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A Brief Tour of GW Invariants

Jun Li
Department of Mathematics
Stanford University
Stanford, CA 94305
and
Gang Tian
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

The machinery of virtual moduli cycle was initiated by authors in 1995-96 to attack a class of topological invariants, especially Gromov-Witten invariant. The development of this machinery went through several stages during which several people made key contribution to its development. This machinery is essential to the study of mathematical problems in super-string theory. It also played key role in symplectic topology and enumerative problems in algebraic geometry. In this article, we will lead a brief tour though the development of this machinery. We will explain the basic idea to its construction and the means to its application. We hope that this will provide a manageable account on how to apply this machinery for non-specialists.

The era of using non-linear analysis to define topological invariants began with Donaldson's celebrated work of his invariants of smooth 4-manifolds. This invariant is a counting invariant that counts ASD-connections of principle G-bundles over 4-manifolds. The prototype of a similar invariant on counting rational curves in a symplectic manifold was proposed by Ruan in early 90's [**Ru1**], inspired by Gromov's work on pseudo-holomorphic maps [**Gr**] and Witten's work on non-linear σ-models in string theory [**Wi1, Wi2**]. This counting began to attract attention in part because of the notion of the Quantum cohomology and the Mirror symmetry conjecture. However, its mathematical foundation was not developed until the end of 1993 when Ruan and the second author constructed the symplectic invariants and proved their basic property for the class of semi-positive symplectic manifolds. This class of invariants are now referred to as Gromov-Witten invariants. Overtime, people began to ponder if similar invariants by counting curves can be defined for any symplectic manifolds and algebraic varieties. The notion of stable morphisms was introduced by Kontsevich [**Ko**] to construct a compactified moduli spaces for this purpose. In 1995, the authors succeeded in constructing such invariants of smooth projective varieties. They reported their construction in the Santa Cruz conference in algebraic geometry that year [**LT3**]. Their method is to construct the virtual moduli cycle of the moduli space of stable morphisms and define Gromov-Witten invariants as the integration of the tautological topological classes against this cycle [**LT1**]. They constructed such cycles by constructing virtual normal cones using Kuranishi maps of the moduli space. In early 1996, Behrend-Fantechi [**BF**] gave an alternative construction of such cycles. They constructed similar cones as Artin stacks over the moduli stack. In 1996, Fukaya-Ono [**FO**] and the authors [**LT2**] independently constructed the Gromov-Witten invariants for general symplectic manifolds, after constructing the virtual moduli cycle in analytic category. Different constructions were achieved later by Siebert [**Si1**] and Ruan [**Ru2**]. Finally, the authors in 1997 [**LT4**] and Siebert in 1998 [**Si2**] proved that the algebraic construction and the analytic construction of Gromov-Witten invariants coincide in case the underlining manifold is a smooth projective variety. All these work concluded the task of constructing Gromov-Witten invariants.

Research on Gromov-Witten invariants over the years has contributed, directly or indirectly, to the progress in other mathematical research. Here we can only single out a few such areas. Among them is the Mirror Symmetry Conjecture for quintic Calabi-Yau manifold, which relates the *virtual* counting of rational curves to variations of Hodge structure of its mirror. After work of many, including Candelas et al [**Ca**], Kontsevich [**Ko**] and Givental [**Gi**], this conjecture was finally proved rigorously by Lian-Liu-Yau [**LLY**] in 1997. Because of the limitation of time and space, we will not be able to cover this and others in any details, suffices to say that they include the development of Floer-homology for symplectic manifolds and the proof of the Arnold conjecture, the localization formula for equivalent virtual moduli cycles, the research for relative GW invariants and the research for Frobenius manifold structure.

We now describe the composition of this paper. The main body of this survey paper consists of three parts, in order they discuss the algebraic construction of the the GW invariants, the symplectic construction of the GW invariants and the equivalence of these two constructions. In the first part, we will describe the geometric motivation of our original construction of virtual moduli cycles. We will explain why the *obstruction theory* of the moduli space is an essential ingredient for this construction and how *Kuranishi maps* and *normal cones* appear naturally in this construction. We will describe briefly the notions that appear in this construction and state the main theorem. We will refer the details of this construction and other related issues to our original paper [**LT1**].

For the construction of symplectic GW invariants, we will present a slightly different approach as to the one presented in [**LT2, LT4**]. The purpose for doing this is to present a construction that requires as little smoothness as possible. We will introduce a class of weakly Fredholm V-bundle and show that they admit Euler classes, just like the usual case. The new ingredient is that to a weakly Fredholm V-bundle $E \to X$, among other things, the base space X is only assumed to have a locally closed partition (called a stratification) into smooth Banach orbifolds, and the bundle E is only assumed to be smooth along each stratum. The benefit of this, as is clear for the GW invariants, is that many moduli spaces admit geometric stratifications of which their strata are easily seen to be smooth. It is to study the normal structures of these strata in the total spaces where the technical difficulty arises. Our approach thus provides a more topological construction that bypass many of the tedious, if not impossible, technical study of singularities. We hope this will be proved to be useful in the study of similar problems in the future.

In the last part, we will briefly mention the equivalence of the above two constructions.

1. GW invariant as counting invariant

We begin with a brief account of GW invariants and its first construction by Ruan and the second author for semi-positive manifolds. Let (X,ω) be a smooth symplectic manifold and J an almost complex structure tame to ω. For non-negative integers g,n and homology class $A \in H_*(X,\mathbb{Z})$ we form the moduli space $\mathfrak{M}_{g,k}(X,A)$ of the equivalence classes of pseudo-holomorphic maps $f:C \to X$ such that C is an n-pointed genus g smooth Riemann surface[1], f is pseudo-holomorphic and $f_*([\Sigma]) = A$. In this article, we will use $f:C \to X$ to denote such a map with marked points on C implicitly understood. In case we want to stress the marked points, we will use $f:D \subset C \to X$ with $D \subset C$ the divisor of the marked points. Here we say that $f: C \to X$ is equivalent to $f': C' \to X$ if there is an isomorphism $\rho: C \to C'$ as pointed Riemann surfaces so that $f' \circ \rho = f$. By choosing ω and J generic, we can make $\mathfrak{M}_{g,k}(X,A)$ smooth (indeed with at most orbifold singularities) of complex dimension

$$r = (3 - \dim_{\mathbb{C}} X)(g-1) + k + A \cdot c_1(X). \tag{1.1}$$

It has an obvious evaluation map

$$\mathrm{ev} : \mathfrak{M}_{g,k}(X,A) \longrightarrow X^k$$

that sends any map $(f, C, x_1, \cdots, x_k)$ to $(f(x_1), \cdots, f(x_k))$. Then for any class $[\alpha] \in H_*(X)$, we choose its cycle representative $\alpha \subset X$ and investigate the set

$$\mathrm{ev}^{-1}(\alpha) \subset \mathfrak{M}_{g,k}(X,A). \tag{1.2}$$

If we choose $[\alpha]$ to be a degree $2r$ class, the above set will be a finite set if various transversality conditions are satisfied. By incorporating the almost complex structure of X, it is standard to show that it is an oriented set. Thus we can algebraically count this set. This counting is the GW invariant of (X,ω).

For the moment, this counting still depends on the choice of the symplectic form ω, the almost complex structure J and the cycle representative α. These combined form the defining data of the set $\mathrm{ev}^{-1}(\alpha)$. To simplify the notation, we will use Λ to denote this set of defining data and use

$$\Phi(\Lambda) \subset \mathfrak{M}_{g,k}(X,A)_\Lambda$$

to denote the pair in (1.2) defined based Λ. We say Λ is generic if various transversality conditions hold in constructing the set $\Phi(\Lambda)$.

We now explain why the algebraic count $\#\Phi(\Lambda)$ is a topological invariant. Let Λ be any generic defining data. Then $\#\Phi(\Lambda)$ is an invariant if it remains constant after perturbation of Λ. In **[RT1]**, Ruan and the second author showed that in case $g = 0$ and (X,ω) is semi-positive[2], then any two generic defining data Λ_0 and Λ_1

[1] We defer the definition to Chapter 3.

[2] A symplectic manifold (X,ω) is semi-positive if for any pseudo-holomorphic map $f:C \to X$ we have $f_*([C]) \cdot c_1(X) \geq 0$.

in the same homotopy class can be connected by a generic path Λ_t such that

$$\Phi(\Lambda_{[0,1]}) = \bigcup_{t\in[0,1]} \Phi(\Lambda_t)$$

form an oriented cobordism between $\Phi(\Lambda_0)$ and $\Phi(\Lambda_1)$. To accomplish this, they first showed that $\Phi(\Lambda_{[0,1]})$ is a smooth manifold with boundary $\Phi(\Lambda_1) - \Phi(\Lambda_0)$. The critical step is to show that it is also closed. For this, they compactified the moduli space $\mathfrak{M}_{0,k}(X,A)_{\Lambda_t}$ and studied the closure of $\Phi(\Lambda_{[0,1]})$ in the union of these compactifications. Relying on semi-positivity condition, they showed that $\Phi(\Lambda_{[0,1]})$ is indeed closed, thus proving the counting $\#\Phi(\Lambda)$ does not depend on the variation of Λ. This defines the GW invariant

$$\gamma^X_{A,0,n} : H^*(X^k,\mathbb{Q}) \longrightarrow \mathbb{Q}$$

that counts rational curves in X. We note that the class of semi-positive manifolds includes all Calabi-Yau manifolds. Later they generalized their construction to cover the GW invariants of all genus for semi-positive manifolds [**RT2**]. However, their approach relied on semi-positivity condition in part because in general the compactification of $\mathfrak{M}_{g,k}(X,A)$ is too large to prove the independence of the counting $\#\Phi(\lambda)$.

2. GW-invariants of algebraic varieties

2.1. Why virtual cycle and what is a virtual cycle. In this section, we will discuss the algebraic construction of virtual moduli cycle and its application to defining GW invariants of smooth projective varieties. In the following, we will use morphisms, schemes and other commonly used terminology in algebraic geometry. For those unfamiliar to such terminologies, they can simply replace morphisms by holomorphic maps and schemes by complex varieties, possibly singular.

The first step to the construction of GW invariants for general varieties (or symplectic manifolds) is to think of it as the outcome of integrating certain tautological topological classes on the moduli space over its fundamental class. For this, we need to work with the compactified moduli space.

Let X be a smooth projective variety and $\mathfrak{M}_{g,k}(X,A)$ the moduli space of stable morphisms[3] $f: D\subset C \to X$ from n-pointed (arithmetic) genus g nodal curve $D\subset C$ to X of degree $f_*([C]) = A$. The notion of stable morphism was introduced by Kontsevich [**Ko**]. The moduli space $\mathfrak{M}_{g,k}(X,A)$ admits a natural evaluation map ev. When the moduli space $\mathfrak{M}_{g,k}(X,A)$ has pure dimension r (in (1.1)) and the subset of maps with smooth domains is dense in $\mathfrak{M}_{g,k}(X,A)$, then the GW invariant is

$$\gamma^X_{A,g,k}(\alpha) = \int_{[\mathfrak{M}_{g,k}(X,A)]} \mathrm{ev}^*(\alpha), \qquad \alpha\in H^*(X^k,\mathbb{Q}). \tag{2.1}$$

The equivalence of this with the one based on counting is apparent from Poincare duality.

This approach has the advantage that even in case the moduli $\mathfrak{M}_{g,k}(X,A)$ has bigger than expected dimension, we can still define the integral (1.1) if we can replace $[\mathfrak{M}_{g,k}(X,A)]$ with an cycle that "imitate" the fundamental class of the

[3] A morphism $f: D\subset C\to X$ is stable if $D\subset C$ is a pre-stable n-pointed nodal curve and if there is no non-trivial vector field of C vanishing at the marked points and the nodal points of C so that $f_*(v) = 0$. An equivalent definition can be found in section 3.5.

moduli space. One way to visualize this cycle is this: We deform X to a family of varieties, say X_t with $X_0 = X$. Then $A \in H_2(X,\mathbb{Z})$ induces $A_t \in H_2(X_t,\mathbb{Z})$, at least for small t (in analytic topology). Then $\mathfrak{M}_{g,n}(X_t, A_t)$ form a family of moduli spaces not necessarily flat. Now imagine for $t \neq 0$ we can find X_t so that for all $t \neq 0$ the spaces $\mathfrak{M}_{g,n}(X_t, A_t)$ have expected dimensions. Then the limit $\lim[\mathfrak{M}_{g,n}(X_t, A_t)]$ when $t \to 0$ will be a cycle in $H_{2r}(\mathfrak{M}_{g,k}(X, A))$. This limit cycle will be the virtual moduli cycle. It is to construct this cycle the machinery of virtual cycles was developed.

2.2. Algebraic construction of virtual cycle. We now discuss the algebraic construction of virtual cycle. We begin with an example that will illustrate this construction. Let

$$\begin{array}{c} E \\ s \uparrow\downarrow \\ W \end{array} \tag{2.2}$$

be a section of a vector bundle over a smooth projective variety W. Let $X = s^{-1}(0)$ be the subscheme of W. We think of W as the ambient space and s the defining equation of X. Let $n = \dim W$ and $m = \operatorname{rank} E$. Since $X \subset W$ was cut out by m equations, the expected dimension

$$\exp.\dim X = n - m.$$

In case X has pure dimension $n - m$, we will take its fundamental class $[X]$ its virtual cycle $[X]^{\mathrm{vir}}$. Note that this is the Euler class of E, viewed as a class in $A_{n-m}W$.[4] In general, we demand $[X]^{\mathrm{vir}} \in A_*X$ to be the class so that it will remain constant if we vary X by varying the section s. This leaves us the only option that $[X]^{\mathrm{vir}}$ is the Euler class $e(E)$. To make it a class in A_*X, we will take the localized Euler class

$$e_{\mathrm{loc}}(E, s) \in A_{n-m}s^{-1}(0) = A_{n-m}X$$

thanks to the section s. The localized Euler class is constructed as follows [**Fu**]: Let $N_{X/W}$ be the normal cone to X in W. It is canonically a subcone of $E|_X$. Let $\eta_E : X \to E|_X$ be the zero section and $\eta_E^* : A_*E|_X \to A_*X$ the Gysin map. Then

$$e_{\mathrm{loc}}(E, s) = \eta_E^*[N_{X/W}] \in A_*X.$$

Analytically, $[N_{X/W}]$ can be viewed as the limit current (cycle) of Γ_{ts} when $t \to 0$, where Γ_s is the graph of s in E and $\eta_E^*[N_{X/W}]$ can be viewed as a class in $H_*(X)$ resulting from intersecting $N_{X/W}$ with a generic section of $E|_X$. This interpretation shows that $\eta_E^*[N_{X/W}]$ is the usual Euler class when viewed as an element in $H_*(W)$.

