Quaternion-Kähler Geometry

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Introduction

Interest in quaternion-Kähler manifolds and metrics has developed during the past decades from at least four separate, originally unrelated, sources: (i) the classification of holonomy groups, (ii) the theory of quaternionic manifolds, (iii) self-duality in 4-dimensions, (iv) σ-models in theoretical physics.

An understanding of (i) leads to the holonomy definition of a quaternion-Kähler manifold that dates back to the 1960's. It needs clarification when the dimension is 4, for which curvature conditions enter explicitly into the definition. This aspect of the theory regards (iii), and involves the decomposition of the Weyl tensor, the significance of which was not fully understood (at least in the Riemannian context) until the end of the 1970's. The net result is that quaternion-Kähler manifolds are always Einstein, though their nature depends very much on the sign (positive, negative or zero) of the scalar curvature s.

One aspect of quaternion-Kähler geometry that is inherently higher-dimensional arises from the representation theory of the simplest non-abelian group $SU(2)$. This is most evident in the description, first given by Wolf, of a class of symmetric spaces that include the only complete examples known for $s > 0$. Despite this limitation in the global theory, there is a surprisingly rich geometry underlying the definitions, and the title and contents of this chapter are meant to reflect that. Some of the underlying constructions that we present are valid for the broader theory of quaternionic manifolds, described in §4, that was developed independently by Bérard Bergery and the author, building on earlier work of many others [**17, 105**].

For the uninitiated, the most important point to be made is that a quaternion-Kähler manifold M is not necessarily Kähler in the usual complex sense. Indeed, locally M has a compatible Kähler metric if and only if $s = 0$, in which case it is hyper-Kähler, and amenable to techniques described elsewhere in this volume. On the other hand, the quaternionic structure of M can always be 'untwisted' by passing to the total space of a suitable fibre bundle, and a quaternion-Kähler manifold does always possess associated higher-dimensional complex manifolds, which are Kähler if $s > 0$. Of particular importance in this case is the twistor space, a contact Fano manifold, whose algebraic geometry provides the best hope for tackling the classification problem. Results of LeBrun and others in this direction are presented in §5.

Curvature conditions follow automatically from torsion ones in dimensions greater than 4, and it is sometimes possible to conclude that a given manifold M is quaternion-Kähler without identifying the Einstein metric explicitly. Each point of M corresponds to a rational curve in its twistor space, and one approach is to reconstruct M from the identification of such rational curves and their deformations. We shall illustrate this process in §6. Many techniques for the construction of quaternion-Kähler metrics also have their origins in the 4-dimensional theory. Important constructions of quaternion-Kähler (equivalently, self-dual Einstein) 4-manifolds that we do not discuss can be found in the works of Hitchin [**57**], Pedersen [**93**] and Tod [**115**].

The fundamental role played by isometry groups in the theory of quaternion-Kähler manifolds pervades the whole chapter, though §6 and §7 are specifically devoted to properties of group actions. This includes Swann's generalization of Wolf spaces and their relationship to complex nilpotent orbits and work of Kronheimer, and the quotient construction of Galicki-Lawson. Morse theory turns up in an essential way in these topics, and leads to a number of open problems. An important tool is that of a moment mapping and, whilst this can be interpreted entirely within the realm of symplectic geometry, it comes in other flavours that are peculiar to the quaternionic setting. The quotient construction produces an abundance of local quaternion-Kähler metrics, though the appearance of orbifolds is inherent in the theory. Four-dimensional quaternion-Kähler metrics themselves give rise to Ricci-flat metrics with holonomy equal to G_2 or $Spin(7)$ [**29**], and are therefore especially relevant to the general search for Einstein metrics.

It would be true to say that this chapter represents only a selection of topics in what is a very active field. A final section is devoted to the topology associated to quaternion-Kähler structures, and reflects the author's own interest. The main applications are to the case of positive scalar curvature and link in with §5. The general philosophy is to try to duplicate results known to hold for the Wolf spaces to arbitrary quaternion-Kähler manifolds with $s > 0$. The theory in §8 also allows one to pose a number of related questions for compact quaternion-Kähler manifolds with $s \leq 0$, or indeed the more general class of quaternionic manifolds.

We conclude by mentioning some other topics that we do not pursue further. In a direction related to (iv) above, there is a large class of solvable Lie groups with quaternion-Kähler metrics with $s < 0$ that were discovered by Alekseevsky [**3**], and have recently been re-classified within the framework of supergravity [**38, 35**]. In this set-up, one considers mappings from a space-time of dimension d into a complete target manifold with a specific geometrical structure, such as Kähler, special Kähler, quaternion-Kähler. In the latter case, topological considerations place one in the realm of negative scalar curvature, though this point of view leads to constructions uniting the various geometries. The theory of special Kähler metrics (Kähler ones admitting a certain type of flat symplectic connection) has recently captured the imagination of mathematicians [**42**], and is likely to lead to further developments in the quaternionic field.

An early result of Gray to the effect that any quaternionic submanifold of a quaternion-kähler manifold is totally geodesic [**51**] puts a stop to any naïve theory of submanifolds. Methods in [**33**] effectively classify quaternionic submanifolds of symmetric spaces, and other types of submanifolds have been considered in [**83**]. There is also a vast literature concerned with the classification of various types of submanifolds of quaternionic projective space. But the most effective direction is the study of certain holomorphic submanifolds of twistor space, and there is an extensive theory of harmonic mappings of surfaces into quaternion-Kähler manifolds [**32, 67**] that generalizes the more familiar situation of mappings into 4-manifolds.

1. Almost-Complex Structures and the Canonical 4-Form

A hyper-Kähler manifold ('HK manifold' for short) can be defined as a Riemannian manifold of dimension $4n \geq 4$ admitting an anti-commuting pair I, J of almost-complex structures, relative to both of which the metric is Kähler. This implies that I, J and therefore $K = IJ$ are parallel or 'constant' relative to the Levi-Civita connection, and define integrable complex structures. The triple of endomorphisms I, J, K behaves like the imaginary quaternions, and if (a, b, c) is a unit vector in $\mathbb{R}^3$ then $aI + bJ + cK$ is also a parallel complex structure. Thus, an HK manifold is endowed with a set of complex structures parametrized by the 2-sphere.

Quaternion-kähler manifolds form a more general class of Riemannian manifolds that incorporate not just hyper-Kähler ones, but also the quaternionic projective space space $\mathbb{HP}^n$. Actually, one can define a Riemannian symmetric space that is quaternion-Kähler with an arbitrary compact simple isometry group, and $\mathbb{HP}^n$ corresponds to the case $Sp(n+1) = C_{n+1}$. On a general quaternion-Kähler manifold ('QK manifold' for short), it is not possible to find individual structures I, J, K that are parallel, but only a bundle V of endomorphisms with fibre isomorphic to the imaginary quaternions which as a whole is preserved by the Levi-Civita connection ∇. Locally, one can therefore find I_1, I_2, I_3 satisfying

$$I_1 I_2 = I_3 = -I_2 I_1, \tag{1.1}$$

and 1-forms α_i such that

$$\begin{array}{rcl} \nabla I_1 & = & \qquad\qquad - \alpha_3 \otimes I_2 + \alpha_2 \otimes I_3, \\ \nabla I_2 & = & \alpha_3 \otimes I_1 \qquad\qquad - \alpha_1 \otimes I_3, \\ \nabla I_3 & = & -\alpha_2 \otimes I_1 + \alpha_1 \otimes I_2. \end{array} \tag{1.2}$$

These equations were considered by Ishihara [**59**].

Identify $\mathbb{R}^{4n}$ with the space $\mathbb{H}^n$ of quaternion column vectors, so that the Euclidean inner product is given by $\langle v, w \rangle = \mathrm{Re}(v^* w)$, where * is the operation of transposing and quaternionically conjugating entries. The group $Sp(n)$ of quaternion $n \times n$ matrices for which $A^* A$ equals the identity then acts isometrically on $\mathbb{R}^{4n}$ by left multiplication. An HK manifold is then the same as a Riemannian manifold whose holonomy reduces to this group. The parallel complex structures I, J, K arise from the right action of the corresponding unit quaternions.

The group $Sp(n)$ is a subgroup of $SO(4n)$, but not a maximal one since it commutes with the action of the group $Sp(1)$ of unit quaternions on $\mathbb{R}^{4n}$ by right multiplication. The enlarged group of transformations

$$v \longmapsto Avq^*, \qquad A \in Sp(n),\ q \in Sp(1) \tag{1.3}$$

is denoted by $Sp(n)Sp(1)$. It is a subgroup of $SO(4n)$ isomorphic to the quotient $Sp(n) \times_{\mathbb{Z}_2} Sp(1)$, where $\mathbb{Z}_2$ is generated by $(-I, -1)$.

DEFINITION 1.1. A QK manifold is a Riemannian manifold of dimension $4n$ whose holonomy group is contained in the group $Sp(n)Sp(1)$.

Since $Sp(1)Sp(1) = SO(4)$, the geometry resulting from this definition generalizes that of oriented Riemannian 4-manifolds. For the moment though, we shall assume that $n \geq 2$.

