T$, which are local polynomials, as residues. In Section 3, we reestablish well-known results on symplectic actions with moment maps, and we give a proof of the formula of Witten and Jeffrey-Kirwan. In Section 4, we construct the canonical line bundle $L$ on the moduli space of $G$-connections on the Riemann surface $\Sigma$ with fixed holonomy at the given marked points. We show that a suitable central extension of the gauge group $\Sigma G$ acts on $L$, and we compute the action of certain stabilizers on $L$ and on a related line bundle $\lambda_p$. In Section 5, we describe the moduli space $M/G$. We show that the formalism of the moment map can be applied to the action of $G$ on $M$. We apply the formula of Witten [**64**] and Jeffrey-Kirwan [**28**] to the moduli spaces associated to semisimple centralizers. Also we give a formula for $c_1(TM/G)$. In Section 6, we apply the theorem of Riemann-Roch-Kawasaki to the orbifold $M/G$, and we give a residue formula for the index $\mathrm{Ind}(D_{p,+})$. In Section 7, we give a residue formula for the Verlinde sums. Finally in Section 8, we compare $\mathrm{Ind}(D_{p,+})$ and the Verlinde formula, and give a number of conditions under which they coincide.

The results contained in this paper were announced in [**12**].

## 1. Simple Lie groups and their centralizers

Let $G$ be a connected and simply connected compact simple Lie group. The purpose of this Section is to give the basic facts which will be needed in the description of the strata of the moduli space of flat $G$-bundles on a Riemann surface $\Sigma$. This involves in particular a complete description of the semisimple centralizers in $G$.

This Section is organized as follows. In Section 1.1, we recall elementary properties of roots and coroots. In Section 1.2, we construct the basic scalar product on the Lie algebra $\mathfrak{g}$ of $G$. In Section 1.3, we compute the volume of a maximal torus $T$. In Section 1.4, we relate the quadratic form attached to a representation to the basic quadratic form. In Section 1.5, we introduce the dual Coxeter number. In Section 1.6, we construct an embedding of the center $Z(G)$ in the Weyl group $W$. In Section 1.7, we give simple properties of the element $\rho/c \in T$, in particular in its relations with the center $Z(G)$. In Section 1.8, we review elementary properties of the characters of $G$. In Sections 1.9-1.12, we describe the semisimple centralizers, and give some of their properties. In Section 1.13, we consider the intersection of an adjoint orbit with such a centralizer. Finally in Section 1.14, we recall various properties of the stabilizers of elements of $\mathfrak{g}$, and we construct the symplectic structure on the coadjoint orbits, and the corresponding line bundles.

**1.1. Roots and coroots.** Let $G$ be a connected simply connected simple compact Lie group of rank $r$. Let $\mathfrak{g}$ be its Lie algebra, let $\mathfrak{g}^*$ be its dual. Let $T$ be a maximal torus in $G$, let $\mathfrak{t}$ be its Lie algebra, let $\mathfrak{t}^*$ be its dual. Let $W$ be the corresponding Weyl group.

We will denote the composition law multiplicatively in $G$, but often additively in $T$.

Let $\Gamma \subset \mathfrak{t}$ be the lattice of integral elements in $\mathfrak{t}$, i.e.

$$\Gamma = \{t \in \mathfrak{t}, \exp(t) = 1 \text{ in } T\}. \tag{1.1}$$

Let $\Lambda = \Gamma^* \subset \mathfrak{t}^*$ be the lattice of weights in $\mathfrak{t}^*$, so that if $h \in \Gamma$, $\lambda \in \Lambda$,

$$\langle \lambda, h \rangle \in \mathbf{Z}. \tag{1.2}$$

Let $R = \{\alpha\} \subset \Lambda$ be the root system of $G$. Then $R$ is a finite family of elements of $\Lambda$, which span $\mathfrak{t}^*$. Let $\overline{R} \subset \Lambda$ be the lattice generated by $R$, let $\overline{R}^* \supset \Gamma$ be the lattice dual to $\overline{R}$.

Let $CR = \{h_\alpha\} \subset \mathfrak{t}$ be the family of coroots attached to $R$. Let $\overline{CR} \subset \mathfrak{t}$ be the lattice generated by $CR$, let $\overline{CR}^* \subset \mathfrak{t}^*$ be the corresponding dual lattice. Since $G$ is simply connected, by [**15**, Theorem V.7.1],

$$\Gamma = \overline{CR}, \qquad \Lambda = \overline{CR}^*. \tag{1.3}$$

Let $Z(G)$ be the center of $G$. By [**15**, Proposition V.7.16],

$$\overline{R}^* / \overline{CR} = Z(G). \tag{1.4}$$

Let $R_s, R_\ell$ and $CR_s, CR_\ell$ be the short, long roots and coroots. Note that $R_s$ corresponds to $CR_\ell$ and $R_\ell$ to $CR_s$. Recall that $G$ is said to be simply laced if all the roots (or coroots) have the same length. In this case, all the roots will be considered as long, and the coroots as short, so that

$$R_s = \emptyset\,, \qquad CR_\ell = \emptyset\,. \tag{1.5}$$

In the sequel, when $G$ is simply laced, any statement concerning $R_s$ or $CR_\ell$ should be disregarded.

Let $\overline{R_\ell}, \overline{R_s}, \overline{CR_\ell}, \overline{CR_s}$ be the lattices generated by $R_\ell$, $R_s$, $CR_\ell$, $CR_s$. It follows from the classification of Lie groups that

$$\overline{CR_s} = \overline{CR}, \qquad \overline{R_s} = \overline{R}\,. \tag{1.6}$$

Note that when $G$ is simply laced, the second equality in (1.6) is empty.

### 1.2. The basic scalar product on the Lie algebra.

DEFINITION 1.1. Let $\langle\ ,\ \rangle$ be the $G$-invariant scalar product on $\mathfrak{g}$ such that if $\|.\|$ is the corresponding norm, if $h_\alpha \in CR_s$,

$$\|h_\alpha\|^2 = 2\,. \tag{1.7}$$

If $G$ is not simply laced, there is one $m \in \mathbf{N}$ (equal to 2 or 3) such that if $h_\alpha \in CR_\ell$,

$$\|h_\alpha\|^2 = 2m\,. \tag{1.8}$$

By [**15**], if $f, f' \in \overline{CR}$,

$$\langle f, f'\rangle \in \mathbf{Z}\,. \tag{1.9}$$

Using $\langle\ ,\ \rangle$, we may and we will identify $\mathfrak{t}$ and $\mathfrak{t}^*$. By [**15**, V, eq. (2.14)], under this identification, if $\alpha \in R$, if $h_\alpha \in CR$ corresponds to $\alpha$, then

$$\alpha = \frac{2}{\|h_\alpha\|^2} h_\alpha\,. \tag{1.10}$$

By (1.10) , we find that

$$\overline{CR} \subset \overline{R} \cap \overline{R}^*\,. \tag{1.11}$$

Also, by (1.6), (1.8), (1.10),

$$\overline{CR} = \overline{R_\ell}, \qquad \overline{CR_\ell} = m\overline{R}\,. \tag{1.12}$$

By (1.12),

$$m\overline{R} \subset \overline{R_\ell} \subset \overline{R}, \qquad m\overline{CR} \subset \overline{CR_\ell} \subset \overline{CR}\,. \tag{1.13}$$

From (1.12),(1.13),

$$m\overline{CR} = m\overline{R_\ell} \subset \overline{CR_\ell} = m\overline{R} \subset \overline{CR} = \overline{R_\ell} \subset \frac{\overline{CR_\ell}}{m} = \overline{R} \subset \frac{\overline{CR}}{m} = \frac{\overline{R_\ell}}{m}\,. \tag{1.14}$$

Also

$$\Gamma = \overline{CR} \subset \overline{R} \cap \overline{R}^* \subset \overline{CR}^* = \Lambda\,. \tag{1.15}$$

Observe that the Weyl group $W$ preserves all the objects which have been constructed above. Also note that

$$T = \mathfrak{t}/\overline{CR}. \tag{1.16}$$

Moreover $T' = \mathfrak{t}/\overline{R}^*$ is a maximal torus in the adjoint group $G' = G/Z(G)$.

**1.3. The volume of $T$.** Let $\mathrm{Vol}(T)$ be the volume of $T$ for the metric $\langle , \rangle$.

PROPOSITION 1.2. *The following identity holds*

$$\mathrm{Vol}(T)^2 = |\overline{CR}^*/\overline{CR}|. \tag{1.17}$$

PROOF. Consider the exact sequence of lattices

$$0 \to \overline{CR} \to \overline{CR}^* \to \overline{CR}^*/\overline{CR} \to 0, \tag{1.18}$$

which induces the exact sequence

$$0 \to \overline{CR}^*/\overline{CR} \to \mathfrak{t}/\overline{CR} \to \mathfrak{t}/\overline{CR}^* \to 0. \tag{1.19}$$

From (1.19), we obtain

$$\mathrm{Vol}(\mathfrak{t}/\overline{CR}) = |\overline{CR}^*/\overline{CR}|\mathrm{Vol}(\mathfrak{t}/\overline{CR}^*). \tag{1.20}$$

Now $\mathfrak{t}/\overline{CR}$ and $\mathfrak{t}^*/\overline{CR}^* \simeq \mathfrak{t}/\overline{CR}^*$ are dual tori. Therefore

$$\mathrm{Vol}(\mathfrak{t}/\overline{CR})\mathrm{Vol}(\mathfrak{t}/\overline{CR}^*) = 1. \tag{1.21}$$

By (1.20), (1.21), we obtain (1.17). The proof of our Proposition is completed. □

PROPOSITION 1.3. *The following identity holds*

$$\mathrm{Vol}(T)^2 = |Z(G)|\ |\overline{R}/\overline{R}_\ell|. \tag{1.22}$$

*In particular, if $G$ is simply laced,*

$$\mathrm{Vol}(T)^2 = |Z(G)|. \tag{1.23}$$

PROOF. Clearly

$$|\overline{CR}^*/\overline{CR}| = |\overline{CR}^*/\overline{R}|\ |\overline{R}/\overline{CR}|. \tag{1.24}$$

Also by (1.4),

$$(\overline{CR}^*/\overline{R})^* = \overline{R}^*/\overline{CR} = Z(G). \tag{1.25}$$

From (1.12), (1.17 ), (1.24), (1.25), we get (1.22) . The identity (1.23) follows. □

**1.4. The quadratic form attached to a representation.** Assume temporarily that $G$ is a compact connected semisimple Lie group, which is not necessarily simply connected. Otherwise, we use the notation of Sections 1.1-1.3.

Let $\sigma : G \to \mathrm{Aut}(V)$ be a finite dimensional representation of $G$.

DEFINITION 1.4. If $A, B \in \mathfrak{g}$, put

$$\langle A, B\rangle^\sigma = \frac{1}{4\pi^2}\mathrm{Tr}^V[\sigma(A)\sigma(B)]. \tag{1.26}$$

Then $\langle\ ,\ \rangle$ is an ad-invariant symmetric bilinear form on $\mathfrak{g}$.

Let $x \in \mathbf{R} \mapsto [x] \in [0,1[$ be the periodic function of period 1 such that for $x \in [0,1[$, $[x] = x$.

PROPOSITION 1.5. *If* $u \in \overline{R}^*$, $h \in \Gamma$, *then*

$$\langle u, h \rangle^\sigma \in \mathbf{Z}. \tag{1.27}$$

PROOF. Let $M \subset \Lambda$ be the set of $T$-weights in the representation $\sigma$. Then

$$\langle u, h \rangle^\sigma = -\sum_{\lambda \in M} \langle \lambda, u \rangle \langle \lambda, h \rangle. \tag{1.28}$$

Also for $\lambda \in M, \langle \lambda, h \rangle \in \mathbf{Z}$. Therefore, mod($\mathbf{Z}$),

$$\begin{aligned} \langle u, h \rangle^\sigma &= -\sum_{\lambda \in M} [\langle \lambda, u \rangle] \langle \lambda, h \rangle \\ &= -\sum_{s \in [0,1[} \langle \sum_{\substack{\lambda \in M \\ [\langle \lambda, u \rangle] = s}} \lambda, h \rangle. \end{aligned} \tag{1.29}$$

Also the image of $u$ in $T = \mathfrak{t}/\Gamma$ lies in $Z(G)$.Therefore the representation $\sigma$ splits into a sum of representations on which $u$ acts like $e^{2i\pi s}, 0 \leq s < 1$. The corresponding T-weights are given by $\{\lambda \in M, [\langle \lambda, u \rangle] = s\}$. Since G is semisimple

$$\sum_{\substack{\lambda \in M \\ [\langle \lambda, u \rangle] = s}} \lambda = 0 \tag{1.30}$$

From (1.29), (1.30), we get (1.27) . The proof of our Proposition is completed. □

**1.5. The dual Coxeter number.** Again we assume that $G$ is a connected and simply connected simple compact Lie group. Also we use the assumptions and notation of Sections 1.1-1.3.

Let $K \subset \mathfrak{t}$ be a Weyl chamber. Let $R_+$ be the corresponding system of positive roots, so that

$$R = R_+ \cup R_-. \tag{1.31}$$

Then

$$K = \{t \in \mathfrak{t}, \text{ for any } \alpha \in R_+, \langle \alpha, t \rangle > 0\}. \tag{1.32}$$

Let $P \subset \mathfrak{t}$ be the alcove in $K$ whose closure contains 0. Then

$$P = \{t \in \mathfrak{t}, \text{ for any } \alpha \in R_+,\ 0 < \langle \alpha, t \rangle < 1\}. \tag{1.33}$$

Since $G$ is simple, the adjoint representation of $G'$ on $\mathfrak{g}$ is irreducible. The corresponding $T$-weights are given by $\{0\} \cup R$. Let $\alpha_0 \in R_+ \cap R_\ell$ be the corresponding highest root. Then

$$P = \{t \in K, \langle \alpha_0, t \rangle < 1\}. \tag{1.34}$$

DEFINITION 1.6. Let $\rho \in \mathfrak{t}^*$ be given by

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha. \tag{1.35}$$

By [**15**, Proposition V.4.12], if $\alpha \in R_+$ is a simple root, $\langle \rho, h_\alpha \rangle = 1$. In particular $\rho \in \overline{CR}^*$, and $\rho \in K$.

DEFINITION 1.7. Let $c \in \mathbf{N}$ be the dual Coxeter number, given by

$$c = \langle \rho, h_{\alpha_0} \rangle + 1. \tag{1.36}$$

Let $\tau : G' \to \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation. Then by [**47**, p 285], [**22**, eq. (1.6.45)],

$$\langle , \rangle^{\tau} = -2c\langle , \rangle. \tag{1.37}$$

PROPOSITION 1.8. *If* $t \in \mathfrak{t}$,

$$ct = \sum_{\alpha \in R_+} \langle \alpha, t\rangle \alpha. \tag{1.38}$$

*In particular*

$$c\overline{R}^* \subset \overline{R}. \tag{1.39}$$

PROOF. Since $\{0\} \cup R$ are the $T$-weights of $\tau$, if $t, t' \in \mathfrak{t}$,

$$\langle t, t'\rangle^{\tau} = -2 \sum_{\alpha \in R_+} \langle \alpha, t\rangle\langle \alpha, t'\rangle. \tag{1.40}$$

Using (1.37), (1.40), we get (1.38). From (1.38), (1.39) follows. □

Recall that we have identified $\mathfrak{t}$ and $\mathfrak{t}^*$ by $\langle , \rangle$. Then $\alpha_0 = h_{\alpha_0}$. In particular, for any $\alpha \in R_+$,

$$0 < \langle \alpha, \rho/c\rangle < 1, \tag{1.41}$$

i.e. $\rho/c \in P$.

DEFINITION 1.9. For $t \in \mathfrak{t}$, put

$$\sigma(t) = \prod_{\alpha \in R_+} (e^{i\pi\langle \alpha, t\rangle} - e^{-i\pi\langle \alpha, t\rangle}). \tag{1.42}$$

Equivalently,

$$\sigma(t) = e^{2i\pi\langle \rho, t\rangle} \prod_{\alpha \in R_+} (1 - e^{-2i\pi\langle \alpha, t\rangle}). \tag{1.43}$$

By (1.43), we find that $\sigma(t)$ descends to a function defined on $T = \mathfrak{t}/\overline{CR}$.
Recall that $W$ is the Weyl group of $G$ . If $w \in W$, set

$$\epsilon_w = \det(w|_{\mathfrak{t}}). \tag{1.44}$$

Then by [**15**, Theorem VI.1.7], if $w \in W, t \in \mathfrak{t}$,

$$\sigma(wt) = \epsilon_w \sigma(t). \tag{1.45}$$

Put

$$\ell = |R_+|. \tag{1.46}$$

Now we recall a result stated in [**3**, Lemma 9.17].

PROPOSITION 1.10. *The following identity holds*

$$(i^{\ell}\sigma(e^{\rho/c}))^2 = |\overline{CR}^*/c\overline{CR}|. \tag{1.47}$$

**1.6. An embedding of the center in the Weyl group.** By [**15**, Theorem V.4.1], $wK \cap K \neq \emptyset$ if and only if $w = 1$.

Take $q \in \mathbf{Z}^*$. Then $\bigcup_{w \in W} w(qP \cap \overline{CR}^*)$ is a disjoint union of finite sets.

PROPOSITION 1.11. *The set* $\bigcup_{w \in W} w(qP \cap \overline{CR}^*)$ *embeds naturally as a subset of* $\overline{CR}^* / q\overline{CR}$. *More precisely,*

$$\bigcup_{w \in W} w(qP \cap \overline{CR}^*) = \{\lambda \in \overline{CR}^* / q\overline{CR}, \sigma^2(\lambda/q) \neq 0\}. \tag{1.48}$$

*Also*

$$cP \cap \overline{CR}^* = \{\rho\}. \tag{1.49}$$

PROOF. Let $W_{\mathrm{aff}} = W \ltimes \overline{CR}$ be the affine Weyl group. By [**15**, Proposition V.7.10], $W_{\mathrm{aff}}$ acts freely and effectively on the set of alcoves in $\mathfrak{t}$. Thus we get (1.48). If $\lambda \in cP \cap \overline{CR}^*$, then $\lambda \in \overline{CR}^* \cap K$. By [**15**, Note V.4.14],

$$\lambda - \rho \in \overline{CR}^*_+ = \overline{CR}^* \cap \overline{K}. \tag{1.50}$$

Also, by (1.36),

$$\langle \rho, h_{\alpha_0} \rangle = c - 1. \tag{1.51}$$

Since $\alpha_0 = h_{\alpha_0}$, by (1.34), since $\lambda \in cP$,

$$\langle \lambda, h_{\alpha_0} \rangle < c. \tag{1.52}$$

By (1.50)-(1.52), we obtain

$$0 \leq \langle \lambda - \rho, h_{\alpha_0} \rangle < 1. \tag{1.53}$$

Since $\langle \lambda - \rho, h_{\alpha_0} \rangle \in \mathbf{N}$, from (1.53), we obtain

$$\lambda = \rho. \tag{1.54}$$

The proof of our Proposition is completed. □

PROPOSITION 1.12. *Let $f$ be a W-invariant function on* $\overline{CR}^* / q\overline{CR}$. *Then*

$$\sum_{\substack{\lambda \in \overline{CR}^* / q\overline{CR} \\ \sigma^2(\lambda/q) \neq 0}} f(\lambda) = |W| \sum_{\lambda \in qP \cap \overline{CR}^*} f(\lambda) \tag{1.55}$$

PROOF. This is a trivial consequence of Proposition 1.11. □

PROPOSITION 1.13. *The set* $qP \cap \overline{CR}^*$ *embeds naturally into* $\{\lambda \in \overline{CR}^* / q\overline{R}^*, \sigma^2(\lambda/q) \neq 0\}$.

PROOF. By (1.33), (1.42), if $\lambda \in qP$, then $\sigma^2(\lambda/q) \neq 0$. Let $\lambda, \lambda' \in qP \cap \overline{CR}^*$, and assume there is $\mu \in \overline{R}^*$ such that

$$\lambda - \lambda' = q\mu. \tag{1.56}$$

By (1.33), for $\alpha \in R_+$,

$$-q < \langle \alpha, \lambda - \lambda' \rangle < q, \tag{1.57}$$

and so

$$-1 < \langle \alpha, \mu \rangle < 1. \tag{1.58}$$

Since $\mu \in \overline{R}^*$, for $\alpha \in R_+, \langle \alpha, \mu \rangle \in \mathbf{Z}$. By (1.58), we get

$$\mu = 0. \tag{1.59}$$

The proof of our Proposition is completed. □

PROPOSITION 1.14. *If* $\lambda \in \overline{CR}^*$, $\sigma^2(\lambda/q) \neq 0$, *there is* $w \in W$, $\lambda' \in qP \cap \overline{CR}^*$ *such that* $w\lambda - \lambda' \in q\overline{R}^*$.

PROOF. Take $\lambda \in \overline{CR}^*$. Then by [**15**, Proposition V.7.10], there exists $w \in W$, $h \in \overline{P}$, $f \in \overline{CR}$, such that

$$\frac{\lambda}{q} = wh + f\,. \tag{1.60}$$

Clearly $qf \in q\overline{CR}$. Also by (1.45),

$$\sigma^2(\lambda/q) = \sigma^2(h)\,. \tag{1.61}$$

So if $\sigma^2(\lambda/q) \neq 0$, then $\sigma^2(h) \neq 0$, so that $h \in P$. The proof of our Proposition is completed. □

Recall that $Z(G) = \overline{R}^*/\overline{CR}$. Also $W$ acts trivially on $Z(G) \subset G$. Equivalently if $f \in \overline{R}^*, w \in W$

$$wf - f \in \overline{CR}\,. \tag{1.62}$$

If $g \in G$, let $Z(g) \subset G$ be the centralizer of $g$, let $\mathfrak{z}(g)$ be its Lie algebra. By [**14**, Corollaire 5.3.1] , since $G$ is simply connected, $Z(g)$ is a connected Lie subgroup of $G$.

An element $t \in T$ is said to be regular (resp. very regular) if $\mathfrak{z}(t) = \mathfrak{t}$ (resp. $Z(t) = T$). By the above, $t \in T$ is regular if and only if $t$ is very regular. Let $T_{\text{reg}} \subset T$ be the set of regular elements in $T$. By [**15**, Proposition V.7.10] , $P$ embeds into $T_{\text{reg}}$. More precisely,

$$T_{\text{reg}} = \bigcup_{w \in W} wP \tag{1.63}$$

and the union in (1.63) is disjoint.

Let $u \in Z(G)$. Then $u + P \subset T$ is another alcove in $T_{\text{reg}}$. Therefore there is a well-defined $w_u \in W$ such that

$$u + P = w_u P\,. \tag{1.64}$$

PROPOSITION 1.15. *The map* $u \in Z(G) \mapsto w_u \in W$ *embeds* $Z(G)$ *as a commutative subgroup of* $W$. *In particular* $|Z(G)|$ *divides* $|W|$.

PROOF. If $u \in Z(G)$, $w_u = 1$, then $u + P = P$ in $T$. Therefore there is $v \in \overline{R}^*$ mapping into $u \in \overline{R}^*/\overline{CR}$ such that $v + P = P$ in $T$. By proceeding as in (1.57), (1.58), we find that $v = 0$, i.e. $u = 1$.

If $u, u' \in Z(G)$, then

$$u + u' + P = u' + w_u P = w_u(u' + P) = w_u w_{u'} P \text{ in } T\,. \tag{1.65}$$

From (1.65) , we get

$$w_{u+u'} = w_u w_{u'}\,. \tag{1.66}$$

The proof of our Theorem is completed. □

By Proposition 1.13 , we can view $qP \cap \overline{CR}^*$ as a subset of $\overline{CR}^*/q\overline{R}^*$, which itself is stable by $W$. In particular if $\lambda \in qP \cap \overline{CR}^*$, $w \in W$, $w\lambda$ will be viewed as an element of $\overline{CR}^*/q\overline{R}^*$. So the equality $w\lambda = \lambda$ says here that $w\lambda - \lambda \in q\overline{R}^*$.

PROPOSITION 1.16. *If $\lambda \in qP \cap \overline{CR}^* \subset \overline{CR}^*/q\overline{R}^*$, $w \in W$, then $w\lambda \in qP \cap \overline{CR}^* \subset \overline{CR}^*/q\overline{R}^*$ if and only if there is $u \in Z(G)$ such that $w = w_u$.*

PROOF. Take $\lambda \in qP \cap \overline{CR}^*$, $u \in \overline{R}^*$ representing an element of $Z(G) = \overline{R}^*/\overline{CR}$. Since $\lambda/q \in P$, by (1.64) , there is $\mu \in P$ such that

$$w_u\lambda/q - \mu - u \in \overline{CR}. \tag{1.67}$$

Therefore

$$w_u\lambda - q\mu \in q\overline{R}^*. \tag{1.68}$$

Then $\lambda' = q\mu \in qP \cap \overline{CR}^*$, and $w_u\lambda - \lambda' \in q\overline{R}^*$.

Conversely, if $\lambda, \lambda' \in qP \cap \overline{CR}^*$, $u \in \overline{R}^*$ are such that

$$w\lambda - \lambda' = qu, \tag{1.69}$$

then

$$w\lambda/q = \lambda'/q + u. \tag{1.70}$$

From (1.70) , since $\lambda/q, \lambda'/q \in P$, we get $w = w_u$. The proof of our Proposition is completed. □

Put

$$h = \frac{|W|}{|Z(G)|}. \tag{1.71}$$

Let $w^1, \dots, w^s \in W$ be distinct representatives in $W$ of the classes of $W/Z(G)$.

THEOREM 1.17. *The set $\bigcup_1^n w^i(qP \cap \overline{CR}^*)$ is a disjoint union, which embeds into $\overline{CR}^*/q\overline{R}^*$. More precisely*

$$\bigcup_1^n w_i(qP \cap \overline{CR}^*) = \{\lambda \in \overline{CR}^*/q\overline{R}^* \ , \ \sigma^2(\lambda/q) \neq 0\}. \tag{1.72}$$

PROOF. Our Theorem follows from Propositions 1.14 and 1.16. □

THEOREM 1.18. *Let $f$ a $W$-invariant function on $\overline{CR}^*/q\overline{R}^*$. Then*

$$\sum_{\substack{\lambda \in \overline{CR}^*/q\overline{R}^* \\ \sigma^2(\lambda/q) \neq 0}} f(\lambda) = \frac{|W|}{|Z(G)|} \sum_{\lambda \in qP \cap \overline{CR}^*} f(\lambda). \tag{1.73}$$

PROOF. Our identity follows from Theorem 1.17. □

REMARK 1.19. Our Theorem is also a consequence of Proposition 1.12. In fact, if $f$ is a $W$-invariant function on $\overline{CR}^*/q\overline{R}^*$,

$$\sum_{\substack{\lambda \in \overline{CR}^*/q\overline{CR} \\ \sigma^2(\lambda/q) \neq 0}} f(\lambda) = \left|\frac{\overline{R}^*}{\overline{CR}}\right| \sum_{\substack{\lambda \in \overline{CR}^*/q\overline{R}^* \\ \sigma^2(\lambda/q) \neq 0}} f(\lambda). \tag{1.74}$$

Now $\overline{R}^*/\overline{CR} \simeq Z(G)$, and so using Proposition 1.12 , we finally obtain (1.73).

**1.7. The element $\rho/c$.**

PROPOSITION 1.20. *If $s \in \mathfrak{t}$, $t \in P$, then*

$$\sum_{\alpha \in R} \langle \alpha, s\rangle [\langle \alpha, t\rangle] = 2\langle ct - \rho, s\rangle . \tag{1.75}$$

PROOF. By (1.35),(1.38),

$$\begin{aligned} \sum_{\alpha\in R} \langle \alpha, s\rangle[\langle\alpha,t\rangle] &= \sum_{\alpha\in R_+} \left(\langle\alpha,s\rangle\langle\alpha,t\rangle - \langle\alpha,s\rangle(1-\langle\alpha,t\rangle)\right) \\ &= \sum_{\alpha\in R} \langle\alpha,s\rangle\langle\alpha,t\rangle - \langle 2\rho, s\rangle \\ &= 2\langle ct-\rho, s\rangle . \end{aligned} \tag{1.76}$$

The proof of our Proposition is completed. □

Let $t \in T_{\text{reg}}$. Then $t$ determines a Weyl chamber $K$ and an alcove $P$ of the above type. In the sequel, we consider $t$ as an element of $P \subset T$. The element $\rho$ is still given by (1.35 ). Of course $\rho$ depends implicitly on $t \in T_{\text{reg}}$.

THEOREM 1.21. *The following identity holds*

$$ct - \rho = \frac{1}{2}\sum_{\alpha\in R} [\langle\alpha,t\rangle]\alpha . \tag{1.77}$$

*In particular the map $t \in T_{\text{reg}} \mapsto ct-\rho \in \mathfrak{t}$ descends to a map $T'_{\text{reg}} = T_{\text{reg}}/Z(G) \to \mathfrak{t}$.*

PROOF. By Proposition 1.20 , we get (1.77) . Also if $u \in Z(G)$, if $t \in T_{\text{reg}}$ is replaced by $t + u \in T_{\text{reg}}$, $[\langle\alpha,t\rangle]$ is unchanged. By (1.77) , we find that $t \in T_{\text{reg}} \mapsto ct - \rho \in \mathfrak{t}$ descends to a map from $T'_{\text{reg}}$ into $\mathfrak{t}$. The proof of our Theorem is completed. □

Observe that since $\rho \in \overline{CR}^*$ and $2\rho \in \overline{R}$, then $u \in Z(G) = \overline{R}^*/\overline{CR} \mapsto \exp(2i\pi\langle\rho,u\rangle) = \pm 1$ is a character of $Z(G)$.

Recall that $K$ is a Weyl chamber, and that $Z(G)$ has been embedded in $W$, by an embedding which depends explicitly on $K$.

THEOREM 1.22. *If $w \in W$, then $w\rho/c - \rho/c \in \overline{CR}$ if and only if $w = 1$. Also if $w \in W$ , then $w\rho/c - \rho/c = u \in \overline{R}^*$ if and only if $w = w_u$, so that $w \in Z(G)$. The map $w \in Z(G) \mapsto w\rho/c - \rho/c \in \overline{R}^*/\overline{CR} = Z(G)$ is a group isomorphism. Also if $u \in Z(G)$,*

$$\exp(2i\pi\langle\rho,u\rangle) = \epsilon_{w_u} . \tag{1.78}$$

PROOF. By (1.41), $\rho/c \in P$. Using [**15**, Proposition V.7.10] , we find that if $w\rho/c - \rho/c \in \overline{CR}$, then $w = 1$. By Theorem 1.21 , if $u \in Z(G) = \overline{R}^*/\overline{CR}$,

$$\rho/c + u = w_u(\rho/c) \text{ in } \mathfrak{t}/\overline{CR} . \tag{1.79}$$

By (1.79) , we find that $w_u(\rho/c) - \rho/c \in \overline{R}^*$. Conversely let $w \in W$ be such that

$$u = w(\rho/c) - \rho/c \in \overline{R}^* . \tag{1.80}$$

By (1.79), (1.80), we get

$$w_u(\rho/c) = w(\rho/c) \quad \text{in} \quad T = \mathfrak{t}/\overline{CR}, \tag{1.81}$$

so that $w = w_u$. Clearly if $u \in Z(G)$, by (1.45),

$$\sigma(w_u e^{\rho/c}) = \varepsilon_{w_u} \sigma(e^{\rho/c}) . \tag{1.82}$$

Also by (1.79) and by the above,

$$\sigma(w_u e^{\rho/c}) = \sigma(e^{\rho/c+u}) = e^{2i\pi\langle\rho,u\rangle}\sigma(e^{\rho/c}) . \tag{1.83}$$

Since $\sigma(e^{\rho/c}) \neq 0$, from (1.82), (1.83), we get (1.78) . The proof of our Theorem is completed. □

Let $\overline{P}$ be the closure of the alcove $P$. By (1.33),

$$\overline{P} = \{t \in \mathfrak{t}, \text{ for any } \alpha \in R_+,\ 0 \leq \langle\alpha, t\rangle \leq 1\}. \tag{1.84}$$

PROPOSITION 1.23. *For $h \in \overline{CR}$, then $\overline{P} \cap (\overline{P} + h) \neq \emptyset$ if and only if $h = 0$.*

PROOF. Recall that $W_{\text{aff}} = W \ltimes \overline{CR}$. By [**15**, Lemma V.4.3], if $f \in \overline{P}$, $v \in W_{\text{aff}}$, then $vf \in \overline{P}$ if and only if $vf = f$. In particular, if $f \in \overline{P}$, $h \in \overline{CR}$, then $f + h \in \overline{P}$ if and only if $h = 0$. The proof of our Proposition is completed. □

PROPOSITION 1.24. *The center $Z(G) = \overline{R}^* / \overline{CR}$ embeds as a finite subset of $\overline{P}$.*

PROOF. If $u \in Z(G) = \overline{R}^* / \overline{CR}$, there is an alcove $Q$ containing 0 such that $u$ is represented in $\mathfrak{t}$ by an element $v \in \overline{Q}$. By [**15**, Theorem V.4.1], there is $w \in W$ such that $wQ = P$. Also since $u \in \overline{R}^* / \overline{CR}$, then $wv - v \in \overline{CR}$. Therefore $wv \in \overline{P}$ still represents $u$. So any element $u \in Z(G)$ has a representative in $\overline{P}$. By Proposition 1.23, this representative is unique. The proof of our Proposition is completed. □

REMARK 1.25. If $u \in Z(G)$, we still denote by $u$ the corresponding representative in $\overline{P}$. Then if $w \in W$, $wu \in w\overline{P}$ is the unique representative of $u$ in $w\overline{P}$. Of course, if $wu \in \overline{P}$, the above implies that $wu = u$. However this also follows from [**15**, Theorem V.4.1].

Let $w_o \in W$ be the unique element of the Weyl group such that $w_o K = -K$.

THEOREM 1.26. *Let $u \in \overline{P}$ be the unique representative in $\overline{P}$ of an element of $Z(G)$. Then if $t \in \overline{P}$, if $[t + u] \in w_u \overline{P}$ represents $t + u \in T$, then*

$$[t + u] = t + w_u u. \tag{1.85}$$

*Also*

$$w_u u = w_o u. \tag{1.86}$$

*In particular*

$$w_u \overline{P} = w_u u + \overline{P}. \tag{1.87}$$

*Moreover*

$$u = \rho/c - w_u \rho/c \text{ in } \mathfrak{t}. \tag{1.88}$$

*In particular, if $t \in \overline{P}$,*

$$w_u \rho - c[t + u] = \rho - ct \text{ in } \mathfrak{t}. \tag{1.89}$$

PROOF. Clearly $w_o(-u)$ lies in $\overline{P}$, and so it is the unique representative in $\overline{P}$ of $u^{-1} \in T$. If $t \in P$, by (1.84), it is clear that $t + w_o u$ lies in some alcove containing 0. Also $t + w_o u$ represents $t + u \in T$. From the above, we get

$$[t+u] = t + w_o u. \tag{1.90}$$

The alcove which contains $t + w_o u$ is necessarily equal to $w_u P$. Therefore $w_u u$ and $w_o u$ both lie in $w_u \overline{P}$, and represent $u \in Z(G)$. By Proposition 1.24, we get (1.86). By (1.86) and (1.90), we obtain (1.85) and (1.87).

Using Theorem 1.21 and (1.85), if $t \in P$,

$$c(t + w_u u) - w_u \rho = ct - \rho \text{ in } \mathfrak{t}. \tag{1.91}$$

From (1.91), we get (1.88) and (1.89). The proof of our Theorem is completed. □

**1.8. Some properties of the characters of $G$.** Put

$$\overline{CR}^*_+ = \overline{CR}^* \cap \overline{K}. \tag{1.92}$$

If $\lambda \in \overline{CR}^*_+$, let $\chi_\lambda$ be the character of $G$ which is the character of the irreducible representation of $G$ with highest weight $\lambda$. By the Weyl character formula, if $t \in T_{\text{reg}}$,

$$\chi_\lambda(t) = \sum_{w \in W} \frac{e^{2i\pi \langle w\lambda, t\rangle}}{\prod_{\alpha \in R_+} (1 - e^{2i\pi \langle w\alpha, t\rangle})}. \tag{1.93}$$

Equivalently

$$\chi_\lambda(t) = \frac{1}{\sigma(t)} \sum_{w \in W} \varepsilon_w e^{2i\pi \langle w(\rho+\lambda), t\rangle}. \tag{1.94}$$

Recall that by [**15**, LemmaVI.1.2], if $\mu \in \overline{CR}^*$ is not included in a Weyl chamber,

$$\sum_{w \in W} \varepsilon_w e^{2i\pi \langle w\mu, t\rangle} = 0. \tag{1.95}$$

If $\mu \in \overline{CR}^*$ lies in $K$, then by [**15**, Note V.4.14], there is $\lambda \in \overline{CR}^*_+$ such that $\mu = \rho + \lambda$ .

If $\lambda \in \overline{CR}^*$, there exists a Weyl chamber $K$ such that $\lambda \in \overline{K}$. In general $K$ is not unique.

PROPOSITION 1.27. *The character $\chi_\lambda$ does not depend on $K$.*

PROOF. Assume that $\lambda \in \overline{K}$, $\lambda \in \overline{K}'$. Then by [**15**, Lemma V.4.2], there is $w' \in W$ such that $w'K = K'$. Let $R_+, R'_+$ be the system of positive roots attached to $K, K'$. Clearly

$$R'_+ = w' R_+ . \tag{1.96}$$

Also since $\lambda \in \overline{K} \cap \overline{K}'$, and $w'\overline{K} = \overline{K}'$, then $\lambda \in \overline{K}, w'^{-1}\lambda \in \overline{K}$. By [**15**, Theorem V.4.1], it follows that

$$w'^{-1}\lambda = \lambda . \tag{1.97}$$

Let $\chi_\lambda^K(t)$, $\chi_\lambda^{K'}(t)$ be the characters attached to $K, K'$, of highest weight $\lambda$. Using (1.93), (1.97) , we get

$$\chi_\lambda^K(t) = \chi_\lambda^{K'}(t)\,. \tag{1.98}$$

Our Proposition follows. □

For $\lambda \in \overline{CR}^*$, we can then write $\chi_\lambda(t)$ without explicitly mentioning a Weyl chamber $K$.

PROPOSITION 1.28. *If* $\lambda \in \overline{CR}^*$, $w \in W$,

$$\chi_{w\lambda} = \chi_\lambda\,. \tag{1.99}$$

PROOF. We may and we will assume that $\lambda \in \overline{CR}^*_+ = \overline{CR}^* \cap \overline{K}$. Then $w\lambda \in \overline{CR}^* \cap wK$. Equation (1.99) now follows from (1.93). □

Recall that if $\lambda \in \overline{CR}^*$, $w \in W$, then $w\lambda - \lambda \in \overline{R}$. It follows that if $\theta_1, \dots, \theta_s \in \overline{CR}^*$, then $\sum_{j=1}^s \theta_j \in \overline{R}$ if and only if, given $(w^1, \dots, w^s) \in W^s$, then $\sum_{j=1}^s w^j\theta_j \in \overline{R}$.

PROPOSITION 1.29. *If* $\sum_{j=1}^s \theta_j \in \overline{R}$, *then* $\prod_{j=1}^s \chi_{\theta_j}$ *descends to a function on the adjoint group* $G' = G/Z'(G)$.

PROOF. By Proposition 1.27 , we may and will assume that the $\theta_j$'s lie in $\overline{CR}^*_+ = \overline{CR}^* \cap K$. By (1.93) , for $t \in T_{\mathrm{reg}}$,

$$\prod_{j=1}^s \chi_{\theta_j}(t) = \sum_{(w^1,\dots,w^s)\in W^s} \frac{e^{2i\pi\langle\sum_{k=1}^s w^k\theta_k,\, t\rangle}}{\prod_{j=1}^s \prod_{\alpha\in R_+} \left(1 - e^{-2i\pi\langle w^j\alpha, t\rangle}\right)}\,. \tag{1.100}$$

If $\sum_{j=1}^s \theta_j \in \overline{R}$, then $\sum_{j=1}^s w^j\theta_j \in \overline{R}$, and so by (1.100), $\prod_{j=1}^s \chi_{\theta_j}(t)$ descends to a function on the adjoint torus $T' = \mathfrak{t}/\overline{R}^*$. The proof of our Proposition is completed. □

PROPOSITION 1.30. *If* $\sum_{j=1}^s \theta_j \notin \overline{R}$, *then*

$$\sum_{\mu\in\overline{R}^*/\overline{CR}} \prod_{j=1}^s \chi_{\theta_j}(e^{t+\mu}) = 0\,. \tag{1.101}$$

PROOF. This follows from (1.100). □

PROPOSITION 1.31. *If* $\lambda \in \overline{CR}^*$, $\lambda \notin \overline{R}$, *then*

$$\chi_\lambda(e^{\rho/c}) = 0\,. \tag{1.102}$$

PROOF. We may and we will assume that $\lambda \in \overline{CR}^*_+$. If $w \in W$, then

$$\chi_\lambda(e^{\rho/c}) = \chi_\lambda(e^{w\rho/c}). \tag{1.103}$$

Take $u \in \overline{R}^*$. Then by Theorem 1.22,

$$w_u\rho/c - \rho/c = u \text{ in } T. \tag{1.104}$$

By (1.103), (1.104),

$$\chi_\lambda(e^{\rho/c}) = \chi_\lambda(e^{\rho/c+u}). \tag{1.105}$$

Now since the representation associated to $\lambda$ is irreducible, the central element $e^u$ acts in the $\lambda$ representation like $e^{2i\pi\langle\lambda,u\rangle}$. From (1.105) , we get

$$\chi_\lambda(e^{\rho/c}) = e^{2i\pi\langle\lambda,u\rangle}\chi_\lambda(e^{\rho/c}). \tag{1.106}$$

If $\lambda \not\in \overline{R}$, it is then clear that (1.102) holds. The proof of our Proposition is completed. □

Now we have the result of Kostant [**36**].

THEOREM 1.32. *If* $\lambda \in \overline{R}$, *then*

$$\chi_\lambda(e^{\rho/c}) = 0,\ +1 \text{ or } -1. \tag{1.107}$$

Take now $p \in \mathbf{N}^*$ and $\lambda \in p\overline{P}$, so that $\lambda \in \overline{CR}^*_+$. Take $u \in Z(G) = \overline{R}^*/\overline{CR}$. Then $\lambda/p + u \in T$ is represented uniquely by an element $[\lambda/p+u] \in w_u\overline{P}$. Observe that by ((1.85),

$$[\lambda/p+u] = \lambda/p + w_u u \text{ in } \mathfrak{t}. \tag{1.108}$$

Also

$$p[\lambda/p+u] - (\lambda + pu) \in p\overline{CR}. \tag{1.109}$$

THEOREM 1.33. *If* $\mu \in \overline{CR}^*/(p+c)\overline{CR}$, $\sigma(\mu/(p+c)) \neq 0$, *then*

$$\chi_{p[\lambda/p+u]}(e^{\mu/(p+c)}) = \varepsilon_{w_u} e^{2i\pi\langle u,\mu\rangle}\chi_\lambda(e^{\mu/(p+c)}). \tag{1.110}$$

*In particular*

$$\chi_{pu}(e^{\mu/(p+c)}) = \epsilon_{w_u} e^{2i\pi\langle u,\mu\rangle}. \tag{1.111}$$

PROOF. By (1.94),

$$\begin{aligned} \chi_\lambda(e^{\mu/(p+c)}) &= \frac{1}{\sigma(e^{\mu/(p+c)})}\sum_{w\in W}\varepsilon_w e^{2i\pi\langle w(\lambda+\rho),\frac{\mu}{p+c}\rangle} \\ &= \frac{1}{\sigma(e^{\mu/(p+c)})}\sum_{w\in W}\varepsilon_w e^{2i\pi\langle w\lambda/p,\mu\rangle} \\ &\quad e^{2i\pi\langle w(\rho-c\lambda/p),\frac{\mu}{p+c}\rangle}. \end{aligned} \tag{1.112}$$

Also $p[\lambda/p+u] \in w_u\overline{P} \cap \overline{CR}^*$. When replacing $\overline{P}$ by $w_u\overline{P}$, $\rho$ is replaced by $w_u\rho$. When replacing $\lambda$ by $p[\lambda/p+u]$, using (1.109), we find that $e^{2i\pi\langle w\lambda/p,\mu\rangle}$ is replaced by $e^{2i\pi\langle w\lambda/p,\mu\rangle}e^{2i\pi\langle wu,\mu\rangle}$. Also by equation (1.89) in Theorem 1.26, we find that when replacing $\lambda$ by $p[\lambda/p+u], \rho - c\lambda/p \in \mathfrak{t}$ is unchanged. Finally $\sigma(e^{\mu/(p+c)})$ is changed into $\varepsilon_{w_u}\sigma(e^{\mu/(p+c)})$. Equation (1.110) follows from the above arguments. Equation (1.111) is a consequence of (1.110). □

REMARK 1.34. If we use (1.110) with $p = 0$, and $\mu = \rho$, we get

$$1 = \epsilon_{w_u} e^{2i\pi\langle\rho,u\rangle}, \tag{1.113}$$

which is precisely equation (1.78).

Theorem 1.33 will be used in Remark 8.4 as a consistency check on our index theoretic computations.

**1.9. The elements whose centralizer is semisimple.** We still assume that $G$ is a connected and simply connected simple compact Lie group.

DEFINITION 1.35. Let $C \subset \mathfrak{t}$ be the set of $u \in \mathfrak{t}$ such that $\{\alpha \in R; \langle\alpha, u\rangle \in \mathbf{Z}\}$ spans $\mathfrak{t}^*$.

Clearly

$$\overline{R}^* \subset C. \tag{1.114}$$

Since by (1.6), (1.12), $\overline{CR} = \overline{CR}_s = \overline{R}_\ell$, then

$$\overline{R}^* \subset \overline{CR}^* \subset C. \tag{1.115}$$

In particular $\overline{R}^* \subset C$ acts by translations on $C$. Also the Weyl group $W$ acts on $C$.

PROPOSITION 1.36. *The set $C/\overline{R}^*$ is a finite subset of the adjoint torus $T' = \mathfrak{t}/\overline{R}^*$, which contains $\overline{CR}^*/\overline{R}^*$. Also $W$ acts on $C/\overline{R}^*$.*

PROOF. Let $R' \subset R$ be such that the elements of $R'$ span $\mathfrak{t}^*$, and let $\overline{R}'$ be the associated lattice. Clearly $\overline{R}/\overline{R}'$ is a finite set. Then

$$C/\overline{R}^* = \bigcup_{R'} \overline{R}'^*/\overline{R}^*. \tag{1.116}$$

From (1.116) , its follows that $C/\overline{R}^*$ is finite. The proof of our Proposition is completed. □

Let now $K$ be a Weyl chamber in $\mathfrak{t}$. Let $R_+$ be the corresponding system of positive roots so that

$$R = R_+ \cup (-R_+). \tag{1.117}$$

DEFINITION 1.37. If $u \in G' = G/Z(G)$, let $Z(u) \subset G$ be the centralizer of $u$.

Recall that by [14, Corollaire 5.3.1], since $G$ is connected and simply connected, $Z(u)$ is a connected Lie subgroup of $G$. Clearly if $u \in T' = \mathfrak{t}/\overline{R}^*$, $T$ is a maximal torus in $Z(u)$.

Also $W$ acts like the identity on $Z(G) = \overline{R}^*/\overline{CR}$. Therefore, if $u \in T' = \mathfrak{t}/\overline{R}^*, w \in W$, then $wu - u$ is well-defined in $T = \mathfrak{t}/\overline{CR}$. Put

$$W_u = \{w \in W\,; wu - u = 0 \text{ in } T = \mathfrak{t}/\overline{CR}^*\}. \tag{1.118}$$

THEOREM 1.38. *If $u \in T' = \mathfrak{t}/\overline{R}^*$, the root system $R_u$ of $Z(u)$ is given by*

$$R_u = \{\alpha \in R, \langle\alpha, u\rangle \in \mathbf{Z}\}. \tag{1.119}$$

*Also $R_{u,+} = R_u \cap R_+$ is a system of positive roots for $Z(u)$. If $Z(Z(u)) \subset Z(u)$ is the center of $Z(u)$, its Lie algebra $\mathfrak{z}(Z(u))$ is given by*

$$\mathfrak{z}(Z(u)) = \{f \in \mathfrak{t}, \text{ for any } \alpha \in R_u, \langle\alpha, f\rangle = 0\}. \tag{1.120}$$

*Also the Weyl group $W_{Z(u)}$ of $Z(u)$ is given by*

$$W_{Z(u)} = W_u. \tag{1.121}$$

*Finally*

$$C/\overline{R}^* = \{u \in T', Z(u) \text{ is semisimple}\}. \tag{1.122}$$

PROOF. Let $\mathfrak{z}(u) \subset \mathfrak{g}$ be the Lie algebra of $Z(u)$. Then

$$\mathfrak{z}(u) = \{f \in \mathfrak{g}, u \cdot f = f\}. \tag{1.123}$$

As a $T$-space, $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ splits as

$$\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = (\mathfrak{t} \otimes_{\mathbf{R}} \mathbf{C}) \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha). \tag{1.124}$$

From (1.124), we get

$$\mathfrak{z}(u) \otimes_{\mathbf{R}} \mathbf{C} = (\mathfrak{t} \otimes_{\mathbf{R}} \mathbf{C}) \oplus (\bigoplus_{\substack{\alpha \in R \\ (\alpha, u) \in \mathbf{Z}}} \mathfrak{g}_\alpha). \tag{1.125}$$

From (1.125), it is clear that (1.119) holds.

Since the forms $\alpha$ in $R$ do not vanish on $K$, the same is true for elements in $R_u$, i.e. $K$ is included in a $Z(u)$ Weyl chamber $K_u$. If $R_{u,+}$ is the corresponding system of positive roots, then

$$R_{u,+} = R_u \cap R_+. \tag{1.126}$$

The identity (1.120) is trivial. Let $N(T) \subset G$ be the normalizer of $T$. Then

$$W = N(T)/T. \tag{1.127}$$

Similarly let $N_u(T)$ be the normalizer of $T$ in $Z(u)$. Then

$$W_{Z(u)} = N_u(T)/T. \tag{1.128}$$

Clearly

$$N_u(T) = N(T) \cap Z(u). \tag{1.129}$$

Therefore $W_{Z(u)}$ is a subgroup of $W$. Since $u \in Z(Z(u))/Z(G)$, if $w \in W_{Z(u)}$,

$$wu - u = 0 \text{ in } T. \tag{1.130}$$

Conversely if $w \in W$, let $w' \in N(T)$ represent $w$. If $wu - u = 0$ in $T$, then

$$w'uw'^{-1} = u \text{ in } T, \tag{1.131}$$

i.e. $w' \in Z(u)$. Therefore $w' \in N_u(T)$, and $w$ lies in $W_{Z(u)}$.

By definition, $Z(u)$ is semisimple if and only if $\mathfrak{z}(Z(u)) = 0$. From (1.120), we get (1.122).

The proof of our Theorem is completed. □

REMARK 1.39. Clearly, if we identify $u$ to a corresponding element in $\mathfrak{t}$, then

$$W_u = \{w \in W, wu - u \in \overline{CR}\}. \tag{1.132}$$

By Theorem 1.38, $W_u = W_{Z(u)}$. Let $\overline{CR}_u$ be the lattice generated by the coroots of $Z(u)$. Since $u \in Z(Z(u))$, if $w \in W_{Z(u)}$, $wu - u \in \overline{CR}_u$. Therefore

$$W_u = \{w \in W, wu - u \in \overline{CR}_u\}. \tag{1.133}$$

PROPOSITION 1.40. *If $n \geq 2$ and $G = \mathrm{SU}(n)$, then*

$$C = \overline{R}^{*}. \tag{1.134}$$

PROOF. By [15, Proposition V.6.3], it is clear that if $\alpha_1, \dots, \alpha_r$ is a basis of $\mathfrak{t}^*$ over $\mathbf{R}$, then $\alpha_1, \dots, \alpha_r$ spans $\overline{R}$. Equation (1.134) follows. □

**1.10. Some properties of semisimple centralizers.** We still assume that $G$ is a connected simply connected simple compact Lie group of rank $r$. Also we use the notation of Sections 1.1-1.3 and 1.9.

Take $u \in C/\overline{R}^{*}$. To the Lie group $Z(u)$, we can associate the objects we considered in Section 1.1 for $G$, with the reservation that since the action of $W_u$ on $\mathfrak{t}$ may be reducible, $Z(u)$ is semisimple and in general not simple.

However, we equip $\mathfrak{z}(u)$ with the scalar product induced from the scalar product of $\mathfrak{g}, \langle\ ,\ \rangle$. Therefore $\mathfrak{t}$ is equipped with the scalar product $\langle\ ,\ \rangle$, and the identification $\mathfrak{t} \simeq \mathfrak{t}^*$ will still be the one we used for $G$.

The objects we considered before which are attached to $Z(u)$ will be denoted with the index $u$. The lattices $\Gamma_u \subset \mathfrak{t}, \Lambda_u = \Gamma_u^* \subset \mathfrak{t}^*$ are given by

$$\Gamma_u = \overline{CR}, \ \Lambda_u = \overline{CR}^{*}. \tag{1.135}$$

The roots $R_u$ have already been described in Theorem 1.38 . Clearly

$$CR_u = \{h_\alpha, \alpha \in R_u\}. \tag{1.136}$$

Note that in general

$$\pi_1(Z(u)) = \overline{CR}/\overline{CR_u}, \tag{1.137}$$

and so $Z(u)$ is in general not simply connected.

If $u \in C/\overline{R}^{*}$, put as in (1.35),

$$\rho_u = \frac{1}{2} \sum_{\alpha \in R_{u,+}} \alpha. \tag{1.138}$$

Then $\rho_u \in \overline{CR}_u^{*}$.

THEOREM 1.41. *For any $u \in C$,*

$$\begin{aligned} 2cu &\in \overline{R}, \\ 2\rho_u &\in \overline{R}. \end{aligned} \tag{1.139}$$

*If $h \in \overline{R}^{*}/\overline{CR}_u$, then $2c\langle u, h\rangle \in \mathbf{Z}$, $2\langle \rho - \rho_u, h\rangle \in \mathbf{Z}$, and moreover*

$$c\langle u, h\rangle = \langle \rho - \rho_u, h\rangle \bmod(\mathbf{Z}). \tag{1.140}$$

*In particular, if $h \in \pi_1(Z(u)) = \overline{CR}/\overline{CR}_u$ , then $2c\langle u, h\rangle \in \mathbf{Z}$, $2\langle \rho_u, h\rangle \in \mathbf{Z}$, and*

$$c\langle u, h\rangle = \langle \rho_u, h\rangle \bmod(\mathbf{Z}). \tag{1.141}$$

PROOF. Let $\tau : G' \to \mathrm{End}(\mathfrak{g})$ be the adjoint representation. Then by (1.37), if $A, B \in \mathfrak{g}$

$$\langle A, B\rangle^{\tau} = -2c\langle A, B\rangle. \tag{1.142}$$

Also $\tau$ induces a representation $Z(u)/Z(G) \to \mathrm{End}(\mathfrak{g})$. Then $\overline{R}^{*} \subset \mathfrak{t}$ is exactly the lattice of integral elements in $\mathfrak{t}$ with respect to $Z(u)/Z(G)$. By Proposition 1.5 and by (1.142), we then find that $2cu \in \overline{R}$. Also $2\rho_u \in \overline{R}_u \subset \overline{R}$.

If $h \in \overline{R}^*$, it follows that $2c\langle u,h\rangle \in \mathbf{Z}$, $2\langle \rho_u, h\rangle \in \mathbf{Z}$. Also since $\overline{CR}_u \subset \overline{R}_u$, and $u \in \overline{R}_u^*$, if $h \in \overline{CR}_u$, $\langle u,h\rangle \in \mathbf{Z}$, and, since $\rho_u \in \overline{CR}_u^*$, then $\langle \rho_u, h\rangle \in \mathbf{Z}$. Therefore mod $(\mathbf{Z})$, if $h \in \overline{R}^*$, $c\langle u,h\rangle$ and $\langle \rho_u, h\rangle$ only depend on the class of $h$ in $\overline{R}^*/\overline{CR}_u$.

Now we establish (1.141). By [**15**, Proposition V.7.10], we may suppose that $u \in C$ is such that for $\alpha \in R$

$$|\langle \alpha, u\rangle| \leq 1\,. \tag{1.143}$$

Take $h \in \overline{R}^*$. Since $Z(u)$ is semisimple, and since the $\alpha \in R$ are the weights of the restriction of $\tau$ to $Z(u)$, then by proceeding as in (1.29),

$$\sum_{\alpha\in R}[\langle \alpha,u\rangle]\langle \alpha,h\rangle = \sum_{s\in[0,1[} s\langle \sum_{\substack{\alpha\in R\\ [\langle\alpha,u\rangle]=s}} \alpha\,, h\rangle = 0\,. \tag{1.144}$$

Also,

$$\begin{aligned}\sum_{\alpha\in R}[\langle \alpha,u\rangle]\langle \alpha,h\rangle &= \sum_{\alpha\in R\setminus R_u}[\langle \alpha,u\rangle]\langle \alpha,h\rangle \\ &= \sum_{\alpha\in R_+\setminus R_{u,+}}(\langle \alpha,u\rangle\langle \alpha,h\rangle - (1-\langle \alpha,u\rangle)\langle \alpha,h\rangle) \\ &= \sum_{\alpha\in R\setminus R_u}\langle \alpha,u\rangle\langle \alpha,h\rangle - \langle \sum_{\alpha\in R_+\setminus R_{u,+}} \alpha, h\rangle\,.\end{aligned} \tag{1.145}$$

If $\alpha \in R_u$, $\langle \alpha\,,u\rangle \in \mathbf{Z}$, and $\langle \alpha,h\rangle \in \mathbf{Z}$. Since roots in $R_u$ come by pairs,by (1.40),(1.142), (1.144), (1.145),

$$2c\langle u,h\rangle = \langle \sum_{\alpha\in R_+\setminus R_{u,+}} \alpha, h\rangle \mod (2\mathbf{Z})\,, \tag{1.146}$$

which is equivalent to (1.140).

If $h \in \overline{CR}$, then $\langle \rho,h\rangle \in \mathbf{Z}$. From (1.140), we get (1.141). The proof of our Theorem is completed. □

REMARK 1.42. Needless to say, the class of $\rho_u$ in $\mathfrak{t}/\overline{R}$ does not depend on the choice of $R_+$. This fact fits with (1.141).

**1.11. The first homotopy group in a semisimple centralizer.** Take $u \in C/\overline{R}^*$. Then $Z(u)$ is a connected semisimple subgroup of $G$. Also by (1.135), (1.137) and by [**15**, Theorem V.7.1],

$$\begin{aligned}\pi_1(Z(u)) &= \overline{CR}/\overline{CR}_u, \\ Z(Z(u)) &= \overline{R}_u^*/\overline{CR}\,.\end{aligned} \tag{1.147}$$

By (1.147),

$$\frac{Z(Z(u))}{Z(G)} = \overline{R}_u^*/\overline{R}^*\,. \tag{1.148}$$

Let $\pi_u : \widetilde{Z}(u) \to Z(u)$ be the universal cover of $Z(u)$. Then by [**15**, Proposition V.7.16],

$$Z(\widetilde{Z}(u)) = \overline{R}_u^*/\overline{CR}_u, \tag{1.149}$$

and $\pi_1(Z(u))$ is a subgroup of $Z(\widetilde{Z}(u))$.

Clearly

$$\frac{Z(Z(u))}{Z(G)} = \overline{R}^*_u/\overline{R}^* \subset C/\overline{R}^* . \tag{1.150}$$

Take $v \in Z(Z(u))/Z(G)$. Then $Z(v) \supset Z(u)$, $CR_v \supset CR_u$. Therefore $\pi_1(Z(u)) = \overline{CR}/\overline{CR}_u$ surjects on $\pi_1(Z(v)) = \overline{CR}/\overline{CR}_v$. Let $\tau_{u,v} : \overline{CR}/\overline{CR}_u \to \overline{CR}/\overline{CR}_v$ be this surjection. Clearly $\overline{R}^*_u/\overline{R}^*$ maps into $\overline{CR}^*_u/\overline{R}^*$.

Let $h \in \pi_1(Z(u)) = \overline{CR}/\overline{CR}_u$. Then $h$ defines a character $\delta_h$ of $Z(Z(u))/Z(G)$ given by

$$\delta_h : v \in Z(Z(u))/Z(G) \mapsto \exp(2i\pi\langle h, v\rangle) . \tag{1.151}$$

Recall that by Proposition 1.36, $\overline{CR}^*/\overline{R}^* \subset C/\overline{R}^*$.

PROPOSITION 1.43. *The element $u \in C/\overline{R}^*$ is such that for any $h \in \overline{CR}/\overline{CR}_u$, $\delta_h(u) = 1$ if and only if $u \in \overline{CR}^*/\overline{R}^*$.*

PROOF. This is trivial. □

PROPOSITION 1.44. *For any $u \in C/\overline{R}^*$, $v \in \dfrac{Z(Z(u))}{Z(G)} = \overline{R}^*_u/\overline{R}^*$, $h \in \pi_1(Z(u)) = \overline{CR}/\overline{CR_u}$,*

$$\exp(2i\pi\langle h, v\rangle) = \exp(2i\pi\langle \tau_{u,v}h, v\rangle) . \tag{1.152}$$

PROOF. We have the exact sequence

$$0 \to \overline{CR}_v/\overline{CR}_u \to \overline{CR}/\overline{CR}_u \overset{\tau_{u,v}}{\to} \overline{CR}/\overline{CR}_v \to 0 . \tag{1.153}$$

Also if $h' \in \overline{CR}_v/\overline{CR}_u$, $\exp(2i\pi\langle h', v\rangle) = 1$. Our Proposition follows. □

REMARK 1.45. Proposition 1.44 will be used in Remark 4.40.

**1.12. Centralizers in a connected and non simply connected semisimple Lie group.** Let $G$ be a compact connected semisimple Lie group. We use otherwise the same notation as in Sections 1.1-1.3 and 1.11.

Let $\pi_1(G)$ be the first homotopy group of $G$. Let $\widetilde{G}$ be the universal cover of $G$. Then $\widetilde{G}$ is a compact connected and simply connected semisimple Lie group, $\pi_1(G)$ is a subgroup of of $Z(\widetilde{G})$ and

$$G = \widetilde{G}/\pi_1(G) \tag{1.154}$$

Let $T$ be a maximal torus in $G$. Let $u \in T' = \mathfrak{t}/\overline{R}^*$. Let $Z(u) \subset G$ be the centralizer of $u$.By [**14**, Corollaire 5.3.1], if $G$ is simply connected, then $Z(u)$ is connected. Let $Z(u)_o$ be the connected component of the identity in $Z(u)$. Then $Z(u)_o$ is a normal subgroup of $Z(u)$, and so $Z(u)_o\backslash Z(u)$ is a finite group. Moreover $T \subset Z(u)_o$, so that $u \in Z(u)_o$.

If $g \in Z(u)$, let $\tilde{g} \in \widetilde{G}$ be a lift of $g$, let $\tilde{u} \in \widetilde{G}$ be a lift of $u$. Then $[\tilde{g}, \tilde{u}] \in \pi_1(G)$ does not depend on $\tilde{g}, \tilde{u}$.

PROPOSITION 1.46. *The map $g \in Z(u) \mapsto [\tilde{g}, \tilde{u}] \in \pi_1(G)$ induces an embedding of $Z(u)_o\backslash Z(u)$ into $\pi_1(G)$. In particular $Z(u)_o\backslash Z(u)$ is commutative.*

PROOF. The Lie group $Z(\tilde{u}) \subset \widetilde{G}$ is connected. Therefore $Z(\tilde{u})/\pi_1(G)$ is a connected subgoup of $Z(u)$.Since $Z(\tilde{u})/\pi_1(G)$ and $Z(u)_o$ have the same Lie algebra, it follows that

$$Z(u)_o = Z(\tilde{u})/\pi_1(G). \tag{1.155}$$

Let $g, g' \in Z(u)$, let $\tilde{g}, \tilde{g'} \in \widetilde{G}$ lift $g, g'$. Let $h, h' \in \pi_1(G)$ be such that

$$\begin{aligned} \tilde{g}\tilde{u}\tilde{g}^{-1} &= \tilde{u}\tilde{h}, \\ \tilde{g'}\tilde{u}\tilde{g'}^{-1} &= \tilde{u}\tilde{h'}. \end{aligned} \tag{1.156}$$

Then

$$\tilde{g'}\tilde{g}\tilde{u}\tilde{g}^{-1}\tilde{g'}^{-1} = \tilde{g'}\tilde{u}h\tilde{g}^{-1} = \tilde{g'}\tilde{u}\tilde{g'}^{-1}h = \tilde{u}h'h. \tag{1.157}$$

If $g \in Z(u)_o$, if $\tilde{g} \in Z(\tilde{u})$ lifts $g$, then

$$[\tilde{g}, \tilde{u}] = 1. \tag{1.158}$$

Conversely if ( 1.158) holds, then $\tilde{g} \in Z(\tilde{u})$ and $g \in Z(u)_o$. The proof of our Proposition is completed. □

Let $N(T) \subset G$ be the normalizer of $T$ in $G$. Then $W = N(T)/T$ is the Weyl group. Put

$$W_u = \{w \in W, wu - u = 0 \text{ in } T = \mathfrak{t}/\Gamma\}. \tag{1.159}$$

PROPOSITION 1.47. *The following identity holds*

$$W_u = (N(T) \cap Z(u))/T. \tag{1.160}$$

PROOF. The proof of our Proposition is the same as in (1.128)-(1.131). □

Let $W_{Z(u)_o}$ be the Weyl group of $Z(u)_o$. Then $W_{Z(u)_o} = W_{Z(\tilde{u})}$. Also $W_{Z(u)_o}$ is a normal subgroup of $W_u$. Then $W_{Z(u)_o}\backslash W_u$ is a finite group.

Let $\overline{CR} \subset \mathfrak{t}$ be the lattice in $\mathfrak{t}$ spanned by the coroots of $\widetilde{G}$. Let $\Gamma \subset \mathfrak{t}$ be the lattice of integrals elements in $\mathfrak{t}$, i.e. whose exponential in $T \subset G$ is equal to 1. Then $\overline{CR} \subset \Gamma$, and

$$\Gamma/\overline{CR} = \pi_1(G) \tag{1.161}$$

We define $\overline{R}, \overline{R}^*$ as in Section 1.1 . Let $\overline{u} \in \mathfrak{t}$ represent $u \in T' = \mathfrak{t}/\overline{R}^*$.Then $w \in W \mapsto w\overline{u} - \overline{u} \in \Gamma/\overline{CR} = \pi_1(G)$ is a well-defined map.

Recall that $Z(\widetilde{G}) = \overline{R}^*/\overline{CR}$, and that $\pi_1(G) = \Gamma/\overline{CR}$ is a subgroup of $Z(\widetilde{G})$. In particular, by Proposition 1.15, $\pi_1(G)$ embeds as a commutative subgroup of $W$.

THEOREM 1.48. *The map $w \in W_u \mapsto w\overline{u} - \overline{u} \in \Gamma/\overline{CR} = \pi_1(G)$ induces an embedding of $W_{Z(u)_o}\backslash W_u$ into $\pi_1(G)$. In particular $W_{Z(u)_o}\backslash W_u$ is commutative. If $w \in W_{Z(u)_o}\backslash W_u$ is identified to the corresponding element $w' \in \pi_1(G) \subset Z(\widetilde{G})$, the image of $w'$ in $W$ lies in $W_u$ and represents $w$ in $W_{Z(u)_o}\backslash W_u$.*

PROOF. Let $w, w' \in W_u$. Let $h, h' \in \Gamma$ be such that

$$w\overline{u} - \overline{u} = h, w'\overline{u} - \overline{u} = h'. \tag{1.162}$$

Then

$$w'w\overline{u} - \overline{u} = w'h + h'. \tag{1.163}$$

Now $\Gamma \subset \overline{R}^*$, and so

$$w'h - h \in \overline{CR}. \tag{1.164}$$

Therefore by (1.163), (1.164)

$$w'w\overline{u} - \overline{u} = h + h' \text{ in } \Gamma/\overline{CR}. \tag{1.165}$$

If $w \in W_{Z(u)_o}$, since $u \in Z(u)_o$ is in the center of $Z(u)_o$, then

$$w\overline{u} - \overline{u} = 0 \text{ in } \Gamma/\overline{CR}. \tag{1.166}$$

Conversely let $w \in W$ be such that

$$w\overline{u} - \overline{u} \in \overline{CR}. \tag{1.167}$$

Let $\tilde{g} \in \widetilde{G}$ represent $w$.By (1.167),

$$[\tilde{g}, \tilde{u}] = 1, \tag{1.168}$$

so that $\tilde{g}$ maps into an element of $Z(u)_o$. In particular $w \in W_{Z(u)_o}$. The proof of our Proposition is completed. □

Observe that

$$W_{Z(u)_o}\backslash W_u = (N(T) \cap Z(u)_o)\backslash(N(T) \cap Z(u)). \tag{1.169}$$

Therefore $W_{Z(u)_o}\backslash W_u$ embeds naturally into $Z(u)_o\backslash Z(u)$.

THEOREM 1.49. *We have the identity*

$$W_{Z(u)_o}\backslash W_u \simeq Z(u)_o\backslash Z(u). \tag{1.170}$$

*This identity is compatible with the embeddings of both groups into* $\pi_1(G) = \Gamma/\overline{CR}$.

PROOF. Let $g \in Z(u)$. Then $gTg^{-1}$ is a maximal torus in $Z(u)_o$.Therefore there is $h \in Z(u)_o$ such that

$$(hg)T(hg)^{-1} = T, \tag{1.171}$$

so that $hg \in N(T) \cap Z(u)$. Therefore the embedding $W_{Z(u)_o}\backslash W_u \to Z(u)_o\backslash Z(u)$ is in fact one to one. It is trivial to verify that the above identification is compatible with the given embeddings into $\pi_1(G)$. The proof of our Theorem is completed. □

**1.13. The intersection of an adjoint orbit with a centralizer.** We make the same assumptions as in Section 1.12.

Let $t \in T$. Let $\mathcal{O}_t$ be the adjoint orbit of $t$ in $G$. By [**15**, Lemma IV.2.5],

$$\mathcal{O}_t \cap T = \{wt\}_{w \in W} \quad . \tag{1.172}$$

More generally, if $H$ is a Lie subgroup of $G$, and $t \in H$, let $\mathcal{O}_H(t)$ be the adjoint orbit of $t$ in $H$. In particular $\mathcal{O}_t = \mathcal{O}_G(t)$.

THEOREM 1.50. *If $G$ is simply connected, if $u \in T'$, then*

$$\mathcal{O}_t \cap Z(u) = \bigcup_{w \in W_u\backslash W} \mathcal{O}_{Z(u)}(wt). \tag{1.173}$$

*If $t$ is regular, the above union is disjoint.*

*If $G$ is not necessarily simply connected, if $t \in T$ is very regular,*

$$\mathcal{O}_t \cap Z(u) = \bigcup_{w \in W_{Z(u)_o} \backslash W} \mathcal{O}_{Z(u)_o}(wt), \tag{1.174}$$
$$\mathcal{O}_t \cap Z(u) = \bigcup_{w \in W_u \backslash W} \mathcal{O}_{Z(u)}(wt),$$

*and the above unions are disjoint.*

PROOF. If $G$ is simply connected, then $Z(u)$ is connected, and $T$ is a maximal torus in $Z(u)$. Let $g \in \mathcal{O}_t \cap Z(u)$. There is $g' \in Z(u)$ such that $g'g{g'}^{-1} \in \mathcal{O}_t \cap T$. By [**15**, Lemma IV.2.5], there is $w \in W$ such that

$$g'g{g'}^{-1} = wt, \tag{1.175}$$

so that

$$g \in \mathcal{O}_{Z(u)}(wt). \tag{1.176}$$

Therefore (1.173) holds. If $t \in T$ is regular, since $G$ is simply connected, $t$ is very regular. Therefore the $\{wt\}$ are distinct in $T$. Moreover two elements in $T$ lie in the same $Z(u)$-orbit if and only if they lie in the same $W_u$-orbit. Using Theorem 1.38, it follows that when $t \in T$ is regular, the union in (1.173) is disjoint.

If $G$ is non necessarily simply connected, if $g \in \mathcal{O}_t \cap Z(u)$, then $u \in Z(g)$. If $t$ is very regular, $Z(g)$ is a maximal torus, which is included in $Z(u)_o$. Therefore $g \in Z(u)_o$. The above argument shows that there is $g' \in Z(u)_o$, and $w \in W$ such that

$$g'g{g'}^{-1} = wt, \tag{1.177}$$

which is equivalent to

$$g \in \mathcal{O}_{Z(u)_o}(wt). \tag{1.178}$$

So we have proved the first identity in (1.174). Since $t$ is very regular in $G$, it is very regular in $Z(u)_o$. So the union in the first identity of (1.174) is disjoint.

Clearly

$$\bigcup_{w \in W_u \backslash W} \mathcal{O}_{Z(u)}(wt) \subset \mathcal{O}_t \cap Z(u), \tag{1.179}$$

and also

$$\bigcup_{w \in W_{Z(u)_o} \backslash W} \mathcal{O}_{Z(u)_o}(wt) \subset \bigcup_{w \in W_u \backslash W} \mathcal{O}_{Z(u)}(wt). \tag{1.180}$$

Therefore the second identity in (1.174) holds.

If $g \in Z(u), w \in W$ are such that $gtg^{-1} = wt$, if $g' \in N(T)$ represents $w$, then ${g'}^{-1}gt({g'}^{-1}g)^{-1} = t$. Since $t$ is very regular, ${g'}^{-1}g \in T$, so that $g \in Z(u) \cap N(T)$.It follows that $w \in W_u$. Therefore the second union in (1.174) is also disjoint.

The proof of our Theorem is completed. □

**1.14. The stabilizer of an element of the Lie algebra, and coadjoint orbits.** Let $G$ be a compact connected semisimple Lie group. We use otherwise the notation of Section 1.1. Recall that $\tau : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation.

DEFINITION 1.51. If $p \in \mathfrak{t}$, put

$$Z(p) = \{g \in G\ ;\ \tau(g) \cdot p = p\}. \tag{1.181}$$

By [**15**, Theorem IV.2.3], $Z(p)$ is a connected Lie subgroup of $G$. Then $T$ is a maximal torus in $Z(p)$. Let $\mathfrak{z}(p)$ be the Lie algebra of $Z(p)$.

THEOREM 1.52. *If $p \in \mathfrak{t}$, then*

$$\mathfrak{z}(p) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in R \\ \langle \alpha, p \rangle = 0}} \mathfrak{g}_\alpha . \tag{1.182}$$

*Also the root system $R_p$ of $Z(p)$ is given by*

$$R_p = \{\alpha \in R\ ;\ \langle \alpha, p \rangle = 0\}. \tag{1.183}$$

*If $Z(Z(p)) \subset Z(p)$ is the center of $Z(p)$, its Lie algebra $\mathfrak{z}(Z(p))$ is given by*

$$\mathfrak{z}(Z(p)) = \{f \in \mathfrak{t},\ \ \textit{for any } \alpha \in R_p, \langle \alpha, f \rangle = 0\}. \tag{1.184}$$

PROOF. The proof of these results is left to the reader. It is essentially the same as the proof of Theorem 1.38. □

DEFINITION 1.53. Let $\pi : \mathfrak{t} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C}$ be the monomial

$$\pi(t) = \prod_{\alpha \in R_+} \langle 2i\pi\alpha, t \rangle . \tag{1.185}$$

By [**15**, Corollary V.4.6 and Lemma V.4.10], if $w \in W$

$$\pi(wt) = \varepsilon_w \pi(t). \tag{1.186}$$

Also if $t \in \mathfrak{t}$, one has the obvious

$$\det \mathrm{Ad}(t)_{|\mathfrak{g}/\mathfrak{t}} = \pi^2(t/i). \tag{1.187}$$

By (1.187), we find that $\pi^2(t/i)$ does not depend on $K$, and lifts to a $G$-invariant function on $\mathfrak{g}$.

DEFINITION 1.54. Set

$$\begin{aligned} \mathfrak{g}_{\mathrm{reg}} &= \{p \in \mathfrak{g}\ ,\ Z(p) \text{ is a maximal torus}\}, \\ \mathfrak{t}_{\mathrm{reg}} &= \{t \in \mathfrak{t}\ ;\ Z(t) = T\}. \end{aligned} \tag{1.188}$$

Clearly

$$\mathfrak{t}_{\mathrm{reg}} = \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{t}. \tag{1.189}$$

PROPOSITION 1.55. *The following identity holds*

$$\begin{aligned} \mathfrak{g}_{\mathrm{reg}} &= \{p \in \mathfrak{g}\ ,\ \pi^2(p/i) \neq 0\}, \\ \mathfrak{t}_{\mathrm{reg}} &= \{t \in \mathfrak{t}\ ;\ \pi^2(t/i) \neq 0\}. \end{aligned} \tag{1.190}$$

PROOF. Take $t \in \mathfrak{t}$. Since $Z(t)$ is connected, $t \in \mathfrak{t}_{\mathrm{reg}}$ if and only if

$$\mathfrak{z}(t) = \mathfrak{t}. \tag{1.191}$$

Using Theorem 1.52, we get the second identity in (1.190). Also by [**15**, Theorem IV.1.6], any $G$-orbit in $\mathfrak{g}$ intersects $\mathfrak{t}$. Our Proposition follows. □

By (1.190), $t \in \mathfrak{t}_{\mathrm{reg}}$ if and only if $t$ lies in a Weyl chamber. The $G$-orbit of $t$ in $\mathfrak{g}$ intersects $\mathfrak{t}$ at $|W|$ distinct elements, which form the $W$-orbit of $t$.

If $p \in \mathfrak{g}$, let $\mathcal{O}_p$ be the $G$-orbit of $p$. Clearly,

$$\pi_p : \ g \in G/Z(p) \mapsto gpg^{-1} \in \mathcal{O}_p \tag{1.192}$$

is a one to one map. Also by [**6**, Lemma 7.22], $\mathcal{O}_p$ is equipped with a canonical symplectic form $\sigma_{\mathcal{O}_p}$. In fact $G$ acts on the left on $\mathcal{O}_p$. If $X \in \mathfrak{g}$, let $X^{\mathcal{O}_p}$ be the corresponding vector field on $\mathcal{O}_p$. Then if $X, Y \in \mathfrak{g}$, $q \in \mathcal{O}_p$,

$$\sigma_{\mathcal{O}_p,q}(X^{\mathcal{O}_p}, Y^{\mathcal{O}_p}) = \langle q, [X,Y]\rangle. \tag{1.193}$$

Let $f_p$ be the left invariant 1-form on $G$

$$f_p = \langle p, g^{-1}dg\rangle\,. \tag{1.194}$$

Let $\pi_p$ be the projection $G \to G/Z(p) \simeq \mathcal{O}_p$.

PROPOSITION 1.56. *The following identity holds*

$$df_p = -\pi_p^* \sigma_p\,. \tag{1.195}$$

*In particular the restriction of $f_p$ to $Z(p)$ is a closed 1-form.*

PROOF. Clearly

$$df_p = \langle p, -\frac{1}{2}[g^{-1}dg, g^{-1}dg]\rangle \tag{1.196}$$

from which (1.195) follows. Since $\pi_p$ maps $Z(p)$ to a constant in $G/Z(p)$, our Proposition follows. □

PROPOSITION 1.57. *If $p \in \overline{CR}^*$, $f_p$ is an integral closed 1-form on $Z(p)$.*

PROOF. Clearly $T$ is a maximal torus in $Z(p)$. Then by [**15**, Proposition V.7.6], $Z(p)/T$ is simply connected. Therefore $\pi_1(T)$ surjects on $\pi_1(Z(p))$. To verify that $f_p$ is an integral 1-form, we only need to check that if $s \in S_1 \mapsto g_s \in T$ is smooth, the integral of $f_p$ on this loop lies in $\mathbf{Z}$. Since $p \in \overline{CR}^*$, this is obvious. The proof of our Proposition is completed. □

Now we assume that $p \in \overline{CR}^*$. Take $g \in Z(p)$. Let $s \in [0,1] \mapsto g_s \in Z(p)$ be a smooth path such that $g_0 = 1, g_1 = g$.

PROPOSITION 1.58. *The map $g \mapsto \exp\left(2i\pi \int_0^1 g_s^* f_p ds\right) \in S_1$ defines a representation $\rho_p$ of $Z(p)$.*

PROOF. This follows from Proposition 1.57. □

DEFINITION 1.59. Let $L_p$ be the Hermitian line bundle on $\mathcal{O}_p$

$$L_p = G \times_{Z(p)} \mathbf{C}\,. \tag{1.197}$$

Clearly the connection

$$d + 2i\pi f_p \tag{1.198}$$

descends to a connection $\nabla^{L_p}$ on $L_p$.

PROPOSITION 1.60. *The following identity holds*

$$c_1(L_p, \nabla^{L_p}) = \sigma_p\,. \tag{1.199}$$

PROOF. This is obvious by (1.195). □

## 2. Fourier analysis on the centralizers of semisimple Lie groups

The purpose of this Section is to express certain Fourier series on $T$ (which will later turn out to be the symplectic volumes of the stratas of the moduli space of flat $G$-bundles on the Riemann surface $\Sigma$, first computed by Witten [**63, 64**] ) as residues of certain holomorphic functions in several complex variables. The main point is that it is then possible to compute explicitly the action of certain differential operators on these Fourier series. The results of this Section will be used in Sections 5, 6 and 7.

This Section is organized as follows. In Section 2.1, we make elementary constructions in linear algebra. In Section 2.2, we apply these constructions to the root system of $G$. In Section 2.3, given $u \in C/\overline{R}^*$ and the corresponding semisimple centralizer $Z(u)$, we consider an associated Fourier series $Q_u(t,x)$, which we express as a simple integral along the fibre of a torus fibration. In Section 2.4, we express $Q_u(t,x)$ in terms of iterated residues. In Sections 2.5-2.8, we introduce other related Fourier series, which are related in particular to the universal cover of semisimple centralizers. In Sections 2.9-2.11, we introduce our symplectic volume Fourier series, which are local polynomials on $T$. We express these Fourier series as residues. Finally, in Section 2.12, we compute the action of any power series of differential operators on these local polynomials.

As explained in the Introduction, residue techniques have been developed by Szenes [**53**], [**54**] to handle the Witten Fourier series [**63, 64**]. The methods of Szenes are more conceptual than ours, which only usesimple linear algebra. It is probable that the results of this Section can be rephrased using Szenes's formalism.

**2.1. Some linear algebra.** Let $V$ be a real vector space of dimension $r$. Let $e_1, \dots, e_r$ be a basis of $V$, let $e^1, \dots, e^r$ be the dual basis of $V^*$.

For $1 \leq i \leq r$, if $u_1, \dots, u_i \in V$, put

$$\langle u_1, \dots, u_i \rangle = \langle u_1 \wedge \dots \wedge u_i, e^1 \wedge \dots \wedge e^i \rangle . \tag{2.1}$$

Equivalently

$$\langle u_1, \dots, u_i \rangle = \frac{u_1 \wedge \dots u_i \wedge e_{i+1} \wedge \dots \wedge e_r}{e_1 \wedge \dots \wedge e_r} . \tag{2.2}$$

Let $f_1, \dots, f_r$ be another basis of $V$, let $f^1, \dots, f^r$ be the corresponding dual basis of $V^*$.

Let $E, E', F, F'$ be the flags in $V$,

$$\begin{aligned} E &: 0 \subset \{e_1\} \subset \{e_1, e_2\} \dots \subset \{e_1, \dots, e_r\} = V, \\ E' &: V = \{e_1, \dots, e_r\} \supset \{e_2, \dots, e_r\} \dots \supset \{e_r\} \supset 0, \\ F &: 0 \subset \{f_1\} \subset \{f_1, f_2\} \dots \subset \{f_1, \dots, f_r\} = V, \\ F' &: \{f_1 \dots, f_r\} \supset \{f_2, \dots, f_r\} \supset \dots \{f_r\} \supset 0 . \end{aligned} \tag{2.3}$$

We assume that $F$ lies in the orbit of $E$. Equivalently for $i$, $1 \leq i \leq r$, $f_1, \dots, f_i$, $e_{i+1}, \dots, e_r$ is a basis of $E$, i.e.

$$\langle f_1, \dots, f_i \rangle \neq 0 . \tag{2.4}$$

Let $A$ be the $(r,r)$ matrix expressing $f_1, \dots, f_r$ on the basis $e_1, \dots, e_r$. Then (2.4) says that the principal minors of $A$ do not vanish.

On $V^*$, we can define the flags $E^*, E'^*, F^*, F'^*$ associated to the basis $e^1, \dots, e^r$ and $f^1, \dots, f^r$. Now $E^*$ lies in the orbit of $F^*$, i.e. for any $i$, $e^1, \dots, e^i, f^{i+1}, \dots, f^r$ is a basis of $V^*$.

For $0 \le i \le r-1$, let $p_i$ be the projection from $V$ on $\{e_{i+1}, \dots, e_r\}$ with kernel $\{f_1, \dots, f_i\}$. Clearly

$$p_{i+1} = p_{i+1} p_i \,. \tag{2.5}$$

Also for $0 \le i \le r-1$, $p_i f_{i+1}, \dots, p_i f_r$ is a basis of $\{e_{i+1}, \dots, e_r\}$. More precisely for $0 \le j \le r-i$, $p_i f_{i+1}, \dots, p_i f_{i+j}, e_{i+j+1}, \dots, e_r$ is a basis of $\{e_{i+1}, \dots, e_r\}$.

Clearly, if $t \in V$,

$$p_1 t = t - \frac{\langle t, e^1 \rangle}{\langle f_1, e^1 \rangle} f_1 \,. \tag{2.6}$$

By the above, there are similar formulas for $p_2, \dots, p_r$. In particular, by (2.5), (2.6), for $1 \le i \le r$, and $t \in V$,

$$p_i t = p_{i-1} t - \frac{\langle p_{i-1} t, e^i \rangle}{\langle p_{i-1} f_i, e^i \rangle} p_{i-1} f_i. \tag{2.7}$$

From (2.7), we get

$$p_i t = t - \sum_{j=1}^{i} \frac{\langle p_{j-1} t, e^j \rangle}{\langle p_{j-1} f_j, e^j \rangle} p_{j-1} f_j. \tag{2.8}$$

Using (2.8) with $i = r$, we obtain

$$t = \sum_{j=1}^{r} \frac{\langle p_{j-1} t, e^j \rangle}{\langle p_{j-1} f_j, e^j \rangle} p_{j-1} f_j. \tag{2.9}$$

For $0 \le i \le r-1$, let $q_i$ be the projection from $V^*$ on $\{f^{i+1}, \dots, f^r\}$ with kernel $\{e^1, \dots, e^i\}$. Then $q_i$ is the transpose of $p_i$, i.e.

$$q_i = \tilde{p}_i. \tag{2.10}$$

Also, as in (2.6), if $x \in V^*$,

$$q_1 x = x - \frac{\langle x, f_1 \rangle}{\langle e^1, f_1 \rangle} e^1 \,. \tag{2.11}$$

Moreover the results which hold for the $p_i$'s also hold for the $q_i$'s.

THEOREM 2.1. *For* $1 \le i \le r$,

$$\sum_{j=1}^{i} \frac{\langle p_{j-1} t, e^j \rangle \langle f_j, q_{j-1} x \rangle}{\langle p_{j-1} f_j, q_{j-1} e^j \rangle} = \langle t, x \rangle - \langle p_i t, q_i x \rangle \,. \tag{2.12}$$

*In particular,*

$$\sum_{i=1}^{r} \frac{\langle p_{i-1} t, e_i \rangle \langle q_{i-1} x, f_i \rangle}{\langle p_{i-1} f_i, q_{i-1} e^i \rangle} = \langle t, x \rangle \,. \tag{2.13}$$

PROOF. Equation (2.12) follows from (2.8), and equation (2.13) from (2.9). □

PROPOSITION 2.2. *If* $t \in V$, $x \in V^*$, *for* $0 \leq i \leq r$,

$$
\begin{aligned}
(1-p_i)t &= \sum_{j=1}^{i} \frac{\langle f_1 \wedge \ldots \wedge f_{j-1} \wedge t \wedge f_{j+1} \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle}{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle} f_j, \\
p_i t &= \sum_{j=i+1}^{r} \frac{\langle f_1 \wedge \ldots \wedge f_i \wedge t, e^1 \wedge \ldots \wedge e^i \wedge e^j \rangle}{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle} e_j \\
&= \sum_{j=i+1}^{r} \Bigg( \langle t, e^j \rangle - \sum_{k=1}^{i} \\
&\quad \frac{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^{k-1} \wedge e^j \wedge e^{k+1} \wedge \ldots \wedge e^i \rangle \langle t, e^k \rangle}{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle} \Bigg) e_j,
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
(1-q_i)x &= \sum_{j=1}^{i} \frac{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^{j-1} \wedge x \wedge e^{j+1} \wedge \ldots \wedge e^i \rangle}{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle} e^j, \\
q_i x &= \sum_{j=i+1}^{r} \frac{\langle f_1 \wedge \ldots \wedge f_i \wedge f_j, e^1 \wedge \ldots \wedge e^i \wedge x \rangle}{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle} f^j \\
&= \sum_{j=i+1}^{r} \Bigg( \langle f_j, x \rangle - \sum_{k=1}^{i} \\
&\quad \frac{\langle f_1 \wedge \ldots \wedge f_{k-1} \wedge f_j \wedge f_{k+1} \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle \langle f_k, x \rangle}{\langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle} \Bigg) f^j .
\end{aligned}
$$

PROOF. Clearly we only need to prove the first two series of identities in (2.14). The first identity and the first part of the second identity are standard linear algebra. Also for $j \geq i+1$,

$$
\begin{aligned}
&\langle f_1 \wedge \ldots \wedge f_i \wedge t, e^1 \wedge \ldots \wedge e^i \wedge e^j \rangle = \langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle \langle t, e^j \rangle \\
&- \sum_{k=1}^{i} (-1)^{k-i} \langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^{k-1} \wedge e^{k+1} \wedge \ldots \wedge e^i \wedge e^j \rangle \langle t, e^k \rangle \\
&= \langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^i \rangle \langle t, e^j \rangle \\
&- \sum_{k=1}^{i} \langle f_1 \wedge \ldots \wedge f_i, e^1 \wedge \ldots \wedge e^{k-1} \wedge e^j \wedge e^{k+1} \wedge \ldots \wedge e^i \rangle \langle t, e^k \rangle .
\end{aligned} \tag{2.15}
$$

The proof of our Proposition is completed. □

PROPOSITION 2.3. *For any* $j$, $1 \leq j \leq r$

$$
\langle f_j, q_{j-1} x \rangle = \langle f_j, x \rangle - \sum_{k=1}^{j-1} \frac{\langle f_1 \wedge \ldots \wedge f_{k-1} \wedge f_j \wedge f_{k+1} \wedge \ldots \wedge f_{j-1}, e^1 \wedge \ldots \wedge e^{j-1} \rangle}{\langle f_1 \wedge \ldots \wedge f_{j-1}, e^1 \wedge \ldots \wedge e^{j-1} \rangle} \langle f_k, x \rangle . \tag{2.16}
$$

PROOF. This is a consequence of the last equality in (2.14). □

REMARK 2.4. From (2.16), we find that $\langle f_j, q_{j-1} x \rangle = \langle p_{j-1} f_j, x \rangle$ depends only on the $\langle f_k, x \rangle$, $1 \leq k \leq j$.

**2.2. The linear algebra of the basis of a root system.** Let $G$ be a compact connected and simply connected simple Lie group of rank $r$. Let $T$ be a maximal torus in $G$. Let $\mathfrak{t}$ be the Lie algebra of $T$. Let $K$ be a Weyl chamber in $\mathfrak{t}$. Otherwise, we use the notation of Section 1.1.

We will also use the notation of Section 2.1, with $V = \mathfrak{t}^*$. We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by the scalar product $\langle\ ,\ \rangle$ of Section 1.2. Then

$$r = \dim \mathfrak{t}. \tag{2.17}$$

Let $e_1, \dots, e_r \subset R_+$ be the simple basis of $\mathfrak{t}^*$ associated to $K$ [**15**, Proposition V.4.5]. Any $\alpha \in R_+$ is a linear combination with non negative integral coefficients of $e_1, \dots, e_r$. Then $e_1, \dots, e_r$ generate $\overline{R}$. Let $e^1, \dots, e^r$ be the corresponding dual basis of $\overline{R}^* \subset \mathfrak{t}$.

Let $\{\alpha_1, \dots, \alpha_\ell\}$ be an ordering of $R_+$, such that

$$\alpha_i = e_i \text{ for } 1 \le i \le r. \tag{2.18}$$

Recall that we use the notation of Section 2.1 . Clearly all the $\langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle$ lie in $\mathbf{Z}$.

If $G$ is not simply laced, $m$ was defined in (1.8) and is equal to 2 or to 3. By convention, if $G$ is simply laced, we take $m = 1$. By (1.12),

$$m\overline{R} \subset \overline{CR}. \tag{2.19}$$

DEFINITION 2.5. Let $d \in \mathbf{N}^*$ be a common multiple of the $m|\langle \alpha_{i_1} \wedge \dots \wedge \alpha_{i_r} \rangle|$.

Observe that since $e_1, \dots, e_r \in R_+$, for any $j$, $1 \le j \le r$, $d$ is a multiple of $m\langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle \in \mathbf{Z}$. Also by (2.19),

$$d\overline{R} \subset \overline{CR}. \tag{2.20}$$

Recall that $C \subset \mathfrak{t}$ was defined in Definition 1.35.

PROPOSITION 2.6. *The following identity holds*

$$dC \subset \overline{R}^*. \tag{2.21}$$

PROOF. Let $u \in C$, let $\alpha_{i_1}, \dots, \alpha_{i_r} \in R_+$ be a basis of $\mathfrak{t}^*$ such that for $1 \le j \le r$, $\langle \alpha_{i_j}, u \rangle \in \mathbf{Z}$. Since $e_1, \dots, e_r$ is a basis of $\overline{R}$, we find that if $H$ is the lattice generated by $\alpha_{i_1}, \dots, \alpha_{i_r}$,

$$d\overline{R} \subset H, \tag{2.22}$$

which is equivalent to

$$dH^* \subset \overline{R}^*. \tag{2.23}$$

Since $u \in H^*$, from (2.23) , we get (2.21). The proof of our Proposition is completed. □

REMARK 2.7. Take $n \ge 2$. Since $\mathrm{SU}(n)$ is simply laced, $m = 1$. Also by [**15**, Proposition V.6.3], all the $|\langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle|$ are equal to 1. Therefore, for $G = \mathrm{SU}(n)$, we can take $d = 1$. Using (1.114) and (2.21), we recover Proposition 1.40.