It is this localized Euler class which we define to be the virtual cycle $[X]^{\mathrm{vir}}$.

To push this construction to a general moduli space, say $\mathbf{M}$, we face the improbable task of embedding it *naturally* into some ambient space $\mathbf{W}$ as the vanishing locus of a section $\mathbf{s}$ of a vector bundle $\mathbf{E}$ over $\mathbf{W}$. Here we stress the naturality of such choice since otherwise the virtual moduli cycle will not be a naturally defined object. For instance, if $\mathbf{M} \subset \mathbf{W}$ is the zero locus of a section $\mathbf{s}$ of $\mathbf{E}$, a section of $\mathbf{E} \oplus \mathbf{C}_{\mathbf{W}}$, then $\mathbf{M}$ will be the zero locus of $\mathbf{s} \oplus \mathbf{0}$, where $\mathbf{0}$ is the zero section of the trivial line bundle $\mathbf{C}_{\mathbf{W}}$. However, $e(\mathbf{E}) \neq e(\mathbf{E} \oplus \mathbf{C}_{\mathbf{W}})$. To satisfies this "naturality" condition, we ask ourselves *what is the canonical defining equation(s) of a moduli*

[4] Z_*W is the group of algebraic cycles and A_*X is Z_*X modulo the rational equivalence relation.

space. The key to this question rests on the notion of *the obstruction theory and its Kuranishi map.*

We recall the notion of an obstruction theory to deformation of a point p in a scheme X.

DEFINITION 2.1. *(X, p) is said to have an obstruction theory with coefficients in V, where V is a finite dimensional $\mathbb{C}$-linear space, if the following holds: For any triple (A, I, φ), where A is an Artin ring, $I \subset A$ is an ideal annihilated by the maximal ideal $\mathfrak{m} \subset A$ and $\varphi : \operatorname{Spec} A/I \to X$ is a morphism such that $\varphi(0) = p$, there is a function $\operatorname{Ob}(A, I, \varphi) \in V \otimes_{\mathbb{C}} I$ such that (1) $\operatorname{Ob}(A, I, \varphi) = 0$ if and only if there is an extension $\tilde{\varphi} : \operatorname{Spec} A \to X$ of φ and (2) this function $\operatorname{Ob}(A, I, \varphi)$ satisfies base change property.*

We skip the exact wording of the base change property since it does not affect our discussion later. It can be found in [**LT1**].

The following example illuminates the notion of obstruction theory.

EXAMPLE 2.2. *Let W be a smooth scheme and $X \subset W$ be a subscheme defined by the vanishing of a section s of a vector bundle E. Let $p \in X$ and*

$$V = \operatorname{Coker}\{ds(p) : T_pW \longrightarrow E|_p\}. \tag{2.3}$$

Then s induces an obstruction theory of (X, p) with coefficients in V.

There is an obvious way to construct this obstruction theory. Since W is smooth, $\varphi : \operatorname{Spec} A/I \to X$ extends to $\psi : \operatorname{Spec} A \to W$. Since $s \circ \psi(\operatorname{Spec} A/I) = 0$ and $\mathfrak{m} \cdot I = 0$, $s \circ \psi$ induces an element in $E|_p \otimes_{\mathbb{C}} I$. We denote by

$$\operatorname{Ob}(A, I, \varphi) \in V \otimes_{\mathbb{C}} I$$

its image in $V \otimes_{\mathbb{C}} I$ under the quotient map $E|_p \otimes_{\mathbb{C}} I \to V \otimes_{\mathbb{C}} I$. It is straightforward to show that this assignment satisfies all requirements of an obstruction theory, hence is an obstruction theory to deformation of p in X.

Now let (X, p) be any scheme endowed with an obstruction theory with coefficients in V. Let $\hat{X}$ be the formal completion of X along p. ($\hat{X}$ sometimes is called the germ of X at p.) Assume $n = \dim T_pX$ and $m = \dim V$. We pick a dual basis $(z) = (z_1, \cdots, z_n)$ of T_pX. Following Kuranishi's construction of his map, there is a pair (ι, Φ), where

$$\iota : \hat{X} \hookrightarrow \operatorname{Spec} \mathbb{C}[[z]] \quad \text{and} \quad \Phi \in V \otimes \mathbb{C}[[z]] \tag{2.4}$$

so that $\hat{X}$ is the vanishing locus of Φ and the obstruction theory induced by Φ coincide with the obstruction theory given. We call (ι, Φ) a Kuranishi map of the obstruction theory.

It is instructive to work out an explicit example. Let $s(z) = 2z^3 + z^4$. It defines the subscheme $X = \{z^3 = 0\}$ in $\mathbf{C}$. For triple $A = \mathbb{C}[t]/(t^3)$, $I = (t^2)$ and $\varphi_2(t) = z : \operatorname{Spec} A/I$ to X, the obstruction class $\operatorname{Ob}(A, I, \varphi) = 0$. Obviously φ_2 extends to $\varphi_3(t) = z : \operatorname{Spec} A \to X$. Now let $A' = \mathbb{C}[t]/(t^4)$ and $I' = (t^3)$. We compute $\operatorname{Ob}(A', I', \varphi_3) = 2$, the coefficient of the leading non-zero term in s. This reveals the fact that the leading non-trivial terms in a Kuranishi map is determined by the obstruction theory, and the *non-uniqueness* of Kuranishi maps comes from the freedom of higher order terms. In this case, any function $\Phi(z) = 2z^3 + O(z^4)$ is a Kuranishi map.

So far we have settled the question of finding natural defining equation of a moduli space: *The moduli space should come with an obstruction theory and the associated Kuranishi maps will serve as its defining equation.* The strategy to construct the virtual cycle is clear now: *We will use the Kuranishi maps to construct a virtual cone in a vector bundle E and use the Gysin map of its zero section to define the virtual cycle.* There remain two technical difficulties: One is that the Kuranishi maps are not unique; The second is that they are local. They only exist as formal functions.

We are still in the set up of (2.2). According to the plan, we intend to reconstruct the normal cone $N_{X/W}$ near p by the Kuranishi maps of the obstruction theory. Let (ι, Φ) be a Kuranishi map of the obstruction theory of $(\hat{X}, p)$, as in (2.2). We consider the normal cone

$$N_\Phi := N_{\hat{X}/\operatorname{Spec}\mathbb{C}[[z]]} \subset V \times \hat{X} \tag{2.5}$$

which is canonically a subcone of $V \times \hat{X}$. The cone N_Φ is closely related to the normal cone $N_{X/W}$ at p.

LEMMA 2.3. *Let the notation be as before and let $\Phi_1, \Phi_2 \in V \otimes \mathbb{C}[[z]]$ be any two Kuranishi maps of $(\hat{X}, p)$. Then there is a vector bundle automorphism ϕ of $V \times \hat{X}$ (as bundle over $\hat{X}$) that restricts to the identity homomorphism at the closed fiber $V \times \{p\}$ such that $\phi^{-1}(N_{\Phi_1}) = N_{\Phi_2}$.*

The upshot of this Lemma is that though Kuranishi maps are not unique, the normal cones they defined are almost unique. The only ambiguity comes from those automorphisms whose restrictions to the closed fiber are the identity automorphisms.

We now investigate the choice of the vector bundle E. It is understood now that the key that makes E the vector bundle of our choice is that the sheaf

$$\mathcal{T} = \operatorname{Coker}\{\Omega_W^\vee|_X \to \mathcal{O}_X(E)\}, \tag{2.6}$$

which has the property that at each point $p \in X$ the space $\mathcal{T} \otimes_{\mathcal{O}_X} \mathbb{C}_p$ is the obstruction space to deformation of p in X, is a quotient sheaf of $\mathcal{O}_X(E)$. We call $\mathcal{T}$ the obstruction sheaf. We remark that not all obstruction theories has this property. Those that do will be called *perfect obstruction theories.*

DEFINITION 2.4. *A scheme Y is said to have a perfect obstruction theory if there is a perfect complex $\mathcal{T}^\bullet = [\mathcal{T}_1 \to \mathcal{T}_2]$ in the derived category $\mathcal{D}^b(X)$ of which the following holds: For any triple (A, I, φ), where A is a $\mathbb{C}$-algebra, $I \subset A$ an ideal so that $I^2 = 0$ and $\varphi : \operatorname{Spec} A/I \to X$ is a morphism, there is a function $\operatorname{Ob}(A, I, \varphi) \in \mathfrak{h}^2(\mathcal{T}^\bullet \otimes_{\mathcal{O}_X} I)$ such that*
(1) $\operatorname{Ob}(A, I, \varphi) = 0$ if and only if there is an extension $\tilde{\varphi}: \operatorname{Spec} A \to X$ of φ.
(2) If $\operatorname{Ob}(A, I, \varphi) = 0$ then the space of all extensions $\tilde{\varphi}$ are parameterized by $\mathfrak{h}^1(\mathcal{T}^\bullet \otimes_{\mathcal{O}_X} I)$.
(3) the function $\operatorname{Ob}(\cdot,\cdot,\cdot)$ satisfies base change property.

This definition is almost the same as Definition (2.1) except that now A can be any $\mathbb{C}$-algebra. Thus this covers deformation of *all* points in the sense of Grothendieck. For those who are unfamiliar with the derived category, they can replace $[\mathcal{T}_1 \to \mathcal{T}_2]$ with a complex of locally free sheaves of $\mathcal{O}_X$ modules $\mathcal{E}^\bullet = [\mathcal{E}_1 \to \mathcal{E}_2]$, and $\mathfrak{h}^j(\mathcal{T}^\bullet \otimes_{\mathcal{O}_X} I)$ by the kernel and the cokernel of $\mathcal{E}_1 \otimes_{\mathcal{O}_X} I \to \mathcal{E}_2 \otimes_{\mathcal{O}_X} I$ for $i = 1$ and 2 respectively.

Let Y be any projective scheme endowed with a perfect obstruction theory with associated complex $\mathcal{T}^\bullet$. Let E be a vector bundle on Y so that the $\mathfrak{h}^2(\mathcal{T}^\bullet)$ is a quotient sheaf of $\mathcal{O}_X(E)$. For any $p \in Y$, we denote by $\hat{Y}$ the formal completion of Y along p and V_p the vector space $\mathfrak{h}^2(\mathcal{T}^\bullet \otimes \mathbb{C}_p)$, the obstruction space to deformation of p in $\hat{Y}$.

BASIC CONSTRUCTION 2.5. *Let the notation be as before. There is a cone scheme N so that for each $p \in Y$ there is a vector bundle homomorphism*

$$\phi : E|_{\hat{Y}} \longrightarrow V_p \times \hat{Y}$$

that extends the given homomorphism $E|_p \to V_p$ so that

$$N|_{\hat{Y}} = \phi^{-1}(N_\Phi). \tag{2.7}$$

In example (2.2), the cone so constructed is exactly the normal cone $N_{X/W}$ when using the obstruction theory induced by the section s with the associated complex $\Omega_W^\vee|X \to \mathcal{O}_X(E)$ and the vector bundle E.

DEFINITION-THEOREM 2.6. *Let X be any projective scheme endowed with a perfect obstruction theory with associated complex $\mathcal{T}^\bullet = [\mathcal{T}_1 \to \mathcal{T}_2]$. Then X has a canonical virtual cycle $[X]^{\mathrm{vir}}$ which is constructed according to the following recipe:*
(1) We pick a vector bundle E over X so that the sheaf cohomology $\mathfrak{h}^2(\mathcal{T}^\bullet)$ is a quotient sheaf of $\mathcal{O}_X(E)$.
(2) We form a cone $N \subset E$ according to the basic construction (2.5).
(3) We define the virtual cycle $[X]^{\mathrm{vir}} = \eta_E^*[N]$.

Here is a few remarks. First, since $\mathcal{T}^\bullet$ is a perfect complex with two terms, its cokernel is a well-defined sheaf of $\mathcal{O}_X$-modules. Because X is projective, the vector bundle E always exists; Second, the existence of the cone N satisfying rule (2.7) does not follow from the existence of the Kuranish maps. The existence is proved by constructing a relative Kuranishi family. Once such cone exists, then it will by unique, following (2.7); Thirdly, the cycle $[X]^{\mathrm{vir}}$ so defined is independent of the choice of E. One caution, the virtual cycle only exists as a class in A_*X. It does not have a canonical cycle representative, just like the Euler class has not canonical submanifold representative.

2.3. GW Invariants. we now show that this machinery can be applied to construct the GW invariants of all smooth projective varieties. Following the blueprint of the construction, we need to describe the canonical obstruction theory of the moduli space $\mathfrak{M}_{g,k}(X, A)$ of stable morphisms, to workout a global vector bundle that makes the obstruction sheaf its quotient sheaf and then use the Gysin map to define the virtual cycle.

Let $f : D \subset C \to X$ be a stable morphism. It associates a complex $\mathcal{A}_f = [f^*\Omega_X \to \Omega_C(D)]$ (in the derived category) of degree -1 and 0. The obstruction theory of f is given by the following Lemma.

LEMMA 2.7. *The space of first order deformation of f is naturally parameterized by the extension group $\mathrm{Ext}^1_C(\mathcal{A}_f, \mathcal{O}_C)$. The deformation functor $\mathfrak{Mor}_f$ admits a canonical obstruction theory with coefficients in the extension group $\mathrm{Ext}^2_C(\mathcal{A}_f, \mathcal{O}_C)$.*

The proof of this Lemma is a standard deformation-obstruction computation, and can be found in [**LT1, Rn**]. Now we let (z) be a multi-variable dual to the first extension group $\mathrm{Ext}^1(\mathcal{A}_f, \mathcal{O}_C)$ and let J be the ideal generated by the components of the Kuranishi map of its obstruction theory. Then over $\operatorname{Spec}\mathbb{C}[[z]]/J$ there is a universal family to deformation of f. Sometimes the scheme $\operatorname{Spec}\mathbb{C}[[z]]/J$ is called the universal family to deformation of f.

The obstruction space fits into the exact sequence of cohomology groups

$$\longrightarrow \operatorname{Ext}^1(\Omega_C(D), \mathcal{O}_C) \longrightarrow H^1(f^*\Omega_X^\vee) \longrightarrow \operatorname{Ext}^2(\mathcal{A}_f, \mathcal{O}_C) \longrightarrow 0.$$

An easy consequence of this is the following smoothness result for convex manifolds[5].

COROLLARY 2.8. *Let X be a smooth convex variety and $f : D \subset C \to X$ be a stable morphism of which C has genus 0. Then there is no obstruction to deformation of f.*

It is known that the germ of $\mathfrak{M}_{g,k}(X, A)$ at f is a quotient of the universal family to deformation of f.