It is an immediate consequence of the above definition that the frame bundle of a QK manifold reduces to a principal bundle with structure group $Sp(n)Sp(1)$. The bundle V defined above is none other than that associated to the homomorphism

$$Sp(n)Sp(1) \to Sp(1)/\mathbb{Z}_2 \cong SO(3),$$

and a local basis $\{I_1, I_2, I_3\}$ of V satisfying (1.1) is determined up to the action of $SO(3)$ at each point. Each almost-complex structure I_i determines a 2-form ω_i by the usual identity $\omega(X,Y) = g(I_i X, Y)$, and the ω_i are analogues of self-dual 2-forms in 4 dimensions. It is easy to see that the 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 \tag{1.4}$$

is independent of the choice of basis, and Ω^n is nowhere zero. This form was introduced by Bonan in [**20**]. An orientation of M can be defined by decreeing that the volume form is a constant positive multiple of Ω^n.

Replacing the I_i's in (1.2) by ω_i's, we see that Ω is parallel, and therefore closed. Since the stabilizer of Ω in $GL(4n,\mathbb{R})$ is exactly $Sp(n)Sp(1)$, the holonomy reduction of a QK manifold is characterized by the existence of a 4-form which is

(i) in the same $GL(4n,\mathbb{R})$-orbit as Ω at each point, and

(ii) parallel.

If M has dimension at least 12, it turns out surprisingly that (ii) is equivalent to requiring that the 4-form be closed [**112**]. The case of 8 dimensions is in a certain sense richer, as there exist metrics at least locally admitting closed but non-parallel 4-forms satisfying (i). This contrasts with the case of $Spin(7)$ holonomy which, as observed in [**28**], is defined by a closed 4-form linearly equivalent to

$$\tilde{\Omega} = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3.$$

On its own, condition (i) defines the class of 'almost QK manifolds'. Let $\mathfrak{h}^\perp$ denote the orthogonal complement of $\mathfrak{h} = \mathfrak{sp}(n) + \mathfrak{sp}(1)$ in $\mathfrak{so}(4n)$; this is in fact an irreducible representation of $Sp(n)Sp(1)$ (denoted $\Lambda_0^2 \otimes \Sigma^2$ below). General principles imply that, on any almost QK manifold, the covariant derivative $\nabla_X \Omega$ (for any tangent vector X) belongs to a subspace of $\bigwedge^4 T^*$ isomorphic to $\mathfrak{h}^\perp$. Numerous classes of almost quaternionic manifolds can be defined by decomposing the space $T^* \otimes \mathfrak{h}^\perp$ and imposing corresponding conditions on Ω [**106, 111**], though we shall adopt a different approach in §4. Analogues of quaternion-Kähler geometry with torsion have been studied by theoretical physicists (see e.g. [**58**]).

Remark. The fact that 4-forms arise automatically from a holonomy reduction may be deduced from the following purely algebraic observation. Let $\mathfrak{h}$ denote the Lie algebra of a compact holonomy group H, regarded as a subspace of $\bigwedge^2 T^*$. Then there is an H-equivariant mapping

$$b\colon S^2\mathfrak{h} \longrightarrow \textstyle\bigwedge^4 T^* \tag{1.5}$$

defined simply by skewing 2-forms together. The symmetric product $S^2\mathfrak{h}$ contains at least a 1-dimensional space of H-invariant elements, so let ρ be such an element. If $b(\rho) = 0$ then ρ is a curvature-like tensor satisfying the first Bianchi identity, and the fundamental work of É. Cartan implies that ρ is the curvature tensor of a Riemannian symmetric space. Otherwise, $b(\rho)$ will determine a non-zero parallel

4-form. In the case of $\mathfrak{h} = \mathfrak{sp}(n) + \mathfrak{sp}(1)$, both these possibilities occur as there is a 2-dimensional space of invariants in $S^2\mathfrak{h}$, spanned by elements ρ_1, ρ_2 with $\rho_1 \in \ker b$. The corresponding symmetric space is $\mathbb{HP}^n$ with curvature tensor ρ_1, and the 4-form $b(\rho_2)$ is proportional to Ω.

A more careful analysis of the 'Bianchi map' (1.5) for $\mathfrak{h} = \mathfrak{sp}(n) + \mathfrak{sp}(1)$ shows that, provided $n \geq 2$, its kernel is spanned only by ρ_1 and the highest-weight component W in the tensor product $\mathfrak{sp}(n) \otimes \mathfrak{sp}(n)$. The summand W contains the curvature tensor of an HK manifold, and since it has no components in common with the space S^2T^* of Ricci tensors, one deduces

COROLLARY 1.2. *Any QK manifold of dimension $4n \geq 8$ is Einstein, and its scalar curvature s vanishes if and only if it is locally HK, i.e. its restricted holonomy group H_0 is a subgroup of $Sp(n)$.*

In this case, the classification of possible holonomy groups H having connected component $H_0 = Sp(n)$ has been carried out by McInnes [**85**], and a related theory of 'locally quaternion-Kähler' manifolds is developed in [**96**]. It also makes sense to talk about QK manifolds with zero curvature.

Definition 1.1 is the traditional one, though we are now able to point out drawbacks in the terminology.

(i) As it stands, a 4-dimensional QK manifold is none other than an oriented Riemannian one. The curvature restrictions that apply from (1.5) in higher dimensions disappear because $\bigwedge^4 T^*$ is just 1-dimensional. However, we may re-impose them by redefining a quaternion-Kähler 4-manifold to be an oriented Riemannian one whose curvature tensor belongs to

$$S^2_0(\textstyle\bigwedge^2_+ T^*) \oplus \langle \rho_1 \rangle,$$

relative to (3.3), where ρ_1 spans $\ker b$. The first component is the 'positive half' W_+ of the Weyl tensor, and the second is a multiple of the constant curvature tensor of $S^4 = \mathbb{HP}^1$. This condition is equivalent to asserting that M is 'self-dual' (meaning that $W_- = 0$) and Einstein.

(ii) A quaternion-Kähler manifold is not in general Kähler, since $Sp(n)Sp(1)$ is not a subgroup of $U(2n)$. (For this reason, the author sees the abbreviation 'QK' as avoiding a certain amount of embarassment.) In fact, if H is a subgroup of

$$(Sp(n)Sp(1)) \cap U(2n) = Sp(n)U(1),$$

then H_0 is a subgroup of $Sp(n)$ [**119**] and M is locally HK. As we have remarked, the two situations are distinguished locally by the Ricci tensor.

(iii) There is a natural tendency to restrict the terminology 'quaternion-Kähler' to the case of non-zero scalar curvature, a situation significantly different from the hyper-Kähler one. This is indeed the approach we adopt in this chapter, although there are further links between the two classes of manifolds that transcend the definitions. For example, we shall see that any QK manifold with positive scalar curvature can be realized as a type of quotient of an HK manifold of 4 dimensions greater. This is disconcerting as it implies that the theory of QK manifolds can logically (at least in the $s > 0$ regime) be subsumed into the theory of HK ones!

(iv) One cannot restrict to the case of non-zero scalar curvature by demanding that the holonomy group should equal $Sp(n)Sp(1)$. For this would exclude most of the symmetric space examples, which have holonomy of the form $K\,Sp(1)$ where K is a subgroup of $Sp(n)$. We study these in the next section.

On a compact Kähler manifold, wedging with the fundamental 2-form ω induces a non-singular map on cohomology in appropriate dimensions; this is the well-known Lefschetz property that relates to formality properties of the de Rham algebra of a Kähler manifold. One of the earliest results for a compact QK manifold M of dimension $4n$ was the analogous result found by Kraines [**73**]. With a slight improvement of Fujiki [**44**], this states that wedging with the 4-form Ω determines an injection

$$H^k(M,\mathbb{R}) \hookrightarrow H^{k+4}(M,\mathbb{R}), \qquad k \le n-1.$$

Refinements were also made in [**21**]. Of course, given that Ω is a closed 4-form, with $\Omega^n \neq 0$, it is also true that $b_{4i} > 0$ for $0 \le i \le n$.

Much more can be said when the scalar curvature is positive. A complete quaternion-Kähler manifold with $s > 0$ is called a 'positive QK manifold', and because of the Einstein condition, completeness is equivalent to compactness. The author proved that a positive QK manifold has vanishing 'odd' Betti numbers $b_{2i+1} = 0$ [**102**]. Building on this, one can show that the differences

$$\beta_{2i} = b_{2i} - b_{2i-4}, \qquad i \le n, \tag{1.6}$$

are all non-negative [**44**]. They are in fact the Betti numbers of an associated 3-Sasakian manifold defined in §5, and feature again after Theorem 8.2(ii).

To conclude this section we quote a result that is relevant to the remarks in (ii) above. Its proof exploits Theorem 5.5(ii) below, and the well-known fact that there is no almost complex structure on $\mathbb{HP}^1 \cong S^4$.