DEFINITION 2.8. A family $I = (i_1, \dots, i_r)$ of distinct indices in $\{1, \dots, \ell\}$ is said to be generic if

$$\langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle \ne 0 \text{ for } 1 \le j \le r. \tag{2.24}$$

Given a generic family $I = (i_1, \dots, i_r)$, we now use the notation in Section 2.1 associated to the given basis $e_1, \dots, e_r$ and $\alpha_{i_1}, \dots, \alpha_{i_r}$ of $\mathfrak{t}^* \simeq \mathfrak{t}$. In particular the operators which appear in Section 2.1 will be denoted with the superscript $I$, to mark their dependence on $I$.

Let $\alpha^{i_1}, \dots, \alpha^{i_r}$ be the basis of $\mathfrak{t}^* \simeq \mathfrak{t}$ which is dual to $\alpha_{i_1}, \dots, \alpha_{i_r}$.

Recall that (2.18) holds.

DEFINITION 2.9. A family $I = (i_1, \dots, i_{j-1})$ of $j-1$ distinct elements of $\{1, \dots, \ell\}$ is said to be generic if $I_{j-1} = \{i_1, \dots, i_{j-1}, j, j+1, \dots, r\}$ is generic.

If $I = (i_1, \dots, i_r)$ is generic, $I_{j-1} = (i_1, \dots, i_{j-1})$ is also generic, and by construction (or by (2.14)),

$$p^I_{j-1} = p^{I_{j-1}}_{j-1}\,, \qquad q^I_{j-1} = q^{I_{j-1}}_{j-1}\,. \tag{2.25}$$

DEFINITION 2.10. If $x \in \mathbf{C}^\ell$, and if $I = (i_1, \dots, i_r)$ is generic, put

$$x^I = \sum_1^r x_{i_j} \alpha^{i_j} \in \mathfrak{t} \simeq \mathfrak{t}^*\,. \tag{2.26}$$

By (2.25),

$$q^I_{j-1} x^I = q^{I_{j-1}}_{j-1} x^I\,. \tag{2.27}$$

Also by Proposition 2.3,

$$\langle \alpha_{i_j}, q^I_{j-1} x^I \rangle = \langle p^I_{j-1} \alpha_{i_j}, x^I \rangle = x_{i_j} - \sum_{k=1}^{j-1} \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{k-1}}, \alpha_{i_j}, \alpha_{i_{k+1}}, \dots, \alpha_{i_{j-1}} \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}} \rangle} x_{i_k}\,. \tag{2.28}$$

Note that the right hand side of (2.28) only depends on $x_{i_1}, \dots, x_{i_j}$.

Assume that $I = (i_1, \dots, i_r)$ is generic. Then by Proposition 2.2, if $t \in \mathfrak{t}$, $y \in \mathfrak{t}^*$

$$\begin{aligned} \langle p^I_{j-1} \alpha_{i_j}, e^j \rangle &= \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}} \rangle}, \\ \langle p^I_{j-1} \alpha_{i_j}, t \rangle &= \frac{\langle \alpha_{i_1} \wedge \dots \wedge \alpha_{i_j}, e^1 \wedge \dots \wedge e^{j-1} \wedge t \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}} \rangle}, \\ \langle p^I_{j-1} y, e^j \rangle &= \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}, y \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}} \rangle}. \end{aligned} \tag{2.29}$$

In particular, by (2.29), we find that

$$\frac{d}{\langle p^I_{j-1} \alpha_{i_j}, e^j \rangle} \in \mathbf{Z}\,. \tag{2.30}$$

Also by (2.28), (2.29), if $x \in \mathbf{Z}^\ell$,

$$d \frac{\langle p^I_{j-1} \alpha_{i_j}, x^I \rangle}{\langle p^I_{j-1} \alpha_{i_j}, e^j \rangle} \in \mathbf{Z}\,. \tag{2.31}$$

DEFINITION 2.11. For $u \in C/\overline{R}^*$, let $I_u \subset \{1, \dots, \ell\}$ be given by

$$I_u = \{i, 1 \le i \le \ell, \alpha_i \in R_{u,+}\}\,. \tag{2.32}$$

Put

$$\ell_u = |I_u|. \tag{2.33}$$

**2.3. Fourier series and integration along the fibre.**

DEFINITION 2.12. For $u \in C/\overline{R}^*$, let $\mathbf{R}^{I_u} \subset \mathbf{R}^\ell$ be given by

$$\mathbf{R}^{I_u} = \{s = (s^1, \dots, s^\ell) \in \mathbf{R}^\ell \ , \ s^i = 0 \ \text{ for } i \notin I_u\} . \tag{2.34}$$

Let $a_u : \mathbf{R}^{I_u} \to \mathfrak{t}^* \simeq \mathfrak{t}$ be given by

$$a_u(s) = \sum_{i \in I_u} s^i \alpha_i . \tag{2.35}$$

The transpose $\widetilde{a}_u : \mathfrak{t} \simeq \mathfrak{t}^* \to \mathbf{R}^{I_u}$ is given by

$$\widetilde{a}_u(t) = (\langle \alpha_i, t \rangle)_{i \in I_u} . \tag{2.36}$$

Set

$$V_u = \ker a_u . \tag{2.37}$$

Then we have the exact sequence

$$0 \to V_u \to \mathbf{R}^{I_u} \xrightarrow[a_u]{} \mathfrak{t} \to 0 \ . \tag{2.38}$$

We define $\mathbf{Z}^{I_u} \subset \mathbf{Z}^\ell$, $(\mathbf{R}/\mathbf{Z})^{I_u} \subset (\mathbf{R}/\mathbf{Z})^\ell$ as before. Then

$$a_u(\mathbf{Z}^{I_u}) = \overline{R}_u . \tag{2.39}$$

Also $a_u$ induces a surjection $(\mathbf{R}/\mathbf{Z})^{I_u} \to \mathfrak{t}/\overline{R}$ . Set

$$K_u = \ker a_u \subset (\mathbf{R}/\mathbf{Z})^{I_u} . \tag{2.40}$$

We have the exact sequence

$$0 \to K_u \to (\mathbf{R}/\mathbf{Z})^{I_u} \xrightarrow[a_u]{} \mathfrak{t}/\overline{R} \to 0 \ . \tag{2.41}$$

Set

$$\gamma_u = \mathbf{Z}^{I_u} \cap V_u . \tag{2.42}$$

Then $K_u$ is a union of $|\overline{R}/\overline{R}_u|$ tori $V_u/\gamma_U$ .

DEFINITION 2.13. For $u \in C/\overline{R}^*$, let $H_u \subset \mathfrak{t}/\overline{R}$ be given by

$$H_u = \{t \in \mathfrak{t}/\overline{R}, t = \sum_{j \in \mathcal{J} \subset I_u} t^j \alpha_j, \ \text{ and } \{\alpha_j, j \in \mathcal{J}\} \ \text{ does not span } \mathfrak{t}^* \simeq \mathfrak{t}\} . \tag{2.43}$$

Then $H_u$ is a finite union of hypertori in $T$. Put

$$H = H_0 . \tag{2.44}$$

Clearly,

$$H = \{t \in \mathfrak{t}/\overline{R}, t = \sum_{j \in \mathcal{J} \subset \{1, \dots, \ell\}} t^j \alpha_j, \ \text{ and } \{\alpha_j, j \in \mathcal{J}\} \ \text{ does not span } \mathfrak{t}^* \simeq \mathfrak{t}\} . \tag{2.45}$$

For any $u \in C/\overline{R}^*$,

$$H_u \subset H . \tag{2.46}$$

PROPOSITION 2.14. *Let $I = (i_1, \dots, i_r) \subset I_u$ be generic. Then if $t \in (\mathfrak{t}/\overline{R}) \setminus H$ is represented by $\bar{t} \in \mathfrak{t}$,*

$$\langle p^I_{j-1}\bar{t}, e^j\rangle \notin \mathbf{Z}. \tag{2.47}$$

PROOF. By (2.29),

$$\langle p^I_{j-1}\bar{t}, e^j\rangle = \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}, \bar{t}\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle}. \tag{2.48}$$

Now for $\bar{t} = e_j$, the expression (2.48) is equal to 1. Also $\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}, \bar{t}\rangle$ vanishes if and only if $\bar{t}$ is a linear combination of $\alpha_{i_1}, \dots, \alpha_{i_{j-1}}, e_{j+1}, \dots, e_r$. Therefore the condition

$$\langle p^I_{j-1}\bar{t}, e^j\rangle \in \mathbf{Z} \tag{2.49}$$

is equivalent to

$$\bar{t} = \sum_{k=1}^{j-1} a^k \alpha_{i_k} + b e_j + \sum_{k=j+1}^{r} c^k e_k \ , \ a^k, c^k \in \mathbf{R}, b \in \mathbf{Z}. \tag{2.50}$$

Then,

$$\bar{t} = \sum_{k=1}^{j-1} a^k \alpha_{i_k} + \sum_{k=j+1}^{r} c^k e_k \ \text{ in } \ \mathfrak{t}/\overline{R}, \tag{2.51}$$

so that $t \in H$.

The proof of our Proposition is completed. □

Let $dt$ be the Lebesgue measure on $\mathfrak{t}$ associated with $\langle\ ,\ \rangle$. We still denote by $dt$ the Lebesgue measure on $\mathfrak{t}/\overline{R}$. We map $L^1(\mathfrak{t}/\overline{R})$ into $\mathcal{D}'(\mathfrak{t}/\overline{R})$ by the map $f \mapsto \frac{f\,dt}{\mathrm{Vol}(\mathfrak{t}/\overline{R})}$. We use the same convention for other tori.

Clearly $a_u$ induces a map $a_{u*}$ from $\mathcal{D}'((\mathbf{R}/\mathbf{Z})^{I_u})$ into $\mathcal{D}'(t/\overline{R})$.

PROPOSITION 2.15. *If $g \in L^1((\mathbf{R}/\mathbf{Z})^{I_u})$, then $a_{u*}g \in L^1(\mathfrak{t}/\overline{R})$. More precisely,*

$$a_{u*}g(t) = \int_{K_u} g(t+s)\frac{ds}{\mathrm{Vol}(K_u)}. \tag{2.52}$$

*Also if $k \in \mathbf{Z}^{I_u}$, then*

$$a_{u*}[e^{2i\pi\langle k,s\rangle}] = e^{2i\pi\langle\lambda,t\rangle} \text{ if there is } \lambda \in \overline{R}^* \text{ such that } k_i = \langle\lambda,\alpha_i\rangle, i \in I_u, \tag{2.53}$$
$$= 0 \qquad \text{otherwise}.$$

PROOF. The proof of this result is trivial. □

DEFINITION 2.16. For $u \in C/\overline{R}^*$, $x \in (\mathbf{C} \setminus 2i\pi\mathbf{Z})^\ell$, $t \in \mathfrak{t}/\overline{R}$, set

$$Q_u(t,x) = \sum_{\lambda\in\overline{R}^*} \frac{\exp(2i\pi\langle\lambda,t\rangle)}{\prod_{i\in I_u}(2i\pi\langle\alpha_i,\lambda\rangle - x_i)}. \tag{2.54}$$

Clearly as a function of $t$, $Q_u(t,x)$ is a well-defined distribution on $\mathfrak{t}/\overline{R}$.

For $M \in \mathbf{N}$, we also consider the partial sums

$$Q_u^M(t,x) = \sum_{\substack{\lambda\in\overline{R}^* \\ |\lambda|\le M}} \frac{\exp(2i\pi\langle\lambda,t\rangle)}{\prod_{i\in I_u}(2i\pi\langle\alpha_i,\lambda\rangle - x_i)}. \tag{2.55}$$

Note that $Q_u(t,x)$ depends only on the projection of $x$ on $\mathbf{C}^{I_u}$.

THEOREM 2.17. *For any* $n \in \mathbf{N}^*$, $u \in C/\overline{R}^*$, $v \in \frac{\overline{R}^*}{n}$, $t \in \mathfrak{t}/\overline{R}$, *the following identity holds*

$$n^{\ell_u - r} \sum_{h \in \overline{R}/n\overline{R}} \exp(2i\pi\langle v, t+h\rangle) Q_u\Big(\frac{t+h}{n}, nx\Big) = Q_u(t, x + 2i\pi\widetilde{a}_u v). \tag{2.56}$$

PROOF. Clearly, if $\lambda \in \overline{R}^*$, $v \in \overline{R}^*/n$,

$$\begin{aligned}\frac{1}{n^r} \sum_{h \in \overline{R}/n\overline{R}} \exp(2i\pi\langle \frac{\lambda}{n} + v, t+h\rangle) &= \exp(2i\pi\langle \frac{\lambda}{n} + v, t\rangle) \ \text{ if } \frac{\lambda}{n} + v \in \overline{R}^*, \\ &= 0 \quad \text{otherwise}.\end{aligned} \tag{2.57}$$

From (2.54),(2.57) , we get

$$\begin{aligned} & n^{\ell_u - r} \sum_{h \in \overline{R}/n\overline{R}} \exp(2i\pi\langle v, t+h\rangle) Q_u(\frac{t+h}{n}, nx) \\ &= \sum_{\lambda \in \overline{R}^*} \frac{\exp(2i\pi\langle\lambda, t\rangle)}{\prod_{i \in I_u} (2i\pi\langle\alpha_i, \lambda\rangle - (x_i + 2i\pi\langle\alpha_i, v\rangle))} \\ &= Q_u(t, x + 2i\pi\widetilde{a}_u v). \end{aligned} \tag{2.58}$$

The proof of our Theorem is completed. □

THEOREM 2.18. *The partial sums* $Q_u^M(t,x)$ *converge uniformly together with their derivatives to* $Q_u(t,x)$ *on compact subsets of* $(\mathfrak{t}/\overline{R}) \setminus H_u \times (\mathbf{C} \setminus 2i\pi\mathbf{Z})^\ell$. *The following identity of distributions holds*

$$Q_u(t,x) = (-1)^{\ell_u} a_{u*} \left[ \frac{1}{\prod_{i \in I_u} \big(\exp(x_i) - 1\big)} \exp(\langle x, s\rangle) \right] \quad \text{in } \mathcal{D}'(\mathfrak{t}/\overline{R}). \tag{2.59}$$

*In particular for* $x \in (\mathbf{C} \setminus 2i\pi\mathbf{Z})^\ell$, $Q_u(t,x)$ *is a distribution in* $L^\infty(\mathfrak{t}/\overline{R})$. *Also (2.59) is an identity of smooth functions on* $(\mathfrak{t}/\overline{R}) \setminus H_u \times (\mathbf{C} \setminus 2i\pi\mathbf{Z})^\ell$.

PROOF. For $x \in \mathbf{C} \setminus 2i\pi\mathbf{Z}$, the Fourier series for $e^{xs}$ ($s \in [0,1]$) is given by

$$e^{xs} = (e^x - 1) \sum_{k \in \mathbf{Z}} \frac{e^{2i\pi ks}}{-2i\pi k + x}. \tag{2.60}$$

Since $e^{xs}$ is smooth on $[0,1]$, the partial sums in (2.60) converge uniformly together with their derivatives to $e^{xs}$ on compact subsets of $\mathbf{R}/\mathbf{Z} \setminus \{0\} \times \mathbf{C} \setminus 2i\pi\mathbf{Z}$.

From Proposition 2.15 and from (2.60) , we get

$$a_{u*}[e^{\langle x, s\rangle}] = \prod_{i \in I_u} (e^{x_i} - 1) \sum_{\lambda \in \overline{R}^*} \frac{e^{2i\pi\langle\lambda, t\rangle}}{\prod_{i \in I_u} (-2i\pi\langle\alpha_i, \lambda\rangle + x_i)}, \tag{2.61}$$

which coincides with (2.59).

Also for $x \neq 0$, the wave front set of the distribution $e^{xs}$ on $S^1$ is just $\{0\} \times \mathbf{R}^*$. By [**26**, Theorem 8.2.13], we see that $(t,p) \in \mathfrak{t}/\overline{R} \times \mathbf{R}^{I_u} \setminus \{0\}$ lies in the wave front set of $Q_u(t,x)$ only if $t = \sum_{i \in I_u} t^i \alpha_i$, and $\langle p, \alpha_i\rangle = 0$ when $t^i \notin \mathbf{Z}$. Therefore $Q_u(t,x)$ is smooth on $(\mathfrak{t}/\overline{R}) \setminus H_u$.

The proof of our Theorem if completed. □

**2.4. Iterated residues and the series $Q_u(t,x)$.** Recall that by Theorem 2.18, for generic values of $x$, $Q_u(t,x)$ is smooth on $(\mathfrak{t}/\overline{R}) \setminus H_u$. Therefore by (2.46), $Q_u(t,x)$ is smooth on $(\mathfrak{t}/\overline{R}) \setminus H$.

Let $x \in \mathbf{R} \mapsto [x] \in [0,1[$ be the periodic function of period 1, such that $[x] = x$ on $[0,1[$.

In the sequel, if $t \in \mathfrak{t}/\overline{R}$, we represent $t$ by a given element in $\mathfrak{t}$, which we also denote by $t$.

Also the map $(f^1, \dots, f^r) \in \{0,1,\dots,d-1\}^r \mapsto f = \sum_1^r f^i e_i \in \overline{R}$ defines a one to one map into $\overline{R}/d\overline{R}$. In the sequel, we will always identify $f \in \overline{R}/d\overline{R}$ to the corresponding element in $\{0,1,\dots,d-1\}^r$.

THEOREM 2.19. *For any $u \in C/\overline{R}^*$, for generic values of $x \in (\mathbf{C} \setminus 2i\pi\mathbf{Z})^\ell$, for $t \in (\mathfrak{t}/\overline{R}) \setminus H$, if we still denote by $t$ a representative in $\mathfrak{t}$,*

$$Q_u(t,x) = (-1)^r \sum_{\substack{I=(i_1,\dots,i_r)\subset I_u \\ I \text{ generic},\ f\in \frac{\overline{R}}{d\overline{R}}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle}$$

$$\exp\left\{ d \sum_{j=1}^r \frac{\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I, \alpha_{i_j}, e^j\rangle} \left[\frac{1}{d}\langle p_{j-1}^I (t+f), e^j\rangle\right]\right\} \tag{2.62}$$

$$\frac{1}{\prod_{i\in I_u\setminus I} (\langle \alpha_i, x^I\rangle - x_i)} \prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle}\right) - 1}.$$

PROOF. Take $1 \le j \le r$, $I = (i_1, \dots, i_{j-1}) \subset I_u$ such that $\langle \alpha_{i_1}\rangle \neq 0, \dots, \langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle \neq 0$. For $\widehat{k} = (k_{j+1}, \dots, k_r) \in \mathbf{Z}^{r-j}$, we identify $\widehat{k}$ with $\sum_{i=j+1}^r k_i e^i \in \mathfrak{t} \simeq \mathfrak{t}^*$. In the sequel, $p_{j-1}^I$ denotes the projection $\mathfrak{t} \to \{e_j, \dots, e_r\}$ with kernel $\{\alpha_{i_1}, \dots, \alpha_{i_{j-1}}\}$. In view of (2.28), for $i \in I_u$, $i \notin I$, we will use the abusive notation

$$\langle p_{j-1}^I \alpha_i, x^I\rangle = x_i - \sum_{k=1}^{j-1} \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{k-1}}, \alpha_i, \alpha_{i_{k+1}}, \dots, \alpha_{i_{j-1}}\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle} x_{i_k}. \tag{2.63}$$

Equation (2.63) will in fact be a definition for the left-hand side. For $s \in \mathbf{R}$, if $1 \le j \le r$, set

$$Q_I = \sum_{k\in\mathbf{Z}} \frac{e^{2i\pi ks}}{\prod_{i\in I_u\setminus I} (\langle p_{j-1}^I \alpha_i, 2i\pi(ke^j + \widehat{k}) - x^I\rangle)}. \tag{2.64}$$

We claim that for at least one $i \in I_u \setminus I$,

$$\langle p_{j-1}^I \alpha_i, e^j\rangle \neq 0. \tag{2.65}$$

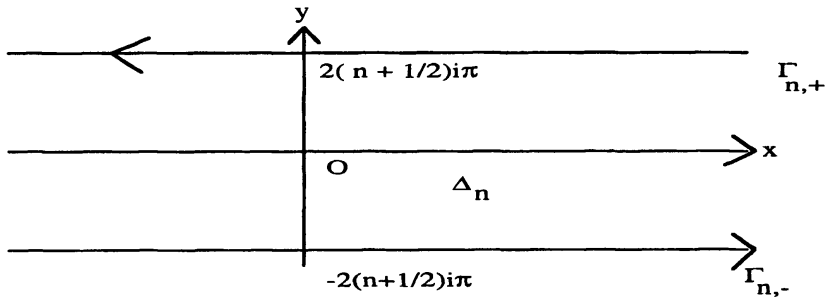


FIGURE 2.1

In fact by (2.29),

$$\langle p_{j-1}^I \alpha_i, e^j \rangle = \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}, \alpha_i \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}} \rangle}. \tag{2.66}$$

Since the elements of $R_{u,+}$ span $\mathfrak{t}^* \simeq \mathfrak{t}$, for at least one $i \in I_u \setminus I$, (2.66) does not vanish.

Clearly

$$Q_I = \frac{1}{d} \sum_{\substack{k \in \mathbf{Z} \\ 0 \le f < d}} \frac{\exp(2i\pi \frac{k}{d}(s+f))}{\prod_{i \in I_u \setminus I} \left(2i\pi \langle p_{j-1}^I \alpha_i, e^j \rangle \frac{k}{d} + \langle p_{j-1}^I \alpha_i, 2i\pi \widehat{k} - x^I \rangle\right)}. \tag{2.67}$$

Then for generic $x \in \mathbf{C}^\ell$, we can write (2.67) in the form

$$Q_I = \frac{1}{d} \sum_{0 \le f < d} \sum_{k \in \mathbf{Z}} \operatorname{Res}_{a = 2i\pi k} \left\{ \frac{e^{a[\frac{s+f}{d}]}}{\left( \prod_{i \in I_u \setminus I} \left( \frac{\langle p_{j-1}^I \alpha_i, e^j \rangle a}{d} + \langle p_{j-1}^I \alpha_i, 2i\pi \widehat{k} - x^I \rangle \right) \right) (e^a - 1)} \right\}. \tag{2.68}$$

Assume that $s \notin \mathbf{Z}$. Then for $f \in \mathbf{N}$, $0 \le f < d$,

$$0 < \left[ \frac{s+f}{d} \right] < 1. \tag{2.69}$$

Also by (2.29) , if $\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}, \alpha_i \rangle \neq 0$, then

$$\frac{d}{\langle p_{j-1}^I \alpha_i, e^j \rangle} \in \mathbf{Z}, \qquad \frac{d \langle p_{j-1}^I \alpha_i, \widehat{k} \rangle}{\langle p_{j-1}^I \alpha_i, e^j \rangle} \in \mathbf{Z}. \tag{2.70}$$

For $n \in \mathbf{N}$, consider the contour $\Gamma_n = \Gamma_{n,+} \cup \Gamma_{n,-}$ given in Figure 2.1, and its interior $\Delta_n$. For $f \in \mathbf{N}$, $0 \leq f < d$, put

$$g_f(a) = \frac{e^{a[\frac{s+f}{d}]}}{\left(\prod_{i \in I_u \setminus I} \left(\frac{\langle p^I_{j-1}\alpha_i, e^j\rangle a}{d} + \langle p^I_{j-1}\alpha_i, 2i\pi\widehat{k} - x^I\rangle\right)\right)(e^a - 1)} . \tag{2.71}$$

By (2.69), as $a \in \Delta_n$, $|a| \to +\infty$,

$$g_f(a) \to 0 . \tag{2.72}$$

Also the integral $\int_{\Gamma_n} g_f(a)da$ converges. We can then use the residue theorem to evaluate $\int_{\Gamma_n} g_f(a)da$. Finally by (2.65), (2.71), as $n \to +\infty$, $\int_{\Gamma_n} g_f(a)da \to 0$. Then we find that for generic $x \in \mathbf{C}^\ell$,

$$\sum_{a \in \mathbf{C}} \mathrm{Res}_a g_f(a) = 0 . \tag{2.73}$$

Now for generic $x \in \mathbf{C}^\ell$, the poles of $g_f(a)$ other than $\{2i\pi k\}_{k \in \mathbf{Z}}$ are simple (this follows in particular from (2.63)), and given by

$$a = -d\frac{\langle p^I_{j-1}\alpha_{i_j}, 2i\pi\widehat{k} - x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}, \ i_j \in I_u \setminus I, \ \langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle \neq 0 . \tag{2.74}$$

In the sequel, we use the notation

$$(I, i_j) = (i_1, \dots, i_j) . \tag{2.75}$$

Observe that if $(i_1, \dots, i_j)$ is generic, if $y \in \mathfrak{t}^*$ for $1 \leq k \leq j-1$,

$$\begin{aligned} &\frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{k-1}}, y, \alpha_{i_{k+1}}, \dots, \alpha_{i_j}\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle} = \\ &\frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle}\Big(\langle \alpha_{i_1}, \dots, \alpha_{i_{k-1}}, y, \alpha_{i_{k+1}}, \dots, \alpha_{i_{j-1}}\rangle \\ &- \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{k-1}}, \alpha_{i_j}, \alpha_{i_{k+1}}, \dots, \alpha_{i_{j-1}}\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle}\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}, y\rangle\Big) . \end{aligned} \tag{2.76}$$

In fact if $y$ lies the vector space spanned by $\alpha_{i_1}, \dots, \alpha_{i_{k-1}}, \alpha_{i_{k+1}}, \dots, \alpha_{i_{j-1}}, \alpha_{i_j}$, $e_{j+1}, \dots, e_r$ both sides of (2.76) vanish, and if $y = \alpha_{i_k}$ both sides are equal to 1. So by (2.6), (2.63), (2.66), (2.68), (2.70), (2.73) and (2.76) with $y = \alpha_i$, $i \notin (I, i_j)$, we get

$$\begin{aligned} Q_I = - &\sum_{\substack{i_j \in I_u \setminus I \\ \langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle \neq 0, 0 \leq f < d}} \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle} \\ &\frac{\exp\left(-2i\pi\frac{\langle p^I_{j-1}\alpha_{i_j}, \widehat{k}\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}(s+f) + \frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{s+f}{d}\right]\right)}{\left(\prod_{i \in I_u \setminus (I, i_j)} \left(\langle p_j^{(I,i_j)}\alpha_i, 2i\pi\widehat{k} - x^I\rangle\right)\right)\left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)} . \end{aligned} \tag{2.77}$$

Now by (2.7), for $j' \geq j+1$,

$$\langle p_j^{(I,i_j)} e_j, e^{j'} \rangle = -\frac{\langle p_{j-1}^I \alpha_{i_j}, e^{j'} \rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j \rangle}. \tag{2.78}$$

Using (2.77), (2.78), we obtain

$$Q_I = -\sum_{\substack{i_j \in I_u \setminus I \\ \langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle \neq 0, 0 \leq f < d}} \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}} \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_j} \rangle} \frac{\exp\left(2i\pi \langle p_j^{(I,i_j)}(s+f)e_j, \widehat{k} \rangle + \frac{d\langle p_{j-1}^I \alpha_{i_j}, x^I \rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j \rangle} \left[\frac{s+f}{d}\right]\right)}{\left(\prod_{i \in I_u \setminus (I,i_j)} \left(\langle p_j^{(I,i_j)} \alpha_i, 2i\pi \widehat{k} - x^I \rangle\right)\right)\left(\exp\left(\frac{d\langle p_{j-1}^I \alpha_{i_j}, x^I \rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j \rangle}\right) - 1\right)}. \tag{2.79}$$

Clearly

$$Q_u(t,x) = \sum_{k=(k_1,\dots,k_r) \in \mathbf{Z}^r} \frac{\exp(2i\pi \sum_{i=1}^r k_i \langle t, e^i \rangle)}{\prod_{i \in I_u} (2i\pi \langle \alpha_i, \sum_{i=1}^r k_i e^i \rangle - x_i)}. \tag{2.80}$$

Also with the notation in (2.63),

$$\langle p_0^\emptyset \alpha_i, x^I \rangle = x_i. \tag{2.81}$$

So using (2.79) with $I = \emptyset$, $s = \langle t, e^1 \rangle$, we find that for $\langle t, e^1 \rangle \notin \mathbf{Z}$,

$$Q_u(t,x) = -\sum_{\substack{i_1 \in I_u \\ \langle \alpha_{i_1} \rangle \neq 0 \\ 0 \leq f^1 < d}} \frac{1}{\langle \alpha_{i_1} \rangle} \sum_{k=(k^2,\dots,k^r) \in \mathbf{Z}^{r-1}} \frac{\exp\left(2i\pi \langle p_1^{(i_1)}(t+f^1 e_1), k \rangle + d\frac{x_{i_1}}{\langle \alpha_{i_1} \rangle} \left[\frac{\langle t+f^1 e_1, e^1 \rangle}{d}\right]\right)}{\left(\prod_{i \in I_u \setminus \{i_1\}} \left(\langle p_1^{(i_1)} \alpha_i, 2i\pi k - x^I \rangle\right)\right)\left(\exp(\frac{dx_{i_1}}{\langle \alpha_{i_1} \rangle}) - 1\right)}. \tag{2.82}$$

Clearly, if $t \in (t/\overline{R}) \setminus H$, then $\langle t, e^1 \rangle \notin \mathbf{Z}$, and so (2.82) holds. More generally, if $t \in (t/\overline{R}) \setminus H$, $f \in \overline{R}$, by Proposition 2.14, $\langle p_{j-1}^I (t+f), e_j \rangle \notin \mathbf{Z}$. Therefore using (2.79), (2.82), we can iterate the procedure. Finally observe that by (2.63), if $I = (i_1, \dots, i_r)$ is generic, for $i \notin I$,

$$\begin{aligned} \langle p_r^I \alpha_i, x^I \rangle &= x_i - \sum_{k=1}^r \frac{\langle \alpha_{i_1}, \dots, \alpha_{i_{k-1}}, \alpha_i, \alpha_{i_{k+1}}, \dots, \alpha_{i_r} \rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_r} \rangle} x_{i_k} \\ &= x_i - \langle \alpha_i, x^I \rangle. \end{aligned} \tag{2.83}$$

Using (2.83), we get (2.62).

The proof of our Theorem is completed. □

REMARK 2.20. If we had assumed instead that $e_1, \dots, e_r$ is a simple basis of $R_{u,+}$, we would have obtained a better equality in (2.62) , with $t \in (\mathfrak{t}/\overline{R}) \setminus H_u$. However, it is essential here that we have use the same simple basis $e_1, \dots, e_r$ of $R_+$ for all the $u \in C/\overline{R}^*$ simultaneously. Finally observe that Theorem 2.19 can possibly be reformulated using the formalism of Szenes [**54**].

REMARK 2.21. Take $n \in \mathbf{N}^*$. Then we have the exact sequence

$$0 \to \overline{R}/d\overline{R} \xrightarrow[n]{} \overline{R}/dn\overline{R} \xrightarrow[p]{} \overline{R}/n\overline{R} \to 0\,. \tag{2.84}$$

In (2.84), the map $n$ is just multiplication by $n$, and $p$ is the obvious projection.

Now in (2.62), we may replace $d$ by $nd$. We get

$$\begin{aligned} Q_u(t,x) = n^{\ell_u - r}(-1)^r & \sum_{\substack{I=(i_1,\dots,i_r)\subset I_u, I \text{ generic} \\ f\in \overline{R}/d\overline{R}, h \in \overline{R}/n\overline{R}}} \\ \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle} \exp\Bigg\{ d \sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, nx^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} & \\ \left[\frac{1}{d}\langle p^I_{j-1}\Big(\frac{(t+h)}{n} + f\Big), e^j\rangle\right]\Bigg\} & \frac{1}{\prod_{i\in I_u\setminus I}(\langle \alpha_i, nx^I\rangle - nx_i)} \\ \frac{1}{\prod_{j=1}^r \left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, nx^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)}. & \end{aligned} \tag{2.85}$$

Comparing (2.62) and (2.85) gives the identity (2.56), with $v = 0$.

By definition, $Q_u(t,x)$ is well-defined on $\mathfrak{t}/\overline{R}$. However it is not entirely clear that the right-hand side of (2.62) is indeed well-defined on $\mathfrak{t}/\overline{R}$. We will now check this fact directly.

THEOREM 2.22. *As a function of $t \in \mathfrak{t}$, the right-hand side of (2.62) descends to a function on $\mathfrak{t}/\overline{R}$.*

PROOF. By Proposition 2.2, $\langle \alpha_{i_1}, \dots, \alpha_{i_k}\rangle p^I_k e_k$ is an integral linear combination of the $(e_\rho)_{\rho \geq k+1}$. Therefore for $k \leq j-1$, $dp^I_{j-1}e_k$ is an integral linear combination of the $(p^I_{j-1}e_\rho)_{\rho\geq k+1}$. So for $k \leq j-1$, $d\langle p^I_{j-1}e_k, e^j\rangle$ is an integral linear combination of the $\langle p^I_{j-1}e_{k'}, e^j\rangle (k+1 \leq k' \leq j)$.

To prove that the right-hand side of (2.62) is well-defined on $(\mathfrak{t}/\overline{R}) \setminus H$, we only need to show that if we add to $t$ an element of $d\overline{R}$, the total expression does not change.

Take $f \in \overline{R}/d\overline{R}$. Then $f$ is uniquely represented by an element we also note $f$,

$$f = \sum_1^r f^k e_k \quad, 0 \leq f^k < d\,. \tag{2.86}$$

Clearly

$$\langle p^I_{j-1}(t+f), e^j\rangle = \langle p^I_{j-1}(t + \sum_1^j f^k e_k), e^j\rangle\,. \tag{2.87}$$

Put

$$A_j = \left[\frac{1}{d}\langle p^I_{j-1}(t+f), e^j\rangle\right]. \tag{2.88}$$

Since $p^I_{j-1}e_j = e_j$, $\langle p^I_{j-1}e_j, e^j\rangle = 1$. Therefore if we add to $f^j$ an integral multiple of $d$, $A_j$ is unchanged.

By the above, if we add to $f^r$ an integral multiple of $d$, the right-hand side of (2.62) is unchanged. If we add to $f^{r-1}$ an integral multiple of $d$, then only the term $A_r$ is possibly affected. However as we saw before, $d\langle p^I_{r-1}e_{r-1}, e^r\rangle$ is an integral multiple of $\langle p^I_{r-1}e_r, e^r\rangle = 1$, i.e.

$$d\langle p^I_{r-1}e_{r-1}, e^r\rangle = q\langle p^I_{r-1}e_r, e^r\rangle \quad , q \in \mathbf{Z}. \tag{2.89}$$

Therefore adding to $f^{r-1}$ an integral multiple of $d$ is equivalent to adding to $f^r$ an integer. This show the right-hand side of (2.62) is invariant under this change.

A trivial downward recursion procedure shows that $Q_u(t,x)$ is indeed well-defined on $\mathfrak{t}/\overline{R}$.

The proof of our Theorem is completed. □

**2.5. The Fourier transform on quotient of lattices.** Let $\Delta, \Delta'$ be lattices in $\mathfrak{t}$, with $\Delta \subset \Delta'$. Then there is a projection $\mathfrak{t}/\Delta \to \mathfrak{t}/\Delta'$. Let $f \in \mathcal{D}'(t/\Delta)$. For $\mu \in \Delta^*/\Delta'^*$, put

$$\widehat{f}_\mu(t) = \frac{1}{\left|\frac{\Delta'}{\Delta}\right|} \sum_{k\in\Delta'/\Delta} e^{-2i\pi\langle\mu,k\rangle} f(t+k). \tag{2.90}$$

Then $\widehat{f}_\mu \in \mathcal{D}'(t/\Delta)$. Moreover if $k \in \Delta'/\Delta$,

$$\widehat{f}_\mu(t+k) = \exp(2i\pi\langle\mu,k\rangle)\widehat{f}_\mu(t). \tag{2.91}$$

Also

$$f(t) = \sum_{\mu\in\Delta^*/\Delta'^*} \widehat{f}_\mu(t). \tag{2.92}$$

By (2.90)

$$f(t+k) = \sum_{\mu\in\Delta^*/\Delta'^*} \exp(2i\pi\langle\mu,k\rangle)\widehat{f}_\mu(t). \tag{2.93}$$

Equation (2.93) is just an aspect of Fourier transform. Note here that if $A \subset \mathfrak{t}/\Delta$ is such that $f$ is $C^\infty$ on $(\mathfrak{t}/\Delta)\setminus A$, then $\widehat{f}_\mu$ is $C^\infty$ on $(\mathfrak{t}/\Delta)\setminus \bigcup_{k\in\Delta'/\Delta}(A+k)$. Then formula (2.93) only expresses $f$ as a function which is smooth on $(\mathfrak{t}/\Delta)\setminus \bigcup_{k\in\Delta'/\Delta}(A+k)$, i.e. there is a loss of regularity in (2.93).

**2.6. Fourier series on T.** In the constructions of Section 2.3 , we may as will replace $R$ by $CR$, $\overline{R}$ by $\overline{CR}$, $\overline{R}^*$ by $\overline{CR}^*$. For $u \in C/\overline{R}^*$, we still define $I_u$, $\mathbf{R}^{I_u}$ as in (2.32), (2.34). However in (2.36), $a_u : \mathbf{R}^{I_u} \to \mathfrak{t}^* \simeq \mathfrak{t}$ is replaced by $b_u : \mathbf{R}^{I_u} \to \mathfrak{t} \simeq \mathfrak{t}^*$ given by

$$b_u(t) = \sum_{i\in I_u} t^i h_{\alpha_i}. \tag{2.94}$$

Clearly

$$b_u(\mathbf{Z}^{I_u}) = \overline{CR_u} \subset \overline{CR}. \tag{2.95}$$

Therefore there is a a surjection $b_u : (\mathbf{R}/\mathbf{Z})^{I_u} \to T = \mathfrak{t}/\overline{CR}$. Then we have an exact sequence

$$0 \to L_u \to (\mathbf{R}/\mathbf{Z})^{I_u} \xrightarrow[b_u]{} T \to 0 \,. \tag{2.96}$$

DEFINITION 2.23. For $u \in C/\overline{R}^*$, let $S_u \subset T = \mathfrak{t}/\overline{CR}$ be given by

$$S_u = \{t \in \mathfrak{t}/\overline{CR},\ t = \sum_{j \in \mathcal{J} \subset I_u} t^j h_{\alpha_j},\ \text{and } \{h_{\alpha_j}, j \in \mathcal{J}\} \text{ do not span } \mathfrak{t}\}. \tag{2.97}$$

Put

$$S = S_0 \,. \tag{2.98}$$

Then

(2.99)

$$S = \{t \in T = \mathfrak{t}/\overline{CR}, t = \sum_{j \in \mathcal{J} \subset \{1,\dots,\ell\}} t^j h_{\alpha_j},\ \text{and } \{h_{\alpha_j}, j \in \mathcal{J}\} \text{ do not span } \mathfrak{t}\}.$$

As in (2.46) , for any $u \in C/\overline{R}^*$,

$$S_u \subset S \,. \tag{2.100}$$

By (1.12) , it is clear that if $\lambda \in \overline{CR}^*$, $\alpha \in R_\ell$, $\beta \in R$,

$$\begin{aligned} &\langle \lambda, \alpha \rangle \in \mathbf{Z}\,, \\ &m\langle \lambda, \beta \rangle \in \mathbf{Z}\,. \end{aligned} \tag{2.101}$$

In the sequel, when $G$ is simply laced, we will make $m = 1$.

DEFINITION 2.24. For $u \in C/\overline{R}^*$, $x \in (\mathbf{C} \setminus (2i\pi\frac{\mathbf{Z}}{m})^\ell$, $t \in T = \mathfrak{t}/\overline{CR}$, put

$$R_u(t,x) = \sum_{\lambda \in \overline{CR}^*} \frac{\exp(2i\pi\langle \lambda, t\rangle)}{\prod_{i \in I_u}(2i\pi\langle \alpha_i, \lambda\rangle - x_i)} \,. \tag{2.102}$$

Clearly, by (1.8), (1.10),

$$R_u(t,x) = m^{|R_{u,+} \cap R_s|} \sum_{\lambda \in \overline{CR}^*} \frac{\exp(2i\pi\langle \lambda, t\rangle)}{\prod_{i \in I_u}(2i\pi\langle h_{\alpha_i}, \lambda\rangle - \frac{\|h_{\alpha_i}\|^2}{2} x_i)} \,. \tag{2.103}$$

Of course in (2.103) , $\frac{\|h_{\alpha_i}\|^2}{2} = 1$ or $m$. From (2.103) , it should now be clear that Theorem 2.18 can be applied to $R_u(t,x)$. In particular on compact subsets of $T \setminus S_u \times (\mathbf{C} \setminus \frac{2i\pi\mathbf{Z}}{m})^\ell$, $R_u(t,x)$ is a smooth function of $(t,x)$.

If $G$ is simply laced, the objects we just constructed are the ones we already obtained in Section 2.3.

Now we use the notation in (2.35). Put

$$a = a_0 \,. \tag{2.104}$$

Then if $s \in \mathbf{R}^\ell$,

$$as = \sum_{i=1}^{\ell} s^i \alpha_i \,. \tag{2.105}$$

Let $\widetilde{a} : \mathfrak{t} \simeq \mathfrak{t}^* \to \mathbf{R}^\ell$ be the transpose of $a$. Then

$$\widetilde{a}t = (\langle \alpha_1, t \rangle, \dots, \langle \alpha_\ell, t \rangle), \tag{2.106}$$

and $\widetilde{a}$ maps $\overline{R}^*$ into $\mathbf{Z}^\ell$.

Now we will use (2.90)-(2.93), with $\Delta = \overline{CR}$, $\Delta' = \overline{R}$. In the sequel, we view $R_u(t,x)$ as an element of $\mathcal{D}'(\mathfrak{t}/\overline{CR})$. If $\mu \in \overline{CR}^*/\overline{R}^*$, we define $(\widehat{R}_u)_\mu(t,x)$ as in (2.90).

PROPOSITION 2.25. *If $\mu \in \overline{CR}^*/\overline{R}^*$, if $\lambda_\mu \in \overline{CR}^*$ represents $\mu$, then*

$$(\widehat{R}_u)_\mu(t,x) = \exp(2i\pi\langle \lambda_\mu, t \rangle) Q_u(t, x - 2i\pi\widetilde{a}\lambda_\mu). \tag{2.107}$$

PROOF. If $\lambda \in \overline{CR}^*$, then

$$\begin{aligned} \widehat{\exp(2i\pi\langle \lambda, t \rangle)}_\mu &= \exp(2i\pi\langle \lambda, t \rangle) \text{ if } \lambda \in \overline{CR}^* \text{ maps to } \mu \in \overline{CR}^*/\overline{R}^*, \\ &= 0 \text{ otherwise }. \end{aligned} \tag{2.108}$$

Then

$$(\widehat{R}_u)_\mu(t,x) = \sum_{\lambda \in \overline{R}^*} \frac{\exp(2i\pi\langle \lambda + \lambda_\mu, t \rangle)}{\prod_{i \in I_u} (2i\pi\langle \alpha_i, \lambda \rangle - (x_i - 2i\pi\langle \alpha_i, \lambda_\mu \rangle))}. \tag{2.109}$$

From (2.109) , we get (2.107). The proof of our Proposition is completed. □

REMARK 2.26. By Proposition 2.25 , we get the otherwise obvious fact that the right-hand side of (2.107) only depends on $\mu$ and not on $\lambda_\mu$.

For any $\mu \in \overline{CR}^*/\overline{R}^*$, we choose $\lambda_\mu \in \overline{CR}^*$ representing $\mu$.

THEOREM 2.27. *The following identity holds*

$$R_u(t,x) = \sum_{\mu \in \overline{CR}^*/\overline{R}^*} \exp(2i\pi\langle \lambda_\mu, t \rangle) Q_u(t, x - 2i\pi\widetilde{a}\lambda_\mu). \tag{2.110}$$

PROOF. This follows from (2.92) and (2.107). □

**2.7. Iterated residues and the series $R_u(t,x)$.** Let $\tau : \mathfrak{t}/\overline{CR} \to t/\overline{R}$ be the obvious projection. For $u \in C/\overline{R}^*$, put

$$\begin{aligned} \overline{S}_u &= \tau^{-1}(H_u), \\ \overline{S} &= \tau^{-1}(H). \end{aligned} \tag{2.111}$$

Clearly $\tau$ maps $S_u$ into $H_u$, $S$ into $H$. Therefore

$$\begin{aligned} S_u &\subset \overline{S}_u, \\ S &\subset \overline{S}. \end{aligned} \tag{2.112}$$

If $G$ is simply laced

$$\begin{aligned} S_u &= \overline{S}_u, \\ S &= \overline{S}. \end{aligned} \tag{2.113}$$

Recall that by (2.20), $d\overline{R} \subset \overline{CR}$.

THEOREM 2.28. *For any $u \in C/\overline{R}^*$, $t \in T \setminus \overline{S}$, then*

$$R_u(t,x) = \left|\frac{\overline{R}}{\overline{\overline{CR}}}\right| (-1)^r \sum_{\substack{I=(i_1,\dots,i_r)\subset I_u \\ I \text{ generic} \\ f \in \overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle} \tag{2.114}$$

$$\frac{1}{\prod_{i\in I_u \setminus I} (\langle \alpha_i, x^I\rangle - x_i)} \exp\left\{ d \sum_{j=1}^r \frac{\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle} \left[\frac{1}{d}\langle p_{j-1}^I(t+f), e^j\rangle\right]\right\}$$

$$\prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle}\right) - 1}.$$

PROOF. We use Theorems 2.19 and 2.27 . Also, if $h \in \mathfrak{t}$,

$$(\tilde{a}h)^I = h. \tag{2.115}$$

So we deduce from (2.115) that if $i \in I_u \setminus I$,

$$\langle \alpha_i, (\tilde{a}\lambda_\mu)^I\rangle - \langle \alpha_i, \lambda_\mu\rangle = 0. \tag{2.116}$$

By (2.19), (2.28), if $\lambda \in \overline{CR}^*$,

$$\frac{d\langle p_{j-1}^I \alpha_{i_j}, \tilde{a}\lambda\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle} \in \mathbf{Z}. \tag{2.117}$$

From (2.13), (2.62), (2.110), (2.116), (2.117), we get (2.114). The proof of our Theorem is completed. □

**2.8. Fourier series for the universal cover of a semisimple centralizer.** Take $u \in C/\overline{R}^*$. Recall that by

$$\pi_1(Z(u)) = \overline{CR}/\overline{CR_u}. \tag{2.118}$$

Also

$$\overline{CR} \subset \overline{R}_u^*. \tag{2.119}$$

Let $\widetilde{Z}(u)$ be the universal cover of $Z(u)$. Then by [**15**, Theorem V.7.1],

$$Z(\widetilde{Z}(u)) = \overline{R}_u^*/\overline{CR_u}. \tag{2.120}$$

Therefore $\pi_1(Z(u)) = \overline{CR}/\overline{CR_u}$ is a subgroup of $Z(\widetilde{Z}(u))$. Also $\overline{CR}_u^*$ is the lattice of weights of $\widetilde{Z}(u)$.

DEFINITION 2.29. For $u \in C/\overline{R}^*$, $t \in \mathfrak{t}/\overline{CR}_u$, $x \in \left(\mathbf{C} \setminus \frac{2i\pi\mathbf{Z}}{m}\right)^\ell$, put

$$\widetilde{R}_u(t,x) = \sum_{\lambda \in \overline{CR}_u^*} \frac{\exp(2i\pi\langle \lambda, t\rangle)}{\prod_{i\in I_u}(2i\pi\langle \lambda, \alpha_i\rangle - x_i)}. \tag{2.121}$$

Clearly

$$\pi_1(Z(u))^* = \overline{CR}_u^*/\overline{CR}^*. \tag{2.122}$$

Also $\overline{R}_u^*/\overline{R}^*$ maps into $\overline{CR}_u^*/\overline{CR}^*$, with kernel $\dfrac{\overline{R}_u^* \cap \overline{CR}^*}{\overline{R}^*}$. In particular $u \in C/\overline{R}^*$ maps to an element of $\pi_1(Z(u))^*$.

DEFINITION 2.30. For $u \in C/\overline{R}^*$, $q \in \mathbf{Z}$, $t \in T = \mathfrak{t}/\overline{CR}$, put

$$R_{u,q}(t,x) = \frac{1}{|\frac{\overline{CR}}{\overline{CR}_u}|} \sum_{h \in \frac{\overline{CR}}{\overline{CR}_u}} \exp(2i\pi\langle qu, h\rangle)\widetilde{R}_u(t+h,x)\,. \tag{2.123}$$

PROPOSITION 2.31. *If $u \in C$, the following identity holds*

$$R_{u,q}(t,x) = R_u(t, x + 2i\pi q\tilde{a}u)\exp(-2i\pi\langle qu, t\rangle)\,. \tag{2.124}$$

PROOF. Since $u \in \overline{CR}^*_u$, our identity follows from (2.123). □

THEOREM 2.32. *For any $u \in C/\overline{R}^*$, $q \in \mathbf{Z}$, $t \in T \setminus \overline{S}$, then*

$$R_{u,q}(t,x) = \left|\frac{\overline{R}}{\overline{CR}}\right|(-1)^r \sum_{\substack{I=(i_1,\dots,i_r)\subset I_u \\ I \text{ generic} \\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1},\dots,\alpha_{i_r}\rangle}\exp(2i\pi(\langle qu, f\rangle)) \tag{2.125}$$

$$\frac{1}{\prod\limits_{i\in I_u\setminus I}(\langle \alpha_i, x^I\rangle - x_i)}\exp\left\{d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}(t+f), e^j\rangle\right]\right\}$$

$$\prod_{j=1}^r \frac{1}{\exp(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}) - 1}\,.$$

PROOF. We use Theorem 2.28 and Proposition 2.31. Since for $i \in I_u$, $\langle u, \alpha_i\rangle \in \mathbf{Z}$, by (2.13), (2.29), (2.116), we get (2.125). The proof of our Theorem is completed. □

### 2.9. Bernouilli polynomials and the Fourier series $P_n(t)$.

DEFINITION 2.33. For $n \in \mathbf{N}$, $t \in T = \mathfrak{t}/\overline{CR}$, put

$$P_n(t) = -\sum_{\substack{\lambda\in\overline{CR}^* \\ \pi(\lambda)\neq 0}} \frac{\exp(2i\pi\langle\lambda, t\rangle)}{[\pi(\lambda)]^n}\,. \tag{2.126}$$

For $M \in \mathbf{N}^*$, we will consider the partial sums

$$P^M_n(t) = -\sum_{\substack{\lambda\in\overline{CR}^* \\ \pi(\lambda)\neq 0 \\ |\lambda|\leq M}} \frac{\exp(2i\pi\langle\lambda, t\rangle)}{[\pi(\lambda)]^n}\,. \tag{2.127}$$

Clearly $P_n(t)$ is a well-defined distribution on $T = \mathfrak{t}/\overline{CR}$. Recall that

$$K = \{t \in \mathfrak{t}, \text{ for } \alpha \text{ in } R_+,\ \langle \alpha, t\rangle > 0\}\,. \tag{2.128}$$

Then by [**15**, Note V.4.14] and (1.186),

$$\begin{aligned} P_n(t) &= -\sum_{\substack{\lambda\in\overline{CR}^*\cap\overline{K} \\ w\in W}} \frac{\varepsilon_w \exp(2i\pi\langle w(\lambda+\rho), t\rangle)}{[\pi(\lambda+\rho)]^n} \quad \text{for } n \text{ odd}, \\ &= -\sum_{\substack{\lambda\in\overline{CR}^*\cap\overline{K} \\ w\in W}} \frac{\exp(2i\pi\langle w(\lambda+\rho), t\rangle)}{[\pi(\lambda+\rho)]^n} \quad n \text{ even}\,. \end{aligned} \tag{2.129}$$

By (1.186), (2.126), for $w \in W$,

$$P_n(wt) = \varepsilon_w^n P(t). \tag{2.130}$$

Since $\pi(t)$ is a polynomial, $\pi\left(\frac{\partial/\partial t}{2i\pi}\right)$ is a differential operator. Then we have the identity of distributions on $T = \mathfrak{t}/\overline{CR}$,

$$\pi\left(\frac{\partial/\partial t}{2i\pi}\right) P_n(t) = P_{n-1}(t) \text{ for } n \geq 1. \tag{2.131}$$

DEFINITION 2.34. For $t \in [0,1]$, $n \geq 0$, put

$$p_n(t) = -\sum_{k \neq 0} \frac{e^{2i\pi kt}}{(2i\pi k)^n}. \tag{2.132}$$

If $G = SU(2)$, then $T \simeq S_1 \simeq \mathbf{R}/\mathbf{Z}$. One verifies easily that the $p_n(t)$'s are exactly the $P_n$'s associated to $G = SU(2)$. Then (2.131) is the equation of distribution on $S_1$

$$p_n'(t) = p_{n-1}(t) \ , \ n \geq 1. \tag{2.133}$$

Also

$$p_0(t) = 1 - \delta_{\{0\}}, \tag{2.134}$$

So by (2.133), (2.134), we get

$$\left(\frac{d}{dt}\right)^n p_n(t) = 1 - \delta_{\{0\}}. \tag{2.135}$$

By (2.135) , it is clear that the $p_n(t)$'s are polynomials on $S_1 \setminus \{0\}$. Also for $n \geq 2$, the series in (2.132) is absolutely convergent on $[0,1]$. For $n \geq 1$, the series in (2.132) converges uniformly together with its derivatives on compact sets of $S_1$, not containing 0.

By [**45**, Appendix B] , the $p_n$'s are exactly the Bernouilli polynomials. In the sequel, we will consider the $p_n$'s as polynomials on $\mathbf{R}$, whose restriction to $]0,1[$ is given by (2.132).

Recall that

$$\mathrm{Td}(x) = \frac{x}{1-e^{-x}}. \tag{2.136}$$

Then

$$\mathrm{Td}(x) - \mathrm{Td}(-x) = 1. \tag{2.137}$$

Finally by [**51**, p147] , if the $B_k$, $k \geq 1$, are the Bernoulli numbers,

$$\mathrm{Td}(x) = 1 + \frac{x}{2} + \sum_{k=1}^{+\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}. \tag{2.138}$$

PROPOSITION 2.35. *For $n \geq 0$, $t \in \mathbf{R}$,*

$$p_n(t) = \mathrm{Res}_{a=0}\left[\frac{e^{ta}}{a^n}\frac{1}{e^a - 1}\right]. \tag{2.139}$$

PROOF. For $0 \le t \le 1$, put

$$f_{t,n}(a) = \frac{e^{ta}}{a^n} \frac{1}{e^a - 1}. \tag{2.140}$$

Clearly $f_{t,n}(a)$ has simple poles at $a = 2i\pi k, k \in \mathbf{Z}^*$, and a pole of order $n+1$ at $a = 0$. Then, for $k \in \mathbf{Z}^*$

$$\operatorname{Res}_{a=2i\pi k} f_{t,n}(a) = \frac{e^{2i\pi kt}}{(2i\pi k)^n}. \tag{2.141}$$

Now we use the Cauchy residue theorem inside a circle of centre 0 and radius $2\pi(M+1/2)$, as $M \to +\infty$. For $n \ge 2$, or for $n = 1$, $t \in ]0,1[$, the integral of $f_{t,n}(a)$ on the circle tends to 0 as $M \to +\infty$. So we find that

$$\sum_{k \in \mathbf{Z}} \operatorname{Res}_{a=2i\pi k} f_{t,n}(a) = 0. \tag{2.142}$$

From (2.142) , we get (2.139) for $n \ge 2$, or $n = 1$, $0 < t < 1$. Then since $p_n(t)$ is a polynomial, (2.139) holds for any $t \in \mathbf{R}$. For $n = 0$, (2.139) is trivial. The proof of our Proposition is completed. □

PROPOSITION 2.36. *For $n \ge 0$, $t \in \mathbf{R}$,*

$$p_n(t) = \operatorname{Td}(-\partial/\partial t)\frac{t^n}{n!}, \tag{2.143}$$
$$p_{n+1}(t+1) - p_{n+1}(t) = \frac{t^n}{n!}.$$

PROOF. Clearly,

$$\frac{\partial}{\partial t} e^{at} = a e^{at}. \tag{2.144}$$

By (2.144) , we find that for $|a| < 2\pi$,

$$\operatorname{Td}(-\partial/\partial t) e^{at} = \operatorname{Td}(-a) e^{at}. \tag{2.145}$$

Using (2.139), (2.145), we get

$$p_n(t) = \operatorname{Res}_{a=0}\left[\frac{e^{ta}}{a^{n+1}}\operatorname{Td}(-a)\right] = \operatorname{Td}(-\partial/\partial t)\operatorname{Res}_{a=0}\left[\frac{e^{ta}}{a^{n+1}}\right] = \operatorname{Td}(-\partial/\partial t)\frac{t^n}{n!}. \tag{2.146}$$

By (2.139)

$$p_{n+1}(t+1) - p_{n+1}(t) = \operatorname{Res}\left[\frac{e^{ta}}{a^{n+1}}\right] = \frac{t^n}{n!}. \tag{2.147}$$

The proof of our Proposition is completed. □

Recall that $R_{s,+}$ denote the set of short positive roots.

THEOREM 2.37. *For $n \ge 1$, as $M \to +\infty$, the partial sums $P_n^M(t)$ converge to $P_n(t)$ uniformly on the compact subsets of $T \setminus S$. The following identity of distributions holds on $T = \mathfrak{t}/\overline{CR}$,*

$$P_n(t) = (-1)^{\ell+1} m^{n|R_{s,+}|} b_* \left[\prod_{i=1}^{\ell} p_n(t^i)\right]. \tag{2.148}$$

*In particular $P_n(t) \in L_\infty(T)$. Also (2.148) is an identity of smooth functions on $T \setminus S$. Finally $P_n(t)$ is a polynomial on $T \setminus S$.*

PROOF. By proceeding as in the proof of Theorem 2.18 , we get (2.148). By the same argument as in Theorem 2.18 , we find that $P_n(t)$ is smooth on $T \setminus S$. Also the uniform convergence results for the $p_n^M(t)$'s and (2.148) imply the corresponding uniform convergence for $P_n^M(t)$.

Now, we will show that $P_n(t)$ is a polynomial on $T \setminus S$. In fact we will prove that for $1 \le i \le \ell$, for $p$ large enough,

$$h^p_{\alpha_i} P_n(t) = 0 \text{ on } T \setminus S. \tag{2.149}$$

Clearly $b_* \partial/\partial t_i = h_{\alpha_i}$. Then using (2.135) , we obtain

$$\begin{aligned} h^n_{\alpha_1} b_*[p_n(t^1)\dots p_n(t^\ell)] = {} & b_*[p_n(t^2)\dots p_n(t^\ell)] \\ & - b_*[\delta_{t_1=0} p_n(t^2)\dots p_n(t^\ell)]. \end{aligned} \tag{2.150}$$

Clearly

$$h_{\alpha_1} b_*[p_n(t^2)\dots p_n(t^\ell)] = 0. \tag{2.151}$$

Let $V_1$ be the vector space in $\mathfrak{t}$ spanned by $h_{\alpha_2}, \dots, h_{\alpha_\ell}$. If $V_1$ is not equal to $\mathfrak{t}$, the support of $b_*[\delta_{t_1=0} p_n(t^2)\dots p_n(t^\ell)]$ is included in $S$. From (2.150) , we then get

$$h^{n+1}_{\alpha_1} b_*[p_n(t^1)\dots p_n(t^\ell)] = 0 \text{ on } T \setminus S. \tag{2.152}$$

If $V_1 = \mathfrak{t}$, we can express $h_{\alpha_1}$ in the form

$$h_{\alpha_1} = \sum_{j=2}^{\ell} a^j h_{\alpha_j}. \tag{2.153}$$

By (2.135),

$$\begin{aligned} h^n_{\alpha_2} b_*[\delta_{t_1=0} p_n(t^2)\dots p_n(t^\ell)] = {} & b_*[\delta_{t_1=0} p_n(t^3)\dots p_n(t^\ell)] \\ & - b_*[\delta_{t_1=0,t_2=0} p_n(t^3)\dots p_n(t^\ell)]. \end{aligned} \tag{2.154}$$

Let $V_2$ be the vector space spanned by $h_{\alpha_2}, \dots, h_{\alpha_\ell}$. If $V_2 \neq \mathfrak{t}$, then by the analogue of (2.152),

$$h^{n+1}_{\alpha_2} q_*[\delta_{t_1=0} p_n(t^2)\dots p_n(t^\ell)] = 0 \text{ on } T \setminus S. \tag{2.155}$$

Now using (2.155) , and iterating the above argument, it should be clear that for $p \in \mathbf{N}$ large enough,

$$h^p_{\alpha_1} P_n(t) = 0 \quad \text{on } T \setminus S. \tag{2.156}$$

In (2.156), we may as well replace the index 1 by the index $i$, $1 \le i \le \ell$. Therefore we have established (2.149).

The proof of our Theorem is completed. □

**2.10. The Fourier series $\tilde{P}_{u,n}(t)$.** Take $u \in C/\overline{R}^*$. Recall that $\widetilde{Z}(u)$ is the universal cover of $Z(u)$. Then $\widetilde{Z}(u)$ is connected and simply connected. Therefore to $\widetilde{Z}(u)$, we can associate the objects we just constructed for $G$. Observe that $\mathfrak{t}/\overline{CR}_u$ is a maximal torus in $\widetilde{Z}(u)$, and $\mathfrak{t}$ is its Lie algebra.

DEFINITION 2.38. Let $\pi_u(t)$ be the function on $\mathfrak{t}$

$$\pi_u(t) = \prod_{\alpha \in R_{u,+}} \langle 2i\pi\alpha, t\rangle. \tag{2.157}$$

Then $\pi_u(t)$ is the analogue of $\pi(t)$.

DEFINITION 2.39. For $u \in C/\overline{R}^*$, $n \in \mathbf{N}$, $t \in \mathfrak{t}/\overline{CR_u}$, put

$$\widetilde{P}_{u,n}(t) = -\sum_{\substack{\lambda \in \overline{CR}^*_u \\ \pi_u(\lambda) \neq 0}} \frac{\exp(2i\pi\langle \lambda, t\rangle)}{[\pi_u(\lambda)]^n}. \tag{2.158}$$

Then $\widetilde{P}_{u,n}(t)$ is the analogue of $P_n(t)$ for $\widetilde{Z}(u)$. Take $w \in W$. We identify $w$ with a representative in $N(T)/T$. Then if $u \in C/\overline{R}^*$,

$$\begin{aligned} Z(wu) &= w\, Z(u) w^{-1}, \\ \overline{CR}_{wu} &= w\overline{CR}_u, \\ R_{Z(wu)} &= wR_u, \\ R_{Z(wu),+} &= wR_u \cap R_+. \end{aligned} \tag{2.159}$$

By (2.159), we get

$$\pi_{wu}(t) = (-1)^{|R_+ \cap w(-R_{u,+})|} \pi_u(w^{-1}t). \tag{2.160}$$

PROPOSITION 2.40. *If $u \in C/\overline{R}^*$, $n \in \mathbf{N}$, $w \in W$, then*

$$\widetilde{P}_{u,n}(t) = (-1)^{n|R_+ \cap w(-R_{u,+})|} \widetilde{P}_{wu,n}(wt). \tag{2.161}$$

PROOF. Clearly by (2.159), (2.160),

$$\begin{aligned} \widetilde{P}_{u,n}(t) &= -\sum_{\substack{\lambda \in \overline{CR}^*_{wu} \\ \pi_u(w^{-1}\lambda) \neq 0}} \frac{e^{2i\pi\langle \lambda, wt\rangle}}{[\pi_u(w^{-1}\lambda)]^n} \\ &= -(-1)^{n|R_+ \cap w(-R_{u,+})|} \sum_{\substack{\lambda \in \overline{CR}^*_{wu} \\ \pi_{wu}(\lambda) \neq 0}} \frac{e^{2i\pi\langle \lambda, wt\rangle}}{[\pi_{wu}(\lambda)]^n} \\ &= (-1)^{n|R_+ \cap w(-R_{u,+})|} \widetilde{P}_{wu,n}(wt), \end{aligned} \tag{2.162}$$

which coincides with (2.161). □

Let $\widetilde{S}_u \subset \mathfrak{t}/\overline{CR_u}$ be the analogue of $S \subset \mathfrak{t}/\overline{CR}$. Of course $\widetilde{S}_u$ projects into $S_u$. Then by Theorem 2.37, $\widetilde{P}_{u,n}(t)$ is polynomial on $(\mathfrak{t}/\overline{CR_u}) \setminus \widetilde{S}_u$.

Let $i_1, \dots, i_{\ell_u}$ be the elements of $I_u = \{i \in \{1, \dots, \ell\}, \alpha_i \in R_{u,+}\}$ arranged in increasing order, i.e. $i_1 < i_2 \dots < i_{\ell_u}$.

If $f(x) = f(x_{i_1}, \dots, x_{i_{\ell_u}})$ is a meromorphic function of $x \in \mathbf{R}^{I_u}$, we denote by $\mathrm{Res}^{I_u}_{x=0} f$ the expression

$$\mathrm{Res}^{I_u}_{x=0} f = \mathrm{Res}_{x_{i_{\ell_u}}=0} \dots \mathrm{Res}_{x_{i_1}=0} f(x_{i_1}, \dots, x_{i_{\ell_u}}). \tag{2.163}$$

It will be of fundamental importance that the order in (2.163) is fixed once and for all.

THEOREM 2.41. *For $u \in C/\overline{R}^*$, $n \geq 1$,*

$$\widetilde{P}_{u,n}(t) = -\mathrm{Res}^{I_u}_{x=0} \frac{1}{(\prod_{i \in I_u} x_i)^n} \widetilde{R}_u(t, x). \tag{2.164}$$

PROOF. Clearly

$$\operatorname{Res}_{x_i=0}\frac{1}{x_i^n}\frac{1}{2i\pi\langle\alpha_i,\lambda\rangle - x_i} = \frac{1}{(2i\pi\langle\alpha_i,\lambda\rangle)^n} \quad \text{if } \langle\alpha_i,\lambda\rangle \neq 0, \qquad = 0 \quad \text{if } \langle\alpha_i,\lambda\rangle = 0. \tag{2.165}$$

Using (2.165) , we get (2.164). The proof of our Theorem is completed. □

REMARK 2.42. The identity (2.164) can be viewed as an identity of distributions on $\mathfrak{t}/\overline{CR_u}$, of smooth functions on $(\mathfrak{t}/\overline{CR_u}) \setminus \widetilde{S}_u$, of elements of $L^\infty(\mathfrak{t}/\overline{CR_u})$. Also for the moment, the ordering in (2.164) is still irrelevant.

**2.11. The function $P_{u,n,q}(t)$.** If $t \in \mathfrak{t}$, we still denote by $t$ the corresponding element in $T = \mathfrak{t}/\overline{CR}$.

DEFINITION 2.43. For $u \in C/\overline{R}^*$, $q \in \mathbf{Z}$, $n \in \mathbf{N}$, $t \in \mathfrak{t}/\overline{CR}$, put

$$P_{u,n,q}(t) = \frac{1}{\left|\frac{\overline{CR}}{\overline{CR_u}}\right|}\sum_{h \in \frac{\overline{CR}}{\overline{CR_u}}} \exp(2i\pi\langle qu, h\rangle)\widetilde{P}_{u,n}(t+h). \tag{2.166}$$

Observe that the function $\exp(2i\pi\langle qu, t\rangle)P_{u,n,q}(t)$ descends to a well defined function on $T = \mathfrak{t}/\overline{CR}$. Equivalently, we may consider $P_{u,n,q}(t)$ as a section of the flat line bundle $L_u$ on $T$, associated to $u \in \mathfrak{t}^*/\overline{CR}^*$.

PROPOSITION 2.44. *If $u \in C/\overline{R}^*$, $q \in \mathbf{Z}$, $n \in \mathbf{N}$, $w \in W$, then*

$$P_{u,n,q}(t) = (-1)^{n|R_+ \cap w(-R_{u,+})|}P_{wu,q,n}(wt). \tag{2.167}$$

PROOF. Clearly $w : \dfrac{\overline{CR}}{\overline{CR_u}} \to \dfrac{\overline{CR}}{\overline{CR_{wu}}}$ is one to one. Then (2.167) follows from Proposition 2.40. □

THEOREM 2.45. *The section $P_{u,n,q}(t)$ is a polynomial on $T \setminus S_u$.*

PROOF. Since $\widetilde{P}_{u,n}(t)$ is polynomial on $(\mathfrak{t}/\overline{CR_u}) \setminus \widetilde{S}_u$, it is clear from (2.166) that $P_{u,n,q}$ is polynomial on $T \setminus S_u$. □

THEOREM 2.46. *For $u \in C/\overline{R}^*$, $n \in \mathbf{N}$, $q \in \mathbf{Z}$, $t \in \mathfrak{t}/\overline{CR}$, the following identity holds*

$$P_{u,n,q}(t) = -\operatorname{Res}^{I_u}_{x=0}\frac{1}{\left(\prod_{i \in I_u} x_i\right)^n}R_{u,q}(t,x). \tag{2.168}$$

PROOF. This follows from (2.123), (2.164), (2.166) . □

DEFINITION 2.47. For $u \in C/\overline{R}^*$, let $\mathcal{I}_u$ be the set of generic $I = (i_1, \dots, i_r) \subset I_u$ such that if $\sigma^I \in \mathcal{S}_r$ is defined by $i_{\sigma^I(1)} < i_{\sigma^I(2)} < \dots < i_{\sigma^I(r)}$, if $j \in I_u \setminus I$, either $j < \sigma^I(1)$, or if $p_j$ is defined by the condition

$$i_{\sigma^I(1)} < \dots < i_{\sigma^I(p_j)} < j < i_{\sigma^I(p_j+1)} < \dots, \tag{2.169}$$

then

$$\alpha_{i_{\sigma^I(1)}} \wedge \dots \wedge \alpha_{i_{\sigma^I(p_j)}} \wedge \alpha_j \neq 0. \tag{2.170}$$

Condition (2.170) is equivalent to

$$\alpha_j \notin \{\alpha_{i_{\sigma^I(1)}}, \dots, \alpha_{i_{\sigma^I(p_j)}}\}. \tag{2.171}$$

If $I \in \mathcal{I}_u$, we define $\operatorname{Res}^I_{x=0}$ by the formula

$$\operatorname{Res}^I_{x=0} = \operatorname{Res}_{x_{i_{\sigma(r)}}=0} \dots \operatorname{Res}_{x_{i_{\sigma(1)}}=0}. \tag{2.172}$$

THEOREM 2.48. *For $u \in C/\overline{R}^*$, $n \geq 1$, $q \in \mathbf{N}$, if $t \in \mathfrak{t}$ represents an element of $T \setminus \overline{S}$, then*

$$P_{u,n,q}(t) = \left|\frac{\overline{R}}{\overline{CR}}\right| (-1)^{r+1} \sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u \\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle} \exp(2i\pi(\langle qu, f\rangle))$$
$$\operatorname{Res}^I_{x=0} \left( \frac{1}{\prod_{\alpha\in R_{u,+}} \langle \alpha, x^I\rangle} \right)^n \exp\left\{ d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \right.$$
$$\left. \left[\frac{1}{d}\langle p^I_{j-1}(t+f), e^j\rangle\right] \right\} \prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1}. \tag{2.173}$$

PROOF. By Theorems 2.32 and 2.46, for $t \in T \setminus \overline{S}$, we get

$$P_{u,n,q}(t) = \left|\frac{\overline{R}}{\overline{CR}}\right| (-1)^{r+1} \sum_{\substack{I=(i_1,\dots,i_r)\subset I_u \\ I\text{ generic} \\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle} \exp(2i\pi(\langle qu, f\rangle))$$
$$\operatorname{Res}^{I_u}_{x=0} \frac{1}{(\prod_{i\in I_u} x_i)^n} \frac{1}{\prod_{i\in I_u\setminus I} (\langle \alpha_i, x^I\rangle - x_i)} \exp\left\{ d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \right. \tag{2.174}$$
$$\left. [\tfrac{1}{d}\langle p^I_{j-1}(t+f), e^j\rangle] \right\} \prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1.}$$

Now $x^I$ depends only on $x_{i_1}, \dots, x_{i_r}$. Therefore in (2.174), the dependence on the $(x_i)_{i\in I_u\setminus I}$ is only via the term $\dfrac{1}{\prod_{i\in I_u\setminus I} x_i^n(\langle \alpha_i, x^I\rangle - x_i)}$.

Take $\epsilon \geq 0$. Let $v \in \mathbf{C}$, $|v| \ll \epsilon$, $w \in \mathbf{C}$, $|w| > \epsilon$. Then by the theorem of residues, for $n \geq 1$,

$$\frac{1}{2\pi i} \int_{\substack{x_j\in\mathbf{C} \\ |x_j|=\epsilon}} \frac{dx_j}{x_j^n(v+w-x_j)} = \frac{1}{(v+w)^n}, \tag{2.175}$$
$$\frac{1}{2\pi i} \int_{\substack{x_j\in\mathbf{C} \\ |x_j|=1}} \frac{dx_j}{x_j^n(v-x_j)} = 0.$$

Take $j \in I_u \setminus I$. If

$$\alpha_j = \sum_{k=1}^r a^k \alpha_{i_{\sigma^I(k)}}, \tag{2.176}$$

then

$$\langle \alpha_j, x^I \rangle = \sum_{k=1}^{r} a^k x_{i_{\sigma^I(k)}}. \tag{2.177}$$

Put

$$v = \sum_{k=1}^{p_j} a^k x_{i_{\sigma^I(k)}} \quad , \quad w = \sum_{k=p_j+1}^{r} a^k x_{i_{\sigma^I(k)}}. \tag{2.178}$$

Observe that $w$ is identically 0 if and only if $a^k = 0$ for $k \geq p_j + 1$, i.e. if $\alpha_j \in \{\alpha_{i_{\sigma^I(1)}}, \dots, \alpha_{i_{\sigma^I(p_j)}}\}$.

To evaluate (2.174), we will use (2.175). Namely take a sequence $\epsilon_1, \dots, \epsilon_{\ell_u}$ in $\mathbf{R}_+$, with $0 < \epsilon_{i_1} < \dots < \epsilon_{\ell_u}$. Then if $f(x_{i_1}, \dots, x_{i_{\ell_u}})$ is a meromorphic function, by definition,

$$\operatorname{Res}^{I_u}_{x=0} f = \int_{\substack{|x_{i_j}|=\epsilon_{i_j} \\ 1 \leq j \leq \ell_u}} f(x_{i_1}, \dots, x_{i_{\ell_u}}) \frac{dx_{i_1}}{2i\pi} \cdots \frac{dx_{i_{\ell_u}}}{2i\pi}. \tag{2.179}$$

If the sequence$\epsilon_1, \dots, \epsilon_{\ell_u}$ is enough decreasing, when taking the $x_{i_j}$ as in (2.179), if $\alpha_j \notin \{\alpha_{i_{\sigma^I(1)}}, \dots, \alpha_{i_{\sigma^I(r)}}\}$, in (2.178), then $|w| > \epsilon_j$.

Using (2.175), (2.179), we find easily that for $j \in I_u \setminus I$,

$$\begin{aligned} \operatorname{Res}_{x_j=0} \frac{1}{x_j^n(\langle \alpha_j, x^I \rangle - x_j)} &= 0 \qquad \text{if } \alpha_j \in \{\alpha_{i_{\sigma^I(1)}}, \dots, \alpha_{i_{\sigma^I(p_j)}}\}, \\ &= \frac{1}{\langle \alpha_j, x^I \rangle^n} \quad \text{if } \alpha_j \notin \{\alpha_{i_{\sigma^I(1)}}, \dots, \alpha_{i_{\sigma^I(p_j)}}\}. \end{aligned} \tag{2.180}$$

A related argument is as follows. Define $v, w$ by (2.178). If $w = 0$, then since the $x_{i_{\sigma^I(k)}}$, $k \leq p_j$ have been made nearly equal to 0 before $x_j$,

$$\frac{1}{v + w - x_j} = -\sum_{k=0}^{+\infty} \frac{v^k}{x_j^{k+1}}. \tag{2.181}$$

From (2.181) , we get

$$\operatorname{Res}_{x_j=0} \left[ \frac{1}{x_j^n} \frac{1}{v + w - x_j} \right] = 0. \tag{2.182}$$

If $w \neq 0$, then

$$\frac{1}{v + w - x_j} = \sum_{k=0}^{+\infty} \frac{(x_j - v)^k}{w^{k+1}}. \tag{2.183}$$

By (2.183) , we get

$$\operatorname{Res}_{x_j=0} \frac{1}{x_j^n} \frac{1}{v + w - x_j} = \sum_{k=n-1}^{+\infty} \frac{C_k^{n-1}(-v)^{k-n+1}}{w^{k+1}} = \frac{1}{(v+w)^n}, \tag{2.184}$$

which gives another proof of (2.180). Finally observe that if $j \in I$,

$$\langle \alpha_j, x^I \rangle = x_j. \tag{2.185}$$

From (2.174), (2.180), (2.185), we get (2.173). The proof of our Theorem is completed. ❑

By Theorem 2.22, we already know that, as a section of a line bundle, $P_{u,n,q}(t)$ is polynomial on $T\backslash\overline{S}$. By Theorem 2.22, the right-hand side of (2.173) descends to a section of the same line bunlde . We will give a direct proof that the right-hand side of (2.173) is polynomial on $T\setminus\overline{S}$.

THEOREM 2.49. *The right-hand side of (2.173) is a polynomial on* $T\setminus\overline{S}$.

PROOF. In the right-hand side of (2.173), up to a locally constant factor on $T\setminus\overline{S}$, we may and will take off the brackets in $\left[\frac{1}{d}\langle p_{j-1}^I(t+f), e^j\rangle\right]$. By (2.13),

$$\sum_{j=1}^{r}\frac{\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}\langle p_{j-1}^I(t+f), e^j\rangle = \langle t+f, x^I\rangle . \tag{2.186}$$

Then

$$\langle t+f, x^I\rangle = \sum_{1}^{r} F^j(t)x_{\sigma^I(j)}\,, \tag{2.187}$$

and the $F^j(t)$ are affine functions of $t\in\mathfrak{t}$. Therefore

$$\exp(\langle t+f, x^I\rangle) = \prod_{j=1}^{r}\left(\sum_{k=0}^{+\infty}\frac{(F^j(t)x_{\sigma^I(j)})^k}{k!}\right). \tag{2.188}$$

Moreover

$$\frac{1}{\exp\left(\frac{d\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}\right)-1} = \frac{\operatorname{Td}\left(\frac{-d\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}\right)}{\frac{d\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}}. \tag{2.189}$$

Also by (2.28), (2.29),

$$\frac{\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle} = \frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_j}\rangle} \left(\langle\alpha_{i_1},\dots,\alpha_{i_{j-1}}\rangle x_{i_j} - \sum_{k=1}^{j-1}\langle\alpha_{i_1},\dots,\alpha_{i_{k-1}},\alpha_{i_j},\alpha_{i_{k+1}},\dots,\alpha_{i_{j-1}}\rangle x_{i_k}\right). \tag{2.190}$$

Finally the $\langle\alpha_j, x^I\rangle$ are linear combinations of $x_{i_1},\dots,x_{i_r}$.

Freeze now $x_{i_{\sigma^I(2)}},\dots,x_{i_{\sigma^I(r)}}$, and consider one of the terms in (2.173) as a function of $x_{i_{\sigma^I(1)}}$. Then it is clear that this term is a meromorphic function of $x_{i_{\sigma^I(1)}}$ with a pole of finite order at $x_{i_{\sigma^I(1)}}=0$. When taking the residue at 0 in the variable $x_{i_{\sigma^I(1)}}$, by the above, it is clear this will introduce a finite power $F^1(t)^k$, i.e. a polynomial function of $t$.

It is now clear that the procedure can be iterated. The proof of our Theorem is completed. □

**2.12. The action of a differential operator on $P_{u,n,q}(t)$.** Let $s\in\mathfrak{t}^*\mapsto F(s)\in C[s]$ be a power series, which converges on a neighborhood of 0. Then $F(\partial/\partial t)$ is a formal power series of differential operators. Since $P_{u,n}(t)$ is a polynomial on $T\setminus S_u$, $F(\partial/\partial t)P_{u,n}(t)$ is a well-defined polynomial on $T\setminus S_u$.

Recall that we have identified $\mathfrak{t}$ and $\mathfrak{t}^*$. Then if $I=(i_1,\dots,i_r)$ is generic, if $x\in\mathbf{C}^\ell$, then $x^I\in\mathfrak{t}\simeq\mathfrak{t}^*$.

THEOREM 2.50. *For $u \in C/\overline{R}^*$, $n \in \mathbf{N}^*$, $q \in \mathbf{N}$, if $t \in \mathfrak{t}$ represents an element of $T \setminus \overline{S}$, then*

$$F(\partial/\partial t)P_{u,n,q}(t) = \left|\frac{\overline{R}}{\overline{CR}}\right| (-1)^{r+1} \sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u \\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle}$$
$$\exp(2i\pi(\langle qu,f\rangle)) \operatorname{Res}^I_{x=0}\left(F(x^I)\frac{1}{(\prod_{j\in I_u}\langle\alpha_j,x^I\rangle)^n}\right.$$
$$\exp\left\{d\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}(t+f),e^j\rangle\right]\right\}$$
$$\left.\frac{1}{\prod_{j=1}^r\left(\exp\left(\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right)-1\right)}\right). \tag{2.191}$$

PROOF. For $y \neq 0$, $[y]-y$ is locally constant. Using (2.186), if $s \in \mathfrak{t}$ is identified with the corresponding vector field, then

$$s\left\{\exp\left(d\sum_{j=1}^r\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}(t+f),e^j\rangle\right]\right)\right\}$$
$$= \langle s,x^I\rangle \exp\left(d\sum_{j=1}^r\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}(t+f),e^j\rangle\right]\right). \tag{2.192}$$

From (2.173), (2.192), we get (2.191).

The proof of our Theorem is completed. □

## 3. Symplectic manifolds and moment maps

In this Section, we recall known results on symplectic manifolds and moment maps. In particular we prove a form of the formula of Witten [64] Jeffrey-Kirwan [28], which expresses the integral of certain characteristic classes on symplectic reductions in terms of the action of differential operators on the symplectic volume. The results of this Section will be used in Section 5, where we will give a formula for the integral of certain characteristic classes on the strata of the moduli space.

This Section is organized as follows. In Section 3.1, we recall elementary facts on orbifolds. In Section 3.2-3.4, we give elementary properties of moment maps. In Section 3.5, we give a direct simple proof that the image of the symplectic volume measure by the moment map can be evaluated in terms of the symplectic volume of the symplectic reductions. In Section 3.6, when 0 is a regular value of the moment map $\mu$, we recall Jeffrey-Kirwan's expression of the volume of the neighbouring fibres in terms of integrals of characteristic classes on the symplectic reduction of $\mu^{-1}(0)$. In Section 3.7, we prove the formula of Witten-Jeffrey-Kirwan. Finally in Section 3.8, we apply the above to the symplectic coadjoint orbits of $G$.

**3.1. Orbifolds.** Let $X$ be a smooth compact manifold. Let $G$ be a compact connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. We assume that $G$ acts on $X$ on the right. If $Y \in \mathfrak{g}$, let $Y^X \in \mathrm{Vect}(X)$ be the corresponding vector field.

We assume that $G$ acts locally freely on $X$, i.e. for any non zero $Y \in \mathfrak{g}$, $Y^X$ is a non vanishing vector field on $X$.

Let $\mathfrak{g}^X$ be the subvector bundle of $TX$ which is the image of $\mathfrak{g}$ by $Y \to Y^X$. Then we have an exact sequence of $G$ vector bundles

$$0 \to \mathfrak{g}^X \to TX \to TX/\mathfrak{g}^X \to 0 . \tag{3.1}$$

The above data define an orbifold $X/G$, and the $G$-bundle $TX/\mathcal{G}^X$ is also called the tangent bundle $TX/G$ to $X/G$. If the $G$-bundle $TX/\mathfrak{g}^X$ is orientable (or equivalently if $TX$ is orientable), we will say that the orbifold $X/G$ is orientable.

If $G$ acts freely on $X$, then $X/G$ is just the standard quotient.

If $y \in X$, let $Z(y) = \{g \in G; yg = y\}$ be the stabilizer of $y$. Then $Z(y)$ is a finite subgroup of $G$. By [24, Proposition 27.4] , there are finitely many conjugacy classes of finite subgroups of $G$, which occur as stabilizers.

Inclusion induces a partial ordering on the set of conjugacy classes of finite subgroup of $G$. On each connected component of $X$, there is a unique minimal conjugacy class of stabilizers $S$, called the generic conjugacy class of stabilizers. This minimal conjugacy class then acts as the identity on the considered connected component. The order $|S|$ of a generic stabilizer is locally constant on $X$.

Let $X_{\mathrm{reg}}$ be the set of $y \in X$ such that $Z(y)$ lies in the minimal conjugacy class. Then $X_{\mathrm{reg}}$ is open in $X$, and $X_{\mathrm{reg}}/G$ is a smooth manifold included in the orbifold $X/G$.

DEFINITION 3.1. A 1-form $\theta : TX \to \mathfrak{g}$ is said to be a connection form on $\pi : X \to X/G$ if

- For $Y \in \mathfrak{g}$,

$$\theta(Y^X) = Y. \tag{3.2}$$

- For $g \in G, Y \in \mathfrak{g}$,

$$g^*\theta = \theta.g. \tag{3.3}$$

One verifies trivially that $G$-connections exist. Then the curvature $\Theta$ of $\theta$ is defined by

$$d\theta = -\frac{1}{2}[\theta,\theta] + \Theta. \tag{3.4}$$

Also if $Y \in \mathfrak{g}, g \in G$,

$$i_{Y^X}\Theta = 0, g^*\Theta = \Theta.g. \tag{3.5}$$

Put

$$T^H X = \{U \in TX, \theta(U) = 0\}. \tag{3.6}$$

Then $T^H X$ is a $G$-invariant subbundle of $TX$ such that

$$TX = T^H X \oplus \mathfrak{g}^X. \tag{3.7}$$

A $G$-invariant form $\alpha$ on $X$ is said to be basic if for $Y \in \mathfrak{g}$, $i_{Y^X}\alpha = 0$. From now on, we suppose that $X/G$ is an oriented orbifold. Then $X_{\text{reg}}/G$ is an oriented manifold. By definition

$$\int_{X/G} \alpha = \int_{X_{\text{reg}}/G} \alpha\,. \tag{3.8}$$

Let $E_1, \ldots, E_n$ be a basis of $\mathfrak{g}$, let $E^1, \ldots, E^n$ be the corresponding dual basis of $\mathfrak{g}^*$. We write the connection $\theta$ in the form

$$\theta = \Sigma_1^n \theta^i E_i\,. \tag{3.9}$$

Then $E^1 \wedge \ldots \wedge E^n$ defines a volume form on $G$. Let $\text{Vol}(G)$ be the corresponding volume of $G$.

We equip $\mathfrak{g}^X$ with the orientation induced by the orientation of $\mathfrak{g}$. Then $TX \simeq TX/\mathfrak{g}^X \oplus \mathfrak{g}^X$ is naturally oriented. If $\alpha$ is a $G$-invariant basic form on $X$,

$$\int_{X/G} \alpha = \frac{|S|}{\text{Vol}(G)} \int_X \alpha \wedge \theta^1 \wedge \ldots \wedge \theta^n\,. \tag{3.10}$$

If $\alpha$ is a $G$-invariant basic form on $X$, $d\alpha$ is also $G$-invariant and basic. Then since $(X/G) \setminus (X_{\text{reg}}/G)$ is a union of submanifolds of codimension $\geq 2$,

$$\int_{X/G} d\alpha = 0\,. \tag{3.11}$$

In fact note that

$$\int_X d\alpha \wedge \theta^1 \wedge \ldots \wedge \theta^n = (-1)^{\deg(\alpha+1)} \int_X \alpha \wedge d(\theta^1 \wedge \ldots \wedge \theta^n)\,. \tag{3.12}$$

By (3.4),

(3.13)

$$\int_X \alpha \wedge d(\theta^1 \wedge \ldots \wedge \theta^n) = \int_X \alpha \wedge \sum_{i=1}^n (-1)^{i-1}\theta^1 \wedge \ldots \wedge \theta^{i-1} \wedge \Theta^i \wedge \ldots \wedge \theta^n = 0,$$

which provides another proof of (3.11).

Let $E \to X$ be a complex $G$-vector bundle on $E$. Let $\nabla^E$ be a $G$-invariant horizontal connection on $X$. Namely suppose first that $G$ acts freely on $X$. Then $\nabla^E$ is just a connection on the vector bundle $X \times_G E$ over $X/G$. More generally, if $G$ only acts locally freely on $X$, the above construction still makes sense. Let $F^E = \nabla^{E,2}$ be the curvature of $\nabla^E$. Then $F^E$ is a $G$-invariant basic 2 form on $X$ with values in $\mathfrak{g}$. Let $P$ be an ad-invariant function on $\mathfrak{g}$. Then $P(E,\nabla^E) = P(\frac{-F^E}{2i\pi})$ is a $G$-invariant basic closed form on $X$. If $X/G$ is an oriented orbifold, the integral $\int_{X/G} P(F,\nabla^E)$ is well-defined and does not depend on $\nabla^E$.

**3.2. Symplectic manifolds and moment maps.** Let $(X,\sigma)$ be a symplectic manifold, so that $\sigma$ is a nondegenerate closed 2-form. Let $H : X \to \mathbf{R}$ be a smooth function. The corresponding hamiltonian vector field $Y_H$ is defined by the equation

$$dH = i_{Y_H}\sigma, \tag{3.14}$$

which we rewrite in the form

$$(d - i_{Y_H})(H + \sigma) = 0. \tag{3.15}$$

From (3.15), we get

$$L_{Y_H}\sigma = 0. \tag{3.16}$$

If $H, H'$ are two smooth functions, put

$$\{H,H'\} = -Y_{H'}.H = Y_H.H' = -\sigma(Y_H, Y_{H'}). \tag{3.17}$$

Then

$$Y_{\{H,H'\}} = [Y_H, Y_{H'}]. \tag{3.18}$$

Let $G$ be a compact connected Lie group acting on the right on $X$, and preserving $\sigma$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Let $\eta : G \to \mathrm{Aut}(\mathfrak{g}^*)$ be the coadjoint representation. If $Y \in \mathfrak{g}, p \in \mathfrak{g}^*$, let $\{Y,p\} \in \mathfrak{g}^*$ be the infinitesimal (left) action of $Y$ on $p$.

We say that $\mu : X \to \mathfrak{g}^*$ is a moment map if

- For $x \in X, g \in G$,

$$\mu(xg) = \mu(x).g. \tag{3.19}$$

- If $Y \in \mathfrak{g}$, if $Y^X$ is the corresponding vector field on $X$, then $\langle \mu, Y\rangle$ is a Hamiltonian for $Y^X$.

In particular, by (3.17),(3.19),

$$\langle \mu, [Y,Y']\rangle = -\sigma(Y^X, Y'^X). \tag{3.20}$$

Also by (3.19),

$$Y^X\mu = -\{Y,\mu\}. \tag{3.21}$$

Assume now that $G$ acts locally freely on $X$. Of course, this never occurs if $X$ is compact. Let $\theta$ be a $G$-connection form on $\pi : X \to X/G$.

PROPOSITION 3.2. *There is a unique closed 2-form $\eta$ on $X/G$ such that*

$$\sigma = \pi^*\eta - d\langle \mu, \theta\rangle. \tag{3.22}$$

PROOF. By (3.3) , ( 3.19) , the 1-form $\langle \mu, \theta\rangle$ is $G$-invariant, so that if $Y \in \mathfrak{g}$,

$$L_{Y^X}\langle \mu, \theta\rangle = 0. \tag{3.23}$$

Also by (3.16),

$$L_{Y^X}\sigma = 0. \tag{3.24}$$

Then by (3.14), (3.23),

$$i_{Y^X}(\sigma + d\langle\mu,\theta\rangle) = i_{Y^X}\sigma + L_{Y^X}\langle\mu,\theta\rangle - di_{Y^X}\langle\mu,\theta\rangle = 0 \tag{3.25}$$

By (3.25), we find that the $G$-invariant 2-form $\sigma + d\langle\mu,\theta\rangle$ is basic. Therefore it is of the form $\pi^*\eta$. The proof of our Proposition is completed. □

Let $\sigma^H$ be the restriction of $\sigma$ to $T^HX$. Since $\sigma^H$ is $G$-invariant, it descends to a 2-form on $X/G$. The same is true for the 2-form $\langle\mu,\Theta\rangle$.

THEOREM 3.3. *The following identity of 2-forms holds on $X/G$,*

$$\sigma^H = \pi^*\eta - \langle\mu,\Theta\rangle. \tag{3.26}$$

PROOF. By (3.4),

$$d\langle\mu,\theta\rangle = \langle d\mu,\theta\rangle + \langle\mu, -\frac{1}{2}[\theta,\theta] + \Theta\rangle. \tag{3.27}$$

From (3.27), we get

$$[d\langle\mu,\theta\rangle]^H = \langle\mu,\Theta\rangle. \tag{3.28}$$

From (3.22) , (3.28), we get (3.26). □

REMARK 3.4. If $X \in \mathfrak{g}$, by (3.15),

$$(d - i_{Y^X})(\sigma + \langle\mu, Y\rangle) = 0. \tag{3.29}$$

Classically [**61**], (3.29) shows that $\sigma^H + \langle\mu,\Theta\rangle$ descends to a closed 2-form on $X/G$. Of course this also follows from Proposition 3.2 and Theorem 3.3.