LEMMA 2.9. *Let $\operatorname{Aut}(f)$ be the automorphism group of f. Then G acts naturally on $\operatorname{Ext}^1(\mathcal{A}_f, \mathcal{O}_C)$ and on the universal family $\operatorname{Spec}\mathbb{C}[[z]]/J$ to the deformations of f. Further, the germ of $\mathfrak{M}_{g,k}(X, A)$ at f is canonically isomorphic to the quotient $(\operatorname{Spec}\mathbb{C}[[z]]/J)/G$.*

Note that it is possible that the local universal family to deformation of f is smooth while the the moduli space is singular at f. In general there are two sources for the singularities of a moduli space: One is from the non-trivial automorphism group of an object and the other is the non-trivial obstruction to deformations.

One essential precondition for constructing virtual cycle is the existence of a perfect obstruction theory. When $S \subset \mathfrak{M}_{g,k}(X, A)$ is an open subset over which a universal family exists, say $f : D \subset C \to X$, then S has a perfect obstruction theory with the associated complex

$$\mathcal{E}xt^1_{C/S}(\mathcal{A}_f, \mathcal{O}_C) \xrightarrow{\times 0} \mathcal{E}xt^2_{C/S}(\mathcal{A}_f, \mathcal{O}_C). \tag{2.8}$$

Because there are maps $f \in \mathfrak{M}_{g,k}(X, A)$ that has non-trivial automorphism groups, the moduli space *does not* admit universally families. Nevertheless, since all such automorphism groups are finite, we have the following

LEMMA 2.10. *For any points $\xi \in \mathfrak{M}_{g,k}(X, A)$ we can find a scheme S acted on by $G = \operatorname{Aut}(f)$ and a G-equivariant family of stable morphisms over S so that the tautological morphism $S/G \to \mathfrak{M}_{g,k}(X, A)$ induced by this family is an open neighborhood (i.e. open embedding) of $\xi \in \mathfrak{M}_{g,k}(X, A)$. Let $f : D \subset C \to X$ be such family over S. Then S admits a canonical perfect obstruction theory with associated complex (2.8).*

The Lemma suggests that we should work with pairs (S, G), called charts of $\mathfrak{M}_{g,k}(X, A)$, and their tautological families. When all charts S are smooth, this is the classical notion of orbifold. For us, we need to deal with singular charts as well. This leads us to the notion of Mumford-Deligne stacks or $\mathbb{Q}$-schemes. (We will call it stack since we will only work with Delign-Mumford stacks.)

The notion of stack is widely accepted nowadays among algebraic geometers working with moduli problems. It is less so for people working in other areas. To completely develope the notion of stacks from scratch is technically demanding.

[5]A complex manifold is said to be convex if for any morphism $f : \mathbf{P}^1 \to X$ we have $H^1(f^*T_X) = 0$.

However, the underlining idea behind it is quite simple. It was developed exactly to study a moduli space like $\mathfrak{M}_{g,k}(X,A)$: *it has a collection* (S_α, G_α), *where* G_α *are finite group acts on* S_α, *so that each* S_α/G_α *is an open subset of* $\mathfrak{M}_{g,k}(X,A)$, *and they together form an open covering of* $\mathfrak{M}_{g,k}(X,A)$. Each (S_α, G_α) in this collection is called a chart of the stack. The informed readers will now suspect that there should be compatibility condition of (S_α, G_α) and (S_β, G_β) in case $S_\alpha \cap S_\beta \neq \emptyset$,[6] etc.. This is certainly the case. For brevity we will not state them here but instead say they satisfy the obvious compatibility condition. The upshot of using charts (S_α, G_α) instead of open subsets S_α/G_α is that the former carry certain important data of which the later does not. For instance, an ordinary orbifold has smooth charts but not smooth neighborhood. Our moduli space $\mathfrak{M}_{g,k}(X,A)$ has charts carrying tautological families.

We are now ready to construct the virtual moduli cycle of $\mathfrak{M}_{g,k}(X,A)$. We pick a collection of charts of $\mathfrak{M}_{g,k}(X,A)$, say $\{(S_\alpha, G_\alpha)\}$, with $f_\alpha : D_\alpha \subset C_\alpha \to X$ the tautological family over S_α. The associated complex of the obstruction theory of f_α is $\mathcal{E}xt^\bullet_{C_\alpha/S_\alpha}(\mathcal{A}_\alpha, \mathcal{O}_{S_\alpha})$, where $\mathcal{A}_\alpha = [f_\alpha^* X \to \Omega_{C_\alpha}(D_\alpha)]$. Following the recipe, it remains to find a global vector bundle over $\mathfrak{M}_{g,k}(X,A)$ that makes the obstruction sheaves its quotient sheaves. Working with stack, it amounts to find a collection of vector bundles E_α over S_α, each is G_α-equivariant, and G_α-equivariant quotient homomorphisms

$$\mathcal{O}_{S_\alpha}(E_\alpha) \longrightarrow \mathcal{E}xt^2_{C_\alpha/S_\alpha}(\mathcal{A}_\alpha, \mathcal{O}_{C_\alpha}),$$

that are compatible over the intersections $S_\alpha \cap S_\beta$. In our case, such a vector bundle can be constructed explicitly **[LT1]**. We still denote it by $\{E_\alpha\}$. Then from the Basic Construction 2.5, we can construct (virtual normal) cone cycle $N_\alpha \subset E_\alpha$. Since such construction is unique, $N_\alpha \subset E_\alpha$ are G_α-equivariant and are compatible.

In the end, we need to intersect this collection of cone-cycles $[N_\alpha] \in Z_* E_\alpha$ with the collection of zero sections of E_α. One way to do this is to descend E_α to a V-vector bundle on $\mathfrak{M}_{g,k}(X,A)$ which contains the descendents of $[N_\alpha]$ as a subcone cycle. The other is to view the collection E_α as a vector bundle over the stack $\mathfrak{M}_{g,k}(X,A)$. We denote $\mathbf{E}$ the collection $\{E_\alpha\}$ either as a V-vector bundle or a vector bundle over a stack. We denote by $[\mathbf{N}] \subset \mathbf{E}$ the corresponding virtual normal cone cycle. As before, if we let $\eta_{\mathbf{E}}$ be the zero section of $\mathbf{E}$, then the virtual moduli cycle should be

$$[\mathfrak{M}_{g,k}(X,A)]^{\mathrm{vir}} = \eta_{\mathbf{E}}^*([\mathbf{N}]) \in A_* \mathfrak{M}_{g,k}(X,A),$$

where $\eta_{\mathbf{E}}^*$ is an appropriately defined Gysin homomorphism of the zero section $\eta_{\mathbf{E}}$. Such Gysin homomorphism is known to exists **[Vi]**. The result of this Gysin homomorphism takes values in the group of cycles with rational coefficient.

Once we have the virtual moduli cycle, we define the GW invariant to be

$$\gamma^X_{A,g,k}(\alpha) = < [\mathfrak{M}_{g,k}(X,A)]^{\mathrm{vir}}, \mathrm{ev}^*(\alpha) > \in \mathbb{Q}, \qquad \alpha \in A^* X^k.$$

This completes the task of constructing the GW invariants of smooth projective varieties.

3. Symplectic GW invariants

The main purpose of this chapter is to construct the virtual moduli cycles for general symplectic manifolds. Accordingly, in this chapter we will work with

[6]The intersection is $S_\alpha \times_{\mathfrak{M}_{g,k}(X,A)} S_\beta$ in Grothendieck's topology.

analytic category. For instance, we will view the moduli space of k-pointed genus g stable curves $\mathfrak{M}_{g,k}$, $2g+k \geq 3$, as a compact orbifold.

Let X be a smooth symplectic manifold with a given symplectic form ω of complex dimension n, and let $A \in H_2(X,\mathbb{Z})$. We state the main theorem of this chapter.

THEOREM 3.1. *Let (X,ω) be a compact symplectic manifold of complex dimension n. Then for each g, k and A, there is a virtual fundamental class*

$$e_{A,g,k}(X) \in H_r(\mathfrak{M}_{g,k} \times X^k, \mathbb{Q}),$$

where $r = 2c_1(X)(A)+2(n-3)(1-g)+2k$. Moreover, this $e_{A,g,k}(X)$ is a symplectic invariant.

As an application, let us define the GW-invariants now. Let $2g+k \geq 3$. We define

$$\psi^X_{A,g,k} : H^*(\mathfrak{M}_{g,k},\mathbb{Q}) \times H^*(X^k,\mathbb{Q}) \mapsto \mathbb{Q}, \tag{3.1}$$

to be the integrals

$$\psi^X_{A,g,k}(\beta,\alpha) = \int_{e_{A,g,k}(X)} \mathrm{pr}_1^*\beta \wedge \mathrm{pr}_2^*\alpha \tag{3.2}$$

where $\beta \in H^*(\mathfrak{M}_{g,k},\mathbb{Q})$, $\alpha \in H^*(X^k,\mathbb{Q})$ and pr_i is the projection of $\mathfrak{M}_{g,k} \times X^k$ to its component. All $\gamma^X_{A,g,k}$ are symplectic invariants of (X,ω).

To construct such cycles, we will introduce the notion of smoothly stratified orbispaces, weakly pseudocycles and weakly Fredholm V-bundles over any smoothly stratified orbispaces. This is what we will do in the first part of this chapter. After this we will sketch the construction of the virtual moduli cycles for weakly Fredholm V-bundles. Finally, we will apply this construction to stable maps and consequently prove the above theorem.

3.1. Smoothly stratified orbispaces. In this section, we introduce a class of topological spaces that admit a class of stratified structures. In this note, all topological spaces are Hausdorff.

We first introduce the notion of *smoothly stratified spaces*.

DEFINITION 3.2. *By a smoothly stratified space we mean a topological space X with a locally finite*[7] *partition $X = \bigcup_{\alpha\in I} X_\alpha$, called a stratification, by locally closed subsets X_α, called strata, such that each X_α is a smooth Banach manifold.*

Note that our definition differs from the usual one in that we do not require knowledge of the normal cone structure along each stratum X_α. Let X and Y be smoothly stratified spaces. We will denote by X^{top} the topological space X without the stratification. A map $f : X \to Y$ between two smoothly stratified spaces is a map $f : X^{\mathrm{top}} \to Y^{\mathrm{top}}$ such that it maps strata of X into strata of Y. f is smooth if $f|_{X_\alpha}$ are smooth. A smooth manifold is viewed as a smoothly stratified space whose stratification consists of a single stratum X. In the following we will call smoothly stratified space simply stratified space if no confusion will arise.

Next we introduce the notion of *smoothly stratified orbispaces* and their *local uniformization charts*. We let X be a topological space with a locally finite, locally closed stratification $X = \bigcup_{\alpha\in I} X_\alpha$.

[7] By this we mean for each $x \in X$ the set $\{\alpha \in I \mid a \in X_\alpha\}$ is finite.

DEFINITION 3.3. *Let X be as before. A local uniformization chart of X consists of a smoothly stratified space $\tilde{\mathbf{U}} = \cup_{\alpha \in I} \tilde{\mathbf{U}}_\alpha$, a finite group $G_{\tilde{\mathbf{U}}}$ acting effectively on $\tilde{\mathbf{U}}$ by smooth maps and a $G_{\tilde{\mathbf{U}}}$-invariant continuous map $\pi_{\mathbf{U}} : \tilde{\mathbf{U}}^{\text{top}} \to X^{\text{top}}$ such that*
(1) $\mathbf{U} = \pi_{\mathbf{U}}(\tilde{\mathbf{U}})$ is an open neighborhood of X;
(2) each stratum $\mathbf{U}_\alpha$ of $\tilde{\mathbf{U}}$ is invariant under $G_{\tilde{\mathbf{U}}}$;
(3) $\pi_{\mathbf{U}}$ induces a covering map from $\tilde{\mathbf{U}}/G_{\tilde{\mathbf{U}}}$ onto $\mathbf{U}$ as stratified spaces that respect the induced stratum $\tilde{\mathbf{U}}_\alpha/G_{\tilde{\mathbf{U}}}$ of $\tilde{\mathbf{U}}/G_{\tilde{\mathbf{U}}}$ and $\tilde{\mathbf{U}} \cap X_\alpha$ of $\mathbf{U}$.

Again, if there is no confusion we will call $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$ a chart and call $\mathbf{U}$ the quotient of the chart $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$. Let $\mathbf{V} \subset \mathbf{U}$ be an open subset, then $(\pi_{\mathbf{U}}^{-1}(\mathbf{V}), G_{\tilde{\mathbf{U}}})$ defines a new chart of X, called the restriction of $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$ to $\mathbf{V}$. For simplicity, we will often denote it by $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})|_{\mathbf{V}}$. When $p \in \mathbf{U}$, we sometimes call $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$ a chart of p. Now let X be as before and let $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$ and $(\tilde{\mathbf{V}}, G_{\tilde{\mathbf{V}}})$ be two charts of X. We say the later is finer than the former if there is a homomorphism $G_{\tilde{\mathbf{V}}} \to G_{\tilde{\mathbf{U}}}$ and an equivariant smooth covering map $\varphi_{\tilde{\mathbf{U}}\tilde{\mathbf{V}}}$ belongs to the square

$$
\begin{tikzcd}
\pi_{\mathbf{V}}^{-1}(\mathbf{U} \cap \mathbf{V}) \arrow[r, "\varphi_{\tilde{\mathbf{U}}\tilde{\mathbf{V}}}"] \arrow[d, "\pi_{\mathbf{V}}"'] & \pi_{\mathbf{U}}^{-1}(\mathbf{U} \cap \mathbf{V}) \arrow[d, "\pi_{\mathbf{U}}"'] \\
\mathbf{U} \cap \mathbf{V} \arrow[r, equal] & \mathbf{U} \cap \mathbf{V}
\end{tikzcd}
\tag{3.3}
$$

Note that if $\mathbf{U} \cap \mathbf{V} = \emptyset$, then $\tilde{\mathbf{U}}$ is automatically finer than $\tilde{\mathbf{V}}$.

We now define the notion of smoothly stratified orbispace.

DEFINITION 3.4. *We say that X is a smoothly stratified orbispace if it can be covered by a collection of charts as in the Definition 3.3 such that for any two charts $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i})$ and $(\tilde{\mathbf{U}}_j, G_{\tilde{\mathbf{U}}_j})$ and $x \in \mathbf{U}_i \cap \mathbf{U}_j$, there is a chart $(\tilde{\mathbf{V}}, G_{\tilde{\mathbf{V}}})$ of $x \in X$ such that $(\tilde{\mathbf{V}}, G_{\tilde{\mathbf{V}}})$ is finer than both $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i})$ and $(\tilde{\mathbf{U}}_j, G_{\tilde{\mathbf{U}}_j})$.*

Later, we will abbreviate smoothly stratified orbispaces to SS-orbispaces. Clearly, each smoothly stratified space is a SS-orbispace. A smoothly stratified space is smooth if all its charts, namely the $\tilde{\mathbf{U}}_i$ in the Definition 3.4, are smooth. In particular, a finite dimensional smooth smoothly stratified space is an orbifold. Note that according to this definition, each stratum X_α of X is a Banach orbifold.