THEOREM 1.3. [**5**] *No positive QK manifold admits a compatible almost complex structure.*

An almost-complex structure is said to be compatible if it is a section of V, so that it can be expressed locally as $aI_1 + bI_2 + cI_3$ with the relations (1.1) and a, b, c functions satisfying $a^2 + b^2 + c^2 = 1$.

2. Symmetric Spaces and Grassmannian Geometry

The essence of quaternion-Kähler geometry is captured by the 1964 paper of Wolf [**119**]. In it, he characterizes the quaternionic structure of a class of symmetric spaces, and discusses what is now known as their twistor fibrations. In a paper [**2**] which coined the term 'Wolf space', Alekseevsky went on to show that any homogeneous QK manifold with $s > 0$ must in fact be one of these symmetric spaces. In this section we begin by listing the spaces in question, and then explain how their existence is related to the theory of 3-dimensional subalgebras.

By a 'Wolf space' we mean a quaternion-Kähler symmetric space with $s > 0$. There are two Wolf spaces of real dimension 4, namely

$$S^4 = \frac{SO(5)}{SO(4)}, \quad \mathbb{CP}^2 = \frac{SU(3)}{S(U(2)\times U(1))}.$$

The curvature of both of these spaces is self-dual and Einstein, in accordance with the revised definition in 4 dimensions.

In general, there is a Wolf space corresponding to each simple compact Lie algebra. In dimension $4n$, there are three families

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}, \quad \mathbb{Gr}_2(\mathbb{C}^{n+2}) = \frac{U(n+2)}{U(n)\times U(2)}, \quad \mathbb{Gr}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\times SO(4)}.$$

There are coincidences $\mathbb{HP}^1 = S^4 = \mathbb{G}r_4(\mathbb{R}^5)$ and $\mathbb{G}r_2(\mathbb{C}^3) = \mathbb{CP}^2$ for $n = 1$, and $\mathbb{G}r_2(\mathbb{C}^4) = \mathbb{G}r_4(\mathbb{R}^6)$ for $n = 2$. In addition, there are the exceptional spaces

$$\frac{G_2}{SO(4)}, \qquad \frac{F_4}{Sp(3)Sp(1)},$$

$$\frac{E_6}{SU(6)Sp(1)}, \quad \frac{E_7}{Spin(12)Sp(1)}, \quad \frac{E_8}{E_7Sp(1)},$$

corresponding to $n = 2, 7, 10, 16, 28$ respectively.

Needless to say, these homogeneous spaces arise as orbits for the action of the isometry group G on a suitable linear space. Any adjoint orbit of G on its Lie algebra $\mathfrak{g}$ is a complex homogeneous space, and the only Wolf space that arises in this way is the complex Grassmannian. Indeed, if we identify $\mathfrak{su}(n+2)$ with the space $\bigwedge_0^{1,1}(\mathbb{C}^{n+2})$ of primitive $(1,1)$-forms on $\mathbb{C}^{n+2}$ then the orbit of an element $\alpha\wedge\overline{\beta}$ (with $\langle\alpha,\beta\rangle = 0$) is isomorphic to $\mathbb{G}r_2(\mathbb{C}^{n+2})$. In the same vein, $\mathbb{HP}^n$ arises as the $Sp(n+1)$-orbit of a suitable element in $\bigwedge_0^2(\mathbb{C}^{2n+2})$, and the Plücker embedding exhibits $\mathbb{G}r_4(\mathbb{R}^{n+4})$ as a $SO(n+4)$-orbit of a simple 4-form inside $\bigwedge^4(\mathbb{R}^{n+4})$.

Remark. The following topological properties of the Wolf spaces reflect general results on QK manifolds that we shall discuss below:

(i) The list consists of exactly those irreducible Riemannian symmetric spaces for which $H^2(M,\mathbb{Z}) \cong \mathbb{Z}_2$, together with $\mathbb{HP}^n$ and $\mathbb{G}r_2(\mathbb{C}^{n+2})$ (which of course have $H^2(M,\mathbb{Z})$ equal to 0 and $\mathbb{Z}$ respectively) [**31**].

(ii) The only Wolf spaces with 4th Betti number $b_4 > 1$ are the Grassmannians $\mathbb{G}r_2(\mathbb{C}^{n+2})$ $(n \geq 2)$ and $\mathbb{G}r_4(\mathbb{R}^{n+4})$ $(n \geq 2)$. The only space with $b_4 > 2$ is $\mathbb{G}r_4(\mathbb{R}^8)$, which has two distinct quaternion-Kähler structures related by an outer automorphism of $SO(8)$, and $b_4 = 3$ [**96**].

(iii) Those with isometry group of type A, D, E have

$$\dim G = 2\chi + b_{2n-2} + b_{2n}, \tag{2.1}$$

where χ denotes the Euler characteristic.

It is conjectured that any positive QK manifold is a Wolf space. The following compilation of results in this direction represents the state of play at the time this article was written.

THEOREM 2.1. *A positive QK manifold M of dimension $4n$ is necessarily symmetric if one of the following is true:*
(i) $n \leq 2$,
(ii) $n \leq 4$ and $b_4 = 1$,
(iii) the isometry group of M has rank at least $n + 1$.

Part (i) for $n = 1$ is Hitchin's theorem asserting that S^4 and $\mathbb{CP}^2$ are the only two compact 4-manifolds with a metric which is self-dual, Einstein, with positive scalar curvature [**55, 17**]. The corresponding result for $n = 2$ was proved in [**98**], and by a different method in [**80**]. A key starting point is the existence of a space of Killing vector fields of sufficiently positive dimension.

The biggest gap in a potential proof of the conjecture is the unknown answer to the conundrum of whether a QK manifold of dimension greater than 16 has any non-zero Killing vector fields. Estimates on the size of the isometry group G are known for $n = 3$ or $n = 4$, and in these cases it is proved in [**49**] that if $b_4 = 1$

then M is isometric to $\mathbb{HP}^n$, whence (ii). The author is confident that a complete analysis will soon be possible in dimension 16 and less.

A valuable bound is that the rank of G cannot exceed $n+1$ [**98**]. Part (iii) is a recent result of Bielawski that is proved by reconstructing the spaces in question as quotients using the techniques hinted at in §7. The only positive QK manifolds with an isometry group of rank $n+1$ are in fact $\mathbb{HP}^n$ and $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$.

The special nature of the isotropy groups of the Wolf spaces is emphasized by the next result. To formulate it, suppose that M is a QK manifold with isometry group G, and let $x \in M$. If $\mathfrak{g}_x$ denotes the Lie algebra of the isotropy subgroup at x, then $\mathfrak{g}_x \subseteq \mathfrak{sp}(n) + \mathfrak{sp}(1)$, and

PROPOSITION 2.2. *If $\mathfrak{g}_x$ contains the summand $\mathfrak{sp}(1)$ of $\mathfrak{sp}(n) + \mathfrak{sp}(1)$ for all x, then M is locally symmetric.*

This follows because ∇R lies in a space ($S^5E \otimes \Sigma^1$ in notation below) that has no non-zero elements invariant by $Sp(1)$.

A non-trivial isotropy subgroup at a single point can even impose quite severe constraints on the curvature tensor [**98**], and an algebraic classification of curvature jets is feasible in certain circumstances. The study of possible isotropy subgroups is important in the classification of QK manifolds with a cohomomogeneity-one group action, that is in progress by the authors of [**36, 18, 6**].

Remark. The symmetric spaces listed above have non-compact duals with $s < 0$, and the embedding of these in their compact partners is discussed in [**119**]. By a theorem of Borel [**22**], the dual of any compact QK symmetric space admits compact quotients. Constraints on the fundamental group of such a quotient arise from work of Corlette [**34**]. He proved that any Riemannian metric with non-positive curvature operator on a compact quotient of quaternionic hyperbolic space $\mathbb{H}H^n$, $n \geq 2$, is the induced one. In this connection, it is known that the homogeneous non-symmetric spaces of [**3**] do not admit compact quotients, and the only known compact examples of QK manifolds with $s < 0$ are locally symmetric.

The tangent space at any point to a Riemannian symmetric space $M = G/H$ can be identified with an orthogonal complement $\mathfrak{m}$ of the Lie algebra $\mathfrak{h}$ in $\mathfrak{g}$. The holonomy group of M then coincides with the linear isotropy group, that is the image of the natural homomorphism $\ell\colon H \to \mathrm{Aut}(\mathfrak{m})$, and is isomorphic to H if G has trivial centre. For each of the Wolf spaces, it is easy to check explicitly that $\ell(H)$ has the form $K \times_{\mathbb{Z}_2} Sp(1)$ where $K \subseteq Sp(n)$, in accordance with (1.3). In fact, H is the normalizer of a 3-dimensional subgroup of G that is generated by a highest root vector of the Lie algegra $\mathfrak{g}$ [**119**]. It follows that G/H parameterizes a 'conjugacy class' of 3-dimensional subalgebras of $\mathfrak{g}$, a fact that we now proceed to generalize.