REMARK 3.5. Let $\langle,\rangle$ be a $G$- invariant scalar product on $\mathfrak{g}$. Then $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified. Let $H \subset G$ be a Lie subgroup of $G$, and let $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Lie algebra. Let $\mathfrak{h}^\perp$ be the orthogonal space to $\mathfrak{h}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ is a $H$-invariant splitting. Let $P^\mathfrak{h} : \mathfrak{g} \to \mathfrak{h}$ be the corresponding projection. Put

$$\theta^\mathfrak{h} = P^\mathfrak{h}\theta,\ \theta^{\mathfrak{h}^\perp} = P^{\mathfrak{h}^\perp}\theta. \tag{3.30}$$

Then $\theta^\mathfrak{h}$ is a connection on $X \to X/H$, whose curvature $\Theta^\mathfrak{h}$ is given by

$$\Theta^\mathfrak{h} = P^\mathfrak{h}\Theta - \frac{1}{2}P^\mathfrak{h}[\theta^{\mathfrak{h}^\perp},\theta^{\mathfrak{h}^\perp}]. \tag{3.31}$$

Take $\mu \in \mathfrak{g}^*$, and suppose that $Z(\mu) = H$. Then

$$\langle\mu, \frac{1}{2}[\theta,\theta]\rangle = \langle\mu, \frac{1}{2}[\theta^{\mathfrak{h}^\perp},\theta^{\mathfrak{h}^\perp}]\rangle = \langle\mu, \frac{1}{2}P^\mathfrak{h}[\theta^{\mathfrak{h}^\perp},\theta^{\mathfrak{h}^\perp}]\rangle. \tag{3.32}$$

Therefore by (3.31), (3.32),

$$\langle\mu, -\frac{1}{2}[\theta,\theta] + \Theta\rangle = \langle\mu,\Theta^\mathfrak{h}\rangle. \tag{3.33}$$

So by (3.4), (3.22),(3.27), (3.33), if $Z(\mu) = H$,

$$\sigma = \pi^*\eta - \langle d\mu,\theta\rangle - \langle\mu,\Theta^\mathfrak{h}\rangle. \tag{3.34}$$

Equation (3.34) is of special interest when $\mu \in \mathfrak{t}^* \simeq \mathfrak{t}$, and $H = T$.

Put

$$E = X \times_G \mathfrak{g}^*. \tag{3.35}$$

Then $E$ is an orbifold vector bundle over $X/G$. Moreover (3.19) says that $\mu$ descends to a section of $E$ over $X/G$.

**3.3. Symplectic reduction.** We make the same assumptions as in Section 3.2. We do no longer assume that $G$ acts locally freely on $X$. If $p \in \mathfrak{g}^*$, put

$$X_p = \mu^{-1}(p). \tag{3.36}$$

Let $Z(p) = \{g \in G, g.p = p\}$ be the stabilizer of $p$ in $G$, and let $\mathfrak{z}(p) = \{X \in \mathfrak{g}, \{X, p\} = 0\}$ be its Lie algebra. By [15, Theorem IV.2.3], $Z(p)$ is a connected Lie subgroup of $G$. Clearly $Z(p)$ acts on $X_p$.

PROPOSITION 3.6. *The element $p \in \mathfrak{g}^*$ is a regular value of $\mu$ if and only if for $x \in X_p, Y \in \mathfrak{g} \mapsto Y^X(x) \in T_xX$ is injective, or equivalently if and only if $Z(p)$ acts locally freely on $X_p$.*

PROOF. By (3.29) if $x \in X_p$, $Y \in \mathfrak{g}$ lies in coker$(d\mu(x))$ if and only if $Y^X(x) = 0$. Also by (3.21), if $\mu(x) = p, Y^X(x) = 0$, then $\{Y, p\} = 0$, i.e. $Y \in \mathfrak{z}(p)$. The proof of our Proposition is completed. □

Assume now that $p$ is a regular value of $\mu$, and that $X_p \neq \emptyset$. Then $X_p$ is a smooth submanifold of X, on which $Z(p)$ acts locally freely, so that $X_p/Z(p)$ is a $Z(p)$-orbifold.

Let $i_p$ be the embedding $X_p \to X$. Then by (3.21), (3.29), for $Y \in \mathfrak{z}(p)$,

$$i_{Y^X} i_p^* \sigma = 0. \tag{3.37}$$

By (3.37) , we find that the $Z(p)$ -invariant closed 2-form $i_p^*\sigma$ descends to a closed 2-form $\sigma_p$ on $X_p/Z(p)$. Then $(X_p/Z(p), \sigma_p)$ is a symplectic orbifold.

If $p \in \mathfrak{g}^*$, $g \in G$ maps $X_p$ into $X_{p.g}$. Also $p \in \mathfrak{g}^*$ is regular if and only if $p.g$ is regular, and there is an obvious symplectomorphism $(X_p/Z(p), \sigma_p) \to (X_{p.g}/Z(p.g), \sigma_{p.g})$.

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. If $p \in \mathcal{O}, Y, Z \in \mathfrak{g}$, put

$$\sigma_{\mathcal{O}}(Y, Z) = \langle p, [Y, Z] \rangle. \tag{3.38}$$

By (1.193), $\sigma_{\mathcal{O}}$ is a $G$-invariant symplectic form on $\mathcal{O}$. Moreover $\mu_{\mathcal{O}} : p \in \mathcal{O} \mapsto -p \in \mathfrak{g}^*$ is a moment map for the right action of $G$ on $\mathcal{O}$.

Put

$$X_{\mathcal{O}} = \mu^{-1}(\mathcal{O}). \tag{3.39}$$

Then $G$ acts on the right on $X_{\mathcal{O}} \subset X$.

Let $p_1, p_2$ be the projections $X \times \mathcal{O} \to X, X \times \mathcal{O} \to \mathcal{O}$. Then $(X \times \mathcal{O}, p_1^*\sigma + p_2^*\sigma_{\mathcal{O}})$ is a symplectic manifold on which $G$ acts symplectically on the right, with moment map $p_1^*\mu + p_2^*\mu_{\mathcal{O}}$.Then

$$(X \times \mathcal{O})_0 = \{(x, p) \in X \times \mathcal{O}, \mu(x) - p = 0\}. \tag{3.40}$$

So, if $p \in \mathcal{O}$,

$$\begin{aligned} &(X \times \mathcal{O}_p)_0 \simeq X_{\mathcal{O}_p}, \\ &(X \times \mathcal{O}_p)_0/G \simeq X_p/Z(p). \end{aligned} \tag{3.41}$$

Moreover $p \in \mathfrak{g}^*$ is a regular value of $\mu$ if and only if 0 is a regular value of $p_1^*\mu + p_2^*\mu_{\mathcal{O}}$. Then one finds easily that (3.41) identifies the symplectic forms on the corresponding orbifolds.

**3.4. Symplectic reduction with respect to a fixed stabilizer.** Let $G$ be a compact connected semisimple Lie group. We use the same notation as in Section 1.14.

Let $\langle , \rangle$ be a $G$-invariant scalar product on $\mathfrak{g}$. So we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$, $\mathfrak{t}$ and $\mathfrak{t}^*$. Let $\eta : G \to \mathrm{Aut}(\mathfrak{g}^*)$ be the coadjoint representation. Of course we can now identify the representations $\tau$ and $\eta$.

Take $p_0 \in \mathfrak{g}^*$. Put

$$\begin{aligned} Z &= Z(p_0), \\ \mathfrak{z} &= \mathfrak{z}(p_0). \end{aligned} \tag{3.42}$$

DEFINITION 3.7. Put

$$\mathfrak{R} = \{p \in \mathfrak{t}^*, Z(p) = Z\}. \tag{3.43}$$

Since the $Z(p)$ are connected,

$$\mathfrak{R} = \{p \in \mathfrak{t}^*, \mathfrak{z}(p) = \mathfrak{z}\}. \tag{3.44}$$

By (1.182), (3.44), $\mathfrak{R}$ is the complement of a finite union of hyperplanes in $\{p \in \mathfrak{t}^*, \text{if } \langle p_0, h_\alpha\rangle = 0, \langle p, h_\alpha \rangle = 0\}$.

Now we make the same assumptions as in Section 3.2. Suppose that $G$ acts locally freely on X. Then by Proposition 3.6 , any $p \in \mathfrak{g}^*$ is a regular value of $\mu$.

DEFINITION 3.8. Put

$$\mathfrak{S} = \mu^{-1}(\mathfrak{R}). \tag{3.45}$$

Then $\mathfrak{S}$ is a submanifold of $X$, on which $Z$ acts locally freely. Let $\sigma_{\mathfrak{S}}$ be the restriction of $\sigma$ to $S$. Then $\sigma_{\mathfrak{S}}$ is a closed 2-form on $\mathfrak{S}$, which in general is not symplectic. Let $j : \mathfrak{g}^* \to \mathfrak{z}^*$ be the obvious projection. Then $j\mu : \mathfrak{S} \to \mathfrak{z}^*$ is a moment map for the action of $Z$ on $\mathfrak{S}$ with respect to $\sigma_{\mathfrak{S}}$. Finally $\mathfrak{S}/Z$ embeds into $X/G$.

Let $P$ be the orthogonal projection $\mathfrak{g} \to \mathfrak{z}$. Let $\theta$ be a connection form on $X \to X/G$. Put

$$\tilde{\theta} = P\theta. \tag{3.46}$$

Then $\tilde{\theta}$ defines a $Z$-connection on $X \to X/Z$. Let $\tilde{\Theta}$ be the curvature of $\theta$. Clearly, over $\mathfrak{S}$, since $\mu \in \mathfrak{z}$,

$$\langle \mu, \theta \rangle = \langle \mu, \tilde{\theta} \rangle. \tag{3.47}$$

By (3.34), over $\mathfrak{S}$,

$$\sigma = \pi^*\eta - (\langle d\mu, \tilde{\theta}\rangle + \langle \mu, \tilde{\Theta}\rangle). \tag{3.48}$$

In particular, if $p \in \mathfrak{R}$,

$$i_p^*\sigma = i_p^*(\pi^*\eta - \langle \mu, \tilde{\Theta}\rangle). \tag{3.49}$$

Recall that the vector bundle $E$ was defined in (3.35). Let $\tilde{\nabla}^E$ be the connection on $E_{|\mathfrak{S}/Z}$ induced by $\tilde{\theta}$. Then over $X_p/Z$,

$$\tilde{\nabla}^E \mu = 0. \tag{3.50}$$

Equation (3.50) explains why over $X_p$, $\sigma$ given by (3.49) is closed. In fact it is the pull-back of the symplectic form $\sigma_p$ on $X_p/Z(p)$.

PROPOSITION 3.9. *Locally, over $\mathfrak{S}$, the cohomology class of $\sigma_p$ depends linearly on $p \in \mathfrak{R}$.*

PROOF. This is obvious by (3.49). □

**3.5. The image of the symplectic volume by the moment map.** Let $G$ be a compact connected semisimple Lie group. We use otherwise the notation of Section 1. Let $(X, \sigma)$ be a compact symplectic manifold, on which $G$ acts symplectically, with moment map $\mu : X \to \mathfrak{g}^*$.

We make the assumption that

$$Z(x) = 1 \text{ for a.e. } x \in X. \tag{3.51}$$

Then a.e., $G$ acts locally freely on $X$. By Proposition 3.6, a.e., $d\mu(x)$ surjects on $\mathfrak{g}^*$. By Sard's theorem, a.e. $p \in \mathfrak{g}^*$ is a regular value of $\mu$.

Let $dp, dt$ be the Lebesgue measures on $\mathfrak{g}, \mathfrak{t}$ with respect to $\langle,\rangle$. Let $dg$ be the Lebesgue measure on $G$. If $p \in \mathfrak{g}$, let $dg_p$ be the Lebesgue measure on $Z(p)$.

Clearly

$$\{e^\sigma\}^{\max} = \frac{\sigma^{\dim X/2}}{(\dim X/2)!}. \tag{3.52}$$

Then $\frac{\sigma^{\dim X/2}}{(\dim X/2)!}$ is a $\dim X$ form on $X$, which does not vanish, and so defines an orientation of $X$. If $f : X \to \mathbf{R}$ is a bounded measurable function, put

$$\int_X f(x)|e^\sigma| = \int_X f(x)e^\sigma, \tag{3.53}$$

The notation (3.53) emphasizes the fact that the left-hand side is an integral with respect to a nonnegative measure. We will use a similar notation over $X_p/Z(p)$.

Now we will compute the disintegration of the symplectic volume on $X$ with respect to the moment map $\mu$. The point of this proof is that it is parallel to a corresponding for moduli spaces given in Theorem 5.45.

Recall that the monomial $\pi : \mathfrak{t} \to \mathbf{C}$ was defined in Definition 1.53.

THEOREM 3.10. *Let $f : X \to \mathbf{R}$ be a bounded measurable function. Then*

$$\begin{aligned} \int_X f(x)|e^\sigma| &= \int_{\mathfrak{t}/W} |\pi(t)|dt \int_{X_t/T} |e^{\sigma_t}| \int_G f(x.g)dg, \\ \int_X f(x)|e^\sigma| &= \int_{\mathfrak{g}} \frac{dp}{|\pi(p)|} \int_{X_p/Z(p)} |e^{\sigma_p}| \int_{Z(p)} f(x.g)dg_p. \end{aligned} \tag{3.54}$$

PROOF. Since $d\mu(x)$ is a.e. surjective, it is clear that $\mu_* \frac{\sigma^{\dim X/2}}{(\dim X/2)!}$ is absolutely continuous with respect to $dp$. Also for a.e. $x$, $d\mu(x)$ is surjective and $\mu(x) \in \mathfrak{g}_{\text{reg}}$.

Let $K \subset \mathfrak{t}$ be a Weyl chamber. Let $q$ be the projection $\mathfrak{g}_{\text{reg}} \to \mathfrak{t}_{\text{reg}}/W \simeq K$. If $x \in X$ is such that $\mu(x) \in \mathfrak{g}_{\text{reg}}$, we have a $G$-equivariant complex

$$(C'_x, \partial) : 0 \to \mathfrak{g} \to T_x X \overset{dq\mu}{\to} \mathfrak{t} \to 0, \tag{3.55}$$

where the map $\mathfrak{g} \to T_x X$ is just $Y \to Y^X(x)$. If $d\mu(x)$ is surjective, by Proposition 3.6, the cohomology of the complex $(C'_x, \partial)$ is concentrated in degree 1. More precisely, one finds easily that in this case,

$$H^1(C'_x, \partial) \simeq T_{\pi(x)} X_{q\mu(x)}/Z(q\mu(x)). \tag{3.56}$$

In particular, by **[35]** and by (3.55), (3.56), we have a canonical isomorphism of real lines

$$(\det C'_x)^{-1} \simeq \det(T_{\pi(x)} X_{q\mu(x)}/Z(q\mu(x))). \tag{3.57}$$

Now $\det(T_x X)$ is equipped with the volume form associated to $\sigma_x$.Also $\mathfrak{g}$ and $\mathfrak{t}$ are equipped with the volume forms $dp$ and $dt$. Therefore $\det(C'_x)$ is equipped with a natural metric. Let $dv_{X_{q\mu(x)}/Z(q\mu(x))}$ be the corresponding volume form on $T_{\pi(x)} X_{q\mu(x)}/Z(q\mu(x))$ via the isomorphism (3.57).

By the formula of change of variables,we get

$$\int_X f(x)|e^\sigma| = \int_{\mathfrak{t}_{\mathrm{reg}}/W} dt \int_{X_t/T} dv_{X_t/T}(x) \int_G f(x.g)dg. \tag{3.58}$$

Take $t \in \mathfrak{t}_{\mathrm{reg}}, x \in X$ such that $\mu(x) = t$. Consider the double complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & T_x X & \xrightarrow{d(q\mu)} & \mathfrak{t} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & T_x X \oplus \mathrm{Im}\,\mathrm{Ad}(t) & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Im}\,\mathrm{Ad}(t) & \longrightarrow & \mathrm{Im}\,\mathrm{Ad}(t) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.59}$$

In (3.59) , the map $\mathfrak{g} \to \mathrm{Im}\,\mathrm{Ad}(t)$ is just $f \mapsto -\mathrm{Ad}(t)f = -[t, f]$, the map $T_x X \to \mathfrak{g}$ is $d\mu(x)$. Also the columns and the lowest row are acyclic. In particular the cohomology groups of the two upper rows are isomorphic. Therefore they are concentrated in degree 1 and equal to $H^1(C'_x, \partial)$.

In the second column of (3.59), we equip $\mathrm{Im}\,\mathrm{Ad}(t) \simeq T_t \mathcal{O}_t$ with the 2-form $\sigma_{\mathcal{O}_t}$ given in (3.38). In the third column of (3.59), we equip $\mathrm{Im}\,\mathrm{Ad}(t) \simeq T_t \mathcal{O}_t$ with the metric induced by the metric of $\mathfrak{g}$. By (3.22), (3.27), the volume induced by the second row on its determinant is just the symplectic volume $\frac{\sigma^{\dim(X_t/T)/2}}{(\dim(X_t/T)/2)!}$ on $\det H^1(C'_x, \partial)$. On the other hand, let $R$ be the lowest row in (3.59). Since it is acyclic, $\det R \simeq \mathbf{R}$ has a canonical section $\eta$. Clearly

$$|\eta| = |\pi(t)|. \tag{3.60}$$

Therefore

$$dv_{X_t/T} = |\pi(t)| \frac{|\sigma_t^{\dim(X_t/T)/2}|}{(\dim(X_t/T)/2)!}. \tag{3.61}$$

By (3.58), (3.61), we get the first identity in (3.54) .

By **[14**, Proposition 6.3.4**]**, if $h : \mathfrak{g} \to \mathbf{R}$ is a bounded measurable function,

$$\int_{\mathfrak{g}} h(p)dp = \frac{1}{\mathrm{Vol}(T)} \int_{\mathfrak{t}/W} |\pi(t)|^2 dt \int_G h(t.g)dg. \tag{3.62}$$

From the first identity in (3.54) and from (3.62), we obtain

$$(3.63)\quad \int_X f(x)|e^\sigma| = \int_{\mathfrak{t}/W} |\pi(t)|dt \int_{X_t/T} |e^{\sigma_t}| \int_{T\backslash G} d\,\dot{g} \int_T f(x.t'\,\dot{g})dt' = \int_{\mathfrak{g}} \frac{dp}{|\pi(p)|} \int_{X_p/Z(p)} |e^{\sigma_p}| \int_{Z(p)} f(x.g)dg_p,$$

which is the second identity in (3.54). The proof of our Theorem is completed. □

DEFINITION 3.11. If $t \in \mathfrak{t}$ is a regular value of $\mu$, let $|V(t)|$ be the absolute value of the symplectic volume of $X_t/Z(t)$ with respect to $\sigma_t$.

THEOREM 3.12. *Let $f : \mathfrak{g}^* \to \mathbf{R}$ be a bounded measurable function. Then*

$$(3.64)\quad \begin{aligned} \int_X f(\mu(x))|e^\sigma| &= \int_{\mathfrak{t}/W} \left[\int_G f(t.g)dg\right] |\pi(t)||V(t)|dt, \\ \int_X f(\mu(x))|e^\sigma| &= \mathrm{Vol}(T) \int_{\mathfrak{g}^*} f(p)\frac{|V(p)|}{|\pi(p)|}dp. \end{aligned}$$

PROOF. Our Theorem follows from Theorem 3.10. □

Clearly, if 0 is a regular value of $\mu$, $\mu_*|e^\sigma|$ has a smooth density with respect to $dp$ near $p = 0$. In particular

$$(3.65)\quad \lim_{\substack{p\in\mathfrak{g}_{reg} \\ p\to 0}} \frac{|V(p)|}{|\pi(p)|}$$

exists and is the value of the above density at $p = 0$. A more precise statement is as follows.

PROPOSITION 3.13. *If $0 \in \mathfrak{g}^*$ is a regular value of $\mu$, and if*

$$(3.66)\quad Z(x) = 1 \,\text{a.e. on}\, X_0,$$

*then*

$$(3.67)\quad \lim_{\substack{p\in\mathfrak{g}_{reg} \\ p\to 0}} \frac{|V(p)|}{|\pi(p)|} = \frac{\mathrm{Vol}(G)}{\mathrm{Vol}(T)}|V(0)|.$$

PROOF. By 3.12, one gets (3.67) easily. □

**3.6. The symplectic volume as a polynomial near the origin.** In the sequel, we will assume that 0 is a regular value of $\mu$, and also that

$$(3.68)\quad Z(x) = 1 \,\text{a.e. on}\, X_0.$$

Let U be a $W$-invariant open neighborhood of 0 in $\mathfrak{t}^*$ consisting of regular values of $\mu$. Let $\mathfrak{g}^*_U \subset \mathfrak{g}^*$ be the corresponding union of coadjoint orbits. Then $\mathfrak{g}^*_U$ is an open neighborhood of 0 in $\mathfrak{g}$.

Classically, $U$ is small enough, there is a $G$-invariant open neighborhood $V$ of $X_0$ in $X$ such that we have the identification of $G$-spaces

$$(3.69)\quad V \simeq X_0 \times \mathfrak{g}^*_U,$$

so that in the right hand-side of (3.69), $\mu$ is just the projection $M \times \mathfrak{g}^*_U \to \mathfrak{g}^*_U$.

By (3.69), if $p \in U$,

$$(3.70)\quad X_{\mathcal{O}_p} \simeq X_0 \times \mathcal{O}_p,$$

so that

$$X_{\mathcal{O}_p}/G \simeq X_0 \times_G \mathcal{O}_p \simeq X_0/Z(p) \tag{3.71}$$

is a $\mathcal{O}_p$-fibre bundle over the orbifold $X_0/G$.

Now we construct an orientation over $X_0/T$. In fact if $t \in T_{\mathrm{reg}}$, then $\mathcal{O}_t \simeq G/T$, and so

$$X_0/T = X_0 \times_G G/T. \tag{3.72}$$

Let $K \subset \mathfrak{t}$ be a Weyl chamber, and let $t \in K$. Then the symplectic form $\sigma_{\mathcal{O}_t}$ orients $G/T \simeq \mathcal{O}_t$, and the corresponding orientation on $G/T$ does not depend on $t \in K$. Also $X_0/G$ carries the symplectic form $\sigma_0$. Therefore once $K$ is fixed, $X_0/T$ carries a canonical orientation.

Take $t \in U$. Let $i_t : X_t \to X$ be the obvious embedding. Then by (3.37), $i_t^*\sigma$ descends to a closed 2-form on $X_t/T$, which we denote $\bar{\sigma}_t$. Note that if $t \in \mathfrak{t}_{\mathrm{reg}}$, $\sigma_t = \bar{\sigma}_t$. If $t \notin \mathfrak{t}_{\mathrm{reg}}$, $T \subset Z(t), T \neq Z(t)$, and $\bar{\sigma}_t$ is in general not symplectic.

By (3.69),

$$X_t \simeq X_0 \times \{t\}. \tag{3.73}$$

We equip $X_t/T \simeq X_0/T$ with the given fixed orientation.

DEFINITION 3.14. Put

$$P(t) = \int_{X_t/T} e^{\bar{\sigma}_t}. \tag{3.74}$$

Then $P(t)$ is a smooth function on $U$.

THEOREM 3.15. *One has the identity*

$$\begin{aligned} |P(t)| &= |V(t)| \text{ if } t \in \mathfrak{t}_{\mathrm{reg}}, \\ &= 0 \text{ if } t \notin \mathfrak{t}_{\mathrm{reg}}. \end{aligned} \tag{3.75}$$

*If $w \in W, t \in U$,*

$$P(wt) = \epsilon_w P(t). \tag{3.76}$$

*Also, near 0, $P(t)$ is a polynomial. More precisely, if $\theta$ is a connection form on $\pi : X_0 \to X_0/G$, and if $\Theta$ is its curvature,*

$$P(t) = \int_{X_0/T} \exp\left(\pi^*\sigma_0 - \langle t, \Theta\rangle + \langle t, \frac{1}{2}[\theta,\theta]\rangle\right). \tag{3.77}$$

*Near 0, $P(t)$ and $\pi(t/i)$ either vanish together or are nonzero, and then they have the same sign. In particular*

$$\pi(t/i)P(t) \geq 0 \text{ near } 0. \tag{3.78}$$

PROOF. By the considerations we made after (3.72), (3.75) holds. If $w \in W$, let $g \in N(T)$ represent $w \in W = N(T)/T$. Since $G$ is connected, it acts on $X_0, \mathfrak{g}, \mathfrak{g}^*$ by orientation preserving maps. Clearly

$$TX_0 = \pi^*(TX_0/G) \oplus \mathfrak{g}^X. \tag{3.79}$$

Also

$$\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{t} \otimes_{\mathbf{R}} \mathbf{C} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\right), \tag{3.80}$$

so that

$$(\mathfrak{g}/\mathfrak{t}) \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha. \tag{3.81}$$

Then $(\mathfrak{g}/\mathfrak{t}) \otimes_{\mathbf{R}} \mathbf{C}$ descends to the complexified tangent space to the fibre $G/T$ over $X_0/G$. Since $g \in N(T)$, $g$ preserves $\mathfrak{t}$ and its orthogonal $\mathfrak{t}^\perp$ in $\mathfrak{g}$. Since $g$ changes the orientation of $\mathfrak{t}$ by the factor $\epsilon_w$, it changes the orientation of $\mathfrak{g}/\mathfrak{t}$ by the same factor.

So $g$ changes the orientation of $X_0/T$ by the factor $\epsilon_w$. Since $\sigma$ is $G$-invariant, and $g$ acts on $X_0/T$,

$$\bar{\sigma}_{gt} = \bar{\sigma}_t. \tag{3.82}$$

From the above, we get (3.76).

By (3.22),(3.27),

$$\sigma = \pi^*\eta - \langle d\mu, \theta \rangle - \langle \mu, \Theta - \frac{1}{2}[\theta, \theta] \rangle. \tag{3.83}$$

Clearly $i_t^* \pi^* \eta$ is cohomologous to $i_0^* \pi^* \eta = \pi^* \sigma_0$. So by (3.83), we get (3.77). From (3.77), it is clear that $P(t)$ is a polynomial near 0.

If $t \in U \backslash \mathfrak{t}_{\text{reg}}$, by the considerations after (3.72), $\bar{\sigma}_t$ is an everywhere degenerate 2-form on $X_t/T$. Therefore

$$P(t) = 0 \text{ on } U \backslash \mathfrak{t}_{\text{reg}}. \tag{3.84}$$

For $t \in U \cap \mathfrak{t}_{\text{reg}}$, $\bar{\sigma}_t$ is a symplectic 2-form on $X_t/T$, so that

$$P(t) \neq 0 \text{ on } U \cap \mathfrak{t}_{\text{reg}}. \tag{3.85}$$

By (1.190), (3.85), $P(t)$ and $\pi(t/i)$ have the same zeroes on $U$.

By (3.38), on $X_0/T$, the form $\langle t, \frac{1}{2}[\theta, \theta] \rangle$ is just the symplectic 2-form along the fibre $G/T \simeq \mathcal{O}_t$. If $t \in K \cap U$ is close to 0, by (3.72), (3.77), $P(t) > 0$. Therefore, on $K \cap U$, $P(t)$ and $\pi(t/i)$ have the same sign. Using (1.186) and (3.76), we get the end of our Theorem, the proof of which is now completed. □

REMARK 3.16. Recall that we have a fibration $X_0/T \overset{G/T}{\to} X_0/G$. Then we can rewrite (3.77) in the form

$$P(t) = \int_{X_0/G} e^{\sigma_0} \int_{G/T} e^{-\langle t, \Theta \rangle + \langle t, \frac{1}{2}[\theta, \theta] \rangle}. \tag{3.86}$$

In (3.86), $\int_{G/T} \cdots$ is an integral along the fibre. In Remark 3.26, we will reinterpret identity (3.86).

**3.7. A formula of Witten-Jeffrey-Kirwan.** We make the same assumptions as in Section 3.6. Also we suppose the set $U$ to be bounded. Let $\langle , \rangle$ be a $G$-invariant scalar product on $\mathfrak{g}$. If $f : \mathfrak{g}^* \to \mathbf{R}$ is a bounded measurable function, let $Mf : \mathfrak{g}^* \to \mathbf{R}$ be given by

$$Mf(p) = \frac{1}{\text{Vol}(G)} \int_G f(p.g) dg. \tag{3.87}$$

Recall that $\mathfrak{g}$ and $\mathfrak{g}^*$ have been identified by $\langle , \rangle$, so that (3.87) also makes sense when $f : \mathfrak{g} \to \mathbf{R}$ is bounded and measurable.

DEFINITION 3.17. If $f : \mathfrak{g}^* \to \mathbf{R}$ is a bounded measurable function with support in $\mathfrak{g}^*_U$, for $u \in \mathfrak{g}$, put

$$\widehat{f}_{\mathfrak{g}}(u) = \int_{\mathfrak{g}^*} f(p) e^{\langle u,p \rangle} dp. \tag{3.88}$$

Clearly

$$M\widehat{f}_{\mathfrak{g}} = \widehat{Mf}_{\mathfrak{g}}. \tag{3.89}$$

Also $\widehat{f}_{\mathfrak{g}}(u)$ is an analytic function of $u$.

If $f : \mathfrak{t}^* \to \mathbf{R}$ is a bounded measurable function with support in $U$, for $t \in \mathfrak{t}$, set

$$\widehat{f}_{\mathfrak{t}}(t) = \int_{\mathfrak{t}^*} f(p) e^{\langle t,p \rangle} dp. \tag{3.90}$$

If $f$ is $W$-invariant, then $\widehat{f}_{\mathfrak{t}}$ is also $W$-invariant.

The maps $f \mapsto f_{\mathfrak{g}}$ and $f \mapsto f_{\mathfrak{t}}$ are injective.

Also by Rossmann's formula [**50**], [**60**], if $f : \mathfrak{g}^* \to \mathbf{R}$ is taken as before and is $G$-invariant, then,

$$\widehat{\pi f}_{\mathfrak{t}} = \pi(\frac{\cdot}{2\pi}) \widehat{f}_{\mathfrak{g}}. \tag{3.91}$$

In the sequel, we orient $X$ by the symplectic form $\sigma$. Now we prove a formula related to a formula of Jeffrey-Kirwan [**28**] in a work where they prove a formula by Witten [**64**]. Our approach is closely related to Vergne [**61**] and especially to Liu [**39, 40**] who worked out similar formulas in the context of moduli spaces.

THEOREM 3.18. *Let $f : \mathfrak{g}^* \to \mathbf{R}$ be a bounded measurable function with support in $\mathfrak{g}^*_U$. Let $\theta$ be a connection form on $X_0 \to X_0/G$, and let $\Theta$ be its curvature. Then*

$$\int_X f(\mu) e^{\sigma} = \mathrm{Vol}(G) \int_{X_0/G} \widehat{Mf}_{\mathfrak{g}}(-\Theta) e^{\sigma_0}. \tag{3.92}$$

PROOF. By Theorem 3.12,

$$\int_X f(\mu) e^{\sigma} = \mathrm{Vol}(G) \int_{\mathfrak{t}/W} Mf(t) |\pi(t)| |V(t)| dt. \tag{3.93}$$

By Theorem 3.15,

$$|\pi(t)| |V(t)| = \pi(t/i) P(t). \tag{3.94}$$

So (3.93) can be rewritten in the form

$$\int_X f(\mu) e^{\sigma} = \mathrm{Vol}(G) \int_{\mathfrak{t}/W} (Mf)(t) \pi(t/i) P(t) dt. \tag{3.95}$$

By (3.77),

$$P(t) = \int_{X_0/T} e^{\pi^* \sigma_0 - \langle t, -\frac{1}{2}[\theta,\theta] + \Theta \rangle}. \tag{3.96}$$

It follows from the above that we may as well assume that over $X_0 \times \mathfrak{g}^*$, $\sigma$ is given by

$$\sigma = \pi^* \sigma_0 - d\langle p, \theta \rangle. \tag{3.97}$$

Equivalently,

$$\sigma = \pi^*\sigma_0 - \langle dp, \theta\rangle - \langle p, -\frac{1}{2}[\theta,\theta] + \Theta\rangle. \tag{3.98}$$

Then

$$\int_X f(\mu)e^\sigma = \int_{X_0\times\mathfrak{g}^*} f(p)e^{\pi^*\sigma_0 - \langle p,\Theta\rangle - \langle dp,\theta\rangle + \frac{1}{2}\langle p,[\theta,\theta]\rangle}. \tag{3.99}$$

Since only the term of top degree in $\Lambda(\mathfrak{g}^*)$ contributes to the integral in the right hand-side of (3.99), we can rewrite (3.99) in the form

$$\int_X f(\mu)e^\sigma = \int_{X_0\times\mathfrak{g}^*} f(p)e^{\pi^*\sigma_0 - \langle p,\Theta\rangle - \langle dp,\theta\rangle}. \tag{3.100}$$

Now recall that $X$ is oriented by $\sigma$, and $X_0/G$ by $\sigma_0$. In particular, near $X_0 \subset X_0\times\mathfrak{g}^*$, $X_0\times\mathfrak{g}^*$ is oriented by $\sigma$. Also $\Theta$ is $G$-equivariant. From (3.100), we get

$$\int_X f(\mu)e^\sigma = \int_{X_0/G} e^{\sigma_0}\int_{\mathfrak{g}^*} e^{-\langle p,\Theta\rangle}\left[\int_G f(p.g)dg\right]dp, \tag{3.101}$$

which coincides with (3.92). The proof of our Theorem is completed. □

REMARK 3.19. In [**28**], Jeffrey-Kirwan use instead the coisotropic embedding theorem [Theorem 39.2] [**24**], which asserts there is a G-equivariant identification $V \simeq X_0\times\mathfrak{g}^*_U$, so that $\sigma$ is exactly given by

$$\sigma = p_1^*\sigma_0 - d\langle p,\theta\rangle. \tag{3.102}$$

Recall that by Theorem 3.15, $P(t)$ is a polynomial near $t=0$. Therefore $P(\frac{\partial}{\partial t})$ is a differential operator.

THEOREM 3.20. *If $f:\mathfrak{t}\to\mathbf{R}$ is a bounded measurable $W$-invariant function with support in $U$, then*

$$\int_X f(\mu)e^\sigma = \frac{\mathrm{Vol}(G)}{|W|}\left[P(\frac{\partial}{\partial t})\widehat{\pi(t/i)f_{|\mathfrak{t}}}(t)\right]_{|t=0}. \tag{3.103}$$

PROOF. By (3.95),

$$\int_X f(\mu)e^\sigma = \frac{\mathrm{Vol}(G)}{|W|}\int_{\mathfrak{t}} f(t)\pi(t/i)P(t)dt. \tag{3.104}$$

From (3.90), (3.104), we get (3.103). The proof of our Theorem is completed. □

Let $\theta$ be a $G$-connection form on $X_0 \to X_0/G$, and let $\Theta$ be its curvature. Let $Q$ be a $G$- invariant $C^\infty$ function defined on $\mathfrak{g}$ with values in $\mathbf{R}$. We define $Q(-\Theta)$ by its Taylor expansion, which only contains a finite number of terms. Then $Q(-\Theta)$ is a closed form on $X_0/G$, and $\int_{X_0/G} Q(-\Theta)e^{\sigma_0}$ does not depend on $\theta$.

Similarly, we define the differential operator $Q(\frac{\partial}{\partial t})$ as the formal power series associated to the Taylor expansion of $Q$. When applying the power series $Q(\frac{\partial}{\partial t})\pi(\frac{\partial/\partial t}{2i\pi})$ to the polynomial $P(t)$, only a finite number of terms in the Taylor expansion contribute, so that $Q(\frac{\partial}{\partial t})\pi(\frac{\partial/\partial t}{2i\pi})P(t)$ is a well-defined polynomial.

THEOREM 3.21. *The following identity holds,*

$$\int_{X_0/G} Q(-\Theta)e^{\sigma_0} = \frac{1}{|W|}\left[Q(\partial/\partial t)\pi(\frac{\partial/\partial t}{2i\pi})P(t)\right]_{|t=0}. \tag{3.105}$$

PROOF. Let $f : \mathfrak{g}^* \to \mathbf{R}$ be a $G$- invariant bounded measurable function with support in $\mathfrak{g}_U^*$. By Theorems 3.18 and 3.20,

$$\int_{X_0/G} \widehat{f}_{\mathfrak{g}}(-\Theta)e^{\sigma_0} = \frac{1}{|W|}\left[P(\partial/\partial t)\widehat{\pi(./i)f_{|\mathfrak{t}}}\right](0). \tag{3.106}$$

By Rossmann's formula (3.91) and by (3.106), we obtain

$$\int_{X_0/G} \widehat{f}_{\mathfrak{g}}(-\Theta)e^{\sigma_0} = \frac{1}{|W|}\left[P(\partial/\partial t)\pi(\frac{t}{2i\pi})\widehat{f}_{\mathfrak{g}}(t)\right](0). \tag{3.107}$$

Equivalently

$$\int_{X_0/G} \widehat{f}_{\mathfrak{g}}(-\Theta)e^{\sigma_0} = \frac{1}{|W|}\left[\widehat{f}_{\mathfrak{g}}(\partial/\partial t)\pi(\frac{\partial/\partial t}{2i\pi})P(t)\right](0). \tag{3.108}$$

Then (3.108) is exactly (3.105) when $Q = \widehat{f}_{\mathfrak{g}}$.

Clearly, it is enough to verify (3.105) when $Q$ is a polynomial. Then $Q$ is the Fourier transform of a distribution whose support is $\{0\}$. Using (3.108) for $f$ with support in $\mathfrak{g}_U^*$, and a simple limit procedure, we get (3.105) when $Q$ is a polynomial. The proof of our Theorem is completed. □

**3.8. The volume of symplectic coadjoint orbits.** Let $t \in \mathfrak{t}$, and let $\mathcal{O}_t \subset \mathfrak{g} = \mathfrak{g}^*$ be the $G$-orbit of $t$. Recall that $\sigma_{\mathcal{O}_t}$ is the canonical symplectic form on $\mathcal{O}_t$ given in 1.193.

Let $p_t : G/T \to \mathcal{O}_t$ be given by

$$p_t g = g.t. \tag{3.109}$$

Then $p_t$ is one to one if and only if $t \in \mathfrak{t}_{\text{reg}}$. If $t \notin \mathfrak{t}_{\text{reg}}$,

$$\dim(G/T) > \dim \mathcal{O}_t. \tag{3.110}$$

We fix a positive Weyl chamber $K \subset \mathfrak{t}$. This defines an orientation on $G/T$, which is fixed once and for all.

DEFINITION 3.22. If $t \in \mathfrak{t}$, $X \in \mathfrak{g}$ , put

$$H(t,X) = \int_{G/T} e^{\langle p_t x, X\rangle + p_t^* \sigma_{\mathcal{O}_t}}. \tag{3.111}$$

Then since $G$ acts on the right on $\mathcal{O}_t$ and preserves $\sigma_{\mathcal{O}_t}$, $H(t,X)$ is a $G$-invariant function of $X \in \mathfrak{g}^*$.

Let $K \subset \mathfrak{t}$ be a Weyl chamber, and let $\pi(t)$ be the corresponding function on $\mathfrak{t}$ defined in (1.185).

PROPOSITION 3.23. *If $t \notin \mathfrak{t}_{\text{reg}}$,*

$$H(t,X) = 0. \tag{3.112}$$

*If $t \in \overline{K}, X \in \mathfrak{t}_{\text{reg}}$, then*

$$H(t,X) = \frac{1}{\pi(X/2i\pi)} \sum_{w\in W} \epsilon_w e^{\langle wt, X\rangle}. \tag{3.113}$$

PROOF. Using (3.110), we get (3.112). Recall that $G$ acts on the right on $\mathcal{O}_t$. Also if $X \in \mathfrak{g}$,

$$d\langle p, X\rangle + i_{X^{\mathcal{O}_t}} \sigma_{\mathcal{O}_t} = 0 \tag{3.114}$$

Then when $t \in K$, we get (3.113) from the localization formula in equivariant cohomology of Duistermaat-Heckman[19], Berline-Vergne [5], [6, Theorem 7.11]. This equality obviously extends to the case where $t \in \bar{K}$. The proof of our Proposition is completed. □

REMARK 3.24. From (3.112), (3.113), we recover the well-known fact [15, Lemma VI.1.2] that if $t \notin \mathfrak{t}_{\text{reg}}$,

$$\sum_{w \in W} \epsilon_w e^{\langle wt, \cdot \rangle} = 0. \tag{3.115}$$

Recall that $H(t,X)$ is a $G$- invariant function of $X \in \mathfrak{g}^*$. Also the function $P(t)$ was defined in Definition 3.14.

PROPOSITION 3.25. *For $t_0 \in U \cap \bar{K}$,*

$$P(t_0) = \frac{1}{|W|} \left[ H(t_0, \partial/\partial t) \pi \left( \frac{\partial/\partial t}{2i\pi} \right) P(t) \right]_{|t=0}. \tag{3.116}$$

PROOF. By (3.113), if $X \in \mathfrak{t}_{\text{reg}}$,

$$H(t_0, X)\pi(\frac{X}{2i\pi}) = \sum_{w \in W} \epsilon_w e^{\langle wt_0, X \rangle}. \tag{3.117}$$

Of course the identity extends to arbitrary $X \in \mathfrak{t}$. In particular, by (3.117), we have the identity of formal power series of differential operators,

$$H(t_0, \partial/\partial t)\pi(\frac{\partial/\partial t}{2i\pi}) = \sum_{w \in W} \epsilon_w e^{\langle wt_0, \partial/\partial t \rangle}. \tag{3.118}$$

Now in view of (3.76),

$$\sum_{w \in W} \epsilon_w e^{\langle wt_0, \partial/\partial t \rangle} P(t)_{|t=0} = |W| P(t_0). \tag{3.119}$$

From (3.118), (3.119), we get (3.116). The proof of our Proposition is completed. □

REMARK 3.26. In view of (3.105), (3.116), we get

$$P(t_0) = \int_{X_0/G} H(t_0, -\Theta) e^{\sigma_0} \tag{3.120}$$

One then verifies that (3.120) is just a reformulation of (3.77).

Recall that we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$, $\mathfrak{t}$ and $\mathfrak{t}^*$ by the scalar product $\langle , \rangle$. For $t \in \mathfrak{t} \simeq \mathfrak{t}^*$, the orbit $\mathcal{O}_t$ is equipped with the symplectic form $\sigma_{\mathcal{O}_t}$.

THEOREM 3.27. *The following identity holds*

$$\left| \int_{G/T} e^{p_t^* \sigma_{\mathcal{O}_t}} \right| = \frac{\text{Vol}(G)}{\text{Vol}(T)} |\pi(t)|, \tag{3.121}$$

$$\frac{\text{Vol}(G)}{\text{Vol}(T)} \frac{1}{|W|} \pi(\frac{\partial/\partial t}{2i\pi}) \pi(t/i) = 1.$$

PROOF. If $t \in \mathfrak{t}_{\text{reg}}$, then $\mathcal{O}_t \simeq G/T$. So the first identity in (3.121) is trivial.

Let $T^*G$ be the cotangent bundle of $G$. We identify $X = T^*G$ to $G \times \mathfrak{g}^*$ via the left action of $G$ on $T^*G$. If $\theta$ is the canonical left-invariant 1-form on G with values in $\mathfrak{g}$, if $p \in \mathfrak{g}^*$, $\langle p, \theta\rangle$ is the canonical real 1-form on $T^*G \simeq G \times \mathfrak{g}^*$, and $-d\langle p, \theta\rangle$ is the canonical symplectic form $\sigma$ on $T^*G$. Also $G$ acts on the right on $T^*G$ and preserves $\sigma$. Then $(g,p) \in G \times \mathfrak{g}^* \mapsto p \in \mathfrak{g}^*$ is a moment map for this action.

Now we use the notation of Section 3.7. Take $t \in \mathfrak{t}$. Then

$$X_t = G \times \{t\}, \tag{3.122}$$

and $X_t/Z(t) \simeq G/Z(t)$ can be identified with $\mathcal{O}_t$ by $g \in G/Z(t) \mapsto g.(t) \in \mathcal{O}_t$. Then one verifies easily that the symplectic form $\sigma_t$ on $X_t/Z(t)$ is just $\sigma_{\mathcal{O}_t}$. By the first identity in (3.121), we get

$$\int_{X_t/T} e^{\tilde{\sigma}_t} = \frac{\text{Vol}(G)}{\text{Vol}(T)}\pi(t/i), \tag{3.123}$$

so that

$$P(t) = \frac{\text{Vol}(G)}{\text{Vol}(T)}\pi(t/i). \tag{3.124}$$

Finally observe that the moment map $\mu : (g,p) \in G \times \mathfrak{g}^* \mapsto p \in \mathfrak{g}^*$ is regular, that $G$ acts freely on $X_0 \simeq G$, and $X_0/G = 1$. We apply Theorem 3.21 with $Q = 1$, use (3.124), and we get

$$\frac{\text{Vol}(G)}{\text{Vol}(T)}\frac{1}{|W|}\left[\pi(\frac{\partial/\partial t}{2i\pi})\pi(t/i)\right]_{|t=0} = 1. \tag{3.125}$$

Also $\pi(\frac{\partial/\partial t}{2i\pi})\pi(t/i)$ is constant. It is now clear that the second equation in 3.121) follows from (3.125). The proof of our Theorem is completed. □

REMARK 3.28. It is of some interest to verify that as should be the case, the right-hand side in the second equation in (3.121) is positive. In fact

$$\pi(\frac{\partial/\partial t}{2i\pi})\pi(t/i) = (2\pi)^\ell \sum_{\sigma\in S_\ell}\langle\alpha_1, \alpha_{\sigma(1)}\rangle \dots \langle\alpha_\ell, \alpha_{\sigma(\ell)}\rangle. \tag{3.126}$$

Put

$$(\alpha_1 \otimes \dots \otimes \alpha_\ell)^\sigma = \frac{1}{l!}\sum_{\sigma\in S_\ell}\alpha_{\sigma(1)} \otimes \dots \alpha_{\sigma(\ell)} \in S^\ell\mathfrak{t}^*. \tag{3.127}$$

Then (3.126) can be written as

$$\pi(\frac{\partial/\partial t}{2i\pi})\pi(t/i) = (2\pi)^\ell\ell!\|(\alpha_1 \otimes \dots \otimes \alpha_\ell)^\sigma\|^2_{S^\ell\mathfrak{t}^*}, \tag{3.128}$$

which is indeed positive.

Recall that $\rho \in K$, so that $\pi(\rho/i) > 0$. We now recover a well-known formula for $\text{Vol}(G/T)$ [**6**, Corollary 7.27].

THEOREM 3.29. *The following identity holds*

$$\frac{\text{Vol}(G)}{\text{Vol}(T)} = \frac{1}{\pi(\rho/i)}. \tag{3.129}$$

PROOF. We proceed as in [6]. Let $\lambda \in \Lambda \cap \overline{K}$. Let $\chi_\lambda$ be the character of highest weight $\lambda$. Then by Kirillov's formula [34], [5], [6, Theorem 8.4], for $t \in \mathfrak{t}$, $|t|$ small enough,

$$\chi_\lambda(e^t) = \frac{\pi(t)}{\sigma(t)} \int_{\mathcal{O}_{\rho+\lambda}} e^{2i\pi\langle\mu,t\rangle + \sigma_{\mathcal{O}_{\rho+\lambda}}}. \tag{3.130}$$

In particular

$$\chi_\lambda(1) = \int_{\mathcal{O}_{\rho+\lambda}} e^{\sigma_{\mathcal{O}_{\rho+\lambda}}}. \tag{3.131}$$

By Theorem 3.27,

$$\int_{\mathcal{O}_{\rho+\lambda}} e^{\sigma_{\mathcal{O}_{\rho+\lambda}}} = \frac{\mathrm{Vol}(G)}{\mathrm{Vol}(T)} \pi((\rho+\lambda)/i). \tag{3.132}$$

Also by Weyl's dimension formula [15, Theorem VI.1.7],

$$\chi_\lambda(1) = \frac{\pi(\rho+\lambda)}{\pi(\rho)}. \tag{3.133}$$

From (3.131)-(3.133), we get (3.129). The proof of our Theorem is completed. ☐

## 4. The affine space of connections

In this Section, we construct a canonical line bundle $L$ on the affine space of connections on the trivial $G$-bundle on a Riemann surface $\Sigma$ with marked points, and an action of a central extension of the gauge group $\Sigma G$ on $L$. We compute the action of the stabilizers of certain connections on the line bundle $L$. We also construct a line bundle $\lambda_p$, which, as we shall see in Section 6.3, will descend to the moduli space of flat $G$-bundles. The main purpose of this Section is to evaluate the angles of the action of the stabilizers on the line bundle $\lambda_p$, in order to apply the theorem of Riemann-Roch-Kawasaki [**32, 33**] to this moduli space, which we will do in Section 6.

Our Section is organized as follows. In Section 4.1, we briefly recall the construction of the central extension $\widetilde{LG}$ of the loop space $LG$. In Section 4.2, we consider the coadjoint orbits of $\widetilde{LG}$, their symplectic form, and the corresponding line bundles. In Section 4.3, we construct a central extension of $(LG)^s$. In Section 4.4, we give a formula for the holonomy of the canonical connection on the $S^1$-bundle $\widetilde{LG} \overset{S^1}{\to} LG$. In Section 4.5, we describe the symplectic affine manifold $\mathcal{A}$ of $G$-connections on $\Sigma$, and a symplectic action of a central extension $\widetilde{\Sigma G}$ of the gauge group $\Sigma G$. In Section 4.6, we construct the line bundle $L$ on $\mathcal{A}$, and an action of $\widetilde{\Sigma G}$ on $L$. In Section 4.7, when $G$ is not simply connected, we classify the $G$-bundles on $\Sigma$. In Section 4.8, we specialize the results of Section 4.7 when $H$ is a connected subgroup of a simply connected group $G$. In Section 4.9, we compute the action of certain elements of $\widetilde{\Sigma G}$ on $L$. Finally, in Section 4.10, we define the line bundle $\lambda_p$ , on which $\Sigma G$ acts, and we compute the action of certain elements of $\Sigma G$ on $\lambda_p$.

**4.1. The central extension of the loop group $LG$.** Let $G$ be a compact connected and simply connected simple Lie group. We will use the notation of Section 1. In particular $\langle , \rangle$ denotes the basic scalar product on $\mathfrak{g}$.

Let $T$ be a maximal torus in $G$, let $\mathfrak{t} \subset \mathfrak{g}$ be its Lie algebra. Let $W$ be the corresponding Weyl group. Then $W$ acts on $\overline{CR}$. Let $W_{\text{aff}}$ be the affine Weyl group

$$W_{\text{aff}} = W \ltimes \overline{CR}. \tag{4.1}$$

Let $S^1 \simeq \mathbf{R}/\mathbf{Z}$, and let $t \in [0,1[$ be the canonical coordinate on $S^1$. Then $\partial/\partial t$ trivalizes $TS^1$ and $dt$ trivializes $T^*S^1$.

Let $LG$ be the loop group of $G$, i.e. the group of smooth maps $s \in S^1 \mapsto g_s \in G$. Let $L\mathfrak{g}$ be the Lie algebra of $LG$, i.e. the set of smooth maps $s \in S^1 \mapsto f_s \in \mathfrak{g}$.

If $\alpha, b \in L\mathfrak{g}$, put

$$\eta(\alpha,\beta) = \int_{S^1} \langle \alpha, d\beta \rangle . \tag{4.2}$$

Then by [**47**, Section 4.2], $\eta$ is a cocycle on $L\mathfrak{g}$. By [**47**, Theorem 4.4.1], there is a unique central extension $\rho : \widetilde{LG} \xrightarrow{S^1} LG$ associated to $\eta$. The Lie algebra of $\widetilde{LG}$ is $\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{R}$. If $(\alpha,a),(\alpha',a') \in \widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{R}$,

$$[(\alpha,a),(\alpha',a')] = \left([\alpha,\alpha'], \int_{S^1} \langle \alpha, d\alpha' \rangle\right) . \tag{4.3}$$

Observe that the Lie group $G \times S^1$ embeds as a Lie subgroup of $\widetilde{L}G$, and that the Lie algebra of $G \times S^1$, $\mathfrak{g} \oplus \mathbf{R}$ (equipped with its standard Lie algebra structure) embeds correspondingly as a Lie subalgebra of $L\mathfrak{g} \oplus \mathbf{R}$.

Clearly $s \in S^1$ acts on $LG$ by $g \in LG \mapsto k_s g = g_{\cdot+s} \in LG$. We can then form the semidirect product $\widehat{L}G = S^1 \ltimes LG$. The Lie algebra $\widehat{L}\mathfrak{g}$ of $\widehat{L}G$ is given by $\widehat{L}\mathfrak{g} = \mathbf{R} \oplus L\mathfrak{g}$, and can be identified to the Lie algebra of differential operators $a\frac{d}{dt} + \alpha$, $a \in \mathbf{R}$, $\alpha \in L\mathfrak{g}$, so that

$$\left[a\frac{d}{dt} + \alpha, a'\frac{d}{dt} + \alpha'\right] = [\alpha, \alpha'] + a\frac{d}{dt}\alpha' - a'\frac{d}{dt}\alpha\,. \tag{4.4}$$

By [47, Theorem 4.4.1], the action of $S^1$ on $LG$ lifts to $\widetilde{L}G$, so that we can form the semidirect product $\widehat{\widetilde{L}}G = S^1 \ltimes \widetilde{L}G$. Its Lie algebra $\widehat{\widetilde{L}}\mathfrak{g}$ is given by

$$\begin{aligned} \widehat{\widetilde{L}}\mathfrak{g} &= \mathbf{R} \oplus L\mathfrak{g} \oplus \mathbf{R} \\ &= \mathbf{R} \oplus \widetilde{L}\mathfrak{g} \\ &= \widehat{L}\mathfrak{g} \oplus \mathbf{R}\,. \end{aligned} \tag{4.5}$$

The Lie bracket in $\widehat{\widetilde{L}}\mathfrak{g}$ is given by

$$[(a,\alpha,b),(a',\alpha',b')] = \left(0, [\alpha,\alpha'] + a\frac{d\alpha'}{dt} - a'\frac{d\alpha}{dt}, \int_{S^1} \langle \alpha, d\alpha' \rangle\right)\,. \tag{4.6}$$

Also $S^1 \times G \times S^1$ embed as a Lie subgroup of $\widehat{\widetilde{L}}\mathfrak{g}$, and $\mathbf{R} \oplus \mathfrak{g} \oplus \mathbf{R}$ (with its standard structure of Lie algebra) embeds correspondingly as a Lie subalgebra of $\mathbf{R} \oplus L\mathfrak{g} \oplus \mathbf{R}$.

Observe that we can reverse the orientation of $S^1$. Namely let $\psi : LG \to LG$ be the morphism $g_s \mapsto g_{-s}$. Then $\psi$ lifts to a morphism $\widehat{\widetilde{L}}G \to \widehat{\widetilde{L}}G$, so that if $(x,y) \in S^1 \times S^1$,

$$\psi(x,y) = (x^{-1}, y^{-1})\,. \tag{4.7}$$

By [47, Section 4.9], on $\widehat{\widetilde{L}}\mathfrak{g}$, there is a $\widehat{\widetilde{L}}G$-invariant symmetric bilinear form

$$\langle (a,\alpha,b),(a',\alpha',b') \rangle = \int_{S^1} \langle \alpha, \alpha' \rangle dt - ab' - ba'\,. \tag{4.8}$$

By (4.8), we have the embedding

$$\mathbf{R} \oplus L\mathfrak{g} \subset (L\mathfrak{g} \oplus \mathbf{R})^*\,. \tag{4.9}$$

Equivalently

$$\widehat{L}\mathfrak{g} \subset (\widetilde{L}\mathfrak{g})^*\,. \tag{4.10}$$

**4.2. The coadjoint orbits of $LG$.** The group $LG$ has a coadjoint action on the right on $(\widetilde{L}\mathfrak{g})^*$. If we restrict this action to $\widehat{L}\mathfrak{g} \subset (\widetilde{L}\mathfrak{g})^*$, for $g \in LG$, we get

$$\left(a\frac{d}{dt} + \alpha\right) \cdot g = a\frac{d}{dt} + ag^{-1}\frac{dg}{dt} + g^{-1}\alpha g\,. \tag{4.11}$$

Let $P = S^1 \times G$ be the trivial $G$-bundle on $S^1$. Let $\mathcal{A}^{S^1}$ be the affine space of $G$-connections on $S^1$. A connection in $\mathcal{A}^{S^1}$ can be written as

$$d + A\,,\ A \in \Omega^1(S^1, \mathfrak{g})\,. \tag{4.12}$$

In the sequel, we identify $d+A \in \mathcal{A}^{S^1}$ with the differential operator $\frac{d}{dt} + A(\partial/\partial t) \in \widehat{L\mathfrak{g}}$. So $\mathcal{A}^{S^1}$ is an affine subspace of $\widehat{L\mathfrak{g}}$.

Clearly $LG$ acts on the right on $\mathcal{A}^{S^1}$, so that if $g \in LG$,

$$d + A \cdot g = g^{-1}(d+A)g \tag{4.13}$$

i.e.

$$A \cdot g = g^{-1}dg + g^{-1}Ag\,. \tag{4.14}$$

By (4.11), (4.14), $\mathcal{A}^{S^1}$ embeds as an affine suspace of $(\widetilde{L\mathfrak{g}})^* = \widehat{L\mathfrak{g}}$ and the action of $LG$ on $(\widetilde{L\mathfrak{g}})^*$ induces the corresponding action of $LG$ on $\mathcal{A}^{S^1}$.

Now we briefly develop the theory of coadjoint orbits for $\widetilde{LG}$.

DEFINITION 4.1. If $A \in L\mathfrak{g}$, let $w \in G$ be the holonomy of the operator $\frac{d}{dt} + A$ on $S^1$, i.e. if $g_t$ is the solution of

$$\frac{dg_t}{dt} g_t^{-1} + A_t = 0\,, \tag{4.15}$$

then

$$w = g_1\,. \tag{4.16}$$

Then Floquet's theory of differential equations (Frenkel [**21**, Section 3.2], Segal [**47**, Proposition 4.3.6]) shows that the orbits of $\frac{d}{dt} + A$ in $\widehat{L\mathfrak{g}}$ can be expressed in terms of the adjoint orbits in $G$. Namely $\frac{d}{dt} + A$ and $\frac{d}{dt} + A'$ lie in the same $LG$-orbit if and only if the corresponding holonomies $w$ and $w'$ lie in the same $G$-orbit. In particular the $LG$ orbits of the differential operator $\frac{d}{dt} + A$ always contain representatives of the form $\frac{d}{dt} + \lambda$, $\lambda \in \mathfrak{t}$, and two $\frac{d}{dt} + \lambda$ lie in the same $LG$-orbit if and only if $\lambda$ and $\lambda'$ lie in the same $W_{\mathrm{aff}}$-orbit.

If $\lambda \in \mathfrak{t}$, let $Z(d/dt + \lambda) \subset LG$ be the stabilizer of $d/dt + \lambda$.

PROPOSITION 4.2. *If* $\lambda \in \mathfrak{t}$,

$$Z(d/dt + \lambda) \simeq Z(e^{-\lambda})\,. \tag{4.17}$$

*Namely if* $g \in Z(e^{-\lambda})$, *the corresponding element of* $Z(d/dt + \lambda)$ *is* $t \in S^1 \mapsto e^{-t\lambda} g e^{t\lambda} \in G$. *If* $e^{-\lambda} \in T_{\mathrm{reg}}$, *then*

$$Z(d/dt + \lambda) = T \subset LG\,. \tag{4.18}$$

PROOF. The proof of this simple result is left to the reader. □

Let $\mathcal{O}_{d/dt+\lambda} \subset \widehat{L\mathfrak{g}}$ be the $LG$ orbit of $d/dt + \lambda$. Clearly the map

$$g \in LG \mapsto g(d/dt + \lambda)g^{-1} \in \widehat{L\mathfrak{g}} \tag{4.19}$$

induces the identification

$$LG/Z(d/dt + \lambda) \simeq \mathcal{O}_{d/dt+\lambda}\,. \tag{4.20}$$

In particular if $\lambda \in P$,

$$LG/T \simeq \mathcal{O}_{d/dt+\lambda}\,. \tag{4.21}$$

Also the theory of coadjoint orbits developed in Section 1.14 tells us that $\mathcal{O}_{d/dt+\lambda}$ is equipped with a symplectic form $\sigma_{\mathcal{O}_{d/dt+\lambda}}$. Namely let $d/dt + A =$

$D^A \in \mathcal{O}_{d/dt+\lambda}$. Recall that $LG$ acts on the right on $\mathcal{O}_{d/dt+\lambda}$. If $\alpha \in L\mathfrak{g}$, let $\alpha^{\mathcal{O}_{d/dt+\lambda}}$ be the corresponding vector field on $\mathcal{O}_{d/dt+\lambda}$. Clearly

$$\alpha_A^{\mathcal{O}_{d/dt+\lambda}} = D^A \alpha . \tag{4.22}$$

Then formula (1.193) for the symplectic form $\sigma_{\mathcal{O}_{d/dt+\lambda}}$ is

$$\sigma(D^A\alpha, D^A\beta) = \langle d/dt + A, [\alpha, \beta] \rangle . \tag{4.23}$$

By (4.3), (4.8), (4.23), we get

$$\sigma(D^A\alpha, D^A\beta) = \int_{S^1} \langle D^A\alpha, \beta \rangle . \tag{4.24}$$

Finally $\widetilde{L}G$ acts symplectically on the left and on the right $\mathcal{O}_{d/dt+\lambda}$, and $d/dt + A \mapsto (d/dt + A) \in (\widetilde{L}\mathfrak{g})^*$ is a moment map for the left action of $\widetilde{L}G$ on $\mathcal{O}_{d/dt+\lambda}$.

If $p \in \mathbf{R}^*$, $\lambda \in \mathfrak{t}$, if $pd/dt + A \in \mathcal{O}_{pd/dt+A}$, then $d/dt + A/p = D^{A/p}$ is a connection, and

$$\sigma_{\mathcal{O}_{pd/dt+\lambda}}(D^A\alpha, D^A\beta) = p \int_{S^1} \langle D^{A/p}\alpha, \beta \rangle . \tag{4.25}$$

Recall that $T \times S^1 \subset \widetilde{L}G$.

DEFINITION 4.3. If $(\lambda, p) \in \overline{CR}^* \times \mathbf{Z}^*$, if $\rho_{(\lambda,-p)}$ is the one dimensional representation of $T \times S^1$ of weight $(\lambda, -p)$, let $H_{(\lambda,p)}$ be the line bundle on $\widetilde{L}G/(T \times S^1) = LG/T$,

$$H_{(\lambda,p)} = \widetilde{L}G \times_{\rho_{(\lambda,-p)}} \mathbf{C} . \tag{4.26}$$

In the same way as the Weyl group $W$ acts on the right on $G/T$, the affine Weyl group $W_{\text{aff}} \simeq W \rtimes \overline{CR}$ acts on the right $LG/T$. In fact if $w \in W$, $w$ acts on $LG/T$ by $g \mapsto gw$ and $\mu \in \overline{CR}$ acts on $LG/T$ by $g \mapsto ge^{\mu t}$. Then $W_{\text{aff}}$ acts on the left on $\overline{CR}^* \times \mathbf{Z}^*$ by the formula

$$\begin{aligned} w(\lambda, p) &= (w\lambda, p) \\ \mu(\lambda, p) &= (\lambda + p\mu, p) . \end{aligned} \tag{4.27}$$

We can then restrict $(\lambda, p)$ to vary in a fundamental domain of the action of $W_{\text{aff}}$. This just says that

$$p \in \mathbf{Z}^* \ , \ \lambda \in |p|\overline{P} . \tag{4.28}$$

Equivalently if $p \in \mathbf{Z}^*$,

$$\text{for } \alpha \in R_+ , \ \ 0 \le \langle \alpha, \lambda \rangle \le \langle \alpha_0, \lambda \rangle \le p . \tag{4.29}$$

Let now $p \in \mathbf{Z}^*$, $\lambda \in |p|P \cap \overline{CR}^*$. Then by (4.18), (4.20),

$$\mathcal{O}_{pd/dt+\lambda} \simeq LG/T . \tag{4.30}$$

Therefore, the line bundle $H_{(\lambda,p)}$ is well defined on $\mathcal{O}_{pd/dt+\lambda}$. Also by proceeding as in Section 1.14, the Hermitian line bundle $H_{(\lambda,p)}$ is equipped with a unitary connection $\nabla^{H_{(\lambda,p)}}$ and

$$c_1\left(H_{(\lambda,p)}, \nabla^{H_{(\lambda,p)}}\right) = \sigma_{\mathcal{O}_{pd/dt+\lambda}} . \tag{4.31}$$

If $p \in \mathbf{Z}^*$, $\lambda \in |p|\overline{P} \cap \overline{CR}^*$, but $\lambda \notin pP$, the theory is slightly more involved.

Now, we will just assume that $\lambda \in \overline{CR}^*$ and explain the construction of the line bundle $H_{(\lambda,p)}$ on $\mathcal{O}_{pd/dt+\lambda}$ in full generality.

By (4.17),

$$Z(pd/dt+\lambda) \simeq Z(e^{-\lambda/p}). \tag{4.32}$$

DEFINITION 4.4. Let $\widetilde{Z}(pd/dt+\lambda)$ be the stabilizer of $pd/dt+\lambda$ in $\widetilde{L}G$.

Then $\widetilde{Z}(pd/dt+\lambda)$ is a central extension of $Z(pd/dt+\lambda)$. Let $\mathfrak{z}(e^{-\lambda/p}), \widetilde{\mathfrak{z}}(pd/dt+\lambda)$ be the Lie algebras of $Z(e^{-\lambda/p})$, $\widetilde{Z}(pd/dt+\lambda)$. Then

$$\widetilde{\mathfrak{z}}(pd/dt+\lambda) \simeq \mathfrak{z}(e^{-\lambda/p}) \oplus \mathbf{R}. \tag{4.33}$$

PROPOSITION 4.5. *If* $(X,a),(Y,b) \in \widetilde{\mathfrak{z}}(pd/dt+\lambda) \simeq \mathfrak{z}(e^{-\lambda/p}) \oplus \mathbf{R}$, *then*

$$[(X,a),(Y,b)] = \left([X,Y], \frac{1}{p}\langle\lambda,[X,Y]\rangle\right). \tag{4.34}$$

PROOF. Clearly, if $X \in \mathfrak{z}(e^{-\lambda/p})$, the corresponding element in the Lie algebra $\mathfrak{z}(pd/dt+\lambda)$ of $Z(pd/dt+\lambda)$ is just $e^{-t\lambda/p}Xe^{t\lambda/p}$. In view of (4.3),

$$\begin{aligned}&\left[(e^{-t\lambda/p}Xe^{t\lambda/p},0)\ ,\ (e^{-t\lambda/p}Ye^{t\lambda/p},0)\right]\\ &\qquad\left(e^{-t\lambda/p}[X,Y]e^{t\lambda/p}\ ,\ \langle X,-[\lambda/p,Y]\rangle\right).\end{aligned} \tag{4.35}$$

Also

$$\langle X,-[\lambda,Y]\rangle = \langle\lambda,[X,Y]\rangle. \tag{4.36}$$

The proof of our Proposition is completed. □

PROPOSITION 4.6. *For any* $\lambda \in \overline{CR}^*$,

$$(X,a) \in \mathfrak{z}(e^{-\lambda/p}) \oplus \mathbf{R} \mapsto (X, \langle\frac{\lambda}{p},X\rangle + a) \in \widetilde{\mathfrak{z}}(pd/dt+\lambda) \tag{4.37}$$

*is an isomorphism of Lie algebras.*

PROOF. This is clear by (4.34). □

REMARK 4.7. If $\lambda \in pP \cap \overline{CR}^*$, so that $e^{-\lambda/p} \in T_{\text{reg}}$, then $\mathfrak{z}(e^{-\lambda/p}) = \mathfrak{t}$. In this case, in (4.34), $\langle\lambda,[X,Y]\rangle = 0$. Therefore $(\lambda,a) \in \mathfrak{z}(e^{-\lambda/p}) \oplus \mathbf{R} \mapsto (X,a) \in \widetilde{\mathfrak{z}}(pd/dt+\lambda)$ is also an isomorphism of Lie algebras.

Observe that $T$ is a maximal torus in $Z(e^{-\lambda/p})$ . By [**14**, Corollaire 5.3.1], $Z(e^{-\lambda/p})/T$ is simply connected.

DEFINITION 4.8. Put

$$R_{e^{-\lambda/p}} = \{\alpha \in \mathbf{R}\ ;\ \langle\alpha,\lambda/p\rangle \in \mathbf{Z}\}. \tag{4.38}$$

We define $CR_{e^{-\lambda/p}}$ as in (1.136). Let $\overline{CR}_{e^{-\lambda/p}} \subset \mathfrak{t}$ be the lattice spanned by $CR_{e^{-\lambda/p}}$.

By [**15**, Theorem V.7.1],

$$\pi_1(Z(e^{-\lambda/p})) = \frac{\overline{CR}}{\overline{CR}_{e^{-\lambda/p}}}. \tag{4.39}$$

Put

$$\overline{R}^*_{e^{-\lambda/p}} = \{t \in \mathfrak{t}\ ;\ \text{if}\ \ \alpha \in R_{e^{-\lambda/p}}\ ,\langle\alpha,t\rangle \in \mathbf{Z}\}. \tag{4.40}$$

By [15, Proposition V.7.16],

$$Z(Z(e^{-\lambda/p})) = \frac{\overline{R}^*_{e^{-\lambda/p}}}{\overline{CR}}. \tag{4.41}$$

In particular

$$-\lambda/p \in \frac{\overline{R}^*_{e^{-\lambda/p}}}{\overline{CR}}. \tag{4.42}$$

Now $\frac{\overline{R}^*_{e^{-\lambda/p}}}{\overline{CR}}$ maps into the group $\pi_1(Z(e^{-\lambda/p}))^*$. In particular the map

$$\psi(pd/dt + \lambda) : h \in \pi_1(Z(e^{-\lambda/p})) \mapsto e^{2i\pi\langle\lambda/p,h\rangle} \in \mathbf{C} \tag{4.43}$$

is well-defined. Since $\lambda \in \overline{CR}^*$, $e^{2i\pi\langle\lambda/p,h\rangle}$ is a $p^{\text{th}}$ root of unity.

Recall that $pd/dt$ is identified to the 1-form $a \in \mathbf{R} \mapsto -pa \in \mathbf{R}$. Also $\lambda \in \overline{CR}^*$ is a left invariant 1-form on $\widetilde{L}G$.

By Proposition 1.56, $pd/dt + \lambda$ is a closed 1-form on $\widetilde{Z}(pd/dt + \lambda)$.

Let $\widetilde{Z}(e^{-\lambda/p})$ be the universal cover of $Z(e^{-\lambda/p})$. Then $pd/dt$ is a closed form on $\widetilde{Z}(e^{-\lambda/p}) \times_{\psi(pd/dt+\lambda)} S^1$.

THEOREM 4.9. *We have the identity of Lie groups*

$$\widetilde{Z}(pd/dt + \lambda) \simeq \widetilde{Z}(e^{-\lambda/p}) \times_{\psi(pd/dt+\lambda)} S^1. \tag{4.44}$$

*In particular $\widetilde{Z}(pd/dt+\lambda)$ contains a $p^{\text{th}}$ cover of $Z(e^{-\lambda/p})$. Under the identification (4.44),*

$$pd/dt + \lambda \simeq pd/dt, \tag{4.45}$$

*and the forms in (4.45) are closed and integral.*

PROOF. Clearly by (4.37), $\widetilde{Z}(pd/dt+\lambda)$ is a locally trivial central extension of $Z(e^{-\lambda/p})$. Let $s \in S^1 \mapsto t_s \in T$ be a smooth loop. The corresponding loop with values in $Z(pd/dt + \lambda)$ is given by $s \in S^1 \mapsto (e^{-t\lambda/p}t_s e^{t\lambda/p}) \in LG$, i.e. by

$$s \in S^1 \mapsto t_s \in G \subset LG. \tag{4.46}$$

Now recall that $G \in \widetilde{L}G$, so that we have a loop

$$s \in S^1 \mapsto t_s \in G \cap \widetilde{Z}(pd/dt + \lambda). \tag{4.47}$$

By (4.37), the horizontal lift of $s \in S^1 \mapsto t_s \in G \subset LG$ in $\widetilde{Z}(pd/dt+\lambda)$ with respect to the flat connection on $\widetilde{Z}(pd/dt + \lambda) \to Z(e^{-\lambda/p})$ is given by

$$s \in S^1 \mapsto t_s \exp\left(2i\pi \int_{S^1} \langle\lambda/p, t_s^{-1}dt_s\rangle\right). \tag{4.48}$$

From (4.48), we get (4.44).

If $S^1_p$ denotes the group of $p^{\text{th}}$ roots of unity in $\mathbf{C}$, $\widetilde{Z}(e^{-\lambda/p}) \times_{\psi(pd/dt+\lambda)} S^1_p$ is a subgroup of $\widetilde{Z}(pd/dt + \lambda)$, which is a $p$-covering of $Z(e^{-\lambda/p})$.

Observe that

$$\langle pd/dt + \lambda, (X, \langle\frac{\lambda}{p}, X\rangle + a)\rangle = -pa = \langle pd/dt, a\rangle. \tag{4.49}$$

From (4.49) we get (4.45).

By Proposition 1.56, the 1-form $pd/dt + \lambda$ is closed on $\widetilde{Z}(pd/dt + \lambda)$. It is even easier to see that $pd/dt$ is closed on $\widetilde{Z}(e^{-\lambda/p}) \times_{\psi(pd/dt+\lambda)} S^1$. Finally it is trivial to verify that $pd/dt$ is integral on $\widetilde{Z}(e^{-\lambda/p}) \times_{\psi(pd/dt+\lambda)} S^1$. The proof of our Theorem is completed. □

By Theorem 4.9, as in Proposition 1.58, the integral 1-form $pd/dt + \lambda$ defines a representation $\rho_1 : \widetilde{Z}(p\frac{d}{dt} + \lambda) \to S^1$. Also $\rho_2 : (g, x) \in \widetilde{Z}(e^{-\lambda/p}) \times_{\psi_{(pd/dt+\lambda)}} S^1 \mapsto x^{-p} \in S^1$ is also a representation.

PROPOSITION 4.10. *Under the isomorphism (4.44),*

$$\rho_1 \simeq \rho_2 . \tag{4.50}$$

PROOF. By (4.45), we get (4.50). □

REMARK 4.11. Assume that $\lambda \in pP \cap \overline{CR}^*$, so that $Z(e^{-\lambda/p}) \simeq T$. Then

$$\widetilde{Z}(e^{-\lambda/p}) \simeq \mathfrak{t} . \tag{4.51}$$

Also the map

$$(f, a) \in \mathfrak{t} \times S^1 \mapsto (f, e^{2i\pi\langle\lambda/p, f\rangle} a) \in \mathfrak{t} \times S^1 \tag{4.52}$$

descends to an isomorphism

$$T \times S^1 \simeq \mathfrak{t} \times_{\psi_{(pd/dt+\lambda)}} S^1 . \tag{4.53}$$

Then the 1-form $pd/dt + \lambda$ on $T \times S^1$ corresponds to $pd/dt$ on $\mathfrak{t} \times_{\psi_{(pd/dt+\lambda)}} S^1$.
In fact in this case,

$$\widetilde{Z}(pd/dt + \lambda) = T \times S^1, \tag{4.54}$$

and (4.53) is a special case of (4.44).

Clearly

$$\frac{\widetilde{L}G}{\widetilde{Z}(pd/dt + \lambda)} = \frac{LG}{Z(pd/dt + \lambda)} \simeq \mathcal{O}_{pd/dt+\lambda} . \tag{4.55}$$

DEFINITION 4.12. Let $H_{(\lambda,p)}$ be the line bundle on $\mathcal{O}_{pd/dt+\lambda}$,

$$H_{(\lambda,p)} = \widetilde{L}G \times_{\rho_1} S^1 . \tag{4.56}$$

Then $H_{(\lambda,p)}$ is a Hermitian line bundle. As in (1.198), we can equip $H_{(\lambda,p)}$ with a unitary connection $\nabla^{H_{(\lambda,p)}}$. By proceeding as in (1.199), we get

$$c_1(H_{(\lambda,p)}, \nabla^{H_{(\lambda,p)}}) = \sigma_{\mathcal{O}_{pd/dt+\lambda}} . \tag{4.57}$$

### 4.3. A central extension of $(LG)^s$.

DEFINITION 4.13. Put

$$P_s = \{(a_1, \cdots, a_s) \in (S^1)^s \, , \, \prod_{j=1}^{s} a_j = 1\} . \tag{4.58}$$

Then $P_s$ is a Lie subgroup of $(S^1)^s$, and its Lie algebra $\mathfrak{p}_s$ is given by

$$\mathfrak{p}_s = \{(b_1, \cdots, b_s) \in \mathbf{R}^s \, , \, \sum_{j=1}^{s} b_j = 0\} . \tag{4.59}$$

Clearly $P_s \subset (S^1)^s$ is a Lie subgroup of $(\widetilde{L}G)^s$.

DEFINITION 4.14. Put

$$(\widetilde{LG})^s = (\tilde{L}G)^s/P_s, \qquad (\widetilde{L\mathfrak{g}})^s = (\tilde{L}\mathfrak{g})^s/\mathfrak{p}_s. \tag{4.60}$$

Then $(\widetilde{LG})^s$ is a central extension of $(LG)^s$ and $(\widetilde{L\mathfrak{g}})^s$ is its Lie algebra. Clearly

$$(\widetilde{L\mathfrak{g}})^s = (L\mathfrak{g})^s \oplus \mathbf{R}. \tag{4.61}$$

Also if $(\alpha_1, \dots, \alpha_s, a), (\alpha'_1, \dots, \alpha'_s, a') \in (\widetilde{L\mathfrak{g}})^s$, then

(4.62)

$$[(\alpha_1, \dots, \alpha_s, a), (\alpha'_1, \dots, \alpha'_s, a')] = \left( [\alpha_1, \alpha'_1], \dots, [\alpha_s, \alpha'_s] \ , \ \sum_{j=1}^{s} \int_{S^1} \langle \alpha_j, d\alpha'_j \rangle \right) .$$

Also $G^s \times S^1$ embeds as a subgroup of $(\widetilde{LG})^s$, and $\mathfrak{g}^s \oplus \mathbf{R}$ embeds as a Lie subalgebra of $(\widetilde{L\mathfrak{g}})^s$. Finally recall that in Section 4.1, we defined an action of $S^1$ on $LG$. Therefore $S^1$ acts on $(LG)^s$.

DEFINITION 4.15. Put

$$(\widehat{LG})^s = S^1 \ltimes (LG)^s, \qquad (\widehat{L\mathfrak{g}})^s = R \oplus (L\mathfrak{g})^s. \tag{4.63}$$

Then $(\widehat{L\mathfrak{g}})^s$ is the Lie algebra of $(\widehat{LG})^s$. If we embed $S^1, \mathbf{R}$ into $(S^1)^s, \mathbf{R}^s$ by the diagonal embeddings, then $(\widehat{LG})^s \subset (\hat{L}G)^s$, and $(\widehat{L\mathfrak{g}})^s$ is a Lie subalgebra of $(\hat{L}\mathfrak{g})^s$. Also by (4.10),

$$(\hat{L}\mathfrak{g})^s \subset (\widetilde{L\mathfrak{g}})^{s*}. \tag{4.64}$$

**4.4. The holonomy of $\tilde{L}G$.** Let $\kappa$ be the closed left invariant 3-form on $G$,

$$\kappa(X,Y,Z) = \frac{1}{2}\langle X, [Y,Z]\rangle . \tag{4.65}$$

Then by [**47**, Proposition 4.4.5],

$$\kappa \in H^3(G, \mathbf{Z}). \tag{4.66}$$

Recall that $\eta$ is the left-invariant 2-form on $LG$,

$$\eta = \frac{1}{2} \int_{S^1} \langle g^{-1}dg, \frac{\partial}{\partial t} g^{-1}dg \rangle dt . \tag{4.67}$$

Put

$$\Delta = \{z \in \mathbf{C} \ , \ |z| \le 1\} . \tag{4.68}$$

Let $g(s,t) : S^1 \times S^1 \to G$ be a smooth map. We identify $g(s,t)$ with the smooth loop in $LG$ $s \in S^1 \mapsto g_s = g(s,\cdot) \in LG$.

Clearly

$$\partial\Delta = S^1 . \tag{4.69}$$

Since $\pi_i(G) = 0$, $0 \le i \le 2$, $g(s,t)$ extends to a smooth map $g(z,t) : \Delta \times S^1 \to G$. Therefore we get a map $\overline{g} : z \in \Delta \mapsto g(z,.) \in LG$.

THEOREM 4.16. *The following identity holds*

$$\frac{1}{2}\int_{S^1\times S^1}\langle g^{-1}\frac{\partial g}{\partial t},g^{-1}\frac{\partial g}{\partial s}\rangle\, dtds - \int_{\Delta\times S^1} g^*\kappa + \int_{\Delta}\overline{g}^*\eta = 0. \tag{4.70}$$

PROOF. Let $h$ be the left-invariant 1-form on $LG$,

$$h = \frac{1}{2}\int_{S^1}\langle g^{-1}\frac{\partial g}{\partial t},g^{-1}dg\rangle dt. \tag{4.71}$$

A straightforward computation shows that

$$dh = -\eta + \frac{1}{2}\int_{S^1}\langle g^{-1}\frac{\partial g}{\partial t},\frac{1}{2}\left[g^{-1}dg,g^{-1}dg\right]\rangle dt. \tag{4.72}$$

Using (4.72), we get (4.70). □

Clearly

$$\widetilde{L\mathfrak{g}} = L\mathfrak{g}\oplus\mathbf{R}. \tag{4.73}$$

The splitting (4.73) defines a connection on the $S^1$ bundle $\widetilde{LG}\xrightarrow{S^1} LG$. Let $s\in S^1\mapsto\widetilde{g}_s\in\widetilde{LG}$ be a horizontal lift of $s\in S^1\mapsto g_s\in LG$. Since $g_0=g_1$, then

$$\widetilde{g}_1\widetilde{g}_0^{-1}\in S^1. \tag{4.74}$$

THEOREM 4.17. *The following identity holds*

$$\begin{aligned}\widetilde{g}_1\widetilde{g}_0^{-1} &= \exp\left(2i\pi\int_{\Delta}\overline{g}^*\eta\right)\\ &= \exp\left(2i\pi\left(-\frac{1}{2}\int_{S^1}\langle g^{-1}\frac{\partial g}{\partial t},g^{-1}\frac{\partial g}{\partial s}\rangle dsdt + \int_{\Delta\times S^1} g^*\kappa\right)\right).\end{aligned} \tag{4.75}$$

PROOF. The first identity is a straightforward consequence of (4.2), (4.67). The second identity follows from Theorem 4.16. □

**4.5. The symplectic space of connections on $\Sigma$.** Let $X$ be a compact Riemann surface of genus $g$. Let $x_1,\dots,x_s$ be $s$ distinct elements of $X$. Let $\Delta_1,\dots,\Delta_s$ be small non intersecting small open disks of center $x_1,\dots,x_s$.

Put

$$\Sigma = X\backslash\bigcup_{j=1}^{s}\Delta_j. \tag{4.76}$$

Then $\Sigma$ is a compact Riemann surface with boundary $\partial\Sigma$. Also $\Sigma$ being oriented, $\partial\Sigma$ is also naturally oriented.

Let $G$ be a compact connected and simply connected compact simple Lie group. We use otherwise the same notation as in Section 1. In particular $\mathfrak{g}$ denotes the Lie algebra of $G$, and $\langle,\rangle$ denotes the basic scalar product on $\mathfrak{g}$ defined in Section 1.2.

Let $P=\Sigma\times G$ be the trivial $G$-bundle on $\Sigma$. Observe that since $G$ is simply connected, a $G$-bundle on $\Sigma$ is necessarily trivial.

DEFINITION 4.18. Let $\mathcal{A}$ be the affine space of $G$-connections on $P$.

Any element in $\mathcal{A}$ can be written in the form $d+A$, with $A \in \Omega^1(\Sigma, \mathfrak{g})$. Therefore $\mathcal{A}$ is an affine space with underlying vector space $\Omega^1(\Sigma, \mathfrak{g})$. Also

$$T\mathcal{A} = \mathcal{A} \times \Omega^1(\Sigma, \mathfrak{g}). \tag{4.77}$$

Let $\Sigma G$, $\Sigma\mathfrak{g}$ be the sets of smooth maps $\Sigma \to G$, $\Sigma \to \mathfrak{g}$. The $\Sigma\mathfrak{g}$ is a Lie algebra, which will be considered as the Lie algebra of $\Sigma G$.

DEFINITION 4.19. If $U, V \in \Omega^1(\Sigma, \mathfrak{g})$, put

$$\sigma(U,V) = \int_\Sigma -\langle U \wedge V \rangle. \tag{4.78}$$

Then $\sigma$ is a symplectic form on $\mathcal{A}$.

Observe that $\Sigma G$ acts on the right on $\mathcal{A}$, so that if $g \in \Sigma G$,

$$d + A \cdot g = g^{-1}(d + A)g, \tag{4.79}$$

i.e.

$$A \cdot g = g^{-1}Ag + g^{-1}dg. \tag{4.80}$$

Clearly $\Sigma G$ preserves the symplectic form $\sigma$.

DEFINITION 4.20. If $\alpha \in \Sigma\mathfrak{g}$, let $\alpha^{\mathcal{A}}$ be the corresponding vector field on $\mathcal{A}$.

Let $\nabla^A$ be the covariant derivative associated to the connection $A \in \mathcal{A}$. Then

$$\alpha^{\mathcal{A}}_A = \nabla^A_{\cdot}\alpha. \tag{4.81}$$

For simplicity, we now fix an oriented parametrization of the $s$ circles in $\partial\Sigma$, so that

$$\partial\Sigma = (S^1)^s. \tag{4.82}$$

DEFINITION 4.21. Let $r$ be the restriction map

$$\begin{aligned} g \in \Sigma G &\mapsto g_{|\partial\Sigma} \in (LG)^s, \\ \alpha \in \Sigma\mathfrak{g} &\mapsto \alpha_{|\partial\Sigma} \in (L\mathfrak{g})^s, \\ d + A \in \mathcal{A} &\mapsto (d+A)_{|\partial\Sigma} \in (\mathcal{A}^{S^1})^s. \end{aligned} \tag{4.83}$$

All these maps are equivariant. By (4.83), we get a map

$$d + A \mapsto \left(\frac{d}{dt} + A\Big(\frac{d}{dt}\Big)\right)_{|\partial\Sigma} \in (\widehat{L\mathfrak{g}})^s. \tag{4.84}$$

We still denote by $\rho$ the projection $(\widetilde{LG})^s \to (LG)^s$.

DEFINITION 4.22. Put

$$\begin{aligned} \widetilde{\Sigma G} &= \{(g, g') \in \Sigma G \times (\widetilde{LG})^s \; ; \; g_{|\partial\Sigma} = \rho g' \text{ in } (LG)^s\}, \\ \widetilde{\Sigma\mathfrak{g}} &= \{(\alpha, \alpha') \in \Sigma\mathfrak{g} \times (\widetilde{L\mathfrak{g}})^s \; ; \; \alpha_{|\partial\Sigma} = \rho\alpha' \text{ in } (L\mathfrak{g})^s\}. \end{aligned} \tag{4.85}$$

Then $\widetilde{\Sigma\mathfrak{g}}$ is the Lie algebra of the Lie group $\widetilde{\Sigma G}$. More precisely

$$\widetilde{\Sigma\mathfrak{g}} = \Sigma\mathfrak{g} \oplus \mathbf{R}. \tag{4.86}$$

Let $\tau$ be the projection $\widetilde{\Sigma\mathfrak{g}} = \Sigma\mathfrak{g} \oplus \mathbf{R} \to \mathbf{R}$. If $(\alpha, a), (\alpha', a') \in \widetilde{\Sigma\mathfrak{g}}$, then

$$[(\alpha, a), (\alpha', a')] = \left([\alpha, \alpha'], \int_{\partial\Sigma} \langle \alpha, d\alpha' \rangle\right). \tag{4.87}$$

Also $G \times S^1$ embeds as a subgroup of $\widetilde{\Sigma G}$ by the map $(g,t) \mapsto (g,(g,\dots,g),t) \subset \Sigma G \times (\widetilde{LG})^s$.

Clearly the restriction map $r$ extend to a map

$$\begin{aligned} \widetilde{\Sigma G} &\to (\widetilde{LG})^s, \\ \widetilde{\Sigma \mathfrak{g}} &\to (\widetilde{L\mathfrak{g}})^s. \end{aligned} \tag{4.88}$$

In particular, there is a dual map

$$(\widetilde{L\mathfrak{g}})^{s*} \to (\widetilde{\Sigma\mathfrak{g}})^*. \tag{4.89}$$

If $A \in \mathcal{A}$, then by (4.64),

$$\frac{d}{dt} + A(\partial/\partial t) \in (\widetilde{L\mathfrak{g}})^{s*}. \tag{4.90}$$

So by (4.89), (4.90), we may view $(d+A)_{|\partial\Sigma}$ as an element of $(\widetilde{\Sigma\mathfrak{g}})^*$.

As we just saw, $\Sigma G$ acts on $\mathcal{A}$ and preserves the symplectic form $\sigma$. Therefore $\widetilde{\Sigma G}$ also acts symplectically on $\mathcal{A}$. Of course $S_1 \subset \widetilde{\Sigma G}$ acts trivially on $\mathcal{A}$. If $\widetilde{\alpha} \in \widetilde{\Sigma\mathfrak{g}}$, let $\widetilde{\alpha}^{\mathcal{A}}$ be the associated vector field on $\mathcal{A}$. If $\alpha = \rho\widetilde{\alpha}$,

$$\widetilde{\alpha}^{\mathcal{A}} = \alpha^{\mathcal{A}} = \nabla^A_{.}\alpha. \tag{4.91}$$

Clearly

$$\Sigma\mathfrak{g} = \Omega^0(\Sigma,\mathfrak{g}). \tag{4.92}$$

Also

$$\Omega^2(\Sigma,\mathfrak{g}) \subset \Omega^0(\Sigma,\mathfrak{g})^* = (\Sigma\mathfrak{g})^*. \tag{4.93}$$

DEFINITION 4.23. If $A \in \mathcal{A}$, let $F^A = dA + \frac{1}{2}[A,A] \in \Omega^2(\Sigma,\mathfrak{g})$ be the curvature of $A$.

By (4.93), if $A \in \mathcal{A}$, $F^A \subset (\Sigma\mathfrak{g})^* \subset (\widetilde{\Sigma\mathfrak{g}})^*$. Also by (4.89), (4.90) if $A \in \mathcal{A}$, $(d+A)_{|\partial\Sigma} \in (\widetilde{\Sigma\mathfrak{g}})^*$.

Recall that in Section 3.2, moment maps were defined with respect to symplectic actions of compact Lie groups on symplectic manifolds. Here we will use the same definition with respect to an action of $\widetilde{\Sigma G}$ on $\mathcal{A}$. We now state an extension of a result of Atiyah-Bott [**2**, Section 9].

THEOREM 4.24. *The map*

$$A \in \mathcal{A} \mapsto \mu(A) = F^A - (d+A)_{|\partial\Sigma} \in (\widetilde{\Sigma\mathfrak{g}})^* \tag{4.94}$$

*is a moment map for the action of* $\widetilde{\Sigma G}$ *on* $\mathcal{A}$.

PROOF. First we will prove that if $\widetilde{\alpha} \in \widetilde{\Sigma\mathfrak{g}}$,

$$d\langle F^A - (d+A)_{|\partial\Sigma}, \widetilde{\alpha}\rangle = i_{\widetilde{\alpha}^{\mathcal{A}}}\sigma. \tag{4.95}$$

Clearly if $\widetilde{\alpha} = (\alpha, a_1, \dots, a_s) \in \widetilde{\Sigma\mathfrak{g}} \oplus \mathbf{R}^s$ is identified with the corresponding element in $\widetilde{\Sigma\mathfrak{g}}$,

$$\langle F^A - (d+A), \widetilde{\alpha}\rangle = \int_\Sigma \langle F^A, \alpha\rangle - \int_{\partial\Sigma}\langle A, \alpha\rangle + \sum_{j=1}^{s} a_j. \tag{4.96}$$

If $B \in T\mathcal{A} \simeq \Omega^1(\Sigma, \mathfrak{g})$, then

$$(4.97)\qquad \begin{aligned} \langle B, d\langle F^A - (d+A)_{|\partial\Sigma}, \widetilde{\alpha}\rangle\rangle &= \int_\Sigma \langle D^A B, \alpha\rangle - \int_{\partial\Sigma}\langle B, \alpha\rangle \\ &= -\int_\Sigma \langle \nabla^A \alpha, B\rangle \\ &= i_{\widetilde{\alpha}^A}\sigma(B)\,. \end{aligned}$$

By (4.97), $\langle F^A - (d+A)_{|\partial\Sigma}, \widetilde{\alpha}\rangle$ is a Hamiltonian for the vector field $\widetilde{\alpha}^A$. Also by definition, if $g \in \widetilde{\Sigma G}$,

$$(4.98)\qquad F^{A\cdot g} - (d + A\cdot g)_{|\partial\Sigma} = (F^A - (d+A)_{|\partial\Sigma})\cdot g\,.$$

We have completed the proof of our Theorem. □

**4.6. The canonical line bundle on $\mathcal{A}$.** Now we describe the construction of the canonical line bundle $L$ on $\mathcal{A}$. In the case where there are no marked points, our construction follows closely earlier work by Ramadas, Singer and Weitsman [**48**].

DEFINITION 4.25. Let $(L, \|\ \|_L)$ be the trivial Hermitian line bundle on $\mathcal{A}$. Let $\nabla^L$ be the unitary connection $(L, \|\ \|_L)$ :

$$(4.99)\qquad \nabla^L = d - i\pi \int_\Sigma \langle ., A\rangle\,.$$

PROPOSITION 4.26. *The following identity holds*

$$(4.100)\qquad c_1(L, \nabla^L) = \sigma\,.$$

PROOF. This follows from (4.99). □

DEFINITION 4.27. If $\widetilde{\alpha} \in \widetilde{\Sigma\mathfrak{g}}$, put

$$(4.101)\qquad L_{\widetilde{\alpha}} = \nabla^L_{\alpha^A} - 2i\pi\langle \mu(A), \widetilde{\alpha}\rangle\,.$$

Clearly for $\widetilde{\alpha} \in \widetilde{\Sigma\mathfrak{g}}$, $L_{\widetilde{\alpha}}$ acts on $C^\infty(\mathcal{A}, L)$.