If X and Y are two SS-orbispaces, a map $f : X \to Y$ is a map $f^{\text{top}} : X^{\text{top}} \to Y^{\text{top}}$ such that for each $x \in X$ there is a chart $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$ of $x \in X$, a chart $(\tilde{\mathbf{V}}, G_{\tilde{\mathbf{V}}})$ of $f(x) \in Y$, a homomorphism $G_{\tilde{\mathbf{U}}} \to G_{\tilde{\mathbf{V}}}$ and a $G_{\tilde{\mathbf{U}}}$-equivariant map $\tilde{f} : \tilde{\mathbf{U}} \to \tilde{\mathbf{V}}$ such that it descends to $f^{\text{top}}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V} \subset X$. We call such $\tilde{f}$ a local representative of f. We say f is smooth if all its local representatives are smooth. A smooth map $f : X \to Y$ with smooth inverse f^{-1} is called an isomorphism.

3.2. Weakly pseudocycles. In this subsection, we will work with a continuous map $f : X \to Y$ between two topological spaces X and Y. All homology theories are with rational coefficients.

We begin with pseudomanifold. An oriented pseudomanifold S of dimension d in X is a pair (ρ, M), where M is an oriented, smooth manifold of dimension d with boundary ∂M with corner[8] and $\rho : M \to X$ is a continuous map. For a

[8] A manifold of dimension d with boundary with corner is is a manifold with boundary except that the neighborhoods of a boundary point are isomorphic to the neighborhoods of a boundary

pseudomanifold $S = (f, M)$ we define ∂S to be $(f, \partial M)$. We define $-S$ to be $(f, \overline{M})$, where $\overline{M}$ is M with the reversed orientation. We define the sum of two pseudomanifolds (f, M) and (g, N) to be $(f \coprod g, M \cup N)$. We then can define the sum $\sum n_i \alpha_i$, where $n_i \in \mathbb{Z}$, of pseudomanifolds inductively. For any (f, M), we say it is 0 if there are two disjoint open subsets U_1 and U_2 of M so that the complement of their union in M is an at most $d-1$ dimensional set[9], such that there is an orientation reversing isomorphism $\rho : U_1 \to U_2$ so that $f|_{U_1} \equiv f \circ \rho$. We can talk about rational pseudomanifolds, which are formal sum of oriented pseudomanifolds with rational coefficients. A rational pseudomanifold is zero if an integer multiple of it is integral and is zero.

Let $f : X \to Y$ be a continuous map between topological spaces. We now define weakly f-pseudocycles.

DEFINITION 3.5. *A rational weakly d-dim'l f-pseudocycle is a triple (ρ, M, K), where (ρ, M) is a rational pseudomanifold in X, K is a closed subset in Y of homological dimension[10] no more than $d-2$ such that the closure of $M - (f \circ \rho)^{-1}(K)$ is compact in M and the rational $(d-1)$-dimensional pseudomanifold $(\rho, \partial M - (f \circ \rho)^{-1}(K))$ is 0.*

Two such cycles (ρ_1, M_1, K_1) and (ρ_2, M_2, K_2) are quasi-cobordant if there is a $(d+1)$-dimensional pseudomanifold (ρ, M) in X, a closed subset $K \subset Y^{\mathrm{top}}$ of homological dimension no more than $d-1$ such that $M - (f \circ \rho)^{-1}(K)$ is compact in M, K_1 and K_2 are contained in K and $(\rho_1, M_1) - (\rho_2, M_2) = (\rho, \partial M)$. Two such cycles S and S' are equivalent if there is a chain of weakly f-pseudocycle $S_0, \cdots, S_l$ so that $S = S_0$, S_i is quasi-cobordant to S_{i+1} and $S_l = S'$.

The usefulness of this definition is illustrated by the following Proposition.

PROPOSITION 3.6. *Let (ρ, M, K) be a weakly d-dim'l f-pseudocycle. Then $f \circ \rho : M \to Y$ defines a unique element in $H_d(Y^{\mathrm{top}}, K) = H_d(Y^{\mathrm{top}})$. Further, two equivalent such cycles define identical elements in $H_d(Y)$.*

PROOF. First, the two homology groups are canonically isomorphic since K has homological dimension $\leq d-2$. Now we show that $f \circ \rho$ defines a cycle in $H_d(Y^{\mathrm{top}}, K)$. Since M is smooth and $M - (f \circ \rho)^{-1}(K)$ is pre-compact in M, we can give M a triangulation so that the boundary of $f \circ \rho(M)$ is contained in K, thus it defines an element in $H_d(Y, K)$. For the same reason, two quasi-cobordant and thus two equivalent such cycles define identical elements in $H_d(Y)$. □

3.3. Weakly Fredholm V-bundles. In this section, we introduce the notion of weakly Fredholm V-bundles over a SS-orbispace $X = \cup_{\alpha \in I} X_\alpha$. We assume throughout this chapter that $0 \in I$ and the stratum X_0 is dense in X. Also, if $\tilde{\mathbf{U}}$ is a chart of X, then $\tilde{\mathbf{U}}$ has a stratification $\cup_{\alpha \in I} \tilde{\mathbf{U}}_\alpha$ so indexed that $\tilde{\mathbf{U}}_\alpha$ corresponds to the stratum $\mathbf{U} \cap X_\alpha$.

DEFINITION 3.7. *Let X be an SS-orbispace. A V-bundle over X is a pair $(\mathbf{E}, \mathbf{P})$, where $\mathbf{E}$ is an SS-orbispace and $\mathbf{P} : \mathbf{E} \to X$ is a projection (of SS-orbispaces) such that X (resp. $\mathbf{E}$) is covered by a set of charts $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i})$ (resp. $(\mathbf{E}_{\tilde{\mathbf{U}}_i}, G_{\tilde{\mathbf{U}}_i})$)*

point of $L = \{x_1 \geq 0, \cdots, x_k \geq 0\}$. The boundary of such manifold is also a manifold with boundary with corner, following the convention that the boundary of L is the disjoint union of k manifolds $L \cap \{x_i = 0\}$.

[9]By this we mean it is the image of $d-1$ dimensional manifold.

[10]By this we mean $\max\{k; H_k(K, \mathbb{Q}) \neq 0\}$.

of which the following hold:
(1) If $\tilde{\mathbf{U}}_i$ is a chart over $\mathbf{U}_i \subset X$ then $\mathbf{E}_{\tilde{\mathbf{U}}_i}$ is a chart over $\mathbf{P}^{-1}(\mathbf{U}_i)$ and there is a projection $\mathbf{P}_{\tilde{\mathbf{U}}_i} : \mathbf{E}_{\tilde{\mathbf{U}}_i} \to \tilde{\mathbf{U}}_i$ representing $\mathbf{P}$ over $\mathbf{U}_i$.
(2) Let $\tilde{\mathbf{U}}_\alpha$ be any stratum of $\tilde{\mathbf{U}}_i$, then restricting to $\tilde{\mathbf{U}}_\alpha$ the bundle $\mathbf{E}_{\tilde{\mathbf{U}}_i}|_{\tilde{\mathbf{U}}_\alpha}$ is a smooth Banach vector bundle over $\tilde{\mathbf{U}}_\alpha$ with a $G_{\tilde{\mathbf{U}}_i}$-linear action. Further the stratum $\mathbf{E}_{\tilde{\mathbf{U}}_i}$ are pull-back of the stratum of $\tilde{\mathbf{U}}_i$.
(3) For any two charts $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i})$ and $(\tilde{\mathbf{U}}_j, G_{\tilde{\mathbf{U}}_j})$ of X (over $\mathbf{U}_i$ and $\mathbf{U}_j$ respectively) and $x \in \mathbf{U}_i \cap \mathbf{U}_j$, there is a chart $(\tilde{\mathbf{U}}_k, G_{\tilde{\mathbf{U}}_k})$ of x finer than both $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i})$ and $(\tilde{\mathbf{U}}_j, G_{\tilde{\mathbf{U}}_j})$ such that there are $G_{\mathbf{U}}$-equivariant isomorphisms

$$\mathbf{E}_{\tilde{\mathbf{U}}_k}|_{\mathbf{U}_i \cap \mathbf{U}_k} \cong \varphi^*_{\tilde{\mathbf{U}}_k \tilde{\mathbf{U}}_i}(\mathbf{E}_{\tilde{\mathbf{U}}_i}) \quad \text{and} \quad \mathbf{E}_{\tilde{\mathbf{U}}_k}|_{\mathbf{U}_j \cap \mathbf{U}_k} \cong \varphi^*_{\tilde{\mathbf{U}}_k \tilde{\mathbf{U}}_j}(\mathbf{E}_{\tilde{\mathbf{U}}_j})$$

preserving the Banach bundle structures along each stratum, where $\varphi_{\tilde{\mathbf{U}}_k \tilde{\mathbf{U}}_i}$ is the map given in (3.3).
(4) Let $\mathbf{O}_{\mathbf{E}_{\tilde{\mathbf{U}}_i}}$ be the zero sections of $\mathbf{E}_{\tilde{\mathbf{U}}_i}$, then they define a SS-orbi-subspace $\mathbf{O}_{\mathbf{E}}$ of $\mathbf{E}$. We require that it is isomorphic to X under the projection $\mathbf{P}$.

Now let $\mathbf{E}$ be a V-fiber bundle over X. A section of $\mathbf{E}$ is a map $\Phi : X \to \mathbf{E}$ as SS-orbispaces so that $\mathbf{P} \circ \Phi : X \to X$ is an isomorphism. We say Φ has compact support if $\Phi^{-1}(O_{\mathbf{E}})$ as a subspace in X^{top} is compact.

We fix a V-bundle over X with a section Φ with compact support. We now describe the weakly Fredholm structure of this bundle with section in terms of its local finite approximations.

DEFINITION 3.8. *A local finite approximation of $(\mathbf{E}, \mathbf{P}, \Phi)$ consists of a chart $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$ of X, a chart $(\mathbf{E}_{\tilde{\mathbf{U}}}, G_{\tilde{\mathbf{U}}})$ of $\mathbf{E}$, a representative $\Phi_{\tilde{\mathbf{U}}} : \tilde{\mathbf{U}} \to \mathbf{E}_{\tilde{\mathbf{U}}}$ of Φ and a finite rank equi-rank $G_{\tilde{\mathbf{U}}}$-linearized vector bundle $\mathbf{F}$ over $\tilde{\mathbf{U}}$ such that:*
(1) $\mathbf{F}$ is a $G_{\tilde{\mathbf{U}}}$-equivariant SS-subspace of $\mathbf{E}_{\tilde{\mathbf{U}}}$. Further restricting to each stratum $\tilde{\mathbf{U}}_\alpha \subset \tilde{\mathbf{U}}$ the inclusion $\mathbf{F}|_{\tilde{\mathbf{U}}_\alpha} \subset \mathbf{E}_{\tilde{\mathbf{U}}}|_{\tilde{\mathbf{U}}_\alpha}$ is a smooth sub-vector bundle.
(3) $U := \Phi^{-1}_{\tilde{\mathbf{U}}}(\mathbf{F}) \subset \tilde{\mathbf{U}}$ is a finite dimension equi-dimension SS-orbispace (with strata $U = \cup_{\alpha \in I} U_\alpha$). Further we require $U_0 = U \cap \tilde{\mathbf{U}}_0$ is dense in U and its complement has codimension at least 2 in U;
(4) $F := \mathbf{F}|_U$ is a continuous vector bundle. Further its restriction to each stratum U_α it is a smooth vector bundle and the map $\phi_F|_{U_\alpha}$, where $\phi_F = \Phi_{\tilde{\mathbf{U}}}|_U : U \to F$, is a smooth section;
(5) At each $w \in \Phi^{-1}(0) \cap \tilde{\mathbf{U}}_0$, the differential

$$d\Phi(w) : T_w \tilde{\mathbf{U}}_0 \longrightarrow \mathbf{E}_{\tilde{\mathbf{U}}_0}|_w \,/\, \mathbf{F}|_w$$

is surjective.

Such local finite approximation will be denoted by $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$.

An orientation of $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ is a $G_{\tilde{\mathbf{U}}}$-invariant orientation of the real line bundle $\wedge^{\mathrm{top}}(TU) \otimes \wedge^{\mathrm{top}}(F)^{-1}$ over U_0. Such orientations are given by $G_{\tilde{\mathbf{U}}}$-invariant non-vanishing sections of $\wedge^{\mathrm{top}}(TU) \otimes \wedge^{\mathrm{top}}(V)^{-1}$ over U_0.

We call $\operatorname{rank} F - \dim U$ the index of $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$.

Now assume that $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'}, \mathbf{E}_{\tilde{\mathbf{U}}'}, \mathbf{F}')$ is another local finite approximation of identical index over $\mathbf{U}' \subset X$.

DEFINITION 3.9. *We say that $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'}, \mathbf{E}_{\tilde{\mathbf{U}}'}, \mathbf{F}')$ is finer than $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ if we have the following:*

(1) $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'})$ *is finer than* $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}})$.
(2) Let $\varphi : \pi_{\tilde{\mathbf{U}}'}^{-1}(\mathbf{U} \cap \mathbf{U}') \to \pi_{\tilde{\mathbf{U}}}^{-1}(\mathbf{U} \cap \mathbf{U}')$ *(which is* $\varphi_{\mathbf{U}\mathbf{U}'}$ *in (3.3)) be the covering map, then* $\varphi^* \mathbf{F} \subset \varphi^* \mathbf{E}_{\tilde{\mathbf{U}}} \equiv \mathbf{E}_{\tilde{\mathbf{U}}'}|_{\mathbf{U}' \cap \mathbf{U}}$ *is a smoothly stratified subbundle of* $\mathbf{F}'$.
(3) Let (U, F) *and* (U', F') *be those defined in the Definition 3.8 for* $\tilde{\mathbf{U}}$ *and* $\tilde{\mathbf{U}}'$ *respectively, then for any* w *in* $U_0' \cap \varphi^{-1}(U)$, *the natural homomorphism*

$$T_w U' / T_{\varphi(w)} U \longrightarrow (F' / \varphi^* F)|_w$$

is an isomorphism.
(4) In case both $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ *and* $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'}, \mathbf{E}_{\tilde{\mathbf{U}}'}, \mathbf{F}')$ *are oriented, then we require that the orientations of* $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ *and* $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'}, \mathbf{E}_{\tilde{\mathbf{U}}'}, \mathbf{F}')$ *are isomorphic through the isomorphism*

$$\wedge^{\text{top}}(T_w U') \otimes \wedge^{\text{top}}(F'|_w)^{-1} \cong \wedge^{\text{top}}(T_{\varphi(w)} U) \otimes \wedge^{\text{top}}(F|_{\varphi(w)})^{-1},$$

induced by the isomorphism in (3), where w *is any point in* $U_0' \cap \varphi^{-1}(U_0)$.