Let G be a compact semisimple Lie group of dimension d, with Lie algebra $\mathfrak{g}$, and let $\mathfrak{su}(2)$ denote the Lie algebra generated by the matrices

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.2}$$

over $\mathbb{R}$, with Lie bracket given by $[A_1, A_2] = A_1A_2 - A_2A_1 = -2A_3$ etc. A non-zero homomorphism of Lie algebras $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ is necessarily injective, and we may regard $\rho(\mathfrak{su}(2))$ is an element of the Grassmannian $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$ of real oriented

3-dimensional subspaces of $\mathfrak{g}$. There is a canonical 3-form on the Lie algebra $\mathfrak{g}$ that defines a real-valued G-invariant function ψ on $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$ defined by setting

$$\psi(V) = -\langle v_1, [v_2, v_3]\rangle,$$

where $V \in \mathbb{G}\mathrm{r}_3(\mathfrak{g})$ is a subspace with an oriented basis $\{v_1, v_2, v_3\}$, orthonormal relative to the Killing form of $\mathfrak{g}$.

LEMMA 2.3. *V is a critical point of ψ with $\psi(V) > 0$ if and only if $V = \rho(\mathfrak{su}(2))$ for some homomorphism ρ.*

To understand this result, recall that the tangent space $T_V\mathbb{G}\mathrm{r}_3(\mathfrak{g})$ can be identified with

$$\mathrm{Hom}(V, V^\perp) \cong V^* \otimes V^\perp, \tag{2.3}$$

where $V^\perp$ is the orthogonal complement of V in $\mathfrak{g}$; if $\alpha \in \mathrm{Hom}(V, V^\perp)$, then the corresponding vector is the one tangent to the curve $t \longmapsto V_t = \mathrm{span}\{v + t\alpha(v) : v \in V\}$ in $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$ at $t = 0$. The gradient of ψ at $V \in \mathbb{G}\mathrm{r}_3(\mathfrak{g})$ is the linear mapping characterized by

$$v_1 \longmapsto -[v_2, v_3] - \psi(V)v_1 \in V^\perp, \tag{2.4}$$

whenever $\{v_1, v_2, v_3\}$ is an oriented orthonormal basis of V.

The orbit of $\rho(\mathfrak{su}(2))$ under the adjoint action of G forms the critical manifold L_ρ, and

$$L_\rho \cong G/N_\rho,$$

where N_ρ denotes the normalizer of $\rho(\mathfrak{su}(2))$. A trajectory or flow line of the vector field gradψ is a curve in $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$ satisfying

$$V'(t) = \mathrm{grad}\,\psi(V(t)). \tag{2.5}$$

It was verified by Burstall that the Hessian of ψ is non-degenerate in normal directions to the critical submanifolds L_ρ, which means that Morse-Bott theory can be applied to the flow lines as in [**66**]. The union of L_ρ and those points on trajectories $V(t)$ with $\lim_{t\to-\infty} V(t) \in L_\rho$ is the so-called unstable manifold M_ρ associated to L_ρ. There are inclusions

$$L_\rho \subseteq M_\rho \subset \mathbb{G}\mathrm{r}_3(\mathfrak{g}),$$

and M_ρ is G-equivariantly diffeomorphic to the total space of the normal bundle to L_ρ in M_ρ. In this way, M_ρ parametrizes a distinguished family of 3-dimensional subspaces of $\mathfrak{g}$ including the subalgebras conjugate to $\rho(\mathfrak{su}(2))$.

THEOREM 2.4. [**114**] *Let $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ be any non-zero Lie algebra homomorphism. Then M_ρ has a G-invariant QK metric with $s > 0$.*

This theorem was proved by Swann by relating (2.5) to Nahm's equations, and then via twistor theory to complex nilpotent codajoint orbits and work of Kronheimer [**75**]. We shall explain in §6 that if $V \in M_\rho$ then the isotropic elements of its complexification V_c are nilpotent, a property that generalizes that enjoyed by 3-dimensional subalgebras. Indeed, the theorem belongs to a select class of results in which a Morse flow is used to classify a family of objects, the most obvious of which correspond to critical points. An analogous example of this in an infinite-dimensional setting appears in the paper [**30**] on harmonic maps.

The infinitesimal quaternionic structure is readily identified at points of L_ρ, and this will help to introduce the algebraic structure of the tangent space of an arbitrary QK manifold. Let us denote the complexification $\rho(\mathfrak{su}(2))_c$ by $\mathfrak{sl}(2,\mathbb{C})$. The latter has a complex $(k+1)$-dimensional irreducible representation that we denote by Σ^k, isomorphic to $S^k\mathbb{C}^2$ and the space of homogeneous polynomials of degree k in two variables. There is a decomposition

$$\mathfrak{g}_c \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \bigoplus_{k\geq 0} \mu_k\Sigma^k = \mathfrak{n}_c \oplus \bigoplus_{k>0} \mu_k\Sigma^k, \tag{2.6}$$

where $\mu_k\Sigma^k$ denotes $\Sigma^k \oplus \cdots \oplus \Sigma^k$, with $\mu_k \geq 0$ summands, and $\mathfrak{n}$ is the Lie algebra of N_ρ. The adjoint representation $\mathfrak{sl}(2,\mathbb{C})$ is itself isomorphic to the space Σ^2 of homogeneous quadratic polynomials.

Fix $V \in L_\rho$, so that $V^\perp \cong \bigoplus_{k\geq 0} \mu_k\Sigma^k$. Using (2.3) and the isomorphism $\Sigma^2 \otimes \Sigma^k \cong \Sigma^{k-2} \oplus \Sigma^k \oplus \Sigma^{k+2}$ (with $k > 0$ and $\Sigma^{-1} = \{0\}$), we obtain

$$T_V \mathbb{G}\mathrm{r}_3(\mathfrak{g}) = T_+ \oplus T_0 \oplus T_-,$$

where

$$\begin{aligned} (T_+)_c &\cong \bigoplus_{k\geq 2} \mu_k\Sigma^{k-2} \\ (T_0)_c &\cong \bigoplus_{k\geq 1} \mu_k\Sigma^{k} \\ (T_-)_c &\cong \bigoplus_{k\geq 1} \mu_k\Sigma^{k+2} \oplus \mu_0\Sigma^2. \end{aligned}$$

Then T_0 coincides with the tangent space $T_V L_\rho$ to the orbit through V, and $T_0 \oplus T_+$ is the tangent space to M_ρ. It has complexification

$$(T_0 \oplus T_+)_c \cong E \otimes_c \Sigma^1, \tag{2.7}$$

where $E = \bigoplus_{k\geq 1} \mu_k\Sigma^{k-1}$. The quaternionic structure of E originates from each summand Σ^k when k is odd, and pairs $\Sigma^k \oplus \Sigma^k$ when k is even. Therefore the action of $SU(2)$ on $T_0 \oplus T_+$ factors through $Sp(n)Sp(1)$, where $2n = \dim_{\mathbb{C}} E = \sum_{k\geq 1} k\mu_k$.

Any root space $\mathfrak{g}_\alpha$ generates such a subalgebra $\rho(\mathfrak{su}(2))$, but a general homomorphism ρ is determined up to conjugacy by assigning an integer in the set $\{0,1,2\}$ to each simple root of $\mathfrak{g}$ according to rules prescribed by Dynkin (this is explained by **[67]** in a useful context). The dimension of N_ρ is as small as possible when $\mathfrak{su}(2)$ is the span of an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathfrak{g}$ where $v_1 + iv_2$ belongs to a highest root space $\mathfrak{g}_\gamma$ of $\mathfrak{g}_c$. We shall call such a subalgebra minimal. The functional ψ attains its maximum value on the Wolf space G/N_ρ, where ρ arises from a highest root. In this case $T_+ = 0$ (equivalently $\mu_k = 0$ for all $k \geq 2$).

3. Representations and the Dirac Operator

The representation of the structure group $Sp(n)Sp(1)$ on the complexified tangent space $(T_x)_c$ of an arbitrary QK manifold is determined by (1.3) and coincides with the right-hand side of (2.7). The structure group of a QK manifold lifts globally to $Sp(n)\times Sp(1)$ if and only if $\varepsilon = 0$, where $\varepsilon \in H^2(M, \mathbb{Z}_2)$ is the class induced by the short exact sequence

$$1 \to \mathbb{Z}_2 \to Sp(n) \times Sp(1) \to Sp(n)Sp(1) \to 1,$$

and introduced explicitly in [**84**]. The significance of this lifting condition was first realized by Sakamoto [**101**], in a study of sectional curvature and pinching.

Over an open set on which the obstruction ε vanishes, it is conventional to write

$$T_c = E \otimes H, \tag{3.1}$$

where E and H now represent complex vector bundles of rank $2n$ and 2 respectively, underlying the standard representations of $Sp(n)$ and $Sp(1)$ on $\mathbb{H}^n$ and $\mathbb{H}$ respectively. Since the latter are self-dual, one can also replace T by T^* in (3.1) without affecting its validity.