PROPOSITION 4.28. *If $\widetilde{\alpha} \in \widetilde{\Sigma\mathfrak{g}}$, then*

$$(4.102)\qquad [L_{\widetilde{\alpha}}, \nabla^L] = 0\,.$$

*Also if $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\Sigma\mathfrak{g}}$,*

$$(4.103)\qquad [L_{\widetilde{\alpha}}, L_{\widetilde{\beta}}] = L_{[\widetilde{\alpha},\widetilde{\beta}]}\,.$$

PROOF. The identities (4.102) and (4.103) are trivial consequences of (3.120) and of Theorem 4.24. □

PROPOSITION 4.29. *If $\widetilde{\alpha} = (\alpha, a) \in \widetilde{\Sigma\mathfrak{g}} = \Sigma\mathfrak{g} \oplus \mathbf{R}$, then*

$$(4.104)\qquad L_{\widetilde{\alpha}} = \nabla_{\alpha^A} - i\pi\int_\Sigma \langle A, d\alpha\rangle - 2i\pi a\,.$$

PROOF. By (4.99), (4.101), we get

$$(4.105)\qquad \begin{aligned} L_{\widetilde{\alpha}} = \nabla_{\alpha^A} \quad - \quad & i\pi\int_\Sigma \langle \nabla^A\alpha, A\rangle - 2i\pi\int_\Sigma \langle F^A, \alpha\rangle \\ & +2i\pi\int_{\partial\Sigma}\langle A, \alpha\rangle - 2i\pi a\,. \end{aligned}$$

Also

$$\begin{aligned}
(4.106)\qquad \int_\Sigma \langle \nabla^A \alpha, A\rangle &= -\int_\Sigma \langle A, d\alpha\rangle - \int_\Sigma \langle [A,A],\alpha\rangle\,,\\
\int_\Sigma \langle F^A,\alpha\rangle &= \int_\Sigma \langle dA + \frac{1}{2}[A,A],\alpha\rangle\\
&= \int_{\partial\Sigma}\langle A,\alpha\rangle + \int_\Sigma (\langle A, d\alpha\rangle + \frac{1}{2}\langle [A,A],\alpha\rangle)\,.
\end{aligned}$$

From (4.100), (4.106), we get (4.104). The proof of our Theorem is completed. □

REMARK 4.30. By (4.104), if $\alpha,\beta \in \Sigma\mathfrak{g}$,

$$(4.107)\qquad [L_\alpha, L_\beta] = L_{[\alpha,\beta]} - 2i\pi \int_{\partial\Sigma} \langle \alpha, d\beta\rangle\,.$$

In view of (4.87), (4.104), (4.107) fits with (4.103).

Now we will show that the action of $\widetilde{\Sigma\mathfrak{g}}$ on $L$ lifts to an action of $\widetilde{\Sigma G}$. This result has been proved by Ramadas, Singer and Weitsman [48] in the case where there are no marked points, so that $\widetilde{\Sigma G}$ is just $\Sigma G$.

THEOREM 4.31. *The action of $\widetilde{\Sigma\mathfrak{g}}$ on $L$ defined in (4.101) lifts uniquely to a unitary action of $\widetilde{\Sigma G}$ on $L$ which preserves $\nabla^L$. In particular if $(g,t) \in G\times S^1 \subset \widetilde{\Sigma G}$, $(g,t)$ acts on $L$ by*

$$(4.108)\qquad f\in L_A \mapsto f\cdot(g,t) = e^{2i\pi t} f \in L_{A\cdot g}\,.$$

PROOF. Let $g\in \Sigma G$. Since $\pi_i(G) = 0$, $0\le i\le 2$, there is a smooth path $s\in[0,1]\mapsto g_s \in \Sigma G$ such that

$$\begin{aligned}
(4.109)\qquad g_0 &= 1\,,\\
g_1 &= g\,.
\end{aligned}$$

Recall that

$$(4.110)\qquad \widetilde{\Sigma\mathfrak{g}} = \Sigma\mathfrak{g}\oplus \mathbf{R}\,.$$

The splitting (4.110) defines a left-invariant connection on the $S^1$-bundle $\widetilde{\Sigma G} \xrightarrow{S^1} \Sigma G$. Let $s\in[0,1]\mapsto \widetilde{g}_s$ be the horizontal lift of $s\in[0,1]\mapsto g_s\in\Sigma G$ with respect to this connection, with $\widetilde{g}_0 = 1$. Put

$$\begin{aligned}
(4.111)\qquad \alpha_s &= g_s^{-1}\frac{dg_s}{ds} \in \Sigma\mathfrak{g}\,,\\
\widetilde{g} &= \widetilde{g}_1\,.
\end{aligned}$$

Then

$$(4.112)\qquad \widetilde{g}_s^{-1}\frac{d\widetilde{g}}{ds} = \alpha_s \in \widetilde{\Sigma\mathfrak{g}}\,.$$

Now we will show that if $A\in\mathcal{A}$, $f\in L_A$,

$$(4.113)\qquad f\cdot\widetilde{g} = \exp\left(i\pi\int_0^1 ds\left[\int_\Sigma \langle A\cdot g_s, d\alpha_s\rangle\right]\right) f \in L_{A\cdot g}$$

is an unambigous formula for the action of $\widetilde{\Sigma G}$ on $L$. Clearly we may assume that $g_1 = g_0 = 1$, so that $s\in S_1 = \mathbf{R}/\mathbf{Z}\mapsto g_s\in\Sigma G$ is a loop in $\Sigma G$.

Recall that by(4.61),

$$(\widetilde{L\mathfrak{g}})^s = (L\mathfrak{g})^s \oplus \mathbf{R}. \tag{4.114}$$

The splitting (4.114) defines a connection on the $S^1$ bundle $(\widetilde{LG})^s \to (LG)^s$. Then by definition $s \in [0,1] \mapsto \widetilde{g}_{s|\partial\Sigma} \in (\widetilde{LG})^s$ is the horizontal lift of $s \in S^1 \mapsto g_s \in (LG)^s$. In particular $\widetilde{g}_{1|\partial\Sigma} \in S^1$.

By (4.104), we only need to verify that in this case,

$$\exp\left(i\pi \int_0^1 ds \left[\int_\Sigma \langle A\cdot g_s, d\alpha_s\rangle\right]\right) = \widetilde{g}_1. \tag{4.115}$$

Observe that

$$A\cdot g_s = g_s^{-1}Ag_s + g_s^{-1}dg_s. \tag{4.116}$$

Let $\theta^g, \theta^d$ be the $\mathfrak{g}$-valued left and right invariant forms on $G$, i.e.

$$\theta^g = g^{-1}dg, \theta^d = dgg^{-1}. \tag{4.117}$$

Then

$$d\theta^g = -\frac{1}{2}[\theta^g,\theta^g] \quad d\theta^d = \frac{1}{2}[\theta^d,\theta^d]. \tag{4.118}$$

From (4.111), (4.118), we get

$$\begin{aligned} d\alpha_s &= \frac{\partial}{\partial s}g_s^*\theta^g + [\alpha_s, g_s^*\theta^g] \\ &= g_s^{-1}\frac{\partial}{\partial s}(g_s^*\theta^d)g_s. \end{aligned} \tag{4.119}$$

Therefore

$$\langle g_s^{-1}Ag_s, d\alpha_s\rangle = \frac{\partial}{\partial s}\langle A, g_s^*\theta^d\rangle. \tag{4.120}$$

Since $g_0 = g_1 = 1$, by (4.120), we obtain

$$\int_0^1 ds \int_\Sigma \langle g_s^{-1}Ag_s, d\alpha_s\rangle = 0. \tag{4.121}$$

By (4.121),

$$\int_0^1 ds \int_\Sigma \langle A\cdot g_s, d\alpha_s\rangle = \int_{S^1} ds \int_\Sigma \langle g_s^{-1}dg_s, d\alpha_s\rangle \tag{4.122}$$

and so (4.122) does not depend on $A$. Observe that this last fact follows from general principles, and that the right-hand side of (4.122) is just the left-hand side evaluated at $A = 0$.

Also

$$\int_\Sigma \langle g_s^{-1}dg_s, d\alpha_s\rangle = -\int_{\partial\Sigma}\langle g_s^*\theta^g, \alpha_s\rangle - \frac{1}{2}\int_\Sigma \langle g_s^*[\theta^g,\theta^g], \alpha_s\rangle. \tag{4.123}$$

In view of (4.65), (4.123), we get

$$\frac{1}{2}\int_0^1 ds \int_\Sigma \langle g_s^{-1}dg_s, d\alpha_s\rangle = -\frac{1}{2}\int_{S^1} ds \int_{\partial\Sigma}\langle g_s^*\theta^g, \alpha_s\rangle - \int_{\Sigma\times S^1} g^*\kappa. \tag{4.124}$$

Recall that $\Sigma$ is obtained from the Riemann surface $X$ by deleting $s$ disks $\Delta_1, \dots, \Delta_s$ centered at $x_1, \dots, x_s$. Since $\pi_i(G) = 0$, $0 \leq i \leq 2$, the smooth map $g : S^1 \times \Sigma \mapsto G$ extends to a smooth map $g : S^1 \times X \to G$ such that

$$g = 1 \text{ near } S^1 \times \{x_j\}\ ,\ 1 \leq j \leq s\,. \tag{4.125}$$

From (4.124), we get

(4.126)
$$\frac{1}{2}\int_0^1 ds \int_\Sigma \langle g_s^{-1}dg_s, d\alpha_s\rangle = -\frac{1}{2}\int_{S^1} ds \int_{\partial\Sigma} \langle g_s^* \theta^g, \alpha_s\rangle + \sum_{j=1}^{s} \int_{S^1\times\Delta_j} g^*\kappa - \int_{S^1\times\Delta_j} g^*\kappa.$$

Since $\kappa \in H^3(G, \mathbf{Z})$,

$$\int_{S^1\times X} g^*\kappa \in \mathbf{Z}. \tag{4.127}$$

For $0 \leq r \leq 1, t \in S^1$, then $rt \in \Delta$. If $(z,t) \in \Delta \times S_j^1$, put

$$h(z,t) = g(\frac{z}{|z|}, |z|t). \tag{4.128}$$

By (4.125), $h$ is a smooth map from $\Delta \times S_j^1$ into $G$. Let $\overline{h} : \Delta \to (LG)^s$ be the smooth map defined the way we did after (4.69).

If we orient $S_j^1$ as a component of $\partial\Sigma$, we get

$$-\frac{1}{2}\int_{S^1} ds \int_{\partial\Sigma} \langle g_s^*\theta^g, \alpha_s\rangle = -\frac{1}{2}\sum_{j=1}^{s} \int_{S^1\times S_j^1} \langle h^{-1}\frac{\partial h}{\partial t}, h^{-1}\frac{\partial h}{\partial s}\rangle dt ds. \tag{4.129}$$

Also for $1 \leq j \leq s$, by orienting $S_j^1$ as before,

$$\int_{S^1\times\Delta_j} g^*\kappa = \int_{\Delta\times S_j^1} h^*\kappa. \tag{4.130}$$

Using Theorem 4.16 and (4.129)-(4.130), we get

$$\begin{aligned} -\frac{1}{2}\int_{S^1} ds \int_{\partial\Sigma} \langle g_s^*\theta^g, \alpha_s\rangle + \textstyle\sum_{j=1}^{s} \int_{S^1\times\Delta_j} g^*\kappa \\ = \textstyle\int_\Delta \overline{h}^*\eta\,. \end{aligned} \tag{4.131}$$

Using now Theorem 4.17 and (4.122), (4.126), (4.127), (4.131), we get (4.115). So formula (4.113) for $f \cdot \widetilde{g}$ is unambiguous. Also by (4.102), $\widetilde{\Sigma G}$ preserves $\nabla^L$.

Assume now that $g \in G$. Let $\alpha \in \mathfrak{g}$ be such that $g = \exp(\alpha)$. Put

$$g_s = \exp(s\alpha)\,. \tag{4.132}$$

Recall that $G \subset \widetilde{\Sigma G}$. Then for $s \in [0,1]$, $g_s \in \widetilde{\Sigma G}$. By (4.113), if $f \in L_A$,

$$f \cdot g = f \in L_{A\cdot g}\,. \tag{4.133}$$

Finally, by (4.104), if $t \in S^1$,

$$f \cdot t = e^{2i\pi t} f\,. \tag{4.134}$$

The proof of our Theorem is completed. ☐

REMARK 4.32. By (4.120), in (4.124), we get

$$\begin{aligned}\frac{1}{2}\int_0^1 ds \int_\Sigma \langle A\cdot g_s, d\alpha_s\rangle = \frac{1}{2}\int_\Sigma \langle A, dg_1 g_1^{-1}\rangle \\ -\frac{1}{2}\int_0^1 ds \int_{\partial\Sigma} \langle g_s^{-1}dg, g_s^{-1}\frac{\partial g}{\partial s}\rangle \\ -\int_{\Sigma\times[0,1]} g^*\kappa\,.\end{aligned} \tag{4.135}$$

So (4.135) makes more explicit the dependence of (4.113) on $A$. Also if $f \in C^\infty(\mathcal{A}, L), \widetilde{g} \in \widetilde{\Sigma G}$, let $\widetilde{g}f \in C^\infty(\mathcal{A}, L)$ be given by

$$\widetilde{g}f(A) = g^{-1}f(A\cdot\widetilde{g})\,. \tag{4.136}$$

Then (4.136) defines a representation of $\widetilde{\Sigma G}$ on $C^\infty(\mathcal{A}, L)$. In this representation $t \in S^1$ acts like $e^{-2i\pi t}$. Also if $\widetilde{\alpha} \in \widetilde{\Sigma G}$,

$$\frac{d}{dt}e^{t\widetilde{\alpha}}\cdot f = L_{\widetilde{\alpha}^{\mathcal{A}}}f\,. \tag{4.137}$$

**4.7. The non simply connected case.** Assume now that $G$ is a connected semisimple compact Lie group, which is not necessarily simply connected. Let $\widetilde{G}$ be the universal cover of $G$. Then $\pi_1(G) \subset Z(\widetilde{G})$ and $G = \widetilde{G}/\pi_1(G)$.

Let $P_{g,s}$ be a $4g + 3s$ polygon covering $\Sigma$. The edges of $P_{g,s}$ are denoted $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g^{-1}, b_g^{-1}, c_1, d_1, c_1^{-1}, \ldots, c_s, d_s, d_s^{-1}$. For $1 \le j \le s$, set $w_j = c_j d_j c_j^{-1}$. Let $q$ be an element of $\Sigma$ which is a common point to the $a_i, b_i, w_j$. Then $\pi_1(q, \Sigma)$ is generated by the circles $a_1, b_1, \ldots, a_g, b_g, w_1, \ldots, w_s$. The case $g = 1, s = 1$ is represented in Figure 5.1.

Let $P \xrightarrow{G} \Sigma$ be a $G$-bundle on $\Sigma$. Take $x \in \Sigma \setminus (\partial\Sigma \bigcup \cup_{i=1}^g (a_i \cup b_i))$. Then $\Sigma\setminus\{x\}$ retracts on $(\cup_{i=1}^g a_i \cup b_i)\bigcup(\cup_{j=1}^s w_j)$. Since $G$ is connected, the restriction of $P$ to the 1-skeleton $(\cup_{i=1}^g a_i \cup b_i)\bigcup(\cup_{j=1}^s w_j)$ is trivial. Therefore $P_{\Sigma\setminus\{x\}}$ is trivial, i.e.

$$P_{|\Sigma\setminus\{x\}} \simeq \Sigma\setminus\{x\}\times G\,. \tag{4.138}$$

Let $\Delta$ be a small disk in $\Sigma$ centered at $x$. We orient $\partial\Delta$ as part of $\partial(\Sigma\setminus\Delta)$. Then

$$P_{|\Delta} \simeq \Delta\times G\,. \tag{4.139}$$

Let $\sigma : \Delta\setminus\{x\} \to G$ be the transition map describing the $G$-bundle $P$ on $\Sigma$, i.e.

$$(y, g) \in P_{|\Delta} \simeq (y, \sigma g) \in P_{|\Sigma\setminus\{x\}}\ , y \in \Delta\setminus\{x\}\,. \tag{4.140}$$

Then the homotopy classes of $G$-bundles are classified by the homotopy classes of maps : $\Delta\setminus\{x\} \to G$. Since $\Delta\setminus\{x\}$ retracts on $\partial\Delta \simeq S^1$, the homotopy classes of $G$ bundles are specified by the element $[P]$ of $\pi_1(G)$ associated to $\sigma_{|\partial\Delta} : S^1 \to G$.

Clearly $[\sigma]$ does not depend on the trivialization of $\pi$ on $\Delta$.

The map $\pi \to [P] \in \pi_1(G)$ depends explicitly on the trivialization of $P$ on $\Sigma\setminus\{x\}$. More precisely the homotopy classes of trivializations of $P$ on the 1-skeleton are given by $\pi_1(G)^{2g+s}$. This just says that if a trivialization of $P$ on $\Sigma\setminus\{x\}$ is given, all the other trivializations are given by the action of $\pi_1(G)^{2g+s}$ on this trivialization.

Let $p$ be the map :

$$p : (a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s) \in \pi_1(G)^{2g+s} \mapsto \sum_{j=1}^{s} c_j \in \pi_1(G). \tag{4.141}$$

Then if given $c \in \pi_1(G)^{2g+s}$, we replace the given trivialization of $P$ on $\Sigma \setminus \{x\}$ by the action of $c$, $[P]$ is changed into $[P] - p(c)$.

Finally, with the above provisos, the map $P \to [P] \in \pi_1(G)$ does not depend on the choice of $x$.

Clearly, if $P \xrightarrow{G} \Sigma$ is trivial, $P$ lifts to a $\widetilde{G}$-bundle. Conversely, if $P \xrightarrow{G} \Sigma$ lifts to a $\widetilde{G}$-bundle $Q \xrightarrow{\widetilde{G}} \Sigma$, since $Q$ is trivial, and $P = Q/\pi_1(G)$, $P$ is also trivial. Therefore $[P] \in \pi_1(G)$ describes the obstruction to a lifting of the $G$-bundle $P$ to a $\widetilde{G}$-bundle $Q$.

As before we fix a trivialization of $P$ on $\Sigma \setminus \{x\}$.

Let $\nabla^P$ be a connection on the $G$-bundle $P \xrightarrow{G} \Sigma$. Since

$$P_{|\Sigma \setminus \{x\}} \simeq \Sigma \setminus \{x\} \times G, \tag{4.142}$$

$P_{|\Sigma \setminus \{x\}}$ lifts to the trivial $\widetilde{G}$-bundle.

$$Q = \Sigma \setminus \{x\} \times \widetilde{G}. \tag{4.143}$$

The connection $\nabla^P$ lifts to a connection $\nabla^Q$ on $Q$. Clearly, this construction of $Q$ depends on the choice of the trivialization of $P$ on $\Sigma \setminus \{x\}$.

We will write that a family of circles $\Delta$ of center $x$ tends to $x$ if their radius tends to 0. Clearly as $\Delta \to x$, the holonomy of $\nabla^P$ around $x$ tends to 1.

PROPOSITION 4.33. *As $\Delta \to x$, the holonomy of $\nabla^Q$ around $\partial\Delta \subset \partial(\Sigma \setminus \Delta)$ tends to $[P] \in \pi_1(G) \subset Z(\widetilde{G})$. In particular if $\nabla^P$ if flat, for $\Delta$ small enough, the holonomy of $\nabla^Q$ around $\partial\Sigma \setminus \Delta$ is equal to $[P]$.*

PROOF. We fix an origin in $S^1 \simeq \partial\Delta$. Let $t \in [0,1] \mapsto \tau_t \in G$, $\tau_0 = 1$ be a horizontal lift of $S^1 \subset \partial(\Sigma \setminus \Delta)$ in $P_{|\partial\Delta}$ with respect to $\nabla^P$, in the trivialization $P_{|\Delta} \simeq \Delta \times G$. Then $t \in [0,1] \mapsto \sigma_t \tau_t \sigma_0^{-1} \in G$ denotes the corresponding horizontal lift of $S^1$ in the trivialization $P_{|\Sigma \setminus \{x\}} \simeq \Sigma \setminus \{x\} \times G$.

Let $t \in [0,1[ \mapsto \widetilde{\sigma}_t \in \widetilde{G}$, $t \in [0,1] \mapsto \widetilde{\tau}_t \in \widetilde{G}$ be lifts of $t \in [0,1[ \mapsto \sigma_t \in G$, $t \in [0,1[ \mapsto \tau_t \in G$, with $\widetilde{\tau}_0 = 1$. Then $t \in [0,1] \mapsto \widetilde{\sigma}_t \widetilde{\tau}_t \widetilde{\sigma}_0^{-1} \in \widetilde{G}$ is a horizontal lift of $S^1 \simeq \partial\Delta$ in $Q_{|\partial\Delta}$ in the trivialization $Q_{|\Sigma \setminus \{x\}} \simeq \Sigma \setminus \{x\} \times \widetilde{G}$, with respect to $\nabla^Q$. In particular, parallel transport along $S^1 \simeq \partial\Delta$ is given by $\widetilde{\sigma}_1 \widetilde{\tau}_1 \widetilde{\sigma}_0^{-1}$. Now recall that by definition

$$\widetilde{\sigma}_1 = [P]\widetilde{\sigma}_0 \ , \ [P] \in \pi_1(G) \subset Z(\widetilde{G}). \tag{4.144}$$

Therefore the above parallel transport is just $[P]\widetilde{\sigma}_0 \widetilde{\tau}_1 \widetilde{\sigma}_0^{-1}$.

Now as $\Delta \to x$, $\tau_1 \to 1$. Therefore $\widetilde{\tau}_1 \to 1$, and the parallel transport tends to $[P]$.

The proof of our Proposition is completed. □

**4.8. The case of a connected subgroup of a simply connected group.** Let now $G$ be a connected and simply connected compact simple Lie group. Let $H \subset G$ be a connected compact semisimple Lie subgroup of $G$.

Let $P \xrightarrow{G} \Sigma$ be a $G$-bundle on $\Sigma$. Then

$$P \simeq \Sigma \times G\,. \tag{4.145}$$

Reducing the $G$-bundle $P$ to a $H$-bundle $Q$ is equivalent to finding a section of $P \times_G G/H \simeq \Sigma \times G/H$. Let $\dot{1} \in G/H$ be the image of $1 \in G$. Then if $\eta \in \Sigma(G/H)$, the corresponding $H$-bundle $Q$ is given by

$$Q = \{(y, g) \in \Sigma \times G\ ,\ g^{-1}\eta = \dot{1}\}\,. \tag{4.146}$$

Therefore homotopy classes of $H$-reductions of $P$ are just homotopy classes in $\Sigma(G/H)$.

Since $\pi_i(G) = 0$, $0 \leq i \leq 2$, we get

$$\begin{aligned} \pi_2(G/H) &\simeq \pi_1(H)\,, \\ \pi_i(G/H) &= 0\ ,\ 0 \leq i \leq 1\,. \end{aligned} \tag{4.147}$$

Take $x \in \Sigma \setminus (\partial\Sigma \bigcup \cup_{i=1}^{g}(a_i \cup b_i))$ as in Section 4.7. Recall that $\Sigma \setminus \{x\}$ is homotopy equivalent to $\cup_{i=1}^{g}(a_i \cup b_i) \bigcup \cup_{j=1}^{g} w_j$. Since $\pi_i(G/H) = 0$, $0 \leq i \leq 1$, there is only one homotopy class of sections $\Sigma \setminus \{x\} \to G/H$.

If $\eta$ is a section of $\Sigma \times G/H$, we may and we will assume that $\eta = \dot{1}$ on $\Sigma \setminus \Delta$. In this case

$$Q_{|\Sigma\setminus\Delta} = \Sigma \times H, \tag{4.148}$$

and $Q$ has a canonical section over $\Sigma \setminus \Delta$.

Therefore homotopy classes of $H$-reductions of $P$ are classified by homotopy classes of maps $\eta : \Delta \to G/H$ such that $\eta_{|\partial\Delta} = \dot{1}$, i.e. by $\pi_2(G/H) \simeq \pi_1(H)$.

Take $z \in \pi_2(G/H) \simeq \pi_1(H)$. Let $Q_z$ be the $H$-reduction of $P$ associated to $z$.

By the above, $Q_{|\Sigma\setminus\{x\}}$ has been canonically trivialized on $\Sigma \setminus \{x\}$. Since $Q$ is a $H$-bundle, the class $[Q_z] \in \pi_1(H)$ is well-defined.

PROPOSITION 4.34. *The following identity holds*

$$[Q_z] = z \text{ in } \pi_1(H)\,. \tag{4.149}$$

PROOF. Let $y \in \Delta \mapsto h(y) \in G/H$ be a smooth map such that $h_{|\partial\Delta} = \dot{1}$, representing $z$ in $\pi_1(H) \simeq \pi_2(G/H)$. Then

$$Q_{z|\Delta} = \{(y, g) \in \Sigma \times G\ ;\ g^{-1}h(y) = \dot{1}\}\,. \tag{4.150}$$

To trivialize $Q_z$ on $\Delta$, we fix a connection $\nabla^{Q_z}$ on $Q_z$. If $(x, g) \in Q_{z,x}$, we parallel transport $g$ along radial lines in $\Delta$. This way, we obtain $g_1 : \partial\Delta \to H$. By definition

$$[g_1] = z \text{ in } \pi_1(H)\,. \tag{4.151}$$

Also $\sigma = g_1$ exactly defines the $H$-bundle $Q_z$ on $\Sigma$ in the sense of (4.140). From (4.151), we get (4.149). The proof of our Proposition is completed. □

**4.9. The action of stabilizers on the line bundle $L$.** Now we use the notation of Section 1, and more especially of Section 1.9.

Let $u \in C/\overline{CR}$. We will specialize the results of Section 4.8 to the case $H = Z(u)$. Clearly the map

$$g \in G/Z(u) \mapsto gug^{-1} \in \mathcal{O}_u \subset G \tag{4.152}$$

is one to one. It maps $\dot{1}$ into $u$.

If $\eta$ is a section of $\Sigma \times \mathcal{O}_u$ over $\Sigma$, the corresponding $Z(u)$-bundle $Q$ is given by

$$Q = \{(x,g) \in \Sigma \times G\ ,\ g^{-1}\eta g = u\}\,. \tag{4.153}$$

Since $Q_{|\Sigma\setminus\{x\}} \simeq \Sigma \setminus \{x\} \times Z(u)$, $Q$ has a section over $\Sigma \setminus \{x\}$. Equivalently, there is $g \in \Sigma \setminus \{x\} \times G$ such that

$$\eta = gug^{-1} \text{ on } \Sigma \setminus \{x\}\,. \tag{4.154}$$

Homotopy classes of trivializations of $Q$ on $\Sigma \setminus \{x\}$ are just homotopy classes of $g \in \Sigma \setminus \{x\}$ such that (4.154) holds. We have already classified these homotopy classes by $\pi_1(Z(u))^{2g+s}$. By (4.147), we know that $\eta_{|\Sigma\setminus\{x\}}$ is homotopy equivalent to the constant $\eta = \dot{1} \simeq u$.

In the sequel, we will assume that $\eta = u$ on $\Sigma \setminus \Delta$ which corresponds to $\eta = \dot{1}$ on $\Sigma \setminus \Delta$ .

Similarly

$$Q_{|\Delta} \simeq \Delta \times Z(u) \tag{4.155}$$

i.e. $Q_{|\Delta}$ has a section. Therefore there exists $g \in \Delta G$ such that

$$\eta = gug^{-1} \text{ on } \Delta\,. \tag{4.156}$$

In particular by (4.156), $g_{|\partial\Delta}$ takes its values in $Z(u)$, so that $g_{|\partial\Delta} \in LZ(u)$. Also there is only one homotopy class of $g \in \Delta G$ such that (4.156) holds.

Then $\sigma = g_{|\partial\Delta} \in LZ(u)$ is exactly the $\sigma$ in (4.140) defining the $Z(u)$-bundle $Q$ on $\Sigma$.

Recall that $[\sigma] \in \pi_1(Z(u)) \simeq \overline{CR}/\overline{CR}_u$. We identify $[\sigma]$ to a given element $[\sigma] \in \overline{CR}$. Let $s \in S^1 \mapsto b_s \in T$ be a loop which represents $[\sigma]$ in $\pi_1(Z(u))$. Put

$$g^0 = (bg^{-1})_{|\partial\Delta} \in LZ(u)\,. \tag{4.157}$$

Then $g^0_{|\partial\Delta}$ is homotopic to 0. Equivalently $g^0_{|\partial\Delta}$ defines an element of $L\widetilde{Z}(u)$. Since $\pi_i(\widetilde{Z}(u)) = 0$, $0 \le i \le 2$, $g^0$ extends to $g^0 \in \Sigma Z(u)$.

By conjugation of $\eta$ by $g^0 \in \Sigma Z(u)$, we may and we will assume in the sequel that

$$\begin{aligned} &\eta = u \text{ on } \Sigma \setminus \Delta\,,\\ &\eta = gug^{-1} \text{ on } \Delta\,,\\ &g \in \Delta G,\ g_{|\partial\Delta} \in LT\,. \end{aligned} \tag{4.158}$$

Clearly $\displaystyle\int_{\partial\Sigma\setminus\Delta} g^{-1}dg \in \overline{CR}$ is the homotopy class of $g \in LT$.

Let now $\eta \in \Sigma G$. Put

$$\mathcal{A}^\eta = \{A \in \mathcal{A}\ ;\ A \cdot \eta = A\}\,. \tag{4.159}$$

Assume that

$$\mathcal{A}^{\eta} \neq \emptyset . \tag{4.160}$$

Put

$$u = \eta(x) . \tag{4.161}$$

In the sequel we assume that

$$u \in C/\overline{CR} . \tag{4.162}$$

Then by (4.159), (4.160), it is clear that $\eta$ is a section of the orbit $\mathcal{O} = \mathcal{O}_u$. In the sequel we assume that the conditions in (4.158) hold.

Let $B \in \mathfrak{t}$ be such that

$$u = \exp(B) . \tag{4.163}$$

For $s \in [0,1]$, $x \in \Sigma \setminus \Delta$, put

$$\overline{\eta}(x,s) = \exp(sB) . \tag{4.164}$$

Then since $\pi_i(G) = 0$, $0 \leq i \leq 2$, the map $\overline{\eta} : \Sigma \setminus \Delta \times [0,1] \to G$ extends to a map $\overline{\eta} : \Sigma \times [0,1] \to G$ such that

$$\begin{aligned} &\bullet\ \overline{\eta}_{|\Sigma\setminus\Delta\times[0,1]} = \exp(sB) , \\ &\bullet\ \overline{\eta}_{|\Sigma\times\{1\}} = \eta . \end{aligned} \tag{4.165}$$

For $s \in [0,1]$, put

$$\eta_s(x) = \overline{\eta}(x,s) . \tag{4.166}$$

Then $s \in [0,1] \mapsto \eta_s \in \Sigma G$ is a smooth path. Let $s \in [0,1] \in \widetilde{\eta}_s \in \widetilde{\Sigma G}$ be the corresponding horizontal lift in $\widetilde{\Sigma G}$.

THEOREM 4.35. *The following identity holds*

$$\widetilde{\eta}_1|_{L|\mathcal{A}^{\eta}} = \exp\left(-2i\pi\langle\int_{\partial\Sigma\setminus\Delta} g^{-1}dg, B\rangle\right) . \tag{4.167}$$

PROOF. We consider the splitting

$$\Sigma = \Sigma \setminus \Delta \cup \Delta . \tag{4.168}$$

Now both $\Sigma \setminus \Delta$ and $\Delta$ are objects like $\Sigma$. They carry the trivial $G$-bundles $P_{|\Sigma\setminus\Delta}$ and $P_{\Delta}$. Therefore to $\Sigma \setminus \Delta$ and $\Delta$, we associate the spaces of $G$-connections $\mathcal{A}^{\Sigma\setminus\Delta}$ and $\mathcal{A}^{\Delta}$, the line bundles $L^{\Sigma/\Delta}$ and $L^{\Delta}$, equipped with the actions of $\widetilde{(\Sigma/\Delta)G}$ and $\widetilde{\Delta G}$.

Clearly $\mathcal{A}$ embeds into $\mathcal{A}^{\Sigma\setminus\Delta} \times \mathcal{A}^{\Delta}$, and $\Sigma G$ embeds into $(\Sigma\setminus\Delta)G \times \Delta G$. First we claim that

$$L = (L^{\Sigma\setminus\Delta} \otimes L^{\Delta})_{|\mathcal{A}} \tag{4.169}$$

and the isomorphism (4.169) identifies the metrics and the connections. In fact this is clear by (4.99).

Put

$$\varepsilon = \{(x,y) \in S^1 \times S^1 \ ,\ xy = 1\} . \tag{4.170}$$

Then $(\widetilde{(\Sigma\setminus\Delta)G} \times \widetilde{\Delta G})/\varepsilon$ acts naturally on $L^{\Sigma\setminus\Delta} \otimes L^{\Delta}$.

We claim that $\widetilde{\Sigma G}$ embeds as a subgroup of $((\widetilde{\Sigma \setminus \Delta)G} \times \widetilde{\Delta G})/\varepsilon$. In fact recall that we orient $\partial\Sigma \setminus \Delta$ as the boundary of $\Sigma \setminus \Delta$. Also in Section 4.1, we defined the morphism $f \in \widetilde{LG} \mapsto \psi f \in \widetilde{LG}$. So if $f \in \widetilde{\Sigma G}$, let $\widetilde{g} \in \widetilde{LG}$ be such that $\rho\widetilde{g} = f_{|\partial\Sigma\setminus\Delta}$. Then to $f \in \widetilde{\Sigma G}$, we associate $((f_{|\Sigma\setminus\Delta G}, \widetilde{g}), (f_{\Delta G}, \psi\widetilde{g})) \in ((\widetilde{\Sigma \setminus \Delta)G} \times \widetilde{\Delta G})/\varepsilon$. Since as we saw in (4.7), if $s \in S^1$, $\psi s = s^{-1}$, we find that this element of $((\widetilde{\Sigma \setminus \Delta)G} \times \widetilde{\Delta G})/\varepsilon$ does not depend on the choice of $\widetilde{g}$.

Now $\widetilde{\Sigma G}$ acts on $L$. Similarly, by the above embedding, $\widetilde{\Sigma G}$ also acts on $(L^{\Sigma\setminus\Delta} \otimes L^{\Delta})_{|\mathcal{A}}$. We claim that both actions coincide. In fact this is obvious by the explicit formula in (4.113).

Let $\eta_s^{\Sigma\setminus\Delta}$ and $\eta_s^{\Delta}$ be the restrictions of $\eta_s$ to $\Sigma \setminus \Delta$ and $\Delta$. Let $s \in [0,1] \mapsto \widetilde{\eta}_s^{\Sigma\setminus\Delta} \in \widetilde{\Sigma \setminus \Delta G}$, $s \in [0,1] \mapsto \widetilde{\eta}_s^{\Delta} \in \widetilde{\Delta G}$ be the corresponding horizontal lifts. By the above, we get the equality in $S^1$,

$$\widetilde{\eta}_{1|L} = \widetilde{\eta}^{\Sigma\setminus\Delta}_{1|L^{\Sigma\setminus\Delta}} \widetilde{\eta}^{\Delta}_{1|L^{\Delta}} . \tag{4.171}$$

Now we will compute both terms in (4.171). By (4.165),

$$\eta_s^{\Sigma/\Delta} = \exp(sB) . \tag{4.172}$$

Using (4.113), we obtain

$$\widetilde{\eta}^{\Sigma\setminus\Delta}_{1|L^{\Sigma\setminus\Delta}} = 1 . \tag{4.173}$$

Consider the path $s \in [0,1] \mapsto \theta_s \in \Delta G$, with

$$\theta_s = g \exp(sB) g^{-1} . \tag{4.174}$$

Clearly by (4.158), (4.165), (4.174),

$$\eta_1^{\Delta} = \theta_1 . \tag{4.175}$$

Moreover since $g_{|\partial\Delta}$ takes its values in $T$,

$$\eta^{\Delta}_{s|\partial\Delta} = \theta_{s|\partial\Delta} . \tag{4.176}$$

Let $s \in [0,1] \mapsto \widetilde{\theta}_s \in \widetilde{\Delta G}$ be the horizontal lift of $s \in [0,1] \mapsto \theta_s \in \Delta G$. By (4.175), (4.176), we get

$$\widetilde{\eta}_1^{\Delta} = \widetilde{\theta}_1 . \tag{4.177}$$

Set

$$\kappa_s = \exp(sB) \in \Delta G . \tag{4.178}$$

Let $s \in [0,1] \mapsto \widetilde{\kappa}_s \in \widetilde{\Delta G}$ be the horizontal lift of $s \in [0,1] \mapsto \kappa_s \in \Delta G$.

Recall that

$$\widetilde{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbf{R} . \tag{4.179}$$

By [**47**, Proposition 4.3.2], we get

$$g(B,0)g^{-1} = \left(gBg^{-1}, -\int_{\partial\Delta} \langle g^{-1}dg, B\rangle\right) . \tag{4.180}$$

Using (4.180), we obtain

$$g\widetilde{\kappa}_1 g^{-1} = \widetilde{\theta}_1 \exp\left(-2i\pi \int_{\partial\Delta} \langle g^{-1}dg, B\rangle\right) . \tag{4.181}$$

Clearly

$$\kappa_1 = u\,, \tag{4.182}$$

and so

$$\theta_1 = g\kappa_1 g^{-1}\,. \tag{4.183}$$

If $A \in \mathcal{A}^{\Delta,\theta_1}$, then $A \cdot g \in \mathcal{A}^{\Delta,\kappa_1}$. By (4.181), we get

$$\widetilde{\kappa}_{1|L^\Delta} = \widetilde{\theta}_{1|L^\Delta} \exp\left(-2i\pi \int_{\partial\Delta} \langle g^{-1}dg, B\rangle\right)\,. \tag{4.184}$$

Finally using (4.113) again, we find

$$\widetilde{\kappa}_{1|L^\Delta} = 1\,. \tag{4.185}$$

From (4.177), (4.184), (4.185), we obtain

$$\begin{aligned} \widetilde{\eta}^{\Delta}_{1|L^\Delta} &= \exp\left(2i\pi \int_{\partial\Delta} \langle g^{-1}dg, B\rangle\right) \\ &= \exp\left(-2i\pi \int_{\partial\Sigma\setminus\Delta} \langle g^{-1}dg, B\rangle\right)\,. \end{aligned} \tag{4.186}$$

By (4.171), (4.173), (4.186), we get (4.167). The proof of our Theorem is completed. □

**4.10. The action of stabilizers on the line bundle $\lambda_p$.** Take now $\theta_1, \dots, \theta_s \in \overline{CR}^*$. Take $p \in \mathbf{Z}^*$. Let $L_1, \dots, L_s$ be the canonical line bundles constructed in Sections 1.14 and 4.2, which are associated to the $\widetilde{L}G$ orbits of $\left(-\frac{pd}{dt} + \theta_1, \dots, -\frac{pd}{dt} + \theta_s\right)$.

DEFINITION 4.36. Put

$$\begin{aligned} \mathcal{A}_p(\theta_1, \dots, \theta_s) &= \{A \in \mathcal{A}, -p\Big(\frac{d}{dt} + A\Big)_{|S^1_1} \in \mathcal{O}_{-p\frac{d}{dt}+\theta_1}, \dots, \\ &\qquad -p\Big(\frac{d}{dt} + A\Big)_{|S^1_s} \in \mathcal{O}_{-p\frac{d}{dt}+\theta_s}\,. \end{aligned} \tag{4.187}$$

Let $\nu_p : \mathcal{A}_p(\theta_1, \dots, \theta_s) \mapsto \prod_{j=1}^{s} \mathcal{O}_{-p\frac{d}{dt}+\theta_j}$ be given by

$$A \mapsto \nu_p(A) = \Big(-p(\frac{d}{dt} + A_{|S^1_1}), \dots, -p(\frac{d}{dt} + A_{|S^1_s})\Big)\,. \tag{4.188}$$

Equivalently, $\mathcal{A}_p(\theta_1, \dots, \theta_s)$ is the set of $A \in \mathcal{A}$ such that the holonomy of $A$ on $S^1_1$ lies in the $G$-orbit of $e^{\theta_1/p}, \dots$ the holonomy of $A$ on $S^1_s$ lies in the $G$-orbit of $e^{\theta_s/p}$.

DEFINITION 4.37. Let $\lambda_p$ be the line bundle on $\mathcal{A}_p(\theta_1, \dots, \theta_s)$,

$$\lambda_p = L^p \otimes \nu_p^*\Big(\otimes_{j=1}^s L_j\Big)\,. \tag{4.189}$$

Clearly $\Sigma G$ acts on the right on $\mathcal{A}_p(\theta_1, \dots, \theta_s)$. By Theorem 4.31, this action lifts to an action on the right of $\widetilde{\Sigma G}$ on $\lambda_p$.

PROPOSITION 4.38. *The action of $\widetilde{\Sigma G}$ on $\lambda_p$ descends to an action of $\Sigma G$ on $\lambda_p$.*

PROOF. We only must show that $S^1 \subset \widetilde{\Sigma G}$ acts trivially on $\lambda_p$. However by (4.108), if $t \in S^1$, $t$ acts on the right on $L^p$ like $e^{2i\pi pt}$. Also $t$ acts on the right on $\otimes_{j=1}^s L_j$ like $e^{-2i\pi pt}$. Our Proposition follows. □

Let $\eta \in \Sigma G$. Assume that

$$\mathcal{A}^\eta \cap \mathcal{A}_p(\theta_1, \dots, \theta_s) \neq \emptyset . \tag{4.190}$$

Put

$$u = \eta(x) . \tag{4.191}$$

By conjugating $\eta$ by $g \in G$, we may and we will assume that $u \in T$. We identify $u$ to a corresponding element in $\mathfrak{t}$, which we still denote by $u$. In the sequel, we assume that

$$u \in C/\overline{CR}, \tag{4.192}$$

so that $Z(u)$ is semisimple.

Now if $A \in \mathcal{A}^\eta$, the $G$-bundle $P$ on $\Sigma$ reduces to a $Z(u)$-bundle $Q$ on $\Sigma$.

Recall that by (4.138),

$$Q_{|\Sigma \setminus \{x\}} \simeq \Sigma \setminus \{x\} \times Z(u) . \tag{4.193}$$

Put

$$\widetilde{Q}_{|\Sigma \setminus \{x\}} = \Sigma \setminus \{x\} \times \widetilde{Z}(u) . \tag{4.194}$$

Then the connection $A$ lifts to a $\widetilde{Z}(u)$ connection on $\widetilde{Q}_{|\Sigma \setminus \{x\}}$. Still two different trivializations of $Q_{|\Sigma \setminus \{x\}}$ produce two distinct $\widetilde{Z}(u)$ bundles $\widetilde{Q}_{|\Sigma \setminus \{x\}}$.

Recall that by (1.173),

$$\mathcal{O}_{\theta_j/p} \cap Z(u) = \bigcup_{w \in W_u \backslash W} \mathcal{O}_{Z(u)}(w\theta_j/p) . \tag{4.195}$$

To normalize $Q_{|\Sigma \setminus \{x\}}$, we impose that the holonomy of the connection $A$ along $S_j^1$ be conjugate in $\widetilde{Z}(u)$ to $e^{w^j \theta_j/p} \in \widetilde{Z}(u)$, with $w^j \in W_u \backslash W$.

Let then $[Q] \in \pi_1(Z(u)) \simeq \frac{\overline{CR}}{\overline{CR}_u}$ be defined in Sections 4.7-4.9.

THEOREM 4.39. *The following identity holds*

$$\eta_{|\lambda_p|\mathcal{A}^\eta \cap \mathcal{A}_p}(\theta_1, \dots, \theta_s) = \exp\left(-2i\pi\langle \sum_{j=1}^s w^j \theta_j + p[Q], u\rangle\right) . \tag{4.196}$$

PROOF. As we saw in (4.158), by conjugating $\eta$ by an element of $\Sigma G$, we may and we will assume that

$$\begin{aligned} \eta &= u \text{ on } \Sigma \setminus \Delta , \\ &= gug^{-1} \text{ on } \Delta \text{ with } g \in \Delta G \ , \ g_{|\partial\Delta} \in T . \end{aligned} \tag{4.197}$$

Then

$$Q_{|\Sigma \setminus \Delta} = \Sigma \setminus \Delta \times Z(u) . \tag{4.198}$$

Let $\nabla^P$ be a connection on $P$ which preserves $\eta$. Then $\nabla^P$ induces a connection $\nabla^Q$, which, in the trivialization associated to (4.198), is given by

$$\nabla^Q = d + A \ , \ A \in \Omega^1(\Sigma \setminus \Delta, \mathfrak{z}(u)) . \tag{4.199}$$

The connection $\nabla^Q$ lifts to a connection $\nabla^{\widetilde{Q}}$ given by

$$\nabla^{\widetilde{Q}} = d + A. \tag{4.200}$$

By our choice of $\widetilde{Q}$, we may and we will assume that the holonomy of $\nabla^{\widetilde{Q}}$ on $S^1_j$ lies in the $\widetilde{Z}(u)$ conjugacy class of $e^{w^j\theta_j/p}$.

By [**21**, Section 3.2], [**47**, Proposition 4.3.6] for $1 \le j \le s$, there is $\widetilde{h}_j \in L\widetilde{Z}(u)$ such that

$$\widetilde{h}_j^{-1}\left(\frac{d}{dt} + A_{|S^1_j}\right)\widetilde{h}_j = \frac{d}{dt} - w_j\frac{\theta_j}{p}. \tag{4.201}$$

Since $\widetilde{Z}(u)$ is simply connected, there is $h \in \Sigma\widetilde{Z}(u)$ such that

$$\widetilde{h}_{|S^1_j} = \widetilde{h}_j \ , \ 1 \le j \le s. \tag{4.202}$$

Let $h \in \Sigma Z(u)$ be the image of $\widetilde{h}$. Then $h_{|\partial\Delta} \in LZ(u)$ is homotopic to the constant loop 1. By proceeding as in (4.157), we may and we will assume that $h_{|\partial\Delta} \in LT$ is homotopic to the constant loop 1 in $Z(u)$.

Now we may as well replace $\eta$ by $h^{-1}\eta h$ and $A \in \mathcal{A}^\eta$ by $A \cdot h \in \mathcal{A}^{h^{-1}\eta h}$. By (4.197),

$$\begin{aligned} h^{-1}\eta h_{|\Delta} &= h^{-1}gug^{-1}h_{|\Delta}, \\ h^{-1}g_{|\partial\Delta} &\in T. \end{aligned} \tag{4.203}$$

Finally the homotopy class of $h^{-1}g_{|S^1_j} \in LZ(u)$ is the same as the homotopy class of $g_{|S^1_j} \in LZ(u)$, $1 \le j \le s$.

So basically, we may and we will assume that $\eta$ verifies the assumptions in (4.197), but we also have the extra assumptions

$$\left(\frac{d}{dt} + A\right)_{|S^1_j} = \frac{d}{dt} - w^j\frac{\theta_j}{p} \ , \ 1 \le j \le s. \tag{4.204}$$

Now by definition, the right action of $u \in \widetilde{L}G$ on $L_{j,-pd/dt+w^j\theta_j}$ is given by $\exp(-2i\pi\langle u, w^j\theta_j\rangle) \in S^1$. Using (4.167), we see that

$$\eta_{|\lambda_p} = \exp\left(2i\pi\langle -\sum_{j=1}^{s} w^j\theta_j - p\int_{\partial\Sigma\setminus\Delta} g^{-1}dg, u\rangle\right). \tag{4.205}$$

By Proposition 4.34,

$$\int_{\partial\Sigma\setminus\Delta} g^{-1}dg = [Q] \in \pi_1(Z(u)). \tag{4.206}$$

From (4.205), (4.206), we get (4.196).

The proof of our Theorem is completed. □

REMARK 4.40. It should be pointed out that the expression (4.196) is natural. In fact it only depends on the $w^j \in W_u\backslash W$. Also as we saw after (4.141), we know how $[Q]$ changes if we change the trivialization of $Q_{|\Sigma\setminus\Delta}$. One verifies easily that (4.196) is compatible with this formula.

Also by Proposition 1.44, one verifies easily that (4.196) is compatible with the fact that $\Sigma G$ acts as a group on $\lambda_p$.

## 5. The moduli space of flat bundles on a Riemann surface

The purpose of this Section is to construct the moduli space $M/G$ of flat $G$-bundles over a Riemann surface $\Sigma$ with marked points. We establish the Witten formula [**63, 64**] for the symplectic volume of $M/G$. Also we show that the formula of Witten [**64**] and Jeffrey-Kirwan [**28**] can be applied to $M/G$. In particular, we express the integrals of certain characteristic classes over $M/G$ in terms of the action of certain differential operators on local polynomials over the maximal torus $T$. Also we give a formula for $c_1(TM/G)$ . All these results will be needed in Section 6, when we apply the Theorem of Riemann-Roch-Kawasaki [**32, 33**] to $M/G$ .

As explained in the introduction , our derivation of the Witten formula is closely related to earlier work by Liu [**39, 40**].

This Section is organized as follows. In Section 5.1, we give the standard combinatorial description of $\Sigma$. In Section 5.2, we introduce the corresponding combinatorial complexes, which compute the absolute and relative cohomology of flat vector bundles over $\Sigma$. In Section 5.3, we introduce the $G$-equivariant map $\phi : X = G^{2g} \times \prod_1^s \mathcal{O}_j \to G$, such that $M = \{x \in X, \phi(x) = 1\}$. Also we relate the differential of $\phi$ to the combinatorial complexes on $\Sigma$ which compute the absolute and relative cohomology of the flat adjoint vector bundle $E$. In Section 5.4, we give natural conditions on the orbits $\mathcal{O}_j$ under which 1 is a regular value of $\phi$, so that $M/G$ is an orbifold. In Section 5.5, we give conditions under which the set of elements in the fibres of $\phi$ or in $X$ with non trivial stabilisers are of codimension $\geq 2$. In Section 5.6, we describe $TM/G$ and the symplectic form $\omega$. In Section 5.7, we show that the symplectic volume form on $M/G$ can be evaluated in terms of the corresponding combinatorial complexes. In Section 5.8, using the results of the previous subsections, we prove Witten's formula [**63, 64**] for the symplectic volume of the reductions of $M/G$.

In Section 5.9, we define logarithms from certain subsets of $G$ into $\mathfrak{g}$. In Section 5.10, we show that the invariant open sets of $X$ where $G$ acts locally freely are naturally equipped with a symplectic form, and that the action of $G$ on these sets has a moment map. In Section 5.11, we compute the integrals of certain characteristic classes on the moduli spaces associated to the centralizers $Z(u), u \in C/\overline{R}^*$. As we will see in Section 6.4, these moduli spaces correspond to the strata of $M/G$. In Section 5.12, we compute certain Euler characteristics. Finally, in Section 5.13, we give a formula for $c_1(TM/G)$.

**5.1. Combinatorial description of the Riemann surface $\Sigma$.** If $a, b$ lie in a group $\Gamma$, put

$$[a,b] = aba^{-1}b^{-1}\,. \tag{5.1}$$

Let $g \in \mathbf{N}$, $s \in \mathbf{N}$, $g + s > 0$. Let $\Gamma$ be the discrete group generated by 1, $u_1, v_1, \ldots, u_g, v_g, w_1, \ldots, w_s$, and the relation

$$\prod_{i=1}^{g}[u_i, v_i] \prod_{j=1}^{s} w_j = 1\,. \tag{5.2}$$

Let $X$ be an oriented connected compact surface of genus $g$. Let $x_1, \ldots, x_s$ be $s$ distinct elements of $X$. Put

$$X' = X \setminus \{x_1, \ldots, x_s\}\,. \tag{5.3}$$

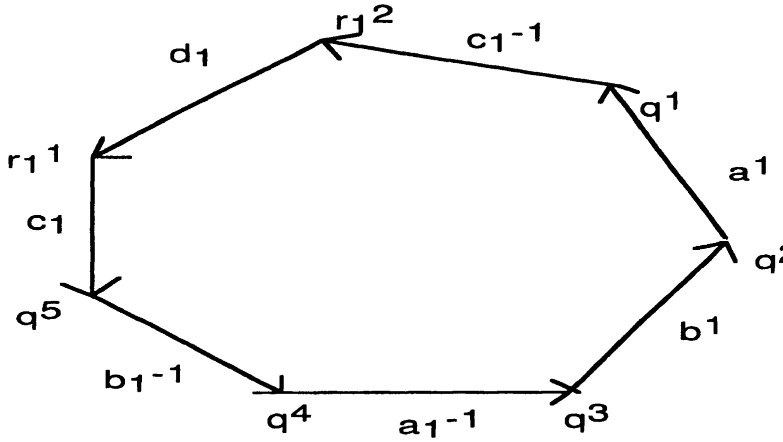


FIGURE 5.1

Let $\Delta_1, \dots, \Delta_s$ be small non intersecting open disks in $X$, centered at $x_1, \dots, x_s$. Let $\Sigma$ be the surface with boundary

$$\Sigma = X \setminus \bigcup_{j=1}^{s} \Delta_j . \tag{5.4}$$

Let $q \in \Sigma$. Then $\Gamma$ can be canonically identified with $\pi_1(q, \Sigma) = \pi_1(q, X')$. Let $P_{g,s}$ be a $4g+3s$ polygon covering $\Sigma$. The edges of $P_{g,s}$ are denoted $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g^{-1}, b_g^{-1}, c_1, d_1, c_1^{-1}, \dots, c_s, d_s, c_s^{-1}$. Then $P_{g,s}$ induces a cell decomposition of $\Sigma$ with one two-cell, $2g + 2s$ one-cells, $1 + s$ zero-cells. The two-cell is the interior $\overset{\circ}{P}_{g,s}$ of $P_{g,s}$. The $2g + 2s$ one-cells are the circles $a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_s$, and the segments $c_1, \dots, c_s$. The $1 + s$ zero-cells consist of $q$ which lies in the $a_i, b_i$, and is a boundary point for the $c_i$, and also $r_1, \dots, r_s$ which lie respectively in $c_1, \dots, c_s$ and in $d_1, \dots, d_s$. The case $g = 1, s = 1$ is represented in Figure 5.1, where $q^1, \dots, q^5$ represent $q$, and $r_1^1, r_2^2$ represent $r_1$. In the description given above, the group $\pi_1(q, \Sigma)$ is generated by the circles $u_1 = a_1, v_1 = b_1, \dots, u_g = a_g, v_g = b_g$, $w_1 = c_1 d_1 c_1^{-1}, \dots, w_s = c_s d_s c_s^{-1}$.

Also the above decomposition induces a cell decomposition of $\partial\Sigma$, with $s$ 1-cells $d_1, \dots, d_s$, and $s$ 0-cells $r_1, \dots, r_s$.

To the above cell-decompositions of $\Sigma$, we associate the corresponding complexes over $\mathbf{Z}$, $(C_{\cdot}^{\Sigma}, \partial)$, $(C_{\cdot}^{\partial\Sigma}, \partial)$ which calculate the homology groups $H_{\cdot}(\Sigma, \mathbf{Z}), H_{\cdot}(\partial\Sigma, \mathbf{Z})$. Let $(C_{\cdot}^{\Sigma,r}, \partial)$ be the quotient complex defined by the exact sequence

$$0 \to (C_{\cdot}^{\partial\Sigma}, \partial) \to (C_{\cdot}^{\Sigma}, \partial) \to (C_{\cdot}^{\Sigma,r}, \partial) \to 0 . \tag{5.5}$$

Then the homology of $(C_{\cdot}^{\Sigma,r}, \partial)$ is the relative homology $H((\Sigma, \partial\Sigma), \mathbf{Z})$.

Note in particular that $C_2^{\Sigma,r}$ is generated by $\overset{\circ}{P}$, $C_1^{\Sigma,r}$ by the $2g + s$ one-cells $a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s$, $C_0^{\Sigma,r}$ by the zero-cell $q$.

**5.2. The combinatorial complexes on $\Sigma$.** Let $V$ be a finite dimensional complex vector space. Let $\tau : \Gamma \to \mathrm{Aut}(V)$ be a represention of $\Gamma$. Equivalently let

$u_i, v_i$ $(1 \leq i \leq g)$, $w_j$ $(1 \leq j \leq s)$ in $\mathrm{Aut}(V)$ such that

$$\prod_{i=1}^{i}[u_i, v_i]\prod_{j=1}^{s} w_j = 1 \tag{5.6}$$

Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the universal cover of $\Sigma$. Put

$$F = \widehat{\Sigma} \times_\Gamma V\,. \tag{5.7}$$

Then $F$ is a flat complex vector bundle on $\Sigma$. Let $\nabla^F$ be the flat connection on $F$.

To the above cell decompositions of $\Sigma, \partial\Sigma$, we associate the corresponding combinatorial complexes $(C^{\Sigma}_{.}(F), \partial)$, $(C^{\partial\Sigma}_{.}(F), \partial)$, $(C^{\Sigma,r}_{.}(F), \partial)$, whose homologies are respectively $H_.(\Sigma, F), H_.(\partial\Sigma, F), H_.((\Sigma, \partial\Sigma), F)$, and which fit in the exact sequence of complexes

$$0 \to (C^{\partial\Sigma}_{.}(F), \partial) \to (C^{\Sigma}_{.}(F), \partial) \to (C^{\Sigma,r}_{.}(F), \partial) \to 0\,. \tag{5.8}$$

For $s > 0$, we have the associated long exact sequence

$$\begin{aligned} 0 \to H_2((\Sigma, \partial\Sigma), F) \to H_1(\partial\Sigma, F) \to H_1(\Sigma, F) \to H_1((\Sigma, \partial\Sigma), F) \\ \to H_0(\partial\Sigma, F) \to H_0(\Sigma, F) \to 0 \end{aligned} \tag{5.9}$$

(in (5.9), we used the fact that if $s > 0$, $H_2(\Sigma, F) = 0$, $H_0((\Sigma, \partial\Sigma), F) = 0$).

Let $p$ be the barycenter of the 2-cell $\overset{\circ}{P}$, let $q^1, \dots, q^{4g+1}, r^1_1, r^2_1, \dots, r^1_s, r^2_s$ be the vertices of $P$ (see Figure 5.1 for the case $g = 1, s = 1$).

Then $C^{\Sigma}_2(F)$ is the space of flat sections of $\pi^* F$ on $\overset{\circ}{P}$, and can be identified to $F_p$. Similarly $C^{\Sigma}_1(F)$ is the space of flat sections of $F$ in the "interior" of the corresponding one-cells. We identify $(\pi^* F)_{q_1}$ to $\pi^* F_p$ by parallel transport with respect to $\pi^*\nabla^F$ along a radial line connecting $p$ to $q^1$. Along $\partial P$ , we identify $\pi^* F$ to $(\pi^* F)_{q_1}$ by clockwise parallel transport with respect to $\pi^*\nabla^F$.

Then we identify $C^{\Sigma}_1(F)$

- over $a_1$, with $(\pi^* F)_{q^1}$ .
- over $b_1$, with $(\pi^* F)_{q^2}$ .
- over $a_2$, with $(\pi^* F)_{q^5}$ .
- over $b_2$, with $(\pi^* F)_{q^6}$ .

  ⋮

- over $c_1$, with $(\pi^* F)_{q^{4g+1}}$ .
- over $d_1$, with $(\pi^* F)_{r^1_1}$

  ⋮

Of course ultimately, $(\pi^* F)_{q^1}$, $(\pi^* F)_{q^2}$, $(\pi^* F)_{q^5}, \dots$ are identified to $\pi^* F_p$ as indicated before.

Finally $C^{\Sigma}_0(F)$ is identified to $(\pi^* F)_{q^1} \oplus (\bigoplus_{j=1}^{s} F_{r^1_j})$, which itself is identified to a sum of $1 + s$ copies of $(\pi^* F)_p$.

So $C^{\Sigma}_2(F)$ is just $(\pi^* F)_p$, $C^{\Sigma}_1(F)$ is a sum of $2g + 3s$ copies of $(\pi^* F)_p$, and $C^{\Sigma}_0(F)$ a sum of $1 + s$ copies of $(\pi^* F)_p$.

Then one verifies easily that if $f \in C_2^{\Sigma}(F) \simeq (\pi^* F)_p$,

$$\begin{aligned}
&\partial f = (1 - u_1 v_1^{-1} u_1^{-1}) f_{|a_1} + (1 - v_1 u_1 v_1^{-1}) u_1^{-1} f_{|b_1} + \\
&(1 - u_2 v_2^{-1} u_2^{-1})[v_1, u_1] f_{|a_2} + (1 - v_2 u_2 v_2^{-1}) u_2^{-1} [v_1, u_1] f_{|b_2} + \dots + \\
&(1 - w_1^{-1})[v_g, u_g] \dots [v_1, u_1] f_{|c_1} + [v_g, u_g] \dots [v_1, u_1] f_{|d_1} \\
&+(1 - w_2^{-1}) w_1^{-1} [v_g, u_g] \dots [v_1, u_1] f_{|c_2} + w_1^{-1} [v_g, u_g] \dots [v_1, u_1] f_{|d_2} + \dots
\end{aligned} \tag{5.10}$$

Similarly $\partial : C_1^{\Sigma}(F) \to C_0^{\Sigma}(F)$ is such that if $f \in (\pi^* F)_p$,

$$\begin{aligned}
\partial(f_{|a_i}) &= (1 - u_i^{-1}) f_{|q^1}, \\
\partial(f_{|b_i}) &= (1 - v_i^{-1}) f_{|q^1}, \\
\partial(f_{|c_j}) &= f_{|q^1} - f_{|r_j^1}, \\
\partial(f_{|d_j}) &= (1 - w_j^{-1}) f_{|r_j^1}.
\end{aligned} \tag{5.11}$$

In view of (5.6), (5.10), (5.11) one verifies that, as should be the case, $\partial^2 = 0$.

By restriction to $\partial\Sigma$, we obtain the chain map $\partial : C_1^{\partial\Sigma}(F) \to C_0^{\partial\Sigma}(F)$ given by

$$\partial f_{|d_j} = (1 - w_j^{-1}) f_{|r_j^1}. \tag{5.12}$$

The complex $(C^{\Sigma,r}(F), \partial)$ is obtained by making formally $f_{|r_j^1} = 0$, $f_{|d_j} = 0$ in (5.10), (5.11).

Note that for $s > 0$, we recover the fact that

$$H_2(\Sigma, F) = 0\,, \ H_0((\Sigma, \partial\Sigma), F) = 0\,. \tag{5.13}$$

PROPOSITION 5.1. *The following identity holds*

$$\begin{aligned}
H_2((\Sigma, \partial\Sigma), F) \simeq \{ f \in (\pi^* F)_p, &(u_1 - 1) f = 0, (v_1 - 1) f = 0, \dots, \\
&(u_g - 1) f = 0, (v_g - 1) f = 0, \\
&(w_1 - 1) f = 0, \dots, (w_s - 1) f = 0 \}.
\end{aligned} \tag{5.14}$$

PROOF. By (5.10), if $f \in C_2^{\Sigma,r}(F)$, the condition $\partial f = 0$, can be written as

$$\begin{aligned}
&u_1 v_1^{-1} u_1^{-1} f = f\,, \\
&[v_1, u_1] f = u_1^{-1} f\,, \\
&u_2 v_2^{-1} u_2^{-1} [v_1, u_1] f = [v_1, u_1] f\,, \\
&[v_2, u_2][v_1, u_1] f = u_2^{-1} [v_1, u_1] f, \\
&\vdots \\
&(1 - w_1^{-1})[v_g, u_g] \dots [v_1, u_1] f = 0\,, \\
&(1 - w_2^{-1}) w_1^{-1} [v_g, u_g] \dots [v_1, u_1] f = 0\,. \\
&\vdots
\end{aligned} \tag{5.15}$$

From the first two equalities in (5.15),

$$[v_1, u_1] f = v_1 f = u_1^{-1} f\,, \tag{5.16}$$

so that

$$v_1^{-1} u_1^{-1} f = f\,. \tag{5.17}$$

From (5.16), (5.17), we find that the first two equalities in (5.15) are equivalent to

$$u_1 f = v_1 f = f\,. \tag{5.18}$$

By proceeding as before, we get from (5.15),

$$u_i f = f\ ,\ v_i f = f\ ,\ 1 \le i \le g\,, \tag{5.19}$$

so that

$$w_j f = f\ ,\ 1 \in j \le s\,. \tag{5.20}$$

The proof of our Proposition is completed. □

Let $F^*$ be the flat vector bundle on $\Sigma$, which is dual to $F$. Then if $V^*$ is dual to $V$, $F^*$ is the flat bundle on $\Sigma$ associated to the dual representation $\tau^* \ : \Gamma \to \mathrm{Aut}(V^*)$.

Let $(C^{\Sigma\cdot}(F), \partial)$, $(C^{\partial\Sigma\cdot}(F), \partial)$, $(C^{\Sigma,r}(F), \partial)$ be the complexes dual to $(C^{\Sigma}_{\cdot}(F^*), \partial)$, $(C^{\partial\Sigma}_{\cdot}(F^*), \partial)$, $(C^{\Sigma,r}_{\cdot}(F^*), \partial)$. Then we have the exact sequence of complexes

$$0 \to (C^{\Sigma,r\cdot}(F), \partial) \to (C^{\Sigma\cdot}(F), \partial) \to (C^{\partial\Sigma\cdot}(F), \partial) \to 0\,. \tag{5.21}$$

The associated cohomology groups are respectively $H^{\cdot}((\Sigma, \partial\Sigma), F)$, $(H^{\cdot}(\Sigma, F)$, $H(\partial\Sigma, F)$. For $s > 0$, the corresponding long exact sequence

$$\begin{aligned} 0 \to H^0(\Sigma, F) \to H^0(\partial\Sigma, F) \to H^1((\Sigma, \partial\Sigma), F) \to H^1(\Sigma, F) \to \\ H^1(\partial\Sigma, F) \to H^2((\Sigma, \partial\Sigma), F) \to 0 \end{aligned} \tag{5.22}$$

is dual to the exact sequence (5.9) for $F^*$.

Now we describe the chain map $\partial$ in the complexes (5.21). We trivialize the flat vector bundle $F$ as indicated above. In particular the complexes in (5.21) are now direct sums of copies of $(\pi^* F)_p$. By (5.10), (5.11), we find that if $f \in \pi^* F_p$, $\partial : C^{\Sigma,0}(F) \to C^{\Sigma,1}(F)$ is such that

$$\begin{aligned} \partial(f|q^1) &= \sum_{i=1}^{g} ((1 - u_i) f|a_i + (1 - v_i) f|b_i) + \sum_{j=1}^{s} f|c_j\,, \\ \partial(f|r_j^1) &= -f|c_j + (1 - w_j) f|d_j\,, \end{aligned} \tag{5.23}$$

and $\partial : C^{\Sigma,1}(F) \to C^{\Sigma,2}(F)$ by

$$\begin{aligned} \partial(f|a_i) &= [u_1, v_1] \dots [u_{i-1}, v_{i-1}](1 - u_i v_i u_i^{-1}) f\,, \\ \partial(f|b_i) &= [u_1, v_1] \dots [u_{i-1}, v_{i-1}] u_i (1 - v_i u_i^{-1} v_i^{-1}) f\,, \\ \partial(f|c_j) &= [u_1, v_1] \dots [u_g, v_g] w_1 \dots w_{j-1} (1 - w_j) f\,, \\ \partial(f|d_j) &= [u_1, v_1] \dots [u_g, v_g] w_1 \dots w_{j-1} f\,. \end{aligned} \tag{5.24}$$

The complex $(C^{\Sigma,r,\cdot}(F), \partial)$ is obtained formally from (5.23), (5.24) by making the components indexed by $d_j$ or $r_j^1$ equal to 0. Finally $\partial : C^{0,\partial\Sigma}(F) \to C^{1,\partial\Sigma}(F)$ is given by

$$\partial(f|r_j^1) = (1 - w_j) f|d_j\,. \tag{5.25}$$

Observe that by (5.23), (5.24), it is clear that for $s > 0$,

$$H^2(\Sigma, F) = 0\ ,\ H^0((\Sigma, \partial\Sigma), F) = 0\,. \tag{5.26}$$

PROPOSITION 5.2. *The following identity holds,*

$$H^0(\Sigma,F)=\{f|q^1+\sum_{j=1}^{s}f|r_j^1\ ,\ \textit{with}\ f\in\pi^*F_p\ \textit{such that}\tag{5.27}$$
$$(1-u_i)f=0,(1-v_i)f=0\ (1\le i\le g)\ ,\ (1-w_j)f=0\ (1\le j\le s)\}\,.$$

PROOF. Our identity follows from (5.23). □

REMARK 5.3. By Poincaré duality

$$H_2((\Sigma,\partial\Sigma),F)\simeq H^0(\Sigma,F)\,.\tag{5.28}$$

Using Propositions 5.1 and 5.2, the identification (5.28) has been made explicit. Another interpretation is to view the complex $(C^{\Sigma\cdot}(F),\partial)$ as the relative homology complex associated to the dual cell decomposition of $\Sigma$.

Suppose that $s\ge 1$. Let $(K^\cdot,\partial)$ be the trivial complex concentrated in degree 1,

$$K^1=\bigoplus_{j=1}^{s}\ker(1-w_j)|c_j\simeq H^0(\partial\Sigma,F)\,.\tag{5.29}$$

Then by (5.24), $(K^\cdot,\partial)$ is a subcomplex of $(C^{\Sigma,r\cdot}(F),\partial)$.

We have the obvious exact sequences

$$0\longrightarrow\ker(1-w_j)\longrightarrow\pi^*F_p\xrightarrow[T_j=1-w_j]{}\mathrm{Im}(1-w_j)\longrightarrow 0\ .\tag{5.30}$$

The exact sequences (5.30) induce an exact sequence of complexes

$$0\longrightarrow(K^\cdot,\partial)\longrightarrow(C^{\Sigma,r\cdot}(F),\partial)\xrightarrow[T]{}(\widehat{C}^{\Sigma\cdot}(F),\partial)\longrightarrow 0\ .\tag{5.31}$$

By (5.23), (5.24),

$$\begin{aligned}\widehat{C}^{\Sigma,k}(F)&=C^{\Sigma,r,k}(F)\,,k=0\text{ or }2\,,\\&=\bigoplus_{i=1}^{g}\pi^*F_{p|a_i}\oplus\pi^*F_{p|b_i}\oplus\bigoplus_{j=1}^{s}\mathrm{Im}(1-w_j)_{|c_j}\,,k=1\,.\end{aligned}\tag{5.32}$$

By (5.23), (5.24), if $f\in\pi^*F_p$, $\partial:\widehat{C}^{\Sigma,0}(F)\to\widehat{C}^{\Sigma,1}(F)$ is given by

$$\partial f=\sum_1^g\left((1-u_i)f_{|a_i}+(1-v_i)f_{|b_i}\right)+\sum_{j=1}^{s}(1-w_j)f_{|c_j}\,,\tag{5.33}$$

and $\partial:\widehat{C}^{\Sigma,1}(F)\to\widehat{C}^{\Sigma,2}(F)$ by

$$\begin{aligned}\partial(f|a_i)&=[u_1,v_1]\dots[u_{i-1},v_{i-1}](1-u_iv_iu_i^{-1})f\,,\\\partial(f|b_i)&=[u_1,v_1]\dots[u_{i-1},v_{i-1}]u_i(1-v_iu_i^{-1}v_i^{-1})f\,,\\\partial(f|c_j)&=[u_1,v_1]\dots[u_g,v_g]w_1\dots w_{j-1}f\,.\end{aligned}\tag{5.34}$$

Let $\widehat{H}^{\cdot}(\Sigma, F)$ be the cohomology of $(\widehat{C}^{\Sigma\cdot}(F), \partial)$. By (5.31), for $s > 0$, we have the long exact sequence

(5.35)

$$0 \longrightarrow \widehat{H}^0(\Sigma, F) \longrightarrow K^1 = H^0(\partial\Sigma, F) \longrightarrow H^1((\Sigma, \partial\Sigma), F)$$

$$\xrightarrow[T]{} \widehat{H}^1(\Sigma, F) \longrightarrow 0 \longrightarrow H^2((\Sigma, \partial\Sigma), F) \longrightarrow \widehat{H}^2(\Sigma, F) \longrightarrow 0.$$

THEOREM 5.4. *For $s > 0$, the following identities hold,*

$$\widehat{H}^0(\Sigma, F) = H^0(\Sigma, F), \qquad (5.36)$$
$$\widehat{H}^2(\Sigma, F) = H^2((\Sigma, \partial\Sigma), F).$$

*Also $\widehat{H}^1(\Sigma, F)$ is the image of $H^1((\Sigma, \partial\Sigma), F)$ in $H^1(\Sigma, F)$. In (5.35), the map $\widehat{H}^0(\Sigma, F) \to K^1$ is just the map $H^0(\Sigma, F) \to H^0(\partial\Sigma, F)$, the map $K^1 \to H^1((\Sigma, \partial\Sigma), F)$ is minus the map $H^0(\partial\Sigma, F) \to H^1((\Sigma, \partial\Sigma), F)$, and $T : H^1((\Sigma, \partial\Sigma), F) \to \widehat{H}^1(\Sigma, F)$ is induced by the map $H^1((\Sigma, \partial\Sigma), F) \to H^1(\Sigma, F)$.*

PROOF. By (5.33),

$$\widehat{H}^0(\Sigma, F) = \{f \in \pi^* F_p, (1 - u_i)f = 0, (1 \leq i \leq g) \qquad (5.37)$$
$$(1 - v_i)f = 0, (1 - w_j)f = 0, 1 \leq j \leq s\}.$$

Using Proposition 5.2, and (5.37), we get the first equality in (5.36). The second equality in (5.36) follows from (5.24), (5.34).

Inspection of (5.23), (5.24), (5.37) shows that the map $\widehat{H}^0(\Sigma, F) \to K^1$ is the canonical $H^0(\Sigma, F) \to H^0(\partial\Sigma, F)$. Take $f \in \ker(1 - w_j)$. By (5.23), if $f|r_j^1$ is viewed as lying in $C^0(\Sigma, F)$, then

$$\partial f|r_j^1 = -f|c_j. \qquad (5.38)$$

Therefore the map $K^1 \to H^1((\Sigma, \partial\Sigma), F)$ is minus the map $H^0(\partial\Sigma, F) \to H^1((\Sigma, \partial\Sigma), F)$.

By (5.35), we have the canonical isomorphism

$$\widehat{H}^1(\Sigma, F) \simeq \frac{H^1((\Sigma, \partial\Sigma), F)}{\mathrm{Im}\ H^0(\partial\Sigma, F)}. \qquad (5.39)$$

By (5.22), (5.39), we find that $\widehat{H}^1(\Sigma, F)$ is exactly the image of $H^1((\Sigma, \partial\Sigma), F)$ in $H^1(\Sigma, F)$.

The proof of our Theorem is completed. □

REMARK 5.5. Assume that $F$ and $F^*$ are naturally isomorphic as flat bundles. This is the case when $F$ is real and carries a flat scalar product. Then $H^0(\Sigma, F)$ and $H^2((\Sigma, \partial\Sigma), F)$ are Poincaré dual, and the image of $H^1((\Sigma, \partial\Sigma), F)$ in $H^1(\Sigma, F)$ is equipped with a non degenerate 2-form, the intersection product. The conclusion is that in this case, the cohomology of $(\widehat{C}^{\Sigma\cdot}(F), \partial)$ exhibits Poincaré duality.

**5.3. The map $\phi$.** Let $G$ be a compact connected semisimple Lie group. Let $Z(G)$ be the center of $G$. Then $Z(G)$ is a finite subgroup of $G$. Let $G' = G/Z(G)$ be the adjoint group. Then $G'$ acts naturally on $G$ by conjugation. If $u \in G'$, $v \in G$, we will write

$$u \cdot v = uvu^{-1}. \qquad (5.40)$$

Similarly $G'$ acts on $G^{2g+s}$ by conjugation. We will use the same notation for the action of $G$ on $G$ or on $G^{2g+s}$ by conjugation.

Take $g, s \in \mathbf{N}$, with $g + s > 0$.

DEFINITION 5.6. Let $\phi : G^{2g+s} \to G$ be the map

$$\phi(u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_s) = \prod_{i=1}^{g} [u_i, v_i] \prod_{j=1}^{s} w_j \,. \tag{5.41}$$

Then $\phi$ is $G'$-equivariant, i.e. if $g \in G'$, $x \in G^{2g+s}$,

$$\phi(g \cdot x) = g \cdot \phi(x) \,. \tag{5.42}$$

Let $\mathfrak{g}$ be the Lie algebra of $G$. If $g \in G$, we identify $T_g G$ to $\mathfrak{g} = T_e G$ via the right multiplication operator $R_{g*}$. Then if $z \in G^{2g+s}$,

$$T_z G^{2g+s} = \mathfrak{g}^{2g+s} \,. \tag{5.43}$$

DEFINITION 5.7. Take $x \in G^{2g+s}$. Let $\delta : \mathfrak{g} \to \mathfrak{g}^{2g+s}$ be the derivative at $g = 1$ of the map $g \in G \mapsto g \cdot x \in G^{2g+s}$. Similarly let $\delta : \mathfrak{g}^{2g+s} \to \mathfrak{g}$ be the derivative at $x \in G^{2g+s}$ of $x' = G^{2g+s} \mapsto \phi(x') \in G$.

Clearly $G$ acts on $\mathfrak{g}$ by the adjoint action.

PROPOSITION 5.8. *Take* $x = (u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_s) \in G^{2g+s}$. *Then if* $f \in \mathfrak{g}$,

$$\begin{aligned}\delta f = ((1 - u_1)f, (1 - v_1)f, \dots, (1 - u_g)f, (1 - v_g)f,\\ (1 - w_1)f, \dots, (1 - w_s)f) \,.\end{aligned} \tag{5.44}$$

*Similarly, if* $(f_1, \dots, f_{2g+s}) \in \mathfrak{g}^{2g+s}$,

$$\begin{aligned}\delta(f_1, \dots, f_{2g+s}) = \sum_{i=1}^{g} [u_1, v_1] \dots [u_{i-1}, v_{i-1}]\\ \left((1 - u_i v_i u_i^{-1}) f_{2i-1} + u_i (1 - v_i u_i^{-1} v_i^{-1}) f_{2i}\right)\\ + \sum_{j=1}^{s} [u_1, v_1] \dots [u_g, v_g] w_1 \dots w_{j-1} f_{2g+j} \,.\end{aligned} \tag{5.45}$$

PROOF. This is a trivial computation, which is left to the reader. □

Let $\mathcal{O} \subset G$ be an orbit in $G$. More precisely, take $g \in G$, and put

$$\mathcal{O} = \{g' \cdot g, g' \in G\} \,. \tag{5.46}$$

Of course $g \in \mathcal{O}$. If $Z(g) \subset G$ is the centralizer of $g$,

$$\mathcal{O} \simeq G/Z(g) \,. \tag{5.47}$$

Clearly the tangent space $T_g \mathcal{O} \subset T_g G \simeq \mathfrak{g}$ is given by

$$T_g \mathcal{O}_g = \operatorname{Im}(1 - g) \,. \tag{5.48}$$

Also $T_e Z(g) \subset \mathfrak{g}$ is given by

$$T_e Z(g) = \ker(1 - g) \,. \tag{5.49}$$

Then the exact sequence

$$0 \to T_e Z(g) \to T_e G \to T_g \mathcal{O}_g \to 0 \tag{5.50}$$

corresponds to the obvious

$$0 \longrightarrow \ker(1-g) \longrightarrow \mathfrak{g} \xrightarrow[1-g]{} \operatorname{Im}(1-g) \longrightarrow 0 \,. \tag{5.51}$$

Let $\mathcal{O}_1, \dots, \mathcal{O}_s$ be $s$ adjoint orbits of $G$ in $G$. Put

$$X = G^{2g} \times \prod_{j=1}^{s} \mathcal{O}_j \,. \tag{5.52}$$

If $x = (u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_s) \in X$, $\delta : \mathfrak{g} \to \mathfrak{g}^{2g+s}$ is in effect a linear map : $\mathfrak{g} \to \mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \operatorname{Im}(1-w_j)$. By restriction we obtain the linear maps

$$(D,\delta) : 0 \longrightarrow \mathfrak{g} \xrightarrow[\delta]{} \mathfrak{g}^{2g} \oplus \bigoplus\nolimits_{j=1}^{s} \operatorname{Im}(1-w_j) \xrightarrow[\delta]{} \mathfrak{g} \longrightarrow 0 \,. \tag{5.53}$$

Note that in general

$$\delta^2 \neq 0 \,. \tag{5.54}$$

Also observe that if $g \in G'$, $g$ maps $(D, \partial)_x$ into $(D, \partial)_{gx}$.

DEFINITION 5.9. If $x \in X$, let $Z(x) \subset G$, $Z'(x) \subset G'$ be the stabilizers of $x$, and let $\mathfrak{z}(x)$ be the Lie algebra of $Z(x)$ and $Z'(x)$.

Of course $Z'(x) = Z(x)/Z(G)$.

PROPOSITION 5.10. *For $x \in X$, then*

$$\{f \in \mathfrak{g}, \delta f = 0\} = \mathfrak{z}(x) \,. \tag{5.55}$$

PROOF. This is obvious by (5.44). □

Let $\langle\ ,\ \rangle$ be a $G$-invariant scalar product on $\mathfrak{g}$. If $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, let $\mathfrak{h}^\perp$ be the orthogonal space to $\mathfrak{h}$ in $\mathfrak{g}$.

PROPOSITION 5.11. *For $x \in X$, then*

$$\left[\delta\Big(\mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \operatorname{Im}(1-w_j)\Big)\right]^\perp = \mathfrak{z}(x) \,. \tag{5.56}$$

PROOF. Let $\delta^* : \mathfrak{g} \to \mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \operatorname{Im}(1-w_j)$ be the adjoint of $\delta : \mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \operatorname{Im}(1-w_j) \to \mathfrak{g}$. Clearly

$$\delta(\mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \operatorname{Im}(1-w_j))^\perp = \ker \delta^* \,. \tag{5.57}$$

Let $P^{\operatorname{Im}(1-w_j)}$ be the orthogonal projection operator $\mathfrak{g} \to \operatorname{Im}(1-w_j)$. Then

$$(1-w_j^{-1}) P^{\operatorname{Im}(1-w_j)} = (1-w_j^{-1}), \tag{5.58}$$

and $1 - w_j^{-1}$ is one to one from $\operatorname{Im}(1-w_j)$ into itself. Using 5.45, we obtain a formula for $\delta^*$ involving the projection $P^{\operatorname{Im}(1-w_j)}$. By (5.58), to calculate $\ker \delta^*$, we may replace $P^{\operatorname{Im}(1-w_j)}$ by $(1-w_j)$. By proceeding as in Proposition 5.1, we get (5.56). □

Recall that $\phi : X \to G$ is said to be regular at $x \in X$ if $d\phi(x) : T_x X \to T_{\phi(x)} G$ is surjective.

THEOREM 5.12. *The map $\phi : X \to G$ is regular at $x \in X$ if and only if $\mathfrak{z}(x) = 0$.*

PROOF. Our Theorem is a trivial consequence of Proposition 5.11. □

DEFINITION 5.13. Put

$$M = \{x \in X,\ \phi(x) = 1\}. \tag{5.59}$$

If $x = (u_1, \ldots, w_s) \in M$, then $x$ defines a morphism $\Gamma \to G$. Also $G$ acts on $\mathfrak{g}$ via the adjoint representation. Let $E$ be the flat real vector bundle on $\Sigma$ constructed as in (5.7), i.e.

$$E = \widehat{\Sigma} \times_\Gamma \mathfrak{g}. \tag{5.60}$$

The following result was first obtained by Liu [**39, 40**].

THEOREM 5.14. *If $x \in M$, then in (5.53), $\delta^2 = 0$, i.e. $(D, \partial)_x$ is a complex. The complex $(D, \delta)_x$ can then be canonically identified to the complex $(\widehat{C}^{\Sigma\cdot}(E), \partial)_x$.*

PROOF. By (5.42), if $x \in M$,

$$\phi(gx) = 1. \tag{5.61}$$

By (5.61), we find that $\delta^2 = 0$. Comparing (5.33), (5.34) and (5.44), (5.45), our Theorem follows. □

REMARK 5.15. By Theorem 5.4, if $x \in M$,

$$\begin{aligned} \widehat{H}^0(\Sigma, E) &= H^0(\Sigma, E) \\ \widehat{H}^2(\Sigma, E) &= H^2((\Sigma, \partial\Sigma), E). \end{aligned} \tag{5.62}$$

In view of Propositions 5.1 and 5.10, we find that if $x \in M$,

$$\begin{aligned} \widehat{H}^0(\Sigma, E) &= \mathfrak{z}(x), \\ \widehat{H}^2(\Sigma, E) &= \mathfrak{z}(x)^*. \end{aligned} \tag{5.63}$$

So when $x \in M$, Proposition 5.11 follows from (5.63).

**5.4. The set of regular values of the map $\phi$.** Recall that $G$ and $G'$ act by the adjoint action on $\mathfrak{g}$. Then one verifies easily that if $x \in M, \mathfrak{g} \in G$,

$$(\widehat{C}^{\Sigma\cdot}_x(E), \partial) \xrightarrow[g]{} (\widehat{C}^{\Sigma\cdot}_{g.x}(E), \partial) \tag{5.64}$$

is an isomorphism of complexes. In particular the induced map

$$\widehat{H}^{\Sigma\cdot}_x(E) \xrightarrow[g]{} \widehat{H}^{\Sigma\cdot}_{g.x}(E) \tag{5.65}$$

is an isomorphism.

Let $T$ be a maximal torus in $G$, let $\mathfrak{t}$ be the Lie algebra of $T$. Let $W$ be the Weyl group of $G$ with respect to $T$. Let $R = \{\alpha\} \subset \mathfrak{t}^*$ be the root system of $G$, let $CR = \{h_\alpha\} \subset \mathfrak{t}$ be the corresponding coroots.

Now we recall the definition of $S \subset T$ given in Definition 2.23.

DEFINITION 5.16. Let $S \subset T$ be given by

$$S = \{t \in T; t = \sum_{\alpha \in R} t^\alpha h_\alpha\,; \text{the } h_\alpha \text{ for which } t^\alpha \neq 0 \text{ do not span } \mathfrak{t}\}. \tag{5.66}$$

Let $t_1, \dots, t_s \in T$. Let $\mathcal{O}_1, \dots, \mathcal{O}_s$ be the orbits of $t_1, \dots, t_s$ in $G$. By [**15**, Lemma IV.2.5],

$$\mathcal{O}_j \cap T = W t_j \ , \ 1 \leq j \leq s \,. \tag{5.67}$$

DEFINITION 5.17. We will say that $(t_1, \dots, t_s)$ verify (A) if for any $(w^1, \dots, w^s) \in W^s$,

$$\sum_1^s w^j t_j \notin S \,. \tag{5.68}$$

THEOREM 5.18. *For $g \geq 1$, the map $\phi : X \to G$ is surjective. For $g \geq 1$, $t \in T$ is a regular value of $\phi$ if and only if $(t_1, t_2, \dots, t_s, -t) \in T^{s+1}$ verify (A).*

*For $g = 0$, $t \in T$ is a regular value of $\phi$ in the following two cases :*

$$\bullet\ t \notin \phi(X). \tag{5.69}$$
$$\bullet\ t \in \phi(X)\,,\ \textit{and}\ (t_1, \dots, t_s, -t)\ \textit{verify}\ (A)\,.$$

PROOF. By [**14**, Corollaire 4.5], since $G$ is semisimple, the map $(u, v) \in G^2 \mapsto [u, v] \in G$ is surjective. Therefore, for $g \geq 1$, the map $\phi$ is surjective.

First we will prove the remainder of our Theorem for $t = 1$. By Theorem 5.12, $1 \in G$ is a non regular value of $\phi$ if and only if there exists $x \in M$ such that $\mathfrak{z}(x) \neq 0$. Equivalently there exists $p \in \mathfrak{g}$, $p \neq 0$ and $x = (u_1, \dots, w_s) \in M$ such that $x \in Z(p)^{2g+s}$. By [**15**, Theorem IV.2.3], $Z(p)$ is a connected Lie subgroup of $G$, which is not semisimple, since $\mathbf{R}p$ is contained in the Lie algebra of its center. Let $T_0 = Z_0(Z(p))$ be the connected component of the identity in $Z(Z(p))$. Then we have the exact sequence

$$1 \to T_0 \to Z(p) \to Z(p)/T_0 \to 1, \tag{5.70}$$

and $Z(p)/T_0$ is semisimple. Put

$$\gamma = \pi_1(Z(p)/T_0)\,, \tag{5.71}$$

and let $\widetilde{Z(p)/T_0}$ be the universal cover of $Z(p)/T_0$. Then (5.70) fits in the diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & 1 & \to & \gamma & \to & \gamma & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & T_0 & \to & \widehat{Z}(p) & \to & \widetilde{Z(p)/T_0} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & T_0 & \to & Z(p) & \to & Z(p)/T_0 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{5.72}$$

In (5.72), $\widehat{Z}(p) \to Z(p)$ is the obvious $\gamma$ covering of $Z(p)$.

Since $\widetilde{Z(p)/T_0}$ is semisimple and simply connected, any central extension of $\widetilde{Z(p)/T_0}$ by a torus is trivial, so that

$$\widehat{Z}(p) = T_0 \times \widetilde{Z(p)/T_0}. \tag{5.73}$$

Therefore $\gamma \subset \widetilde{Z(p)/T_0}$ embeds as a finite subgroup of $T_0 \times \widetilde{Z(p)/T_0}$.

Let $\widetilde{T}(p)$ be a maximal torus in $\widetilde{Z(p)/T_0}$. Then $T_0 \times \widetilde{T}(p)$ is a maximal torus in $\widehat{Z}(p)$. From (5.72), (5.73), we get the complex

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & \gamma & \longrightarrow & \gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_0 & \longrightarrow & T_0 \times \widetilde{T}(p) & \longrightarrow & \widetilde{T}(p) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_0 & \longrightarrow & T & \longrightarrow & T/T_0 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{5.74}$$

Now we use the notation of Theorem 1.52. Let $\mathfrak{t}_0 = \mathfrak{z}(Z(p))$ be the Lie algebra of $T_0$. Let $\mathfrak{t}_1 \subset \mathfrak{t}$ be the vector subspace of $\mathfrak{t}$ spanned by the $\{h_\alpha\}_{\alpha \in R_p}$. Clearly

$$\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1. \tag{5.75}$$

Also $\mathfrak{t}_1$ is the Lie algebra of $T/T_0$ or of $\widetilde{T}(p)$.

Let $[\widehat{Z}(p), \widehat{Z}(p)]$ be the commutator subgroup $\widehat{Z}(p)$, i.e. the group spanned by commutators in $\widehat{Z}(p)$. By (5.73), since $\widetilde{Z(p)/T_0}$ is semisimple,

$$[\widehat{Z}(p), \widehat{Z}(p)] = \widetilde{Z(p)/T_0}. \tag{5.76}$$

Clearly $[\widehat{Z}(p), \widehat{Z}(p)]$ maps onto $[Z(p), Z(p)]$. Therefore $[Z(p), Z(p)]$ is a compact connected subgroup of $Z(p)$, and so it is a connected Lie subgroup of $Z(p)$.

Let now $t \in T \cap [Z(p), Z(p)]$. By the above, there is $a \in \widetilde{Z(p)/T_0}$, $b = (b_0, b_1) \in T_0 \times \widetilde{T}(p)$ which map to $t$. Therefore, if $c = ab^{-1}$, then $c \in \gamma$. Clearly

$$c = (b_0^{-1}, ab_1^{-1}). \tag{5.77}$$

By (5.74), (5.77),

$$ab_1^{-1} \in \gamma \subset \widetilde{T}(p). \tag{5.78}$$

From (5.77), we get

$$a \in \widetilde{T}(p). \tag{5.79}$$

We thus find that if $t \in T \cap [Z(p), Z(p)]$, $t$ is the projection of an element of $\widetilde{T}(p)$. Equivalently $t$ can be represented by $\overline{t} \in \mathfrak{t}_1$.

Recall that

$$\prod_{i=1}^{g}[u_i,v_i]\prod_{j=1}^{s}w_j=1\,, \tag{5.80}$$

and $u_i, v_i \in Z(p)$, $1 \leq i \leq g$, $w_j \in Z(p)$, $1 \leq j \leq s$. Clearly $T$ is a maximal torus in $Z(p)$. So there is $g \in Z(p)$ such that $gw_jg^{-1} \in T$. Since $gw_jg^{-1} \in T \cap \mathcal{O}_{t_j}$, by (5.67), there is $w^j \in W$ such that

$$gw_jg^{-1}=w^jt_j\,. \tag{5.81}$$

Therefore

$$w_j=w^jt_j \text{ in } Z(p)/[Z(p),Z(p)]\,. \tag{5.82}$$

So by (5.80), (5.82), we get

$$\sum_{j=1}^{s}w^jt_j\in T\cap[Z(p),Z(p)]\,. \tag{5.83}$$

By the above, we find that

$$\sum_{j=1}^{s}w^jt_j=\sum_{\alpha\in R_p}s^\alpha h_\alpha \text{ in } T\,. \tag{5.84}$$

Now since $p \neq 0$, $\{h_\alpha, \alpha \in R_p\}$ does not span $\mathfrak{t}$, i.e. $(t_1,\dots,t_s)$ does not verify (A).

Conversely, suppose that $g \geq 1$, and $(t_1,\dots,t_s)$ does not verify (A). Then there is $(w^1,\dots,w^s) \in W^s$, such that $\sum_{j=1}^{s} w^jt_j$ can be expressed as a linear combination of $\{h_\alpha\}$ which do not span $\mathfrak{t}$. Let $p \in \mathfrak{t}$, $p \neq 0$ be orthogonal to the corresponding $\{\alpha\}$. By the above,

$$\sum_{j=1}^{s}w^jt_j\in[Z(p),Z(p)]\,. \tag{5.85}$$

Using the fact that in a compact semisimple Lie group, any element is a commutator [**14**, Corollaire 4.5], and also the considerations after (5.76), by (5.85), there is $u_g, v_g \in Z(p)$ such that

$$[u_g,v_g]\prod_{j=1}^{s}w^jt_j=1\,. \tag{5.86}$$

If $x = (1,\dots,1,u_g,v_g,w^1t_1,\dots,w^st_s)$, then $x \in M$, and $p \in \mathfrak{z}(x)$, so that $\mathfrak{z}(x) \neq \{0\}$. By Theorem 5.12, we find that 1 is not a regular value of $\phi$.