Note that from (5) of the Definition 3.8, $U_0' \cap \varphi^{-1}(U)$ is a locally closed smooth submanifold of codimension $\operatorname{rank} F' - \operatorname{rank} F$ in U'. Hence the homomorphism (3) in the above Definition is well-defined.

Now let $\mathfrak{A} = \{(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i}, \mathbf{E}_{\tilde{\mathbf{U}}_i}, \mathbf{F}_i)\}_{i \in \mathcal{K}}$ be a collection of oriented finite approximations of $(X, \mathbf{E}, \Phi)$. In the following, we will denote by $\mathbf{U}_i$ the open subsets of X such that $\Lambda_i = (\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i})$ is a uniformization chart over $\mathbf{U}_i$. We say $\mathfrak{A}$ covers $\Phi^{-1}(0)$ if $\Phi^{-1}(0)$ is contained in the union of $\mathbf{U}_i$ in X.

DEFINITION 3.10. *An index* r *oriented pre-weakly Fredholm structure of the triple* $(\mathbf{B}, \mathbf{E}, \Phi)$ *is a collection* $\mathfrak{A} = \{(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i}, \mathbf{E}_{\tilde{\mathbf{U}}_i}, \mathbf{F}_i)\}_{i \in \mathcal{K}}$ *of index* r *oriented local finite approximations such that* $\mathfrak{A}$ *covers* $\Phi^{-1}(0)$ *and that for any* $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i}, \mathbf{E}_{\tilde{\mathbf{U}}_i}, \mathbf{F}_i)$ *and* $(\tilde{\mathbf{U}}_j, G_{\tilde{\mathbf{U}}_j}, \mathbf{E}_{\tilde{\mathbf{U}}_j}, \mathbf{F}_j)$ *in* $\mathfrak{A}$ *with* $p \in \mathbf{U}_i \cap \mathbf{U}_j$, *there is a local finite approximation* $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F}) \in \mathfrak{A}$ *such that* $p \in \mathbf{U}$ *and* $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ *is finer than both* $(\tilde{\mathbf{U}}_i, G_{\tilde{\mathbf{U}}_i}, \mathbf{E}_{\tilde{\mathbf{U}}_i}, \mathbf{F}_i)$ *and* $(\tilde{\mathbf{U}}_j, G_{\tilde{\mathbf{U}}_j}, \mathbf{E}_{\tilde{\mathbf{U}}_j}, \mathbf{F}_j)$.

It follows easily from this definition that for any number of local finite approximations $\{(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)\}_{1 \le i \le l}$ in $\mathfrak{A}$ and $x \in \cap_{i=1}^{l} \mathbf{U}_i$, there is a local finite approximation $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ which contains x and is finer than all $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)$.

Let $\mathfrak{A}'$ be another index r oriented pre-weakly Fredholm structure of $(X, \mathbf{E}, \Phi)$. We say that $\mathfrak{A}'$ is finer than $\mathfrak{A}$ if for any $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F}) \in \mathfrak{A}$ over $\mathbf{U} \subset X$ and $p \in \mathbf{U} \cap \Phi^{-1}(0)$, there is a $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'}, \mathbf{E}_{\tilde{\mathbf{U}}'}, \mathbf{F}') \in \mathfrak{A}'$ over $\mathbf{U}'$ such that $p \in \mathbf{U}'$ and $(\tilde{\mathbf{U}}', G_{\tilde{\mathbf{U}}'}, \mathbf{E}_{\tilde{\mathbf{U}}'}, \mathbf{F}')$ is finer than $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$. We say that these two structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are equivalent if $\mathfrak{A}_1$ is finer than $\mathfrak{A}_2$ and vice verse.

DEFINITION 3.11. *A* V*-vector bundle with section* $(X, \mathbf{E}, \Phi)$ *is weakly Fredholm* V*-bundle if it admits an oriented pre-weakly Fredholm structure such that* $\Phi^{-1}(0)$ *is compact and is contained in finitely many strata of* X.

We will abbreviate a weakly Fredholm V-bundle to WFV-bundle.

DEFINITION 3.12. *Two WFV-bundles* $(X, \mathbf{E}, \Phi)$ *and* $(X, \mathbf{E}', \Phi')$ *are homotopic if there is a WFV-bundle* $(X \times [0,1], \overline{\mathbf{E}}, \Psi)$ *such that it restricts to* $(X, \mathbf{E}, \Phi)$ *over* $X \times \{0\}$ *and to* $(X', \mathbf{E}', \Phi')$ *over* $X \times \{1\}$.

3.4. Construction of virtual moduli cycles. The following is the main result on constructing virtual moduli cycles.

Let X be any SS-orbispace, let Y be a smooth orbifold and $f: X^{\mathrm{top}} \to Y^{\mathrm{top}}$ be a smooth map. We index the stratification $X = \cup_{\alpha \in I} X_\alpha$ and $Y = \cup_{\alpha \in I} Y_\alpha$ so that $f(X_\alpha) \subset Y_\alpha$.

THEOREM 3.13. *Let $(X, \mathbf{E}, \Phi)$ be a WFV-bundle of relative index r. Let $\mathfrak{A}$ be the collection in the definition of WFV-bundle. We assume further that for any $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F}_{\tilde{\mathbf{U}}})$ in $\mathfrak{A}$ and $U = \cup_{\alpha \in I} U_\alpha$ the stratification, the representative $f_{\tilde{\mathbf{U}},\alpha}: U_\alpha \to Y_\alpha$ is a submersion. Then we can assign to $(X, \mathbf{E}, \Phi)$ an equivalence class of weakly f-pseudocycle $e(X, \mathbf{E}, \Phi)$, called the Euler class of $(X, \mathbf{E}, \Phi)$. It depends only on the homotopy class of $(X, \mathbf{E}, \Phi)$. Further, the image of this class under f_* is a well-defined homology class in $H_r(Y, \mathbb{Q})$.*

The class $e(X, \mathbf{E}, \Phi)$ or its image under f_* is the virtual cycle we set to construct.

Sometimes it is desirable to include the case where Y is a more general topological space. For instance if we want the Euler class to be a class in $H_r(X^{\mathrm{top}})$, then we can take $f = \mathrm{id}_X : X \to X$ and hence $Y = X$ will be an infinite dimensional space. The proof of Theorem 3.13 can be generalized to such case if $f : X \to Y$ satisfies certain topological condition. However, when Y is infinite dimensional, we have not found a satisfactory condition that is general enough to include many interesting examples. We will leave this to our future investigation.

In the remainder of this subsection, we will briefly sketch how to use the WFV structure of $(X, \mathbf{E}, \Phi)$ to construct the required Euler class. The complete account of this approach is a modification of the construction in [**LT2**, **LT4**] and will appear elsewhere [**LT5**].

The idea of the construction is as follows: Given a local finite approximation $(\tilde{\mathbf{U}}, G_{\tilde{\mathbf{U}}}, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ over $\mathbf{U} \subset X$, we can associate to it a finite dimensional model consisting of a finite dimensional SS-orbispace U, a smoothly stratified finite rank V-bundle F over U and a smooth section $\phi : U \to F$. Following the topological construction of Euler class, we like to perturb ϕ to a new section $\tilde{\phi}: U \to F$ so that $\tilde{\phi}$ is as transversal to the zero section of F as possible. Here the difficulty with this is that F is only smooth along, not near, strata of U. Thus the notion of transversality is ill-defined near points where F is not smooth. Here is our solution: We find a homological dimension $r-2$ set $K \subset U$ so that we can perturb ϕ to $\tilde{\phi}$ so that $\tilde{\phi}$ is transversal to the zero section of F away from K. Since the dimension of the zero locus of $\tilde{\phi}$ is r, the set K will not affect the construction of an r dimensional cycle. In the case we work with many local finite approximations, we will choose these perturbations so that their vanishing locus patch together to form a well-defined weakly f-pseudocycle in X. The Euler class of $(X, \mathbf{E}, \Phi)$ is the equivalent class it represents.

As is clear from this description, the main difficulty of this construction is to make sure that these perturbed vanishing locus patch together. We will do perturbation one chart at a time, and in the mean time make sure that the perturbation on the current chart agrees with the perturbations chosen in the preceding charts. One difficulty arises in that the intersection of charts are not open. However if the current chart is finer than all the preceding charts, then the intersection will be an open subset in the preceding charts and a locally closed subset in the current chart. Hence our first order is to pick a good set of charts to begin with. Let

$\mathfrak{A} = \{(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)\}_{i \in \mathcal{K}}$ be the weakly smooth structure of $(X, \mathbf{E}, \Phi)$. For simplicity, we will denote the projection $\pi_{\mathbf{U}_i}: \tilde{\mathbf{U}}_i \to \mathbf{U}_i$ by π_i, denote $\Phi_i^{-1}(\mathbf{F}_i)$ by U_i, denote $\mathbf{F}_i|_{U_i}$ by F_i and denote the restriction $\Phi_i|_{U_i}: U_i \to \tilde{\mathbf{E}}_i|_{U_i}$ by $\phi_i: U_i \to F_i$. Without loss of generality, we can assume that for any approximation $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i) \in \mathfrak{A}$ over $\mathbf{U}_i$ and any $\mathbf{U}' \subset \mathbf{U}_i$, the restriction $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)|_{\mathbf{U}'}$ is also a member in $\mathfrak{A}$.

The following lemma enables us to pick a good set of charts.

LEMMA 3.14. *There is a finite collection $\mathcal{L} \subset \mathcal{K}$ and a total ordering of $\mathcal{L}$ of which the following holds:*
(1) The set $\Phi^{-1}(0)$ is contained in the union $\cup\{U_i \mid i \in \mathcal{L}\}$;
(2) For any pair $i < j \in \mathcal{L}$, the approximation $(\tilde{\mathbf{U}}_j, G_j, \mathbf{E}_j, \mathbf{F}_j)$ is finer than the approximation $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)$.

PROOF. We now outline the proof, which is elementary.

We will use the stratified structure of $(X, \mathbf{E}, \Phi)$ as we did in [LT2]. Let X_α, $\alpha \in I_0$ be all the strata of X that intersect $\Phi^{-1}(0)$. We give an order to I_0 so that for each $\alpha \in I_0$, $\Phi^{-1}(0) \cap \bigcup_{\beta \geq \alpha} X_\beta$ is compact. For convenience, we assume $I_0 = \{1, \cdots, k\}$.

To prove the lemma, we suffice to construct subsets $\mathcal{L}_\alpha^l \subset \mathcal{K}$, where $\alpha \in I_0$ and $0 \leq l \leq l_\alpha$, such that
$\mathbf{A}_1$: For any distinct $i, j \in \mathcal{L}_\alpha^l$, $\mathbf{U}_i \cap \mathbf{U}_j = \emptyset$;
$\mathbf{A}_2$: For each $\alpha \in I_0$ and $l \leq l_\alpha$, the set

$$\mathbf{Z}_\alpha^l = \Phi^{-1}(0) \cap X_\alpha - \cup_{j \geq l} \cup_{\beta \in \mathcal{L}_\alpha^j} \mathbf{U}_\beta$$

is contained in the image of finitely many manifolds of dimension less than l in X_α;
$\mathbf{A}_3$: For any pair of distinct $(i, j) \in \mathcal{L}_\alpha^l \times \mathcal{L}_{\alpha'}^{l'}$ with $(\alpha, l) < (\alpha', l')$, i.e. $\alpha \leq \alpha'$ or $\alpha = \alpha'$ and $l \leq l'$, the chart $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)$ is finer than $(\tilde{\mathbf{U}}_j, G_j, \mathbf{E}_j, \mathbf{F}_j)$.
In particular, all $\mathbf{U}_i$ with $i \in \bigcup_{0 \leq l \leq l_\alpha} \mathcal{L}_\alpha^l$ cover $\Phi^{-1}(0) \cap X_\alpha$.

We will construct $\mathcal{L}_\alpha^l$ inductively, starting from the largest $\alpha \in I_0$. Note that $\mathcal{L}_\alpha^l$ may be empty. We now assume that for some $\alpha \in I_0$ we have constructed $\mathcal{L}_{\alpha+1}^l, \mathcal{L}_{\alpha+2}^l, \cdots$ for all l. We now construct $\mathcal{L}_\alpha^l$. We first pick a finite $\mathcal{L}' \subset \mathcal{K}$ such that $\{\mathbf{U}_i | i \in \mathcal{L}'\}$ covers $\mathbf{Z}_{\alpha+1}^0$. This is possible since it is compact. Let l_α be the maximum of $\{\dim(U_i \cap X_\alpha) | i \in \mathcal{L}'\}$. By assumption, $l_\alpha > 0$. Since $\tilde{\mathbf{U}}_i$ is a SS-space with strata $\pi_i^{-1}(\mathbf{U}_i \cap X_\alpha)$, we can find open subsets $\mathbf{U}_i' \subset \mathbf{U}_i$ such that (1) let $\tilde{\mathbf{U}}_i' = \pi_i^{-1}(\mathbf{U}_i)$ then $\tilde{\mathbf{U}}_i'^- \cap \pi_i^{-1}(X_\alpha)$ is the same as the closure of $\tilde{\mathbf{U}}_i' \cap \pi_i^{-1}(X_\alpha)$ in $\pi_i^{-1}(X_\alpha)$; (2) Let $R = \tilde{\mathbf{U}}_i' \cap U_i \cap \pi_i^{-1}(X_\alpha)$, then $R^- - R$ (in $U_i \cap \pi_i^{-1}(X_\alpha)$) is a smooth submanifold in $U_i \cap \pi_i^{-1}(X_\alpha)$ of dimension less than l_α. After fixing an ordering on $\mathcal{L}'$, we define $\mathcal{L}_\alpha^{l_\alpha}$ to be the collection of charts

$$(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)|_{\mathbf{U}_i' \setminus \bigcup_{j<i} \mathbf{U}_j'^-}, \qquad i \in \mathcal{L}'. \tag{3.4}$$