Given the well-known isomorphism $Sp(1) \times Sp(1) \cong Spin(4)$ over an oriented Riemannian 4-manifold, E and H are in this case the same as the spin bundles, denoted V_+ and V_- in [**8**]. Thus E and H exist globally over S^4, and indeed over $\mathbb{HP}^n$ for all $n \geq 1$ since $H^2(\mathbb{HP}^n, \mathbb{Z}_2) = 0$. Regarded as a quaternionic line bundle, H is simply the tautological bundle whose fibre H_x at a point $x \in \mathbb{HP}^n$ is the line represented by that point, and E_x can be identified with the complement $H^\perp$ in $\mathbb{H}^{n+1}$. The resulting decomposition

$$\mathbb{H}^{n+1} = E_x \oplus H_x$$

characterizes the action of the isotropy group $Sp(n) \times Sp(1)$ on $\mathbb{H}^{n+1}$.

Example. The algebra underlying the 4-dimensional situation is also relevant when one examines the 8-dimensional space $G_2/SO(4)$. Identifying $SO(4) = Sp(1)Sp(1)$, its inclusion in G_2 is described by the decomposition $\mathbb{C}^7 = S^2V_- \oplus (V_- \otimes V_+)$ of the standard representation of G_2. This equation provides the well-known link between self-duality in dimension 4 and G_2-structures in 7. Furthermore,

$$(\mathfrak{g}_2)_c \cong S^2V_- \oplus S^2V_+ \oplus (S^3V_- \otimes V_+),$$

and the last summand is effectively the isotropy representation $\mathfrak{m}$. So we may take $E = S^3V_+$ and $H = V_-$; these are not globally defined bundles as $H^2(G_2/SO(4), \mathbb{Z}) \cong \mathbb{Z}_2$ is generated by ε.

The use of the locally-defined bundles E and H is a very convenient tool in describing exterior forms and other natural tensors on a QK manifold. For example, anticipating the notation below, the bundle of 2-forms can be written

$$\begin{aligned} \textstyle\bigwedge^2 T_c^* &\cong \textstyle\bigwedge^2(E \otimes H) \\ &\cong (S^2E \otimes \textstyle\bigwedge^2 H) \oplus (\textstyle\bigwedge^2 E \otimes S^2H) \\ &\cong S^2E \oplus S^2H \oplus (\textstyle\bigwedge_0^2 E \otimes S^2H) \\ &\cong \Lambda_1^2 \oplus \Sigma^2 \oplus (\Lambda_0^2 \otimes \Sigma^2). \end{aligned} \tag{3.2}$$

It has three irreducible real components, corresponding to $\mathfrak{h} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ and $\mathfrak{h}^\perp$. This generalizes the celebrated decomposition

$$\textstyle\bigwedge^2 T^* = \bigwedge_+^2 \oplus \bigwedge_-^2 = \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- \tag{3.3}$$

on an oriented Riemannian 4-manifold, and leads to extensions of Yang-Mills theory (see §4).

It is natural to ask how the standard representation Δ of $Spin(4n)$ of dimension 2^{2n} decomposes relative to the natural homomorphism $Sp(n) \times Sp(1) \to Spin(4n)$

for $n \geq 2$. To proceed, one needs to distinguish certain representations of $Sp(n)$. Choosing standard coordinates on the Lie algebra of a maximal torus, we may identify irreducible $Sp(n)$-modules with n-tuples of integers

$$(a_1, a_2, \dots, a_n), \qquad a_1 \geq a_2 \geq \cdots \geq a_n \geq 0,$$

corresponding to dominant weights (this is explained in [**106**]). In particular, E corresponds to $(1, 0, \dots, 0)$, and the symmetric power $S^m E$ corresponds to $(m, 0, \dots, 0)$ and is irreducible.

We shall be more interested in the summands

$$\Lambda^m_q = (\underbrace{2, \dots, 2}_{q}, \underbrace{1, \dots, 1}_{m-2q}, 0 \dots, 0), \qquad 0 \leq q \leq \lfloor m/2 \rfloor. \tag{3.4}$$

of the m-fold tensor product $\bigotimes^m E$. The space Λ^m_0 is isomorphic to the so-called primitive or effective summand of $\bigwedge^m E$, and the representations (3.4) all arise from tensor products of primitive ones:

LEMMA 3.1. $\Lambda^m_0 \otimes \Lambda^n_0 \cong \bigoplus_{k=0}^{\min(m,n)} \Lambda^{m+n}_k$.

Recall that the irreducible complex representations of $Sp(1)$ are merely the symmetric powers $S^q H$ which we denote by Σ^q. The spin representation of M is given by combining the primitive summands with these symmetric powers.

PROPOSITION 3.2. [**10, 116**] $\Delta \cong \bigoplus_{q=0}^{n} \Lambda^{n-q}_0 \otimes \Sigma^q$.

The summands of Δ arise from representations of $Sp(n)Sp(1)$ (rather than just $Sp(n) \times Sp(1)$) if and only if n is even, and in this case they determine vector bundles defined globally on M. (We usually denote these associated bundles by the same symbol as the representation, relying on the context to make the meaning clear.) In the case in which M is hyper-Kähler, Σ^q becomes a trivial bundle of dimension $q+1$. Moreover, the vector bundle E exists globally and is isomorphic to the holomorphic tangent bundle $T^{1,0}$ relative to any compatible complex structure. ($T^{1,0}$ is also isomorphic to its dual $\Lambda^{1,0}$ by means of the appropriate holomorphic symplectic form.) It is well known that in this case Δ is the full exterior algebra on E, and this is consistent with the above proposition.

COROLLARIES 3.3. *(i) A QK manifold of even quaternionic dimension n is always spin.*
(ii) An HK manifold of quaternionic dimension n has a complex $(n+1)$-dimensional space of harmonic spinors.

On an HK manifold, the Dirac operator can be identified with $\overline{\partial} + \overline{\partial}^*$ acting on the full exterior algebra of E, and the relevant operators can be 'strung out' into the usual Dolbeault complex. A similar phenomenon occurs on a QK $4n$-manifold for which n is even or $\varepsilon = 0$. Namely there is an elliptic complex of the form

$$0 \to \Lambda^n_0 \overset{D}{\to} \Lambda^{n-1}_0 \otimes \Sigma^1 \overset{D}{\to} \Lambda^{n-2}_0 \otimes \Sigma^2 \to \cdots \to \Sigma^n \to 0 \tag{3.5}$$

(of course, to make sense of the notation, the objects between the arrows are now sheaves or sections of the corresponding vector bundles). This complex has the remarkable property that it can be coupled to any vector bundle V with a connection whose curvature lies in the space $\mathfrak{h} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ without destroying the property that $D^2 = 0$. This fact leads to one possible generalization of the Sieberg-Witten equations to a quaternionic context.

Analogues of the Dirac complex (3.5) that do not require a Riemannian metric for their definition are studied in [**105, 11**]. The most obvious such complexes are those obtained by tensoring (3.5) by Σ^{k+n} and rearranging the pieces to give

$$(3.6) \qquad 0 \to \Sigma^k \stackrel{\overline{D}_k}{\to} E \otimes \Sigma^{k+1} \to \bigwedge^2 E \otimes \Sigma^{k+2} \to \cdots \to \bigwedge^n E \otimes \Sigma^{k+n} \to 0.$$

The reappearance of full exterior powers of E ensures that they can indeed be defined relative to the G-structure used in the definition of a quaternionic manifold (see §4). A recent application of (3.6) for $k = 0$ is in the definition of a quaternionic version of analytic torsion [**81**].

For $k \geq 1$, the $\overline{D}_k$ are 'twistor operators'; each is overdetermined and has locally a finite-dimensional space of solutions that have special geometrical significance. For example, a solution of $\overline{D}_1 \zeta = 0$ determines a hypercomplex structure, and a solution of $\overline{D}_2 \zeta = 0$ a 'quaternionic complex structure' [**60**].

Remark. The work of Friedrich and others [**15**] on eigenvalues of the Dirac operator on a compact spin manifold has motivated work on the problem of finding a lower bound λ of the Dirac operator on a QK manifold of dimension $4n$ with n even [**54, 74**]. The result is that

$$\lambda^2 \geq \frac{n+3}{n+2}\frac{s}{4},$$

with equality occurs if and only if $M = \mathbb{HP}^n$ and the eigenvector lies in the summand $\Lambda_0^n \oplus (\Lambda_0^{n-1} \otimes \Sigma^1)$. An analogous sharp lower bound is known for the Laplacian acting on functions on a QK manifold [**79, 4**].

The 'Fueter complex'

$$(3.7) \qquad 0 \to H \stackrel{D}{\to} E \stackrel{\Delta}{\to} \bigwedge^3 E \to \bigwedge^4 E \otimes \Sigma^1 \to \cdots \to \bigwedge^{n+3} E \otimes \Sigma^n \to 0$$

is a version of (3.6) for $k = -3$, and incorporates a natural analogue D of the $\overline{\partial}$-operator in complex analysis. All the operators in this complex are first order, except for Δ which is second order. Local sections of H solving the equation $Df = 0$ in flat space $\mathbb{H}^n$ are the so-called quaternionic regular functions defined by Fueter, and twistor theory can be used to show that (3.7) is a resolution of the associated sheaf, so that for example $g = Df$ is locally solvable if and only if $\Delta g = 0$. Explicit expressions for the operators can be found in [**1**]. Much of what can be done in flat space extends to the class of hypercomplex manifolds, and Joyce has developed an extensive programme aimed, amongst other things, at reconstructing HK metrics from their function theory [**63, 99**].