Let $t = t_{s+1}^{-1} \in T$. Let $\mathcal{O}_{s+1} \subset G$ be the $G$-orbit of $t_{s+1}$. Put $X_{+1} = G^{2g} \times \prod_{j=1}^{s+1}\mathcal{O}_j$. More generally we denote with the index $+1$ the objects we constructed above, which are associated to $X_{+1}$. Clearly

$$X_{+1}=X\times\mathcal{O}_{s+1}\,. \tag{5.87}$$

Also if $(x, w_{s+1}) \in X_{+1}$,

$$\phi_{+1}(x,w_{s+1})=\phi(x)w_{s+1}\,. \tag{5.88}$$

Clearly $\phi(x) = t$ if and only if $(x, t_{s+1}) \in M_{+1}$. By (5.88), if $\phi$ is regular at $x$, $\phi_{+1}$ is regular at $(x, t_{s+1})$. Conversely, suppose that $\phi_{+1}$ is regular at $(x, t_{s+1})$. By (5.88),

$$\operatorname{Im}(d\phi_{+1})(x, t_{s+1}) = \operatorname{Im}(d\phi\phi^{-1})_x + (t_{s+1} - 1)(\mathfrak{g}) . \tag{5.89}$$

By (5.42), we find that

$$(t_{s+1} - 1)(\mathfrak{g}) \subset \operatorname{Im}(d\phi\phi^{-1})_x . \tag{5.90}$$

From (5.89), (5.90), we find that $\phi$ is regular at $t$ if and only if $\phi_{+1}$ is regular at $(x, t_{s+1})$. Now we use our Theorem for $t = 1$ and we obtain the stated result in full generality. □

REMARK 5.19. Instead of studying the case $t = 1$ first, we could as well have used Theorem 5.12 and proceeded directly.

### 5.5. The stabilizers $Z'(x)$.

THEOREM 5.20. *If $G$ is simply connected, under one of the following three conditions, if $t \in T$ is a regular value of $\phi$, $\{x \in \phi^{-1}\{t\}, Z'(x) \neq 1\}$ is included in a union of submanifolds of $\{x \in \phi^{-1}(t)\}$ of codimension $\geq 2$ :*

- *For $g \geq 2$.*
- *For $g = 1$, if one of the $t_j$'s or $t$ is regular.*
- *For $g = 0$, $s = 1$, in which case $\phi^{-1}\{t\}$ is empty, or if at least 3 of the elements $\{t_1, \dots, t_s, t\}$ are regular.*

*If $G$ is only connected, if $t_1, \dots, t_s$ are very regular, if $t \in T$ is a regular value of $\phi$, $\{x \in \phi^{-1}(t), Z'(x) \neq 1\}$ is included in a union of submanifolds of $\{x \in \phi^{-1}(t)\}$ of codimension $\geq 2$ :*

- *For $g \geq 2$.*
- *For $g = 1$, if $s \geq 1$ or if $t$ is very regular.*
- *For $g = 0$, $s = 1$, in which case $\phi^{-1}(t)$ is empty, if $s = 2$ and $t$ is very regular, or if $s \geq 3$.*

PROOF. We will prove our Theorem in various stages.

- *The case where $G$ is simply connected and $t = 1$.*

If 1 is a regular value of $\phi$, either $M = \phi^{-1}\{1\}$ is empty, or it is a smooth submanifold of $X$.

By Theorem 5.12, when $M$ is non empty, the group $G$ acts locally freely on $M$. Therefore $M/G$ is an orbifold. If $\dim(M/G)$ is the dimension of the maximal statum of $M/G$, then

$$\dim M/G = (2g - 2)\dim(\mathfrak{g}) + \sum_{j=1}^{s} \dim(\mathcal{O}_j) . \tag{5.91}$$

Of course the same observation applies to the action of $G'$ on $M$. Clearly $M/G \simeq M/G'$. In the sequel we will use the notation $M/G$ or $M/G'$ indifferently. However in Section 6.3, we will construct an orbifold $G$-line bundle on $M/G$, which may well not be an orbifold $G'$-line bundle.

Take now $x \in M$ such that $Z'(x) \neq 1$. Let $u \in Z'(x)$, $u \neq 1$. If $x = (u_1, \dots, w_s)$, then $u_1, \dots, w_s \in Z(u)$. By conjugation, we may as well assume that $u \in T' = T/Z(G)$. Since $G$ acts locally freely on $M$, the group $Z(u)$ is semisimple. By Theorem 1.38, $u \in C/\overline{R}^*$. Of course here $u \neq 0$.

Since $G$ is simply connected, $Z(u)$ is connected. By Theorem 1.50, for any $t \in T$,

$$\mathcal{O}(t) \cap Z(u) = \bigcup_{w \in W_u \backslash W} \mathcal{O}_{Z(u)}(wt) . \tag{5.92}$$

For $1 \leq j \leq s$, let $w^j \in W$ such that

$$w_j \in \mathcal{O}_{Z(u)}(w^j t_j), \ 1 \leq j \leq s . \tag{5.93}$$

Put

$$X^u = Z(u)^{2g-2} \times \prod_{j=1}^{s} \mathcal{O}_{Z(u)}(w^j t_j) . \tag{5.94}$$

We define $\phi_u : X^u \to Z(u)$ as in (5.41). Let $M^u = \phi_u^{-1}\{1\}$. Then $x \in M^u$. Moreover $Z(u)$ acts locally freely on $M^u$. Therefore, by Theorem 5.12, 1 is a regular value of $\phi_u$, i.e. $M^u$ is a smooth submanifold of $M$. Note that this also follows from the fact that $M$ is a smooth manifold, $G$ acts on $M$, and $M^u$ is a component of the fixed point set of $u$ in $M$. Then

$$\dim M^u/Z(u) = (2g-2) \dim \mathfrak{z}(u) + \sum_{j=1}^{s} \dim \mathcal{O}_{Z(u)}(w^j t_j) . \tag{5.95}$$

Now by Theorem 1.52,

$$\dim \mathfrak{z}(u) \leq \dim(\mathfrak{g}) - 2 . \tag{5.96}$$

Also

$$\dim \mathcal{O}_{Z(u)}(w^j t_j) \leq \dim \mathcal{O}_j . \tag{5.97}$$

Now we consider 3 cases :

- If $g \geq 2$, by (5.91), (5.95)-(5.97),

$$\dim M^u/Z(u) \leq \dim M/G - 2 . \tag{5.98}$$

By (5.98), $M^u/Z(u)$ maps to a submanifold of codimension $\geq 2$ in $M/G$. Therefore the $G$-orbit of $M^u$ in $M$ is a submanifold of codimension $\geq 2$ in $M$.

- If $g = 1$, assume that $t_1 \in T$ is regular. Then

$$\begin{aligned} \dim \mathcal{O}_{t_1} &= \dim(\mathfrak{g}) - \dim(\mathfrak{t}) , \\ \dim \mathcal{O}_{Z(u)}(w^1 t_1) &= \dim \mathfrak{z}(u) - \dim(\mathfrak{t}) . \end{aligned} \tag{5.99}$$

By (5.91), (5.95), (5.96), (5.97), (5.99), we find again that

$$\dim M^u/Z(u) \leq \dim M/G - 2 . \tag{5.100}$$

- If $g = 0$, if $s = 2$, then

$$\dim \mathcal{O}_{t_j} \leq \dim(\mathfrak{g}) - \dim(\mathfrak{t}) ; \ s = 1, 2 , \tag{5.101}$$

so that using (5.91),

$$\dim M/G < 0, \tag{5.102}$$

i.e. $M = \emptyset$.

If $s \geq 3$, assume that $t_1, t_2, t_3$ are regular. Then

$$\dim M/G = \dim(\mathfrak{g}) - 3\dim(\mathfrak{t}) + \sum_{j=4}^{s} \dim(\mathcal{O}_j), \tag{5.103}$$

$$\dim M^u/Z(u) = \dim \mathfrak{z}(u) - 3\dim(\mathfrak{t}) + \sum_{j=4}^{s} \dim \mathcal{O}_{Z(u)}(w^j t_j).$$

By (5.96), (5.97), (5.103)

$$\dim M^u/Z(u) \leq \dim M/G - 2. \tag{5.104}$$

So we have proved our Theorem in this case.

- *The case where G is non simply connected and $t = 1$.*

First we proceed as above. In this case for $u \in C/\overline{R}^*$, $u \neq 1$, $Z(u)$ is in general non connected. However by (1.174), since $t_1, \ldots, t_s$ are very regular,

$$\mathcal{O}_j \cap Z(u) = \bigcup_{w^j \in W_{Z(u)_o} \backslash W} \mathcal{O}_{Z(u)_o}(w^j t_j). \tag{5.105}$$

For $1 \leq j \leq s$, let $w^j \in W$ be such that

$$w_j \in \mathcal{O}_{Z(u)_o}(w^j t_j). \tag{5.106}$$

Again we define $X^u$ by (5.94), and we define $\phi_u$ as before. Put $M^u = \phi_u^{-1}\{1\}$. By the same argument as before, 1 is a regular value of $\phi_u$. Then (5.95) still holds. If $g \geq 2$, (5.104) is still true. If $g = 1$, since $t_1$ is very regular in $G$, then $t_1$ lies in $Z(u)_0$ and is very regular in $Z(u)$, so that (5.100) still holds.

If $g = 0$, under the stated assumptions, the argument in the proof above can still be reproduced.

The proof of our Theorem is completed in this case.

- *The case where $t \in T$ is an arbitrary regular value of $\phi$.*

Let $t = t_{s+1}^{-1} \in T$. We use the notation in the proof of Theorem 5.18. Then $t$ is a regular value of $\phi$ if and only if 1 is a regular value of $\phi_{+1}$. By the above, we obtain our Theorem in full generality.

The proof of our Theorem is completed. □

THEOREM 5.21. *If G is simply connected, under one of the following conditions, $\{x \in X; Z'(x) \neq 1\}$ is included in a union of submanifolds of codimension $\geq 2$:*

- $g \geq 1$.
- $g = 0$, $s \geq 2$, *and at least 2 of the $t_j$'s are regular.*

*If G is non necessarily simply connected, if all the $t_j$'s are very regular, then $\{x \in X, Z'(x) \neq 1\}$ is included in a union of submanifolds of codimension $\geq 2$ if*

- $g \geq 1$.
- $g = 0$, $s \geq 2$, *and at least 2 of the $t_j$'s are very regular.*

PROOF. By [**24**, Proposition 27.4], the set of conjugacy classes of stabilizers $Z'(x)$ is finite. Take $x \in X$ and assume that $Z'(x) \neq e$. Let $G'(x)$ be the orbit of $G'$ through $x$. Then

$$G'(x) \simeq G'/Z'(x). \tag{5.107}$$

Let $N_x$ be the normal bundle to $G'(x)$ at $x$. Then $Z'(x)$ acts on $N_x$. By [**24**, Proposition 27.2], there is a $G'$-invariant open neighborhood of $G'(x)$ in $X$ which can be identified as a $G$-space to a neighborhood of the zero section in $N = G' \times_{Z'(x)} N_x$.

Let $N_x^f$ be the invariant part of $N_x$ under $Z'(x)$. Then $N_x^f$ extends to a vector subbundle of the vector bundle $N$ on $G'/Z'(x)$, of the form $G'/Z'(x) \times N_x^f$.

If $y \in N_x$, then $Z'(y) \subset Z'(x)$ and $Z'(y) = Z'(x)$ if and only if $y \in N_x^f$. In particular $Z'(y)$ is conjugate to $Z'(x)$ if and only if $y \in N_x^f$, in which case $Z'(y) = Z'(x)$. It follows that near $G'(x)$, the elements of $X$ whose stabilizer is conjugate to $Z'(x)$ form a neighborhood of the zero section of $N^f$.

Let $u \in Z'(x)$, $u \neq 1$. We may and we will assume that $u \in T$. Let $(G'/Z'(x))^u$ be the fixed point set of $u$ in $G'/Z'(x)$. Let $\operatorname{codim}((G'/Z'(x))^u, G'/Z'(x))$ be the codimension of $(G'/Z'(x))^u$ in $G'/Z'(x)$. Clearly

$$\operatorname{codim}((G'/Z'(x))^u, G'/Z'(x)) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{z}(u)) . \tag{5.108}$$

Let $N_x^u$ be the vector subspace of $N_x$ fixed under $u$. Then $N_x^f \subset N_x^u$. If $N^u$ is the fixed point set of $N$ under $u$, then $N^u$ is a vector bundle over $(G'/Z'(x))^u$, with fibre modelled on $N_x^u$.

Let $\dim(N^f) = \dim(G'/Z'(x)) + \dim N_x^f$ be the dimension of the total space of $N^f$. Similarly let $\dim(N^u)$ be the dimension of the total space of $N_u$. By (5.108), we get

$$\dim(N^f) \leq \dim(N^u) + \dim(\mathfrak{g}) - \dim \mathfrak{z}(u) . \tag{5.109}$$

Clearly

$$\dim X = 2g \dim(\mathfrak{g}) + \sum_{j=1}^{s} \dim(\mathcal{O}_j) . \tag{5.110}$$

Let $X^u$ be the fixed point set of $X$ under $u$. Then

$$X^u = Z(u)^{2g} \times \prod_{j=1}^{s} (\mathcal{O}_j \cap Z(u)), \tag{5.111}$$

so that

$$\dim(X^u) = 2g \dim \mathfrak{z}(u) + \sum_{j=1}^{s} \dim(\mathcal{O}_j \cap Z(u)) . \tag{5.112}$$

By (5.109), (5.112),

$$\dim(N^f) \leq (2g-1) \dim \mathfrak{z}(u) + \dim \mathfrak{g} + \sum_{j=1}^{s} \dim(\mathcal{O}_j \cap Z(u)) . \tag{5.113}$$

If $g \geq 1$, using (5.96), (5.110), (5.113),

$$\dim(N^f) \leq \dim(X) - 2 . \tag{5.114}$$

Assume now that $g = 0$, that $s \geq 2$, that $G$ is simply connected and $t_1, t_2$ are regular, or more generally that $t_1, t_2$ are very regular. By Theorem 1.50,

$$\begin{aligned} \dim \mathcal{O}_j &= \dim(\mathfrak{g}) - \dim(\mathfrak{t}) , \\ \dim \mathcal{O}_j \cap Z(u) &= \dim \mathfrak{z}(u) - \dim \mathfrak{t} . \end{aligned} \tag{5.115}$$

So if $g = 0$, using (5.96), (5.110), (5.113), (5.115), we get

$$\begin{aligned}(5.116)\quad \dim(N^f) &\leq \dim \mathfrak{z}(u) + \dim \mathfrak{g} - 2\dim(\mathfrak{t}) + \sum_{j=3}^{s} \dim(\mathcal{O}_j \cap Z(u)) \\ &\leq \dim(X) - 2.\end{aligned}$$

From (5.114), (5.116), our Theorem follows. □

**5.6. The tangent bundle to the moduli space and its symplectic form.** We make the same assumptions as in Sections 5.1-5.4.

If $x \in X$, let $G'(x)$ be the orbit of $G'$ at $x$, and let $T_xG'(x)$ be the tangent space to $G'(x)$ at $x$. Recall that $T_xX \subset \mathfrak{g}^{2g+s}$, so that $T_xG'(x) \subset \mathfrak{g}^{2g+s}$.

PROPOSITION 5.22. *If $x \in M$,*

$$(5.117)\qquad T_xG'(x) = \partial(\widehat{C}_x^{\Sigma,0}(E)) \subset \widehat{C}_x^{\Sigma,1}(E).$$

*If $\phi$ is regular at $x \in M$, then $M$ is a submanifold of $X$ near $x$, and*

$$(5.118)\qquad T_xM = \ker \partial_{|\widehat{C}^{\Sigma,1}(E) \to \widehat{C}^{\Sigma,2}(E)}.$$

PROOF. By Theorem 5.14, the first part of our Proposition is clear. If $\phi$ is regular at $x \in M$,

$$(5.119)\qquad T_xM = \ker d\phi(x).$$

Using Theorem 5.14 and (5.119), we get (5.118). □

REMARK 5.23. Of course, if $\phi$ is regular at $x$, then $T_xG'(x) \subset T_xM$. This fits with (5.117), (5.118), because $\partial^2 = 0$. By Proposition 5.10 and Theorem 5.12, $T_xG'(x)$ has dimension $\dim \mathfrak{g}$.

Let $\pi : M \to M/G$ be the obvious projection. If $\phi$ is regular at $x \in M$, $Z'(x)$ is a finite group.

THEOREM 5.24. *If $\phi$ is regular at $x \in M$, $M$ is smooth near $x$, and $M/G$ is an orbifold near $\pi(x)$. More precisely, near $\pi(x) \in M/G$,*

$$(5.120)\qquad TM/G = M \times_G \widehat{H}^{\Sigma,1}(E).$$

*Also near $\pi(x)$,*

$$(5.121)\qquad M/G \simeq \widehat{H}_x^1(\Sigma, E)/Z'(x).$$

PROOF. These are classical facts from the theory of orbifolds (see [?, Proposition 27.7]. □

REMARK 5.25. By Theorems 5.4 and 5.24, $TM/G$ is the image of $H^1((\Sigma, \partial\Sigma), E)$ into $H^1(\Sigma, E)$. This fact was also observed in Guruprasad, Huebschamnn,Jeffrey and Weinstein [**25**].

Now we assume temporarily that $G$ is simply connected. We use the notation of Section 4.

DEFINITION 5.26. Put

$$(5.122)\qquad \mathcal{A}^{\text{flat}}(t_1, \dots, t_s) = \{A \in \mathcal{A}, F^A = 0, -(\frac{d}{dt} + A)_{S_j^1} \in \mathcal{O}_{-\frac{d}{dt}+t_j}\}, 1 \leq j \leq s.$$

Equivalently, $\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)$ is the set of flat connections $A$ such that for $1 \le j \le s$, the holonomy $w_j$ of $A$ along $S_j^1$ lies in the orbit $\mathcal{O}_j$.

Clearly $\Sigma G$ acts on the right on $\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)$.

DEFINITION 5.27. Let $\Sigma_q G$ be the group of smooth maps $g : \Sigma \to G$ such that $g_q = 1$.

It is easy to see that $\Sigma_q G$ acts freely on $\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)$ . Also there is an obvious continuous map $f : \mathcal{A}^{\text{flat}}(t_1,\dots,t_s) \mapsto M$ which, to $A$, associates $(u_1, v_1, \dots w_1, \dots,$
$w_s)$, the holonomies of $A$ along the circles $a_1, b_1, \dots, c_1 d_1 c_1^{-1}, \dots, c_s d_s c_s^{-1}$.

We equip $\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)/\Sigma_q G$ with the quotient topology.

PROPOSITION 5.28. *The map $f$ induces an identification of compact spaces*

$$\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)/\Sigma_q G \simeq M\,. \tag{5.123}$$

PROOF. Clearly $f$ descends to a one to one map $\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)/\Sigma_q G \simeq M$. This map is continuous. By [**18**, Proposition 2.2.3], $f$ is a pointwise identification. The fact that the topologies coincide follow for instance from the techniques of [**18**, Section 4.2], in a much simpler context. □

Assume now that $(t_1,\dots,t_s)$ verify (A). Then by Theorem 5.18, $M$ is a smooth manifold, possibly empty if $g=0$. Using the techniques of [**18**, Section 4.2], we find that $\mathcal{A}^{\text{flat}}(t_1,\dots,t_s)$ is a smooth manifold. Note that here, we use explicitly the fact that for any $x \in M$, $\widehat{H}^2(\Sigma, E) = 0$. By [**18**, Proposition 4.2.23], $\mathcal{A}^{\text{flat}}(t_1,\dots t_s)/\Sigma G$ is a smooth manifold. Also $M \to M/G$ is a $G$-orbifold. By the above, $\Sigma G$ acts locally freely on $\mathcal{A}^{\text{flat}}(t_1,\dots t_s)$.

PROPOSITION 5.29. *We have the identification of orbifolds,*

$$\mathcal{A}^{\text{flat}}(t_1,\dots t_s)/\Sigma G \simeq M/G. \tag{5.124}$$

PROOF. By proceeding as in [**18**, Proposition 4.2.23], we find easily that the identification $f$ in Proposition 5.28 is an identification of smooth manifolds. Our Proposition follows. □

We do no longer assume $G$ to be simply connected.

Let $\langle , \rangle$ be a $G$-invariant bilinear symmetric form on $\mathfrak{g}$. Then $E$ is equipped with the corresponding flat bilinear symmetric form. Recall that by Theorem 5.4, $\widehat{H}^1(\Sigma, E)$ is the image of $H^1((\Sigma, \partial\Sigma), E)$ into $H^1(\Sigma, E)$.

DEFINITION 5.30. If $x \in M, \alpha, \alpha' \in \widehat{H}^1_x(\Sigma, E)$, put

$$\omega_x(\alpha, \alpha') = \int_\Sigma -\langle \alpha, \alpha' \rangle. \tag{5.125}$$

Then $\omega_x$ is an intersection form, so that it is non degenerate.

THEOREM 5.31. *The 2-form $\omega$ is $G$-invariant. It descends to a symplectic 2-form on the orbifold $M/G$.*

PROOF. Suppose first that $G$ is simply connected. By Proposition 5.28, the space $\mathcal{A}^{\text{flat}}(t_1,\dots t_s)/\Sigma G$ is an orbifold. Then the results of Section 4.1 and Theorem 4.24 show that $\mathcal{A}^{\text{flat}}(t_1,\dots t_s)/\Sigma G$ is a symplectic reduction of the symplectic manifold $\mathcal{A}$. Since it is a symplectic reduction of the symplectic affine space $\mathcal{A}$, it carries a symplectic form. One verifies easily that $\omega$ is just this form. Therefore $\omega$ is closed.

If $G$ is not simply connected, the homotopy types of $G$-bundles have been classified in Section 4.7. Let $\widetilde{G}$ be the universal cover of $G$. By introducing an extra holonomy $h \in Z(\widetilde{G})$, we can then replace $G$ by $\widetilde{G}$. The fact that $\omega$ is closed is now a consequence of the corresponding result for $\widetilde{G}$.

The proof of our Theorem is completed. □

Recall that $G$ acts on the right and on the left on $X$. If $x \in M, a \in \Gamma$, then $x(a) \in G$. We define a right action of $\Gamma$ on $X$ by the formula

$$x.a = x(a).x,\ x \in M,\ a \in \Gamma \tag{5.126}$$

Then if $a \in \Gamma, g \in G, x \in M$,

$$(x.g).a = (x.g)(a).(x.g) = (x.a).g, \tag{5.127}$$

i.e. the actions of $\Gamma$ and $G$ on $M$ commute.

Also $\Gamma \times G$ acts on the right on $\widehat{\Sigma}$ (of course, the factor $G$ acts trivially on $\widehat{\Sigma}$). Therefore $\Gamma \times G$ acts on the right on $M \times \widehat{\Sigma}$. Also $\Gamma \times G$ acts locally freely on $M$, and $\Gamma$ acts freely on $\widehat{\Sigma}$.

Let $V$ be a complex vector space. Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of $G$. Then $\rho$ induces a representation $\Gamma \times G \to \mathrm{Aut}(V)$. Put

$$F = (M \times \widehat{\Sigma}) \times_{\Gamma \times G} V. \tag{5.128}$$

Then $F$ is an orbifold vector bundle on $M/G \times \Sigma$. It is obtained via the identification

$$(x, \sigma, f) \simeq (x.g, \sigma.a, g^{-1}x(a)^{-1}f), (x, \sigma, f) \in M \times \widehat{\Sigma} \times V, (a, g) \in \Gamma \times G. \tag{5.129}$$

For a given $x \in M$, the restriction of $F$ to the fibre $\Sigma$ is exactly the flat vector bundle considered in Section 5.2.

If $\sigma \in \Sigma$, then $\sigma^* F$ is an orbifold vector bundle on $M/G$. If $\sigma, \sigma' \in \Sigma$, if $t \in [0,1] \mapsto \sigma_t \in \Sigma$ is a smooth path, with $\sigma_0 = \sigma, \sigma_1 = \sigma'$, parallel transport with respect to the flat connection identifies $\sigma^* F$ and $\sigma'^* F$.

Over $\Sigma$, there are two distinguished points $p$ and $q$. Recall that $G$ acts on $\mathfrak{g}$ by conjugation. When $V = \mathfrak{g}$, let $E$ be the corresponding real vector bundle on $M/G \times \Sigma$.

DEFINITION 5.32. Let $\mathcal{E}$ be the vector bundle on $M/G$,

$$\mathcal{E} = p^* E. \tag{5.130}$$

Also $G$ acts by conjugation on $G$ and on the $\mathcal{O}_j$'s. Put

$$\widehat{G} = M \times_G G,\ \widehat{\mathcal{O}}_j = M \times_G \mathcal{O}_j. \tag{5.131}$$

Then $\widehat{G} \to M/G$ is a $G$-bundle, and $\widehat{\mathcal{O}}_j \to M/G$ is a $\mathcal{O}_j$-bundle. Also $u_1, \dots, v_g$ are sections of $\widehat{G}$ and $w_1, \dots, w_s$ are sections of $\widehat{\mathcal{O}}_1, \dots, \widehat{\mathcal{O}}_s$.

Elements of $\widehat{G}$ act naturally on $\mathcal{E}$. As explained in Section 5.2, $u_1$ can be considered as the parallel transport operator along the closed curve $a_1$ ...

For $1 \le j \le s$, let $T_{w_j}\widehat{\mathcal{O}}_j/(M/G)$ be the relative tangent bundle to the fibre $\widehat{\mathcal{O}}_j$ . As we saw in (5.48),

$$T_{w_j}\widehat{\mathcal{O}}_j/M = \mathrm{Im}(1 - w_j) \subset \mathcal{E}. \tag{5.132}$$

As explained in Theorem 5.14, on $M/G$, we have the bundle of complexes

$$0 \to \widehat{C}^{\Sigma,0}(E) \to \widehat{C}^{\Sigma,1}(E) \to \widehat{C}^{\Sigma,2}(E) \to 0, \tag{5.133}$$

which, by (5.53), (5.132) can also be written as

$$0 \to \mathcal{E} \to \mathcal{E}^{2g} \oplus \Big(\bigoplus_{j=1}^{s} T_{w_j}(\widehat{\mathcal{O}}_j/(M/G))\Big) \to \mathcal{E} \to 0. \tag{5.134}$$

THEOREM 5.33. *On $M/G$, we have the identity*

$$TM/G = \mathcal{E}^{2g} \oplus \Big(\bigoplus_{j=1}^{s} T_{w_j}\widehat{\mathcal{O}}_j/(M/G)\Big) \ominus \mathcal{E}^2 \text{ in } K(M/G). \tag{5.135}$$

PROOF. By (5.120),

$$TM/G = \widehat{H}^1(\Sigma, E). \tag{5.136}$$

Also over $M$, by (5.63),

$$\widehat{H}^j(\Sigma, E) = 0,\ j = 0, 2. \tag{5.137}$$

Hence, over $M/G$, by (5.134), (5.136), (5.137),

$$\widehat{H}^1(\Sigma, E) = \mathcal{E}^{2g} \oplus \Big(\bigoplus_{j=1}^{s} T_{w_j}\widehat{\mathcal{O}}_j/(M/G)\Big) \ominus \mathcal{E}^2 \text{ in } K(M/G). \tag{5.138}$$

From (5.137), (5.138), we get (5.135). □

THEOREM 5.34. *If the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_s$ are very regular (resp. regular), then*

$$TM/G = \mathcal{E}^{2(g-2)+s} \ominus \mathbf{R}^{s \dim \mathfrak{t}} \text{ in } K(M/G) \text{ (resp. in } K(M/G) \otimes_{\mathbf{Q}} \mathbf{Z}.) \tag{5.139}$$

PROOF. Let $t \in T$ be very regular, let $\mathcal{O}$ be the orbit of $t$ in $G$. Then $Z(t) = T$. If $g \in G$,

$$Z(g.t) = g.T,\ \mathfrak{z}(g.t) = g.\mathfrak{t}. \tag{5.140}$$

If $w \in \mathcal{O}$, we have the splitting

$$\mathfrak{g} = \operatorname{Im}(1-w) \oplus \ker(1-w), \tag{5.141}$$

which corresponds to the splitting

$$\mathcal{E} = T_w\mathcal{O} \oplus N_{\mathcal{O}/\mathcal{E}}. \tag{5.142}$$

By (5.140), the vector bundle $\mathfrak{z}(w) = \ker(1-w)$ is trivial on $\mathcal{O}$. The action of $G$ on the orbit $\mathcal{O}$ lifts to an action on the vector bundle $\mathfrak{z}(w)$. From (5.141), (5.142), we find that the normal bundle $N_{\mathcal{O}/\mathcal{E}}$ is equivariantly trivial. From (5.135), (5.142), we get (5.139).

If $t \in T$ is only regular, then $\mathfrak{z}(t) = \mathfrak{t}$. Also $\mathcal{O} = G/Z(t)$. Then if $w = g.t$, $g \in G \mapsto g.\mathfrak{z}(t) = \mathfrak{z}(w)$ induces a $G$-invariant flat connection on $\mathfrak{z}(w) \simeq N_{\mathcal{O}/\mathfrak{g}}$. Consider the splitting of vector bundles

$$\mathcal{E} = T_{w_j}(\widehat{\mathcal{O}}_j/(M/G)) \oplus \ker(1-w_j). \tag{5.143}$$

If the orbit $\mathcal{O}_j$ is very regular, $\ker(1-w_j)$ is a trivial vector bundle on $M/G$. Therefore

$$T_{w_j}\widehat{\mathcal{O}}_j/(M/G) = \mathcal{E} \ominus \mathbf{R}^{\dim \mathfrak{t}} \text{ in } K(M/G). \tag{5.144}$$

If the orbit $\mathcal{O}_j$ is only regular, then one finds easily by the above that $\ker(1-w_j)$ is equipped over $M/G$ with a flat connection. We thus get (5.139) in this case. The proof of our Theorem is completed. □

**5.7. A metric on the determinant of $TM/G$.** If $\lambda$ is a line, let $\lambda^{-1}$ be the dual line. If $E$ is a vector space, set

$$\det(E) = \Lambda^{\max}(E)\,. \tag{5.145}$$

If $E = \bigoplus_{i=0}^{m} E_i$ is a $\mathbf{Z}$ graded vector space, set

$$\det(E) = \bigotimes_{i=0}^{m} (\det E_i)^{(-1)^i}\,. \tag{5.146}$$

Let $G$ be a compact connected semisimple Lie group.

DEFINITION 5.35. If $x \in M$, let $\lambda_x$ be the real line

$$\lambda_x = (\det \widehat{H}_x(\Sigma, E))^{-1}. \tag{5.147}$$

Let $\langle,\rangle$ be a $G$-invariant scalar product on $\mathfrak{g}$. Then $E$ is equipped with a fibrewise flat scalar product. By Theorem 5.4 and Remark 5.5,

$$\widehat{H}^2(\Sigma, E) \simeq (\widehat{H}^0(\Sigma, E))^*,\ \widehat{H}^1(\Sigma, E) \simeq (\widehat{H}^1(\Sigma, E))^*. \tag{5.148}$$

Observe that in (5.148), the identifications depend explicitly on $\langle,\rangle$. By (5.148), for $x \in M$,

$$\lambda_x^2 \simeq \mathbf{R}. \tag{5.149}$$

Now $\mathbf{R}$ carries a canonical metric $\| \ \|_{\mathbf{R}}$ such that $\|1\|_{\mathbf{R}} = 1$.

DEFINITION 5.36. Let $\| \ \|_{\lambda_x}$ be the metric on $\lambda_x$ such that the identification (5.149) is an isometry.

In the sequel, we assume that $(t_1, \dots, t_s)$ verify (A).

PROPOSITION 5.37. *If $x \in M$, the metric $\| \ \|_{\lambda_x}$ defines a volume element on $\Lambda^{\max} T_x(M/G)$, which is exactly the volume associated to the symplectic form $\omega_x$.*

PROOF. If $x \in M$, then $\widehat{H}^0_x(\Sigma, E) = 0$, $\widehat{H}^1_x(\Sigma, E) = 0$, so that

$$\lambda_x = \det \widehat{H}^1_x(\Sigma, E)\,. \tag{5.150}$$

By (5.120),

$$T_x M/G = \widehat{H}^1_x(\Sigma, E)\,. \tag{5.151}$$

Finally in (5.148), the identification $\widehat{H}^1_x(\Sigma, E) \simeq (\widehat{H}^1_x(\Sigma, E))^*$ is done via the symplectic form $\omega_x$. Our Proposition follows. □

Now recall that $\widehat{H}_x(\Sigma, E)$ is the cohomology of the complex $(\widehat{C}^{\Sigma}_x(E), \partial)$. Put

$$\widehat{\lambda}_x = (\det(\widehat{C}^{\Sigma}_x(E)))^{-1}. \tag{5.152}$$

Then by [**35**], there is a canonical isomorphism

$$\lambda_x \simeq \widehat{\lambda}_x. \tag{5.153}$$

Also, since $E$ is equipped with a flat scalar product, $\widehat{\lambda}_x$ is also naturally equipped with a metric $\| \ \|_{\widehat{\lambda}_x}$, which also depends on $\langle,\rangle$. Let $\| \ \|^{\widehat{}}_{\lambda_x}$ be the corresponding metric on $\lambda_x$ via the canonical isomorphism (5.153). .

DEFINITION 5.38. For $1 \leq j \leq s$, put

$$|\sigma_{\mathcal{O}_j}| = |\det(1-w_j)_{|\mathrm{Im}(1-w_j)}|^{1/2},\ w_j \in \mathcal{O}_j. \tag{5.154}$$

As the notation indicates, $|\sigma_{\mathcal{O}_j}|$ is a constant on $\mathcal{O}_j$. We make the convention that if $\mathrm{Im}(1-w_j) \subset \mathfrak{g}$ is reduced to 0,

$$|\sigma_{\mathcal{O}_j}| = 1 \tag{5.155}$$

REMARK 5.39. Recall that the function $\sigma : G \to \mathbf{C}$ was defined in (1.42). If $\mathcal{O}_j$ is regular, if $w_j \in \mathcal{O}_j$,

$$|\sigma_{\mathcal{O}_j}| = |\sigma(w_j)|. \tag{5.156}$$

The following result has been proved by Witten [**63**, Section 4], who exhibited the role of the Reidemeister torsion [**49**] in this context.

THEOREM 5.40. *For any $x \in M$,*

$$\|\cdot\|_{\widehat{\lambda_x}} = \prod_{j=1}^{s} |\sigma_{\mathcal{O}_j}| \|\cdot\|_{\lambda_x}. \tag{5.157}$$

PROOF. By (5.31),

$$\det(\widehat{C}^{\Sigma}(E)) \simeq \det(C^{\Sigma,r}(E)) \otimes (\det(K))^{-1}, \tag{5.158}$$

which can also be written in the form,

$$\det(\widehat{C}^{\Sigma}(E)) \simeq \det(C^{\Sigma,r}(E)) \otimes \det(H^0(\partial\Sigma, E)). \tag{5.159}$$

By (5.29),

$$H^0(\partial\Sigma, E) = \bigoplus_{j=1}^{s} \ker(1-w_j). \tag{5.160}$$

Let $\|.\|_{\det H^0(\partial\Sigma,E)}$ be the metric on $\det H^0(\partial\Sigma, E)$ induced by the scalar product of $E$. By (5.31) , we find that under the isomorphism (5.159),

$$\| \ \|_{\det \widehat{C}^{\Sigma}(E)} = \| \ \|_{\det C^{\Sigma,r}(E)} \| \ \|_{\det H^0(\partial\Sigma,E)} \left[ \prod_{j=1}^{s} \det(1-w_j)_{|\mathrm{Im}(1-w_j)} \right]^{-1}. \tag{5.161}$$

Also by (5.21), there is a canonical isomorphism

$$\det C^{\Sigma}(E) \simeq \det C^{\Sigma,r}(E) \otimes \det C^{\partial\Sigma}(E). \tag{5.162}$$

If $\| \ \|_{\det C^{\Sigma}(E)}$ denote the obvious metric on $\det(C^{\Sigma}(E))$, by (5.21), (5.162), we get

$$\| \ \|_{\det C^{\Sigma}(E)} = \| \ \|_{\det C^{\Sigma,r}(E)} \| \ \|_{\det C^{\partial\Sigma}(E)}. \tag{5.163}$$

Also, we have a canonical isomorphism

$$\det(C^{\partial\Sigma}(E)) \simeq \det(H(\partial\Sigma, E)). \tag{5.164}$$

By Poincaré duality,

$$H^1(\partial\Sigma, E) = (H^0(\partial\Sigma, E))^*, \tag{5.165}$$

so that by (5.164), (5.165),

$$\det(C^{\partial\Sigma}(E)) \simeq \det(H^0(\partial\Sigma, E))^2. \tag{5.166}$$

By (5.25), (5.166),

$$(5.167) \qquad \| \ \|_{\det(C^{\partial\Sigma}(E))} = \| \ \|^2_{\det H^0(\partial\Sigma,E)} \left[ \prod_{j=1}^{s} \det(1-w_j)_{|\mathrm{Im}(1-w_j)} \right]^{-1} .$$

Using (5.159)-(5.166),we get

$$(5.168) \qquad (\det \widehat{C}^{\Sigma}(E))^2 \simeq \det C^{\Sigma}(E) \otimes \det C^{\Sigma,r}(E).$$

Also under the isomorphism (5.168),

$$(5.169) \qquad \| \ \|^2_{\det \widehat{C}^{\Sigma}(E)} = \| \ \|_{\det C^{\Sigma}(E)} \| \ \|_{\det C^{\Sigma,r}(E)} \left[ \prod_{j=1}^{s} \det(1-w_j)_{|\mathrm{Im}(1-w_j)} \right]^{-1} .$$

Now recall that by [**35**], there are canonical isomorphisms

$$(5.170) \qquad \begin{aligned} \det(C^{\Sigma}(E)) &\simeq \det(H(\Sigma, E)), \\ \det(C^{\Sigma,r}(E)) &\simeq \det(H((\Sigma, \partial\Sigma), E). \end{aligned}$$

By Poincaré duality,

$$(5.171) \qquad \det H(\Sigma, E) \otimes \det H((\Sigma, \partial\Sigma), E) \simeq \mathbf{R}.$$

So by (5.170), (5.171),

$$(5.172) \qquad \det C^{\Sigma}(E) \otimes \det C^{\Sigma,r}(E) \simeq \mathbf{R}.$$

Let $\| \ \|_{\mathbf{R}}$ be the trivial metric on $\mathbf{R}$, such that $\|1\|_{\mathbf{R}} = 1$. We claim that under the Poincaré duality isomorphism (5.172),

$$(5.173) \qquad \| \ \|_{\det C^{\Sigma}(E)} \| \ \|_{\det C^{\Sigma,r}(E)} = \| \ \|_{\mathbf{R}} .$$

In fact (5.173) is a simple consequence of the existence of the Reidemeister torsion [**49**], or more precisely of the Reidemeister metric [**13**, Section 1] on $\det H(\Sigma, E))$ and $\det H((\Sigma, \partial\Sigma), E)$. In fact given any cell decomposition of the manifold with boundary $\Sigma$, one can construct Reidemeister metrics on $\det H(\Sigma, E)$ and $\det H((\Sigma, \partial\Sigma), E)$ by the procedure indicated in [**13**]. The basic fact is that these metrics do not depend on the choice of the cell decomposition. By applying this argument to a cell decomposition and the corresponding dual decomposition, one deduces immediately that the Reidemeister metrics on $\det H(\Sigma, E)$ and $\det H((\Sigma, \partial\Sigma), E)$ correspond by Poincaré duality. As a consequence, (5.173) follows.

From (5.169), (5.173), we get (5.157). The proof of our Theorem is completed. □

From (5.156), (5.157), it follows that if $\mathcal{O}_1, \dots, \mathcal{O}_s$ are regular, if $x \in B$,

$$(5.174) \qquad \| \ \|^{\widehat{}}_{\lambda_x} = \prod_{j=1}^{s} |\sigma(w_j)| \| \ \|_{\lambda_x} .$$

**5.8. The Witten formula for the symplectic volume distribution.** Let $G$ be a compact connected semisimple Lie group. Let $\langle\ ,\ \rangle$ be a $G$-invariant scalar product on $\mathfrak{g}$. Let $dg$ be the Haar measure on $G$ associated to $\langle\ ,\ \rangle$. We us otherwise the same notation as in Section 5.7. In particular $T \subset G$ is a maximal torus in $G$.

Let $\mathcal{O}_1, \dots, \mathcal{O}_s$ be $s$ adjoint orbits of $G$ in $G$. Take $w_j \in \mathcal{O}_j$. Let $Z(w_j)$ be the centralizer of $w_j$. Then the map

$$a \in G/Z(w_j) \mapsto a \cdot w_j \in \mathcal{O}_j \tag{5.175}$$

is an identification of smooth manifolds.

The scalar product $\langle\ ,\ \rangle$ induces a volume form $dv_{\mathcal{O}_j}$ on $\mathcal{O}_j \subset G$.

DEFINITION 5.41. Let $dv_X$ be the volume form on $X = G^{2g} \times \prod_{j=1}^{s} \mathcal{O}_j$,

$$dv_X(u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_s) = \prod_{i=1}^{2g} du_i dv_i \prod_{j=1}^{s} dv_{\mathcal{O}_j}(w_j)\,. \tag{5.176}$$

Let $dt$ be the Lebesgue measure on $\mathfrak{t}$ induced by $\langle , \rangle$.

DEFINITION 5.42. If $g \in G$, let $X_{\mathcal{O}_g} \subset G^{2g} \times \prod_{j=1}^{s} \mathcal{O}_j \times \mathcal{O}_g$ be the set $M$ in (5.59) associated to the orbits $\mathcal{O}_1, \dots, \mathcal{O}_s, \mathcal{O}_g$.

More precisely

$$X_{\mathcal{O}_g} = \Big\{(u_1, v_1, \dots, w_1, \dots, w_s, w) \in G^{2g} \times \prod_{j=1}^{s} \mathcal{O}_j \times \mathcal{O}_g\,, \tag{5.177}$$

$$\prod_{i=1}^{g}[u_i, v_i] \prod_{j=1}^{s} w_j w = 1\Big\}\,. \tag{5.178}$$

Clearly, $X_{\mathcal{O}_g}$ can be identified with $\{x \in X,\ \phi(x) \in \mathcal{O}_{g^{-1}}\}$. Set

$$X_g = \{x \in X; \phi(x) = g^{-1}\}\,. \tag{5.179}$$

Now $G$ acts naturally on $X_{\mathcal{O}_g}$, and $Z(g)$ acts on $X_g$. Then one has the obvious

$$X_{\mathcal{O}_g}/G \simeq X_g/Z(g)\,. \tag{5.180}$$

Clearly, if $x \in X$, $\int_G f(x \cdot g) dg$ depends only on $\pi x \in X/G$.

Let $G_{\text{vreg}}$ (resp. $G_{\text{reg}}$ ) be the set of very regular (resp. regular) elements in $G$. If $g \in G_{\text{vreg}}$, let $dt_g$ be the Haar measure on the maximal torus $Z(g)$, which is associated to $\langle\ ,\ \rangle$.

By Sard's theorem, a.e. every $g \in G$ is a regular value of $\phi$. For such a $g \in G$, $X_{\mathcal{O}_g}$ is smooth, and $G$ acts locally freely on $X_{\mathcal{O}_g}$. Also by Theorem 5.31, the orbifold $X_{\mathcal{O}_g}/G$ is equipped with a symplectic 2 form $\omega_g$. Let $dv_{X_{\mathcal{O}_g}/G}$ be the corresponding volume element on $X_{\mathcal{O}_g}/G$. Since $X_{\mathcal{O}_g}/G = X_g/Z(g)$, let $dv_{X_g/Z(g)}$ be the associated volume on $X_g/Z(g)$. Observe here that $dv_{X_g/G}$ is unsensitive to orientation. In particular the integral of a nonnegative function with respect to $dv_{X_{\mathcal{O}_g}/G}$ is nonnegative.

In the sequel, we make the following assumptions:

- $G$ is simply connected, and one of the following assumptions is verified :
  - $g \geq 2$.
  - $g \geq 1$, $s \geq 1$ and at least one of the $t_j$'s is regular.
  - $g = 0$, $s \geq 3$, and at least 3 of the $t_j$'s are regular.

or

- $G$ is connected, the $t_j$'s are very regular , and either
  - $g \geq 2$.
  - $g \geq 1,\ s \geq 1$.
  - $g = 0,\ s \geq 3$.

Now we will get an analogue of Theorem 3.10.

THEOREM 5.43. *Let $f : X \to \mathbf{R}$ be a bounded measurable function. Then*

$$\int_X f(x) dv_X(x) = \frac{\prod_{j=1}^{s} |\sigma_{O_j}|}{|Z(G)|} \int_{T/W} |\sigma(t)| dt \int_{X_t/T} dv_{X_t/T}(x) \int_G f(x \cdot g) dg, \tag{5.181}$$

$$\int_M f(x) dv_X(x) = \frac{\prod_{j=1}^{s} |\sigma_{O_j}|}{|Z(G)|} \int_G \frac{dg}{|\sigma(g)|} \int_{X_g/Z(g)} dv_{X_g/Z(g)}(x) \int_{Z(g)} f(x \cdot t) dt_g .$$

PROOF. By Theorem 5.21, we know that a.e., $Z'(x) = 1$. By Theorem 5.12, for a.e. $x \in X, d\phi(x)$ is surjective. Therefore the image $\phi_* dv_X$ of $dv_X$ by $\phi$ is absolutely continuous with respect to $dg$. Also on the set $\{x \in X,\ \phi$ is regular at $x,\ Z'(x) = 1\}$, which has full measure and is stable by $G$, we can use the implicit function theorem, and also integrate along the fibres of the action of $G$ , which are diffeomorphic to $G$.

Let $p : G_{\text{vreg}} \to T_{\text{vreg}}/W$ be the obvious projection. Since $\phi_* dv_X$ is absolutely continuous with respect to $dg$, $\phi^{-1}(G_{\text{vreg}})$ is an open set in $X$, whose complement is $dv_X$ negligible. Then $x \in \phi^{-1}(G_{\text{vreg}}) \mapsto p\phi(x) \in T_{\text{vreg}}/W$ is a smooth map. Also if $x \in \phi^{-1}(G_{\text{vreg}})$, $g \in G$,

$$p\phi(x \cdot g) = p\phi(x) . \tag{5.182}$$

By (5.182), if $x \in \phi^{-1}(G_{\text{vreg}})$, the obvious analogue of $(D, \delta)$ in (5.53) is the complex

$$(C'_x, \partial) : 0 \longrightarrow \mathfrak{g} \xrightarrow[\delta]{} \mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \operatorname{Im}(1 - w_j) \xrightarrow[d(p\phi)_x]{} \mathfrak{t} \longrightarrow 0 . \tag{5.183}$$

By Proposition 5.10 and Theorem 5.12, if $x \in G$ is such that $Z'(x) = 1$, the cohomology of the complex (5.183) is concentrated in degree 1, and the first cohomology group of (5.183) is isomorphic to $T_{\pi(x)} X_{p\phi(x)}/T$. In particular,

$$(\det C'_x)^{-1} \simeq \det(T_{\pi(x)} X_{p\phi(x)}/T) . \tag{5.184}$$

Now the complex $C'_x$ is equipped with a scalar product. Let $\| \ \|_{\det(C'_x)}$ be the obvious metric on $\det(C'_x)$, and let $\| \ \|_{\det(T_{\pi(x)} X_{p\phi(x)}/T)}$ be the corresponding metric on $\det(T_{\pi(x)} X_{p\phi(x)}/T)$. Then $\| \ \|_{\det(T_{\pi(x)} X_{p\phi(x)}/T)}$ defines a smooth volume form on $(X_{p\phi(x)}/T)_{\text{reg}}$, which will be denoted $\widetilde{dv}_{X_{p\phi(x)}/T}$. Then the formula of change of variable asserts that

$$\int_X f(x) dv_X(x) = \frac{1}{|Z(G)|} \int_{T/W} dt \int_{X_t/T} \widetilde{dv}_{X_t/T}(x) \int_G f(x \cdot g) dg . \tag{5.185}$$

If $x \in \phi^{-1}(T_{\mathrm{vreg}})$, put $t = [\phi(x)]^{-1} \in T_{\mathrm{vreg}}$. Then we have the double complex

(5.186)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \mathrm{Im}(1-w_j) & \xrightarrow{d(p\phi)} & \mathfrak{t} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}^{2g} \oplus \bigoplus_{j=1}^{s} \mathrm{Im}(1-w_j) \oplus \mathrm{Im}(1-t) & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Im}(1-t) & \xrightarrow{t^{-1}} & \mathrm{Im}(1-t) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $x \in \Sigma \backslash \partial\Sigma$, and let $\Delta$ be a small disk of center $x$. Put $\Sigma_{+1} = \Sigma \backslash \Delta$. In (5.186), starting from below, the first row is trivial. The second row is just the complex $(\widehat{C}^{\Sigma_{+1}}_{(x,t)}(E), \partial)$ we constructed in (5.31) (with $s$ replaced by $s+1$). The third row is the complex $(C'_x, \partial)$. Also the columns are acyclic.

Let us now explain why the diagram commutes. By (5.34), if $x = (u_1, v_1, \dots, w_s)$,

$$\begin{aligned}(5.187)\qquad \partial(f_1, \dots, f_{2g+s+1}) = \sum_{i=1}^{g}([u_1, v_1] \dots [u_{i-1}, v_{i-1}](1 - u_i v_i u_i^{-1}) f_{2i-1} \\ + u_i(1 - v_i u_i v_i^{-1}) f_{2i}) + \sum_{j=1}^{s} [u_1, v_1] \dots [u_g, v_g] w_1 \dots w_{j-1} f_{2g+j} + \\ \prod_{i=1}^{g} [u_i, v_i] \prod_{j=1}^{s} w_j f_{2g+s+1} \, .\end{aligned}$$

By construction,

$$(5.188)\qquad \prod_{i=1}^{g} [u_i, v_i] \prod_{j=1}^{s} w_j = t^{-1} \, .$$

So in (5.187),

$$(5.189)\qquad \partial(f_{2g+s+1}) = t^{-1} f_{2g+s+1} \, .$$

So from (5.189), we find the diagram (5.186) commutes.

We equip the vector spaces which appear in (5.186) with the scalar products induced by the scalar product of $\mathfrak{g}$. The determinants of the columns of (5.186) are canonically trivial, and the norms of the associated canonical sections are equal to 1. Also since the first row is acyclic, the cohomology groups of the second and third rows are canonically identified, using the obvious long exact sequence. Finally the first row is acyclic, and the norm of the canonical section of the determinant of the first row is equal to 1.

By an obvious extension of [**11**, Theorem 1.10], it follows that if $x \in \phi^{-1}(T_{\text{vreg}})$ is regular, then the metric on $\det(\widehat{H}^1(\Sigma_{+1}, E))$ (where $\widehat{H}^1(\Sigma_{+1}, E)$ is the first cohomology group of the complexes $(C'_x, \partial)$ or $(\widehat{C}^{\Sigma_{+1}}_{(x,t)}(E), \partial)$) induced by the complexes $(C'_x, \partial)$ or $(\widehat{C}^{\Sigma_{+1}}_{(x,t)}(E), \partial)$ are identical. Using now (5.156) and Theorem 5.40, we get

$$\widetilde{dv}_{X_t/T} = \prod_{j=1}^{s} |\sigma_{O_j}| |\sigma(t)| dv_{X_t/T} . \tag{5.190}$$

By (5.185), (5.190), we obtain the first equality in (5.181).

The scalar product $\langle\ ,\ \rangle$ induces a scalar product on $\mathfrak{g}/\mathfrak{t} = \mathfrak{t}^\perp$. Let $d\dot{g}$ be the corresponding volume element on $T/G$. Tautologically if $k : G \to \mathbf{R}$ is bounded and measurable

$$\int_G k(g)dg = \int_{T\backslash G} d\dot{g} \int_T k(tg)dt . \tag{5.191}$$

Also Weyl's integration formula [**15**, Theorem IV.1.11] asserts that

$$\int_G k(g)dg = \frac{1}{\text{Vol}(T)} \int_{T/W} |\sigma|^2(t)dt \int_G f(t \cdot g)dg . \tag{5.192}$$

Using the first identity in (5.181) and (5.191), (5.192), we obtain

$$\begin{aligned} \int_X f(x)dv_X(x) = {} & \frac{\prod_{j=1}^{s} |\sigma_{O_j}|}{|Z(G)|} \int_{T/W} |\sigma(t)|dt \\ & \int_{X_t/T} dv_{X_t/T}(\dot{x}) \int_{T\backslash G} d\dot{g} \int_T f(x \cdot t'g)dt' \\ = {} & \frac{\prod_{j=1}^{s} |\sigma_{O_j}|}{|Z(G)|} \int_G \frac{dg}{|\sigma(g)|} \int_{X_g/Z(g)} dv_{X_g/Z(g)}(x) \int_{Z(g)} f(xt)dt_g . \end{aligned} \tag{5.193}$$

The proof of our Theorem is completed. □

DEFINITION 5.44. If $t \in T$, let $|V(t)|$ be the absolute value of volume of $X_t/Z(t)$ with respect to $\omega_t$.

Then $V(t)$ is a $W$-invariant function on $T$, which extends to a central function on $G$.

Now we prove a result which was already essentially proved by Liu [**39, 40**].

THEOREM 5.45. *Let $f : G \to \mathbf{R}$ be a bounded measurable function. Then*

$$\begin{aligned} \int_X f(\phi^{-1}(x))dv_X(x) &= \frac{\prod_{j=1}^{s} |\sigma_{O_j}|}{|Z(G)|} \int_{T/W} \Big[\int_G f(g \cdot t)dg\Big] |\sigma(t)||V(t)|dt , \\ \int_X f(\phi^{-1}(x))dv_X(x) &= \frac{\text{Vol}(T) \prod_{j=1}^{s} |\sigma_{O_j}|}{|Z(G)|} \int_G \frac{f(g)|V(g)|}{|\sigma(g)|} dg . \end{aligned} \tag{5.194}$$

PROOF. Our formula follows from Theorem 5.43. □

REMARK 5.46. In [**39, 40**], Liu uses an essentially similar argument in his proof of the Witten formula [**63, 64**]. In fact let $p_t(g)$ be the convolution heat kernel on $G$. Liu considers the quantity $\int_X p_t(\phi^{-1}(x))dv_X$, and following Witten [**63**], he studies its limit as $t \to 0$. The arguments he uses to evaluate the limit as being (up to a normalization constant) the absolute value of the symplectic volume of $M/G$ are essentially the ones which are used in the proofs of Theorems 5.43 and 5.45.

Let $A$ be the set of regular values of $\phi^{-1}$ in $G$. Then $A$ is an open dense set, such that ${}^cA$ is negligible. Also on $A$, $\phi_* dv_X$ has a smooth density with respect to $dg$. By Theorem 5.45, it follows that $\frac{|V(g)|}{|\sigma(g)|}$ is smooth on $A \cap G_{\mathrm{vreg}}$.

Assume temporarily that $(t_1, \ldots, t_s)$ verify $(A)$. By Theorem 5.18, 1 is a regular value of $\phi$. By Theorem 5.20, $Z'(x) = 1$ a.e. on $M = X_1$. Using Theorem 5.45, and proceeding as in the proof of Theorem 3.13, we get

$$\lim_{\substack{g \in G_{\mathrm{vreg}} \\ g \to 1}} \frac{|V(g)|}{|\sigma(g)|} = \frac{\mathrm{Vol}(G)}{\mathrm{Vol}(T)} V(1). \tag{5.195}$$

Using (5.195) we find that

$$\lim_{t \to 0} \int_X p_t(\phi^{-1}(x)) dv_X = \frac{\mathrm{Vol}(G) \prod_{j=1}^s |\sigma_{\mathcal{O}_j}|}{|Z(G)|} V(1), \tag{5.196}$$

which is a formula obtained by Liu [**39, 40**] by arguments essentially similar to the ones we gave in our proof of Theorems 5.43 and 5.45.

Let $K$ be a positive Weyl chamber in $\mathfrak{t}$. Let $\Lambda_+ = \Lambda \cap \overline{K}$ be the set of nonnegative weights. Then the irreducible representations of $G$ are indexed by $\Lambda_+$. If $\lambda \in \Lambda_+$, let $\chi_\lambda$ be the character of the corresponding representation of $G$.

By Theorem 5.45, $\frac{|V(g)|}{|\sigma(g)|}$ defines a $L_1$ $G$-invariant function on $G$. Therefore, it defines an invariant distribution on $G$, which can be expanded as a linear combination of the characters $\chi_\lambda$ of $G$.

Let $w_j \in \mathcal{O}_j$. Then $\mathcal{O}_j \simeq G/Z(w_j)$. Also $\mathrm{Vol}(Z(w_j))$ does not depend on the choice of $w_j$. Finally $\chi_\lambda$ takes the complex value $\chi_\lambda(t_j)$ on the orbit $\mathcal{O}_j$.

Now we prove a result of Witten [**63, 64**].

THEOREM 5.47. *The following identity of G-invariant distributions holds on G,*

$$\frac{|V(g)|}{|\sigma(g)| \prod_{j=1}^s |\sigma_{\mathcal{O}_j}|} = |Z(G)| \frac{\mathrm{Vol}(G)^{2g+s-1}}{\mathrm{Vol}(T) \prod_{j=1}^s \mathrm{Vol}(Z(t_j))} \sum_{\lambda \in \Lambda_+} \frac{\prod_{j=1}^s \chi_\lambda(t_j) \chi_\lambda(g)}{\chi_\lambda(1)^{2g+s-1}}. \tag{5.197}$$

PROOF. By Theorem5.45, we get

$$\frac{1}{\mathrm{Vol}(G)} \int_G \overline{\chi}_\lambda(g) \frac{|V(g)|}{|\sigma(g)|} dg = \frac{|Z(G)|}{\mathrm{Vol}(G)\mathrm{Vol}(T)} \frac{1}{\prod_{j=1}^s |\sigma_{\mathcal{O}_j}|} \int_X \chi_\lambda(\phi(x)) dv_X(x). \tag{5.198}$$

To evaluate the integral in the right-hand side of (5.198), we follow Witten [**63**]. By [**15**, Theorem II.4.5], if $a, b \in G$,

$$\int_G \chi_\lambda(agbg^{-1}) dg = \frac{\mathrm{Vol}(G)}{\chi_\lambda(1)} \chi_\lambda(a) \chi_\lambda(b), \tag{5.199}$$

$$\int_G \chi_\lambda(ag) \chi_{\lambda'}(g^{-1}b) dg = \mathrm{Vol}(G) \delta_{\lambda,\lambda'} \frac{\chi_\lambda(ab)}{\chi_\lambda(1)}.$$

Finally, if $h : \mathcal{O}_j \to \mathbf{R}$ is a bounded measurable function, we have the easy formula

$$\int_{\mathcal{O}_j} h(g) dv_{\mathcal{O}_j}(g) = \frac{|\sigma_{\mathcal{O}_i}|^2}{\mathrm{Vol}(Z(t_j))} \int_G h(g t_j g^{-1}) dg \,. \tag{5.200}$$

By (5.199), (5.200), we obtain

$$\int_X \chi_\lambda(\phi(x)) dv_X(x) = \frac{\mathrm{Vol}(G)^{2g+s} \prod_{j=1}^s |\sigma_{\mathcal{O}_j}|^2}{\prod_{j=1}^s \mathrm{Vol}(Z(t_j))} \frac{\chi_\lambda(t_1)\dots\chi_\lambda(t_s)}{\chi_\lambda^{2g+s-1}(1)} \,. \tag{5.201}$$

So by (5.198), (5.201), we get

$$\begin{aligned} &\frac{1}{\mathrm{Vol}(G)} \int_G \overline{\chi}_\lambda(g) \frac{|V(g)|}{|\sigma(g)|} dg \\ &= |Z(G)| \frac{\mathrm{Vol}(G)^{2g+s-1}}{\mathrm{Vol}(T)} \prod_{j=1}^s \frac{|\sigma_{\mathcal{O}_j}|}{\mathrm{Vol}(Z(t_j))} \chi_\lambda(t_j) \frac{1}{\chi_\lambda(1)^{2g+s-1}} \,. \end{aligned} \tag{5.202}$$

Now (5.202) is exactly the $\lambda$-Fourier coefficient of the invariant distribution $\frac{|V(g)|}{|\sigma(g)|}$. The proof of our Theorem is completed. □

Clearly the distribution $\frac{|V(g)|}{|\sigma(g)|}$ in $C^\infty$ at the regular values of the function $\phi$. Now we will make a crude analysis of the Sobolev regularity of $|V(g)|$.

PROPOSITION 5.48. *If $p < 2g - 1 - \dim(\mathfrak{t})/2$, the invariant distribution in the right-hand side of (5.197) lies in $H^p(G)$. If $2g - 2 - \dim(\mathfrak{t}) > 0$, this distribution is continuous.*

PROOF. Let $\rho$ be the half-sum of the positive roots. By Weyl's dimension formula [**15**, Theorem VI.1.7], if $R_+$ is the set of positive roots,

$$\chi_\lambda(1) = \prod_{\alpha \in R_+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \,. \tag{5.203}$$

By [**15**, Proposition V.4.12], if $\alpha \in R_+$, $\langle \rho, \alpha \rangle > 0$. Therefore there is $c > 0$ such that if $\alpha \in R_+$, $\lambda \in \Lambda_+$,

$$\langle \lambda + \rho, \alpha \rangle \geq \sup(c, \langle \lambda, \alpha \rangle) \,. \tag{5.204}$$

By (5.203), (5.204), we find that since the $\alpha \in R_+$ form a basis of $\mathfrak{t}^*$, there is $C > 0$ such that

$$\chi_\lambda(1) \geq C \|\lambda\| \,. \tag{5.205}$$

Also

$$|\chi_\lambda| \leq \chi_\lambda(1) \,. \tag{5.206}$$

Let $\Delta_G$ be the Casimir operator on $G$ [**47**, Section 9.4]. By [**47**, Proposition 9.4.2],

$$\Delta_G \chi_\lambda = \frac{1}{2}(\|\lambda + \rho\|^2 - \|\rho\|^2) \chi_L \,. \tag{5.207}$$

Using (5.207), we get

$$(\Delta^G)^{p/2} \sum_{\lambda\in\Lambda_+} \frac{\prod_{j=1}^s \chi_\lambda(t_j)\chi_\lambda(g)}{(\chi_\lambda(1))^{2g+s-1}} = \frac{1}{2^{p/2}} \sum_{\lambda\in\Lambda_+} \frac{(\|\lambda+\rho\|^2 - \|\rho\|^2)^{p/2}}{\chi_\lambda(1)^{2g+s-1}} \prod_{j=1}^s \chi_\lambda(t_j)\chi_\lambda(g). \tag{5.208}$$

By (5.205), (5.206),

$$\frac{|(\|\lambda+\rho\|^2 - \|\rho\|^2)^{p/2} \prod_1^s \chi_\lambda(t_j)|}{\chi_\lambda(1)^{2g+s-1}} \leq C(1+|\lambda|)^{-(2g-1-p)}. \tag{5.209}$$

Also

$$\int_{\mathfrak{t}} \frac{d\lambda}{(1+|\lambda|)^{2(2g-1-p)}} < +\infty \tag{5.210}$$

if and only if $p < 2g - 1 - \frac{\dim(\mathfrak{t})}{2}$. From (5.209), (5.210), we get the first part of our Proposition. Also,

$$\int_{\mathfrak{t}} \frac{d\lambda}{(1+|\lambda|)^{2g-2}} < +\infty \tag{5.211}$$

if and only if $2g - 2 - \dim(\mathfrak{t}) > 0$. By (5.210), we obtain the second part of our Proposition. □

**5.9. Logarithms.** Let $U$ be a nonempty open set in $G$, stable by the adjoint action of $G$. We assume that there is a well-defined logarithm $\log : U \to \mathfrak{g}$, i.e. a smooth function $U \to \mathfrak{g}$ such that if $g \in U$, $g' \in G$,

$$\begin{aligned} g &= \exp(\log(g)), \\ \log(g'gg'^{-1}) &= g' \cdot \log(g). \end{aligned} \tag{5.212}$$

In particular by (5.212), $\log(g)$ is $Z(g)$-invariant.

EXAMPLE 5.49. If $U$ is a small ad-invariant open neighborhood of a central element in $G$, a logarithm is well-defined on $U$.

EXAMPLE 5.50. Let $\mathcal{O} \subset G$ be a very regular orbit. Then $\mathcal{O} \cap T$ consists of $|W|$ distinct elements. Let $t \in \mathcal{O} \cap T$, and let $h \in \mathfrak{t}$ such that $\exp(h) = t$. Then $t \mapsto h$ extends into a well defined logarithm $\mathcal{O} \to \mathcal{G}$.

EXAMPLE 5.51. Suppose that $G$ is simply connected. Let $K$ be a Weyl Chamber in $\mathfrak{t}$, let $P$ be the alcove in $K$ whose closure contains 0. Then by [**15**, Proposition V.7.10]

$$W\backslash T_{\text{reg}} = P. \tag{5.213}$$

Also by [**15**, Proposition V.7.11], the map

$$(g,t) \in G/T \times P \mapsto g\exp(t)g^{-1} \in G_{\text{reg}} \tag{5.214}$$

is one to one. Let $\log : G_{\text{reg}} \to \mathfrak{g}$ be the $G$-invariant function on $G_{\text{reg}}$ such that if $t \in P$,

$$\log(\exp(t)) = t \in P. \tag{5.215}$$

Then $\log : G_{\text{reg}} \to \mathfrak{g}$ is a logarithm.

We still assume that $G$ is simply connected and simple. Let $u \in C/\overrightarrow{R}^{*}$. Let $Z(u) \subset G$ be the centralizer of $u$. Let $Z(u)_{\text{vreg}}$ be the set of very regular elements in $Z(u)$. Clearly

$$Z(u) \cap G_{\text{reg}} \subset Z(u)_{\text{vreg}}. \tag{5.216}$$

Also the function log maps $Z(u) \cap G_{\text{reg}}$ into $\mathfrak{z}(u)$, and gives a logarithm on $Z(u)_{\text{vreg}}$.

Let $\mathcal{B}_U$ be the set of connections on the trivial $G$ bundle over $S_1$, whose holonomy $w$ lies in $U$. Needless to say, in a given trivialization, any element of $\mathcal{B}_u$ can be written in the form

$$\frac{D}{Dt} = \frac{d}{dt} + a_t \ , \ a_t \in L\mathfrak{g}_t. \tag{5.217}$$

Let $\tau_t^0$ be the parallel transport operator along $s \in [0,t]$, so that $w = \tau_1^0$. Set

$$w_t = \tau_t^0 \tau_1^0 (\tau_t^0)^{-1}. \tag{5.218}$$

Then $w_t$ is just $w$ under translation of the origin in $S_1$ by $t$.

Clearly $\log(w_t)$ is well-defined, and

$$\log(w_t) = \tau_t^0 \log(w)(\tau_t^0)^{-1}. \tag{5.219}$$

Also

$$\frac{D}{Dt}\log(w_t) = 0. \tag{5.220}$$

By (5.219),

$$e^{t\log(w_t)} = \tau_t^0 e^{t\log(w)}(\tau_t^0)^{-1}. \tag{5.221}$$

Put

$$\frac{{}^0D}{Dt} = e^{-t\log(w_t)}\frac{D}{Dt}e^{t\log(w_t)}. \tag{5.222}$$

By (5.220), (5.222),

$$\frac{{}^0D}{Dt} = \frac{D}{Dt} + \log(w_t). \tag{5.223}$$

From (5.220), (5.223), we get

$$\frac{{}^0D}{Dt}\log(w_t) = 0. \tag{5.224}$$

The parallel transport operator ${}^0\tau_t^0$ for $\frac{{}^0D}{Dt}$ is given by

$${}^0\tau_t^0 = e^{-t\log(w_t)}\tau_t^0, \tag{5.225}$$

so that

$${}^0\tau_1^0 = 1. \tag{5.226}$$

By (5.226), the parallel transport trivialization with respect to $\frac{{}^0D}{Dt}$ is globally defined on $S^1$, and in this trivialization,

$$\frac{{}^0D}{Dt} = \frac{d}{dt}. \tag{5.227}$$

**5.10. A symplectic structure on an open set in $X$.** In the sequel, we assume that the assumptions before Theorem 5.43 are in force, and also that $(t_1, \dots, t_s)$ verify (A).

Let $U_1, \dots, U_s$ be ad-invariant open subsets of $G$ on which a logarithm is well defined. Then $G$ acts on the right on $G^{2g} \times \prod_{j=1}^s U_j$.

Recall that $x \in G^{2g+s} \mapsto \phi(x) \in G$ was defined in (5.41). Let $x \in G^{2g} \times \prod_{j=1}^s U_j$ be such that $\phi(x) = 1$, and that $G$ acts locally freely at $x$. Let $U$ be an open neighbhorhood of 1 in $G$, such that a logarithm is still well-defined on $U$.

Let $\mathcal{V}$ be an open neighbhorhood of $x$ in $G^{2g} \times \prod_{j=1}^s U_j$ such that $h(\mathcal{V}) \subset U$. Then we can find an open neighborhood $A \subset G^{2g} \times \prod_{j=1}^s U_j$ of $x$, which is $G$-invariant, such that $h(A) \subset U$, and on which $G$ acts locally freely.

Let $\phi_{+1} : G^{2g+s+1} \to G$ be given as in (5.41), with $s$ replaced by $s+1$. Set

$$A' = \{x' \in G^{2g+s+1}; \phi_{+1}(x') = 1\}. \tag{5.228}$$

Clearly $x \in G^{2g+s} \mapsto (x, \phi^{-1}(x)) \in A'$ is a one to one $G$-equivariant map. Let $\widetilde{A} \subset A'$ be the image of $A$ by this map.

Let $\Sigma_{+1}$ be the Riemann surface $\Sigma$ with $s+1$ small disks deleted. Equivalently $\Sigma_{+1}$ is obtained from $\Sigma$ by deleting an extra small disk. Let $\widehat{\Sigma}_{+1}$ be the universal cover of $\Sigma_{+1}$. Put $\Gamma_{+1} = \pi_1(\Sigma_{+1})$. Then $\Gamma_{+1}$ is generated by $u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_{s+1}$, and the relation

$$\prod_{i=1}^{g} [u_i, v_i] \prod_{j=1}^{s+1} w_j = 1. \tag{5.229}$$

As in (5.126), we find that $\Gamma_{+1}$ and $G$ both act on the right on $\widetilde{A}$, and these actions commute. Also $\Gamma_{+1} \times G$ acts on the right on $\widehat{\Sigma}_{+1}$ (and the action of $G$ on $\widehat{\Sigma}_{+1}$ is trivial). Then $\Gamma_{+1} \times G$ acts on the right on $\widetilde{A} \times \widehat{\Sigma}_{+1}$. Since $\Gamma_{+1}$ acts freely on $\widehat{\Sigma}_{+1}$, $\Gamma_{+1} \times G$ acts locally freely on $\widetilde{A} \times \widehat{\Sigma}_{+1}$.

Set

$$C = (\widetilde{A} \times \widehat{\Sigma}_{+1}) / \Gamma_{+1}. \tag{5.230}$$

Then $C$ is a fibre bundle over $\Sigma_{+1}$ with fibre $\widetilde{A}$. Also $G$ acts locally freely on $\widetilde{A}$ and this action descends to a locally free action on $C$. We can then form the orbifolds $\widetilde{A}/G$ and $C/G = \widetilde{A}/G \times \Sigma_{+1}$.

Let $V$ be a complex vector space. Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of $G$. We still denote by $\rho$ the corresponding representation of $\Gamma_{+1} \times G$ in $\mathrm{Aut}(V)$.

Put

$$F = \widetilde{A} \times \widehat{\Sigma}_{+1} \times_{\Gamma_{+1} \times G} V. \tag{5.231}$$

Then $F$ is a vector bundle on $\widetilde{A}/G \times \Sigma_{+1}$. Also,verifies easily that

$$F = C \times_G V. \tag{5.232}$$

Moreover $F$ is obained by the identification in (5.129), i.e.

$$(x, \sigma, f) \simeq (xg, \sigma a, g^{-1} x(a^{-1}) f), \; x \in \widetilde{A}, \quad \sigma \in \widetilde{\Sigma}_{+1}, \; f \in V, \; g \in G, \; a \in \Gamma_{+1}. \tag{5.233}$$

Recall that $r_1, \ldots, r_{s+1}$ are the origin in $S_1^1, \ldots, S_{s+1}^1$. For $1 \leq j \leq s+1$, put

$$\frac{{}^0D_j}{Dt} = \frac{D}{Dt} + \log(w_{j,t}). \tag{5.234}$$

Then $\frac{{}^0D_j}{Dt}$ is a connection on the $G$-bundle $P$ on $S_j$, with holonomy 1.

For $1 \leq j \leq s+1$, let $\nabla_j$ be a connection on the $G$-bundle $\widetilde{A} \to \widetilde{A}/G$. Consider the $G$-bundle $\widetilde{A} \times \widehat{\Sigma}_{+1} \xrightarrow{G} \widetilde{A}/G \times \widehat{\Sigma}_{+1}$. Along the fibres $\widehat{\Sigma}_{+1}$, we can equip this $G$-bundle with the trivial connection. This connection is $\Gamma_{+1}$ invariant. Therefore it descends to a $G$-connection on $(\widetilde{A} \times \widehat{\Sigma}_{+1})/\Gamma_{+1} \xrightarrow{G} \widetilde{A}/G \times \Sigma_{+1}$ along the fibres $\Sigma_{+1}$. This connection along $\Sigma_{+1}$ is exactly the flat $G$-connection associated to the given element of $\widetilde{A}$.

Along $S_j^1$, $1 \leq j \leq s+1$, we trivialize the $G$-bundle $(\widetilde{A} \times \widehat{\Sigma}_{+1})/\Gamma_{+1} \xrightarrow{G} \widetilde{A}/G \times \Sigma_{+1}$ with respect to $\frac{{}^0D_j}{Dt}$. Then over $\widetilde{A}/G \times S_j^1$, $1 \leq j \leq s+1$, the connection $\nabla_j$ induces a $G$-connection on $(\widetilde{A} \times \widehat{\Sigma}_{+1})/\Gamma_{+1} \xrightarrow{G} \widetilde{A}/G \times \Sigma_{+1}$ along $\widehat{A}/G$. Since the $\widetilde{A} \times S_j^1$ are disjoint, we can extend this connection to a $G$-connection on $(\widetilde{A} \times \widehat{\Sigma}_{+1})/\Gamma_{+1} \xrightarrow{G} \widetilde{A}/G \times \widehat{\Sigma}_{+1}$ along $\widetilde{A}/G$.

Ultimately the $G$-bundle $(\widetilde{A} \times \widehat{\Sigma}_{+1})/\Gamma_{+1} \xrightarrow{G} \widetilde{A}/G \times \Sigma_{+1}$ is equipped with a $G$-connection $\nabla$. Let $F$ be the curvature of this connection. Then $F$ is a $G$-equivariant basic 2-form on $(\widetilde{A} \times \widehat{\Sigma}_{+1})/\Gamma_{+1}$ with values in $\mathfrak{g}$.

Let $E$ be the orbifold vector bundle on $\widetilde{A}/G \times \Sigma_{+1}$ associated to the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g})$. Let $\nabla^E$ be the induced connection on $E$. Then $F$ descends to a 2-form on $\widetilde{A}/G \times \Sigma_{+1}$ with values in $E$. Set

$$E_j = r_j^* E \; ; \; 1 \leq j \leq s+1. \tag{5.235}$$

Then $E_j$ is a vector bundle on $\widetilde{A}/G$. Let $\nabla^{E_j}$ be the connection induced by $\nabla^E$ (or $\nabla_j$) on $E_j$. Observe that for $1 \leq j \leq s+1$, $\log(w_j)$ is a section of $E_j$.

Recall that $C/G = \widetilde{A}/G \times \Sigma_{+1}$. Then

$$\Lambda(T^*C/G) = \Lambda(T^*\widetilde{A}/G)\widehat{\otimes}\Lambda(T^*\Sigma_{+1}). \tag{5.236}$$

If $\omega \in \Lambda(T^*C)$, we can write $\omega$ in the form

$$\omega = \Sigma\omega^{(p,q)} \;,\; \omega^{(p,q)} \in \Lambda^p(T^*\widetilde{A}/G)\widehat{\otimes}\Lambda^q(T^*\Sigma_{+1}). \tag{5.237}$$

Let $k$ be the embedding $\widetilde{A}/G \times \partial\Sigma_{+1} \to C/G$. Let $\Theta_j$ be the curvature of $\nabla_j$.

THEOREM 5.52. *The following identities hold*

$$\begin{aligned} F^{(0,2)} &= 0, \\ k^*F &= \Theta_j - \nabla^{E_j}\log(w_j)dt \text{ on } \widetilde{A}/G \times S_j^1 \;,\; 1 \leq j \leq s+1. \end{aligned} \tag{5.238}$$

PROOF. By construction the first identity holds in (5.238). Also on $\widetilde{A}/G \times S_j^1$,

$$k^*F = k^*\left(\nabla^{E_j} + dt\Big(\frac{{}^0D}{Dt} - \log(w_j)\Big)\right)^{2,(1,1)} = \Theta_j - \nabla^{E_j}\log(w_j)dt. \tag{5.239}$$

The proof of our Theorem is completed. □

Let $\langle\, ,\, \rangle$ be a $G$-invariant bilinear symmetric form on $\mathfrak{g}$. Clearly $F^2 \in \Lambda^4(T^*C) \otimes \mathfrak{g} \otimes \mathfrak{g}$. Then $\langle F^2 \rangle \in \Lambda^4(T^*C)$.

Let $\pi, \pi'$ be the projections $\widetilde{A}/G \times \Sigma_{+1} \to \widetilde{A}/G$, $\widetilde{A}/G \times \partial\Sigma_{+1} \to \widetilde{A}/G$. Let $\rho$ be the projection $\widetilde{A} \to \widetilde{A}/G$.

For $1 \leq j \leq s+1$, let $\theta_j$ be the connection 1-form on the $G$-bundle $\widetilde{A} \to \widetilde{A}/G$ associated to the connection $\nabla_j$. Then $\theta_j$ is a 1 form with values in $\mathfrak{g}^*$.

DEFINITION 5.53. Let $\alpha$ be the 2-form on $\widetilde{A}$,

$$\alpha = \rho^* \left[ \pi_* \left[ \frac{\langle F^2 \rangle}{2} \right] + \sum_{j=1}^{s+1} \langle \log(w_j), \Theta_j \rangle \right] - \sum_{j=1}^{s+1} d\langle \log(w_j), \theta_j \rangle . \tag{5.240}$$

THEOREM 5.54. *The 2-form $\alpha$ is $G$-invariant and closed. It does not depend on the choices made in its construction.*

PROOF. By Chern-Weil theory, the form $\frac{\langle F^2 \rangle}{2}$ is closed on $\widetilde{A}/G \times \Sigma_{+1}$. Using Stokes formula, we get

$$d\pi_* \left[ \frac{\langle F^2 \rangle}{2} \right] = \pi'_* \left[ \frac{k^* \langle F^2 \rangle}{2} \right] . \tag{5.241}$$

By (5.238), (5.241), we get

$$d\pi_* \left[ \frac{\langle F^2 \rangle}{2} \right] = - \sum_{j=1}^{s+1} \langle \nabla^{E_j} \log(w_j), \Theta_j \rangle . \tag{5.242}$$

Also by Bianchi's identity,

$$d\langle \log(w_j), \Theta_j \rangle) = \langle \nabla^{E_j} \log(w_j), \Theta_j \rangle . \tag{5.243}$$

By (5.242), (5.243), we find that the form $\alpha$ is closed.

Now we replace $\widehat{\Sigma}_{+1}$ by $\widehat{\Sigma}_{+1} \times \mathbf{R}$. We still denote by $\pi$ the projection $\widetilde{A} \times \widehat{\Sigma}_{+1} \times \mathbf{R} \to \widetilde{A}/G \times \mathbf{R}$ with fibre $\widehat{\Sigma}_{+1}$. The $G$- bundle $\widetilde{A} \to \widetilde{A}/G$ is replaced by $\widetilde{A} \times \mathbf{R} \to \widetilde{A}/G \times \mathbf{R}$. We consider a smooth family of data, which are used to construct the form $\alpha_\ell$. In particular the connections $\nabla_j$ depends on $\ell$. We extend these fibrewise connections: $\nabla_j$ $\widetilde{A} \to \widetilde{A}/G$ to a connection $d\ell \frac{\partial}{\partial \ell} + \nabla_j$ on $\widetilde{A} \times \mathbf{R} \to \widetilde{A}/G \times \mathbf{R}$. We will denote with a $\widetilde{\ }$ the analogue of the above objects over $\widetilde{A} \times \widehat{\Sigma}_{+1} \times \mathbf{R}$. This way, we obtain a form $\widetilde{\alpha}$ on $\widetilde{A} \times \mathbf{R}$ such that

$$\widetilde{\alpha} = \alpha_\ell + d\ell \wedge \beta_\ell , \tag{5.244}$$

where $\beta_\ell$ is a 1-form on $\widetilde{A}/G$. By the above arguments, $\widetilde{\alpha}$ is closed.

Now we will show that

$$\beta_\ell = 0 . \tag{5.245}$$

This will imply that

$$\frac{\partial \alpha_\ell}{\partial \ell} = 0 . \tag{5.246}$$

So we will have established our claim that $\alpha_\ell$ does not depend on $\ell$.

Recall that along the fibres $\widehat{\Sigma}_{+1}$, the connection on the considered $G$-bundle does not depend on $\ell \in \mathbf{R}$, and is flat. It then follows easily that $\pi_* \left[ \frac{\langle \widetilde{F}^2 \rangle}{2} \right]$ does not contain $d\ell$.

Also the connection form $\widetilde{\theta}_j$ does not contain $d\ell$, i.e. it is of the form

$$\widetilde{\theta}_j = \theta_{j,\ell}\,. \tag{5.247}$$

Then

$$\widetilde{\Theta}_j = \Theta_{j,\ell} + d\ell\frac{\partial}{\partial\ell}\theta_{j,\ell}\,. \tag{5.248}$$

Therefore

$$\begin{aligned} \langle\log(w_j), \widetilde{\Theta}_j\rangle &= \langle\log(w_j), \widetilde{\Theta}_{j,\ell} + d\ell\frac{\partial}{\partial\ell}\theta_{j,\ell}\rangle, \\ \widetilde{d}\langle\log(w_j), \widetilde{\theta}_j\rangle &= \langle d\log(w_j), \theta_{j,\ell}\rangle + \langle\log(w_j), d\theta_{j,\ell}\rangle, \\ &\quad +\langle\log(w_j), d\ell\frac{\partial}{\partial\ell}\theta_{j,\ell}\rangle\,. \end{aligned} \tag{5.249}$$

From (5.142), we find the sum of the last terms in (5.240) does not contain $d\ell$ either.

The proof of our Theorem is completed. □

Now we fix $t_1 \in T\cap U_1, \ldots, t_s \in T\cap U_s$. For $1\le j\le s$, let $\mathcal{O}_j \subset G$ be the orbit of $t_j$.

Put

$$\widehat{X} = \{x\in \widetilde{A}\, w_j \in \mathcal{O}_j\ ,\ 1\le j\le s\}\,. \tag{5.250}$$

Then $\widehat{X}\subset X$ is stable under $G$. Let $m$ be one of the embeddings $\widehat{X}\to\widetilde{A}$, $\widehat{X}/G\to \widetilde{A}/G$.

For $1\le j\le s$, let $\widetilde{\mathcal{O}}_j\subset \mathfrak{g}$ be the $G$-orbit of $\log(t_j)$. Since there is a well-defined logarithm on $U_j$, $\mathcal{O}_j$ and $\widetilde{\mathcal{O}}_j$ are in one to one correspondance.

For $1\le j\le s$, let $\sigma_j$ be the canonical symplectic form on the orbit $\widetilde{\mathcal{O}}_j$. If $Y\in\mathfrak{g}$, let $Y^{\widetilde{\mathcal{O}}_j}$ be the corresponding vector field associated to the right action of $G$ on $\widetilde{\mathcal{O}}_j$. Then if $Y, Y'\in\mathfrak{g}$, $p\in\widehat{\mathcal{O}}_j$, as in (1.193),

$$\sigma_{j,p}(Y,Y') = \langle p, [Y,Y']\rangle\,. \tag{5.251}$$

DEFINITION 5.55. Put

$$\sigma = m^*\alpha + \sum_{j=1}^{s}\log(w_j)^*\sigma_j\,. \tag{5.252}$$

Clearly $\sigma$ is a $G$-invariant closed 2-form on $\widehat{X}$. We will calculate $\sigma$.

For $1\le j\le s$, the $G$-bundle $\widehat{X}\to\widehat{X}/G$ can be reduced to the $Z(t_j)$-bundle $\{x\in\widehat{X}, w_j = t_j\}\mapsto\{x\in\widehat{X}, w_j=t_j\}/Z(t_j)$. Let $\nabla_j$ be a $Z(t_j)$-connection on this last bundle. This connection lifts to a $G$-connection on $\widehat{X}\to\widehat{X}/G$. We will make this choice of $\nabla_j$ in our construction of $\sigma$. Then for $1\le j\le s$,

$$\begin{aligned} \nabla_j w_j &= 0\,, \\ \nabla_j \log(w_j) &= 0\,. \end{aligned} \tag{5.253}$$

THEOREM 5.56. *The following identity holds*

$$\begin{aligned} \sigma = m^*\rho^*\Big(\pi_*\frac{\langle F^2\rangle}{2}\Big) + \langle\log(w_{s+1}), \Theta_{s+1}\rangle\Big) \\ -d\langle\log(w_{s+1}), \theta_{s+1}\rangle\,. \end{aligned} \tag{5.254}$$

PROOF. The second identity in (5.253) can be written in the form

$$d\log(w_j) + [\theta_j, \log(w_j)] = 0\,. \tag{5.255}$$

By (5.251), we get

$$\log(w_j)^*\sigma_j = \langle \log(w_j), \frac{1}{2}[\theta_j, \theta_j]\rangle\,. \tag{5.256}$$

Also

$$d\theta_j = -\frac{1}{2}[\theta_j, \theta_j] + \Theta_j\,. \tag{5.257}$$

From (5.255)-(5.257), we obtain for $1 \leq j \leq s$,

$$\begin{aligned} &\rho^*\langle \log(w_j), \Theta_j\rangle - d\langle \log(w_j), \theta_j\rangle + \log(w_j)^*\sigma_j \\ &= -\langle d\log(w_j), \theta_j\rangle + \langle \log w_j, [\theta_j, \theta_j]\rangle = 0\,. \end{aligned} \tag{5.258}$$

The proof of our Theorem is completed. □

Set

$$\widehat{M} = \{x \in \widehat{X},\, w_{s+1} = 1\}\,. \tag{5.259}$$

Our notation in (5.259) is compatible with (5.59). By Propositions 5.10 and 5.11, we know that $\widehat{M}$ is a submanifold of $\widehat{X}$, and that $G$ acts locally freely on $\widehat{M}$. Let $i : \widehat{M} \to \widehat{X}$, $\widehat{M}/G \to \widehat{X}/G$ be the obvious embeddings.

By Theorem 5.31, $\widehat{M}/G$ carries a symplectic form $\omega$. Recall that $\mathfrak{g}$ is equipped with the scalar product $\langle , \rangle$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by this scalar product.

THEOREM 5.57. *If $U$ is small enough, $\sigma$ is a symplectic form on $\widehat{X}$. Also $x = (u_1, v_1, \ldots, u_g, v_g, w_1, \ldots, w_{s+1}) \in \widehat{X} \mapsto \log(w_{s+1}) \in \mathfrak{g}$ is a moment map for the action of $G$ on $X$ with respect to the 2-form $\sigma$. The associated symplectic form on the symplectic reduction $\widehat{M}/G$ coincides with $i^*\pi_*(\frac{\langle F^2\rangle}{2})$ and with $\omega$.*

PROOF. By (5.254),

$$i^*\sigma = i^*\rho^*\pi_*\left(\frac{\langle F^2\rangle}{2}\right)\,. \tag{5.260}$$

Let $\nabla^H, \nabla^V$ be the components of $\nabla$ along $\widetilde{A}/G, \Sigma_{+1}$. Since $\nabla$ is flat along $\Sigma_{+1}$,

$$F = \nabla^{H,2} + [\nabla^H, \nabla^V]. \tag{5.261}$$

From (5.261) we get

$$\pi_*(\frac{\langle F^2\rangle}{2}) = \pi_*(\frac{[\nabla^H, \nabla^V]}{2}). \tag{5.262}$$

Moreover by (5.238), (5.253),

$$k^*F^{(1,1)} = 0. \tag{5.263}$$

From (5.261)-(5.263), we get easily

$$i^*\pi_*(\frac{\langle F^2\rangle}{2}) = \rho_*\omega. \tag{5.264}$$

From (5.260), (5.264), we obtain

$$i^*\sigma = \rho^*\omega. \tag{5.265}$$

Now by Theorem 5.31, $\omega$ is a symplectic form. Therefore by (5.254), (5.265), if $U$ is small enough, $\sigma$ is also a symplectic form.

If $Y \in \mathfrak{g}$, let $Y^{\widehat{X}}$ be the corresponding vector field on $\widehat{X}$. Then by (5.254),

$$\begin{aligned} i_{Y^X}\sigma &= -i_{Y^{\widehat{X}}} d\langle \log(w_{s+1}), \theta_{s+1}\rangle \\ &= d i_{Y^{\widehat{X}}} \langle \log(w_{s+1}), \theta_{s+1}\rangle \\ &= d\langle \log(w_{s+1}), Y\rangle . \end{aligned} \tag{5.266}$$

From (5.266), we find that $x \in X \mapsto \log(w_{s+1}) \in \mathfrak{g} \simeq \mathfrak{g}^*$ is a moment map for the action of $G$ on $\widehat{X}$.

The proof of our Theorem is completed. □

By Theorem 5.12, since $G$ acts locally freely on $\widehat{X}$, the derivative of $x \in \widehat{X} \mapsto w_{s+1}^{-1} \in G$ is surjective.

Take $t \in U$ close enough to 0. Put

$$X_t = \{x \in \widehat{X}, w_{s+1} = t\} . \tag{5.267}$$

Then $Z(t)$ acts locally freely on $X_t$ . So $X_t/Z(t)$ can be equipped with the symplectic form $\sigma_t$, the reduction of the symplectic form $\sigma$. Also recall that by Theorem 5.31, $M_t/Z(t)$ is equipped with a symplectic form $\omega_t$.

THEOREM 5.58. *For* $t \in U$,

$$\sigma_t = \omega_t . \tag{5.268}$$

PROOF. Clearly the $G$-bundle $\widetilde{A} \to \widetilde{A}/G$ reduces to $X_t \to X_t/Z(t)$. Let $\nabla_{s+1}$ be a $Z(t)$ connection on this last bundle. Then $\nabla_{s+1}$ lifts to a $G$-connection on $\widetilde{A} \to \widetilde{A}/G$. We will use $\nabla_{s+1}$ to calculate the restriction of $\sigma$ to $X_t$. On $X_t$,

$$\begin{aligned} &\nabla^{E_{s+1}} \log(w_{s+1}) = 0 , \\ &d \log(w_{s+1}) + [\theta_{s+1}, \log(w_{s+1})] = 0 . \end{aligned} \tag{5.269}$$

Over $\log(w_{s+1}) = -\log(t)$, we get

$$[\theta_{s+1}, \log(w_{s+1})] = 0 . \tag{5.270}$$

Then over $\log(w_{s+1}) = -\log(t)$, using (5.257), (5.269), (5.270),

$$\begin{aligned} d\langle \log(w_{s+1}), \theta_{s+1}\rangle &= \langle \log(w_{s+1}), \Theta_{s+1}\rangle - \langle \log(w_{s+1}), \frac{1}{2}[\theta_{s+1}, \theta_{s+1}]\rangle \\ &= \langle \log(w_{s+1}), \Theta_{s+1}\rangle . \end{aligned} \tag{5.271}$$

From (5.254), (5.271), we find that over $X_t$,

$$\sigma = \rho^* \pi_* \left(\frac{\langle F^2\rangle}{2}\right) . \tag{5.272}$$

Using (5.272) and proceeding as in the proof of Theorem 5.57, our Theorem follows. □

Take $j$, $1 \le j \le s$. Put

$$\widehat{X}_j = \{x \in \widetilde{A}; w_{j'} \in \mathcal{O}_{j'}, j' \neq j, w_j \in U_j, w_{s+1} = 1\} . \tag{5.273}$$

Let $n_j$ be the embedding $\widehat{X}_j \to \widetilde{A}$.

DEFINITION 5.59. Put

$$\kappa_j = n_j^*\alpha + \sum_{j'=1\ ,\ j'\neq j}^{s} \log(w_j)^*\sigma_j\,. \tag{5.274}$$

We will choose the connection $\nabla_{j'}$, $1 \leq j \leq s, j' \neq j$ as in (5.253). Then by proceeding as in the proof of Theorem 5.56, we get

$$\begin{aligned}\kappa_j &= n_j^*\rho^*\left(\pi_*\left(\frac{\langle F^2\rangle}{2}\right) + \langle \log(w_j), \Theta_j\rangle\right) \\ &\quad -d\langle \log(w_j), \theta_j\rangle\,.\end{aligned} \tag{5.275}$$

THEOREM 5.60. *If $U_j$ is small enough, $\kappa_j$ is a symplectic form on $\widehat{X}_j$. Also $x = (u_1, v_1, \dots, u_g, v_g, w_1, \dots, w_s) \in \widehat{X}_j \mapsto \log(w_j) \in \mathfrak{g}$ is a moment map for the action of $G$ on $\widehat{X}_j$ with respect to $\kappa_j$. Finally for $t_j \in U_j$, the symplectic form on the symplectic reduction $\{x \in \widehat{X}_j, w_j = t_j\}/Z(t_j)$ is the symplectic form defined in Theorem 5.31.*

PROOF. The proof of our Theorem is the same as the proof of Theorems 5.57 and 5.58. □

REMARK 5.61. If $t_1, \dots, t_s$ are restricted to be very regular, we may and will assume that $\theta_1, \dots, \theta_s$ are $T$-connections. By Theorems 5.58 and 5.60, we find that $\pi_*(\frac{\langle F^2\rangle}{2})$ restricts to the symplectic form of Theorem 5.31. Also by (5.275), the cohomology class of

$$\omega + \sum_{j=1}^{s} \langle \log(w_j), \Theta_j\rangle \tag{5.276}$$

is locally constant, which is a consequence of the theory of the moment map for torus actions obtained by Duistermaat-HeckmanDH.

**5.11. The integral of certain characteristic classes on the strata of $M/G$.** Let $G$ be a compact connected and simply connected simple Lie group. We use the notation of Section 1. In particular, $\langle,\rangle$ denotes the basic scalar product on $\mathfrak{g}$.

Let $t_1, \dots, t_s$ be regular elements in $T$. Then $t_1, \dots, t_s$ are very regular.