Clearly, $\mathcal{L}_\alpha^{l_\alpha}$ satisfies $\mathbf{A}_1 - \mathbf{A}_3$. Now we assume that $\mathcal{L}_\alpha^{l_\alpha}, \cdots, \mathcal{L}_\alpha^l$ have been constructed. We want to construct $\mathcal{L}_\alpha^{l-1}$. For each $x \in \mathbf{Z}_\alpha^l$, we can find a finite approximation $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)$ such that it contains x and it is finer than all approximations in $\cup_{l' \geq l} \mathcal{L}_\alpha^{l'}$. We denote this collection by $\mathcal{L}'$ again. Since $\mathbf{Z}_\alpha^l$ is compact, we can pick a finite subcollection, still denote by $\mathcal{L}'$, such that $\{\mathbf{U}_i | i \in \mathcal{L}'\}$ covers $\mathbf{Z}_\alpha^l$. By our choice of the charts, $\mathbf{Z}_\alpha^l$ is contained in the image of finitely many submanifolds in X_α of dimension less than l. Next we choose $\mathbf{U}_i' \subset \mathbf{U}_i$ such that $\pi_i^{-1}(\mathbf{U}_i')$, $i \in \mathcal{L}'$, still cover $\mathbf{Z}_\alpha^l$ and that if we let $R = \pi_i^{-1}(\mathbf{U}_i') \cap \mathbf{Z}_\alpha^l$, then $R^- - R$ is a finite union of smooth submanifolds of dimensions $\leq l-1$. Now we fix an ordering on $\mathcal{L}'$, as before, and define $\mathcal{L}_\alpha^{l-1}$ to be the collection (3.4). By continuing this procedure for $l-2,\cdots,0$, we obtain the desired $\mathcal{L}_\alpha^l$ for $0 \leq l \leq l_\alpha$. This completes the proof of the Lemma. $\square$

We now briefly sketch the strategy of the proof of the Theorem 3.13. Let $(\tilde{\mathbf{U}}_i, G_i, \mathbf{E}_i, \mathbf{F}_i)_{\alpha\in\mathcal{L}}$ be the collection of local finite approximations given by Lemma 3.14. We denote by $\phi_i : U_i \to F_i$ the corresponding finite models. As stratified set $U_i = \cup_{\alpha\in I_0} U_{i,\alpha}$, where I_0 is the ordered index set in the proof of Lemma 3.14. Since $\Phi^{-1}(0)$ is compact, we can find open $\tilde{\mathbf{W}}_i \subset \tilde{\mathbf{U}}_i$ with $\pi_i^{-1}(\pi_i(\tilde{\mathbf{W}}_i)) = \tilde{\mathbf{W}}_i$ such that $\cup\{\pi_i(\tilde{\mathbf{W}}) \mid i \in \mathcal{L}\}$ covers $\Phi^{-1}(0)$ and $\tilde{\mathbf{W}}_i \cap U_i$ is precompact in U_i.[11] Note that by assumption, we can assume that for each $\alpha \in I_0$, $f|_{U_{i,\alpha}} : U_{i,\alpha} \to Y_\alpha$ is a submersion.

Before we perturb the sections ϕ_i, we need to clarify by which we mean a perturbation is generic. We begin with the general situation. Let $p: U \to V$ be a smooth map between two open smooth manifolds. Let $K \subset U$ be a pre-compact subset and $s \in \Gamma_U(E)$ is a smooth section of a smooth vector bundle over U. We say t is a p-generic perturbation of s relative to K with compact support if the following holds. (1) The support of $s-t$ is pre-compact in U and (2) there is a precompact open neighborhood K^{nbfd} of $K^- \subset U$ so that the graph of t is transversal to the zero section in K^{nbhd} and $p: K^{\text{nbhd}} \cap t^{-1}(0) \to V$ is a strong immersion. Here by a strong immersion $p: A \to B$ between open manifolds we mean p is an immersion and further for any $x, y \in A$ so that $p(x) = p(y)$ then $p_* T_x A$ is transversal to $p_* T_y A$.

LEMMA 3.15. *Let the situation be as before. Suppose dim* $U-$ *rank* $E <$ *dim* V *and* $f: U \to V$ *is a submersion. Then for any pre-compact* $K \subset U$ *and* $s \in \Gamma_U(E)$, *the space of* f*-generic perturbations of* s *relative to* K *with compact support, denoted* B_0, *is dense and open in the space* $B = \{h \in \Gamma_U(E) \mid h - s \text{ is compact}\}$ *with Whitney* C^∞ *topology.*

PROOF. The proof is an easy application of the ordinary transversality theorem. It is clear that $B_0 \subset B$ is open. It remains to show it is dense. First, by the usual transversality theorem we can find a small perturbation t of s with compact support $t-s$ such that the graph of t is transversal to the zero section in a neighborhood of K^-. We then can find a diffeomorphism $\varphi: U \to U$ sufficiently close to Id_U with $\{x \in U \mid \varphi(x) \neq x\}$ precompact in U so that $p: \varphi(t^{-1}(0)) \to V$ is a strong submersion in a neighborhood of K^-. We pick an isomorphism $\phi: E \cong \varphi^* E$ so that it is close to $E \equiv E$ and away from a compact set $L \supset K$ of U the isomorphism ϕ reduces to $E \equiv E$. Here of course we assume $\varphi|_{U-L} = \mathrm{Id}_{U-L}$. Then the corresponding section $\phi^*(t)$ is a small perturbation of t and is in B_0. This proves the Lemma. $\square$

We now show how to construct the perturbations of ϕ_i one at a time. Without loss of generality, we assume $r \leq dimY$, since otherwise $H_r(Y) = 0$. We begin with the first member in $\mathcal{L}$, which is 1. To make the notation easy to follow, we denote the finite model $\phi_1 : U_1 \to F_1$ by $\phi: U \to F$, $\tilde{\mathbf{W}}_1 \subset \tilde{\mathbf{U}}_1$ by $\tilde{\mathbf{W}}$ and $U_{1,\alpha}$ by U_α. We first look at the largest $\alpha \in \mathcal{I}$ so that $U_\alpha \neq \emptyset$. Assume it is not dense in U. Recall $f: X \to Y$ is the map given and $f: U_\alpha \to Y_\alpha$ is a submersion. We then pick a small f-generic perturbation of $\phi|_{U_\alpha}$ relative to $U_\alpha \cap \tilde{\mathbf{W}}^-$ of compact

[11] By this we mean its closure is compact.

support, denoted by $\phi_\alpha : U_\alpha \to Y_\alpha$. Since U_α is not dense in U, by assumption, it has codimension at least 2. Thus

$$\dim f(\phi_\alpha^{-1}(0) \cap \tilde{\mathbf{W}}^-) \leq r - 2.$$

Since Y is a smooth orbifold, we can find a closed neighborhood $L_\alpha \subset Y$ of $f(\phi_\alpha^{-1}(0) \cap \tilde{\mathbf{W}}^-)$ such that it has homological dimension $\leq r-2$ and its boundary is a smooth orbifold in Y. Let $M_\alpha = f^{-1}(L_\alpha)$.

Let $\beta \in \mathcal{L}$ is the next largest element so that $U_\beta \neq \emptyset$. We next extend ϕ_α to U_β. Since the vanishing locus of ϕ_α in $\tilde{\mathbf{W}}^- \cap U_\beta$ is contained in the interior point of $M_\alpha \cap U_\beta$, we can extend ϕ_α to a neighborhood of $U_\alpha \subset U_\alpha \cup U_\beta$ so that its zero locus in $\tilde{\mathbf{W}}$ is still in $M_\alpha \cap (U_\alpha \cup U_\beta)$. Hence, we can find a small perturbation of $\phi|_{U_\alpha \cup U_\beta}$ so that it is an extension of ϕ_α, it is an f-generic perturbation of $\phi|_{U_\alpha \cup U_\beta}$ relative to $\tilde{\mathbf{W}} - M_\alpha$ with compact support. Let the new section be $\phi_\beta \in \Gamma_{U_\alpha \cup U_\beta}(F)$. Hence $f : \phi_\beta^{-1}(0) \cap \tilde{\mathbf{W}}_\alpha^- \to Y$ is a strong immersion away from L_α. Without loss of generality, we can assume that $f(\phi_\beta^{-1}(0))$ intersects the boundary of L_α transversely, in the sense of orbifold. Since it has dimension $\leq r-2$, assuming U_β is not dense either, we can find a closed strong retraction neighborhood L'_β of $f(\phi_\beta^{-1}(0) \cap \tilde{\mathbf{W}}^-) - L_\alpha$ in Y. Hence $L_\alpha \cup L'_\beta$ will have homological dimension $\leq r-2$ as well. We let $L_\beta = L_\alpha \cup L'_\beta$ and $M_\beta = f^{-1}(L_\beta)$. Continue this procedure, we can find a closed neighborhood $L \subset Y$ of homological dimension $\leq r-2$ and an f-generic perturbation of ϕ relative to $\tilde{\mathbf{W}} \cap U - f^{-1}(L)$ of compact support. Let the perturbed section be $\tilde{\phi}_1$. Let $U_1^n \subset U = U_1$ be a pre-compact neighborhood containing $U \cap \tilde{\mathbf{W}}^-$ so that $\tilde{\phi}$ is transversal to the zero section in $U_1^n - L$. Let $n_1 =$ the product of the number of the sheets of the covering $\tilde{\mathbf{U}}_1/G_2 \to \mathbf{U}_1$ with the order of G_1. We define $\Delta_1 = (1/n_1)[\pi_1(\tilde{\phi}_1^{-1}(0))]$, considered as a rational pseudomanifold.

Next, we continue this procedure to U_2. Let U_{12} be $\varphi_{12}^{-1}(U_1)$, where φ_{12} is defined in (3.3). Then $\varphi_{12}^{-1}(U_1 \cap U_1^n - L_1)$ is a smooth submanifold of U_2. Hence, we can extend $\varphi_{12}^*(\tilde{\phi})$ to a neighborhood of R of $U_{12} \subset U_2$, say ϕ', so that it is a small f-generic perturbation of ϕ_2 relative to $R - f^{-1}(L)$ with compact support. We then extend this small perturbation to whole U_2, say $\tilde{\phi}_2$, so that there is a subset $L_2 \subset Y$ of homological dimension $\leq r-2$ so that $\tilde{\phi}_2$ is an extension of $\varphi_{12}^*(\tilde{\phi}_1)$, it is an f-generic perturbation of ϕ_2 relative to $\tilde{\mathbf{W}}_2^- - f_2^{-1}(L_2)$ with compact support. Let $\Delta = (1/n_2)[\pi_2(\tilde{\phi}_2^{-1}(0))]$. Clearly, $\Delta_1 = \Delta_2$ when restricted to $\mathbf{U}_1 \cap \mathbf{U}_2 - f^{-1}(L_2)$.

Continue this procedure until we reach the largest $m \in \mathcal{L}$. We let $L_m \subset Y$ be the corresponding closed neighborhood of homological dimension $\leq r-2$. Then by our construction, for each $i, j \in \mathcal{L}$,

$$\Delta_i|_{\mathbf{U}_i \cap \mathbf{U}_j - f^{-1}(L_m)} = \Delta_j|_{\mathbf{U}_i \cap \mathbf{U}_j - f^{-1}(L_m)}$$

as pseudomanifolds. We let Δ be the patch together of these Δ_i. Since $\{\tilde{\mathbf{W}}_i \mid i \in \mathcal{L}\}$ covers $\Phi^{-1}(0)$, if we choose the perturbations $\tilde{\phi}_i$ sufficiently close to ϕ_i for all i, then the pair (Δ, L_m) will be a weakly f-pseudocycle. This cycle is the Euler class of $(X, \mathbf{E}, \Phi)$.

3.5. Stable maps. In this section, we review the definition of stable maps.

Let X be a smooth symplectic manifold with a given symplectic form ω, and let $A \in H_2(X, \mathbb{Z})$ and let $g, n \in \mathbb{Z}$ be fixed once and for all. In the following, by a k-pointed prestable curve, we mean a connected complex curve with k marked points and only normal crossing singularity.

We recall the notion of stable C^2-maps [**LT2**, Definition 2.1].

DEFINITION 3.16. *A k-pointed stable map is a collection $(f, \Sigma, x_1 \dots, x_k)$, where $(\Sigma, x_1, \dots, x_k)$ is a k-pointed connected prestable curve and $f: \Sigma \to X$ is a continuous map such that*
(1) the composite $f \circ \pi$ is C^2-smooth, where $\pi: \tilde{\Sigma} \to \Sigma$ is the normalization of Σ and
(2) any component $R \subset \tilde{\Sigma}$ satisfying $(f \circ \pi)_*([R]) = 0 \in H_2(X, \mathbb{Z})$ must have $3 \geq 2g(R) +$ the number of distinguished points on R, where the set of distinguished points on $\tilde{\Sigma}$ consists of all preimages of the marked points and the nodal points of Σ.

For convenience, we will abbreviate $(f, \Sigma, x_1, \dots, x_k)$ to $(f, \Sigma, (x_i))$ or to (f, Σ) or simply to f, if no confusion will arise. We call Σ the domain of f, with marked points understood. Two stable maps $(f, \Sigma, (x_i))$ and $(f', \Sigma', (x'_i))$ are said to be equivalent if there is an isomorphism $\rho: \Sigma \to \Sigma'$ such that $f' \circ \rho = f$ and $x'_i = \rho(x_i)$. When $(f, \Sigma) \equiv (f', \Sigma')$, such a ρ is called an automorphism of (f, Σ). We will denote by Aut_f the group of automorphisms of (f, Σ).

We let $\mathbf{B}^X_{A,g,k}$ be the space of equivalence classes $[f, \Sigma]$ of C^2-stable maps (f, Σ) such that the arithmetic genus of Σ is g and $f_*([\Sigma]) = A \in H_2(X; \mathbb{Z})$. With X, A, g and k understood, we will simply write $\mathbf{B}^X_{A,g,k}$ as $\mathbf{B}$. There is a natural topology on $\mathbf{B}$, which we will define at Section 3.7. However, $\mathbf{B}$ is not a smooth Banach manifold as one hopes. It does admit a stratification with smooth strata, which we now describe.

Given any almost complex structure J compatible with ω, one can define a generalized bundle $\mathbf{E}$ over $\mathbf{B}$ as follows. Let (f, Σ) be any stable map and let $\tilde{f}: \tilde{\Sigma} \to X$ be the composite of f with $\pi: \tilde{\Sigma} \to \Sigma$. We define $\Lambda^{0,1}_f$ to be the space of all C^1-smooth sections of $(0,1)$-forms of $\tilde{\Sigma}$ with values in $\tilde{f}^*TX$. Here, by a $\tilde{f}^*TX$-valued $(0,1)$-form we mean a section ν of $T\tilde{\Sigma} \otimes \tilde{f}^*TX$ over $\tilde{\Sigma}$ such that $J \cdot \nu = -\nu \cdot \mathbf{j}$, where $\mathbf{j}$ denotes the complex structure of $\tilde{\Sigma}$. Assume that (f, Σ) and (f', Σ') are two equivalent stable maps with the associated isomorphism $\rho: \Sigma \to \Sigma'$, then there is a canonical isomorphism $\Lambda^{0,1}_{f'} \cong \Lambda^{0,1}_f$. It follows that the automorphism group Aut_f acts on $\Lambda^{0,1}_f$ and the quotient $\Lambda^{0,1}_f / \mathrm{Aut}_f$ is independent of the choice of the representative in the equivalence class of (f, Σ). So we can define $\Lambda^{0,1}_{[f]}$ to be $\Lambda^{0,1}_f / \mathrm{Aut}_f$. Put

$$\mathbf{E} = \bigcup_{[f] \in \mathbf{B}} \Lambda^{0,1}_{[f]}.$$

It has an obvious projection $\mathbf{P}: \mathbf{E} \to \mathbf{B}$ whose fibers are finite quotients of infinite dimensional linear spaces.