A decomposition of the exterior forms on a QK manifold can in theory be determined by the formula

$$\Delta \otimes \Delta \cong \bigoplus_{k=0}^{4n} \bigwedge^k T^*.$$

Lemma 3.1 implies that each summand has the form $\Lambda_q^p \otimes \Sigma^r$. To illustrate this, we quote without proof that

$$\bigwedge^4 T^* \cong (\Lambda_2^4 \oplus \Lambda_0^2 \oplus \mathbb{R}) \oplus ((\Lambda_1^4 \oplus \Lambda_1^2 \oplus \Lambda_0^2) \otimes \Sigma^2) \oplus ((\Lambda_0^4 \oplus \Lambda_0^2 \oplus \mathbb{R}) \otimes \Sigma^4).$$

On the other hand, for $\mathfrak{h} = \mathfrak{sp}(n) + \mathfrak{sp}(1)$,

$$\begin{aligned} S^2\mathfrak{h} &\cong S^2(S^2E) \oplus (S^2E \otimes \Sigma^2) \oplus S^2(\Sigma^2) \\ &\cong (S^4E \oplus \Lambda_2^4 \oplus \Lambda_0^2 \oplus \mathbb{R}) \oplus (\Lambda_1^2 \otimes \Sigma^2) \oplus \Sigma^4 \oplus \mathbb{R}. \end{aligned}$$

All these summands occur in $\bigwedge^4 T^*$ apart from a trivial summand and $W = S^4E$, and this can be used to justify the remarks before Corollary 1.2.

COROLLARY 3.4. *The curvature tensor of a QK manifold equals* $R_Q + s\rho_1$*, where* R_Q *takes values in* S^4E *and* ρ_1 *is* $Sp(n)Sp(1)$*-invariant.*

On a HK manifold, the space of curvature tensors is isomorphic to S^4E. A choice of complex structure on an HK manifold yields an identification between E and the holomorphic cotangent bundle T^*, so we may regard R as a smooth section of $T^* \otimes S^3E$. The second Bianchi identity can be used show that R actually determines an element of the sheaf cohomology group $H^1(M, \mathcal{O}(S^3T^*))$. This last result is relevant to theory resulting from the so-called Witten-Rosansky invariants [**100, 64**], and some of the relevant representation theory appears in [**50**].

Corollary 3.4 leads to the idea due originally to Roček that, in certain circumstances, HK metrics can be constructed as the limit of a sequence of QK metrics with scalar curvature tending to zero. Although no general theory for such a phenomenon as yet exists, this idea has led to the whole programme relating hyper-Kähler to quaternion-Kähler described later in this chapter.

4. Quaternionic Manifolds and Bundles

The inability to choose a global basis of complex structures on a quaternion-Kähler manifold M can be overcome by passing to the total space of an associated bundle. This approach is however best viewed within the wider context of quaternionic manifolds, which we now describe.

DEFINITION 4.1. A quaternionic manifold is a smooth manifold of dimension $4n \geq 8$ admitting a G-structure and a torsion-free G-connection, where G denotes the subgroup $GL(n,\mathbb{H})GL(1,\mathbb{H})$ of $GL(4n,\mathbb{R})$.

The group G is defined as in (1.3), but without reference to an inner product on $\mathbb{R}^{4n}$. Thus, $GL(n,\mathbb{H})$ is the commutator of the group $GL(1,\mathbb{H})$ of transformations $v \longmapsto vq^*$, q a non-zero quaternion, and

$$G = GL(n,\mathbb{H})Sp(1) \cong GL(n,\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1).$$

To complete the definition, it is logical to define a quaternionic manifold of real dimension 4 to be one with a self-dual conformal structure.

Let M be a quaternionic manifold. The homomorphism

$$G \to Sp(1)/\mathbb{Z}_2 \cong SO(3)$$

given by projection to the second factor allows one to define bundles over M associated to various representations of $SO(3)$. First, let $SO(V)$ denote the principal $SO(3)$ bundle parametrizing triples $\{I_1, I_2, I_3\}$ of almost-complex structures satisfying (1.1), whose existence does not require a Riemannian metric.

Let F be any space (linear or otherwise) on which $SO(3)$ acts, and let $\mathbf{F}$ denote the fibre bundle associated to $SO(V)$ with fibre F. Here are some obvious candidates for F:

(i) $SO(3)$, acted on by itself by left translation;

(ii) the standard representations $\mathbb{R}^3$ and $\mathbb{C}^3$;
(iii) the 2-sphere S^2 in $\mathbb{R}^3$;
(iv) $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_2$, where $\mathbb{C}^2$ is the standard representation of $Sp(1)$;
(v) $S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$, where S^3 is the set of unit vectors in the above $\mathbb{C}^2$.

In each case the bundle $\mathbf{F}$ has been well studied. In (i) it is simply $SO(V)$ itself, and in (ii) we recover the bundle V of endomorphisms defining the quaternionic structure and its complexification V_c. In (iii) $\mathbf{F}$ is the subset of unit vectors in V; it is denoted by Z and called the twistor space of M. In (iv) we shall see that $\mathbf{F}$ can be identified with the total space, minus its zero section, of a complex line bundle L^* over Z. Finally, (v) coincides with (i) as $SO(V)$ may be also be identified with the set of 'unit' vectors in L^*.

The geometry of M is simplified to a greater or lesser extent when passing to the total space of each of the above bundles, and there are pros and cons to focussing on each case. However, it is Z that encodes the underlying quaternionic structure of M most directly into complex geometry. We shall always denote the projection $Z \to M$ by π, and a fibre $\pi^{-1}(x)$ by Z_x. Then each point $z \in Z_x$ is an almost-complex structure on T_xM of the form $a_1I_1 + a_2I_2 + a_3I_3$, where $\{I_1, I_2, I_3\}$ is a local orthonormal basis of V. Thus, a section s of Z over an open set M' of M can itself be regarded as an almost-complex structure I_s on M'. Let $\sigma\colon Z \to Z$ denote the 'antipodal mapping' $I \longmapsto -I$ defined on each S^2 fibre, and with no fixed points.

THEOREM 4.2. [**17, 104**] *Over a quaternionic manifold M, the total space Z admits a complex structure with the property that (i) its fibres are rational curves with normal bundle $2n\mathcal{O}(1)$, (ii) σ is anti-holomorphic, and (iii) a local section $s(M')$ is a complex submanifold if and only if I_s is an integrable complex structure.*

Here, $2n\mathcal{O}(1)$ is short for $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$, where $\mathcal{O}(1)$ denotes the hyperplane line bundle; more generally $\mathcal{O}(k)$ will denote the tensor power $\mathcal{O}(1)^{\otimes k}$. It is a corollary that the quaternionic structure of M is always generated locally by a complex structure I_1 and an almost-complex structure I_2 anti-commuting with I_1. If I_2 is also integrable then the resulting structure is hypercomplex (see below).

We shall often call the fibres of Z over M the 'twistor lines'. By identifying a vector $I_1 \in Z_x$ with the projective class of $I_2 + iI_3$, one may also regard $C = Z_x$ as the conic of null lines in the projective plane $\mathbb{P}((V_x)_c)$. In many situations the vector spaces V_x are explicitly realized as subspaces of a 'universal' vector space $\mathbb{V}$. In any case, since

$$H^0(C, 2n\mathcal{O}(1)) \cong \mathbb{C}^{4n}, \quad H^1(C, 2n\mathcal{O}(1)) = \{0\}, \tag{4.1}$$

Kodaira's theory implies that C belongs to a complex $4n$-dimensional family of rational curves. The existence of such a curve C with the given normal bundle thus captures the essential geometry of a twistor space.

The complex structure J on Z characterized by Theorem 4.2(iii) may be defined by first identifying Z locally with the complex projective bundle

$$\mathbb{P}(H) = \mathbb{P}(H \otimes A^\lambda), \tag{4.2}$$

where A is the real line bundle arising from the standard representation of the centre $\mathbb{R}^*$ of G. The point is that the the twistor operator $\overline{D}_1$ defined in (3.6) is only invariantly defined if Σ^1 is replaced by $\tilde{H} = H \otimes A^\lambda$ for an appropriate value of the 'weight' λ (computed in [**95**] to equal $n/(n+1)$). The integrability of J may then be deduced by applying the proof of [**8**, Theorem 4.1] and results on the curvature of quaternionic manifolds from [**105**]. Because Z is now complex analytically a projective bundle, it is a corollary of this approach that there exists a holomorphic line bundle L over Z which restricts to $\mathcal{O}(2)$ on each fibre. The existence of a torsion-free connection is precisely the condition that guarantees the integrability of (Z, J).