We assume that $s \geq 1$, and that

- $(t_1, \dots, t_s)$ verify (A).
- If $g = 0$, then $s \geq 3$.

In particular

$$2g - 2 + s \geq 1 \qquad . \tag{5.277}$$

Let $u \in C/\overline{R}^*$. Recall that $\pi_u : \widetilde{Z}(u) \to Z(u)$ is the universal cover of $Z(u)$, with fibre $\pi_1(Z(u)) \simeq \overline{CR}/\overline{CR}_u \subset Z(\widetilde{Z}(u))$. Also $\widetilde{T}_u = \mathfrak{t}/\overline{CR}_u$ is a maximal torus in $\widetilde{Z}(u)$.

Remember that $T_{\mathrm{reg}}$ is the set of regular elements in $T$ with respect to $G$. Let $t \in T_{\mathrm{reg}}$. Then $Z(t) = T$, $t$ is very regular in $Z(u)$, and $\mathcal{O}_{Z(u)}(t) \simeq Z(u)/T$.

Let $\tilde{t} \in \widetilde{T}_u$ be a lift of $t$ in $\widetilde{Z}(u)$. Then $\tilde{t}$ is still regular in $\widetilde{Z}(u)$. Since $Z(t) = T$, the centralizer $Z(\tilde{t})$ of $\tilde{t}$ in $\widetilde{Z}(u)$ is just $Z(\tilde{t}) = \mathfrak{t}/\overline{CR}_u$. Then $\mathcal{O}_{\widetilde{Z}(u)}(\tilde{t}) \simeq$

$\widetilde{Z}(u)/(\mathfrak{t}/\overline{CR}_u) \simeq Z(u)/T$. Equivalently the projection $\mathcal{O}_{\widetilde{Z}(u)}(\tilde{t}) \to \mathcal{O}_{Z(u)}(t)$ is one to one.

As before, we identify $t_1, \dots, t_s$ with corresponding elements of $G$-alcoves in $\mathfrak{t}$ whose closure contains 0. This way, we get elements of $\widetilde{T}_u = \mathfrak{t}/\overline{CR}_u$, which lift $t_1, \dots, t_s$ unambiguously in $\widetilde{Z}(u)$. We still denote these elements by $t_1, \dots, t_s$.

Clearly

$$\widetilde{Z}'(u) = Z'(u). \tag{5.278}$$

Set

$$\begin{aligned} X_u &= Z(u)^{2g} \times \prod_{j=1}^{s} \mathcal{O}_{Z(u)}(t_j), \\ \widetilde{X}_u &= \widetilde{Z}(u)^{2g} \times \prod_{j=1}^{s} \mathcal{O}_{\widetilde{Z}(u)}(t_j). \end{aligned} \tag{5.279}$$

Then $Z(u)$ acts on $X_u$ and on $\widetilde{X}_u$. Moreover $(\overline{CR}/\overline{CR}_u)^{2g}$ acts freely on $\widetilde{X}_u$. Namely if $f = (\alpha_1, \beta_1, \dots \alpha_g, \beta_g) \in (\overline{CR}/\overline{CR}_u)^{2g}$, if $x = (\tilde{u}_1, \tilde{v}_1, \dots \tilde{u}_g, \tilde{v}_g, \tilde{w}_1, \dots \tilde{w}_s) \in \widetilde{X}_u$,, put

$$f.x = (\tilde{u}_1\alpha_1, \tilde{v}_1\beta_1, \dots, \tilde{u}_g\alpha_g, \tilde{v}_g\beta_g, \tilde{w}_1, \dots \tilde{w}_s). \tag{5.280}$$

The actions of $Z(u)$ and of $(\overline{CR}/\overline{CR}_u)^{2g}$ commute. Also the map $\pi_u : \widetilde{Z}(u) \to Z(u)$ extends to a map $\widetilde{X}_u \to X_u$. Clearly $\pi_u$ is $Z(u)$-equivariant. Also if $f \in (\overline{CR}/\overline{CR}_u)^{2g}$,

$$\pi_u f = \pi_u. \tag{5.281}$$

More precisely $\pi_u$ is a $(\overline{CR}/\overline{CR}_u)^{2g}$ cover.

DEFINITION 5.62. Let $\tilde{\phi}_u : \widetilde{X}_u \to \widetilde{Z}(u)$ and $\phi_u : X_u \to Z(u)$ be the analogues of $\phi$ defined in (5.41).

Clearly

$$\phi_u \pi_u = \pi_u \tilde{\phi}_u. \tag{5.282}$$

Also $\tilde{\phi}_u$ and $\phi_u$ are $Z(u)$- equivariant .

Clearly $(t_1, \dots, t_s)$ verify (A) with respect to $Z(u)$ or $\widetilde{Z}(u)$. Therefore by Theorem 5.18, 1 is a regular value of $\phi_u$. Equivalently, by Theorem 5.18, 1 is a regular value of $\phi$. By Theorem 5.12, $G$ acts locally freely on $M = \phi^{-1}(1)$, and so by Theorem 5.12, 1 is a regular value of $\phi_u$.

PROPOSITION 5.63. *Any $h \in \overline{CR}/\overline{CR}_u$ is a regular value of $\tilde{\phi}_u$.*

PROOF. Since 1 is a regular value of $\phi_u$, any $h \in \pi_u^{-1}(1)$ is a regular value of $\tilde{\phi}_u$. □

Recall that the Lie algebra $\mathfrak{z}(u)$ is equipped with the scalar product induced by the scalar product $\langle , \rangle$ on $\mathfrak{g}$.

Let $U$ be a $G$- invariant open neighborhood of 1 in $G$, such that a logarithm $\log : U \to \mathfrak{g}$ is defined, with

$$\log(1) = 0. \tag{5.283}$$

Put

$$U_u = U \cap Z(u),\ \widetilde{U}_u = \pi_u^{-1}(U_u). \tag{5.284}$$

Clearly log maps $U_u$ into $\mathfrak{z}(u)$. Then $\widetilde{U}_u$ is an open neighborhood of $\overline{CR}/\overline{CR}_u$ in $\widetilde{Z}(u)$, which only consists of regular values of $\tilde{\phi}_u$.

In the sequel, we will view $\exp(\log(\phi_u(x))$ as an element of $\widetilde{Z}(u)$. Observe that if $\tilde{x} \in \widetilde{X}_u$ is such that $\tilde{x} \in \tilde{\phi}_u^{-1}(\widetilde{U}_u)$,

$$\pi_u \left[\tilde{\phi}_u(\tilde{x}) \exp\left(-\log(\phi_u \pi_u(\tilde{x}))\right)\right] = 1. \tag{5.285}$$

Therefore

$$\tilde{\phi}_u(\tilde{x}) \exp\left(-\log(\phi_u \pi_u)(\tilde{x}))\right) \in \overline{CR}/\overline{CR}_u. \tag{5.286}$$

Finally note that if $x \in \phi_u^{-1}(U)$, if $\tilde{x}$ is such that $\pi_u(\tilde{x}) = x$, then $\tilde{\phi}_u(\tilde{x})$ does not depend on $\tilde{x}$. Therefore $\tilde{\phi}_u(\tilde{x}) \exp\left(-\log(\phi_u \pi_u(\tilde{x}))\right) \in \overline{CR}/\overline{CR}_u$ does not depend on $\tilde{x}$.

DEFINITION 5.64. If $h \in \overline{CR}/\overline{CR}_u$, let $\phi_u^{-1}(U_u)_h \subset \phi_u^{-1}(U)$ be given by

$$\phi_u^{-1}(U_u)_h = \{x \in \phi_u^{-1}(U_u),\ \text{if } \pi_u(\tilde{x}) = x, \tilde{\phi}_u(\tilde{x}) \exp\left(-\log(\phi_u \pi_u(\tilde{x}))\right) h = 1\}. \tag{5.287}$$

Similarly, let $\tilde{\phi}_u^{-1}(\widetilde{U}_u)_h \subset \tilde{\phi}_u^{-1}(\widetilde{U}_u)$ be given by

$$\tilde{\phi}_u^{-1}(\widetilde{U}_u)_h = \{\tilde{x} \in \tilde{\phi}_u^{-1}(\widetilde{U}_u), \tilde{\phi}_u(\tilde{x}) \exp\left(-\log(\phi_u \pi_u(\tilde{x}))\right) h = 1\}. \tag{5.288}$$

Then we have the disjoint union

$$\phi_u^{-1}(U_u) = \bigcup_{h \in \overline{CR}/\overline{CR}_u} \phi_u^{-1}(U_u)_h, \tag{5.289}$$

$$\tilde{\phi}_u^{-1}(\widetilde{U}_u) = \bigcup_{h \in}$$

$\overline{CR}/\overline{CR}_u \tilde{\phi}_u^{-1}(\widetilde{U}_u)_h$,and for any $h \in \overline{CR}/\overline{CR}_u, \pi_u : \tilde{\phi}_u^{-1}(\widetilde{U}_u)_h \to \phi_u^{-1}(U_u)_h$ is a $\left(\overline{CR}/\overline{CR}_u\right)^{2g}$-cover. Moreover $Z(u)$ preserves $\tilde{\phi}_u^{-1}(\widetilde{U}_u)_h$ and $\phi_u^{-1}(U_u)_h$, and $(\overline{CR}/\overline{CR}_u)^{2g}$ acts freely on $\tilde{\phi}_u^{-1}(\widetilde{U}_u)_h$.

If $U$ is small enough, there is an open neignborhood $\widetilde{V}_u$ of 1 in $\widetilde{Z}(u)$ such that $\pi_u : (\widetilde{V}_u) \to U_u$ is one to one. Also if $t \in U$, and $\tilde{t} \in \widetilde{V}_u$ is the lift of $t$, then $\pi_u$ is one to one from $\mathcal{O}_{\widetilde{Z}(u)}(\tilde{t})$ into $\mathcal{O}_{Z(u)}(t)$.

DEFINITION 5.65. If $h \in \overline{CR}/\overline{CR}_u, t \in U_u, \tilde{t} \in \widetilde{V}_u$, with $\pi_u(\tilde{t}) = t$, put

$$\begin{aligned} M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h) &= \{x \in \phi_u^{-1}(U_u)_h, \phi_u(x) = t^{-1}\}, \\ M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h) &= \{\tilde{x} \in \tilde{\phi}_u^{-1}(\widetilde{U}_u)_h, \tilde{\phi}_u(\tilde{x}) = \tilde{t}^{-1}h^{-1}\}. \end{aligned} \tag{5.290}$$

Then $\pi_u : M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h) \to M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h)$ is a $|\overline{CR}/\overline{CR}_u|^{2g}$ cover. If $Z_u(t)$ is the centralizer of $t$ in $Z(u)$, $\pi_u$ is a $Z_u(t)$ equivariant map.

By Theorem 5.12 and Proposition 5.63, $Z_u(t)$ acts locally freely on $M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)$ and on $M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h)$.

PROPOSITION 5.66. *The map* $\pi_u : M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)/Z_u(t) \to M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h)/Z_u(t)$ *is a* $|\overline{CR}/\overline{CR}_u|^{2g}$ *cover on the regular part of the orbifold* $M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)/Z_u(t)$.

PROOF. Let $\tilde{x} \in M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)$, and let $f \in \left(\overline{CR}/\overline{CR}_u\right)^{2g}, g \in Z_u(t)$ be such that

$$f.\tilde{x} = \tilde{x}.g. \tag{5.291}$$

Then by (5.281), (5.291),

$$\pi_u(\tilde{x}) = \pi_u(\tilde{x}).g. \tag{5.292}$$

If $x = \pi_u(x)$ is in the regular part of $M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h)$, from (5.292), we get

$$g = 1. \tag{5.293}$$

So by (5.291), (5.293),

$$f = 1. \tag{5.294}$$

The proof of our Proposition is completed. □

For $t \in U_u$, let $\sigma_t^u$ be the symplectic form on $M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h)/Z_u(t)$, and let $\tilde{\sigma}_t^u$ be the symplectic form on $M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)/Z_u(t)$.

DEFINITION 5.67. For $u \in C/\overline{R}^*, t \in U_u$, put

$$V_u(t_1, \dots t_s, t, h) = \left| \int_{M_t(Z(u), \mathcal{O}_{Z(u)}(t_1), \dots, \mathcal{O}_{Z(u)}(t_s), h)/Z'_u(t)} e^{\sigma_t^u} \right|, \tag{5.295}$$

$$\widetilde{V}_u(t_1, \dots t_s, t, h) = \left| \int_{M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)/Z'_u(t)} e^{\tilde{\sigma}_t^u} \right|.$$

PROPOSITION 5.68. *The following identity holds*

$$\tilde{\sigma}_t^u = \pi_u^* \sigma_t^u. \tag{5.296}$$

*Moreover*

$$|V_u|(t_1, \dots t_s, t, h) = \frac{1}{|\overline{CR}/\overline{CR}_u|^{2g}} |\widetilde{V}_u|(t_1 \dots t_s, t, h). \tag{5.297}$$

PROOF. Equation (5.296) is trivial. Using (5.296), we get (5.297). □

Let $K \subset \mathfrak{t}$ be a Weyl chamber for $G$ which is fixed once and for all. Then if $u \in C/\overline{R}^*$, $K$ is included in a unique Weyl chamber $K_u$ for $Z(u)$ or $\widetilde{Z}(u)$. Put

$$\overline{CR}_{u,+}^* = \overline{CR}_u^* \cap \overline{K}_u. \tag{5.298}$$

Then the irreducible representations of $\widetilde{Z}(u)$ are parametrized by $\overline{CR}_{u,+}^*$. If $\lambda \in \overline{CR}_{u,+}^*$, let $\chi_\lambda^{\widetilde{Z}(u)}$ be the representation of $\widetilde{Z}(u)$ with highest weight $\lambda$. For $t \in \widetilde{T}_u = \mathfrak{t}/\overline{CR}_u$, put

$$\sigma_{Z(u)}(\widetilde{T}) = \prod_{\alpha \in R_{u,+}} \left( e^{i\pi\langle\alpha, t\rangle} - e^{-i\pi\langle\alpha, t\rangle} \right). \tag{5.299}$$

Then $|\sigma_{Z(u)}(t)|$ is well defined on $\mathfrak{t}/\overline{R}^*$, and so is well defined on $T = \mathfrak{t}/\overline{CR}$.

THEOREM 5.69. *For any $u \in C/\overline{R}^*, h \in \overline{CR}/\overline{CR}_u$, the following identity of $Z(u)$-invariant distributions in the variable $t \in U$ holds,*

$$\frac{|\widetilde{V}_u|(t_1, \dots t_s, h, t)}{|\sigma_{Z(u)}(t+h)\prod_{j=1}^s |\sigma_{Z(u)}(t_j)|} = |Z(\widetilde{Z}(u))| \mathrm{Vol}(\mathfrak{t}/\overline{CR}_u)|^{2g-2} \left|\frac{\mathrm{Vol}(\widetilde{Z}(u)}{\mathrm{Vol}(\widetilde{T}_u)}\right|^{2g+s-1} \sum_{\lambda \in \overline{CR}^*_{u,+}} \frac{\prod_{j=1}^s \chi_\lambda^{\widetilde{Z}(u)}(t_j)\chi_\lambda^{\widetilde{Z}(u)}(t+h)}{\chi_\lambda^{\widetilde{Z}(u)}(1)^{2g+s-1}}. \tag{5.300}$$

PROOF. Recall that $t \in U$ is identified to the corresponding element in $\widetilde{V}_u \subset \widetilde{Z}(u)$. Since $t_j \in \widetilde{Z}(u)$ is very regular, the centralizer of $t_j$ in $\widetilde{Z}(u)$ is equal to $\widetilde{T}_u$. Then by (5.156) and by Theorem 5.47, we get (5.300) . The proof of our Theorem is completed. □

REMARK 5.70. Note that $\overline{CR}^* \cap \overline{K}_u$ is exactly the set of nonnegative weights for $Z(u)$. Also since $h \in \overline{CR}/\overline{CR}_u$ lies in $Z(\widetilde{Z}(u))$,

$$\chi_\lambda^{\widetilde{Z}(u)}(t+h) = e^{2i\pi\langle\lambda,h\rangle}\chi_\lambda^{\widetilde{Z}(u)}(t). \tag{5.301}$$

From (5.300), (5.301), we get easily,

$$\frac{1}{|\overline{CR}/\overline{CR}_u|^{2g}} \frac{\sum_{h\in\overline{CR}/\overline{CR}_u} |\widetilde{V}_u|(t_1, \dots, t_s, h, t)}{|\sigma_{Z(u)}(t)| |\prod_{j=1}^s |\sigma_{Z(u)}(t_j)|} = |Z(\widetilde{Z}(u))| |\mathrm{Vol}(\mathfrak{t}/\overline{CR}_u)|^{2g-2} \left|\overline{CR}/\overline{CR}_u\right|^{1-2g} \left|\frac{\mathrm{Vol}(\widetilde{Z}_u)}{\mathrm{Vol}(\widetilde{T}_u)}\right|^{2g+s-1} \sum_{\lambda\in\overline{CR}^*\cap\overline{K}_u} \frac{\prod_{j=1}^s \chi_\lambda^{Z(u)}(t_j)\chi_\lambda^{Z(u)}(t)}{\chi_\lambda^{Z(u)}(1)^{2g+s-1}}. \tag{5.302}$$

Clearly

$$\begin{aligned} &Z(Z(u)) = Z(\widetilde{Z}(u))/(\overline{CR}/\overline{CR}_u), \\ &\mathrm{Vol}(\mathfrak{t}/\overline{CR}_u) = \mathrm{Vol}(T)|\overline{CR}/\overline{CR}_u|, \\ &\frac{\mathrm{Vol}(\widetilde{Z}(u))}{\mathrm{Vol}(\widetilde{T}_u)} = \frac{\mathrm{Vol}(Z(u))}{\mathrm{Vol}(T)}. \end{aligned} \tag{5.303}$$

From (5.302), (5.303), we get

$$\frac{1}{|\overline{CR}/\overline{CR}_u|^{2g}} \frac{\sum_{h\in\overline{CR}/\overline{CR}_u} \widetilde{V}_u(t_1, \dots, t_s, h, t)}{|\sigma_{Z(u)}(t)| |\prod_{j=1}^s |\sigma_{Z(u)}(t_j)|} = |Z(Z(u))| \mathrm{Vol}(\mathfrak{t}/\overline{CR})^{2g-2} \left|\frac{\mathrm{Vol}(Z(u)}{\mathrm{Vol}(T)}\right|^{2g+s-1} \sum_{\lambda\in\overline{CR}^*\cap\overline{K}_u} \frac{\prod_{j=1}^s \chi_\lambda^{Z(u)}(t_j)\chi_\lambda^{Z(u)}(t)}{\chi_\lambda^{Z(u)}(1)^{2g+s-1}}. \tag{5.304}$$

Identity (5.304) fits with (5.197) and with (5.297).

Since $t_1, \dots, t_s$ are very regular in $\widetilde{Z}(u)$, there are $Z(u)$-invariant open neighborhoods $U_1^u, \dots, U_s^u$ of $t_1, \dots, t_s$ in $\widetilde{Z}(u)$ on which a logarithm is well defined.

Since $Z(u)$ acts locally freely on $M_0(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)$, all the results of Section 5.10 can be used in this situation.

In particular, by Theorem 5.57, there is a $Z(u)$-invariant open neighborhood $\widehat{X}_{u,h}$ of $M_0(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)$ in $\widetilde{X}_u$, equipped with a symplectic form $\tilde{\sigma}^u$, such that the $\tilde{\sigma}^u_t$ are the symplectic reductions of $\tilde{\sigma^u}$. So we may use the results of Section 3 in this situation.

Let $T'_u = \mathfrak{t}/\overline{R^*_u}$ be the obvious maximal torus in $Z'(u) = Z(u)/Z(Z(u))$. Now we will use the notation in Section 3.6. As in Section 3.6, the choice of a Weyl chamber$K$ and of the corresponding Weyl chamber $K_u$ for $Z(u)$ defines an orientation on the $M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)/T'_u$, for $t \in T \cap U_u$.

DEFINITION 5.71. For $t \in T \cap U_u$, put

$$H_{(u,g,s)}(t_1, \dots t_s, t) = \int_{M_t(\widetilde{Z}(u), \mathcal{O}_{\widetilde{Z}(u)}(t_1), \dots, \mathcal{O}_{\widetilde{Z}(u)}(t_s), h)/T'_u} e^{\tilde{\sigma}^u_t}. \tag{5.305}$$

Using Theorems 3.15, 5.57 and 5.60 , we know that $H_{(u,g,s)}(t_1, \dots t_s, t)$ is locally a polynomial in $t_1, \dots, t_s, t$. Also recall that $\widetilde{P}_{u,2g+s-1}(t)$, $t \in \widetilde{T}_u$ was defined in Definition 2.39. By Theorem 2.37, $\widetilde{P}_{u,2g+s-1}(t)$ is a polynomial on $\widetilde{T}_u \backslash \widetilde{S}_u$. Finally remember that in (2.33) , we set $\ell_u = |R_{u,+}|$.

THEOREM 5.72. *The following identity of local polynomials in $(t_1, \dots, t_s, t)$ holds*

$$\begin{aligned} H_{(u,g,s)}(t_1, \dots, t_s, h, t) = (-1)^{\ell_u(g-1)+1} |Z(\widetilde{Z}(u))| \\ |\mathrm{Vol}(\mathfrak{t}/\overline{CR}_u)|^{2g-2} \prod_{j=1}^{s} \mathrm{sgn}\big((-i)^{\ell_u} \sigma_{Z(u)}(t_j)\big) e^{-2i\pi\langle \rho_u, h\rangle} \\ \sum_{(w'^1, \dots, w'^s) \in W_u^s} \prod_{j=1}^{s} \epsilon_{w'^j} \widetilde{P}_{u,2g+s-1}(t + h + \sum_{j=1}^{s} w'^j t_j). \end{aligned} \tag{5.306}$$

PROOF. Clearly, if $t \in U_u \cap T_{\mathrm{reg}}$, then $Z'(t) = T'$, so that $Z_u(t) = T'_u$. By (5.295), (5.305),

$$|H_{(u,g,s)}(t_1, \dots, t_s, h, t)| = \widetilde{V}_u(t_1, \dots t_s, h, t) \text{ on } U_u \cap T_{\mathrm{reg}}. \tag{5.307}$$

Also

$$|\sigma_{Z(u)}(t)| = \mathrm{sgn}\big((-i)^{\ell_u} \sigma_{Z(u)}(t)\big)(-i)^{\ell_u} \sigma_{Z(u)}(t). \tag{5.308}$$

Moreover, by (1.43), if $h \in \overline{CR}/\overline{CR}_u$,

$$\sigma_{Z(u)}(t + h) = e^{2i\pi\langle \rho_u, h\rangle} \sigma_{Z(u)}(t), \tag{5.309}$$

By Theorem 1.41,

$$e^{2i\pi\langle \rho_u, h\rangle} = \pm 1. \tag{5.310}$$

Moreover for $t$ close enough to 0, $(-i)^{\ell_u} \sigma_{Z(u)}(t)$ and $\pi_u(t/i)$ have the same sign. Also by (3.129), (3.133),

$$\frac{\chi_\lambda^{\widetilde{Z}(u)}(1)}{\mathrm{Vol}(\widetilde{Z}(u))/\mathrm{Vol}(\widetilde{T}_u)} = \pi_u(\frac{\rho_u + \lambda}{i}). \tag{5.311}$$

Finally by Theorem 3.15, for $t$ close enough to 0, $H_{(u,g,s)}(t)$ and $\pi_u(t/i)$ either vanish together, or they are nonzero, and then they have the same sign. Using (5.300), (5.310), (5.311) , we get the identity of distributions

$$H_{(u,g,s)}(t_1,\dots,t_s,h,t) = (-1)^{\ell_u(g-1)}|Z(\widetilde{Z}(u))||\mathrm{Vol}(\mathfrak{t}/\overline{CR}_u)|^{2g-2} \prod_{j=1}^{s} \mathrm{sgn}\big((-i)^{\ell_u}\sigma_{Z(u)}(t_j)\big)e^{-2i\pi\langle\rho_u,h\rangle} \sum_{\lambda\in\overline{CR}^*_{u,+}} \frac{\prod_{j=1}^{s}\big(\sigma_{Z(u)}(t_j)\chi_\lambda^{\widetilde{Z}(u)}(t_j)\big)\sigma_{Z(u)}(t+h)\chi_\lambda^{\widetilde{Z}(u)}(t+h)}{\big(\pi_u(\rho_u+\lambda)\big)^{2g+s-1}}. \tag{5.312}$$

By (1.94), Theorem 1.38 and by (5.312), we obtain

$$H_{(u,g,s)}(t_1,\dots,t_s,h,t) = (-1)^{\ell_u(g-1)}|Z(\widetilde{Z}(u))|\mathrm{Vol}(\mathfrak{t}/\overline{CR}_u)|^{2g-2} \prod_{j=1}^{s}\mathrm{sgn}\big((-i)^{\ell_u}\sigma_{Z(u)}(t_j)\big)e^{-2i\pi\langle\rho_u,h\rangle} \sum_{\lambda\in\overline{CR}^*_{u,+}} \sum_{(w'^1,\dots,w'^s,w)\in W_u^{s+1}} \frac{\prod_{j=1}^{s}\epsilon_{w'^j}\epsilon_w^{s-1}xp(2i\pi\langle w(\lambda+\rho),t+h+\sum_{j=1}^{s}w'^jt_j\rangle)}{\big(\pi_u(\rho_u+\lambda)\big)^{2g+s-1}} \tag{5.313}$$

Now by (1.186), if $w\in W_u$,

$$\pi_u(w(\rho_u+\lambda)) = \epsilon_w\pi_u(\rho_u+\lambda). \tag{5.314}$$

Also by using in particular [**15**, Note V.4.14],

$$\{w(\rho_u+\lambda)\}_{\substack{w\in W_u,\\ \lambda\in\overline{CR}^*_{u,+}}} = \{\lambda\in\overline{CR}^*_u, \pi_u(\lambda)\neq 0\}. \tag{5.315}$$

Using (2.158), ((5.313)-(5.315), we get the identity of distributions on $U_u$,

$$H_{(u,g,s)}(t_1,\dots,t_s,h,t) = (-1)^{\ell_u(g-1)+1}|Z(\widetilde{Z}(u))||\mathrm{Vol}(\mathfrak{t}/\overline{CR}_u)|^{2g-2} \prod_{j=1}^{s}\mathrm{sgn}\big((-i)^{\ell_u}\sigma_{Z(u)}(t_j)\big)e^{-2i\pi\langle\rho_u,h\rangle} \sum_{(w'^1,\dots,w'^s)\in W_u^s}\prod_{j=1}^{s}\epsilon_{w'^j}\tilde{P}_{u,2g+s-1}(t+h+\sum_{j=1}^{s}w'^jt_j). \tag{5.316}$$

Now by Theorems 2.37, 3.15, 5.57, 5.60, we know that both sides of (5.316) are local polynomials of $(t_1,\dots,t_s,t)$. Therefore (5.316) extends to an identity of local polynomials. The proof of our Theorem is completed. □

Put

$$M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h) = M_0(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h). \tag{5.317}$$

Then $M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)\subset M$, and $Z(u)$ acts locally freely on $M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)$.

Let $\theta$ be a $Z(u)$ connection form on the $Z(u)$-bundle $M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)\to M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)/Z(u)$, and let $\Theta$ be its curvature. Let $\theta_1,\dots,\theta_s$ be connection forms taken as in Section 5.10, and such that (5.253) holds, and let $\Theta_1,\dots,\Theta_s$ be their curvatures. Then $\Theta_1,\dots,\Theta_s$ take their values in $\mathfrak{t}$.

Let $Q$ be a $Z(u)$-invariant $C^\infty$ function on $\mathfrak{z}(u)$, let $Q_1, \dots, Q_s$ be $C^\infty$ functions on $\mathfrak{t}$. Recall that $\omega$ is the canonical symplectic form on $M/G$ which is associated to the basic scalar product $\langle , \rangle$ on $\mathfrak{g}$.

PROPOSITION 5.73. *If $p \in \mathbf{R}^*$, the following identity holds*

$$\int_{M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)/Z(u)} Q(-\Theta)\prod_{j=1}^s Q_j(-\Theta_j)e^{p\omega} = \frac{1}{|\overline{CR}/\overline{CR}_u|^{2g}}\frac{p^{(g-1)\dim \mathfrak{z}(u)+\frac{s}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})}}{|W_u|} Q(\frac{\partial/\partial t}{p})\prod_{j=1}^s Q_j(\frac{\partial/\partial t_j}{p})\pi_u(\frac{\partial/\partial t}{2i\pi})H_{(u,g,s)}(t_1,\dots,t_s,h,t). \tag{5.318}$$

PROOF. Clearly

$$\int_{M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)/Z(u)} Q(-\Theta)e^{p\omega}\prod_1^s Q_j(-\Theta_j) = p^{(\dim M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)/Z(u))/2} \int_{M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)/Z(u)} Q(-\Theta/p)\prod_1^s Q_j(-\Theta_j/p)e^{\omega}. \tag{5.319}$$

Also by Theorem 5.57 and Proposition 5.66,

$$\int_{M(Z(u),\mathcal{O}_{Z(u)}(t_1),\dots,\mathcal{O}_{Z(u)}(t_s),h)/Z(u)} Q(-\Theta/p)e^{\omega} = \frac{1}{|\overline{CR}/\overline{CR}_u|^{2g}}\int_{M(\widetilde{Z}(u),\mathcal{O}_{\widetilde{Z}(u)}(t_1),\dots,\mathcal{O}_{\widetilde{Z}(u)}(t_s),h)/Z(u)Z(u)} Q(-\Theta/p)e^{\bar{\sigma}_0^u}. \tag{5.320}$$

Finally our assumptions on the $t_j$'s, Theorems 5.20 and 5.21 guarantee that $Z'_u(x) = 1$ a.e. on $M(\widetilde{Z}(u),\mathcal{O}_{\widetilde{Z}(u)}(t_1),\dots,\mathcal{O}_{\widetilde{Z}(u)}(t_s),h)$. We can then apply Theorem 3.21 to (5.320) and get (5.318). The proof of our Theorem is completed. □

REMARK 5.74. In **[39, 40]**, Liu derived the above formulas for the intersection numbers of the corresponding moduli spaces.

**5.12. An evaluation of certain Euler characteristics.** Recall that $x \in \mathbf{R} \mapsto [x] \in [0,1[$ is the periodic function of period 1 such that for $x \in [0,1[$, $[x] = x$.

PROPOSITION 5.75. *Let $m \in \mathbf{N}$. Put $z = e^{\frac{2i\pi}{m}}$. Then for $\ell \in \mathbf{Z}$,*

$$\frac{1}{m}\sum_{k=1}^{m-1}\frac{z^{k\ell}}{1-z^{-k}} = \frac{1}{2} - \left[\frac{\ell}{m}\right] - \frac{1}{2m}. \tag{5.321}$$

PROOF. Take $\rho \in ]-1,+1[$, and $\ell$ with $0 \le \ell < m$. Then

$$\begin{aligned}\sum_{k=1}^{m-1}\frac{z^{-k\ell}}{1-\rho z^k} &= \sum_{n=0}^{+\infty}\sum_{k=1}^{m-1}\rho^n z^{k(n-\ell)} = \sum_{n=0}^{+\infty}\rho^n\left(\sum_{k=0}^{m-1}z^{k(n-\ell)}-1\right) \\ &= \sum_{n=0}^{+\infty}\rho^n m 1_{m|(n-\ell)} - \frac{1}{1-\rho} \\ &= \frac{m\rho^\ell}{1-\rho^m} - \frac{1}{1-\rho}.\end{aligned} \tag{5.322}$$

By making $\rho \to 1$ in (5.322), we get

$$\sum_{k=1}^{m-1} \frac{z^{-k\ell}}{1-z^k} = \frac{m-1}{2} - \ell, \tag{5.323}$$

which is equivalent to

$$\frac{1}{m}\sum_{k=1}^{m-1} \frac{z^{k\ell}}{1-z^{-k}} = \frac{1}{2} - \left[\frac{\ell}{m}\right] - \frac{1}{2m}. \tag{5.324}$$

So we have established (5.321) for $0 \le \ell < m$. Similarly if $-m \le \ell < 0$, using (5.324), we obtain

$$\frac{1}{m}\sum_{k=1}^{m-1} \frac{z^{k\ell}}{1-z^{-k}} = \frac{1}{m}\sum_{k=1}^{m-1} \frac{z^{k(m+\ell)}}{1-z^{-k}} = \frac{1}{2} - \left[1+\frac{\ell}{m}\right] - \frac{1}{2m} \tag{5.325}$$
$$\frac{1}{2} - \left[\frac{\ell}{m}\right] - \frac{1}{2m}.$$

The proof of our Proposition is completed. □

Let $\Sigma_g$ be a Riemann surface of genus $g$. Here, we have fixed a complex structure on $\Sigma_g$. Let $x_1, \dots, x_s$ be $s$ distinct points in $\Sigma_g$. Let $D$ be the divisor

$$D = \sum_{j=1}^{s} x_j. \tag{5.326}$$

Let $[D]$ be the corresponding holomorphic line bundle on $\Sigma_g$. Let $\sigma_D$ be the canonical section of $[D]$. Clearly

$$\int_{\Sigma_g} c_1([D]) = s. \tag{5.327}$$

Let $m \in \mathbf{N}$ such that $m$ divides $s$. Let $\lambda$ be a holomorphic line bundle on $\Sigma_g$ such that

$$\lambda^m = [D]. \tag{5.328}$$

Put

$$\Sigma_g^b = \{t \in \lambda\,,\ \sigma_D = t^m\}. \tag{5.329}$$

Then $p = \Sigma_g^b \to \Sigma_g$ is a branched covering of order $m$, with branching points $x_1, \dots, x_s$. Also by Hurwitz's formula, the genus $g'$ of $\Sigma_g^b$ is given by

$$g' = mg + \frac{1}{2}(m-1)(s-2). \tag{5.330}$$

If $\ell \in \mathbf{Z}/m\mathbf{Z}, t \in \Sigma_g^b$, put

$$\ell(t) = e^{\frac{2i\pi\ell}{m}} t. \tag{5.331}$$

Then (5.331) defines an action of $\mathbf{Z}/m\mathbf{Z}$ over $\Sigma_g^b$ such that $p\ell = p$. Also if $\ell \in \mathbf{Z}/m\mathbf{Z}$, $l \neq 0$, then $x_1, \dots, x_s \in \Sigma_g^b$ are the only fixed points of $\ell$.

Let $\Sigma$ be the Riemann surface with boundary, which is obtained from $\Sigma_g$ by deleting $s$ small disks centered at $x_1, \dots, x_s$. Set

$$\Sigma^b = p^{-1}\Sigma. \tag{5.332}$$

Then $\Sigma^b$ is also a Riemann surface with boundary, obtained from $\Sigma_g^b$ by deleting $s$ small disks $\Delta_1, \dots, \Delta_s$ centered at $x_1, \dots, x_s$.

Let $G$ be a compact connected and simply connected simple Lie group.

DEFINITION 5.76. We will say that $g \in G$ is of order $m$ if $g^m = 1$. More generally if $\mathcal{O} \subset G$ is an adjoint orbit, $\mathcal{O}$ will be said to be of order $m$ if for one (or any) element g$\in \mathcal{O}$, $g^m = 1$.

In the sequel, we assume that $\mathcal{O}_1, \dots \mathcal{O}_s$ are of order $m$.

Let $x \in M$. Then the trivial $G$-bundle $P$ over $\Sigma$ is equipped with the corresponding flat $G$-connection. Therefore the $G$-bundle $p^*P \to \Sigma^b$ is equipped with the corresponding flat connection. Moreover $\mathbf{Z}/m\mathbf{Z}$ acts naturally on this $G$-bundle and preserves the flat connection.

Observe that for $1 \leq j \leq s$, the holonomy of the flat connection over the circle $p^{-1}(S_j^1)$ —which is a $m$ cover of $S_j$— is $w_j^m = 1$, i.e. $\pi^*P$ has trivial holonomy around $\pi^{-1}(S_j)$. Therefore the flat connection on $\Sigma^b$ extends to a flat connection on the trivial $G$-bundle $p^*P \to \Sigma_g^b$.

We claim the action of $\mathbf{Z}/m\mathbf{Z}$ extends to the bundle $p^*P \to \Sigma_g^b$. To define the flat bundle $p^*P$ near $x_j$, we use the identification by parallel transport along the circle $S_j^1$. In fact recall that for $1 \leq j \leq s$, $w_j$ is the holonomy of the flat connection along $S_j^1$ considered as lying in $\partial\Sigma$. The holonomy along $S_j$ considered as the boundary of the disk $\Delta_j$ is $w_j^{-1}$. If $\ell \in \mathbf{Z}/m\mathbf{Z}$, $t \in \Sigma$, $f \in P$, then

$$\ell(t,f) = \left(e^{\frac{2i\pi\ell}{m}}t, f\right). \tag{5.333}$$

However using the trivialization of parallel transport along $p^*S_j^1$, in this trivialization

$$\ell(t,f) = \left(e^{\frac{2i\pi\ell}{m}}t, w_j^\ell f\right). \tag{5.334}$$

In particular the action of $\ell$ on $p^*P_{x_j}$ is given by $f \in P \mapsto w_j^\ell f \in P$.

Let $V$ be a complex vector space. Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of $G$. Let $F$ be the flat vector bundle on $\Sigma$

$$F = P \times_G V. \tag{5.335}$$

Then by the above construction, $p^*F$ extends to a flat vector bundle on $\Sigma_g^b$, on which $\mathbf{Z}/m\mathbf{Z}$ acts. In particular $\mathbf{Z}/m\mathbf{Z}$ acts on $H^\cdot(\Sigma_g^b, p^*F)$. Let $[H(\Sigma_g^b, p^*F)]^{\mathbf{Z}/m\mathbf{Z}}$ be the invariant part of $H(\Sigma_g^b, p^*F)$ under the action of $\mathbf{Z}/m\mathbf{Z}$.

Let $\langle\ ,\ \rangle$ be a $G$-invariant bilinear symmetric form on $V$. Let $\langle\ ,\ \rangle_{\widehat{H}^1(\Sigma,F)}$ and $\langle\ ,\ \rangle_{H^1(\Sigma_g^b,p^*F)}$ be the coresponding intersection forms on $\widehat{H}^1(\Sigma, F)$ and $H^1(\Sigma_g^b, p^*F)$.

PROPOSITION 5.77. *The following identity holds*

$$\widehat{H}^\cdot(\Sigma, F) = [H^\cdot(\Sigma_g^b, p^*F)]^{\mathbf{Z}/m\mathbf{Z}}. \tag{5.336}$$

*Under the identification (5.336), if* $\alpha, \alpha' \in \widehat{H}^1(\Sigma, F)$,

$$\langle\alpha, \alpha'\rangle_{\widehat{H}^1(\Sigma,F)} = \frac{1}{m}\langle\alpha, \alpha'\rangle_{H^1(\Sigma_g^b,p^*F)}. \tag{5.337}$$

PROOF. Clearly, $[H^0(\Sigma_g^b, p^*F)]^{\mathbf{Z}/m\mathbf{Z}}$ consists of flat $\mathbf{Z}/m\mathbf{Z}$-invariant sections of $p^*F$ on $\Sigma_g^b$. In particular by for $1 \leq j \leq s$, $w_j f_{|x_j} = f_{|x_j}$. Therefore these sections descend to flat sections of $F$ on $\Sigma$. Using (5.27) and (5.36), (5.336) holds in degree 0. By Theorem 5.4, using Poincaré duality, (5.336) holds in degree 2.

Let $\alpha$ be a $\mathbf{Z}/m\mathbf{Z}$-invariant closed form in $\Omega^1(\Sigma_g^b, p^*F)$ representing $[\alpha] \in [H^1(\Sigma_g^b,$
$p^*F)]^{\mathbf{Z}/m\mathbf{Z}}$. We may and will assume that $\alpha$ vanish near $x_1, \dots, x_s$. Then $\alpha$ descends to a smooth closed 1 form on $\Sigma$, which vanishes on $\partial\Sigma$. Also $\alpha$ is defined up to the coboundary of a $\mathbf{Z}/m\mathbf{Z}$-invariant form in $\Omega^0(\Sigma_g^b, p^*F)$. Using Theorem 5.4, we find there is a well-defined map $[H^1(\Sigma_g^b, p^*F)]^{\mathbf{Z}/m\mathbf{Z}} \to \widehat{H}^1(\Sigma, F)$. Conservely, if $\beta$ is closed in $\Omega^1(\Sigma, F)$ and vanishes near $\partial\Sigma$, then $p^*\beta$ is a smooth $\mathbf{Z}/m\mathbf{Z}$-invariant closed 1 form in $\Omega^1(\Sigma_g^b, p^*F)$ . So we have defined a map $\widehat{H}^1(\Sigma, F) \to [H^1(\Sigma_g^b, p^*F)]^{\mathbf{Z}/m\mathbf{Z}}$. It is now easy to verify these two maps are inverse to each other. The identity (5.337) follows trivially.

The proof of our Theorem is completed. □

Let $\widehat{\chi}(F)$ be the Euler characteristic of the complex $(\widehat{C}^{\Sigma}(F), \partial)$. Then

$$\widehat{\chi}(F) = \sum_{i=0}^{2} (-1)^i \dim(\widehat{H}^i(\Sigma, F)) = \sum_{i=0}^{2} (-1)^i \dim(\widehat{C}^{\Sigma,i}(F)) . \tag{5.338}$$

So by (5.32), (5.338),

$$\widehat{\chi}(F) = (2-2g)\dim F - \sum_{j=1}^{s} \dim(1-w_j)(F)) . \tag{5.339}$$

Let $\chi^{\mathbf{Z}/m\mathbf{Z}}(p^*F)$ be the invariant Euler characteric of $p^*F$ on $\Sigma_g^b$, i.e.

$$\chi^{\mathbf{Z}/m\mathbf{Z}}(p^*F) = \sum_{i=0}^{2} (-1)^i \dim[H^i(\Sigma_g^b, p^*F)]^{\mathbf{Z}/m\mathbf{Z}} . \tag{5.340}$$

By Proposition 5.77,

$$\widehat{\chi}(F) = \chi^{\mathbf{Z}/m\mathbf{Z}}(p^*F) . \tag{5.341}$$

We will prove (5.341) again using the Theorem of Riemann-Roch-Kawasaki [**32, 33**], stated in Theorem 6.8. We get

$$\chi^{\mathbf{Z}/m\mathbf{Z}}(p^*F) = \frac{1}{m}\left[\dim(F)\int_{\Sigma_g^b} e(T\Sigma_g^b) + \sum_{j=1}^{s} \mathrm{Tr}^F\left[\sum_{k=1}^{m-1} w_j^k\right]\right] . \tag{5.342}$$

Now using (5.330), we get

$$\int_{\Sigma_g^b} e(T\Sigma_g^b) = 2 - 2g' = m(2-2g) - (m-1)s . \tag{5.343}$$

Moreover

$$\frac{1}{m}\sum_{k=1}^{m-1} w_j^k = \frac{1}{m}\sum_{k=0}^{m-1} w_j^k - \frac{1}{m} . \tag{5.344}$$

Now $\frac{1}{m}\sum_1^{m-1} w_j^k$ is a projection on $\{f \in F, w_j f = f\}$. So by (5.344),

$$\frac{1}{m}\mathrm{Tr}^F\left[\sum_1^{m-1} w_j^k\right] = \dim\ker((w_j - 1)_{|F}) - \frac{\dim F}{m}. \tag{5.345}$$

By (5.342)-(5.345),

$$\chi^{\mathbf{Z}/m\mathbf{Z}}(p^*F) = (2-2g)\dim F - \sum_{j=1}^{s}\dim((1-w_j)(F)), \tag{5.346}$$

which fits with (5.339), (5.341).

Now we assume that $\rho : G \to \mathrm{Aut}(V)$ is a real representation of $G$. We will now give another proof of (5.346). Let $\chi^{h,\mathbf{Z}/m\mathbf{Z}}(p^*F)$ be the holomorphic invariant Euler characteristic

$$\chi^{h,\mathbf{Z}/m\mathbf{Z}}(p^*F) = \sum_{i=0}^{1}\dim\left[H^{0,i}(\Sigma_g^b, p^*F)\right]^{\mathbf{Z}/m\mathbf{Z}}. \tag{5.347}$$

By Hodge theory,

$$\chi^{\mathbf{Z}/m\mathbf{Z}}(p^*F) = 2\chi^{h,\mathbf{Z}/m\mathbf{Z}}(p^*F). \tag{5.348}$$

Also by the Riemann-Roch-Kawasaki theorem [**32**, **33**],

$$\chi^{h,\mathbf{Z}/m\mathbf{Z}}(p^*F) = \frac{1}{m}\Big[\dim(F)\int_{\Sigma_g^b}\frac{1}{2}c_1(T\Sigma_g^b) + \sum_{j=1}^{s}\sum_{k=1}^{m-1}\frac{1}{1-e^{-2i\pi k/m}}\mathrm{Tr}^F[w_j^k]. \tag{5.349}$$

By (5.343),

$$\frac{1}{m}\int_{\Sigma_g^b}\frac{1}{2}c_1(T\Sigma_g^b) = 1 - g - \frac{1}{2}(1-\frac{1}{m})s. \tag{5.350}$$

The eigenvalues of $w_{j|V}$ have absolute value 1. Since $V$ is real, they are either $\pm 1$, or they come as complex conjugate pairs. By Proposition 5.75, we get

$$\frac{1}{m}\sum_{k=1}^{m-1}\frac{1}{1-e^{-2i\pi k/m}}\mathrm{Tr}^V[w_j^k] = -\frac{1}{2m}\dim F + \frac{1}{2}\dim\ker(w_j - 1)_{|F}. \tag{5.351}$$

By (5.349)-(5.351), we obtain

$$\chi^{h,\mathbf{Z}/m\mathbf{Z}}(p^*F) = (1-g)\dim F - \frac{1}{2}\sum_1^{s}\dim((1-w_j)(F)), \tag{5.352}$$

which fits with (5.346).

**5.13. Evaluation of $c_1(TM/G)$.** In the sequel, we assume that $G$ is a connected simply connected compact simple Lie group. Otherwise we use the notation in Section 1. In particular $\langle\ ,\ \rangle$ denotes the basic scalar product on $\mathfrak{g}$ defined in Section 1.2.

As in Section 5.10, we construct a connection on the $G$-bundle $\frac{M\times\widehat{\Sigma}}{\Gamma} \xrightarrow{G} M/G \times \Sigma$ , such that the assumptions after (5.252) hold. In particular, for $1 \leq j \leq s$,

$$\begin{aligned} \nabla_{j.} w_j &= 0, \\ \nabla_{j.} \log(w_j) &= 0. \end{aligned} \tag{5.353}$$

Also since $t_j$ lies in an alcove $P$, it determines $\rho_j \in \overline{CR}^* \cap P$ by formula (1.35). Over $M/G$, $t_j, \rho_j$ descend to sections of $E_j$. By (5.353), for $1 \leq j \leq s$,

$$\nabla_j \rho_j = 0. \tag{5.354}$$

Let $\theta_j$ be the connection form on associated to $\nabla_j$, let $\Theta_j$ be its curvature. Then $\theta_j$ can be considered as a t-connection, and $\Theta_j$ is a t-valued 2 form on $M/G$.

Recall that $TM/G$ is a symplectic vector bundle. Let $J^{TM/G}$ be any almost complex structure polarizing $\sigma$, i.e. $\sigma$ is $J^{TM/G}$ invariant, and $U, V \in TM/G \mapsto \sigma(J^{TM/G}U, V)$ is a scalar product. Such $J^{TM/G}$ exist and are homotopic. Therefore $c_1(TM/G)$ is a well-defined element of $H^2(M/G, \mathbf{Q})$.

Recall that $c$ is the dual Coxeter number defined in Definition 1.7.

THEOREM 5.78. *The following identity hold,*

$$c_1(TM/G) = 2(c\omega + \sum_{j=1}^{s} \langle ct_j - \rho_j, \Theta_j \rangle). \tag{5.355}$$

PROOF. First we assume that m$\in \mathbf{N}$, that $t_1, \ldots, t_s$ are of order $m$, and $s|m$. Then the $G$-bundle $(M \times \widehat{\Sigma})/\Gamma \xrightarrow{G} M/G \times \Sigma$ lifts to a $G$-bundle $Q \xrightarrow{G} M/G \times \Sigma_g^b$ on which $\mathbf{Z}/m\mathbf{Z}$ acts naturally. In fact, we only need to make the lifting constructions of Section 5.12 fibrewise.

Recall that a complex structure has been fixed on $\Sigma_g$ and $\Sigma_g^b$, and that $\mathbf{Z}/m\mathbf{Z}$ acts holomorphically on $\Sigma_g^b$.

By (5.120), we have the identity

$$T_{\mathbf{R}}M/G = \widehat{H}^1(\Sigma, E). \tag{5.356}$$

Using Proposition 5.77 and (5.356), we get

$$T_{\mathbf{R}}M/G = \left[H^1(\Sigma_g^b, p^*E)\right]^{\mathbf{Z}/m\mathbf{Z}}. \tag{5.357}$$

Now

$$H^1(\Sigma_g^b, p^*E) \otimes_{\mathbf{R}} \mathbf{C} = H^{(1,0)}(\Sigma_g^b, p^*E) \oplus H^{(0,1)}(\Sigma_g^b, p^*E), \tag{5.358}$$

and the splitting (5.358) is $\mathbf{Z}/m\mathbf{Z}$ invariant. By (5.356)-(5.359), we get

$$\widehat{H}^1(\Sigma, E) \otimes_{\mathbf{R}} \mathbf{C} = \left[H^{(1,0)}(\Sigma_g^b, p^*E)\right]^{\mathbf{Z}/m\mathbf{Z}} \oplus \left[H^{(0,1)}(\Sigma_g^b, p^*E)\right]^{\mathbf{Z}/m\mathbf{Z}}. \tag{5.359}$$

Let $J$ be the complex structure on $H^1(\Sigma_g^b, p^*E)$ which is $i$ on $H^{(1,0)}(\Sigma_g^b, p^*E)$, $-i$ on $H^{(0,1)}(\Sigma_g^b, p^*E)$. We claim that $J$ polarizes the symplectic form $\omega$ on $\widehat{H}^1(\Sigma, E) = T_{\mathbf{R}}M/G$. In fact by if $\alpha, \alpha' \in \widehat{H}^1(\Sigma, E)$ are represented by the forms $\eta, \eta' \in \Omega^1(\Sigma_j^b, p^*E)$ which are closed and $\mathbf{Z}/m\mathbf{Z}$ invariant, then by Proposition 5.77,

$$\omega(\alpha, \alpha') = -\frac{1}{m} \int_{\Sigma_g^b} \langle \eta, \eta' \rangle. \tag{5.360}$$

It is now trivial to verify that $J$ polarizes $\omega$.

So by (5.356), (5.359),

$$c_1(TM/G) = c_1\left(\left[H^{(1,0)}(\Sigma_g^b, p^*E)\right]^{\mathbf{Z}/m\mathbf{Z}}\right), \tag{5.361}$$

which is equivalent to

$$c_1(TM/G) = -c_1\left(\left[H^{(0,1)}(\Sigma_g^b, p^*E)\right]^{\mathbf{Z}/m\mathbf{Z}}\right). \tag{5.362}$$

Let $\theta$ be a $\mathbf{Z}/m\mathbf{Z}$ invariant connection on the $G$-bundle $Q \xrightarrow{G} \Sigma_g^b \times M/G$. Observe that $(\{x_j\} \times M/G)_{1\leq j\leq s}$ are exactly the fixed point of the action of $\mathbf{Z}/m\mathbf{Z}$ over $\Sigma_g^b \times M/G$. Since the connection $\theta$ is $\mathbf{Z}/m\mathbf{Z}$-invariant, and since, for $1 \leq j \leq s$, $1 \in \mathbf{Z}/m\mathbf{Z}$ acts on $E_{x_j}$ like $w_j$, we find that over $M/G$

$$\nabla w_{j_{|x_j}} = 0. \tag{5.363}$$

By (5.63),

$$H^0(\Sigma, E) = 0. \tag{5.364}$$

By (5.336), (5.364),

$$[H^0(\Sigma_g^b, p^*E)]^{\mathbf{Z}/m\mathbf{Z}} = 0. \tag{5.365}$$

Now we will use an equivariant version of the curvature theorem of Bismut and Freed [**10**]. Namely we equip $T\Sigma_g^b$ with a $\mathbf{Z}/m\mathbf{Z}$-invariant metric. Recall that $p^*E$ is also equipped with a $\mathbf{Z}/m\mathbf{Z}$-invariant metric. Since $\mathbf{Z}/m\mathbf{Z}$ is a finite group, the construction sof [**9**, Definition 2.2] provide us with a like metric on the line bundle $\det([H^{(0,\cdot)}(\Sigma_g^b, p^*E)]^{\mathbf{Z}/m\mathbf{Z}})$. By proceeding as in [**10**], we also obtain a unitary connection on this line bundle. Incidently, observe that since all our data are holomorphic, we could instead use the holomorphic constructions of [**11**] in an equivariant context. The curvature of the connection on $\det([H^{(0,\cdot)}(\Sigma_g^b, p^*E)]^{\mathbf{Z}/m\mathbf{Z}})$ is obtained by applying the techniques of [**10**] or [**11**]. An important technical point is to prove an equivariant version of the local families index theorem of [**8**]. A large part of the steps which are needed in extending the results of [**8**] is already done in [**9**]. By mixing the techniques used in Lefschetz fixed point theory and in the prooof of the local families index theorem as in [**8**], [**6**], [**9**], one finds easily that the curvature is given by a differential form version of the theorem of Riemann-Roch-Kawasaki [**32, 33**].

Using (5.365) and the above considerations, we obtain

$$c_1\left(\left[H^{(0,1)}(\Sigma_g^b, E)\right]^{\mathbf{Z}/m\mathbf{Z}}\right) = -\frac{1}{m}\Bigg(\int_{\Sigma_g^b} \mathrm{Td}(T\Sigma_g^b)\mathrm{ch}(p^*E, \nabla^{p^*E}) + \sum_{j=1}^{s}\sum_{k=1}^{m-1} \frac{1}{1-e^{-2i\pi k/m}}\mathrm{Tr}^E\left[w_j^k \exp\left(-\frac{\Theta_j}{2i\pi}\right)\right]\Bigg)^{(2)}. \tag{5.366}$$

By (1.37) and by Theorems 5.55 and 5.56,

$$\left(\frac{1}{m}\int_{\Sigma_g^b} \mathrm{Td}(T\Sigma_g^b)\mathrm{ch}(p^*E, \nabla^{p^*E})\right)^{(2)} = 2c\omega. \tag{5.367}$$

Recall that

$$\mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}=\mathfrak{t}\oplus\left(\bigoplus_{\alpha\in\mathbf{R}}\mathfrak{g}_\alpha\right). \tag{5.368}$$

Let $S_j$ be the operator acting on $\mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$,

$$S_j=\frac{1}{m}\sum_{k=1}^{m-1}\frac{1}{1-e^{-2i\pi k/m}}\tau(t_j^k). \tag{5.369}$$

By Proposition 5.75, we find that

$$\begin{aligned} S_{j|\mathfrak{t}} &= \frac{1}{2}-\frac{1}{2m}, \\ S_{j|\mathfrak{g}_\alpha} &= \frac{1}{2}-[\langle\alpha,t_j\rangle]-\frac{1}{2m},\ \alpha\in R. \end{aligned} \tag{5.370}$$

By conjugation by an element $G$, we may and we will assume that $w_j=t_j$. Then

$$\left(\sum_{k=1}^{m-1}\frac{1}{1-e^{-2i\pi k/m}}\mathrm{Tr}^E\left[w_j^k\exp\left(-\frac{\Theta_j}{2i\pi}\right)\right]\right)^{(2)}=-\mathrm{Tr}^{\mathfrak{g}}\left[S_j\frac{\Theta_j}{2i\pi}\right]. \tag{5.371}$$

Also because $w_j=t_j$, $\nabla_{x_j}^{E,2}$ is a 2 form on $M/G$ with values in $\mathfrak{t}$. From (5.370), we get easily

$$\begin{aligned} -\mathrm{Tr}^{\mathfrak{g}}\left[S_j\frac{\Theta_j}{2i\pi}\right] &= -\sum_{\alpha\in\mathbf{R}}\left(\frac{1}{2}-[\langle\alpha,t_j\rangle]\right)\langle\alpha,\Theta_j\rangle \\ &= \sum_{\alpha\in\mathbf{R}}[\langle\alpha,t_j\rangle]\langle\alpha,\Theta_j\rangle. \end{aligned} \tag{5.372}$$

Now by Proposition 1.20, and by (5.372), we get

$$-\mathrm{Tr}^{\mathfrak{g}}\left[S_j\frac{\nabla_{x_j}^{E,2}}{2i\pi}\right]=2\langle ct_j-\rho_j,\Theta_j\rangle. \tag{5.373}$$

From (5.362), (5.366), (5.367), (5.371)-(5.373), we get

$$c_1(TM/G)=2(c\omega+\sum_{j=1}^{s}\langle ct_j-\rho_j,\Theta_j\rangle). \tag{5.374}$$

We claim now that in the special case when the orbits $\mathcal{O}_1,\dots,\mathcal{O}_m$ are of order $m$, (5.374) is exactly (5.355). In spite of the formal simultanities, the objects introduced in both equation are not exactly of the same kind. However we leave to the reader the verification that they indeed coincide .

Now we establish (5.355) in full generality. Clearly if $(t_1,\dots,t_s)$ are regular and verify (A), they can be approximated by a sequence of regular elements $(t_1^m,\dots,t_s^m)$ which verify (A) and are of order $m$. Let $s_m\geq s$ be such that $m|s_m$. Then we consider the above situation, with $s$ replaced by $s_m$. At $x_1,\dots,x_s$, we assume that the holonomies are $w_1,\dots,w_s$ and at $x_{s+1},\dots,x_{s_m}$, they are 1. Needless to say, since 1 is far from being regular, we do not impose any restriction on the connection $\nabla_j, j\geq s+1$. Since for $j\geq s+1$, $\mathrm{Tr}^{\mathfrak{g}}[\Theta_j]=0$, it is clear that in the final formula, the $j\geq s+1$ do not contribute, so that (5.374) still hold. Then by the above (5.355) holds. A trivial limit procedure shows that (5.355) still holds in full generality. □

REMARK 5.79. By Remark 5.66, the cohomology class of $\omega + \sum_{j=1}^{s} \langle \log(w_j), \Theta_j \rangle$ is locally constant. Observe that in (5.355), $\langle \rho_j, \Theta_j \rangle$ is a closed form on $M/G$ whose cohomology class does not depend locally on $t_1, \dots, t_s$. So Theorem 5.78 fits with the above considerations.

Assume temporarily that some $t_j$ lie in $Z(G) = \overline{R}^* / \overline{CR}$. By (5.372), we find that such $t_j$ do not contribute to formula (5.355) for $c_1(TM/G)$. In other words if the $t_j$ are either regular or central, formula (5.355) still holds, where in the right-hand side, the summation is limited to a sum over the regular orbits.

## 6. The Riemann-Roch-Kawasaki formula on the moduli space of flat bundles

The purpose of this Section is to give formula for the index of a Dirac operator on the moduli space $M/G$ of flat vector $G$ bundles, by using the theorem of Riemann-Roch-Kawasaki [**32, 33**]. To do this, we describe the strata of $M/G$ and we express the contribution of each stratum as a residue in several variables, using the results of Sections 2 and 5. The results of this Section were already obtained by Szenes [**53**] for $G = \mathrm{SU}(3)$ and Jeffrey-Kirwan [**30**] in the case $G = \mathrm{SU}(n), s = 1$, with a central holonomy at the marked point for which $M/G$ is smooth.

This Section is organized as follows. In Section 6.1, we describe the strata of a general orbifold, and we introduce various associated characteristic classes. In Section 6.2, we state the theorem of Riemann-Roch-Kawasaki for almost complex orbifolds. In Section 6.3, we construct the orbifold line bundle $\lambda^p$ on the orbifold $M/G$. In Section 6.4, we describe the strata of the moduli space $M/G$ as moduli spaces associated to semisimple centralizers in $G$. In Section 6.5, we compute the Atiyah-Bott-Lefschetz-Todd class of a given stratum. In Section 6.6, we compute the dimension of certain vector spaces which appear naturally in the evaluation of the Atiyah-Bott-Lefschetz class.

Then we make a number of genericity assumptions on the $t_j$'s. In Section 6.7, we compute the contribution of a stratum to the Riemann-Roch-Kawasaki formula in terms of differential operators acting on symplectic volumes. In Section 6.8, we briefly show that under an obvious condition on the holonomies $t_j, 1 \leq j \leq s$, the index of the considered Dirac operator vanishes identically. In Section 6.9, we give a residue formula for the index. In Section 6.10, we give another related asymptotic formula for $|p|$ large.

Then we drop the genericity assumptions. In Section 6.11, we compute the index of a Dirac operator on a perturbed moduli space, for which the genericity assumptions hold. The point is that, as we shall see in Section 7, the index of the Dirac operator for the perturbed moduli space is exactly given by the Verlinde formula. Under genericity assumptions, this is only true asymptotically for the given moduli space $M/G$.

### 6.1. Almost complex orbifolds.

Let $M$ be a smooth compact manifold. Let $G$ be a compact connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. We assume that $G$ acts on $M$ on the left. If $X \in \mathfrak{g}$, let $X^M \in \mathrm{Vect}(M)$ be the corresponding vector field.

We assume that $G$ acts locally freely on $M$, i.e. for any non zero $X \in \mathfrak{g}$, $X^M$ is a non vanishing vector field on $M$.

We will use here the notation of Section 3.1, with $X$ replaced by $M$. Then $M/G$ is an orbifold.

DEFINITION 6.1. If $g \in G$, put

$$M^g = \{x \in M\,, gx = x\}\,. \tag{6.1}$$

Set

$$H = \{g \in G\,; M^g \neq \phi\}\,. \tag{6.2}$$

Then H is a finite union of conjugacy classes in $G$. Let $(H)$ be the corresponding finite set of conjugacy classes.

If $g \in G$, then $Z(g)$ acts locally freely on $M^g$. We can then apply the above constructions to $M^g$. Let $H_g \subset Z(g)$ be the generic stabilizer of $M^g$.

Observe that if $g' \in G$ is conjugate to $g$, the above constructions correspond by conjugation.

Take $g \in H$. Let $N_{M^g/M} \simeq TM/TM^g$ be the normal bundle to $M^g$ in $M$. Then we have the complex of $Z(g)$ vector bundles over $M^g$,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{g}^M/\mathfrak{z}^M(g) & \longrightarrow & N_{M^g/M} & \longrightarrow & N_{M^g/M}/(\mathfrak{g}^M/\mathfrak{z}^M(g)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{g}^M & \longrightarrow & TM & \longrightarrow & TM/\mathfrak{g}^M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{z}^M(g) & \longrightarrow & TM^g & \longrightarrow & TM^g/\mathfrak{z}^M(g) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{6.3}
$$

and the rows and columns in (6.3) are acyclic.

Clearly if $g' \in G$, $g'$ maps $M^g$ into $M^{g'gg'^{-1}}$. Also $g'$ acts on the complex (6.3).

Put

$$
N^g = N_{M^g/M}/(\mathfrak{g}^M/\mathfrak{z}^M(g)). \tag{6.4}
$$

Then the third column in (6.3) is the exact sequence of $Z(g)$ vector bundles on $M^g$,

$$
0 \to TM^g/Z(g) \to TM/G \to N^g \to 0. \tag{6.5}
$$

Equivalently $N^g$ is the "normal bundle" to $TM^g/Z(g)$ into $M/G$.

Clearly $g$ acts on each of the vector bundles in (6.3). Then $g$ acts like 1 on $\mathfrak{z}^M(g)$, $TM^g$ and $TM^g/Z(g)$. In particular , there are locally constants $\theta, 0 < \theta < \pi$ on $M^g$ such that

$$
N_{M^g/M} \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{0<\theta<\pi} (N^{\theta}_{M^g/M} \oplus N^{-\theta}_{M^g/M}) \oplus N^{\pi}. \tag{6.6}
$$

In (6.6), the $\theta$'s are distinct, $g$ acts on the left on $N^{\theta}_{M^g/M}, N^{-\theta}_{M^g/M}$ by multiplication by $e^{i\theta}, e^{-i\theta}$, and on $N^{\pi}$ by multiplication by $-1$. Therefore $N^g \otimes_{\mathbf{R}} \mathbf{C}$ splits as

$$
N^g \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{0<\theta<\pi} (N^{g,\theta} \oplus N^{g,-\theta}) \oplus N^{g,\pi}. \tag{6.7}
$$

DEFINITION 6.2. The orbifold $M/G$ will be said to be almost complex if $TM/G = TM/\mathfrak{g}^M$ is equipped with a $G$-invariant almost complex structure $\mathcal{J}^{TM/G}$.

Since in (6.5), $TM^g/Z(g)$ is the $+1$ eigenspace of $g$ on $TM/G$, we find that for any $g \in G$, $M^g/Z(g)$ is an almost complex orbifold. Therefore $N^g$ is also equipped with an almost complex structure.

Now we denote with the superscript $(1,0)$ the $+i$ eigenspace of the given complex structure. In particular $T^{(1,0)}M^g/Z(g)$ is well defined, and $N^{g,(1,0)}$ splits as

$$N^{g,(1,0)} = \bigoplus_{\theta \in ]-\pi,\pi]\setminus\{0\}} N^{g,(1,0),\theta}. \tag{6.8}$$

In the sequel, we will assume that the orbifold $M/G$ is almost complex.

**6.2. The Theorem of Riemann-Roch-Kawasaki.** Let $\nabla^{T^{(1,0)}M_g/Z(g)}, \dots, \nabla^{N^{g,(1,0),\theta}}$ be $Z(g)$ invariant horizontal connections on $T^{(1,0)}M_g/Z(g), N^{g,(1,0),\theta}$.

Let $E$ be a complex G-vector bundle on $M$, equipped with a $G$-invariant horizontal connection $\nabla^E$. Let $F^E$ be the curvature of $\nabla^E$. Then $(E,\nabla^E)_{|M^g}$ is a $Z(g)$-vector bundle equipped with a $Z(g)$ invariant connection.

DEFINITION 6.3. Let $\mathrm{ch}_g(E,\nabla^E)$ be the closed form on $M^g/Z(g)$

$$\mathrm{ch}_g(E,\nabla^E) = \mathrm{Tr}\left[g\exp\left(\frac{-F^E}{2i\pi}\right)\right]. \tag{6.9}$$

In (6.9), $g$ denotes the left action of the given element of $G$ on on $E$. Let $\mathrm{ch}_g(E)$ be the cohomology class associated to the form $\mathrm{ch}_g(E,\nabla^E)$.

DEFINITION 6.4. If $B$ is a square matrix, put

$$\begin{aligned} \mathrm{Td}(B) &= \det\left(\frac{B}{1-e^{-B}}\right), \\ \widehat{\mathrm{Td}}(B) &= \det\left(\frac{1}{1-e^{-B}}\right), \\ \widehat{A}(B) &= \det\left(\frac{B}{2\sinh(B/2)}\right). \end{aligned} \tag{6.10}$$

Observe that

$$\mathrm{Td}(B) = \widehat{A}(B)e^{\frac{1}{2}\mathrm{Tr}[B]} \tag{6.11}$$

DEFINITION 6.5. Put

$$\begin{aligned} \mathrm{Td}(T^{(1,0)}M_g/Z_g, \nabla^{T^{(1,0)}M_g/Z_g}) &= \mathrm{Td}\left(-\frac{F^{T^{(1,0)}M_g/Z_g}}{2i\pi}\right), \\ \widehat{\mathrm{Td}}(N^{g(1,0)}, \nabla^{N^{g(1,0)}}) &= \prod_{\theta\in]-\pi,\pi]\setminus\{0\}} \widehat{\mathrm{Td}}\left(i\theta - \frac{F^{N^{g,(1,0),\theta}}}{2i\pi}\right). \end{aligned} \tag{6.12}$$

Then the forms in (6.12) are closed on $M_g/Z(g)$. Let $\mathrm{Td}(T^{(1,0)}M_g/Z_g), \widehat{\mathrm{Td}}(N^{g,(1,0)})$ be the corresponding cohomology classes.

DEFINITION 6.6. Put

$$L(g,E) = \mathrm{Td}(T^{(1,0)}M_g/Z_g)\widehat{\mathrm{Td}}(N^{g,(1,0)})\mathrm{ch}_g(E). \tag{6.13}$$

Then $\int_{M^g/Z(g)} L(g,E)$ depends only on the conjugacy class of $g$ in $G$.

Let $h^{T^{(1,0)}M}, h^E$ be $G$-invariant metrics on $T^{(1,0)}M, E$. Let $\nabla^{T^{(1,0)}M/G}, \nabla^E$ be $G$-invariant unitary horizontal connections on $T^{(1,0)}M, E$. Then $\Lambda T^{*(1,0)}M/G\otimes E$ is naturally equipped with a $G$-invariant unitary horizontal connection $\nabla^{\Lambda T^{*(1,0)}M/G\otimes E}$.

Recall that by [**6**, p 135], $\Lambda T^{*(1,0)}M/G \otimes E$ is a $TM/G$ Clifford module. If $X \in TM(G)$, let $c(X)$ be the corresponding Clifford multiplication operator.

Let $dv$ be the volume element on $M_{\mathrm{reg}}/G$ associated to $h^{T^{(1,0)}M/G}$.

Let $K = K_+ \oplus K_-$ be the vector space of $G$-invariant $C^\infty$ sections of $\Lambda(T^{*(0,1)}M/G) \otimes E = (\Lambda^{\mathrm{even}}(T^{*(0,1)}M/G) \otimes E) \oplus (\Lambda^{\mathrm{odd}}(T^{*(0,1)}M/G) \otimes E)$. We equip $H$ with the Hermitian product

$$(6.14) \qquad s, s' \in H \mapsto \langle s, s' \rangle = \int_{M_{\mathrm{reg}}/G} \langle s, s' \rangle_{\Lambda(T^{*(0,1)}M/G)\otimes E} dv .$$

Let $e_1, \dots, e_n$ be an orthonormal basis of $TM/G$.

DEFINITION 6.7. Let $D^M$ be the Dirac operator acting on $K$

$$(6.15) \qquad D^M = \sum_1^n c(e_i) \nabla_{e_i}^{\Lambda(T^{*(0,1)}M/G)\otimes E} .$$

Then $D^M$ is a formally self-adjoint operator which exchanges $K_+$ and $K_-$. Let $D_+^M$ be the restriction of $D^M$ to $K_+$. Then we write $D^M$ in matrix form as

$$(6.16) \qquad D^M = \begin{bmatrix} 0 & D_-^M \\ D_+^M & 0 \end{bmatrix}$$

Then by [**32, 33**], $D_+^M$ is a Fredholm operator. Its index $\mathrm{Ind}(D_+^M)$ is given by

$$(6.17) \qquad \mathrm{Ind}(D_+^M) = \dim\ker D_+^M - \dim\ker D_-^M .$$

If $\gamma \in (H)$, let $g_\gamma \in \gamma$ be any representative in $G$ of the conjugacy class $\gamma$. Now we state the theorem of Riemann-Roch-Kawasaki [**32, 33**].

THEOREM 6.8. *The following identity holds*

$$(6.18) \qquad \mathrm{Ind}(D_+^M) = \sum_{\gamma \in (H)} \int_{M^{g_\gamma}/Z(g_\gamma)} \frac{1}{|H_{g_\gamma}|} L(g_\gamma, E) .$$

**6.3. The line bundle $\lambda^p$.** From now on, we suppose that all the assumptions of Section 5.11 are in force. Also we fix once and for all a positive Weyl chamber $K$ and the corresponding alcove $P \subset K$ whose closure contains 0. Finally we may and we will assume that

$$(6.19) \qquad t_j \in P,\ 1 \le j \le s.$$

DEFINITION 6.9. Let $\mathbf{M} \in \mathbf{Z}$ be given by

$$(6.20) \qquad \mathbf{M} = \{p \in \mathbf{Z}, pt_1, \dots, pt_s \in \overline{CR}^*\}.$$

In the sequel we assume that $\mathbf{M}$ is not reduced to 0. Then there is $p_0 \in \mathbf{N}^*$ such that

$$(6.21) \qquad \mathbf{M} = p_0 \mathbf{Z}.$$

Let $p \in \mathbf{M}$. Put

$$(6.22) \qquad \theta_j = pt_j,\ 1 \le j \le s.$$

Recall that the set of connections $\mathcal{A}_p(\theta_1, \dots, \theta_s)$ was defined in Definition 4.36. Also the Hermitian line bundle with unitary connection $(\lambda_p, \nabla^{\lambda_p})$ on $\mathcal{A}_p(\theta_1, \dots, \theta_s)$ was defined in Definition 4.37. By Proposition 4.38, $\Sigma G$ acts on the right on $\lambda_p$ and preserves $\nabla^{\lambda_p}$. Finally $\mathcal{A}^{\mathrm{flat}}(t_1, \dots, t_s)$ was defined in Definition 5.26.

Clearly if $p \in \mathbf{M}$, if $\theta_1, \dots \theta_s$ are given by (6.22), then

$$\mathcal{A}^{\text{flat}}(t_1, \dots, t_s) \subset \mathcal{A}_p(\theta_1, \dots, \theta_s). \tag{6.23}$$

Therefore the line bundle $\lambda_p$ restricts to $\mathcal{A}^{\text{flat}}(t_1, \dots, t_s)$. Also $\Sigma G$ acts on $\mathcal{A}^{\text{flat}}(t_1, \dots, t_s)$. Finally by Proposition 5.28,

$$\mathcal{A}^{\text{flat}}(t_1, \dots, t_s)/\Sigma_q G \simeq M. \tag{6.24}$$

DEFINITION 6.10. Let $\lambda^p, \nabla^{\lambda^p}$ be the Hermitian line bundle with unitary connection over $M$ of the $\Sigma_q G$-invariant sections of $\lambda_p$ on $\mathcal{A}^{\text{flat}}(t_1, \dots, t_s)$.

The notation for $\lambda^p$ is justified by the fact that if $p \in \mathbf{M}, p' \in \mathbf{Z}$, then

$$\lambda^{pp'} = (\lambda^p)^{\otimes p'}. \tag{6.25}$$

It is then clear that the action of $G$ on $M$ lifts to $\lambda^p$. Recall that $\omega$ is the canonical symplectic form on $M/G$ which was defined in Definition 5.30, which is associated to the basic scalar product $\langle , \rangle$ on $\mathfrak{g}$.

PROPOSITION 6.11. *The following identity of closed* 2*-forms holds on* $M$,

$$c_1(\lambda^p, \nabla^{\lambda^p}) = p\omega. \tag{6.26}$$

PROOF. Let $A \in \mathcal{A}^{\text{flat}}(t_1, \dots, t_s)$. Let $\alpha, \alpha'$ be 2 closed forms in $\Omega^1(\Sigma, E)$, which are exact on $\partial\Sigma$, i.e.. there are $\beta, \beta' \in \Omega^0(\partial\Sigma, E)$ such that $\alpha_{|\partial\Sigma} = \nabla^A \beta, \alpha'_{|\partial\Sigma} = \nabla^A \beta'$. By (4.24),(4.78), (4.100), (4.189),

$$c_1(\lambda^p, \nabla^{\lambda^p})(\alpha, \alpha') = p \int_\Sigma -\langle \alpha, \alpha' \rangle + p \int_{\partial\Sigma} \langle \beta, \nabla^A \beta' \rangle. \tag{6.27}$$

If $[\alpha], [\alpha']$ are the classes of $\alpha, \alpha'$ in $\widehat{H}^1(\Sigma, E)$, from (6.27), we get

$$c_1(\lambda^p, \nabla^{\lambda^p})(\alpha, \alpha') = p\omega([\alpha], [\alpha']). \tag{6.28}$$

The proof of our Theorem is completed. □

Since $\omega$ is a symplectic form, there is an almost complex structure $J$ on $TM/G$ which polarizes $\omega$, i.e. $\omega(JX, Y)$ is a Riemannian metric on $TM/G$. Also $J$ is unique up to homotopy. In the sequel, we will always equip $TM/G$ with such a complex structure.

Then we will apply the theorem of Riemann-Roch-Kawasaki [**32, 33**] the orbifold $M/G$ and the orbifold line bundle $\lambda^p$. Up to now, we have made $G$ act on $M$ or on $\lambda^p$ on the right. However to fit with the formalism of Sections 6.1 and 6.2 , we will now make $G$ act on the left by setting $gx = xg^{-1}$.

**6.4. The Theorem of Riemann-Roch-Kawasaki on the moduli space of flat bundles.** Recall that

$$M = \{x \in G^{2g} \times \prod_{j=1}^{s} \mathcal{O}_j \,, h(x) = 1\}\,. \tag{6.29}$$

We will use the notation

$$M = M(G, \mathcal{O}_1, \dots, \mathcal{O}_s)\,. \tag{6.30}$$

Let $\tau$ be projection $T = \mathfrak{t}/\overline{CR} \to T' = \mathfrak{t}/\overline{R}^*$.

THEOREM 6.12. *The following identity holds*

$$(H) = W\backslash C/\overline{CR}. \tag{6.31}$$

*Also if* $v \in C/\overline{CR}$, *if* $u = \tau v \in C/\overline{R}^{*}$, $M^v = M^u$. *Moreover*

$$M^u = \bigcup_{(w^1,\dots,w^s)\in(W_u\backslash W)^s} M(Z(u), \mathcal{O}_{Z(u)}(w^1t_1),\dots,\mathcal{O}_{Z(u)}(w^st_s)), \tag{6.32}$$

*and the union in (6.32) is disjoint.Finally, if* $v \in C/\overline{CR}$,

$$H_v = Z(Z(v)). \tag{6.33}$$

PROOF. Clearly, if $u \in G'$, then $ugu^{-1} = g$ if and only if $g \in Z(u)$. It is then clear that if $u \in G'$,

$$M^u = \{x \in Z(u)^{2g} \times \prod_{j=1}^{s} (\mathcal{O}_j \cap Z(u)),\ h(x) = 1\}. \tag{6.34}$$

Since $G$ acts locally freely on $M$, if $M^u \neq$, $Z(u)$ is semisimple. By Theorem 1.38, we get (6.31). If $u \in C/\overline{CR}$, using Theorem 1.50 and (6.34), we get (6.32). Finally by Theorem 5.20, we obtain (6.33). The proof of our Theorem is completed. □

REMARK 6.13. By Proposition 1.40, if $G = \mathrm{SU}(n)$, $n \geq 2$, if $u \in C$, then $u \in \overline{R}^{*}$. From Theorem 6.12, it follows that $G$ acts freely on $M$, so that $M/G$ is a smooth manifold.

Clearly $Z(G) = \overline{R}^{*}/\overline{CR} \subset T$ is fixed by $W$. Therefore if $v \in T$, $W_v$ depends only on $u = \tau v \in \mathfrak{t}/\overline{R}^{*}$. We will then write $W_u$ instead of $W_v$.

We use the notation of Section 1. In particular $\pi_u : \widetilde{Z}(u) \to Z(u)$ is the universal cover of $Z(u)$.

Let $u \in C/\overline{R}^{*}$, let $t \in T = \mathfrak{t}/\overline{CR}$ be regular. Then $Z(t) = T$, and $\mathcal{O}_{Z(u)}(t) \simeq Z(u)/T$.

Let $\widetilde{t} \in \mathfrak{t}/\overline{CR}_u$ be a lift of $t$ in $\widetilde{Z}(u)$. Then $\widetilde{t}$ is still regular in $\widetilde{Z}(u)$. Since $Z(t) = T$, the centralizer $Z(\widetilde{t})$ in $\widetilde{Z}(u)$ is just $Z(\widetilde{t}) = \mathfrak{t}/\overline{CR}_u$. Then $\mathcal{O}_{\widetilde{Z}(u)}(\widetilde{t}) \simeq \widetilde{Z}(u)\big/(\mathfrak{t}/\overline{CR}_u) \simeq Z(u)/T$. Equivalently the projection $\mathcal{O}_{\widetilde{Z}(u)}(\widetilde{t}) \to \mathcal{O}_{Z(u)}(t)$ is one to one.

Take $u \in C/\overline{R}^{*}$, $(w^1,\dots,w^s) \in W^s$. Recall that here $t_1,\dots,t_s$ are also considered as elements of $\mathfrak{t}$, so that $w^1t_1,\dots,w^st_s \in \mathfrak{t}$. Ultimately, we may consider $w^1t_1,\dots,w^st_s$ as element of $\widetilde{Z}(u)$. Then by the above, $\mathcal{O}_{\widetilde{Z}(u)}(w^1t_1),\dots,\mathcal{O}_{\widetilde{Z}(u)}(w^st_s)$ lift $\mathcal{O}_{Z(u)}(w^1t_1),\dots,\mathcal{O}_{Z(u)}(w^st_s)$, and the projection $\pi_u$ identifies the corresponding orbits.

Let $x = (u_1,v_1,\dots,u_g,v_g,w_1,\dots,w_s) \in M(Z(u),\mathcal{O}_{Z(u)}(w^1t_1),\dots,\mathcal{O}_{Z(u)}(w^st_s))$. Let $\widetilde{u}_1,\widetilde{v}_1,\dots,\widetilde{u}_g,\widetilde{v}_g \in \widetilde{Z}(u)$ be lifts of $u_1,v_1,\dots,u_g,v_g \in Z(u)$. Also $w_1,\dots,w_s \in \mathcal{O}_{Z(u)}(w^1t_1),\dots,\mathcal{O}_{Z(u)}(w^st_s)$ lift uniquely to $\widetilde{w}_1,\dots,\widetilde{w}_s \in \mathcal{O}_{\widetilde{Z}(u)}(w^1t_1),\dots,\mathcal{O}_{\widetilde{Z}(u)}(w^st_s)$.

PROPOSITION 6.14. *The element* $\prod_{i=1}^{g}[\widetilde{u}_i,\widetilde{v}_i]\prod_{j=1}^{s}\widetilde{w}_j \in \widetilde{Z}(u)$ *does not depend on* $\widetilde{u}_1,\dots,\widetilde{v}_g$. *It lies in* $\pi_1(Z(u)) = \overline{CR}/\overline{CR}_u$.

PROOF. Since $\pi_1(Z(u)) \subset Z(\widetilde{Z}(u))$, the first part of the Proposition is trivial. Also,

$$\pi_u \left[ \prod_{i=1}^{g} [\widetilde{u}_i, \widetilde{v}_i] \prod_{j=1}^{s} \widetilde{w}_j \right] = \prod_{i=1}^{g} [u_i, v_i] \prod_{j=1}^{s} w_j = 1 . \tag{6.35}$$

From (6.35), we get the second part of the Proposition. □

Using Proposition 6.14 , we can now define:

DEFINITION 6.15. If $u \in C/\overline{CR}$, $(w^1, \ldots, w^s) \in W^s$, $h \in \pi_1(Z(u)) = \overline{CR}/\overline{CR_u}$ put

$$\begin{aligned} &M(Z(u), \mathcal{O}_{Z(u)}(w^1 t_1), \ldots, \mathcal{O}_{Z(u)}(w^s t_s), h) = \\ &\Big\{ x = (u_1, \ldots, u_g, v_g, w_1, \ldots, w_s) \in M(Z(u), \mathcal{O}_{Z(u)}(w^1 t_1), \ldots, \mathcal{O}_{Z(u)}(w^s t_s)), \\ &\qquad \textstyle\prod_{i=1}^{g} [\widetilde{u}_i, \widetilde{v}_i] \prod_{j=1}^{s} \widetilde{w}_j h = 1 \Big\} . \end{aligned} \tag{6.36}$$

Clearly, we have the disjoint union

$$\begin{aligned} &M(Z(u), \mathcal{O}_{Z(u)}(w^1 t_1), \ldots, \mathcal{O}_{Z(u)}(w^s t_s)) = \\ &\textstyle\bigcup_{h \in \overline{CR}/\overline{CR}_u} M(Z(u), \mathcal{O}_{Z(u)}(w^1 t_1), \ldots, \mathcal{O}_{Z(u)}(w^s t_s), h) . \end{aligned} \tag{6.37}$$

Also since $\overline{CR}/\overline{CR}_u \subset Z(\widetilde{Z}(u))$, $Z'(u)$ preserves each $M(Z(u), \mathcal{O}_{Z(u)}(w^1 t_1), \ldots, \mathcal{O}_{Z(u)}(w^s t_s), h)$.

By Definition 6.10, if $p \in \mathbf{M}$, there is a well-defined $G$-orbifold line bundle $\lambda^p$ on $M/G$. Let $D_p$ be the corresponding Dirac operator acting on smooth sections of $\Lambda(T^{*(0,1)} M/G) \otimes \lambda^p$ over $M/G$.

THEOREM 6.16. *For $p \in \mathbf{M}$, the following identity holds*

$$\operatorname{Ind}(D_{p,+}) = \sum_{u \in C/\overline{CR}} \frac{|W_u|}{|W|} \frac{1}{|Z(Z(u))|} \int_{M^u/Z(u)} L(u, \lambda^p) . \tag{6.38}$$

PROOF. By Theorems 6.8 and 6.12,

$$\operatorname{Ind}(D_{p,+}) = \sum_{u \in W \backslash C/\overline{CR}} \frac{1}{|Z(Z(u))|} \int_{M^u/Z(u)} L(u, \lambda^p) \tag{6.39}$$

from which (6.38) follows. The proof of our Theorem is completed. □

**6.5. Evaluation of the Atiyah-Bott-Lefschetz Todd class on a stratum of the moduli space.** Now we take $u \in C/\overline{R}^*$, $x \in M^u$. Then $u$ descends to a flat section of the $G$ bundle $\widehat{G}$, which acts naturally on the left on the vector bundle $E$ as a flat section of $\operatorname{Aut}(E)$. In particular, on $M^u$, the vector bundle $\mathcal{E} \otimes_{\mathbf{R}} \mathbf{C}$ splits as a direct sum of vector bundles

$$\mathcal{E} \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{0<\theta<\pi} (\mathcal{E}^\theta \oplus \mathcal{E}^{-\theta}) \oplus \mathcal{E}^\pi . \tag{6.40}$$

In (6.40), for $-\pi < \theta \leq \pi$, $u$ acts on $\mathcal{E}^\theta$ like $e^{i\theta}$.

For $0 < \theta < \pi$, $\mathcal{E}^\theta$ is a complex vector bundle on $M^u/Z'(u)$. Let $\prod\Big(e^{\frac{x+i\theta}{2}} - e^{-\frac{x+i\theta}{2}}\Big)(\mathcal{E}^\theta$ be the corresponding characteristic class. Also $\mathcal{E}^\pi$ is a real vector bundle on $M^u/Z'(u)$. Moreover $2i\cosh(\frac{x}{2})$ is an even function of $x$. Let $\prod\Big(e^{\frac{x+i\pi}{2}} -$

$e^{-\frac{x+i\pi}{2}}\Big)(\mathcal{E}^\pi) = \prod 2i\cosh(\frac{x}{2})(\mathcal{E}^\pi)$ be the corresponding Pontryagin class of $\mathcal{E}^\pi$. Needless to say, if $\mathcal{E}^\pi$ has a complex structure , then this class is exactly $\prod 2i\cosh(\frac{x}{2})(\mathcal{E}^{\pi,(1,0)})$.

THEOREM 6.17. *For $u \in C/\overline{R}^*$, the following identity of characteristic classes holds on $M^u/Z'(u)$,*

$$\operatorname{Td}(T^{(1,0)}M^u/Z'(u)) = \widehat{A}^{2g-2+s}(\mathcal{E}^0)e^{\frac{1}{2}c_1(T^{(1,0)}M^u/Z'(u))}, \tag{6.41}$$

$$\widehat{\operatorname{Td}}(N^{u(1,0)}) = \left[\frac{1}{\prod_{0<\theta\leq\pi}\Pi\left(e^{\frac{x+i\theta}{2}} - e^{\frac{-(x+i\theta)}{2}}\right)(\mathcal{E}^\theta)}\right]^{2g-2+s}$$
$$e^{\frac{1}{2}c_1(N^{u(1,0)})}e^{\frac{i}{2}\sum_{\theta\in]-\pi,\pi]\setminus\{0\}}\theta\dim(N^{u(1,0),\theta})}(-1)^{\sum_{-\pi<\theta<0}\dim(N^{u(1,0),\theta})}.$$

PROOF. Using Theorem 5.34 and (6.11), we get the first identity in (6.41). Clearly

$$\widehat{\operatorname{Td}}(N^{u(1,0)}) = \left[\prod_{\theta\in]-\pi,\pi]\setminus\{0\}}\left(e^{\frac{1}{2}(x+i\theta)} - e^{-\frac{1}{2}(x+i\theta)}\right)(N^{u(1,0),\theta})\right]^{-1} \tag{6.42}$$
$$c^{\frac{1}{2}c_1(N^{u(1,0)})+\frac{i}{2}\sum_{\theta\in]-\pi,\pi]\setminus\{0\}}\theta\dim(N^{u(1,0),\theta})}.$$

Also, for $-\pi < \theta < 0$,

$$N^{u(1,0),\theta} = \overline{N^{u(0,1),-\theta}} \tag{6.43}$$

From (6.43), we get

$$\prod_{\theta\in]-\pi,\pi]\setminus\{0\}}\left(e^{\frac{1}{2}(x+i\theta)} - e^{-\frac{1}{2}(x+i\theta)}\right)(N^{u(1,0),\theta}) \tag{6.44}$$
$$= \prod_{0<\theta\leq\pi}\left(e^{\frac{1}{2}(x+i\theta)} - e^{-\frac{1}{2}(x+i\theta)}\right)(N^{u(1,0),\theta})$$
$$\prod_{0<\theta<\pi}\left(e^{-\frac{1}{2}(x+i\theta)} - e^{\frac{1}{2}(x+i\theta)}\right)(N^{u(0,1),\theta})$$
$$= \prod_{0<\theta<\pi}\left(e^{\frac{1}{2}(x+i\theta)} - e^{-\frac{1}{2}(x+i\theta)}\right)(N^{u,\theta})\prod(ie^{x/2} + ie^{-x/2})(N^{u(1,0),\pi})$$
$$(-1)^{\sum_{-\pi<\theta<0}\dim(N^{u(1,0),\theta})}.$$

Also one finds easily that over $M^u/Z(u)$, equality (5.139) of Theorem 5.34 can be split according to the values of $\theta$. So we find that for $0 < \theta < \pi$,

$$\Pi\left(e^{\frac{1}{2}(x+i\theta)} - e^{-\frac{1}{2}(x+i\theta)}\right)(N^{u,\theta}) \tag{6.45}$$
$$= \left[\Pi\left(e^{\frac{1}{2}(x+i\theta)} - e^{-\frac{1}{2}(x+i\theta)}\right)(\mathcal{E}^\theta)\right]^{2g-2+s}.$$

Also $2i\cosh(x/2)$ is an even function of $x$. Then

$$\prod(ie^{x/2} + ie^{-x/2})(N^{u(1,0),\pi}) = \prod(2i\cosh(x/2))(N^{u,\pi}) \tag{6.46}$$
$$= \left[\prod(2i\cosh(\tfrac{x}{2}))\right]^{2g-2+s}(\mathcal{E}^\pi) = \prod\left(e^{\frac{x+i\pi}{2}} - e^{-\frac{(x+i\pi)}{2}}\right)^{2g-2+s}(\mathcal{E}^\pi).$$

By (6.42)-(6.46), we get the second identity in (6.41). The proof of our Theorem is completed. □

**6.6. The dimensions of the splitting of the normal bundle to the strata.** Let $J$ be a $G$-invariant almost complex structure on $TM/G$ which polarizes the symplectic form $\omega$. We know that $J$ is unique up to homotopy.

Take $u \in C/\overline{R}^*$. Let $x \in M^u$. Then by (5.120),

$$TM/G = \widehat{H}^1(\Sigma, E)\,. \tag{6.47}$$

Recall that $u$ defines a flat section of the bundle $\widehat{G} \xrightarrow{G} \Sigma$. Clearly

$$\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = (\mathfrak{z}(u) \otimes_{\mathbf{R}} \mathbf{C}) \oplus (\bigoplus_{\alpha \in R \setminus R_u} \mathfrak{g}_\alpha)\,. \tag{6.48}$$

Let $x \in \mathbf{R} \mapsto [x]' \in ]-1/2, +1/2]$ be the function periodic of period 1, such that

$$[x]' = x \text{ for } x \in ]-1/2, +1/2]\,. \tag{6.49}$$

Clearly $u$ acts on the left on $\mathfrak{z}(u)$ like the identity and on $\mathfrak{g}_\alpha$ like $e^{2i\pi\langle\alpha,u\rangle}$. For $-\pi < \theta \leq \pi$, put

$$\mathfrak{g}^\theta = \bigoplus_{\substack{\alpha \in R \setminus R_u \\ [\langle\alpha,u\rangle]' = \frac{\theta}{2\pi}}} \mathfrak{g}^\alpha\,, \tag{6.50}$$

with the convention that if $\theta = 0$, $\mathfrak{g} = \mathfrak{z}(u)$.

Then (6.48) can be written as

$$\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{-\pi < \theta \leq \pi} \mathfrak{g}^\theta\,. \tag{6.51}$$

Also (6.51) is a $Z(u)$-invariant splitting. It induces a corresponding flat splitting of the flat bundle $E$ of the form

$$E = \bigoplus_{-\pi < \theta \leq \pi} E^\theta\,. \tag{6.52}$$

In (6.52), $E^0$ is just the analogue of $E$ when replacing $G$ by $Z(u)$.

By (6.52), we get

$$\widehat{H}^1(\Sigma, E) \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{-\pi < \theta \leq \pi} \widehat{H}^1(\Sigma, E^\theta)\,. \tag{6.53}$$

Also $TM/G = \widehat{H}^1(\Sigma, E)$, and $J$ acts on $\widehat{H}^1(\Sigma, E)$. Since $J$ is $G$-invariant, $J$ commutes with $u$. In particular $J$ acts on each $\widehat{H}^1(\Sigma, E^\theta)$. Let $\widehat{H}^{(1,0)}(\Sigma, E^\theta)$, $\widehat{H}^{(0,1)}(\Sigma, E^\theta)$ be the $+i, -i$ eigenspaces of $J$, so that

$$\widehat{H}^1(\Sigma, E^\theta) = \widehat{H}^{(1,0)}(\Sigma, E^\theta) \oplus \widehat{H}^{(0,1)}(\Sigma, E^\theta)\,. \tag{6.54}$$

Observe that since $\widehat{H}^1(\Sigma, E^\theta)$ is equipped with the $u$-invariant symplectic form $\omega$, then

$$\begin{aligned} [\widehat{H}^{(1,0)}(\Sigma, E^\theta)]^* &= \widehat{H}^{(0,1)}(\Sigma, E^\theta)\,, \text{ if } \theta = 0, \pi, \\ [\widehat{H}^{(1,0)}(\Sigma, E^\theta)]^* &= \widehat{H}^{(0,1)}(\Sigma, E^{-\theta})\,,\ \theta \in ]-\pi, \pi[\,. \end{aligned} \tag{6.55}$$

Since $J$ is unique up to homology, the dimensions of the vector spaces which appear in (6.55) do not depend on $J$.