There is a natural map

$$\Phi_J : \mathbf{B} \longrightarrow \mathbf{E} \quad \text{satisfying} \quad \mathbf{P} \circ \Phi_J = \mathrm{Id}_{\mathbf{B}},$$

defined as follows: For any stable map f, we define $\Phi_J(f)$ to be the image of $d\tilde{f} + J \cdot d\tilde{f} \cdot \mathbf{j} \in \Lambda^{0,1}_f$ in $\Lambda^{0,1}_{[f]}$. Obviously, we have $\Phi_J(f) = \Phi_J(f')$ if f and f' are equivalent. Thus Φ_J descends to a map $\mathbf{B} \to \mathbf{E}$, which we still denote by Φ_J.

We will denote by $\mathfrak{M}^X_{A,g,k}$ the moduli space of J-holomorphic stable maps with homology class A, having k-marked points and whose domains have arithmetic genus g. Clearly, $\mathfrak{M}^X_{A,g,k} = \Phi_J^{-1}(0)$.

3.6. Stratifying the space of stable maps. In this section, we give a natural stratification of $\mathbf{B} = \mathbf{B}^X_{A,g,n}$ and study its basic properties.

Given a stable map (f, Σ), we can associate to it a dual graph Γ_f as follows: Each irreducible component Σ_α of Σ corresponds to a vertex v_α in Γ_f together with a marking (g_α, a_α), where g_α is the geometric genus of Σ_α and a_α is the homology class $f_\alpha([\Sigma_\alpha])$ in X; For each marked point x_i of Σ_α we attach a leg to v_α; For each intersection point of distinct components Σ_α and Σ_β we attach an edge joining v_α and v_β and for each self-intersection point of Σ_α we attach a loop to the vertex v_α. In the following, we will denote by $\mathrm{Ver}(\Gamma)$ the set of all vertices of Γ and $\mathrm{Ed}(\Gamma)$ the set of all edges of Γ.

Clearly, the dual graph Γ_f is independent of the representatives in $[f]$, so we can simply denote the dual graph of $[f]$ by $\Gamma_{[f]} = \Gamma_f$. Moreover, the genus g of $[f]$ is the same as the genus of $\Gamma_{[f]}$, which is defined to be the sum of g_α for all $\alpha \in \mathrm{Ver}(\Gamma)$ and the number of holes in the graph $\Gamma_{[f]}$. Also, the homology class a of $f_*([\Sigma])$ is the sum of all a_α, which is defined to be the homology class of the graph $\Gamma_{[f]}$.

Given any graph Γ with genus g and homology class a, we let $\mathbf{B}(\Gamma)$ be the space of all equivalence classes $[f]$ of stable maps in $\mathbf{B}$ with $\Gamma_{[f]} = \Gamma$. Clearly, $\mathbf{B}$ is a disjoint union of all $\mathbf{B}(\Gamma)$.

The main result of this section is the following.

PROPOSITION 3.17. *For each dual graph* Γ *the space* $\mathbf{B}(\Gamma)$ *is a smooth Banach orbifold and the restriction of* $\mathbf{E}$ *to* $\mathbf{B}(\Gamma)$ *is a smooth orbifold bundle. Further the restriction of* Φ_J *to* $\mathbf{B}(\Gamma)$ *is a smooth orbifold section.*

PROOF. We first prove that $\mathbf{B}(\Gamma)$ is a Banach orbifold.

It suffices to construct the neighborhoods of $[f]$ in $\mathbf{B}(\Gamma)$. Let $f = (f, \Sigma, (x_i))$ and let $\tilde{\Sigma}$ be the normalization of Σ. Then

$$\tilde{\Sigma} = \bigcup \left\{ \tilde{\Sigma}_\alpha \mid \alpha \in \mathrm{Ver}(\Gamma) \right\}, \tag{3.5}$$

where $\tilde{\Sigma}_\alpha$ is the normalization of the component Σ_α corresponding to α. Recall that the distinguished points of $\tilde{\Sigma}_\alpha$ are preimages of marked points and the nodal points of Σ. Note that each edge $e \in Ed(\Gamma)$ corresponds to a node in Σ, thus has two distinguished points in $\tilde{\Sigma}$. We denote them by z_{e1} and z_{e2}.

To avoid any confusion, we denote by $\tilde{\Sigma}^{\mathrm{top}}$ the underlining real 2-dimensional manifold of $\tilde{\Sigma}$. A complex structure on $\tilde{\Sigma}^{\mathrm{top}}$ is given by an almost complex structure which is a homomorphism $\mathbf{j} : T\tilde{\Sigma}^{\mathrm{top}} \mapsto T\tilde{\Sigma}^{\mathrm{top}}$ with $\mathbf{j}^2 = -\mathrm{Id}$. Two almost complex structures $\mathbf{j}$ and $\mathbf{j}'$ give rise to the same complex structure if and only if there is a diffeomorphism ϕ of $\tilde{\Sigma}^{\mathrm{top}}$ such that $\mathbf{j}' = d\phi \cdot \mathbf{j} \cdot (d\phi)^{-1}$. Let $\mathbf{j}_t$ be a family of almost complex structures such that $\mathbf{j}_0$ is the given complex structure of $\tilde{\Sigma}$. Then the derivative $\mathbf{v}(\mathbf{j}_0) := (d/dt\, \mathbf{j}_t)_{t=0}$ is a $T\tilde{\Sigma}$-valued (0,1)-form. If $\mathbf{j}'_t$ is another family of almost complex structures such that each $\mathbf{j}'_t$ induces the same complex structure as $\mathbf{j}_t$ does, then the corresponding (0,1)-form $\mathbf{v}(\mathbf{j}'_0)$ can be written as $\mathbf{v}(\mathbf{j}_0) + \bar{\partial} u$ for some section u of $T\tilde{\Sigma}$. This shows that local complex deformations of $\tilde{\Sigma}$ are parameterized by an open subset of $H^1_{\tilde{\Sigma}}(T\tilde{\Sigma})$. Thus a local universal family $\tilde{\mathcal{U}}$

of marked curve $(\tilde{\Sigma},(\tilde{x}_i),(z_{e1}.z_{e2}))$, where $\tilde{x}_i \in \tilde{\Sigma}$ is the preimage of $x_i \in \Sigma$ and $e \in \mathrm{Ed}(\Gamma)$, is of the form

$$Q \times \prod_i P_i \times \prod_e P_{e1} \times P_{e2}, \tag{3.6}$$

where Q, P_i, P_{e1} and P_{e2} are small neighborhoods of 0 in $H^1_{\tilde{\Sigma}}(T\tilde{\Sigma})$ and small neighborhoods of $\tilde{x}_i$, z_{e1} and z_{e2} in $\tilde{\Sigma}$, respectively. Here as usual $(\tilde{x}_i)$ is the k-marked points and (z_{e1}, z_{e2}) denotes $2 \cdot \#\mathrm{Ed}(\Gamma)$-marked points, after fixing an ordering of $\mathrm{Ed}(\Gamma)$.

By the Serre duality, we have $H^1_{\tilde{\Sigma}}(T\tilde{\Sigma}) = H^0_{\tilde{\Sigma}}(T^*\tilde{\Sigma}^{\otimes 2})$. The latter is the space of holomorphic quadratic differential forms.

Let $\mathcal{V}$ be a small neighborhood of 0 in the space of $\tilde{f}^*TX$-valued vector fields along $\tilde{\Sigma}$ in the C^2-topology. Here $\tilde{f}: \tilde{\Sigma} \to X$. For each $v \in \mathcal{V}$, we can associate to it a unique C^2-map $\exp_{\tilde{f}}(v)$ from $\tilde{\Sigma}$ into X, where exp is the exponential map of a fixed metric on X.

We denote by G_f the automorphism group of f. It is a finite group and acts naturally on $H^1_{\tilde{\Sigma}}(T\tilde{\Sigma})$. This is because each $\sigma \in G_f$ lifts naturally to an automorphism of $\tilde{\Sigma}$, so it induces an action on $H^1_{\tilde{\Sigma}}(T\tilde{\Sigma})$. Without loss of generality, we can assume that both Q and $\prod_i P_i \times \prod_e P_{e1} \times P_{e2}$ are G_f-invariant. Now we define a natural G_f-action on $\tilde{\mathcal{U}} \times \mathcal{V}$: Any point in $\tilde{\mathcal{U}} \times \mathcal{V}$ is of the form

$$(q,(x_i'),(z_{e1}',z_{e2}'),v), \ q \in Q, x_i' \in P_i, (z_{e1}', z_{e2}') \in P_{e1} \times P_{e2} \text{ and } v \in \mathcal{V}.$$

Given any $\sigma \in G_f$, we define

$$\sigma(q,(x_i'),(z_{e1}',z_{e2}'),v) = (\sigma_*(q),(\sigma(x_i')),(\sigma(z_{e1}'),\sigma(z_{e2}')),v \cdot \sigma^{-1}).$$

Next we construct a product manifold X_Γ as follows: for each edge or loop e of Γ, we associate to it two copies of X, say X_{e1}, X_{e2}. Define $X_\Gamma = \prod_e X_{e1} \times X_{e2}$. We also define $\Delta_\Gamma = \prod_e \Delta_e$, where Δ_e is the diagonal of $X_{e1} \times X_{e2}$. We can define a smooth evaluation map $\pi: \tilde{\mathcal{U}} \times \mathcal{V} \to X_\Gamma$ by

$$\pi(q,(x_i'),(z_{e1}',z_{e2}'),v) = \prod_e \big(\exp_{\tilde{f}}(v)(z_{e1}), \exp_{\tilde{f}}(v)(z_{e2})\big).$$

One can easily show that π is transversal to Δ_Γ. So $\tilde{\mathbf{W}} = \pi^{-1}(\Delta_\Gamma)$ is smooth. The finite group G_f acts on $\tilde{\mathbf{W}}$ smoothly, thus $\tilde{\mathbf{W}}/G_f$ is a neighborhood of f in $\mathbf{B}(\Gamma)$. This proves the first part of the Proposition.

We now prove the remainder part of the Proposition. Note that each point in $\tilde{\mathbf{W}}$ is represented by a stable map $(f,\Sigma,(x_i))$ and $(\tilde{\Sigma},(x_i),(z_{e1},z_{e2})) \in \tilde{\mathcal{U}}$. Let $\tilde{\mathbf{E}}_f$ be the set of all C^1-smooth, $\tilde{f}^*TX$-valued (0,1)-forms over $\tilde{\Sigma}$. Put

$$\tilde{\mathbf{E}}_{\tilde{\mathbf{W}}} = \bigcup_{f \in \tilde{\mathbf{W}}} \tilde{\mathbf{E}}_f.$$

Clearly, $\tilde{\mathbf{E}}_{\tilde{\mathbf{W}}}$ is a smooth Banach bundle, and G_f, which acts on $\tilde{\mathbf{W}}$, lifts to a linear action on $\tilde{\mathbf{E}}_{\tilde{\mathbf{W}}}$ such that the natural projection $\tilde{\mathbf{E}}_{\tilde{\mathbf{W}}} \to \tilde{\mathbf{W}}$ is G_f-equivariant. Moreover, $\mathbf{E}|_{\mathbf{W}} = \tilde{\mathbf{E}}_{\tilde{\mathbf{W}}}/G_f$, so $\mathbf{E}$ is an orbifold bundle over $\mathbf{B}$.

Finally, the section Φ_J lifts to a section $\tilde{\Phi}_{\mathbf{W}}$ over $\tilde{\mathbf{W}}$ defined by

$$\tilde{\Phi}_{\mathbf{W}}(f) \;=\; d\tilde{f} + J \cdot d\tilde{f} \cdot \mathbf{j}_\Sigma \in \tilde{\mathbf{E}}_f.$$

Obviously, it is smooth. This proves the Proposition. □

3.7. Topology of the space of stable maps. Now it is time to describe the topology of $\mathbf{B}$. Let $[f]$ be any stable map in $\mathbf{B}$ represented by a stable map $(f, \Sigma, (x_i))$. We want to construct the neighborhoods of $[f]$ in $\mathbf{B}$. Let Γ be the dual graph of f. For each $\alpha \in \mathrm{Ver}(\Gamma)$, we define k_α to be the number of edges and legs attached to it. Here we count a loop to α as two edges. As before we still denote by Σ_α the corresponding component and $\tilde{\Sigma}_\alpha$ its normalization. We define $\mathrm{Ver}_u(\Gamma) \subset \mathrm{Ver}(\Gamma)$ to be $\{\alpha \in \mathrm{Ver}(\Gamma) \mid k_\alpha < 3 \text{ and } g_\alpha = 0\}$ and define $\mathrm{Ver}_s(\Gamma)$ the complement of $\mathrm{Ver}_u(\Gamma)$. When $\alpha \in \mathrm{Ver}_u(\Gamma)$, then $\tilde{\Sigma}_\alpha$ contains one or two distinguished points. We add two or one marked point(s) to Σ_α according to whether $\tilde{\Sigma}_\alpha$ contains one or two distinguished point. We also require that at these added points the curve Σ is smooth and the differential df is injective. Note that this is always possible since $f_*([\Sigma_\alpha]) \neq 0$, by the stability of f. We denote by $(y_j)_{1 \leq j \leq l}$ the set of all added points to Σ. Then $(\Sigma, (x_i, y_j))$ is a Deligne-Mumford stable curve with $k + l$ marked points.

As a stable curve with $k+l$ marked points, local deformations of $(\Sigma, (x_i, y_j))$ are parameterized by admissible quadratic differentials on Σ. An admissible quadratic differential q is a meromorphic quadratic differential with at most simple poles at x_i or y_j and double poles at nodes satisfying: if w_1, w_2 are local coordinates of Σ near a node, i.e., Σ is defined by $w_1 w_2 = 0$ in $\mathbb{C}^2$ near such a node, then

$$\lim_{w_1 \to 0, w_2 = 0} w_1^2 \frac{q}{dw_1^2} = \lim_{w_2 \to 0, w_1 = 0} w_2^2 \frac{q}{dw_2^2}\,.$$

Neighborhoods of $(\Sigma, (x_i, y_j))$ in $\overline{\mathfrak{M}}_{g,k+l}$ can be constructed as follows: let $\tilde{G}_f$ be the automorphism group of $(\Sigma, (x_i, y_j))$, then a neighborhood $\mathcal{U}$ of $(\Sigma, (x_i, y_j))$ in $\overline{\mathfrak{M}}_{g,k+l}$ is of the form $\tilde{\mathcal{U}}/\tilde{G}_f$, where $\tilde{\mathcal{U}}$ is a small neighborhood of the origin in the space of admissible quadratic differentials.