A special case of a quaternionic manifold is a manifold with a torsion-free connection $\overline{\nabla}$ preserving a $GL(n,\mathbb{H})$-structure. In this case V is trivial and there exist globally-defined triples of parallel complex structures $\{I_1, I_2, I_3\}$. Such a manifold is called hypercomplex, and bears the same relationship to quaternionic that hyper-Kähler bears to quaternion-Kähler (see [**106**] and references therein). In fact, a hypercomplex structure is uniquely specified by two anti-commuting complex structures I_1, I_2, for in this case $I_3 = I_1 I_2$ is also complex, and the ('Obata') connection $\overline{\nabla}$ is uniquely determined.

If M is a hypercomplex manifold then the twistor space Z is trivial as a smooth bundle, and the projection $Z \to \mathbb{CP}^1$ is holomorphic. Points of M correspond to sections of π with normal bundle $2n\mathcal{O}(1)$. This point of view has proved particularly valuable in the construction of non-compact HK manifolds [**17**], and in classifying deformations of hypercomplex manifolds [**94**].

One reason for including Definition 4.1 in this chapter is that curvature can be defined in this more general context.

PROPOSITION 4.3. [**105**] *(i) A quaternionic manifold has a tensor R_Q that is the component of the curvature of a torsion-free G-connection ∇ independent of the choice of ∇.*
(ii) On a hypercomplex manifold, the curvature of $\overline{\nabla}$ equals $R_Q + R_\kappa$, where $R_\kappa \in \Lambda^2_1$ represents the curvature 2-form of $\kappa = \bigwedge^{2n,0}$.

The canonical bundle κ in (ii) is the complexification of a real bundle arising from a homomorphism $GL(n,\mathbb{H}) \to \mathbb{R}^*$, and is therefore independent of the complex structure chosen to define $\bigwedge^{2n,0}$. Observe that $\Lambda^2_1 \cong S^2E$ is a subspace of $\bigwedge^2 T^*M$ defined by (3.2) by the G-structure. It can be identified with the intersection of the spaces $\bigwedge^{1,1}$ of (1,1)-forms relative to each almost-complex structure $I \in Z_x$. A 2-form is called self-dual if it takes values in this subspace, which coincides with $\bigwedge^2_+$ when $n = 1$.

As the notation implies, on a QK manifold R_Q can be identified with the non-trivial component of the Riemann tensor defined by Corollary 3.4, and in 4 dimensions, it would just be the non-vanishing half W_+ of the Weyl tensor. A compact simply-connected quaternionic manifold with $R_Q \equiv 0$ is necessarily isomorphic to $\mathbb{HP}^n$.

A hypercomplex manifold for which $R_Q = 0 = R_\kappa$ is covered by coordinate charts with constant quaternionic linear transition functions. This class of manifolds was considered by Sommese in the paper [**109**], which contains one of the earliest references to the concept of the twistor space. Such affine flat examples include $S^{4n-1} \times S^1$, and the abelian hypercomplex nilmanifolds considered in [**40**] which are quotients of $\mathbb{H}^n$. The tensor R_κ is a type of skew-symmetric Ricci tensor, and less trivial examples with $R_\kappa = 0$ include metrics which are conformally HK. A compact hypercomplex 4-manifold M necessarily has $R_\kappa = 0$, and Boyer effectively used this to show that either M admits a HK metric (and is therefore a torus or K3 surface), or else is diffeomorphic to a Hopf surface [**25, 65**].

Example. Suppose that M is a QK manifold with a compatible hypercomplex structure. Let ∇ represent its Levi-Civita connection, R the Riemann tensor, and $\overline{R}$ the curvature of $\overline{\nabla}$. The difference $\nabla - \overline{\nabla}$ may be regarded as a 1-form α with the property that $\alpha_i = I_i\alpha$ in (1.2), and

$$S = R - \overline{R} = \nabla\alpha + \tfrac{1}{2}\alpha \wedge \alpha.$$

Since R_Q must coincide with that component of R in S^4E, it follows that the symmetric part of S is proportional to the Riemannian metric g, and its skew part $d\alpha$ a self-dual 2-form. This approach is used in [**5**] to show that with certain additional assumptions M must be quaternionic hyperbolic space.

Let F be a complex vector bundle over a quaternionic manifold M, and suppose that ∇ is a connection on F. The curvature R_∇ of ∇ is a 2-form with values in $\mathrm{End}F$, and referring to (3.2) we record the

DEFINITION 4.4. [**105, 82, 90, 48**] The connection ∇ is called quaternionic, of type B_2 or c_2-self-dual if R_∇ is self-dual as a 2-form so that $R_\nabla \in \mathrm{End}F \otimes \Lambda^2_1$.

These connections satisfy the Yang-Mills equations. Although their moduli spaces are known in some special cases with $\dim M \geq 8$ [**82, 88**], 'quaternionic Yang-Mills theory' is still in its infancy. The self-duality condition on the curvature of ∇ enables the complexes of differential operators described in §3 to be extended by tensoring by F, and a number of cohomological results are known [**89**].

On a hypercomplex manifold, the connection $\overline{\nabla}$ induces covariant derivatives on all vector bundles associated (even locally) to the $GL(n,\mathbb{H})$-structure. The same is true on a QK manifold equipped with its Levi-Civita connection ∇. The following result is related to Proposition 4.3(ii).

LEMMA 4.5. *On a hypercomplex or QK manifold, the connection induced on E is c_2-self-dual.*

Now suppose that M is a quaternionic manifold with $\varepsilon = 0$, and that ∇ is self-dual. Then there exists a twistor operator

$$\overline{D}_1 \colon F \otimes \tilde{H} \to F \otimes \tilde{E} \otimes S^2\tilde{H},$$

where the tildes represent appropriate weights. If F has complex rank $2r$ and ∇ preserves a $GL(r,\mathbb{H})$-structure on F then $\overline{D}_1$ is an operator between real vector bundles of rank $4r$ and $12nr$ respectively.

THEOREM 4.6. [**105**] *With the above hypotheses, the real $(4n+4r)$-dimensional total space M_F of $F \otimes \tilde{H}$ is a quaternionic manifold.*

The twistor space of M_F can be identified with the total space of the complex vector bundle $(\pi^{-1}F) \otimes L^{1/2}$ over Z, where $L^{1/2}$ is a holomorphic square root of L determined by the smooth splitting $\pi^{-1}H = L^{1/2} \oplus \overline{L^{1/2}}$. Indeed, the fact that $\pi^{-1}F$ is a holomorphic vector bundle over Z follows from the fact that the curvature of the pulled-back connection $\pi^{-1}\nabla$ has no $(0,2)$-component, and a celebrated integrability theorem of Atiyah [**8, 39**]. For example, if M is hypercomplex, then ∇ is c_2-self-dual if and only if ∇ is 'triholomorphic'.

Example. Let M be a QK manifold. Lemma 4.5 implies that the total space M_E of the tangent bundle TM of a QK manifold is itself quaternionic. This applies in particular to S^4 and $\mathbb{CP}^2$. In the latter case $E \cong \kappa^{1/2} \oplus \overline{\kappa}^{-1/2}$ where κ denotes the canonical line bundle whose curvature is a multiple of the Kähler-form $\omega \in \bigwedge^+$.

Over an arbitrary quaternionic manifold one can take $F = \mathbb{C}^2 = \mathbb{H}$, and ∇ the trivial connection. Then M_F can be identified with the total space of the quaternionic line bundle $\tilde{H}$. Let $\tilde{H}^*$ denote $\tilde{H}$ with its zero section removed; this may be regarded as a principal $\mathbb{H}^*$-bundle over M. The proof of Theorem 4.2 implies that $\tilde{H}^*$ admits a complex structure J, and the orbit of J under $\mathbb{H}^*$ is a 2-sphere $\{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}$ of complex structures satisfying (1.1). This endows $\tilde{H}^*$ and

$$U = \tilde{H}^*/\mathbb{Z}_2 \tag{4.3}$$

with a hypercomplex structure. Natural though the definition of the twistor space Z is, there is a sense in which it involves the choice of a complex structure, and the construction of U overcomes this objection.

Example. The homogeneous space $SU(n+2)/SU(n)$ that fibres over the Wolf space $M = \mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ is closely related to U, although of course one has compact fibre and the other not. Let $U' \cong U/\mathbb{Z}$ denote the bundle associated to U with fibre $U(1) \times SO(3)$. Then there is a principal $U(1)$-bundle P over M such that $SU(n+2)/SU(n)$ double covers the quotient of $U' \times P$ by the diagonal action of $U(1)$. This action preserves the hypercomplex structure, and the latter persists because P has a c_2-self-dual connection which enables its complexififcation to be viewed as a holomorphic bundle over the twistor space Z [**60, 12**]. The case $n = 1$ relates to the fact that $SU(3)$ is itself hypercomplex.