Since $Z(u)$ is semisimple, for $-\pi < \theta \leq \pi$,

$$\sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=\frac{\theta}{2\pi}}} \alpha = 0\,. \tag{6.56}$$

Therefore for any $t \in \mathfrak{t}$,

$$\sum_{[\langle\alpha,u\rangle]'=\frac{\theta}{2\pi}} [\langle\alpha,t\rangle] \in \mathbf{Z}\,. \tag{6.57}$$

THEOREM 6.18. *For $-\pi < \theta \leq \pi$, over $M^u$,*

$$\dim \widehat{H}^{(0,1)}(\Sigma, E^\theta) = (g-1)\dim \mathfrak{g}^\theta + \sum_{j=1}^{s} \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=\frac{\theta}{2\pi}}} [\langle\alpha,t_j\rangle]\,. \tag{6.58}$$

PROOF. First assume that $m \in \mathbf{N}$, that $t_1,\ldots,t_s$ are of order $m$, and $m|s$. Then we make the construction in Section 5.12. In particular $u$ lifts to a $\mathbf{Z}/m\mathbf{Z}$-invariant parallel section over $\Sigma_g^b$. Then we have the $\mathbf{Z}/m\mathbf{Z}$ invariant splitting

$$H^1(\Sigma_g^b, E)\otimes_{\mathbf{R}} \mathbf{C} = H^{(1,0)}(\Sigma_g^b, E) \oplus H^{(0,1)}(\Sigma_g^b, E)\,. \tag{6.59}$$

By arguing as in the proof of Theorem 5.78, when taking the $\mathbf{Z}/m\mathbf{Z}$ invariant part of (6.59), we get a complex structure on $\widehat{H}^1(\Sigma, E)$ which polarizes $\sigma$. It is then feasible to take

$$\widehat{H}^{(0,1)}(\Sigma, E^\theta) = [H^{(0,1)}(\Sigma, E^\theta)]^{\mathbf{Z}/m\mathbf{Z}}\,. \tag{6.60}$$

By construction,

$$[\widehat{H}^0(\Sigma, E^\theta)] = 0\,. \tag{6.61}$$

Therefore

$$\dim[H^{0,1}(\Sigma, E^\theta)]^{\mathbf{Z}/m\mathbf{Z}} = -\chi^{h,\mathbf{Z}/m\mathbf{Z}}(\Sigma_g^b, p^*E^\theta)\,. \tag{6.62}$$

Using (5.321) and the theorem of Riemann-Roch-Kawasaki [**32, 33**] as in (5.349), (5.350), and also (6.62), we get (6.58).

Let us now consider the general case. As in the proof of Theorem 5.78, we approximate $(t_1,\ldots,t_s)$ by regular $(t_1^m,\ldots,t_s^m)$ which are of order $m$ and such that (A) holds. Recall that over $\Sigma_g^b$, we still have a $Z(u)$ bundle, so that we may and we will assume that at $x_j$ $(j \geq s+1)$, we have in fact a $Z(u)$ connection. Since $Z(u)$ is semisimple, for $j \geq s+1$,

$$\mathrm{Tr}^{E_\theta}[\Theta_j] = 0\,. \tag{6.63}$$

Again the extra points $x_{s+1},\ldots,x_{s_m}$ do not contribute to the computation, so that (6.58) still holds. Since for $1 \leq j \leq s$, $[\langle\alpha,t_j^m\rangle] \to [\langle\alpha,t_j\rangle]$, we get (6.58) in full generality.

The proof of our Theorem is completed. □

THEOREM 6.19. *For $\theta \in ]-\pi,\pi]$,*

$$\dim \widehat{H}^{(1,0)}(\Sigma, E^\theta) = (g-1)\dim \mathfrak{g}^\theta + \sum_{j=1}^{s} \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=\frac{\theta}{2\pi}}} (1-[\langle\alpha,t_j\rangle])\,. \tag{6.64}$$

PROOF. By (6.55), for $\theta \in ]-\pi,\pi[$,

$$\dim H^{1,0}(\Sigma, E^{\theta}) = \dim H^{0,1}(\Sigma, E^{-\theta}). \tag{6.65}$$

So by Theorem 6.18, we get

$$\dim \widehat{H}^{1,0}(\Sigma, E^{\theta}) = (g-1)\dim \mathfrak{g}^{-\theta} + \sum_{j=1}^{s} \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]' = \frac{-\theta}{2\pi}}} [\langle\alpha, t_j\rangle]. \tag{6.66}$$

Now for $\theta \in ]-\pi,\pi[$,

$$\dim \mathfrak{g}^{\theta} = \dim \mathfrak{g}^{-\theta}. \tag{6.67}$$

Also since for $1 \le j \le s$, $\alpha \in R$, then $\langle\alpha, t_j\rangle \notin \mathbf{Z}$,

$$\sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]' = \frac{-\theta}{2\pi}}} [\langle\alpha, t_j\rangle] = \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]' = \frac{\theta}{2\pi}}} [\langle-\alpha, t_j\rangle] = \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]' = \frac{\theta}{2\pi}}} (1-[\langle\alpha, t_j\rangle]). \tag{6.68}$$

From (6.66), (6.68), we get (6.64) when $\theta \in ]-\pi,\pi[$.

Also $E^{\pi}$ is a real vector bundle. Then

$$\dim \widehat{H}^{(1,0)}(\Sigma, E^{\pi}) = \dim \widehat{H}^{(0,1)}(\Sigma, E^{\pi}). \tag{6.69}$$

By Theorem 6.18, we get

$$\dim H^{(1,0)}(\Sigma, E^{\pi}) = (g-1)\dim \mathfrak{g}^{\pi} + \sum_{j=1}^{s} \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]' = \frac{1}{2}}} [\langle\alpha, t_j\rangle]. \tag{6.70}$$

Observe that if $\alpha \in R$, then $[\langle\alpha,u\rangle]' = \frac{1}{2}$ if and only if $[-\langle\alpha,u\rangle]' = \frac{1}{2}$. Also when changing $\alpha$ into $-\alpha$, $[\langle\alpha,t_j\rangle]$ is changed into $1-[\langle\alpha,t_j\rangle]$. So for $\theta = \pi$, we find that (6.70) is still equivalent to (6.64).

The proof of our Theorem is completed. □

THEOREM 6.20. *For $u \in C/\overline{R}^{*}$, $\theta$, $\theta \in ]-\pi,\pi]\backslash\{0\}$,*

$$\dim N^{u(1,0),\theta} = (g-1)\dim \mathfrak{g}^{\theta} + \sum_{j=1}^{s} \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]' = \frac{\theta}{2\pi}}} (1-[\langle\alpha, t_j\rangle]). \tag{6.71}$$

PROOF. Recall that

$$TM/G = \widehat{H}^{1}(\Sigma, E). \tag{6.72}$$

Also (6.72) is an identification of $u$-spaces. Then our Theorem is just another formulation of Theorem 6.19. □

**6.7. The contribution of a stratum of the moduli space.** Recall that by Theorem 1.41,

$$h \in \overline{CR}/\overline{CR}_u \mapsto \exp(2i\pi\langle\rho_u, h\rangle) \in S_1 \tag{6.73}$$

is a character with values in $\pm 1$.

By Theorem 5.18 and by Proposition 5.63, for $g \geq 1$, $M(Z(u), \mathcal{O}_{Z(u)}(w^1t_1), \ldots, \mathcal{O}_{Z(u)}(w^st_s), h)$ is a non empty smooth manifold, and for $g = 0$, $M(Z(u), \mathcal{O}_{Z(u)}(w^1t_1), \ldots, \mathcal{O}_{Z(u)}(w^st_s), h)$ is either non empty and smooth, or it is empty.

In the sequel, it will be understood that any geometric statement about the empty set is empty.

Recall that by [**34**], [**6**, Lemma 7.22] and (1.193), the orbits $\mathcal{O}(w^jt_j)$ carry a natural complex structure, which polarizes the canonical symplectic form $\sigma_{\mathcal{O}(w^jt_j)}$ on $\mathcal{O}(w^jt_j)$. Let $N^{(1,0)}_{\mathcal{O}_{Z(u)}(w^jt_j)/\mathcal{O}(w^jt_j)}$ be the holomorphic normal bundle to $\mathcal{O}_{Z(u)}(w^jt_j)$ in $\mathcal{O}(w^jt_j)$.

As we saw after (5.305), $H_{u,g,s}(t_1, \ldots, t_s, t)$ is locally a polynomial of $(t_1, \ldots, t_s, t)$.

THEOREM 6.21. *For $u \in C/\overline{CR}$, $p \in \mathbf{M}, p \neq -c$, $(w^1, \ldots, w^s) \in W^s$, $h \in \frac{\overline{CR}}{\overline{CR}_u}$,*

$$\begin{aligned}
&\int_{M(Z(u),\mathcal{O}_{Z(u)}(w^1t_1),\ldots,\mathcal{O}_{Z(u)}(w^st^s),h)/Z(u)} L(u,\lambda^p) = (p+c)^{(g-1)\dim\mathfrak{z}(u)+\frac{s}{2}\dim\mathfrak{z}(u)/t} \\
&\frac{(-1)^{\frac{(g-1)}{2}(\dim(\mathfrak{g})-\dim(\mathfrak{z}(u))}}{|W_u|} \prod_{j=1}^{s} e^{i\pi\sum_{\alpha\in R_+}[\langle w^j\alpha,u\rangle]+2i\pi\langle w^jpt_j+ph,u\rangle} \\
&\Bigg[\prod_{\alpha\in R_{u,+}} \widehat{A}\left(\langle\alpha, \frac{\partial/\partial t}{p+c}\rangle\right) \\
&\prod_{\alpha\in R_+\setminus R_{u,+}} \frac{1}{2\sinh\left(\frac{1}{2}(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle + 2i\pi[\langle\alpha,u\rangle])\right)}\Bigg]^{2g-2+s} \\
&\prod_{j=1}^{s} e^{\frac{1}{p+c}\langle(\rho-ct_j),\partial/\partial t_j\rangle}\pi_u\left(\frac{\partial/\partial t}{2\pi i}\right)\frac{1}{|\frac{\overline{CR}}{\overline{CR}_u}|^{2g}}H_{u,g,s}(w^1t_1,\ldots,w^st_s,h,t)\,.
\end{aligned}\tag{6.74}$$

*Also*

$$\begin{aligned}
&\left\{\widehat{\mathrm{Td}}(N^{u(1,0)})_{|M(Z(u),\mathcal{O}_{Z(u)}(w^1t_1),\ldots,\mathcal{O}_{Z(u)}(w^st_s))/Z'(u)}\right\}^{(0)} = \\
&\prod_{\alpha\in R_+\setminus R_{u,+}}\left(\frac{1}{4\sin^2(\pi\langle\alpha,u\rangle)}\right)^{g-1}\prod_{j=1}^{s}\frac{1}{\det(1-u^{-1})_{|N^{(1,0)}_{\mathcal{O}_{Z(u)}(w^jt_j)/\mathcal{O}(w^jt_j)_{|w^jt_j}}}}\,.
\end{aligned}\tag{6.75}$$

PROOF. First we consider the case where $g = 0$, and $M(Z(u), \mathcal{O}_{Z(u)}(w^1t_1), \ldots, \mathcal{O}_{Z(u)}(w^st_s), h) = \emptyset$. By definition, the left-hand side of (6.74) is 0. Also for $t \in T$ close enough to 1, $M_t(Z(u), \mathcal{O}_{Z(u)}(w^1t_1, \ldots, \mathcal{O}_{Z(u)}(w^st_s), h)$ is also empty. By (5.305), it follows that $H_{u,g,s}(t_1, \ldots, t_s, t)$ vanishes identically near $t = 1$. Then (6.74) is just the identity $0 = 0$.

Now we assume that $M(Z(u), \mathcal{O}_{Z(u)}(w^1t_1), \ldots, \mathcal{O}_{Z(u)}(w^st_s), h)$ is non empty. If $w = (w^1, \ldots, w^s) \in W^s$, put

$$M^u_{w,h} = M(Z(u), \mathcal{O}_{Z(u)}(w^1t_1), \ldots, \mathcal{O}_{Z(u)}(w^st_s), h)\,. \tag{6.76}$$

We use the notation of Section 4.7. If $x \in M^u_{w,h}$, the corresponding flat connection reduces the $G$-bundle $P$ to a $Z(u)$-bundle we still denote by $P$. This $Z(u)$-bundle lifts to a $\widetilde{Z}(u)$-bundle $Q$. By Proposition 4.33, and by (6.36),

$$[P] = e^h. \tag{6.77}$$

By Theorem 4.39, the right action of $u$ on $\lambda^p$ over $M^u_{w,h}$ is given by $e^{-2i\pi p\langle\sum_{j=1}^s w^j t_j + h, u\rangle}$. The corresponding left action is then given by $e^{2i\pi p\langle\sum_{j=1}^s w^j t_j + h, u\rangle}$. By (6.41) , and by the above result,we get

$$\begin{aligned}\int_{M^u_{w,h}/Z(u)} L(u,\lambda^p) = \int_{M^u_{w,h}/Z(u)} &\left[\widehat{A}(\mathcal{E}^0)\frac{1}{\prod\limits_{0<\theta\leq\pi} \Pi\left(e^{\frac{x+i\theta}{2}} - e^{-\frac{(x+i\theta)}{2}}\right)(\mathcal{E}^\theta)}\right]^{2g-2+s} \\ &e^{\frac{1}{2}c_1(T^{1,0}M/G)+p\omega}e^{\frac{i}{2}\sum_{\theta\in]-\pi,\pi]\setminus\{0\}}\theta\dim(N^{u(1,0),\theta})} \\ &(-1)^{\sum_{-\pi<\theta<0}\dim(N^{u(1,0),\theta})}\prod_{j=1}^s e^{2i\pi\langle w^j p t_j, u\rangle}e^{2i\pi\langle pu,h\rangle}.\end{aligned} \tag{6.78}$$

Now we use Theorem 5.78, which gives us a formula for $c_1(TM/G)$. By Proposition 5.73 and by (6.78),

$$\begin{aligned}\int_{M^u_{w,h}/Z(u)} L(u,\lambda^p) = {}& \frac{1}{|W_u|}e^{\frac{i}{2}\sum_{\theta\in]-\pi,\pi]\setminus\{0\}}\theta\dim(N^{u(1,0),\theta})} \\ &(-1)^{\sum_{-\pi<\theta<0}\dim(N^{u(1,0),\theta})}(p+c)^{(g-1)\dim\mathfrak{z}(u)+\frac{s}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})}\left[\prod_{\alpha\in R_{u,+}}\widehat{A}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle\right)\right. \\ &\left.\prod_{\substack{\alpha\in R\\ 0<[\langle\alpha,u\rangle]<\frac{1}{2}}}\frac{1}{2\sinh\left(\frac{1}{2}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle+2i\pi[\langle\alpha,u\rangle]\right)\right)}\prod_{\substack{\alpha\in R_+\\ [\langle\alpha,u\rangle]=\frac{1}{2}}}\frac{1}{2i\cosh\left(\frac{1}{2}\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle\right)}\right]^{2g-2+s} \\ &\prod_{j=1}^s e^{\frac{1}{p+c}\langle\rho-ct_j,\partial/\partial t_j\rangle}\pi_u\left(\frac{\partial/\partial t}{2\pi i}\right)\frac{1}{\left|\frac{CR}{CR_u}\right|^{2g}}H_{u,g,s}(w^1t_1,\dots,w^st_s,h,t)\prod_{j=1}^s e^{2i\pi\langle w^jpt_j,u\rangle}e^{2i\pi\langle ph,u\rangle}.\end{aligned} \tag{6.79}$$

Observe that if $\alpha \in R\setminus R_u$, when changing $\alpha$ into $-\alpha$, $2\sinh\left(\frac{1}{2}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle + 2i\pi[\langle\alpha,u\rangle]\right)\right)$ is unchanged. Therefore

$$\begin{aligned}\prod_{\substack{\alpha\in R\\ 0<[\langle\alpha,u\rangle]<\frac{1}{2}}}\frac{1}{2\sinh\left(\frac{1}{2}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle+2i\pi[\langle\alpha,u\rangle]\right)\right)}&\prod_{\substack{\alpha\in R_+\\ [\langle\alpha,u\rangle]=\frac{1}{2}}} \\ \frac{1}{2i\cosh\frac{1}{2}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle\right)} &= \prod_{\alpha\in R_+\setminus R_{u,+}}\frac{1}{2\sinh\left(\frac{1}{2}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle+2i\pi[\langle\alpha,u\rangle]\right)\right)}.\end{aligned} \tag{6.80}$$

By Theorem 6.20,

$$\begin{aligned}\exp(i/2\sum_{\theta\in]-\pi,\pi]\setminus\{0\}}\theta\dim(N^{u(1,0),\theta})) &= \\ \exp\left(i/2\left(\sum_{\theta\in]-\pi,\pi]}\theta(\dim\mathfrak{g}^\theta(g-1)+\sum_{j=1}^s\sum_{\substack{\alpha\in R\\ [\langle\alpha,u\rangle]'=\theta/2\pi}}(1-[\langle\alpha,w^jt_j\rangle]))\right)\right)&.\end{aligned} \tag{6.81}$$

Now the $\theta \in ]-\pi,\pi[\backslash\{0\}$ come by opposite pairs. Therefore

$$\sum_{\theta\in]-\pi,\pi]\backslash\{0\}} \theta \dim \mathfrak{g}^{\theta} = \pi \dim \mathfrak{g}^{\pi}\,, \tag{6.82}$$

and so

$$e^{\frac{i}{2}\Sigma\theta \dim \mathfrak{g}^{\theta}(g-1)} = (-1)^{\frac{\dim \mathfrak{g}^{\pi}}{2}(g-1)}\,. \tag{6.83}$$

For $1 \leq j \leq s$, put

$$\begin{aligned} t'_j &= w^j t_j \\ R_{j,+} &= w^j R_+ \end{aligned} \tag{6.84}$$

Then $R_{j,+}$ is the positive root system associated to $t'_j$. For $1 \leq j \leq s$,

$$\begin{aligned} &\textstyle\sum_{s\in]-1/2,1/2]} s \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=s}} (1-[\langle\alpha,t'_j\rangle]) \\ &= \textstyle\sum_{\alpha\in R}[\langle\alpha,u\rangle]'(1-[\langle\alpha,t'_j\rangle]) \\ &= \textstyle\sum_{\alpha\in R_{j,+}} \left([\langle\alpha,u\rangle]'(1-\langle\alpha,t'_j\rangle) + [-\langle\alpha,u\rangle]'\langle\alpha,t'_j\rangle\right) \\ &= -\textstyle\sum_{\alpha\in R}[\langle\alpha,u\rangle]'\langle\alpha,t'_j\rangle + \sum_{\alpha\in R_{j,+}}[\langle\alpha,u\rangle]'\,. \end{aligned} \tag{6.85}$$

Clearly,

$$\sum_{\alpha\in R}[\langle\alpha,u\rangle]'\langle\alpha,t'_j\rangle = \sum_s s\langle \sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=s}} \alpha, t'_j\rangle\,. \tag{6.86}$$

Now for a given $s$, the $\{\alpha \in R, [\langle\alpha,u\rangle]' = s\}$ are exactly the weights of the representation of $Z(u)$ on $\{f \in \mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}, uf = e^{2i\pi s}f\}$. Since $Z(u)$ is semisimple,

$$\sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=s}} \alpha = 0\,. \tag{6.87}$$

From (6.81)-(6.87) , we get

(6.88)

$$\exp\Big(\frac{i}{2}\sum_{\theta\in]-\pi,\pi]\backslash\{0\}} \theta \dim(N^{u(1,0),\theta})\Big) = (-1)^{\frac{\dim \mathfrak{g}^{\pi}}{2}(g-1)} \prod_{j=1}^{s} \exp\Big(i\pi \sum_{\alpha\in R_+} [\langle w^j\alpha,u\rangle]'\Big)\,.$$

Also by Theorem 6.20, using (6.87) , we get

$$\begin{aligned} &\textstyle\sum_{-\pi<\theta<0} \dim N^{u(1,0),\theta} \\ &= (g-1)\textstyle\sum_{-\pi<\theta<0} \dim \mathfrak{g}^{\theta} + \sum_{j=1}^{s}\sum_{\substack{\alpha\in R \\ -1/2<[\langle\alpha,u\rangle]'<0}} (1-[\langle\alpha,t'_j\rangle]) \\ &= (g-1)|\{\alpha\in R, -\tfrac{1}{2} < [\langle\alpha,u\rangle]' < 0\}| \\ &+\textstyle\sum_{j=1}^{s}\Big(-\sum_{-1/2<s<0}\langle\sum_{\substack{\alpha\in R \\ [\langle\alpha,u\rangle]'=s}} \alpha, t'_j\rangle + |\{\alpha\in R, -\tfrac{1}{2} < [\langle\alpha,u\rangle]' < 0, \langle\alpha,t'_j\rangle > 0\}|\Big) \\ &= (g-1)|\{\alpha\in R, -\tfrac{1}{2} < [\langle\alpha,u\rangle]' < 0\}| \\ &+\textstyle\sum_{j=1}^{s} |\{\alpha\in R, -\tfrac{1}{2} < [\langle\alpha,u\rangle]' < 0, \langle\alpha,t'_j\rangle > 0\}| \\ &= (g-1)|\{\alpha\in R, -\tfrac{1}{2} < [\langle\alpha,u\rangle]' < 0\}| + \textstyle\sum_{j=1}^{s} |\{\alpha\in R_+, -\tfrac{1}{2} < [\langle w^j\alpha,u\rangle]' < 0\}\,. \end{aligned} \tag{6.89}$$

So by (6.88),(6.89),

$$\exp\Big(\frac{i}{2}\sum_{\theta\in]-\pi,\pi]\setminus\{0\}}\theta\dim(N^{u(1,0),\theta})\Big)(-1)^{\sum_{-\pi<\theta<0}\dim(N^{u,(1,0),\theta})} = (-1)^{(g-1)(|\{\alpha\in R, 0<[\langle\alpha,u\rangle]'<\frac{1}{2}\}|+\frac{1}{2}|\{\alpha\in R,0<[\langle\alpha,u\rangle]'=\frac{1}{2}\}|)} \prod_{j=1}^{s}\exp\Big(i\pi\sum_{\alpha\in R_+}[\langle w^j\alpha,u\rangle]\Big). \tag{6.90}$$

Moreover

$$|\{\alpha\in R, 0<[\langle\alpha,u\rangle]'<\frac{1}{2}\}|+\frac{1}{2}|\{\alpha\in R,[\langle\alpha,u\rangle]'=\frac{1}{2}\}| = \frac{1}{2}(\dim(\mathfrak{g})-\dim\mathfrak{z}(u)). \tag{6.91}$$

From (6.79), (6.80), (6.90), (6.91) , we get (6.74).

Using (6.74) , or by proceeding directly, we get

$$\Big\{\widehat{\mathrm{Td}}(N^{u(1,0)})_{|M^u_{w,h}/Z'(u)}\Big\}^{(0)} = (-1)^{(g-1)(\dim\mathfrak{g}-\dim\mathfrak{z}(u))/2}\prod_{j=1}^{s}\exp\Big(i\pi\sum_{\alpha\in R_+}[\langle w^j\alpha,u\rangle]\Big) \Big(\prod_{\alpha\in R_+\setminus R_{u,+}}\frac{1}{2\sinh(i\pi[\langle\alpha,u\rangle])}\Big)^{2g-2+s}. \tag{6.92}$$

Using the invariance of $\sinh(i\pi[\langle\alpha,u\rangle])$ when $\alpha\in R\setminus R_u$ is changed into $-\alpha$, from (6.92), we get

$$\begin{aligned}\Big\{\widehat{\mathrm{Td}}(N^{u(1,0)})_{|M^u_{w,h}/Z'(u)}\Big\}^0 &= \Big(\prod_{\alpha\in R_+\setminus R_{u,+}}\frac{1}{4\sin^2(\pi\langle\alpha,u\rangle)}\Big)^{g-1} \prod_{j=1}^{s}\prod_{\substack{\alpha\in R_+\\ \alpha\notin w_j^{-1}R_u}}\frac{e^{i\pi[\langle\alpha,(w^j)^{-1}u\rangle]}}{2\sinh(i\pi[\langle\alpha,(w^j)^{-1}u\rangle])} = \\ &= \prod_{\alpha\in R_+\setminus R_{u,+}}\Big[\frac{1}{4\sin^2(\pi\langle\alpha,u\rangle)}\Big]^{g-1}\prod_{j=1}^{s}\prod_{\substack{\alpha\in w_jR_+,\\ \alpha\notin R_u}}\frac{1}{1-e^{-2i\pi\langle\alpha,u\rangle}} = \\ &= \prod_{\alpha\in R_+\setminus R_{u,+}}\Big[\frac{1}{4\sin^2(\pi\langle\alpha,u\rangle)}\Big]^{g-1}\prod_{j=1}^{s}\frac{1}{\det(1-u^{-1})_{|N^{(1,0)}_{O_{Z(u)}(w^jt_j)/O(w^jt_j)}|w^jt_j}},\end{aligned} \tag{6.93}$$

which is just (6.75). The proof of our Theorem is completed. □

Recall that

$$\begin{aligned}\ell &= |R_+| \\ \ell_u &= |R_{u,+}|, \\ r &= \dim\mathfrak{t}.\end{aligned} \tag{6.94}$$

Our assumptions on $g,s$ guarantee that $2g-2+s\geq 1$. Also by (2.100) and by Theorem 2.45, the function $P_{u,p+c,2g-2+s}(t)$ is a polynomial on $T\setminus S$.

THEOREM 6.22. *For $u \in C/\overline{CR}$, $p \in \mathbf{M}, p \neq -c$, then*

$$\int_{M^u/Z(u)} L(u,\lambda^p) = \mathrm{Vol}(T)^{2g-2}\frac{|Z(Z(u))|}{|W_u|}(p+c)^{(g-1)\dim(\mathfrak{z}(u))+\frac{s}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})} \tag{6.95}$$
$$(-1)^{\ell(g-1)+1}\prod_{\alpha\in R_{u,+}}$$
$$\left[\widehat{A}\left(\langle\alpha,\frac{\partial/\partial t\rangle}{p+c}\right)\prod_{\alpha\in R_+\setminus R_{u,+}}\frac{1}{2\sinh\left(\frac{1}{2}\left(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle+2i\pi[\langle\alpha,u\rangle]\right)\right)}\right]^{2g-2+s}$$
$$\sum_{(w^1,\dots,w^s)\in W^s}\prod_{j=1}^s\left[\mathrm{sgn}((-i)^{\ell_u}\sigma_{Z(u)}(w^jt_j))e^{\langle w^j(\rho-ct_j),\frac{\partial/\partial t}{p+c}\rangle+}\right.$$
$$\left.+i\pi\sum_{\alpha\in R_+}[\langle w^j\alpha,u\rangle]+2i\pi\langle w^jpt_j,u\rangle\right]P_{u,2g-2+s,p+c}(t)_{|t=\sum_1^s w^jt_j}$$

PROOF. By (6.32), (6.37),

$$\int_{M^u/Z(u)} L(u,\lambda^p) = \tag{6.96}$$
$$\sum_{\substack{(w^1,\dots,w^s)\in(W_u\setminus W)^s\\ h\in\overline{CR}/\overline{CR_u}}}\int_{M(Z(u),\mathcal{O}_{Z(u)}(w^1t_1),\dots,\mathcal{O}_{Z(u)}(w^st_s),h)/Z(u)} L(u,\lambda^p)\,.$$

We use Theorem 6.21 to calculate the term in the right-hand side of (6.96) . Also by (2.131) and by Theorem 5.72,

$$\pi_u\left(\frac{\partial/\partial t}{2i\pi}\right)H_{u,g,s}(w^1t_1,\dots,w^st_s,h,t) = (-1)^{\ell_u(g-1)+1} \tag{6.97}$$
$$|Z(\widetilde{Z}(u))||\mathrm{Vol}(\mathfrak{t}/\overline{CR_u})|^{2g-2}\sum_{(w'^1,\dots,w'^s)\in W_u^s}$$
$$\prod_{j=1}^s\varepsilon_{w'^j}\mathrm{sgn}((-i)^{\ell_u}\sigma_{Z(u)}(w^jt_j))e^{-2i\pi\langle\rho_u,h\rangle}\widetilde{P}_{u,2g-2+s}(t+h+\sum_{j=1}^s w'^jw^jt_j)\,.$$

Clearly

$$Z(Z(u)) = \frac{Z(\widetilde{Z}(u))}{\overline{CR}/\overline{CR_u}}, \tag{6.98}$$

and so

$$|Z(\widetilde{Z}(u))| = |Z(Z(u)|\left|\frac{\overline{CR}}{\overline{CR_u}}\right|. \tag{6.99}$$

Moreover

$$\mathrm{Vol}(\mathfrak{t}/\overline{CR_u}) = \mathrm{Vol}(\mathfrak{t}/\overline{CR})\left|\frac{\overline{CR}}{\overline{CR_u}}\right| = \mathrm{Vol}(T)\left|\frac{\overline{CR}}{\overline{CR_u}}\right|. \tag{6.100}$$

By (6.97)-(6.100) , we get

$$\pi_u\left(\frac{\partial/\partial t}{2i\pi}\right)\frac{H_{u,g,s}}{\left|\frac{\overline{CR}}{\overline{CR_u}}\right|^{2g}}(w^1t_1,\dots,w^st_s,h,t) = (-1)^{\ell_u(g-1)+1} \tag{6.101}$$
$$|Z(Z(u))|\mathrm{Vol}(T)^{2g-2}\sum_{(w'^1,\dots,w'^s)\in W^s(u)}\prod_{j=1}^s\varepsilon_{w'^j}((-i)^{\ell u}\sigma_{Z(u)})(w^jt_j)$$
$$\exp(-2i\pi\langle\rho_u,h\rangle)\frac{\widetilde{P}_{u,2g-2+s}}{\left|\frac{\overline{CR}}{\overline{CR_u}}\right|}(t+h+\sum_{j=1}^s w'^jw^jt_j)\,.$$

Also by Theorem 1.41 , for $h \in \overline{CR}/\overline{CR_u}$,

$$\exp(-2i\pi\langle\rho_u,h\rangle) = \exp(2i\pi\langle cu,h\rangle)\,. \tag{6.102}$$

Now observe that if $w'^j \in W_u$, by (1.45),

$$\varepsilon_{w'^j} = \frac{\sigma_{Z(u)}(w'^j w^j t_j)}{\sigma_{Z(u)}(w^j t_j)}, 1 \leq j \leq s. \tag{6.103}$$

Therefore

$$\operatorname{sgn}\left((-i)^{\ell_u}\sigma_{Z(u)}(w^j t_j)\right)\varepsilon_{w'^j} = \operatorname{sgn}\left((-i)^{\ell_u}\sigma_{Z(u)}(w'^j w^j t_j)\right). \tag{6.104}$$

Also

$$\ell_u + (\dim(\mathfrak{g}) - \dim(\mathfrak{z}(u)))/2 = \ell. \tag{6.105}$$

From (2.166), (6.74), (6.96), (6.101)-(6.105), we get (6.95). The proof of our Theorem is completed. □

REMARK 6.23. As shown in Section 2.11, the function $\exp(2i\pi\langle qu, t\rangle)P_{u,2g-2+s,p+c}(t)$ descends to a function which is well-defined on $T = \mathfrak{t}/\overline{CR}$. Ultimately, this explains why equation (6.95) is unambiguous.

Recall that the function $\sigma(t) = \prod_{i=1}^{\ell}\left(e^{i\pi\langle\alpha,t\rangle} - e^{-i\pi\langle\alpha,t\rangle}\right)$ is well-defined on $T = \mathfrak{t}/\overline{CR}$, and the function $\dfrac{e^{2i\pi\langle\rho,t\rangle}}{\sigma(t)}$ is well-defined on $T' = \mathfrak{t}/\overline{R}^*$.

PROPOSITION 6.24. *For any $u \in C/\overline{R}^*$, for any $s \in \mathfrak{t}_{\text{reg}}$, $x \in P$, $w \in W$,*

$$\operatorname{sgn}((-i)\sigma_{Z(u)}(wx)) \prod_{\alpha\in R_+} \frac{1}{2\sinh(\frac{1}{2}(\langle\alpha, s\rangle + 2i\pi[\langle\alpha, u\rangle]))} \tag{6.106}$$
$$\exp(i\pi\sum_{\alpha\in R_+}[\langle w\alpha, u\rangle]) = \varepsilon_w \prod_{\alpha\in R_+} \frac{1}{2\sinh(\frac{1}{2}\langle\alpha, s + 2i\pi u\rangle)} e^{2i\pi\langle w\rho, u\rangle}.$$

PROOF. As we just saw, both sides are unchanged when replacing $u$ by $u + \gamma$, $\gamma \in \overline{CR}$. By [**15**, Proposition V.7.10], we may as well assume that for any $\alpha \in R$,

$$|\langle\alpha, u\rangle| \leq 1. \tag{6.107}$$

If $\alpha \in R \backslash R_u$, $\sinh(\frac{1}{2}(\langle\alpha, s\rangle + 2i\pi[\langle\alpha, u\rangle]))$ is unchanged when $\alpha$ is replaced by $-\alpha$. Therefore using (1.45), we get

$$\begin{aligned}
&\textstyle\prod_{\alpha\in R_+} \frac{1}{2\sinh(\frac{1}{2}(\langle\alpha,s\rangle+2i\pi[\langle\alpha,u\rangle]))} = (-1)^{|R_+\cap w^{-1}(-R_{u,+})|} \\
&\textstyle\prod_{\alpha\in wR_+} \frac{1}{2\sinh(\frac{1}{2}(\langle\alpha,s\rangle+2i\pi[\langle\alpha,u\rangle]))} = \\
&(-1)^{|R_+\cap w^{-1}(-R_{u,+})|+|\{\alpha\in R_+,-1<\langle w\alpha,u\rangle<0\}|+|\{\alpha\in R_+,\langle w\alpha,u\rangle=\pm1\}|} \\
&\textstyle\prod_{\alpha\in R_+} \frac{1}{2\sinh(\frac{1}{2}\langle\alpha,w^{-1}(s+2i\pi u)\rangle)} = \\
&\varepsilon_w(-1)^{|R_+\cap w^{-1}(-R_{u,+})|+|\{\alpha\in R_+,-1<\langle w\alpha,u\rangle<0\}|+|\{\alpha\in R_+,\langle w\alpha,u\rangle=\pm1\}|} \\
&\textstyle\prod_{\alpha\in R_+} \frac{1}{2\sinh(\frac{1}{2}\langle\alpha,s+2i\pi u\rangle)}.
\end{aligned} \tag{6.108}$$

Moreover, for $x \in P$,

$$\operatorname{sgn}((-i)^{\ell_u}\sigma_{Z(u)}(wx)) = (-1)^{|R_+\cap w^{-1}(-R_{u,+})|}. \tag{6.109}$$

Also by (6.107),

$$(6.110)\quad \exp(i\pi\sum_{\alpha\in R_+}[\langle w\alpha,u\rangle]) = \exp(i\pi\langle w\sum_{\alpha\in R_+}\alpha,u\rangle)(-1)^{|\{\alpha\in R_+,-1<\langle w\alpha,u\rangle<0\}|+|\{\alpha\in R_+,\langle w\alpha,u\rangle=\pm1\}|} = e^{2i\pi\langle w\rho,u\rangle}(-1)^{|\{\alpha\in R_+,-1<\langle w\alpha,u\rangle<0\}|+|\{\alpha\in R_+,\langle w\alpha,u\rangle=\pm1\}|}.$$

From (6.108)-(6.110) , we get (6.106). The proof of our Proposition is completed. □

REMARK 6.25. It is clear that the right-hand side of (6.106) only depends on the class of $u$ in $C/\overline{R}^*$.

Observe that the function

$$(6.111)\qquad s\in\mathfrak{t}\mapsto\frac{\prod_{\alpha\in R_{u,+}}\langle\alpha,s\rangle e^{2i\pi\langle w\rho,u\rangle}}{\prod_{\alpha\in R_+}2\sinh(\frac{1}{2}\langle\alpha,s+2i\pi u\rangle)}\in\mathbf{C}$$

is a well-defined holomorphic function near $s=0$.

THEOREM 6.26. *For any* $u\in C/\overline{CR}, p\in\mathbf{M}, p\neq -c$, *the following identity holds,*

$$(6.112)\quad \int_{M^u/Z(u)}L(u,\lambda^p)=\mathrm{Vol}(T)^{2g-2}\frac{|Z(Z(u))|}{|W_u|}(p+c)^{(g-1)\dim(\mathfrak{z}(u))+\frac{s}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})}(-1)^{\ell(g-1)+1}\left[\frac{\prod_{\alpha\in R_{u,+}}\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle}{\prod_{\alpha\in R_+}2\sinh\left(\frac{1}{2}\langle\alpha,\frac{\partial/\partial t}{p+c}+2i\pi u\rangle\right)}\right]^{2g-2+s}$$
$$\sum_{(w^1,\dots,w^s)\in W^s}\prod_{j=1}^s\varepsilon_{w^j}\exp(\langle w^j(\rho-ct_j),\tfrac{\partial/\partial t}{p+c}\rangle+2i\pi\langle w^j(\rho+pt_j),u\rangle)P_{u,2g-2+s,p+c}(t)_{|t=\sum_{j=1}^s w^jt_j}.$$

PROOF. Clearly, since $2g-2$ is even,

$$(6.113)\quad \left[\prod_{\alpha\in R_{u,+}}\widehat{A}\Big(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle\Big)\prod_{\alpha\in R_+\setminus R_{u,+}}\frac{1}{2\sinh(\frac{1}{2}(\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle+2i\pi[\langle\alpha,u\rangle]))}\right]^{2g-2}=\left[\frac{\prod_{\alpha\in R_{u,+}}\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle}{\prod_{\alpha\in R_+}2\sinh(\frac{1}{2}\langle\alpha,\frac{\partial/\partial t}{p+c}+2i\pi u\rangle)}\right]^{2g-2}$$

Also, since $t_j \in P$, by Proposition 6.24 , we get

$$\prod_{\alpha\in R_{u,+}} \widehat{A}\Big(\langle\alpha, \tfrac{\partial/\partial t}{p+c}\rangle\Big) \prod_{\alpha\in R_+\setminus R_{u,+}} \frac{1}{2\sinh(\frac{1}{2}(\langle\alpha, \frac{\partial/\partial t}{p+c}\rangle + 2i\pi[\langle\alpha,u\rangle]))}$$

$$\operatorname{sgn}((-i)^{\ell_u}\sigma_{Z(u)}(w^j t_j))e^{i\pi\sum_{\alpha\in R_+}[\langle w^j\alpha,u\rangle]} = \varepsilon_w \frac{\prod\limits_{\alpha\in R_{u,+}} \langle\alpha, \frac{\partial/\partial t}{p+c}\rangle}{\prod\limits_{\alpha\in R_+} 2\sinh(\frac{1}{2}\langle\alpha, \frac{\partial/\partial t}{p+c} + 2i\pi u\rangle)} e^{2i\pi\langle w^j\rho,u\rangle}. \tag{6.114}$$

By Theorem 6.22 and by (6.113),(6.114) , we get (6.112). The proof of our Theorem is completed. □

REMARK 6.27. Suppose temporarily that $t_1, \dots, t_{s-1}$ verify assumption (A), and that $\epsilon > 0$ is small enough so that if $|t_s| < \epsilon$, $(t_1, \dots, t_s)$ still verify (A). Then (6.112) can be written in the form

$$\begin{aligned}
&\int_{M^u/Z(u)} L(u,\lambda^p) = \\
&\operatorname{Vol}(T)^{2g-2}\frac{|Z(Z(u))|}{|W_u|}\,(p+c)^{(g-1)\dim(\mathfrak{z}(u))+\frac{s}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})} \\
&(-1)^{\ell(g-1)+1}\left[\frac{\prod\limits_{\alpha\in R_{u,+}}\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle}{\prod\limits_{\alpha\in R_+} 2\sinh\left(\frac{1}{2}\langle\alpha,\frac{\partial/\partial t}{p+c}+2i\pi u\rangle\right)}\right]^{2g-2+s} \\
&\sum_{w\in W}\varepsilon_w e^{\langle w(\rho+pt_s),\frac{\partial/\partial t}{p+c}+2i\pi u\rangle}\sum_{(w^1,\dots,w^{s-1})\in W^{s-1}} \\
&\prod_{j=1}^{s-1}\varepsilon_{w^j}e^{\langle w^j(\rho-ct_j),\frac{\partial/\partial t}{p+c}\rangle+2i\pi\langle w^j(\rho+pt_j),u\rangle}P_{u,2g-2+s,p+c}(t)_{|t=\sum_{j=1}^{s-1}w^jt_j}.
\end{aligned} \tag{6.115}$$

Using (1.94), (2.131), since $pt_s = \theta_s \in \overline{CR}^*_+$, we may rewrite (6.115) in the form

$$(6.116)\quad \int_{M^u/Z(u)} L(u,\lambda^p) =$$
$$\mathrm{Vol}(T)^{2g-2}\frac{|Z(Z(u))|}{|W_u|}(p+c)^{(g-1)\dim\mathfrak{z}(u)+\frac{(s-1)}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})}$$
$$(-1)^{\ell(g-1)+1}\left[\frac{\prod_{\alpha\in R_{u,+}}\langle\alpha,\frac{\partial/\partial t}{p+c}\rangle}{\prod_{\alpha\in R_+}2\sinh\left(\frac{1}{2}\langle\alpha,\frac{\partial/\partial t}{p+c}+2i\pi u\rangle\right)}\right]^{2g-2+s-1}\chi_{pt_s}\left(e^{\frac{\partial/\partial t}{2i\pi(p+c)}+u}\right)$$
$$\sum_{(w^1,\dots,w^{s-1})\in W^{s-1}}\prod_{j=1}^{s-1}\varepsilon_{w^j}e^{\langle w^j(\rho-ct_j),\frac{\partial/\partial t}{p+c}\rangle+2i\pi\langle w^j(\rho+pt_j),u\rangle}$$
$$P_{u,2g-2+s-1,p+c}(t)_{|t=\sum_{j=1}^{s-1}w^jt_j}\,.$$

With respect to (6.112) , $s$ has been replaced by $s-1$, and we have the extra differential operator $\chi_{pt_s}\left(e^{\frac{\partial/\partial t}{2i\pi(p+c)}+u}\right)$.

Let $M'$ be the manifold attached to $G, \mathcal{O}_1,\dots,\mathcal{O}_{s-1}$. Let $F$ be the vector bundle over $M'/G$ associated to the representation of highest weight $pt_s\subset\overline{CR}^*$ of $G$. Then if we still denote by $L(u,\lambda^p)$ the corresponding class (6.13) over $M'^u/Z(u)$, in view of (5.318), we can rewrite (6.115),(6.116) in the form

$$(6.117)\qquad \int_{M^u/Z(u)} L(u,\lambda^p) = \int_{M'^u/Z(u)} L(u,\lambda^p)\mathrm{ch}_u(F)\,.$$

Also by Theorem 5.57, for $|t_s|$ small enough, the orbifold $M/G$ fibres over $M'/G$ with fibre $G/T$. Let $\rho$ be the projection $M/G\to M'/G$. Then one verifies easily that

$$(6.118)\qquad \rho_*L(u,\lambda^p) = L(u,\lambda^p)\mathrm{ch}_u(E),$$

which makes (6.117) tautological.

REMARK 6.28. If $u\in C/\overline{CR}$, $w\in W$, one should have the equality

$$(6.119)\qquad \int_{M^u/Z(u)} L(u,\lambda^p) = \int_{M^{wu}/Z(wu)} L(wu,\lambda^p)\,.$$

We will briefly explain why the right-hand side of (6.112) is unchanged when replacing $u$ by $wu$. Put

$$(6.120)\quad C = \mathrm{Vol}(T)^{2g-2}\frac{|Z(Z(u))|}{|W_u|}(p+c)^{(g-1)\dim\mathfrak{z}(u))+s/2\dim(\mathfrak{z}(u)/\mathfrak{t})}(-1)^{\ell(g-1)+1}\,.$$

Then using (1.45), (2.167), (6.112), we get

$$\int_{M^{wu}/Z(wu)} L(wu,\lambda^p) = C \left[ \frac{\prod_{\alpha\in R_{u,+}} \frac{\langle w\alpha, \partial/\partial t\rangle}{p+c}}{\prod_{\alpha\in R_+} 2\sinh\left(\frac{1}{2}\langle w\alpha, \frac{\partial/\partial t}{p+c} + 2i\pi wu\rangle\right)} \right]^{2g-2+s}$$

$$\left(\varepsilon_w (-1)^{|R_+\cap w(-R_{u,+})|}\right)^s \sum_{(w^1,\dots,w^s)\in W^s} \prod_{j=1}^s \varepsilon_{w^j} e^{\langle w^j(\rho - ct_j), \frac{\partial/\partial t}{p+c}\rangle_+ + 2i\pi\langle w^j(\rho+pt_j), wu\rangle}$$

$$P_{wu,2g-2+s,p+c}(t)_{|t=\sum_1^s w^j t_j} = C \left[ \frac{\prod_{\alpha\in R_{u,+}} \frac{\langle \alpha, \partial/\partial t\rangle}{p+c}}{\prod_{\alpha\in R_+} 2\sinh\left(\frac{1}{2}\langle \alpha, \frac{\partial/\partial t}{p+c} + 2i\pi u\rangle\right)} \right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s} \prod_{j=1}^s \varepsilon_{w^j} e^{\langle w^j(\rho - ct_j), \frac{\partial/\partial t}{p+c}\rangle_+ + 2i\pi\langle w^j(\rho+pt_j), u\rangle}$$

$$P_{u,2g-2+s,p+c}(t)_{|t=\sum_1^s w^j t_j} = \int_{M^u/Z(u)} L(u,\lambda^p)\,. \tag{6.121}$$

So we find that (6.112) is compatible with (6.119).

**6.8. The case where $\sum_{j=1}^s pt_j \notin \overline{R}$.** Recall that in Section 1.8, we saw that if $\sum_{j=1}^s pt_j \in \overline{R}$ , for any $(w^1,\dots,w^s) \in W^s$, then $\sum_{j=1}^s w^j pt_j \in \overline{R}$.

THEOREM 6.29. *If $\sum_1^s pt_j \notin \overline{R}$, if $v \in C/\overline{R}^*$, then*

$$\sum_{\substack{u\in C/\overline{CR} \\ \tau u = v}} \int_{M^u/Z(u)} L(u,\lambda^p) = 0. \tag{6.122}$$

*In particular, if $p \in \mathbf{M}, p \neq -c$,*

$$\operatorname{Ind}(D_{p,+}) = 0 \tag{6.123}$$

PROOF. To establish (6.122), we use Theorem 6.26. In fact, in the right hand-side of formula (6.112), we observe that the various terms depend on the image $v = \tau u$, with the exception of $\prod_{j=1}^s \exp(2i\pi\langle w^j pt_j, u\rangle)$. Also

$$\sum_{v\in\overline{R}^*/\overline{CR}} \exp\left(2i\pi\langle\sum_{j=1}^s w^j pt_j, v\rangle\right) = 0 \qquad \text{if} \sum_{j=1}^s w^j pt_j \notin \overline{R}, \tag{6.124}$$

$$= \left|\frac{\overline{R}^*}{\overline{CR}}\right| \qquad \text{if} \sum_{j=1}^s w^j pt_j \in \overline{R}. \tag{6.125}$$

It is now clear that (6.122) holds. Equation (6.123) follows from Theorem 6.16 and from (6.122). □

**6.9. A residue formula for the index.** In view of Theorem 6.29, we may and we will assume that

$$\sum_{j=1}^s pt_j \in \overline{R}. \tag{6.126}$$

We now use the notation of Section 2. Recall that $\overline{S} \subset \mathfrak{t}/\overline{CR}$ has been defined in (2.111).

DEFINITION 6.30. We will say that $(t_1, \dots, t_s)$ verify assumption $(\overline{A})$ if for any $(w^1, \dots, w^s) \in W^s$,

$$\sum_{j=1}^{s} w^j t_j \notin \overline{S}. \tag{6.127}$$

Clearly assumption $(\overline{A})$ is stronger than assumption $(A)$ of Definition 5.17. In the sequel, we assume that $(t_1, \dots, t_s)$ verify $(\overline{A})$.

THEOREM 6.31. *For any $u \in C/\overline{R}^*$, $p \in \mathrm{M}, p \neq -c$, the following identity holds*

$$\int_{M^u/Z(u)} L(u, \lambda^p) = \left|\frac{\overline{R}}{\overline{CR}}\right| \mathrm{Vol}(T)^{2g-2} \frac{|Z(Z(u))|}{|W_u|}(p+c)^{(g-1)r}(-1)^{\ell(g-1)+r}$$

$$\sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u \\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle} \mathrm{Res}^I_{x=0}\left[\prod_{\alpha\in R_+}\frac{1}{2\sinh\left(\frac{1}{2}\langle\alpha, \frac{x^I}{p+c}+2i\pi u\rangle\right)}\right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s}\varepsilon_{w^1}\dots\varepsilon_{w^s}\exp\Big(\langle\textstyle\sum_{j=1}^{s} w^j(\rho-ct_j), \frac{x^I}{p+c}\rangle$$
$$+2i\pi\langle\textstyle\sum_{j=1}^{s} w^j(\rho+pt_j)+(p+c)f, u\rangle\Big)\exp\Big(d\textstyle\sum_{j=1}^{r}\frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}$$
$$\Big[\frac{1}{d}\langle p^I_{j-1}(\textstyle\sum_{k=1}^{s} w^k t_k+f), e^j\rangle\Big]\Big)\frac{1}{\prod_{j=1}^{r}\exp\left(d\frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right)-1}. \tag{6.128}$$

PROOF. If $(w^1, \dots, w^s) \in W^s, \sum_{j=1}^{s} w^j t_j \notin \overline{S}$. Then we use Theorems 2.50 and 6.26 and we get (6.128). □

THEOREM 6.32. *If $p \in \mathrm{M}, p \neq -c$, then*

$$\mathrm{Ind}(D_{p,+}) = \left|\frac{\overline{R}}{\overline{CR}}\right| \mathrm{Vol}(T)^{2g-2} \frac{|Z(G)|}{|W|}(p+c)^{(g-1)r}(-1)^{\ell(g-1)+r} \sum_{u\in C/\overline{R}^*}$$

$$\sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u \\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle} \mathrm{Res}^I_{x=0}\left[\prod_{\alpha\in R_+}\frac{1}{2\sinh\left(\frac{1}{2}\langle\alpha, \frac{x^I}{p+c}+2i\pi u\rangle\right)}\right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s}\varepsilon_{w^1}\dots\varepsilon_{w^s}\exp\Big(\langle\sum_{j=1}^{s} w^j(\rho-ct_j), \frac{x^I}{p+c}\rangle$$
$$+2i\pi\sum_{j=1}^{s} w^j(\rho+pt_j)+(p+c)f, u\rangle\Big)\exp\Big(d\sum_{j=1}^{r}\frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}$$
$$\Big[\frac{1}{d}\langle p^I_{j-1}(\sum_{k=1}^{s} w^k t_k+f), e^j\rangle\Big]\Big)\frac{1}{\prod_{j=1}^{r}\left(\exp\left(d\frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right)-1\right)}. \tag{6.129}$$

PROOF. Recall that $Z(G) = \overline{R}^*/\overline{CR}$. Using Theorems 6.16, (6.124) and 6.31, we get (6.129). The proof of our Theorem is completed. □

REMARK 6.33. As we saw in Proposition 1.40 and Remark 2.7, for $n \geq 2$ and $G = \mathrm{SU}(n)$, then $C/\overline{R}^* = 0$, and we can choose $d = 1$. Then formula (6.129) has an especially simple form.

## 6.10. A formula for the index for large $p$.

THEOREM 6.34. *For $p \in \mathbf{M}$, and $|p|$ large enough, the following identity holds*

$$\mathrm{Ind}(D_{p,+}) = \left|\frac{\overline{R}}{\overline{CR}}\right| \mathrm{Vol}(T)^{2g-2} \frac{|Z(G)|}{|W|}(p+c)^{(g-1)r}(-1)^{\ell(g-1)+r} \sum_{u \in C/\overline{R}^*}$$

$$\sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f \in \overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle} \mathrm{Res}^I_{x=0} \left[ \prod_{\alpha \in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha, \frac{x^I}{p+c} + 2i\pi u\rangle\right)}\right]^{2g-2+s}$$

$$(6.130) \quad \sum_{(w^1,\dots,w^s)\in W} \varepsilon_{w^1}\dots\varepsilon_{w^s} \exp\Big(\langle \sum_{j=1}^s w^j \frac{\rho}{p+c}, x^I\rangle$$

$$+2i\pi\langle \sum_{j=1}^s w^j(\rho + pt_j + (p+c)f, u\rangle\Big)$$

$$\exp\Big(d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\Big[\frac{1}{d}\langle p^I_{j-1}(\sum_{k=1}^s \frac{w^k p t_k}{p+c} + f), e^j\rangle\Big]\Big)$$

$$\frac{1}{\prod_{j=1}^r \left(\exp\left(d\frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)}.$$

PROOF. Clearly, for $p \to +\infty$, $\sum_1^s \frac{w^j c t_j}{p+c} \to 0$. Also by Proposition 2.14, since $\sum_{j=1}^s w^j t_j \notin \overline{S}$, if $f \in \overline{R}$,

$$(6.131) \qquad \langle p^I_{j-1}\Big(\sum_{k=1}^s w^k t_k + f\Big), e^j\rangle \notin \mathbf{Z}.$$

Therefore by (2.13) , for $|p|$ large enough,

$$(6.132) \quad d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s \frac{w^k p t_k}{p+c} + f\Big), e^j\rangle\right] =$$

$$d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s w^k t_k + f), e^j\rangle\right] - \langle \sum_{j=1}^s \frac{c w^j t_j}{p+c}, x^I\rangle.$$

By (2.115), (6.132) , we find that for $|p|$ large enough,

$$\exp\left(\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_i, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s \frac{w^k p t_k}{p+c} + f\Big), e^j\rangle\right]\right)$$

$$(6.133) \qquad = \exp\Big(\langle -\sum_{j=1}^s \frac{w^j c t_j}{p+c}, x^I\rangle\Big) \quad .$$

$$\exp\left(d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s w^k t_k + f), e^j\rangle\right]\right).$$

From (6.129), (6.133) , we get (6.130). The proof of our Theorem is completed. □

REMARK 6.35. The formulas in Theorems 6.32 and 6.34 are essentially identical.The point is that, as we will see in Theorems 7.23 and 8.1, the right-hand side of (6.130) is just Verlinde's formula.

**6.11. A perturbation of the index problem.** We still assume that $s \geq 1$ and that if $g = 0$, then $s \geq 3$, so that $2g - 2 + s \geq 1$. Note that these assumptions are almost irrelevant, since by adding as many marked points as one wishes with holonomy equal to 1, one can make $s$ arbitrarily large.

Recall that, by Proposition 1.23, $\overline{P}$ embeds into $T$. Also the orbit under the Weyl group $W$ of any element in $T$ always intersects $\overline{P}$.

Let $t_1, \dots, t_s$ be $s$ elements in $T$. We may and we will assume that they all lie in $\overline{P}$. In the sequel we will consider $t_1, \dots, t_s$ as elements of $\mathfrak{t}$. We still define $\mathbf{M} \subset \mathbf{Z}$ as in (6.20), and we assume that $\mathbf{M}$ is not reduced to 0. For $p \in \mathbf{M}$, set

$$\theta_j = p\theta_j,\ 1 \leq j \leq s. \tag{6.134}$$

Then $\theta_j \in \overline{CR}^*$, $1 \leq j \leq s$.

Let $\delta = (\delta_1, \dots, \delta_s) \in \mathfrak{t}^s$ and assume that $t_1 + \delta_1, \dots, t_s + \delta_s$ lie in $P$ and verify $(\overline{A})$. Put

$$X^\delta = G^{2g} \times \prod_{j=1}^{s} \mathcal{O}_{t_j + \delta_j}, \tag{6.135}$$

and let $M^\delta$ correspond to $X^\delta$ as in (5.59). Then $M^\delta$ is a smooth submanifold of $X^\delta$.

We will briefly show how to equip $M^\delta$ with an orbifold line bundle $\lambda^p$. We use the notation of Section 4.10. For $1 \leq j \leq s$, consider the orbit $\mathcal{O}_{-pd/dt + p(t_j+\delta_j)} \subset \widehat{L\mathfrak{g}}$. By (4.30),

$$\mathcal{O}_{-pd/dt+p(t_j+\delta_j)} \simeq LG/T. \tag{6.136}$$

Also as in Definition 4.3, $\widetilde{L}G/(T \times S^1) \simeq LG/T$ can be equipped with the line bundle $H_{(\theta_j, -p)}$ of weight $(\theta_j, p)$. Therefore $\mathcal{O}_{-pd/dt+p(t_j+\delta_j)}$ can be equipped with the corresponding line bundle $L_j$. Recall that $pt_j = \theta_j$. Then we define the line bundle $\lambda_p$ as in (4.189), and the line bundle $\lambda^p$ on $M^\delta$ as in Definition 6.10.

We can then define a Dirac operator $D_p^{(t_j+\delta_j)}$ in the same way as in Sections 6.3 and 6.4.

THEOREM 6.36. *For $p \in \mathbf{M}, p \neq -c$,*

$$
\begin{aligned}
(6.137) \quad & \operatorname{Ind}(D_{p,+}^{(t_j+\delta_j)}) = \left|\frac{\overline{R}}{\overline{CR}}\right| \operatorname{Vol}(T)^{2g-2} \frac{1}{|W|}(p+c)^{(g-1)r} \\
& (-1)^{\ell(g-1)+r} \sum_{u \in C/\overline{CR}} \sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f\in \overline{R}/\overline{dR}}} \frac{1}{\langle \alpha_{i_1},\dots,\alpha_{i_r}\rangle} \\
& \operatorname{Res}^I_{x=0}\left[\prod_{\alpha\in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha, \frac{x^I}{p+c} + 2i\pi u\rangle\right)}\right]^{2g-2+s} \\
& \sum_{(w^1,\dots,w^s)\in W^s} \varepsilon_{w^1}\dots\varepsilon_{w^s} \exp\Big(\langle \sum_{j=1}^s w^j\Big(\frac{\rho - ct_j}{p+c} - \delta_j\Big), x^I\rangle \\
& + 2i\pi\langle \sum_{j=1}^s w^k(\rho + pt_j) + (p+c)f, u\rangle\Big) \\
& \exp\left(d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\left(\sum_{k=1}^s w^k(t_k+\delta_k) + f\right), e^j\rangle\right]\right) \\
& \frac{1}{\prod_{j=1}^r \left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)}.
\end{aligned}
$$

PROOF. The proof of our Theorem follows the same steps as the proof of Theorem 6.32, where the case $\delta = 0$ was considered. We briefly describe the main steps where the proof of our more general result differs.

- Instead of (6.26) in Proposition 5.11, we have

$$
c_1(\lambda^p, \nabla^{\lambda^p}) = p(\omega + \sum_{j=1}^s \langle \delta_j, \Theta_j\rangle). \tag{6.138}
$$

  This follows from an easy computation which is left to the reader.
- In formula (6.74) in Theorem 6.21, the main point is in fact that $\prod_{j=1}^s e^{2i\pi\langle w^j pt_j + ph, u\rangle}$ is unchanged. The argument is in fact the same as in the proof of Theorem 6.21. Also using (6.138) instead of Theorem 5.78, one finds that in formula (6.74) in Theorem 6.21, $\prod_{j=1}^s e^{\frac{1}{p+c}\langle(\rho - ct_j), \partial/\partial t_j\rangle}$ should be replaced by $\prod_{j=1}^s e^{\langle \frac{\rho - ct_j}{p+c} - \delta_j, \partial/\partial t_j\rangle}$.

By proceeding otherwise as before, we get (6.137). The proof of our Theorem is completed. □

REMARK 6.37. Using (6.137) and proceeding as in the proof of Theorem 6.29, we find that (6.123) is still true. So in the sequel, we may and we will assume that (6.126) holds.

Now we will slightly modify the statement of Theorem 6.36. Recall that $t_1, \dots, t_s$ all lie in $\overline{P}$. Therefore for $p \in \mathbf{M}, p \geq 0$, for $1 \leq j \leq s$, $pt_j/(p+c)$ still lies in $\overline{P}$.

THEOREM 6.38. *For $p \in \mathbf{M}, p \geq 0$, if $\varepsilon_1, \dots, \varepsilon_s \in \mathfrak{t}$ are such that $\frac{pt_1}{p+c} + \varepsilon_1, \dots \frac{pt_s}{p+c} + \varepsilon_s \in P$, and that, for any $(w^1, \dots, w^s) \in W^s$, $\sum_{j=1}^s w^j \left(\frac{pt_j}{p+c} + \varepsilon_j\right) \notin \overline{S}$, then*

$$\operatorname{Ind}(D_{p,+}^{(\frac{pt_j}{p+c}+\varepsilon_j)}) = \left|\frac{\overline{R}}{\overline{CR}}\right| \operatorname{Vol}(T)^{2g-2} \frac{|Z(G)|}{|W|} (p+c)^{(g-1)r}$$

$$(-1)^{\ell(g-1)+r} \sum_{u \in C/\overline{R}^*} \sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f \in \overline{R}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle}$$

$$\operatorname{Res}^I_{x=0} \left[\prod_{\alpha \in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha, \frac{x^I}{p+c} + 2i\pi u\rangle\right)}\right]^{2g-2+s} \tag{6.139}$$

$$\sum_{(w^1,\dots,w^s)\in W^s} \varepsilon_{w^1} \dots \varepsilon_{w^s} \exp\Big(\langle \sum_{j=1}^s w^j \Big(\frac{\rho}{p+c} - \varepsilon_j\Big), x^I\rangle$$

$$+2i\pi\langle \sum_{j=1}^s w^j(\rho + pt_j) + (p+c)f, u\rangle\Big)$$

$$\exp\left(d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\left(\sum_{k=1}^s w^k\Big(\frac{pt_k}{p+c} + \varepsilon_k\Big) + f\right), e^j\rangle\right]\right)$$

$$\frac{1}{\prod_{j=1}^r \left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)}.$$

PROOF. In Theorem 6.36, we take

$$\delta_j = \frac{-ct_j}{p+c} + \varepsilon_j\ ,\ 1 \leq j \leq s\,. \tag{6.140}$$

Then we get (6.139). □

PROPOSITION 6.39. *For $p \in \mathbf{M}, p \geq 0$, there is $\varepsilon_1, \dots, \varepsilon_s \in \mathfrak{t}$ such that for $\ell \in ]0,1]$, $1 \leq j \leq s$, $\frac{pt_j}{p+c} + \ell\varepsilon_j \in T$ is regular, and for any $(w^1, \dots, w^s) \in W^s, \ell \in ]0,1]$, $\sum_{j=1}^s w^j \left(\frac{pt_j}{p+c} + \ell\varepsilon_j\right) \notin \overline{S}$.*

PROOF. Recall that $\overline{S}$ is a union of hypertori in $T$. Therefore $\overline{S} \cap \overline{P}$ is the intersection of $\overline{P} \subset \mathfrak{t}$ with a union of hyperplanes in $\mathfrak{t}$. Our Proposition now follows easily. □

Let $[x]_+, [x]_-$ be the functions defined on $\mathbf{R}$ with values in $[0,1]$, which are periodic of period 1 and such that

$$\begin{aligned} [x]_+ &= x \text{ for } x \in [0,1[, \\ [x]_- &= x \text{ for } x \in ]0,1]\,. \end{aligned} \tag{6.141}$$

Observe that by 2.14 , for $\ell \in ]0,1]$, $f \in \overline{R}/d\overline{R}$, $1 \leq j \leq r$,

$$\langle p^I_{j-1}\left(\sum_{k=1}^s w^k\left(\frac{pt_k}{p+c} + \ell\varepsilon_k\right) + f\right), e_j\rangle \notin \mathbf{Z}\,. \tag{6.142}$$

From (6.142) , it follows that either

$$\langle p_{j-1}^I \left( \sum_{j=1}^{s} w^j \left( \frac{pt_j}{p+c} \right) + f \right), e^j \rangle \notin \mathbf{Z}, \tag{6.143}$$

or

$$\begin{aligned} &\langle p_{j-1}^I \left( \sum_{j=1}^{s} w^j \left( \frac{pt_j}{p+c} \right) + f \right), e^j \rangle \in \mathbf{Z} \\ &\langle p_{j-1}^I \left( \sum_{j=1}^{s} w^j \varepsilon^j \right), e^j \rangle \neq 0 . \end{aligned} \tag{6.144}$$

Definition 6.40. Given $(w^1, \dots, w^s) \in W^s$, for $f \in \overline{R}/d\overline{R}$, $I = (i_1, \dots, i_r)$, put

$$\begin{aligned} \eta_j^{(w^1,\dots,w^s)}(f, I) &= + \text{ if } \langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \left( \frac{pt_j}{p+c} \right) + f \right), e^j \rangle \notin \mathbf{Z}, \\ &\quad \text{or if } \langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \left( \frac{pt_k}{p+c} \right) + f \right), e^j \rangle \in \mathbf{Z}, \\ &\quad \langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \varepsilon_k \right), e^j \rangle > 0 \\ &= - \text{ if } \langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \left( \frac{pt_k}{p+c} \right) + f \right), e^j \rangle \in \mathbf{Z}, \\ &\quad \langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \varepsilon_k \right), e^j \rangle < 0 . \end{aligned} \tag{6.145}$$

Observe that if $f = \sum_{k=1}^{r} f^j e_j$

$$\langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \left( \frac{pt_k}{p+c} + f \right), e^j \rangle = \langle p_{j-1}^I \left( \sum_{k=1}^{s} w^k \frac{pt_k}{p+c} + \sum_{k=1}^{j} f^k e_k \right), e^j \rangle . \tag{6.146}$$

By (6.146), it is clear that $\eta_j^{(w^1,\dots,w^s)}(f, I)$ depend only on $f^1, \dots, f^j, i_1, \dots, i_{j-1}$.

THEOREM 6.41. *Given $p \in \mathbf{M}, p \geq 0$, the integer* Ind $D_{p,+}^{(\frac{pt_j}{p+c}+\ell\varepsilon_j)}$ *does not depend on $\ell \in ]0,1]$. More precisely*

$$\text{Ind } D_{p,+}^{(\frac{pt_i}{p+c}+\ell\varepsilon_i)} = \left|\frac{\overline{R}}{\overline{CR}}\right| \text{Vol}(T)^{2g-2}\frac{|Z(G)|}{|W|}(p+c)^{(g-1)r} \tag{6.147}$$

$$(-1)^{\ell(g-1)+r}\sum_{u\in C/\overline{R}^*}\sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u\\ f\in\overline{CR}/d\overline{R}}}\frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle}$$

$$\text{Res}^I_{x=0}\left[\prod_{\alpha\in R_+}\frac{1}{2\sinh\left(\frac{1}{2}\langle\alpha,\frac{x^I}{p+c}+2i\pi u\rangle\right)}\right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s}\varepsilon_{w^1}\dots\varepsilon_{w^s}\exp\Big(\langle\sum_{j=1}^s w^j\left(\frac{\rho}{p+c}\right),x^I\rangle$$

$$+2i\pi\langle\sum_1^s w^j(\rho+pt_j)+(p+c)f,u\rangle\Big)$$

$$\exp\left(d\sum_{j=1}^s\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right.$$

$$\left.\left[\frac{1}{d}\langle p^I_{j-1}\left(\sum_{k=1}^s w^k\left(\frac{pt_k}{p+c}\right)+f\right),e_j\rangle\right]_{\eta_j^{(w^1,\dots,w^s)}(f,I)}\right)$$

$$\frac{1}{\prod_{j=1}^s\left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right)-1\right)}.$$

PROOF. Clearly, for $\ell \in ]0,1]$,

$$\left[\frac{1}{d}\langle p^I_{j-1}\sum_{k=1}^s w^k\left(\frac{pt_k}{p+c}+\ell\varepsilon_k\right)+f,e_j\rangle\right] \tag{6.148}$$

depends continuously on $\ell \in ]0,1]$, and moreover as $\ell \to 0$, it converges to

$$\left[\frac{1}{d}\langle p^I_{j-1}\left(\sum_1^s\frac{w^jt_j}{p+c}+f\right),e_j\rangle\right]_{\eta_j^{(w^1,\dots,w^s)}(f,I)}. \tag{6.149}$$

By (6.139) and the above, Ind $D_{p,+}^{(\frac{pt_j}{p+c}+\ell\varepsilon_j)}$ depends continuously on $\ell \in ]0,1]$, and so remains constant. This last fact should also be clear by the construction of $D_{p,+}^{(\frac{pt_j}{p+c}+\ell\varepsilon_j)}$.

Using (6.139), (6.149) , we get (6.147). The proof of our Theorem is completed. □

REMARK 6.42. Observe that in our direct index theoretic computations, we have avoided introducing central holonomies, because they are non generic. We dealt with this case by a perturbation argument. However we may assume that one of the $t_j$, say $t_s$, is equal to $h_o \in Z(G)$, while the other holonomies are generic.

By Proposition 1.24, we can identify $h_o$ to a unique element in $\overline{P}$, which we still denote $h_o$, so that $h_o \in \overline{P} \cap \overline{R}^*$. When the moduli space $M/G$ is an orbifold (which is the case when condition $(\overline{A})$ holds), all the computations we did in Sections 6.1-6.9 can be done directly. By proceeding as in Remark 6.27, it is obvious that in both cases, we compute the same index. This is less obvious at the level of explicit computations. Using in particular equations (1.78) and (1.140), we find that equation (6.112) is now replaced by

$$\begin{aligned} &\int_{M^u/Z(u)} L(u,\lambda^p) = \epsilon_{w_{h_o}} \operatorname{Vol}(T)^{2g-2} \frac{|Z(Z(u))|}{|W_u|} \\ &(p+c)^{(g-1)\dim(\mathfrak{z}(u)) + \frac{(s-1)}{2}\dim(\mathfrak{z}(u)/\mathfrak{t})} \\ &(-1)^{\ell(g-1)+1} \left[ \frac{\prod_{\alpha \in R_{u,+}} \langle \alpha, \frac{\partial/\partial t}{p+c} \rangle}{\prod_{\alpha\in R_+} 2\sinh\left(\frac{1}{2}\langle \alpha, \frac{\partial/\partial t}{p+c} + 2i\pi u\rangle\right)} \right]^{2g-2+s-1} \\ &\sum_{(w^1,\dots,w^{s-1})\in W^{s-1}} \prod_{j=1}^{s-1} \varepsilon_{w^j} \exp(\langle w^j(\rho - ct_j), \tfrac{\partial/\partial t}{p+c}\rangle \\ &\quad + 2i\pi\langle w^j(\rho+pt_j), u\rangle) \exp(2i\pi\langle (p+c)h_o, u\rangle) \\ &P_{u,2g-2+s-1,p+c}(t)_{|t=\sum_{j=1}^{s-1} w^j t_j + h_o} . \end{aligned} \tag{6.150}$$

The fact that ultimately, the two explicit index formulas coincide will be verified via the Verlinde formulas in Remark 8.4, by using in particular Theorem 1.33.

## 7. Residues and the Verlinde formula

In this Section, we apply residues techniques to the finite Verlinde sum, we express it as a residue in several complex variables, and we prove that for $p$ large enough, the Riemann-Roch number of $M/G$ is given by the Verlinde formula. Without any condition on $p$, the Verlinde formula is the Riemann-Roch number of some perturbation of the moduli space of $M/G$.

As indicated in the introduction, higher cohomology groups on $M/G$ may well not vanish. This would account for the discrepancy between the index and the Verlinde formula for small $p$.

This Section is organized as follows. In Section 7.1, we give a residue formula for a Fourier series over $\overline{R}^*/q\overline{R}^*$, $L_q(t,x)$. In Section 7.2, we consider a related series $M_q(t,x)$, which we also evaluate as a sum of residues, the sum being indexed by the semisimple centralizers $Z(u)$. In Section 7.3, we introduce the Verlinde sums $V_{g,q}(\theta_1,\dots,\theta_s)$. In Section 7.4, we express the Verlinde sums as residues.

Finally in Sections 7.5-7.7, we "localize" the Verlinde formulas to put them a form which is very similar to the form we obtained in Section 6 for $\text{Ind}(D_{p,+})$. This is done using the techniques we already developed in Section 2.

### 7.1. A residue formula for $L_q(t,x)$.

DEFINITION 7.1. For $q \in \mathbf{N}^*$, $t \in \overline{R}/q\overline{R}$, $x \in (\mathbf{C}\backslash 2i\pi\mathbf{Z})^\ell$, put

$$L_q(t,x) = \sum_{\lambda\in\overline{R}^*/q\overline{R}^*} \frac{\exp(2i\pi\langle\lambda,t/q\rangle)}{\prod_{i=1}^\ell 2q\tanh\left(\frac{2i\pi\langle\alpha_i,\lambda\rangle - x_i}{2q}\right)}. \tag{7.1}$$

If $t \in \overline{R}/q\overline{R}$, then $t/q \in (\overline{R}/q)\big/\overline{R} \subset \mathfrak{t}/\overline{R}$. Recall that the subset $H$ of $\mathfrak{t}/\overline{R}$ was defined in (2.44), (2.45). Then by (2.45),

$$\{t\in\overline{R}/q\overline{R}, t/q\in H\} = \Big\{t\in\overline{R}/q\overline{R}, t = \sum_{j\in\mathcal{J}\subset\{1,\dots,r\}} t^j\alpha_j \text{ in } \mathfrak{t}/q\overline{R}, \tag{7.2}$$
$$\text{and the } \{\alpha_j, j\in\mathcal{J}\} \text{ do not span } \mathfrak{t}^*\simeq\mathfrak{t}\Big\}.$$

As in Section 2.4 , we identify $\overline{R}/d\overline{R}$ to $\{0,1,\dots,d-1\}^r$, i.e. $f\in\overline{R}/d\overline{R}$ is identified to $\sum_1^r f^ie_i, f^i\in\{0,\dots,d-1\}$.

By (2.14), if $I=(i_1,\dots,i_r)$ is generic, if $x\in\mathbf{Z}^\ell$, then $d\langle e_j,x^I\rangle\in\mathbf{Z}$, $1\le j\le r$, and so for $1\le i\le\ell$, $d\langle\alpha_i,x^I\rangle\in\mathbf{Z}$.Therefore the function $x\in\mathbf{C}^\ell\mapsto \tanh\left(\frac{\langle\alpha_i,x^I\rangle-x_i}{2q}\right)$ is periodic of period $(2i\pi qd,\dots,2i\pi qd)$.

Similarly if $t\in\overline{R}$, we claim that the function

$$x\in\mathbf{C}^\ell\mapsto\exp\Big(d\sum_{j=1}^r\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\Big[\frac{1}{d}\langle p^I_{j-1}t,\frac{e^j}{q}\rangle\Big]\Big) \tag{7.3}$$

is periodic of period $(2i\pi qd,\dots,2i\pi qd)$. In fact by (2.13), (2.28), (2.29), if $x\in(2i\pi\mathbf{Z})^\ell$,

$$\exp\left(d\sum_{j=1}^r\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\Big[\frac{1}{d}\langle p^I_{j-1}t,\frac{e^j}{q}\rangle\Big]\right) = \exp\left(\langle\frac{t}{q},x^I\rangle\right). \tag{7.4}$$

Also, as we just saw, if $t \in \overline{R}$, $z \in \mathbf{Z}^\ell$, $d\langle t, x^I\rangle \in \mathbf{Z}$. From (7.4), it follows that if $x \in (2i\pi qd\mathbf{Z})^\ell, t \in \overline{R}$, (4.4) is equal to 1, which proves the periodicity of (7.3).

If $g = (g_1, \dots, g_r) \in \mathbf{Z}^r$, if $I = (i_1, \dots, i_r) \subset \{1, \dots, \ell\}$, let $g_I \subset \mathbf{C}^\ell$ be such that

$$(7.5) \qquad \begin{aligned} (g_I)_{i_j} &= g_j, \quad 1 \le j \le r, \\ (g_I)_i &= 0, \quad i \notin I. \end{aligned}$$

THEOREM 7.2. *For generic values of $x \in (\mathbf{C}\backslash 2i\pi\mathbf{Z})^\ell$, for $t \in \overline{R}/q\overline{R}$, $t/q \notin H$, if we still denote by $t$ a representative of $t$ in $\overline{R}$, then*

$$(7.6) \qquad L_q(t,x) = \frac{(-1)^r}{d^r} \sum_{\substack{I=(i_1,\dots,i_r)\subset\{1,\dots,\ell\}, I \text{ generic} \\ f\in\overline{R}/d\overline{R}, g\in(\mathbf{Z}/d\mathbf{Z})^r}} \operatorname{sgn}\left(\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle\right)$$
$$\left[\exp\left(d\sum_{j=1}^r \frac{\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p_{j-1}^I(t+qf), \frac{e^j}{q}\rangle\right]\right)\right.$$
$$\left.\frac{1}{\prod_{i\in {}^cI} 2q\tanh\left(\frac{\langle \alpha_i, x^I\rangle - x_i}{2q}\right)}\right](x + 2i\pi q g_I) \prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle}\right) - 1}.$$

PROOF. Take $1 \le j \le r$, $I = (i_1, \dots, i_{j-1}) \subset \{1, \dots, \ell\}$ such that $\langle \alpha_{i_1}\rangle \neq 0$, $\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle \neq 0$. We use otherwise the notation and the conventions in the proof of Theorem 2.19, in particular in equation (2.63).

For $\widehat{k} = (k_{j+1}, \dots, k_r) \subset \mathbf{Z}^{r-j}$, we identify $\widehat{k}$ to $\sum_{i=j+1}^r k^i e_i \in \overline{R}^*$. For $a \in \mathbf{C}$, $x \in \mathbf{C}^\ell$, $t \in \overline{R}$, put

$$(7.7) \qquad S_I(a,x) =$$
$$\sum_{\substack{0\le g_p<d \\ 1\le p\le j-1}} \left\{\exp\left(a\langle p_{j-1}^I t, \frac{e^j}{q}\rangle + d\sum_{n=1}^{j-1} \frac{\langle p_{n-1}^I \alpha_{i_n}, x^I\rangle}{\langle p_{n-1}^I \alpha_{i_n}, e^n\rangle}\left[\frac{1}{d}\langle p_{n-1}^I t, \frac{e^n}{q}\rangle\right]\right)\right.$$
$$\left.\frac{1}{\prod_{i\in {}^cI} 2q\tanh\left(\frac{\langle p_{j-1}^I \alpha_i, ae^j + 2i\pi\widehat{k} - x^I\rangle}{2q}\right)}\right\}\left(x + 2i\pi q(g_1, \dots, g_{j-1})_I\right).$$

We claim that as a function of $x \in \mathbf{C}^\ell$, each of the expressions $\{\quad\}$, with $(g_1, \dots, g_{j-1}) = 0$, is periodic of period $2i\pi qd, \dots, 2i\pi qd$. In fact by (2.63), if $x \in \mathbf{Z}^\ell$,

$$(7.8) \qquad d\langle p_{j-1}^I, \alpha_i, x^I\rangle \in \mathbf{Z}.$$

Also by (2.28), (2.63), for $1 \le n \le j-1$, $x \in \mathbf{Z}^\ell$

$$(7.9) \qquad \frac{d\langle p_{n-1}^I \alpha_{i_n}, x^I\rangle}{\langle p_{n-1}^I \alpha_{i_n}, e^n\rangle} \in \mathbf{Z},$$

so that if $x \in \mathbf{Z}^\ell$, using (2.12),

$$\exp\left(2i\pi d\sum_{n=1}^{j-1}\frac{\langle p^I_{n-1}\alpha_{i_n}, x^I\rangle}{\langle p^I_{n-1}\alpha_{i_n}, e^n\rangle}\left[\frac{1}{d}\langle p^I_{n-1}t, \frac{e^n}{q}\rangle\right]\right) \tag{7.10}$$

$$= \exp\left(\frac{2i\pi}{q}\sum_{n=1}^{j-1}\frac{\langle p^I_{n-1}\alpha_{i_n}, x^I\rangle}{\langle p^I_{n-1}\alpha_{i_n}, e^n\rangle}\langle p^I_{n-1}t, e^n\rangle\right)$$

$$\exp\left(\frac{2i\pi}{q}\left(\langle t, x^I\rangle - \langle p^I_{j-1}t, x^I\rangle\right)\right).$$

Also by (2.14), (2.63), if $t \in \overline{R}$, $x \in \mathbf{Z}^\ell$,

$$\begin{aligned} &d\langle t, x^I\rangle \in \mathbf{Z}, \\ &d\langle p^I_{j-1}t, x^I\rangle \in \mathbf{Z}. \end{aligned} \tag{7.11}$$

By (7.11), it follows that if $x \in qd\mathbf{Z}^\ell$, (7.10) is equal to 1.

Ultimately, we find that indeed, each of the expression $\{\ \}$ in (7.7) is periodic in $x \in \mathbf{C}^\ell$ of period $2i\pi qd, \dots, 2i\pi qd$. Therefore, we will write (7.7) in the form

$$S_I(a,x) = \sum_{g\in(\mathbf{Z}/d\mathbf{Z})^{j-1}}\left\{\exp\Big(a\langle p^I_{j-1}t, \frac{e^j}{q}\rangle + d\right.$$

$$\sum_{n=1}^{j-1}\frac{\langle p^I_{n-1}\alpha_{i_n}, x^I\rangle}{\langle p^I_{n-1}\alpha_{i_n}, e^n\rangle}\Big[\frac{1}{d}\langle p^I_{n-1}t, \frac{e^n}{q}\rangle\Big]\Big)$$

$$\left.\frac{1}{\prod_{i\in{}^cI} 2q\tanh\Big(\frac{\langle p^I_{j-1}\alpha_i, ae^j+2i\pi\widehat{k}-x^I\rangle}{2q}\Big)}\right\}(x+2i\pi q g_I). \tag{7.12}$$

In particular $S_I(a,x)$ is periodic in $x \in \mathbf{C}^\ell$ of period $2i\pi q, \dots, 2i\pi q$.

Now we claim that $S_I(a,x)$ is periodic in $a \in \mathbf{C}$, with period $2i\pi q$. In fact, since $e^j = (\widetilde{a}e^j)^I$,

$$\langle p^I_{j-1}\alpha_i, e^j\rangle = \langle p^I_{j-1}\alpha_i, (\widetilde{a}e^j)^I\rangle. \tag{7.13}$$

Also by (2.12),

$$\langle p^I_{j-1}t, e^j\rangle = \langle t, e^j\rangle - \sum_{n=1}^{j-1}\frac{\langle p^I_{n-1}\alpha_{i_n}, (\widetilde{a}e^j)^I\rangle}{\langle p^I_{n-1}\alpha_{i_n}, e^n\rangle}\langle p^I_{n-1}t, e^n\rangle. \tag{7.14}$$

Since $t \in \overline{R}$, then $\langle t, e^j\rangle \in \mathbf{Z}$ and so, by (7.14), we get

$$\langle p^I_{j-1}t, e^j\rangle = -\sum_{n-1}^{j-1}\frac{\langle p^I_{n-1}\alpha_{i_n}, (\widetilde{a}e^j)^I\rangle}{\langle p^I_{n-1}\alpha_{i_n}, e^n\rangle}\langle p^I_{n-1}t, e^n\rangle \mod (\mathbf{Z}). \tag{7.15}$$

From (7.12), (7.13), (7.15), we find that

$$\begin{aligned} S_I(a+2i\pi q, x) &= S_I(a, x-2i\pi q\widetilde{a}e^j) \\ &= S_I(a,x), \end{aligned} \tag{7.16}$$

i.e. $S_I(a,x)$ is periodic in $a$ with period $2i\pi q$.

Put

$$L_I = \sum_{k\in\mathbf{Z}/q\mathbf{Z}} S_I(2i\pi k, x). \tag{7.17}$$

Clearly

$$L_I = \frac{1}{d} \sum_{k \in \mathbf{Z}/dq\mathbf{Z}} S_I(2i\pi k, x). \tag{7.18}$$

From (7.18), we obtain

$$L_I = \frac{1}{d^2} \sum_{\substack{0 \le f < d \\ k \in \mathbf{Z}/d^2q\mathbf{Z}}} \exp\left(\frac{2i\pi f k}{d}\right) S_I\left(\frac{2i\pi k}{d}, x\right). \tag{7.19}$$

By (7.12), for generic $x$, we rewrite (7.19) in the form

$$\begin{aligned} L_I = \frac{1}{d^2} \sum_{k \in \mathbf{Z}/d^2q\mathbf{Z}} \Bigg\{ \sum_{\substack{0 \le f < d \\ g \in (\mathbf{Z}/d\mathbf{Z})^{j-1}}} \operatorname{Res}_{a=2i\pi k} \Bigg[ \exp\Bigg( a \left[ \frac{\langle p_{j-1}^I (t + qfe_j), e^j/q \rangle}{d} \right] \\ + d \sum_{n=1}^{j-1} \frac{\langle p_{n-1}^I \alpha_{i_n}, x^I \rangle}{\langle p_{n-1}^I \alpha_{i_n}, e^n \rangle} \left[ \frac{1}{d} \langle p_{n-1}^I t, e^n/q \rangle \right] \Bigg) \\ \frac{1}{\prod_{i \in {}^c I} 2q \tanh\left( \frac{\langle p_{j-1}^I \alpha_i, ae^j/d + 2i\pi \widehat{k} - x^I \rangle}{2q} \right)} \frac{1}{e^a - 1} \Bigg] \Bigg\} (x + 2i\pi q g_I). \end{aligned} \tag{7.20}$$

Now we just saw that the function of $a, \sum_{\substack{0 \le f < d \\ g \in (\mathbf{Z}/d\mathbf{Z})^{j-1}}} \dots$, which appears in the right-hand side of(7.20), is periodic of $2i\pi qd$.

We claim that for $f$, $0 \le f < d$, $g \in (\mathbf{Z}/d\mathbf{Z})^{j-1}$, the function of $a$,

$$h_{f,g}(a) = \frac{\exp\left( a \left[ \frac{p_{j-1}^I (t + qfe_j), e^j/q \rangle}{d} \right] \right)}{\prod_{i \in {}^c I} 2q \tanh\left( \frac{\langle p_{j-1}^I \alpha_i, ae^j/d + 2i\pi \widehat{k} - x^I \rangle}{2q} \right)} \frac{1}{e^a - 1} \tag{7.21}$$

is periodic of period $2i\pi d^2 q$. This follows from the fact that since $t \in \overline{R}$,

$$\begin{aligned} &d \langle p_{j-1}^I (t + qfe_j, e^j \rangle \in \mathbf{Z}, \\ &d \langle p_{j-1}^I \alpha_i, e^j \rangle \in \mathbf{Z}. \end{aligned} \tag{7.22}$$

By Proposition 2.14, if $t/q \notin H$,

$$0 < \left[ \frac{\langle p_{j-1}^I (t + qfe^j), e^j/q \rangle}{d} \right] < 1. \tag{7.23}$$

Then for given $f, 0 \le f < d$, $g \in (\mathbf{Z}/d\mathbf{Z})^{j-1}$, we will apply the theorem of residues to the function $h_{f,g}(a)$ on the domain given in Figure 7.1. By (2.65), for at least one $i_j \notin I$, $\langle p_{j-1}^I \alpha_{i_j}, e^j \rangle \neq 0$. For generic $x$, mod $(2i\pi d^2 q)$, the poles of the function $h_{f,g}(a)$ other than the $2i\pi k$ are simple and given by

$$\begin{aligned} &a = \frac{d}{\langle p_{j-1}^I \alpha_{i_j}, e^j \rangle} \left( \langle p_{j-1}^I \alpha_{i_j}, x^I - 2i\pi \widehat{k} \rangle + 2i\pi g^j q \right), \\ &0 \le g^j < d |\langle p_{j-1}^I \alpha_{i_j}, e^j \rangle| , \ i_j \notin I, \langle p_{j-1}^I \alpha_{i_j}, e^j \rangle \neq 0. \end{aligned} \tag{7.24}$$

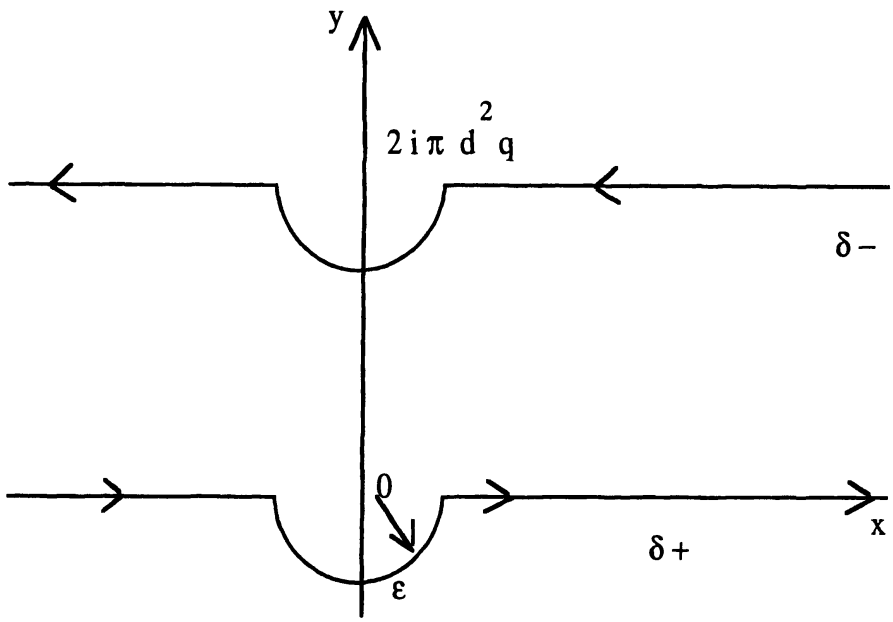


FIGURE 7.1

Needless to say, in (7.24), $d\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle \in \mathbf{Z}$. In view of (2.28) , we rewrite (7.24) in the form

$$a = \frac{d}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle}\langle p_{j-1}^I \alpha_{i_j}, \left(x + 2i\pi q (g^j)_{I,i_j}\right)^I - 2i\pi \widehat{k}\rangle\,, \qquad 0 \leq g^j < d|\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle|\,. \tag{7.25}$$

By periodicity, the integrals of $h_{g,j}(a)$ over $\delta_+$ and $\delta_-$ cancel each other.

Using the theorem of residues and (2.70), (2.76), (2.78), (7.20), (7.23), (7.25), we get

$$\begin{aligned} L_I = -\frac{1}{d} & \sum_{\substack{0\leq f<d\ ,g\in(\mathbf{Z}/d\mathbf{Z})^{j-1} \\ i_j:(\alpha_{i_1},\dots,\alpha_{i_j})\neq 0\ ,0\leq g^j<d|\langle p_{j-1}^I\alpha_{i_j},e^j\rangle|}} \\ & \left\{\frac{1}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle} \exp\left(2i\pi \langle p_j^{(I,i_j)} e_j, \widehat{k}\rangle \langle p_{j-1}^I (t + qfe_j), e^j/q\rangle \right.\right. \\ & \left. + d\sum_{n=1}^{j} \frac{\langle p_{n-1}^I \alpha_{i_n}, x^I\rangle}{\langle p_{n-1}^I \alpha_{i_n}, e^n\rangle}\left[\frac{1}{d}\langle p_{n-1}^I(t+qfe_j), e^n/q\rangle\right]\right) \\ & \left.\frac{1}{\prod_{i\notin(I,i_j)} 2q\tanh\left(\frac{\langle p_j^{(I,i_j)}\alpha_i, 2i\pi\widehat{k} - x^I\rangle}{2q}\right)}\right\} (x + 2i\pi q(g, g_j)_{I,i_j}) \\ & \frac{1}{\exp\left(\frac{d\langle p_{j-1}^I \alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I \alpha_{i_j}, e^j\rangle}\right) - 1}\,. \end{aligned} \tag{7.26}$$

We claim that as functions of $x_{i_j}$, the terms in the right-hand side of (7.26) are periodic of period $2i\pi qd$. In fact this follows from what we saw after (7.7), by replacing $j-1$ by $j$. We also claim these functions, are periodic as functions of $x_{i_j}$, with period $2i\pi qd\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle$. In fact, using (2.29),

$$(7.27)\qquad \frac{d\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle} = \frac{d}{\langle \alpha_{i_1}, \dots, \alpha_{j-1}\rangle} \in \mathbf{Z}\,.$$

In view of (2.63), (7.27) takes care of the terms $\tanh\left(\frac{\langle p^{(I,i_j)}_{j-1}\alpha_i, 2i\pi\widehat{k} - x^I\rangle}{2q}\right)$. Also among the term $\langle p^I_{n-1}\alpha_{i_n}, x^I\rangle$, $n \le j$, only $\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle$ depends on $x_{i_j}$. More precisely by (2.63),

$$(7.28)\qquad \frac{\partial}{\partial x_{i_j}}\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle = 1\,.$$

Clearly,

$$(7.29)\qquad \begin{aligned} &qd\frac{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}d\left[\frac{1}{d}\langle p^I_{j-1}(t+qfe_j), e^j/q\rangle\right] = \\ &d\langle p^I_{j-1}(t+qfe_j), e^j\rangle \mod (\mathbf{Z})\,. \end{aligned}$$

Using (2.29), we find that (7.29) is 0 mod ($\mathbf{Z}$), i.e. the left-hand side of (7.29) lies in $\mathbf{Z}$. Ultimately, this guarantees the periodicity of the functions appearing in (7.26) in $x_{i_j}$, with period $2i\pi qd\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle$.