For each y_i, we choose a codimension two submanifold $H_j \subset X$ such that H_j intersects $f(\Sigma)$ uniquely and transversely at $f(y_j)$. We orient H_j so that it has positive intersection with $f(\Sigma)$.

We fix a compact set $K \subset \Sigma \backslash Sing(\Sigma)$ containing all marked points (x_i, y_j). We may assume that K is G_f-invariant. Let $\mathcal{CU}$ be the universal curve over $\tilde{\mathcal{U}}$. We then choose a diffeomorphism ϕ from a neighborhood of $K \times \tilde{\mathcal{U}}$ into $\mathcal{CU}$ such that F preserves fibers over $\tilde{\mathcal{U}}$ and restricts to the identity map on $K \times \{(\Sigma, (x_i, y_j))\}$. We also fix a $\delta > 0$.

To each collection of $\mathcal{U}$, H_j, K, δ and ϕ given as above, we can associate to it a neighborhood $\mathbf{U} = \mathbf{U}(\mathcal{U}, H_j, K, \delta, \phi)$ as follows: define $\tilde{\mathbf{U}} = \tilde{\mathbf{U}}(\mathcal{U}, H_j, K, \delta, \phi)$ to be the set of all tuples $(f', \Sigma', (x_i', y_j'))$ satisfying: (1) $(\Sigma', (x_i', y_j'))$ is in $\tilde{\mathcal{U}}$; (2) f' is a continuous map from Σ' into X with $f'(y_j) \in H_j$; (3) f' lifts to a C^2-map $\tilde{\Sigma}' \to X$; (4) $||f' \cdot \phi - f||_{C^2(K)} < \delta$; (5) $d_X(f(\Sigma), f'(\Sigma')) < \delta$, where d_X denotes the distance function of a fixed Riemannian metric on V. Note that the topology of Σ' may be different from that of Σ. We define $\tilde{\mathbf{U}}(\mathcal{U}, H_j, K, \delta, \phi)$ to be a local uniformization chart of $\mathbf{B}$.

Given any tuple $(f', \Sigma', (x_i', y_j'))$ in $\tilde{\mathbf{U}}$, we call $\mathbf{f}' = (f', \Sigma', (x_i'))$ its descendant. One can show that $\mathbf{f}'$ is a stable map and gives rise to a point $[\mathbf{f}']$ in $\mathbf{B}$. Let $\mathbf{U}$ be the set of all equivalence classes of stable maps descended from tuples in $\tilde{\mathbf{U}}$. Then $\mathbf{U}$ is a neighborhood of $[f]$.

The topology of $\mathbf{B}$ is generated by all such neighborhoods $\mathbf{U}(\mathcal{U}, H_j, K, \delta, \phi)$.

Now assume that δ and $\tilde{\mathcal{U}}$ are sufficiently small. Then there is a natural action of G_f on $\tilde{\mathbf{U}}$: let $\sigma \in G_f$ and $\tilde{\mathbf{f}}' = (f', \Sigma', (x'_i, y'_j)) \in \tilde{\mathbf{U}}$, then σ acts on $\tilde{\mathcal{U}}$ by sending Σ' to $\sigma(\Sigma')$ and x_i to $\sigma(x_i)$. We define

$$\sigma(\tilde{\mathbf{f}}') = (f' \cdot \sigma^{-1}, \sigma(\Sigma'), (\sigma(x'_i), y''_j)),$$

where for each j, y''_j is the unique point in $\sigma(\Sigma')$ near y_j such that $f'(\sigma^{-1}(y''_j)) \in H_j$. The existence and uniqueness of such y''_j is assured by the assumption that H_j are transversal to $f(\Sigma)$. Note that y''_j may not be y'_j. Clearly, $\tilde{\mathbf{f}}'$ and $\sigma(\tilde{\mathbf{f}}')$ descend to the identical $[\mathbf{f}']$ in $\mathbf{B}$. Conversely, if $\tilde{\mathbf{f}}' = (f', \Sigma', (x'_i, y'_j))$ and $\tilde{\mathbf{f}}'' = (f'', \Sigma'', (x''_i, y''_j))$ descend to the same $[\mathbf{f}']$ in $\mathbf{U}$, then there is a biholomorphic map $\tau : \Sigma' \mapsto \Sigma''$ such that $f' = f'' \circ \tau$ and $\tau(x'_i) = x''_i$. When δ and $\tilde{\mathcal{U}}$ are sufficiently small, we may assume that τ induces a biholomorphic map, denoted by σ, of $(\Sigma, (x_i))$. This σ acts on $\tilde{\mathbf{U}}$ as defined above. Thus one can easily see that $\sigma(\mathbf{f}') = \mathbf{f}''$. It follows that $\mathbf{U}$ is of the form $\tilde{\mathbf{U}}/G_f$. Therefore, the neighborhoods $\mathbf{U}(\mathcal{U}, H_j, K, \delta, \phi)$ are the quotients of local uniformization charts $\tilde{\mathbf{U}}(\mathcal{U}, H_j, K, \delta, \phi)$.
This proves

THEOREM 3.18. *Equiped with the topology described above,* $\mathbf{B}$ *is a SS-orbispace.*

Moreover, we have

THEOREM 3.19. *Assume that* (X, ω) *is a compact manifold with a compatible almost complex structure* J. *Then the moduli space* $\mathfrak{M}^X_{A,g,k}$ *is compact in* $\mathbf{B}$ *in the above topology.*

The compactness theorem of this sort first appeared in the work of Gromov [Gr], further studied and extended by Parker and Wolfson, Pansu and Ye. A proof was also given in [RT1] for holomorphic maps of any genus, following Sacks and Uhlenbeck on harmonic maps in early 80's.

Given a local uniformization chart $\tilde{\mathbf{U}}$, we define

$$\mathbf{E}_{\tilde{\mathbf{U}}} = \bigcup_{(f', \Sigma', (x'_i, y'_j)) \in \tilde{\mathbf{U}}} \tilde{\mathbf{E}}_{(f', \Sigma', (x'_i, y'_j))},$$

where $\tilde{\mathbf{E}}_{(f', \Sigma', (x'_i, y'_j))}$ consists of all C^1-smooth, f'^*TX-valued (0,1)-forms over the normalization of Σ'. All such $\mathbf{E}_{\tilde{\mathbf{U}}}$'s form charts of $\mathbf{E}$. One can show that the conditions in Definition 3.7 are all satisfied. Therefore, we have proved

THEOREM 3.20. $(\mathbf{B}, \mathbf{E}, \Phi_J)$ *is a* V*-vector bundle.*

4. Proof of the main theorem

By Theorem 3.13, in order to prove our main theorem, we need to show that $(\mathbf{B}, \mathbf{E}, \Phi_J)$ is a WFV-bundle. By the results in the last section, we suffice to construct local finite approximations of $(\mathbf{B}, \mathbf{E}, \Phi_J)$.

We continue to use the notations developed so far. Let $\tilde{\mathbf{U}}$ be a chart of $\mathbf{B}$ with the corresponding group G, and let $\mathbf{E}_{\tilde{\mathbf{U}}}$ be the corresponding chart of $\mathbf{E}$ over $\tilde{\mathbf{U}}$. We know that $\tilde{\mathbf{U}}$ is of the form $\tilde{\mathbf{U}}(\mathcal{U}, H_j, K, \delta, \phi)$. Let $\tilde{\mathcal{U}}$ be the local uniformization of $\mathcal{U}$ and $\mathcal{CU}$ be the universal curve over $\tilde{\mathcal{U}}$. We may assume that $\mathcal{U}$ is sufficiently small.

Recall that a TX-valued (0,1)-form on $\mathcal{CU} \times X$ is an endomorphism $v : T\mathcal{CU} \mapsto TX$ such that $J \cdot v = -v \cdot \mathbf{j}_{\mathcal{CU}}$, where $\mathbf{j}_{\mathcal{CU}}$ is the complex structure on $\mathcal{CU}$. Let $\Lambda^{0,1}(\mathcal{CU}, TX)_0$ be the space of all C^∞-smooth TX-valued (0,1)-forms on $\mathcal{CU} \times X$ which vanish near $\mathrm{Sing}(\mathcal{CU})$. Here $\mathrm{Sing}(\mathcal{CU})$ denotes the set of the singularities in the fibers of $\mathcal{CU}$ over $\tilde{\mathcal{U}}$.

Given each $v \in \Lambda^{0,1}(\mathcal{CU}, TX)_0$, we can associate a section of $\mathbf{E}_{\tilde{\mathbf{U}}}$ as follows: For each $\mathbf{f} = (f, \Sigma, (x_i, y_j)) \in \tilde{\mathbf{U}}$, we define $v|_{\mathbf{f}}$ by

$$v|_{\mathbf{f}}(x) = v(x, f(x)), \ x \in \Sigma.$$

Clearly, $v|_{\mathbf{f}}$ is a section on the fiber of $\mathbf{E}_{\tilde{\mathbf{U}}}$ over $\mathbf{f}$. This way we obtain a section $\mathbf{f} \mapsto v|_{\mathbf{f}}$ over $\tilde{\mathbf{U}}$. To avoid introducing new notations, we still denote this section by v. Now assume G_f is non-trivial. Then for $\sigma \in G_f$ the pull-back $\sigma^*(v_i)$ is a section over $\tilde{\mathbf{U}}$. Without loss of generality, we can assume the $l \cdot |G_f|$ sections

$$\{\sigma^*(v_i) \mid 1 \le i \le l, \sigma \in G_f\}$$

of $\mathbf{E}_{\tilde{\mathbf{U}}}$ is linearly independent everywhere. We define $\mathbf{F} = \mathbf{F}(v_1, \cdots, v_l)$ to be the subbundle in $\mathbf{E}_{\tilde{\mathbf{U}}}$ generated by the above $l \cdot |G_f|$ sections. $\mathbf{F}$ is a trivial vector bundle and is a G_f-equivariant subbundle of $\mathbf{E}_{\tilde{\mathbf{U}}}$.

LEMMA 4.1. *Suppose $\mathbf{f} = (f, \Sigma, (x_i, y_j)) \in \tilde{\mathbf{U}}$ and $\mathrm{Sp}(v_1, \cdots, v_l)|_{\mathbf{f}}$ are transverse to L_f, where L_f is the linearization of the Cauchy-Riemann equation at f. We further assume that in the definition of $\tilde{\mathbf{U}}$, δ is sufficiently small and K is sufficiently big. Then $\Phi_J^{-1}(\mathbf{F})$ is a smooth manifold of dimension $r + l$, where r is the index of L_f which can be computed in terms of $c_1(X)$, the homology class of $f(\Sigma)$, the genus of Σ and the number of marked points.*

Using this lemma, one can easily show that $(\tilde{\mathbf{U}}, G_f, \mathbf{E}_{\tilde{\mathbf{U}}}, \mathbf{F})$ is a local finite approximation. Furthermore, one can assign it a natural orientation by the canonical orientation on the determinant line $\det(L_f)$ of L_f. Note that L_f is different from a J-invariant $\bar\partial$-operator $\bar\partial_f$ by an zero-order operator, so $\det(L_f) = \det(\bar\partial_f)$, hence, the canonical orientation is given by the canonical one on $\det(\bar\partial_f)$. Lastly, it is easy to see that strata of the finite model U of $(\tilde{\mathbf{U}}, \mathbf{F})$ mapped submersively to strata of $\mathfrak{M}_{g,k} \times^k$, if we choose enough sections $v_1, \cdots, v_l$. This completes the proof of the main theorem.

5. Final remark

We now make a few remark to tie up the lose end of our presentation.

The first is the equivalence of the algebraic and the symplectic definition of GW invariants. This was generaly believed to be true and was proved in [**LT4, Si2**].

THEOREM 5.1. *Let X be any smooth projective variety with a Kähler form ω. Then the algebraically constructed GW-invariants of X coincide with the analytically constructed GW-invariants of the symplectic manifold $(X^{\mathrm{top}}, \omega)$.*

The proof relied on the fact that we can find holomorphic finite approximation $\phi : U \to F$, in the symplectic construction of GW invariants, so that the obstruction theory induced by ϕ is identical to the canonical obstruction of $\mathfrak{M}_{g,k}(X, A)$. We refer the details to our paper above.

The second is some basic properties of the GW invariants. One property of virtual moduli cycle we stressed earlier is that if X_T is a smooth family of peojective varieties with smooth base and if A_t extends to a global constant family in $\cup_t A_* X_t$, it exists after a base change of T if necessary, then the GW invariant $\gamma^{X_t}_{A_t,g,k}$ should be a local constant. This is indeed the case. It is proved by showing that the virtual cycle $[\mathfrak{M}_{g,k}(X_t, A_t)]^{\mathrm{vir}}$ is the image of the Gysin map $\eta_t^*[\mathfrak{M}_{g,k}(X_T, A_T)]^{\mathrm{vir}}$, where η is defined by the square

$$\begin{array}{ccc} \mathfrak{M}_{g,k}(X_t, A_t) & \xrightarrow{\eta} & \mathfrak{M}_{g,k}(X_T, A_t) \\ \downarrow & & \downarrow \\ \{t\} & \longrightarrow & T \end{array}$$

The second is that the GW invariants are expected to satisfy several important relations, among them the associativity (or composition) law. This again can be proved by studying some geometric (co)cycle Δ in the moduli of curves $\mathfrak{M}_{g,k}$ and the virtual cycle of

$$\mathfrak{M}_{g,k}(X, A) \times_{\mathfrak{M}_{g,k}} \Delta.$$

We should also mention that the alternative construction of Behrend-Fantechi [**BF**] used the notion of cotangent complex to represent the obstruction theory of the moduli stack $\mathfrak{M}_{g,k}$, and then constructed a cone, called virtual normal cone, as an Artin stack over the D-M stack $\mathfrak{M}_{g,k}(X, A)$. In the end, they use a vector bundle E over $\mathfrak{M}_{g,k}(X, A)$, similar to the one mentioned earlier, to obtain a cone in E and hence obtain the virtual cycle using the Gysin map of the zero section of E. Recently this last step of using vector bundle E was relaxed by the work of Kresch [**Kr**] after he developed the intersection theory on Artin stack. It is interesting to note our symplectic construction of GW invariants *did not* use the existence of such global vector bundle, thus it can serve as a hint that the global bundle E is not essential to the construction of virtual cycle. We note that the symplectic constructions by Siebert [**Si1**] and Ruan [**Ru2**] both relied on such global vector bundles.

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