Now by (2.29),

$$(7.30)\qquad d\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle = \frac{d\langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle}{\langle \alpha_{i_1}, \dots, \alpha_{i_{j-1}}\rangle}\,.$$

From (7.30), we find that

$$(7.31)\qquad d\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle \mid d^2\,.$$

Since the functions in (7.26) are periodic in $x_{i_j}$ of period $2i\pi qd\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle$, using (7.31), in (7.26), we may and we will replace the condition $0 \le g^j < d|\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle|$ by the condition $0 \le g^j < d^2$, and this introduces a correcting factor $\frac{|\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle|}{d}$ in the right-hand side of (7.26). Using the periodicity in $x_{i_j}$ of period $2i\pi qd$, we may finally replace the condition $0 \le g^j < d^2$ by $0 \le g^j < d$, with a new correcting factor $d$. Ultimately in (7.26), we replace the condition $0 \le g^j < d|\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle|$ by $0 \le g^j < d$, with a correcting factor $|\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle|$. Also by (7.30),

$$(7.32)\qquad \operatorname{sgn}\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle = \operatorname{sgn}\langle \alpha_{i_1}, \dots, \alpha_{i_j}\rangle \operatorname{sgn}\langle \alpha_{i_1}, \dots, \alpha_{j-1}\rangle\,.$$

So, from (7.26), (7.32), we get

$$(7.33)\qquad L_I = \frac{-1}{d} \sum_{\substack{0\le f<d,\, g\in(\mathbf{Z}/d\mathbf{Z})^j \\ i_j:\langle\alpha_{i_1},\dots,\alpha_{i_j}\rangle\neq 0}} \operatorname{sgn}(\langle\alpha_{i_1},\dots,\alpha_{i_{j-1}}\rangle\langle\alpha_{i_1},\dots,\alpha_{i_j}\rangle)$$

$$\Bigg\{\exp\Bigg(2i\pi\langle p_j^{(I,i_j)}e_j,\widehat{k}\rangle\langle p_{j-1}^I(t+qfe_j),e^j\rangle + d\sum_{n=1}^{j}\frac{\langle p_{n-1}^I\alpha_{i_n},x^I\rangle}{\langle p_{n-1}^I\alpha_{i_n},e^n\rangle}$$

$$\left[\frac{1}{d}\langle p_{n-1}^I(t+qfe_j),e^n/q\rangle\right]\Bigg)$$

$$\frac{1}{\prod_{i\notin(I,i_j)} 2q\tanh\left(\frac{\langle p_j^{(I,i_j)}\alpha_i,2i\pi\widehat{k}-x^I\rangle}{2q}\right)}\Bigg\}(x+2i\pi q g_{(I,i_j)})$$

$$\frac{1}{\exp\left(\frac{d\langle p_{j-1}^I\alpha_{i_j},x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j},e^j\rangle}\right)-1}.$$

Clearly

$$(7.34)\qquad L_q(t,x) = \sum_{k=(k_1,\dots,k_r)\in(\mathbf{Z}/q\mathbf{Z})^r} \frac{\exp(2i\pi\sum_{j=1}^r k_j\langle t,e^j/q\rangle)}{\prod_{i=1}^{\ell} 2q\tanh\left(\frac{2i\pi\langle\alpha_i,\sum_{j=1}^r k_je^j\rangle-x_i}{2q}\right)}.$$

Also with the notation in (2.63),

$$(7.35)\qquad \langle p_0^{\phi}\alpha_i,x^I\rangle = x_i\,.$$

Then we use (7.33), so that if $t\in\overline{R}/q\overline{R}$, $t/q\notin H$,

$$(7.36)\qquad L_q(t,x) = \frac{-1}{d}\sum_{\substack{0\le f^1<d,\, g\in\mathbf{Z}/d\mathbf{Z} \\ \langle\alpha_{i_1}\rangle\neq 0,\, k=(k^2,\dots,k^r)\in(\mathbf{Z}/q\mathbf{Z})^{r-1}}}$$

$$\operatorname{sgn}\langle\alpha_{i_1}\rangle\exp\left(2i\pi\langle p_1^{(i_1)}(t+qf^1e_1),k\rangle + d\frac{x_{i_1}}{\langle\alpha_{i_1}\rangle}\left[\frac{\langle t+qf'e_1,e^1/q\rangle}{d}\right]\right)$$

$$\frac{1}{\prod_{i\neq i_1} 2q\tanh\left(\frac{\langle p_1^{(i_1)}\alpha_i,2i\pi k-x^I\rangle}{2q}\right)}(x+2i\pi q g_{i_1})\frac{1}{\exp\left(\frac{dx_{i_1}}{\langle\alpha_{i_1}\rangle}\right)-1}.$$

Using (7.33), (7.36) and an obvious iteration, in view of (2.83), we obtain (7.6). The proof of our Theorem is completed. □

**7.2. A residue formula for $M_q(t,x)$.**

DEFINITION 7.3. For $q\in\mathbf{N}^*$, $t\in\overline{R}/q\overline{CR}$, $x\in(\mathbf{C}\backslash 2i\pi\mathbf{Z}/m)^{\ell}$, put

$$(7.37)\qquad M_q(t,x) = \sum_{\lambda\in\overline{CR}^*/q\overline{R}^*} \frac{\exp(2i\pi\langle\lambda,t/q\rangle)}{\prod_{i=1}^{\ell} 2q\tanh\left(\frac{2i\pi\langle\alpha_i,\lambda\rangle-x_i}{2q}\right)}.$$

Now we will adapt the formalism of Sections 2.5 and 2.6, with $\mathfrak{t}/\overline{CR}$ replaced by $\overline{R}/q\overline{CR}$. If $f(t)$ is a function on $\overline{R}/q\overline{CR}$, if $\mu\in\overline{CR}^*/\overline{R}^*$, put

$$(7.38)\qquad \widehat{f}_\mu(t) = \frac{1}{|\overline{R}/\overline{CR}|}\sum_{v\in\overline{R}/\overline{CR}} e^{-2i\pi\langle\mu,v\rangle}f(t+qv)\,.$$

Then by (2.92),

$$f(t) = \sum_{\mu \in \overline{CR}^*/\overline{R}^*} \widehat{f}_\mu(t). \tag{7.39}$$

PROPOSITION 7.4. *If $\mu \in \overline{CR}^*/\overline{R}^*$, if $\lambda_\mu \in \overline{CR}^*$ represents $\mu$, for $t \in \overline{R}/q\overline{CR}$,*

$$(\widehat{M_q})_\mu(t,x) = \exp(2i\pi\langle\lambda_\mu, t/q\rangle)L_q(t, x - 2i\pi\widetilde{a}\lambda_\mu). \tag{7.40}$$

PROOF. If $\lambda \in \overline{CR}^*/q\overline{R}^*$, then

$$\begin{aligned} \exp(2\widehat{i\pi\langle\lambda, t/q\rangle})_\mu \;&=\; \exp(2i\pi\langle\lambda, t/q\rangle) \text{ if } \lambda \text{ maps to } \mu \\ &\qquad\qquad \text{in } \overline{CR}^*/\overline{R}^* \\ &\;\;\; 0 \quad \text{otherwise}. \end{aligned} \tag{7.41}$$

Then

$$(\widehat{M_q})_\mu(t,x) = \sum_{\lambda \in \overline{R}^*/q\overline{R}^*} \frac{\exp(2i\pi\langle\lambda + \lambda_\mu, t/q\rangle}{\prod_{i=1}^{\ell} 2q \tanh\left(\frac{2i\pi\langle\alpha_i,\lambda\rangle - (x_i - 2i\pi\langle\alpha_i,\lambda_\mu\rangle)}{2q}\right)}, \tag{7.42}$$

which is equivalent to (7.40). □

PROPOSITION 7.5. *For $q \in \mathbf{N}^*$, $t \in \overline{R}/q\overline{CR}$, $x \in (\mathbf{C}\backslash 2i\pi\mathbf{Z}/m)^\ell$,*

$$M_q(t,x) = \sum_{\mu \in \overline{CR}^*/\overline{R}^*} \exp(2i\pi\langle\lambda_\mu, t/q\rangle)L_q(t, x - 2i\pi\widetilde{a}\lambda_\mu). \tag{7.43}$$

PROOF. This follows from (7.39), (7.40). □

Recall that $\tau$ is the projection $\mathfrak{t}/\overline{CR} \to \mathfrak{t}/\overline{R}$. Then $\tau$ induces the projection $\overline{R}/q\overline{CR} \to \overline{R}/q\overline{R}$. Also recall that the set $\overline{S} \subset \mathfrak{t}/\overline{CR}$ was defined in (2.111). By (2.45), (2.111),

$$\begin{aligned} \{t \in \overline{R}/q\overline{CR}, t/q \in \overline{S}\} \;&=\; \{t \in \overline{R}/q\overline{CR}, \tau t = \sum_{j \in \mathcal{J}} t^j \alpha_j, \\ &\qquad \{\alpha_j, j \in \mathcal{J}\} \text{ does not span } \mathfrak{t}^*\}. \end{aligned} \tag{7.44}$$

THEOREM 7.6. *For generic values of $x \in (\mathbf{C}\backslash 2i\pi\mathbf{Z}/m)^\ell$, $t \in \overline{R}/q\overline{CR}$, $t/q \notin \overline{S}$, if we still denote by $t$ a representative of $t$ in $\overline{R}$, then*

$$\begin{aligned} M_q(t,x) = \;& \\ &\frac{(-1)^r}{d^r} \sum_{\substack{I=(i_1,\dots,i_r)\subset\{1,\dots,\ell\}, I \text{ generic} \\ f \in \overline{R}/d\overline{R}, g \in (\mathbf{Z}/d\mathbf{Z})^r, \mu \in \overline{CR}^*/\overline{R}^*}} \operatorname{sgn}(\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle)\exp(2i\pi\langle\lambda_\mu, t/q\rangle) \\ &\left[\exp\left(d\sum_{j=1}^{r} \frac{\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p_{j-1}^I(t+qf), e^j/q\rangle\right]\right)\right. \\ &\left.\frac{1}{\prod_{i \in {}^cI} 2q\tanh\left(\frac{\langle\alpha_i, x^I\rangle - x_i}{2q}\right)}\right](x + 2i\pi q g_I - 2i\pi\widetilde{a}\lambda_\mu) \\ &\frac{1}{\prod_{j=1}^{r}\left(\exp\left(\frac{d\langle p_{j-1}^I\alpha_{i_j}, x^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}\right) - 1\right)}(x - 2i\pi\widetilde{a}\lambda_\mu). \end{aligned} \tag{7.45}$$

PROOF. This follows from Theorem 7.2 and Proposition 7.5. □

DEFINITION 7.7. If $I = (i_1, \dots, i_r)$ is generic, let $H^I$ be the lattice in $\mathfrak{t}$ generated by $\alpha^{i_1}, \dots, \alpha^{i_r}$.

Recall that $C \subset \overline{R}^*$ was defined in Definition 1.35.

PROPOSITION 7.8. *The following identity holds*

$$C = \bigcup_{I \text{ generic}} H^I. \tag{7.46}$$

*In particular for $I$ generic,*

$$dH^I \subset \overline{R}^*. \tag{7.47}$$

PROOF. The identity (7.46) follows from the definition of $C$. Then (7.47) follows from (2.21), (7.46). The proof of our Proposition is completed. □

By Proposition 7.8, we have a natural projection

$$h^I : H^I/dH^I \to H^I/\overline{R}^*, \tag{7.48}$$

whose kernel is just

$$\ker h^I = \overline{R}^*/dH^I. \tag{7.49}$$

PROPOSITION 7.9. *For any generic $I$,*

$$|\overline{R}^*/dH^I| = \frac{d^r}{|\langle \alpha_{i_1}, \dots, \alpha_{i_r} \rangle|}. \tag{7.50}$$

PROOF. Recall that $e_1, \dots, e_r$ span $\overline{R}$, and $e^1, \dots, e^r$ span $\overline{R}^*$. Therefore

$$\begin{aligned} |\overline{R}^*/dH^I| &= d^r |\langle \alpha^{i_1} \wedge \dots \wedge \alpha^{i_r}, e_1 \wedge \dots \wedge e_r \rangle| \\ &= \frac{d^r}{|\langle \alpha_{i_1} \wedge \dots \wedge \alpha_{i_r} \rangle|}. \end{aligned} \tag{7.51}$$

The proof of our Proposition is completed. □

THEOREM 7.10. *For generic values of $x \in (\mathbf{C} \backslash 2i\pi \mathbf{Z}/m)^\ell$, for $t \in \overline{R}/q\overline{CR}$, $t/q \notin \overline{S}$, if we still denote by $t$ a representative of $t$ in $\overline{R}$, then*

$$\begin{aligned} M_q(t,x) = \left| \frac{\overline{R}}{\overline{CR}} \right| (-1)^r \sum_{u \in C/\overline{R}^*} \sum_{\substack{I = (i_1, \dots, i_r) \subset I_u, I \text{ generic} \\ f \in \overline{CR}/d\overline{R}}} \\ \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r} \rangle} \frac{\exp\left(2i\pi\left(\langle u, t+qf \rangle\right)\right)}{\prod_{i \in {}^c I} 2q \tanh\left(\frac{\langle \alpha_i, x^I + 2i\pi q u \rangle - x_i}{2q}\right)} \\ \exp\left(d \sum_{j=1}^r \frac{\langle p^I_{j-1} \alpha_{i_j}, x^I \rangle}{\langle p^I_{j-1} \alpha_{i_j}, e^j \rangle} \left[ \frac{1}{d} \langle p^I_{j-1}(t+qf), e^j/q \rangle \right] \right) \\ \frac{1}{\prod_{j=1}^r \left( \exp\left( \frac{d \langle p^I_{j-1} \alpha_{i_j}, x^I \rangle}{\langle p^I_{j-1} \alpha_{i_j}, e^j \rangle} \right) - 1 \right)}. \end{aligned} \tag{7.52}$$

PROOF. Clearly the map $g \in (\mathbf{Z}/d\mathbf{Z})^\ell \mapsto g_I^I \in H^I/dH^I$ is one to one. Also if $t \in \overline{R}$, $f \in \overline{R}$, by (2.13), (2.28), (2.29),

$$\exp\left(d\sum_{j=1}^{r}\frac{\langle p_{j-1}^I\alpha_{i_j}, 2i\pi q g_I^I\rangle}{\langle p_{j-1}^I\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p_{j-1}^I(t+qf), e^j/q\rangle\right]\right) = \exp(2i\pi\langle t+qf, g_I^I\rangle) = \exp(2i\pi\langle t+qf, h^I g_I^I\rangle)\,. \tag{7.53}$$

Moreover by (2.115),

$$\langle\alpha_i, (\widetilde{a}\lambda_\mu)^I\rangle - (\widetilde{a}\lambda_\mu)_i = 0\,. \tag{7.54}$$

Also if $i \notin I$, using (7.47),

$$\langle\alpha_i, g_I^I\rangle - (g_I)_i = \langle\alpha_i, g_I^I\rangle\,, \qquad q\langle\alpha_i, g_I^I\rangle = q\langle\alpha_i, h^I g_I^I\rangle \text{ in } \mathbf{Z}/q\mathbf{Z}\,. \tag{7.55}$$

Finally by Proposition 7.8, the map $h^I : H^I/dH^I \to C/\overline{R}^*$ surjects on $\{u \in C/\overline{R}^*, I \subset I_u\}$, and by (7.49), (7.50), the fibre has $\frac{d^r}{|\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle|}$ elements.

From (2.13), (2.119A), (7.45), (7.53)-(7.55), and proceeding as in the proof of Theorem 2.28, we get (7.52). The proof of our Theorem is completed. □

**7.3. The Verlinde sums.** Recall that by Proposition 1.27, if $\lambda \in \overline{CR}^*$, the character $\chi_\lambda$ of $G$ does not depend on the choice of a Weyl chamber $K$ such that $\lambda \in \overline{K}$. Also by Proposition 1.28, if $w \in W$

$$\chi_{w\lambda} = \chi_\lambda\,. \tag{7.56}$$

Moreover the function $(i^\ell\sigma)^2(t)$ is well defined on $T' = \mathfrak{t}/\overline{R}^*$.

Now we introduce the Verlinde sums [62], [3].

DEFINITION 7.11. For $g \in \mathbf{N}$, $q \in \mathbf{N}^*$, $\theta_1, \dots, \theta_s \in \overline{CR}^*$, put

$$V_{g,q}(\theta_1,\dots,\theta_s) = \left|\frac{\overline{CR}^*}{q\overline{CR}}\right|^{g-1}\frac{1}{|W|}\sum_{\substack{\lambda\in\overline{CR}^*/q\overline{CR}\\ \sigma^2(\lambda/q)\neq 0}}\frac{1}{(i^\ell\sigma(\lambda/q))^{2g-2}}\prod_{j=1}^{s}\chi_{\theta_j}(e^{\lambda/q})\,. \tag{7.57}$$

By Proposition 1.12, we get

$$V_{g,q}(\theta_1,\dots,\theta_s) = \left|\frac{\overline{CR}^*}{q\overline{CR}}\right|^{g-1}\sum_{\lambda\in qP\cap\overline{CR}^*}\frac{1}{(i^\ell\sigma(\lambda/q))^{2g-2}}\prod_{j=1}^{s}\chi_{\theta_j}(e^{\lambda/q})\,. \tag{7.58}$$

Recall that in Section 1.8, it was shown that $\sum_{j=1}^s \theta_j \in \overline{R}$, if and only if for $(w^1,\dots,w^s) \in W^s$, $\sum_{j=1}^s w^j\theta_j \in \overline{R}$.

By Proposition 1.29, observe that if $\sum_{j=1}^s \theta_j \in \overline{R}$, then $\prod_{j=1}^s \chi_{\theta_j}(e^{\lambda/q})$ is a well-defined function of $\lambda \in \overline{CR}^*/q\overline{R}^*$.

THEOREM 7.12. *If $\sum_{j=1}^s \theta_j \notin \overline{R}$, then*

$$V_{g,q}(\theta_1,\dots,\theta_s) = 0\,. \tag{7.59}$$

*If* $\sum_{j=1}^{s} \theta_j \in \overline{R}$, *then*

$$V_{g,q}(\theta_1, \dots, \theta_s) = \left| \frac{\overline{CR}^*}{q\overline{CR}} \right|^{g-1} \frac{|Z(G)|}{|W|} \sum_{\substack{\lambda \in \overline{CR}^*/q\overline{R}^* \\ \sigma^2(\lambda/q) \neq 0}} \frac{1}{(i^\ell \sigma(\lambda/q))^{2g-2}} \prod_{j=1}^{s} \chi_{\theta_j}(e^{\lambda/q}) . \tag{7.60}$$

PROOF. Clearly

$$V_{g,q}(\theta_1, \dots, \theta_s) = \left| \frac{\overline{CR}^*}{q\overline{CR}} \right|^{g-1} \frac{1}{|W|} \sum_{\substack{\lambda \in \overline{CR}/q\overline{R}^* \\ \sigma^2(\lambda/q) \neq 0}} \frac{1}{(i^\ell \sigma(\lambda/q))^{2g-2}} \sum_{\mu \in \overline{R}^*/\overline{CR}} \prod_{j=1}^{s} \chi_{\theta_j}(e^{\lambda/q+\mu}) . \tag{7.61}$$

Using Propositions 1.29 and 1.30, (7.61), and the fact that $Z(G) = \overline{R}^*/\overline{CR}$, we get our Theorem. □

THEOREM 7.13. *The following identity holds,*

$$V_{g,c}(\theta_1, \dots, \theta_s) = \prod_{j=1}^{s} \chi_{\theta_j}(e^{\rho/c}) . \tag{7.62}$$

*If one of the* $\theta_j$*'s does not lie in* $\overline{R}$,

$$V_{g,c}(\theta_1, \dots, \theta_s) = 0 . \tag{7.63}$$

*More generally*

$$V_{g,c}(\theta_1, \dots, \theta_s) = 0 \ , \ +1 \ or \ -1 . \tag{7.64}$$

PROOF. By Proposition 1.10 and 1.11, we get (7.62). By Proposition 1.31 or by Theorem 7.12 when $s = 1$ and by (7.62), we get (7.63). By (7.62) and by Theorem 1.32, (7.64) follows. The proof of our Theorem is completed. □

Let $K$ be a Weyl chamber.

DEFINITION 7.14. If $\theta \in \overline{CR}^*$, put

$$\overline{\chi}_\theta^K(t) = \frac{1}{\sigma(t)} \sum_{w \in W} \varepsilon_w e^{2i\pi \langle w(\rho+\theta), t \rangle} . \tag{7.65}$$

By (3.115) , if $\rho + \theta$ does not lie in a Weyl chamber,

$$\overline{\chi}_\theta^K(t) = 0 . \tag{7.66}$$

If $\rho + \theta$ lies in the Weyl chamber $w_0 K$, then

$$\overline{\chi}_\theta^K(t) = \chi_{\rho+\theta-w_0\rho}(t) . \tag{7.67}$$

Also if $w \in W$,

$$\overline{\chi}_{w\theta}^{wK}(t) = \overline{\chi}_\theta^K(t) . \tag{7.68}$$

DEFINITION 7.15. For $g \in \mathbf{N}$, $q \in \mathbf{Z}^*$, $\theta_1, \dots, \theta_s, \theta_{s+1} \in \overline{CR}^*$, put

$$\overline{V}^K_{g,q}(\theta_1, \dots, \theta_s, \theta_{s+1}) = \left|\frac{\overline{CR}^*}{q\overline{CR}}\right|^{g-1} \frac{1}{|W|} \sum_{\substack{\lambda \in \overline{CR}^*/q\overline{CR} \\ \sigma^2(\lambda/q) \neq 0}} \frac{1}{(i^\ell \sigma(\lambda/q))^{g-1}} \prod_{j=1}^{s} \chi_{\theta_j}(e^{\lambda/q}) \overline{\chi}^K_{\theta_{s+1}}(e^{\lambda/q}). \tag{7.69}$$

THEOREM 7.16. *For $g \in \mathbf{N}$, $q \in \mathbf{N}^*$, the following identity holds,*

$$V_{g,q}(\theta_1, \dots, \theta_s) = \frac{(-1)^\ell}{\left|\frac{\overline{CR}^*}{q\overline{CR}}\right|} \sum_{(v^1, v^2) \in W^2} \varepsilon_{v^1} \varepsilon_{v^2} \overline{V}^K_{g+1,q}(\theta_1, \dots, \theta_s, v^1\rho + v^2\rho). \tag{7.70}$$

PROOF. Recall that

$$\chi_0 = 1. \tag{7.71}$$

By (7.57), (7.71),

$$V_{g,q}(\theta_1, \dots, \theta_s) = \left|\frac{\overline{CR}^*}{q\overline{CR}}\right|^{g-1} \frac{1}{|W|} \sum_{\substack{\lambda \in \overline{CR}^*/q\overline{CR} \\ \sigma^2(\lambda/q) \neq 0}} \frac{1}{(i^\ell \sigma(\lambda/q))^{2g-2}} \prod_{j=1}^{s} \chi_{\theta_j}(e^{\lambda/q}) \chi_0^3(e^{\lambda/q}). \tag{7.72}$$

By (7.56), we may and we will assume that $\theta_1, \dots, \theta_s \in \overline{CR}^*_+ = \overline{CR}^* \cap \overline{K}$.
Using (1.94), (7.72) we obtain

$$\begin{aligned} V_{g,q}(\theta_1, \dots, \theta_s) &= \frac{\left|\frac{\overline{CR}^*}{q\overline{CR}}\right|^{g-1}}{|W|} \sum_{\substack{\lambda \in \overline{CR}^*/q\overline{CR} \\ \sigma^2(\lambda/q) \neq 0}} \frac{1}{(i^\ell \sigma(\lambda/q))^{2g-2}} \frac{1}{(\sigma(\lambda/q))^{s+3}} \sum_{(w^1, \dots, w^s) \in W^{s+3}} \prod_{j=1}^{s+3} \varepsilon_{w^j} \\ &\quad \exp\left(2i\pi\langle \sum_{j=1}^{s} w^j(\rho + \theta_j) + \sum_{j=s+1}^{s+3} w^j\rho, \lambda/q\rangle\right) \\ &= (-1)^\ell \frac{|\overline{CR}^*/q\overline{CR}|^{g-1}}{|W|} \sum_{\substack{\lambda \in \overline{CR}^*/q\overline{CR} \\ \sigma^2(\lambda/q) \neq 0}} \frac{1}{(i^\ell \sigma(\lambda/q))^{2g}} \frac{1}{(\sigma(\lambda/q))^{s+1}} \\ &\quad \sum_{(v^1, v^2) \in W^2} \varepsilon_{v^1} \varepsilon_{v^2} \sum_{(w^1, \dots, w^{s+1}) \in W^{s+1}} \prod_{j=1}^{s+1} \varepsilon_{w^j} \\ &\quad \exp\left(2i\pi\langle \sum_{j=1}^{s} w^j(\rho + \theta_j) + w^{s+1}(\rho + v^1\rho + v^2\rho), \lambda/q\rangle\right), \end{aligned} \tag{7.73}$$

which is just (7.70).

The proof of our Theorem is completed. □

REMARK 7.17. If $w \in W$, $\rho + w\rho \in \overline{R}$. Therefore (7.59) and (7.70) are compatible. Also observe that the fusion rules [**3**], [**62**] express $V_{g,q}(.)$ in terms of $V_{g-1,q}(.)$. Here equation (7.70) goes the opposite way.

**7.4. A residue formula for the Verlinde sums.** In the sequel, we fix a Weyl chamber $K \subset \mathfrak{t}$, and we use the notation in Section 2.

Also by (7.56) and (7.59), we may and we will assume that

$$\theta_j \in \overline{CR}^*_+ = \overline{CR}^* \cap \overline{K} \ , \ 1 \leq j \leq s, \qquad \sum_{j=1}^{s} \theta_j \in \overline{R}. \tag{7.74}$$

DEFINITION 7.18. If $w \in W$, let $\sigma_w \in \mathcal{S}_\ell$, let $\varepsilon_w(1), \dots, \varepsilon_w(\ell) \in \{-1, +1\}$ be such that for $1 \leq i \leq \ell$,

$$w^{-1}\alpha_i = \varepsilon_w(i)\alpha_{\sigma_w(i)} \,. \tag{7.75}$$

By [**15**, Corollary V.4.6 and Lemma V.4.10] , we know that

$$\varepsilon_w = \prod_{i=1}^{\ell} \varepsilon_w(i) \,. \tag{7.76}$$

DEFINITION 7.19. For $x \in \mathbf{C}^\ell$, $w \in W$, put

$$\varphi_w(x) = \prod_{i=1}^{\ell} \frac{1}{1 - e^{-\varepsilon_w(i) x_i}} , . \tag{7.77}$$

Equivalently, by (7.76),

$$\varphi_w(x) = \varepsilon_w \prod_{i=1}^{\ell} \frac{1}{2\sinh(\frac{x_i}{2})} e^{\frac{1}{2}\sum_1^\ell \varepsilon_w(i) x_i} \,. \tag{7.78}$$

THEOREM 7.20. *For $g \geq 2$, $q \in \mathbf{N}^*$, the following identity holds,*

$$\begin{aligned} V_{g,q}(\theta_1, \dots, \theta_s) = {} & \operatorname{Vol}(T)^{2g-2} q^{(g-1)r} \frac{|Z(G)|}{|W|} \\ & (-1)^{\ell(g-1)} \operatorname{Res}_{x=0}^{(1,\dots,\ell)} \frac{1}{\left[\prod_{i=1}^{\ell} 2\sinh(\frac{x_i}{2q})\right]^{2g-2}} \sum_{(w^1,\dots,w^s) \in W^s} \prod_{j=1}^{s} \varphi_{w^j}(x/q) \\ & M_q\left(\sum_{j=1}^{s} w^j \theta_j, x\right) . \end{aligned} \tag{7.79}$$

*For $g \geq 0$, for $s$ even, $2g - 2 + s \geq 1$ and $q \in \mathbf{N}^*$, the following identity holds,*

$$\begin{aligned} V_{g,q}(\theta_1, \dots, \theta_s) = {} & \operatorname{Vol}(T)^{2g-2} q^{(g-1)r} \frac{|Z(G)|}{|W|} \\ & (-1)^{\ell(g-1)} \operatorname{Res}_{x=0}^{(1,\dots,\ell)} \frac{1}{\left[\prod_{i=1}^{\ell} 2\sinh(\frac{x_i}{2q})\right]^{2g-2+s}} \sum_{(w^1,\dots,w^s) \in W^s} \prod_{j=1}^{s} \epsilon_{w^j} \\ & M_q\left(\sum_{j=1}^{s} w^j(\rho + \theta_j), x\right). \end{aligned} \tag{7.80}$$

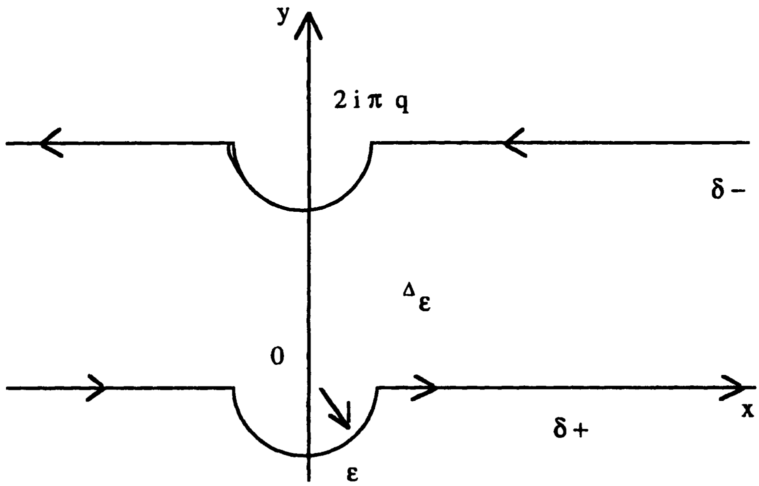


FIGURE 7.2

PROOF. First we assume that $g \geq 2$. For $\lambda \in \overline{CR}^* / q\overline{R}^*$, $w = (w^1, \ldots, w^s) \in W^s$, $x \in (\mathbf{C} \backslash 2i\pi\mathbf{Z}/m)^\ell$, put

$$U_{\lambda,w}(x) = \frac{1}{\left[\prod_{i=1}^{\ell} 2\sinh(\frac{x_i}{2q})\right]^{2g-2}} \prod_{j=1}^{s} \varphi_{w^j}(x/q) \tag{7.81}$$

$$\prod_{i=1}^{\ell} \frac{1}{2q \tanh\left(\frac{2i\pi\langle\lambda,\alpha_i\rangle - x_i}{2q}\right)}.$$

Then $U_{\lambda,w}(x)$ is a meromorphic function of $x_1, \ldots, x_\ell$, which is periodic of period $2i\pi q, \ldots, 2i\pi q$.

For $\varepsilon > 0$, let $\Delta_\varepsilon$ be the domain in $\mathbf{C}$ given in Figure 7.2. For $\varepsilon > 0$ small enough, as a function of $x_i \in \Delta_\varepsilon$, the function $U_{\lambda,w}(x)$ has poles at 0, and also at $2i\pi q \left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]$. Here $q\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]$ is the real number in $[0, q[$ which represents $\langle\lambda, \alpha_i\rangle$ mod $q$. For $\varepsilon > 0$ small enough, the poles do not meet the boundary $\partial\Delta_\varepsilon = \delta_- \cup \delta_+$.

Assume first that $\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right] \neq 0$. Then $2i\pi q\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]$ is a simple pole of $U_{\lambda,w}(x)$. We now use the theorem of residues on $\Delta_\varepsilon$. Since $g \geq 2$, as $x_i \to \pm\infty$ inside $\Delta_\varepsilon$, the function $U_{\lambda,w}(x)$ tends to 0. Also since $U_{\lambda,w}(x)$ is periodic of period $2i\pi q$ in the variable $x_i$, the boundary integrals cancel each other. Therefore

$$\mathrm{Res}_{x_i=0} U_{\lambda,w}(x) + \mathrm{Res}_{x_i = 2i\pi q\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]} U_{\lambda,w}(x) = 0\,. \tag{7.82}$$

Clearly

$$\text{(7.83)}\quad \operatorname{Res}_{x_i=2i\pi q\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]} U_{\lambda,w}(x) =$$

$$-\left[\frac{1}{2i\sin\left(\frac{\langle\lambda,\alpha_i\rangle\pi}{q}\right)}\prod_{j\neq i}\frac{1}{2\sinh\left(\frac{x_j}{2q}\right)}\right]^{2g-2}$$

$$\prod_{j=1}^{s}\varphi_{w^j}\left(\frac{x_1}{q}\dots\frac{x_{i-1}}{q},\frac{2i\pi\langle\lambda,\alpha_i\rangle}{q},\dots,\frac{x_\ell}{q}\right)\prod_{j\neq i}\frac{1}{2q\tanh\left(\frac{2i\pi\langle\lambda,\alpha_i\rangle-x_i}{2q}\right)}\,.$$

Assume now that $\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]=0$. Then as a function of $x_i$, the function $U_{\lambda,w}(x)$ has a single pole at $x_i=0$. By the theorem of residues used as before, we get

$$\text{(7.84)}\qquad \operatorname{Res}_{x_i=0}U_{\lambda,w}(x)=0\,.$$

By (7.83), (7.84), we obtain

$$\text{(7.85)}\qquad (-1)^{\ell(g-1)}\operatorname{Res}_{x=0}^{1,\dots,\ell}\frac{1}{\left[\prod_1^\ell 2\sinh\left(\frac{x_i}{2q}\right)\right]^{2g-2}}\sum_{(w^1,\dots,w^s)\in W^s}\varphi_{w^j}\left(\frac{x}{q}\right)$$

$$M_q\left(\sum_{j=1}^{s}w^j\theta_j,x\right)=\sum_{\substack{\lambda\in\overline{C\mathbf{R}^*}/q\mathbf{R}^*,\sigma^2(\lambda/q)\neq 0\\(w^1,\dots,w^s)\in W^s}}\frac{1}{\left(i^\ell\sigma(\lambda/q)\right)^{2g-2}}$$

$$\prod_{j=1}^{s}\varphi_{w^j}\left(2i\pi\frac{\langle\lambda,\alpha_1\rangle}{q},\dots,2i\pi\frac{\langle\lambda,\alpha_\ell\rangle}{q}\right)\exp\left(2i\pi\langle w^j\theta_j,\lambda/q\rangle\right)\,.$$

Now by (1.93), (7.75), (7.78), for $1\leq j\leq s$,

$$\text{(7.86)}\qquad \sum_{w^j\in W}\varphi_{w^j}\left(2i\pi\frac{\langle\lambda,\alpha_i\rangle}{q},\dots,2i\pi\frac{\langle\lambda,\alpha_1\rangle}{q}\right)\exp\left(2i\pi\langle w^j\theta_j,\lambda/q\rangle\right)$$

$$=\chi_{\theta_j}(e^{\lambda/q})\,.$$

From (1.17), (7.85), (7.86), we get (7.79).

Now we consider the case where $g\geq 0$, $s$ is even and $2g-2+s\geq 1$. Put

$$\text{(7.87)}\quad H_{\lambda,w}(x)=\frac{1}{\left[\prod_{i=1}^\ell 2\sinh(\frac{x_i}{2q})\right]^{2g-2+s}}\prod_{i=1}^{\ell}\frac{1}{2q\tanh\left(\frac{2i\pi\langle\lambda,\alpha_i\rangle-x_i}{2q}\right)}\,.$$

Then $H_{\lambda,w}(x)$ is a meromorphic function of $x_1,\dots,x_\ell$. Since $s$ is even, it is periodic of period $2i\pi q,\dots,2i\pi q$ in $x_1,\dots,x_\ell$.

Assume that $\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]\neq 0$. Since $2g-2+s\geq 1$, as $x_i$ tends to $\pm\infty$ inside $\Delta_\epsilon$, $H_{\lambda,w}(x)$ tends to 0. We thus find that the analogue of (7.82) holds, with $U_{\lambda,w}$ replaced by $H_{\lambda,w}$. The analogue of (7.83) is now

$$\text{(7.88)}\quad \operatorname{Res}_{x_i=2i\pi q\left[\frac{\langle\lambda,\alpha_i\rangle}{q}\right]}H_{\lambda,w}(x)=-\left[\frac{1}{2i\sin\left(\frac{\langle\lambda,\alpha_i\rangle\pi}{q}\right)}\prod_{j\neq i}\frac{1}{2\sinh\left(\frac{x_j}{2q}\right)}\right]^{2g-2+s}\prod_{j\neq i}\frac{1}{2q\tanh\left(\frac{2i\pi\langle\lambda,\alpha_i\rangle-x_i}{2q}\right)}\,.$$

Also the analogue of (7.84) still holds. So we get the analogue of (7.85),

$$(7.89)\quad (-1)^{\ell(g-1)}\mathrm{Res}^{1,\dots,\ell}_{x=0}\frac{1}{\left[\prod_1^\ell 2\sinh\left(\frac{x_i}{2q}\right)\right]^{2g-2+s}}\sum_{(w^1,\dots,w^s)\in W^s}\epsilon_{w^1}\dots\epsilon_{w^s}$$
$$M_q\left(\sum_{j=1}^s w^j(\rho+\theta_j),x\right)=\sum_{\substack{\lambda\in\overline{CR}^*/q\overline{R}^*,\sigma^2(\lambda/q)\neq 0\\(w^1,\dots,w^s)\in W^s}}\frac{1}{(i^\ell\sigma(\lambda/q))^{2g-2}}\frac{1}{(\sigma(\lambda/q))^s}$$
$$\prod_{j=1}^s\epsilon_{w^j}\exp\left(2i\pi\langle w^j(\rho+\theta_j),\lambda/q\rangle\right).$$

Now by (1.94), for $1\leq j\leq s$,

$$(7.90)\qquad \frac{1}{\sigma(\lambda/q)}\sum_{w^j\in W}\epsilon_{w^j}\exp(2i\pi\langle w^j(\rho+\theta_j),\lambda/q\rangle)=\chi_{\theta_j}(e^{\lambda/q}).$$

From (1.17), (7.89), (7.90), we get (7.80).
The proof of our Theorem is completed. □

REMARK 7.21. It is crucial that in (7.80), $s$ is even. In fact under the given conditions on the $\theta_j$'s, $\sum_{j=1}^s w^j\theta_j\in\overline{R}$, but we know that $\sum_{j=1}^s w^j\rho\in\overline{R}$ only if $s$ is even.

### 7.5. The generic case with $g\geq 2$.

DEFINITION 7.22. We will say that $(\theta_1,\dots,\theta_s)$ verify assumption $(\overline{A}_q)$ if for any $(w^1,\dots,w^s)\in W^s$, $\frac{1}{q}\sum_{j=1}^s w^j\theta_j\notin\overline{S}$.

In the sequel, if $u\in C/\overline{R}^*$, we choose once and for all a representative of $u$ in $C$, which we still denote by $u$.

THEOREM 7.23. *If $q \in \mathbf{N}^*$, $g \geq 2$, if $(\theta_1, \dots, \theta_s)$ verify $(\overline{A}_q)$, then*

$$V_{g,q}(\theta_1, \dots, \theta_s) = \left|\frac{\overline{R}}{\overline{CR}}\right| \mathrm{Vol}(T)^{2g-2} \frac{|Z(G)|}{|W|} q^{(g-1)r} (-1)^{\ell(g-1)+r}$$

$$\sum_{u \in C/\overline{R}^*} \sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f \in \overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle}$$

$$\mathrm{Res}^I_{x=0} \left[ \prod_{\alpha \in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha, x^I/q + 2i\pi u\rangle\right)} \right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s} \varepsilon_{w^1} \dots \varepsilon_{w^s} \exp\left( \langle \sum_{j=1}^{s} w^j \rho, x^I/q\rangle \right. \tag{7.91}$$

$$\left. + 2i\pi \langle \sum_{j=1}^{s} w^j(\rho + \theta^j) + qf, u\rangle \right)$$

$$\exp\left( d \sum_{j=1}^{r} \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \left[ \frac{1}{d} \langle p^I_{j-1}\Big( \sum_{k=1}^{s} w^k \theta_k/q + f\Big), e^j\rangle \right] \right)$$

$$\frac{1}{\prod_{j=1}^{r} \left( \exp\left( \frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \right) - 1 \right)}.$$

PROOF. Take $t \in \overline{R}$. By Theorem 7.10, if $t \in \overline{R}$, $t/q \notin \overline{S}$, then

$$\mathrm{Res}^{(1,\dots,\ell)}_{x=0} \frac{1}{\left[ \prod_{i=1}^{\ell} 2\sinh\left(\frac{x_i}{2q}\right) \right]^{2g-2}} \prod_{j=1}^{s} \varphi_{w^j}\left(\frac{x}{q}\right) M_q(t, x)$$

$$= \left|\frac{\overline{R}}{\overline{CR}}\right| (-1)^r \sum_{u \in C/\overline{R}^*} \sum_{\substack{I=(i_1,\dots,i_r)\subset \mathcal{I}_u, I \text{ generic} \\ f \in \overline{CR}/d\overline{R}}}$$

$$\mathrm{Res}^{(1,\dots,\ell)}_{x=0} \left\{ \frac{1}{\left[ \prod_{i=1}^{\ell} 2\sinh\left(\frac{x_i}{2q}\right) \right]^{2g-2}} \prod_{j=1}^{s} \varphi_{w^j}(x/q) \right. \tag{7.92}$$

$$\frac{1}{\prod_{i \in {}^cI} 2q \tanh\left( \frac{\langle \alpha_i, x^I + 2i\pi q u\rangle - x_i}{2q} \right)}$$

$$\frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle} \exp(2i\pi \langle u, t + qf\rangle)$$

$$\exp\left( d \sum_{j=1}^{r} \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \left[ \frac{1}{d} \langle p^I_{j-1}(t + qf), \frac{e^j}{q}\rangle \right] \right)$$

$$\frac{1}{\prod_{j=1}^{r} \left( \exp\left( \frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \right) - 1 \right)}.$$

Observe that in the right-hand side of (7.92), the expressions starting after $\exp(2i\pi\langle u,t+qf\rangle)$ does not depend on the $x_i$, $i\notin I$.

Now we explain how to evaluate the residue in (7.92). We use the notation in Definition 2.47.

1. *The case $i\notin I$, $i\notin\mathcal{I}_u$ or $i\notin I$, $i\in\mathcal{I}_u$, $\alpha_i\notin\{\alpha_{\sigma^I(1)},\dots,\alpha_{\sigma^I(p_i)}\}$.*

Note that the above condition just says that $i\notin I$, and $\langle\alpha_i,u\rangle\in\mathbf{Z}$, $\alpha_i\notin\{\alpha_{\sigma^I(1)},\dots,\alpha_{\sigma^I(p_i)}\}$, or $\langle\alpha_i,u\rangle\notin\mathbf{Z}$. If $\langle\alpha_i,u\rangle\in\mathbf{Z}$, $\alpha_i\notin\{\alpha_{\sigma^I(1)},\dots,\alpha_{\sigma^I(p_i)}\}$, for at least one $k\in I$, $k>i$, then $\langle\alpha_i,\alpha^k\rangle\neq 0$. So since the $\{x_k\}_{k>i}$ have not yet been made "small", for generic $x$, $\langle\alpha_i,x^I\rangle$ is "large". So we find that under the assumptions we made in 1., for generic $x$, mod $(2i\pi q\mathbf{Z})$, $\langle\alpha_i,x^I+2i\pi qu\rangle$ should be considered as different of 0. Then by proceeding as in (7.82) and using the fact that $g\geq 2$, we get

$$(7.93)\qquad \operatorname{Res}_{x_i=0}\left[\frac{1}{\left[2\sinh\left(\frac{x_i}{2q}\right)\right]^{2g-2+s}}e^{\frac{1}{2q}\sum_{j=1}^{s}\varepsilon_{w^j}(i)x_i}\frac{1}{2q\tanh\left(\frac{\langle\alpha_i,x^I+2i\pi qu\rangle-x_i}{2q}\right)}\right]=\left[\frac{1}{2\sinh\left(\frac{\langle\alpha_i,x^I+2i\pi qu\rangle}{2q}\right)}\right]^{2g-2+s}e^{\frac{1}{2q}\langle\sum_{j=1}^{s}\varepsilon_{w^j}(i)\alpha_i,x^I+2i\pi qu\rangle}.$$

2. *The case where $i\notin I$, $i\in\mathcal{I}_u$ and $\alpha_i\in\{\alpha_{\sigma^I(1)},\dots,\alpha_{\sigma^I(p_i)}\}$.*

In this case $\langle\alpha_i,u\rangle\in\mathbf{Z}$, and so

$$(7.94)\qquad \frac{1}{2q\tanh\left(\frac{\langle\alpha_i,x^I+2i\pi qu\rangle-x_i}{2q}\right)}=\frac{1}{2q\tanh\left(\frac{\langle\alpha_i,x^I\rangle-x_i}{2q}\right)}=-\frac{1}{2q}\frac{1-\tanh\left(\frac{x_i}{2q}\right)\tanh\left(\frac{\langle\alpha_i,x^I\rangle}{2q}\right)}{\tanh\left(\frac{x_i}{2q}\right)\left(1-\frac{\tanh\left(\frac{\langle\alpha_i,x^I\rangle}{2q}\right)}{\tanh\left(\frac{x_i}{2q}\right)}\right)}.$$

Now since $\alpha_i\in\{\alpha_{\sigma^I(1)},\dots,\alpha_{\sigma^I(p_i)}\}$, $\langle\alpha_i,x^I\rangle$ is a linear combination of the $x_{\sigma^I(k)}$, $k\leq p_i$, which have been made "small". Therefore we get the expression

$$(7.95)\qquad \frac{1}{1-\frac{\tanh\left(\frac{\langle\alpha_i,x^I\rangle}{2q}\right)}{\tanh(x_i/2q)}}=1+\frac{\tanh\left(\frac{\langle\alpha_i,x^I\rangle}{2q}\right)}{\tanh(x_i/2q)}+\frac{\tanh^2\left(\frac{\langle\alpha_i,x^I\rangle}{2q}\right)}{\tanh^2(x_i/2q)}+\dots.$$

In view of (7.92), (7.94), (7.95), for $k\geq 1$, we should evaluate

$$(7.96)\qquad \operatorname{Res}_{x_i=0}\left[\frac{1}{\left[2\sinh(x_i/2q)\right]^{2g-2+s}}e^{\frac{1}{2q}\sum_{j=1}^{s}\varepsilon_{w^j}(i)x_i}\frac{1}{\tanh^k(x_i/2q)}\right].$$

The function of $x_i$ which appears in (7.96) is periodic of period $2i\pi q$. Recall that the contour $\Delta_\varepsilon$ was defined in the proof of Theorem 7.20. For $\varepsilon>0$, 0 is the only pole inside $\Delta_\varepsilon$. By using the theorem of residues as in (7.84), and the fact that $g\geq 2$, we find that (7.96) vanishes.

From (7.94)-(7.96), we see that if $i \notin I$, $i \in \mathcal{I}_u$, $\alpha_i \in \{\alpha_{\sigma^I(1)}, \dots, \alpha_{\sigma^I(p_i)}\}$, then

$$\operatorname{Res}_{x_i=0}\left[\frac{1}{\left[2\sinh(x_i/2q)\right]^{2g-2+s}}\, e^{\frac{1}{2q}\sum_{j=1}^{s}\varepsilon_{w^j}(i)x_i}\, \frac{1}{2q\tanh\left(\frac{\langle\alpha_i, x^I+2i\pi q u\rangle - x_i}{2q}\right)}\right] = 0\,. \tag{7.97}$$

Using (7.79) in Theorem 7.20 and (7.92), (7.93), (7.97), we get

$$V_{g,q}(\theta_1,\dots,\theta_s) = \left|\frac{\overline{R}}{\overline{CR}}\right| \operatorname{Vol}(T)^{2g-2}\frac{|Z(G)|}{|W|}q^{(g-1)r}(-1)^{\ell(g-1)+r} \sum_{u\in C/\overline{R}^*}\ \sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u\\ f\in\overline{CR}/d\overline{R}}} \frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle}$$
$$\operatorname{Res}^I_{x=0}\Bigg\{\left[\prod_{i\in I}\frac{1}{2\sinh(\frac{x_i}{2q})}\prod_{i\notin I}\frac{1}{2\sinh\left(\frac{\langle\alpha_i,x^I+2i\pi q u\rangle}{2q}\right)}\right]^{2g-2+s}$$
$$\sum_{(w^1,\dots,w^s)\in W^s}\varepsilon_{w^1}\dots\varepsilon_{w^s}\exp\left(\frac{1}{2}\sum_{j=1}^{s}\left(\langle\sum_{i\notin I}\varepsilon_{w^j}(i)\alpha_i,\frac{x^I}{q}\rangle + \frac{\sum_{i\in I}\varepsilon_{w^j}(i)x_i}{q}\right)\right)$$
$$\exp\left(2i\pi\langle u, \sum_{j=1}^{s}\left(\frac{\sum_{i\notin I}\varepsilon_{w^j}(i)\alpha_i}{2} + w^j\theta_j\right) + qf\rangle\right)$$
$$\exp\left(d\sum_{j=1}^{r}\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^{s}w^k\theta_k+qf\Big),e^j/q\rangle\right]\right)$$
$$\frac{1}{\prod_{j=1}^{r}\left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right)-1\right)}\Bigg\}. \tag{7.98}$$

Now observe that if $i \in I$,

$$\langle\alpha_i, x^I\rangle = x_i\,. \tag{7.99}$$

Therefore

$$\sum_{i\notin I}\langle\varepsilon_{w^j}(i)\alpha_i,x^I\rangle + \sum_{i\in I}\varepsilon_{w^j}(i)x_i = \langle\sum_{i=1}^{\ell}\varepsilon_{w^j}(i)\alpha_i,x^I\rangle = \langle w^j\sum_{i=1}^{\ell}\alpha_i,x^I\rangle = 2\langle w^j\rho,x^I\rangle\,. \tag{7.100}$$

Also if $i \in I \subset \mathcal{I}_u$, then $\langle\alpha_i,u\rangle\in\mathbf{Z}$, and so

$$\left[\frac{1}{2\sinh\left(\frac{\langle\alpha_i,x^I+2i\pi q u\rangle}{2q}\right)}\right]^{2g-2} = \left[\frac{1}{2\sin(\frac{x_i}{2q})}\right]^{2g-2}, \qquad \frac{1}{2\sinh\left(\frac{\langle\alpha_i,x^I+2i\pi q u\rangle}{2q}+i\pi\right)}\exp\left(\frac{2i\pi\langle u,\varepsilon_{w^j}(i)\alpha_i\rangle}{2}\right) = \frac{1}{2\sinh\left(\frac{x_i}{2q}\right)}\,. \tag{7.101}$$

From (7.98)-(7.101), we get (7.91). The proof of our Theorem is completed. □

REMARK 7.24. Clearly,

$$\left[\frac{1}{2\sin\left(\frac{\langle\alpha_i, x^I/q+2i\pi u\rangle}{2}\right)}\right]^{2g-2} \tag{7.102}$$

does not depend on the representative $u$ in $C$. Also

$$\prod_{\alpha\in R_+}\frac{1}{2\sinh\left(\langle\alpha_i, x^I/q+2i\pi u\rangle\right)}\varepsilon_{w^j}\exp\left(\langle w^j\rho, x^I/q+2i\pi u\rangle\right) = \prod_{\alpha\in R_+}\frac{1}{1-\exp\left(-\langle w^j\alpha, x^I/q+2i\pi u\rangle\right)} \tag{7.103}$$

does not depend either on the representative $u \in C$. This explains why the terms in the right-hand side of (7.91) do not depend on the choice of the representatives of $u$.

**7.6. The non generic case for $g \geq 2$.** Recall that the functions $[x]_+, [x]_-$ were defined in (6.141).

Let $\eta_1, \dots, \eta_r$ be $r$ functions from $\overline{R}/d\overline{R} \times \{I, I \text{generic}\}$ into $\{+,-\}$ such that if $f = \sum_1^r f^j e_j \in \overline{R}/d\overline{R}$, $I = (i_1, \dots, i_r)$, then $\eta_j(f, I)$ depends only on $f^1, \dots, f^j, i_1, \dots, i_{j-1}$.

First, we will extend Theorem 7.2 to the case where $t \in \overline{R}/q\overline{R}$ is arbitrary.

THEOREM 7.25. *For any $t \in \overline{R}/q\overline{R}$, there are meromorphic functions $\varphi_1(t, x_2, \dots, x_\ell), \dots, \varphi_i(t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell) \dots$ which vanish identically when $t/q \notin H$, such that*

$$\begin{aligned} L_q(t,x) = \frac{(-1)^r}{d^r} &\sum_{\substack{I=(i_1,\dots,i_r)\subset\{I,\dots,\ell\}, I \text{ generic} \\ f\in\overline{R}/d\overline{R}, g\in(\mathbf{Z}/d\mathbf{Z})^r}} \operatorname{sgn}\left(\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle\right) \\ &\left[\exp\left(d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}(t+qf), \frac{e^j}{q}\rangle\right]_{\eta_j(f,I)}\right)\right. \\ &\left.\frac{1}{\prod_{i\in{}^cI} 2q\tanh\left(\frac{\langle\alpha_i,x^I\rangle - x_i}{2q}\right)}\right](x+2i\pi q g_I)\prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right)-1} \\ &+\sum_{i=1}^{\ell}\varphi_i(t, x_1,\dots,x_{i-1},x_{i+1},\dots,x_\ell). \end{aligned} \tag{7.104}$$

PROOF. We use the notation in the proof of Theorem 7.2. Let $\eta_1, \dots, \eta_{j-1} \in \{+,-\}$. We define $S_I(a,x)$ as in (7.7), except that $\left[\frac{1}{d}\langle p^I_{i_n-1}t, \frac{e^n}{q}\rangle\right]$ is replaced by $\left[\frac{1}{d}\langle p^I_{i_n-1}t, \frac{e^n}{q}\rangle\right]_{\eta_n}$, $1 \leq n \leq j-1$.

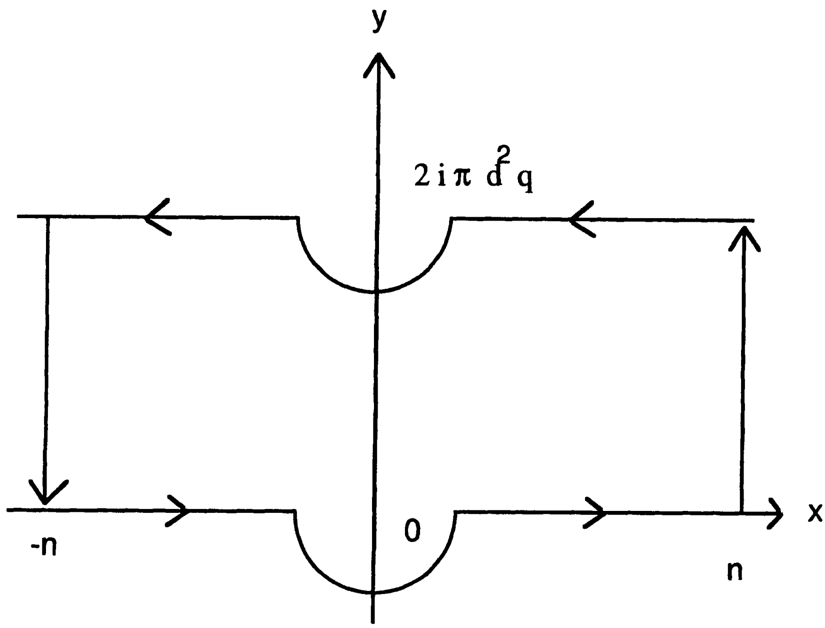


FIGURE 7.3

Take $\eta_j(f) : \{0, \dots, d-1\} \mapsto \{+,-\}$. Then we rewrite (7.20) in the form

$$
\begin{aligned}
(7.105)\qquad L_I = \frac{1}{d^2} \sum_{k\in \mathbf{Z}/d^2q\mathbf{Z}} \Bigg\{ \sum_{\substack{0\le f<d \\ g\in (\mathbf{Z}/d\mathbf{Z})^{j-1}}} & \\
\operatorname{Res}_{a=2i\pi k} \Bigg[ \exp\Bigg( a \left[ \frac{\langle p^I_{j-1}(t+qfe_j), e^j/q\rangle}{d} \right]_{\eta_j(f)} & \\
+ d \sum_{n=1}^{j-1} \frac{\langle p^I_{n-1}\alpha_{i_n}, x^I\rangle}{\langle p^I_{n-1}\alpha_{i_n}, e^n\rangle} \left[ \frac{1}{d}\langle p^I_{n-1}t, e^n/q\rangle \right]_{\eta_n} \Bigg) & \\
\frac{1}{\prod_{i\in {}^cI} 2q\tanh\left(\frac{\langle p^I_{j-1}\alpha_i, ae^j/d+2i\pi\widehat{k}-x^I\rangle}{2q}\right)} \frac{1}{e^a-1}\Bigg]\Bigg\}(x+2i\pi q g_I). &
\end{aligned}
$$

Instead of (7.21), we now set

$$
(7.106)\qquad h_{f,g}(a) = \frac{\exp\left( a \left[ \frac{\langle p^I_{j-1}(t+qfe_j), \frac{e^j}{q}\rangle}{d} \right]_{\eta_j(f)} \right)}{\prod_{i\in {}^cI} 2q\tanh\left(\frac{\langle p^I_{j-1}\alpha_i, \frac{ae^j}{d}+2i\pi\widehat{k}-x^I\rangle}{2q}\right)} \frac{1}{e^a-1}.
$$

The key point is that we do no longer assume that (7.23) holds. Equivalently $\frac{\langle p^I_{j-1}(t+qfe_j), e^j/q\rangle}{d}$ may now well be an integer.

For $n \in \mathbf{N}$, we now replace the contour in Figure 7.1 by the contour of Figure 7.3. We apply the theorem of residues to $h_{f,g}(a)$ on this contour, and let $n \to +\infty$.

Clearly, if $\frac{\langle p^I_{j-1}(t+q/e_j),e^j/q\rangle}{d}$ is an integer, if $\eta_j(f)=\pm$, then

$$\lim_{a\to\pm\infty} h_{f,g}(a)=0, \tag{7.107}$$

$$\lim_{a\to\mp\infty} h_{f,g}(a)=\mp\frac{1}{\prod\limits_{\substack{i\in{}^cI\\ \langle p^I_{j-1}\alpha_i,e^j\rangle=0}}\tanh\left(\frac{\langle p^I_{j-1}\alpha_i,2i\pi\widehat{k}-x^I\rangle}{2q}\right)}$$

$$\frac{1}{(2q)^{|{}^cI|}}(-1)^{|\{i\in{}^cI,\langle p^I_{j-1}\alpha_i,e^j\rangle\neq 0,\ \mathrm{sgn}\langle p^I_{j-1}\alpha_i,e^j\rangle=\pm 1\}|}.$$

Observe here that by (2.65), $\{i\in{}^cI;\langle p^I_{j-1}\alpha_i,e^j\rangle\neq 0\}$ is non empty. Then by (2.63), the right-hand side of the second equation in (7.107) does not depend on $\{x_i\}_{i\in{}^cI\ ,\langle p^I_{j-1}\alpha_i,e^j\rangle\neq 0}$.

So by proceeding as in (7.21)-(7.25), we find that

- if $\frac{\langle p^I_{j-1}(t+qfe_j),e^j/q\rangle}{d}$ is not an integer, (7.26) still holds, with $[\ ]$ replaced by $[\ ]_{\eta_1},\dots,[\ ]_{\eta_{j-1}},[\ ]_{\eta_j(f)}$.
- if $\frac{\langle p^I_{j-1}(t+qfe_j),e^j/q\rangle}{d}$ is an integer, writing $\eta_j$ instead of $\eta_j(f)$, then (7.26) is replaced by

$$L_I=-\frac{1}{d}\sum_{\substack{0\le f<d\ ,g\in(\mathbf{Z}/d\mathbf{Z})^{j-1}\\ i_j:(\alpha_{i_1},\dots,\alpha_{i_j})\neq 0\ ,0\le g^j<d|\langle p^I_{j-1}\alpha_{i_j},e^j\rangle|}} \tag{7.108}$$

$$\left\{\frac{1}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\exp\left(2i\pi\langle p_j^{(I,i_j)}e_j,\widehat{k}\rangle\langle p^I_{j-1}(t+qfe_j),e^j/q\rangle\right.\right.$$

$$\left.+d\sum_{n=1}^{j}\frac{\langle p^I_{n-1}\alpha_{i_n},x^I\rangle}{\langle p^I_{n-1}\alpha_{i_n},e^n\rangle}\left[\frac{1}{d}\langle p^I_{n-1}(t+qfe_j),e^n/q\rangle\right]_{\eta_n}\right)$$

$$\left.\frac{1}{\prod_{i\notin(I,i_j)}2q\tanh\left(\frac{\langle p_j^{(I,i_j)}\alpha_i,2i\pi\widehat{k}-x^I\rangle}{2q}\right)}\right\}(x+2i\pi q(g,g_j)_{I,i_j})$$

$$\frac{1}{\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right)-1}+q\sum_{\substack{0\le f<d\\ g\in(\mathbf{Z}/d\mathbf{Z})^{g-1}}}$$

$$\left[\exp\left(d\sum_{n=1}^{j-1}\frac{\langle p^I_{n-1}\alpha_{i_n},x^I\rangle}{\langle p^I_{n-1}\alpha_i,e^n\rangle}\left[\frac{1}{d}\langle p^I_{n-1}t,e^n/q\rangle\right]_{\eta_n}\right)\right.$$

$$\left.\frac{1}{\prod\limits_{\substack{i\in{}^cI\\ \langle p^I_{j-1}\alpha_i,e^j\rangle=0}}\tanh\left(\frac{\langle p^I_{j-1}\alpha_i,2i\pi\widehat{k}-x^I\rangle}{2q}\right)}\right](x+2i\pi qg_I)\frac{1}{(2q)^{|{}^cI|}}$$

$$(-1)^{|\{i\in{}^cI,\langle p^I_{j-1}\alpha_i,e^j\rangle\neq 0,\ \mathrm{sgn}\langle p^I_{j-1}\alpha_i,e^j\rangle=\eta_j(f)\}|},$$

the key point being that the last sum in (7.108) is a sum of functions which do not depend on the $x_i$, $i\in{}^cI$, $\langle p^I_{j-1}\alpha_i,e_j\rangle\neq 0$).

Using (7.108) and proceeding by recursion as in the proof of Theorem 7.2, we get (7.104). The proof of our Theorem is completed. □

REMARK 7.26. The last term is the right-hand side of (7.108) depends in a non trivial way on the choice of $\eta_j$. More precisely, it is a sum of terms depending on $\eta_j$ up to sign. As a corollary, we find that the function $\sum_1^\ell \varphi_i(t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell)$ also depends on the choice of $\eta_1, \dots, \eta_r$ in a non trivial way.

Now we extend Theorem 7.6 to the general case. We still take $\eta_1, \dots, \eta_r$ as in Theorem 7.25.

THEOREM 7.27. *For any* $t \in \overline{R}/q\overline{R}$, *there are meromorphic functions* $\psi_1(t, x_2, \dots, x_\ell), \dots, \psi_i(t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell)$ *which vanish identically when* $t/q \notin \overline{S}$, *such that*

$$
(7.109) \qquad M_q(t,x) = \frac{(-1)^r}{d^r} \sum_{\substack{I=(i_1,\dots,i_r)\subset\{1,\dots,\ell\}, I \text{ generic} \\ f\in\overline{R}/d\overline{R}, g\in(\mathbb{Z}/d\mathbb{Z})^r, \mu\in\overline{CR}^*/\overline{R}^*}} \operatorname{sgn}(\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle) \exp(2i\pi\langle\lambda_\mu, t/q\rangle)
$$
$$
\left[\exp\left(d\sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}(t+qf), e^j/q\rangle\right]_{\eta_j(f,I)}\right]
$$
$$
\frac{1}{\prod_{i\in {}^cI} 2q\tanh\left(\frac{\langle\alpha_i, x^I\rangle - x_i}{2q}\right)}(x + 2i\pi q g_I - 2i\pi\widetilde{a}\lambda_\mu)
$$
$$
\prod_{j=1}^r \frac{1}{\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1}(x - 2i\pi q\widetilde{a}\lambda_\mu) +
$$
$$
\sum_{i=1}^\ell \psi_i(t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell).
$$

PROOF. Using Proposition 7.5 and Theorem 7.25, we get (7.109). ☐

Given $(w^1, \dots, w^s) \in W^s$, let $\eta_1^{(w^1,\dots,w^s)}, \dots, \eta_s^{w^1,\dots,w^s)}$ be $s$ functions from $\overline{R}/d\overline{R} \times \{I, I \text{ generic}\}$ into $\{+,-\}$ having the properties listed before Theorem 7.25.

THEOREM 7.28. *If $q \in \mathbf{N}^*$, $g \geq 2$, then*

$$V_{g,q}(\theta_1, \dots, \theta_s) = \left|\frac{\overline{R}}{\overline{CR}}\right| \mathrm{Vol}(T)^{2g-2} \frac{|Z(G)|}{|W|} q^{(g-1)r} (-1)^{\ell(g+1)+r} \tag{7.110}$$

$$\sum_{u \in C/\overline{R}^*} \sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f \in \overline{CR}/d\overline{R}}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle}$$

$$\mathrm{Res}^I_{x=0} \left[\prod_{\alpha \in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha_i, x^I/q + 2i\pi u\rangle\right)}\right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s} \varepsilon_{w^1} \dots \varepsilon_{w^s} \exp\Bigg(\langle \sum_{j=1}^s w^j \rho, x^I/q\rangle$$

$$+2i\pi\langle \sum_{j=1}^s w^j(\rho + \theta^j) + qf, u\rangle\Bigg)$$

$$\exp\left(d \sum_{j=1}^r \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s \frac{w^k\theta_k}{q} + f\Big), e^j\rangle\right]_{\eta_j^{(w^1,\dots,w^s)}(f,I)}\right)$$

$$\frac{1}{\prod_{j=1}^r \left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)}.$$

PROOF. We use (7.79) in Theorem 7.20 and 7.27, and also the arguments in the proof of Theorem 7.23. We will show that for $1 \leq i \leq \ell$,

$$\mathrm{Res}^{1,\dots,\ell}_{x=0} \frac{1}{\left[\prod_1^\ell 2\sinh\left(\frac{x_i}{2q}\right)\right]^{2g-2}} \prod_{j=1}^s \varphi_{w^j}(x/q) \psi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell) = 0 \tag{7.111}$$

from which (7.110) follows. To establish (7.111), we only need to show that

$$\mathrm{Res}_{x_i=0} \frac{1}{\left[2\sinh\left(\frac{x_i}{2q}\right)\right]^{2g-2}} \prod_{j=1}^s \varphi_{w^j}(x/q) = 0\,. \tag{7.112}$$

The proof of (7.112) is the same as the proof of (7.84).

We have thus established our Theorem. □

**7.7. The general case.** We will now establish another form of Theorem 7.28.

THEOREM 7.29. *If $g \geq 0$, if $s$ is even and if $2g-2+s \geq 1$, the following identity holds*

$$(7.113)\quad V_{g,q}(\theta_1,\dots,\theta_s) = \left|\frac{\overline{R}}{\overline{CR}}\right| \operatorname{Vol}(T)^{2g-2}\frac{|Z(G)|}{|W|}q^{(g-1)r}(-1)^{\ell(g+1)+r}$$

$$\sum_{u\in C/\overline{R}^*}\sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u\\ f\in\overline{CR}/d\overline{R}}}\frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle}$$

$$\operatorname{Res}^I_{x=0}\left[\prod_{\alpha\in R_+}\frac{1}{2\sinh\left(\frac{1}{2}\langle\alpha_i,x^I/q+2i\pi u\rangle\right)}\right]^{2g-2+s}$$

$$\sum_{(w^1,\dots,w^s)\in W^s}\varepsilon_{w^1}\dots\varepsilon_{w^s}\,\exp(2i\pi\langle\sum_{j=1}^s w^j(\rho+\theta^j)+qf,u\rangle)$$

$$\exp\left(d\frac{\sum_{j=1}^r\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s\frac{w^k(\rho+\theta_k)}{q}+f\Big),e^j\rangle\right]_{\eta_j^{(w^1,\dots,w^s)}(f,I)}\right)$$

$$\frac{1}{\prod_{j=1}^r\left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right)-1\right)}.$$

PROOF. We proceed as in the proof of Theorems 7.23 and 7.28. Instead of (7.79) in Theorem 7.20, we use (7.80) and we still use Theorem 7.27. In particular the obvious analogues of (7.93) and (7.97) still hold because $s$ is even and $2g-2+s \geq 1$. For the same reason, the analogue of (7.111), (7.112), with the $\varphi_{w^j}$ replaced by 1 still holds.

The proof of our Theorem is completed. □

## 8. The Verlinde formulas

In this Section, we prove the Verlinde formulas, in the restricted sense which was described in the Introduction. Namely, we show essentially that for $p$ large enough, the Riemann-Roch number of $M/G$ is given by the Verlinde formulas. Also we show that the Riemann-Roch numbers of suitable perturbations of $M/G$ are given by the Verlinde formulas.

This Section is organized as follows. In Section 8.1, we establish our results under a suitable genericity assumption on the holonomies. In Sections 8.2 and 8.3, we consider suitable perturbations of the moduli space.

### 8.1. The generic case.

Here we will assume that $s \geq 1$, $2g - 2 + s \geq 1$, that $t_1, \dots, t_s$ are regular and lie in the alcove $P$, and that $(t_1, \dots, t_s)$ verify assumption $(\overline{A})$ of Section 6.9. Namely we assume that for any $(w^1, \dots, w^s) \in W^s$, $\sum_{j=1}^{s} w^j t_j \notin \overline{S}$.

Recall that $\mathbf{M} \subset \mathbf{Z}$ was defined in (6.20). Here we assume that $\mathbf{M}$ is not reduced to 0.

THEOREM 8.1. *For $p \in \mathbf{M}$, $p \geq 0$ large enough,*

$$\operatorname{Ind}(D_{p,+}) = V_{g,p+c}(pt_1, \dots, pt_s). \tag{8.1}$$

PROOF. By Theorem 6.29 and by Theorem 7.12, we know that if $\sum_{j=1}^{s} pt_j \notin \overline{R}$, then both sides of (8.1) are equal to 0. So we may as well assume that $\sum_{j=1}^{s} pt_j \in \overline{R}$.

First we consider the case where $g \geq 2$. Observe that for $p \in \mathbf{M}$ large enough, $(pt_1, \dots, pt_s)$ verify $(\overline{A}_{p+c})$. By comparing formula (6.130) in Theorem 6.34 and formula (7.91) in Theorem 7.23, we get (8.1).

Now we consider the case $g = 1$. By Theorem 7.16, we get

(8.2)
$$V_{1,p+c}(pt_1, \dots, pt_s) = \frac{(-1)^{\ell}}{\left|\frac{\overline{CR}^*}{(p+c)\overline{CR}}\right|} \sum_{(v^1,v^2) \in W^2} \epsilon_{v^1} \epsilon_{v^2} \overline{V}^K_{2,p+c}(pt_1, \dots, pt_s, v^1\rho + v^2\rho).$$

Observe that for $p$ large enough, $(pt_1, \dots, pt_s, v^1\rho + v^2\rho)$ verify $(\overline{A}_{p+c})$. Using (1.17), Theorem 7.23 with $g = 2$ and (8.2), we obtain

$$(8.3)\qquad V_{1,p+c}(pt_1, \dots, pt_s) = \frac{|Z(G)|}{|W|}(-1)^r$$

$$\sum_{u \in C/\overline{R}^*} \sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f\in \overline{R}/d\overline{R}, \mu \in \overline{CR}^*/\overline{R}^*}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r}\rangle}$$

$$\mathrm{Res}^I_{x=0}\left[\prod_{\alpha\in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha, x^I/q + 2i\pi u\rangle\right)}\right]^{s+3} \sum_{(w^1,\dots,w^{s+3})\in W^{s+3}} \varepsilon_{w^1}\dots\varepsilon_{w^{s+3}}$$

$$\exp\left(\langle \sum_{j=1}^{s+1} w^j\rho, x^I/(p+c)\rangle + 2i\pi\langle \sum_{j=1}^{s} w^j pt_j/(p+c) + \sum_{j=s+2}^{s+3} w^j\rho/(p+c), \lambda_\mu\rangle\right.$$

$$\left. +2i\pi\langle \sum_{j=1}^{s} w^j(\rho + pt^j) + \sum_{j=s+1}^{s+3} w^j\rho + (p+c)f, u\rangle\right)$$

$$\left\{\exp\left(d\sum_{j=1}^{r} \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^{s} w^k pt_k/(p+c) + \right.\right.\right.$$

$$\left.\left.\sum_{k=s+2}^{s+3} w^k\rho/(p+c) + f\Big), e^j\rangle\right]\right)$$

$$\left.\frac{1}{\prod_{j=1}^r\left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle}\right) - 1\right)}\right\}(x - 2i\pi\tilde{a}\lambda_\mu).$$

Since $(t_1, \dots, t_s)$ verify $(\overline{A})$, by Proposition 2.14,

$$(8.4)\qquad \langle p^I_{j-1}\sum_{k=1}^{s} w^k t_k + f, e^j\rangle \notin \mathbf{Z}.$$

By proceeding as in (6.132) and using (8.4), we get for $p \in \mathbf{M}$ large enough,

$$(8.5)\qquad [\frac{1}{d}\langle p^I_{j-1}(\sum_{k=1}^{s} w^k pt_k/(p+c) + \sum_{k=s+2}^{s+3} w^k\rho/(p+c) + f), e^j\rangle] =$$

$$[\frac{1}{d}\langle p^I_{j-1}(\sum_{k=1}^{s} w^k pt_k/(p+c) + f), e^j\rangle] + \frac{1}{d}\langle p^I_{j-1}\sum_{k=s+2}^{s+3} w^k\rho/(p+c), e^j\rangle.$$

Also by (2.13),

$$(8.6)\qquad \sum_{j=1}^{s} \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, ej\rangle}\langle p^I_{j-1}\sum_{k=s+2}^{s+3} w^k\rho/(p+c), e^j\rangle = \langle \sum_{k=s+2}^{s+3} w^k\rho/(p+c), x^I\rangle.$$

So by (8.3), (8.5), (8.6), we get

$$
\begin{aligned}
&(8.7)\; V_{1,p+c}(pt_1, \dots, pt_s) = \frac{|Z(G)|}{|W|}(-1)^r \\
&\sum_{u \in C/\overline{R}^*} \sum_{\substack{I=(i_1,\dots,i_r)\in \mathcal{I}_u \\ f\in \overline{R}/d\overline{R}, \mu \in \overline{CR}^*/\overline{R}^*}} \frac{1}{\langle \alpha_{i_1}, \dots, \alpha_{i_r} \rangle} \\
&\operatorname{Res}^I_{x=0} \left[ \prod_{\alpha \in R_+} \frac{1}{2\sinh\left(\frac{1}{2}\langle \alpha, x^I/(p+c) + 2i\pi u\rangle\right)} \right]^{s+3} \sum_{(w^1,\dots,w^{s+3})\in W^{s+3}} \varepsilon_{w^1} \dots \varepsilon_{w^{s+3}} \\
&\exp\left( \langle \sum_{j=1}^{s} w^j \rho, x^I/(p+c)\rangle + \langle \sum_{j=s+1}^{s+3} w^j \rho, \frac{x^I}{p+c} + 2i\pi u \rangle + \right. \\
&\left. 2i\pi \langle \sum_{j=1}^{s} w^j pt_j/(p+c), \lambda_\mu \rangle + 2i\pi \langle \sum_{j=1}^{s} w^j(\rho + pt^j) + (p+c)f, u \rangle \right) \\
&\left\{ \exp\left( d \sum_{j=1}^{r} \frac{\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \left[ \frac{1}{d} \langle p^I_{j-1} \Big( \sum_{k=1}^{s} w^k pt_k/(p+c) + f \Big), e^j \rangle \right] \right) \right. \\
&\left. \frac{1}{\prod_{j=1}^{r} \left( \exp\left( \frac{d\langle p^I_{j-1}\alpha_{i_j}, x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j}, e^j\rangle} \right) - 1 \right)} \right\} (x - 2i\pi \widetilde{a} \lambda_\mu).
\end{aligned}
$$

Now by (1.94), (7.71),

$$
(8.8) \qquad \frac{1}{\prod_{\alpha \in R_+} \sinh\left(\frac{1}{2}\langle \alpha, \frac{x^I}{p+c} + 2i\pi u \rangle\right)} \sum_{w \in W} \epsilon_w \exp(\langle w\rho, \frac{x^I}{p+c} + 2i\pi u \rangle) = 1.
$$

By (8.7), (8.8), we obtain

$$(8.9)\qquad V_{1,p+c}(pt_1,\dots,pt_s)=\frac{|Z(G)|}{|W|}(-1)^r$$

$$\sum_{u\in C/\overline{R}^*}\ \sum_{\substack{I=(i_1,\dots,i_r)\in\mathcal{I}_u\\ f\in\overline{R}/d\overline{R},\mu\in\overline{CR}^*/\overline{R}^*}}\frac{1}{\langle\alpha_{i_1},\dots,\alpha_{i_r}\rangle}$$

$$\mathrm{Res}^I_{x=0}\left[\prod_{\alpha\in R_+}\frac{1}{2\sinh\left(\frac{1}{2}\langle\alpha,x^I/(p+c)+2i\pi u\rangle\right)}\right]^s\sum_{(w^1,\dots,w^s)\in W^s}\varepsilon_{w^1}\dots\varepsilon_{w^s}$$

$$\exp\Bigg(\langle\sum_{j=1}^s w^j\rho,x^I/(p+c)\rangle+2i\pi\langle\sum_{j=1}^s w^jpt_j/(p+c),\lambda_\mu\rangle$$

$$+2i\pi\langle\sum_{j=1}^s w^j(\rho+pt^j)+(p+c)f,u\rangle\Bigg)$$

$$\left\{\exp\left(d\sum_{j=1}^r\frac{\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\left[\frac{1}{d}\langle p^I_{j-1}\Big(\sum_{k=1}^s w^k\frac{pt_k}{p+c}+f\Big),e^j\rangle\right]\right)\right.$$

$$\left.\frac{1}{\prod_{j=1}^r\left(\exp\left(\frac{d\langle p^I_{j-1}\alpha_{i_j},x^I\rangle}{\langle p^I_{j-1}\alpha_{i_j},e^j\rangle}\right)-1\right)}\right\}(x-2i\pi\tilde{a}\lambda_\mu)\,.$$

By comparing (6.130) and (8.9), we get (8.1) for g = 1.

A similar proof can be given for the case $g=0$. The proof of our Theorem is completed. □

**8.2. The perturbed case at $\frac{pt_j}{p+c}$.** Now we make the same assumptions as in Section 6.11. Namely take $p\in\mathbf{M},p\geq 0$. We assume that $\epsilon_1,\dots,\epsilon_s\in\mathfrak{t}$ are such that form $\ell\in]0,1]$, $1\leq j\leq s$, $\frac{pt_j}{p+c}+\ell\epsilon_j\in T$ is regular, and for any $(w^1,\dots,w^s)\in W^s,\ell\in]0,1]$, $\sum_{j=1}^s w^j\frac{pt_j}{p+c}+\ell\epsilon_j\notin\overline{S}$.

THEOREM 8.2. *For $g\geq 2$,*

$$(8.10)\qquad \mathrm{Ind}(D_{p,+}^{(\frac{pt_j}{p+c}+\ell\epsilon_j)})=V_{g,p+c}(pt_1,\dots,pt_s).$$

*For $g=0$ or $g=1$, if $(t_1,\dots,t_s)$ verify $(\overline{A})$, for $p\in\mathbf{M}$ and $p\geq 0$ large enough, equation (8.10) still holds.*

PROOF. By Remark 6.37, we find that if $\sum_{j=1}^s pt_j\notin\overline{R}$, then the left-hand side of (8.10) vanishes. By (7.59) in Theorem 7.12, the right hand-side also vanishes. As in the proof of Theorem 8.1, we may and we will assume that $\sum_{j=1}^s pt_j\in\overline{R}$.

First we consider the case $g\geq 2$. Then (8.10) follows by comparing (6.147) in Theorem 6.41, and (7.110) in Theorem 7.28.

In the case $0\leq g\leq 1$, we proceed as in the proof of Theorem 8.1, and use (8.10) in the case $g\geq 2$.

The proof of our Theorem is completed. □

**8.3. The perturbed case at $\frac{\rho+pt_j}{p+c}$.** Observe that if $t \in \overline{P}$, for $p \in \mathbf{N}$, $\frac{\rho+pt}{p+c} \in P$, i.e. $\frac{\rho+pt}{p+c}$ is regular. This follows follows from (1.33), (1.34), (1.36).

The same argument as in the proof of Proposition 6.39 shows that if $s \geq 1$, given $p \geq 0, p \in \mathbf{M}$, there exist $\epsilon_1, \dots, \epsilon_s \in \mathfrak{t}$ such that for $\ell \in ]0,1], 1 \leq j \leq s$, $\frac{pt_j+\rho}{p+c} + \ell\epsilon_j \in T$ is regular, and for any $(w^1, \dots, w^s) \in W^s, \ell \in ]0,1]$, $\sum_{j=1}^s w^j(\frac{pt_j}{p+c} + \ell\epsilon_j) \notin \overline{S}$.

In Theorem 6.36 we now take

$$\delta_j = \frac{\rho - ct_j}{p+c} + \ell\epsilon_j. \tag{8.11}$$

By the construction of Section 6.11, we get a Dirac operator $D_{p,+}^{\frac{\rho+pt_j}{p+c}}$.

THEOREM 8.3. *For $g \geq 0$, and $2g-2+s \geq 1$, for $\ell \in ]0,1]$,*

$$\operatorname{Ind}(D_{p,+}^{(\frac{\rho+pt_j}{p+c}+\ell\epsilon_j)}) = V_{g,p+c}(pt_1, \dots, pt_s). \tag{8.12}$$

PROOF. As in the proof of Theorems 8.1 and 8.2, we may restrict ourselves to the case where $\sum_{j=1}^s pt_j \in \overline{R}$. First assume that $s$ is even. Then (8.12) follows from (6.137) in Theorem 6.36, with $\delta_j$ given by (8.11), and from (7.113) in Theorem 7.29. When $s$ is odd, first we perturb the $s$ holonomies as indicated before. Then we add an extra marked point $x_{s+1}$, with holonomy equal to 1. We perturb the holonomy 1 to a generic holonomy, and we use the result we just proved with $s+1$ marked points. We can then gently make our holonomy $t_{s+1}$ tend to 1. This is in fact possible because the perturbation of the first $s$ holonomies are generic and verify condition $(\overline{A})$. The proof of our Theorem is completed. □

REMARK 8.4. Suppose temporarily that the assumptions of Remark 6.42 are satisfied. In particular we assume the holonomy $t_s$ to be central and represented by $h_o \in \overline{P}$. A direct treatment would show that an analogue of (8.12) would still hold, where, in the Verlinde sum (7.57), the term $\chi_{ph_o}(e^{\lambda/(p+c)}$ should apparently be replaced by $\epsilon_{w_{h_o}} \exp(2i\pi\langle h_o, \lambda\rangle)$. However Theorem 1.33 tells us that indeed, this is just $\chi_{ph_o}(e^{\lambda/(p+c)}$, so that we recover the standard Verlinde formula.

REMARK 8.5. By Teleman's vanishing results [**55, 56**], if $(t_1, \dots, t_s)$ verify $(A)$, for any $p \in \mathbf{M}$, one should have

$$\operatorname{Ind}(D_{p,+}) = \dim H^0(\mathcal{M}, \lambda^p). \tag{8.13}$$

Since $\dim H^0(\mathcal{M}, \lambda^p)$ is given by Verlinde's formula, then one should have for any $p \in \mathbf{M}$,

$$\operatorname{Ind}(D_{p,+}) = V_{g,p+c}(pt_1, \dots, pt_s). \tag{8.14}$$

The arguments of Teleman also show that more generally, if $\delta_1, \dots, \delta_s$ are taken as in Theorem 6.36, for any $p \in \mathbf{M}$,

$$\operatorname{Ind}(D_{p,+}^{t_j+\delta_j}) = V_{g,p+c}(pt_1, \dots, pt_s). \tag{8.15}$$

Here, Theorem 8.1 only asserts that if $(t_1, \dots, t_s)$ verify $(\overline{A})$, for $p$ large enough, (8.13) holds. For a given $p \in \mathbf{M}$, Theorems 8.2 and 8.3 give a corresponding equality for a suitable perturbation of the moduli space, depending on $p$, the perturbation being smaller as $p \to +\infty$. A proof of (8.14), (8.15) for any $p \in \mathbf{M}$ would be possible if one could modify Theorem 7.20, so that in (7.79) or (7.80), the term where the function $M_q$ appears would be instead $M_q(\frac{q}{p}\sum_{j=1}^s w^j\theta_j, x)$. To establish such a result, one would need to prove the vanishing of a certain residue. This vanishing result is true for $g$ large enough, but not obvious for small values of $g$.

## References


[1] Alekseev A., Malkin A., Meinrenken E. : Lie group valued moment maps. *dg-gal*/9707021 (1997).

[2] Atiyah M.F., Bott R. : The Yang-Mills equations over Riemann surfaces. *Phil. Trans. Roy. Soc. London*, **A 308**, 523-615 (1983).

[3] Beauville A. : Conformal blocks, fusion rules and the Verlinde formula. In *Proceedings of the Hirzebruch 65 Conference in Algebraic Geometry*, Ramat-Gan 1993, 75-96. Israël Math. Conf. Proc. **9**. Ramat-Gan: Bar-Ilan University 1996.

[4] Beauville A., Laszlo Y. : Conformal blocks and generalized theta functions. *Comm. Math. Phys.*, **164**, 385-419 (1994).

[5] Berline N., Vergne M. : Zéros d'un champ de vecteurs et classes caractéristiques équivariantes. *Duke Math. J.*, **50**, 539-549 (1983).

[6] Berline N., Getzler E., Vergne M. : *Heat kernels and Dirac operators*. Grundl. Math. Wiss. **298**. Berlin-Heldelberg-New-York: Springer 1992.

[7] Bertram A., Szenes A. : Hilbert polynomials of moduli spaces of rank 2 vector bundles. II. *Topology*, **32**, 599-609 (1993).

[8] Bismut, J.-M. : The index Theorem for families of Dirac operators : two heat equation proofs. Invent. Math. ,**83**, 91-151 (1986).

[9] Bismut J.-M. : Equivariant immersions and Quillen metrics. *J. Diff. Geom.*, **41**,53-157 (1995).

[10] Bismut, J.-M., Freed D. : The analysis of elliptic families. I. Metrics and connections on determinant bundles. *Comm. Math. Phys.*, **106**, 159-176 (1986). II. Eta invariants and the holonomy theorem. *Comm. Math. Phys.* , **107**, 103-163 (1986).

[11] Bismut J.-M., Gillet H., Soulé C. : Analytic torsion and holomorphic determinant bundles I. *Comm. Math. Phys.*, **115**, 49-78 (1988). II. *Comm. Math. Phys.*, **115**, 79-126 (1988). III. *Comm. Math. Phys.*, **115**, 301-351 (1988).

[12] Bismut J.-M., Labourie F. : Formules de Verlinde pour les groupes simplement connexes et géométrie symplectique. *C.R. Acad. Sci. Paris*, **325**, Série I, 1009-1014 (1997).

[13] Bismut J.-M., Zhang W.P. : *An extension of a theorem by Cheeger and Müller*. Astérisque **205**. Paris SMF 1992.

[14] Bourbaki N. : *Eléments de Mathématique*. Chapitre 9. Paris: Masson 1982.

[15] Bröcker T., Dieck T. : *Representations of compact Lie groups*. Graduate texts in Mathematics **98**. Berlin-Heidelberg-New-York: Springer 1995.

[16] Chang S.X.: Fixed point formulas and loop group actions. *alg-geom*/9812148 (1996).

[17] Donaldson S.K. : Gluing techniques in the cohomology of moduli spaces. In *Topological methods in modern mathematics*, L. Goldberg and A. Phillips eds., 137-170. Houston: Publish or Perish 1993.

[18] Donaldson S.K., Kronheimer P. : *The geometry of Four-Manifolds*. Oxford: Clarendon Press 1990.

[19] Duistermaat J.J., Heckman G. : On the variation of the cohomology of the reduced phase space. *Invent. Math.*, **69**, 256-268 (1982). Addendum , **72**, 153-158 (1983).

[20] Faltings G. : A proof of the Verlinde formula. *J. of Alg. Geom.* **3**, 347-374 (1994).

[21] Frenkel I.B. : Orbital theory for affine Lie algebras. *Invent. Math.*, **77**, 301-352 (1984).

[22] Fuchs J. : *Affine Lie Algebras and Quantum Groups*. Cambridge Monographs in Math. Phys.. Cambridge: Cambridge Univ. Press 1992.

[23] Guillemin V., Sternberg S. : Geometric quantization and multiplicities of group representations. *Invent. Math.*, **67**, 515-538 (1982).

[24] Guillemin V., Sternberg S. : *Symplectic techniques in Physics*. Cambridge: Cambridge University Press 1990.

[25] Guruprasad K., Huebschmann J., Jeffrey L., Weinstein A.: Group systems, groupoids, and moduli spaces of parabolic bundles. *Duke Math. J.*, **89**, 377-412 (1997).

[26] Hörmander L. : *The analysis of linear partial differential operators, Vol. 1*, Grundl. der Math. Wiss. Band **256**. Berlin-Heidelberg-New-York: Springer 1983.

[27] Jeffrey L. : Extended moduli spaces of flat connections on Riemann surfaces. *Math. Annalen* **298**, 667-692 (1994).

[28] Jeffrey L., Kirwan F. : Localization for non abelian group actions. *Topology*, **34**, 291-327 (1995).

[29] Jeffrey-Kirwan F. : Localization and the quantization conjecture. *Topology*, **36**, 647-693 (1977).

[30] Jeffrey L., Kirwan F. : Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface. *alg-geom*/9608029 (1996).

[31] Jeffrey L., Weitsman J. : Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula. *Comm. Math. Phys.* **150**, 593-630 (1992).

[32] Kawasaki T. : The Riemann-Roch theorem for complex $V$-manifolds. *Osaka J. Math.*, **16**, 151-159 (1979).

[33] Kawasaki T. : The index of elliptic operators on $V$-manifolds. *Nagoya Math. J.*, **9**, 135-157 (1981).

[34] Kirillov A.A. : *Elements of the theory of representations.* Grundl. Math. Wiss. **220**. Berlin-Heidelberg-New-York: Springer 1976.

[35] Knudsen, F.F., Mumford, D. : The projectivity of the moduli space of stable curves, I: Preliminaries on "det" and "div". *Math Scand.* ,**39**, 19-55 (1976).

[36] Kostant B. : On Macdonald's $\eta$-function formula, the Laplacian and generalized exponents. *Adv. in Math.*, **20**, 179-212 (1976).

[37] Kumar S., Narasimhan M.S. : Picard group of the moduli space of $G$-bundles. *Math. Annal.*, **308**, 155-173 (1997).

[38] Kumar S., Narasimhan M.S., Ramanathan A. : Infinite Grassmannian and moduli spaces of $G$-bundles. *Math. Annal.*, **300**, 41-75 (1994).

[39] Liu K. : Heat kernel and moduli spaces . *Math. Res. Letters*, **3**, 743-762 (1996).

[40] Liu K. : Heat kernel and moduli spaces II. *Math. Res. Letters*, **4**, 569-588 (1997).

[41] Meinrenken E. : On Riemann-Roch formulas for multiplicities. *J. Am. Math. Soc.* , **9**, 373-389 (1996).

[42] Meinrenken E. : Symplectic surgery and the Spin$^c$ Dirac operator. *Adv. in Math.*, to appear.

[43] Meinrenken E., Woodward C. : A symplectic proof of Verlinde factorization. *dg-ga*/9612018 (1996).

[44] Meinrenken E., Woodward C. : Fusion of Hamiltonian loop group manifolds and cobordism. Preprint, July 1997.

[45] Milnor J.M., Stasheff J.D. : *Characteristic classes.* Annals of Mathematics Studies **76**. Princeton: Princeton University Press 1974.

[46] Narasimhan M.S., Seshadri C.S. : Stable and unitary vector bundles on a compact Riemann surface. *Ann. Math.*, **82**, 540-567 (1965).

[47] Pressley A., Segal G. : *Loop groups.* Oxford Mathematical Monographs. Oxford: Clarendon Press 1986.

[48] Ramadas T.R., Singer I.M., Weitsman J. : Some comments on Chern-Simons theory. *Comm. Math. Phys.* **126**, 409-420 (1989).

[49] Reidemeister K. : Homotopieringe und Linsenraüme. *Hamburger Abhandl.*, **11**, 102-109 (1935).

[50] Rossmann W. : Kirillov's character formula for reductive Lie groups. *Invent. Math.*, **48**, 207-220 (1978).

[51] Serre J.P. : *Cours d'arithmétique.* Paris: Presses Universitaires de France 1970.

[52] Szenes A. : Hilbert polynomials of moduli spaces of rank 2 vector bundles. I. *Topology*, **32**, 587-597 (1993).

[53] Szenes A. : The combinatorics of the Verlinde formula. In *Vector bundles in Algebraic geometry*, 241-253. LMS Lecture Notes Series **208**. Cambridge: Cambridge University Press 1995.

[54] Szenes A. : Iterated residues and multiple Bernouilli polynomials. *Preprint* MIT 1998.

[55] Teleman C. : Borel-Weil-Bott theory on the moduli stack of $G$-bundles over a curve. *Invent. Math.*, **134**, 1-57 (1998).

[56] Teleman C. : The quantization conjecture revisited. *alg-geom*/9808029 v2 (1998).

[57] Thaddeus M. : Stable pairs, linear systems and the Verlinde formula. *Inv. Math.* **117**, 317-353 (1994).

[58] Tian Y., Zhang W. : An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg. *Invent. Math.*, **132**, 229-259 (1998).

[59] Tsuchiya, A., Ueno K., Yamada Y. : Conformal field theory and universal family of stable curves with gauge symmetry. *Adv. Studies in Pure Math.*, **19**, 459-566 (1989).

[60] Vergne M. : On Rossmann's character formula for discrete series. *Invent. Math.*, **54**, 11-14 (1979).
[61] Vergne M. : A note on the Jeffrey-Kirwan-Witten localization formula. *Topology*, **35**, 243-266 (1996).
[62] Verlinde E. : Fusion rules and modular transformations in $2D$-conformal field theory. *Nuclear Physics B*, **300**, 360-376 (1988).
[63] Witten E. : On quantum gauge theories in two dimensions. *Comm. Math. Phys.* **141**, 153-209 (1991).
[64] Witten E. : Two dimensional gauge theories revisited. *J. Geom. Phys.* **9**, 303-368 (1992).

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