There are many ways in which the last example can be generalized to construct compact hypercomplex structures on Stiefel manifolds [**23**], and Lie groups [**110, 61**]. We conclude this section by summarizing the latter, which proceeds by extending the decomposition (2.6) by a sequence of minimal 3-dimensional subalgebras.

Given a 3-dimensional subalgebra $\mathfrak{sp}(1) = \mathfrak{sp}(1)_1$ of $\mathfrak{g} = \mathfrak{h}_0$ generated by a highest root, one may regard its centralizer $\mathfrak{h} = \mathfrak{h}_1$ as a Lie algebra in its own right. If this is not abelian it will contain a minimal 3-dimensional subalgebra $\mathfrak{sp}(1)_2$ (see the end of §2), and we may write $\mathfrak{h}_{i-1} = \mathfrak{sp}(1)_i \oplus \mathfrak{h}_i \oplus \mathfrak{m}_i$ for $i \geq 1$. Hence,

$$\mathfrak{g} = \mathfrak{h}_k \oplus \bigoplus_{i=1}^{k} (\mathfrak{sp}(1)_i \oplus \mathfrak{m}_i),$$

where $\mathfrak{m}_i \cong \mathbb{C}^{2m_i} \otimes \Sigma^1$, as an $\mathfrak{sp}(1)_i$-module, but lies in the centralizer of $\mathfrak{sp}(1)_j$ when $j > i$. The subalgebra $\mathfrak{h}_i$ is the centralizer of $\mathfrak{sp}(1)_i$ in $\mathfrak{h}_{i-1}$, and the process can be continued unless $\mathfrak{h}_k$ is abelian. Whether or not this is the case, $U(1)^k \times (G/H_k)$ has a hypercomplex structure with tangent space isomorphic to

$$\mathbb{R}^k \oplus (\mathfrak{g}/\mathfrak{h}_k) \cong \bigoplus_{i=1}^{k} (\mathbb{R} \oplus \operatorname{Im}\mathbb{H} \oplus \mathbb{H}^{m_i}).$$

One can also replace H_j by its semisimple part by adjusting the number of extra $U(1)$ factors required. Taking k maximal then yields a hypercomplex structure on $U(1)^k \times G$ for some integer k no greater than the rank of G (or 3 if G has rank 2).

5. Fano Twistor Spaces

The Levi-Civita connection on M determines a horizontal distribution on all the associated bundles considered in the last section. By general principles [**17, 117**] that are discussed elsewhere in this volume, all these total spaces admit families of Einstein metrics. The situation is particularly important when the scalar curvature is positive.

THEOREM 5.1. [**26, 102, 113**] *If M is a QK manifold with $s > 0$ then*
(i) Z has a Kähler-Einstein metric,
(ii) U has an HK metric, and distinct QK metrics, and
(iii) $SO(V)$ has a 3-Sasakian metric.

We might add that a Ricci-flat metric can be defined on V using the techniques of [**29**]. This metric will be irreducible and (it almost follows) will have holonomy group equal to $SO(4n+3)$, a fact that is significant as there are few known examples of Ricci-flat metrics without reduced holonomy.

We shall now comment on additional structures that exist on these manifolds, spending most time on Z. The horizontal space at a point $z \in Z_x$ can be identified with T_xM and therefore has a natural complex structure determined by z. We let D denote the corresponding bundle of complexified horizontal vectors of type $(1,0)$. The isomorphism (4.2) leads to an interpretation of the relevant structures in terms of the 'EH' formalism. The almost-complex structure determined by the projective line $[h]$ with $h \in H$ has $E \otimes [h]$ as its subspace of $(1,0)$-vectors in $(T_xM)_c$. The quotient TZ/D can be identified with the holomorphic line bundle L introduced after (4.2). It follows that

$$D \cong \pi^*E \otimes L^{1/2},$$

and this bundle acquires a holomorphic structure over Z, by Proposition 4.5, reflecting the Einstein curvature of M.

The following result appears in [**102**] (though what is here called L is there called L^2).

PROPOSITION 5.2. *If $s \neq 0$, D is a holomorphic contact distribution on Z.*

To explain this, observe that the exact sequence

$$0 \to D \to TZ \to L \to 0 \tag{5.1}$$

determines a holomorphic 1-form $\theta \in H^0(Z, \mathcal{O}(T^*Z \otimes L))$ with values in L. Although $d\theta$ itself can only be calculated by choosing a local section of L, its restriction to $\bigwedge^2 D$ is independent of this choice and gives rise to an element of $H^0(\bigwedge^2 D^* \otimes L)$ that is non-degenerate provided $s \neq 0$. This is the contact condition, and corresponds to the distribution D being 'maximally non-integrable'.

Theorem 4.2 with Proposition 5.2 combine to give a powerful encryption of QK metrics. LeBrun has shown that a complex contact manifold (Z, D) with a fixed-point free anti-involution σ and a family of rational curves transverse to D with normal bundle $2n\mathcal{O}(1)$ is the twistor space of a pseudo-Riemannian metric with holonomy in $Sp(p, n-p)Sp(1)$ for some p [**77**]. This inversion theorem also led him to prove that the moduli space of complete such metrics on $\mathbb{R}^{4n}$ is infinite-dimensional [**78**]. We shall exemplify a family of rational curves with the stated properties in §6.

If $\kappa = \bigwedge^{2n+1} T^*Z$ denotes the canonical bundle of Z, the well-defined section

$$\theta \wedge (d\theta)^n \in H^0(Z, \mathcal{O}(\kappa \otimes L^{n+1}))$$

yields an isomorphism $\kappa^* \cong L^{n+1}$. If $s > 0$, the 2-form defining the Kähler-Einstein metric of Z is proportional to the curvature of a natural connection on L [**102**], and L is an ample line bundle. Since the same is true of the anticanonical bundle κ^*, Z is by definition a Fano manifold. The algebraic geometry of Fano 3-folds from the twistor space point of view can be found in [**56**] and [**92**].

It is an open problem to determine conditions on a contact Fano manifold to ensure that it is the twistor space of a positive QK manifold. In this direction,

THEOREM 5.3. [**79, 86**] *If Z is a compact Kähler-Einstein manifold with a holomorphic contact structure then Z is the twistor space of some QK manifold M.*

The space U of (4.3) can be identified with the total space of L^* over Z with its zero section removed. The contact form θ pulls back (and then evaluates) to a genuine 1-form on U. The contact condition ensures that the exterior derivative

$$\omega = d\theta \tag{5.2}$$

is in fact a holomorphic symplectic form on U. Indeed, any contact manifold has a 'symplectification', and given Z, U is it. In the case in which $s > 0$, the total space of L^* has a natural Kähler metric, and the HK structure on U is generated by an action of $Sp(1)/\mathbb{Z}_2 = SO(3)$.

Conversely, suppose that N is a hyper-Kähler manifold with a free action of $SO(3)$ inducing a transitive action on the 2-sphere S^2 of complex structures. If, furthermore, IX_I is independent of $I \in S^2$ (where X_I is the vector field generated by the circle subgroup preserving I) then N is locally isometric to the bundle U of some QK manifold. This fact allows one to construct the join of QK manifolds. If M_1 and M_2 are both QK, then the product $U_1 \times U_2$ has a HK structure. It follows that the manifold $M_1 * M_2 = (U_1 \times U_2)/\mathbb{H}^*$ (locally isomorphic to an open set of the quaternionic projective bundle $\mathbb{P}(H_1 \oplus H_2)$) is quaternion-Kähler. Taking $M = M_1$ to have $s > 0$ and $M_2 = \{x\}$ to be a point establishes the existence of a QK metric on $M * \{x\} \cong U$ with positive scalar curvature. More details, as well as a related discussion of HK potentials, can be found in [**113**].

The bundle U can also be viewed as a cone over $SO(V)$, and its HK structure reflects the 3-Sasakian structure of $SO(V)$ in accordance with the theory of Killing spinors [**9**]. When M is a self-dual Einstein 4-manifold then $SO(V)$ actually carries an Einstein metric with so-called 'weak holonomy G_2', and the quaternionic line bundle H associated metrics with holonomy $Spin(7)$ [**43, 49**]. The great significance of the bundle $SO(V)$ is that it may be a manifold even in situations in which M has orbifold singularities; this has led to some surprisingly rich classification questions [**27**], that are presented elsewhere in this volume. The manifold $SO(V)$ also has an underlying 'quaternionic contact structure', a notion exploited in [**19**] for the local construction of QK metrics.

There are many general results that apply to a Fano contact manifold Z without the assumption that it fibres over a QK manifold. The exterior powers of the 'DTL' sequence (5.1) provide important information. Associated long exact sequences relate the Dolbeault cohomology spaces $H^q(Z, \mathcal{O}(\bigwedge^p D \otimes L^p))$ and

$H^{p,q}(Z,\mathcal{O})$. This allows one to deduce that the Hodge numbers $h^{p,q}$ of Z vanish if $p \neq q$, and derive the following formula for holomorphic Euler characteristics.

LEMMA 5.4. [**102**] $\chi(Z,\mathcal{O}(\bigwedge^p D\otimes L^{-p-r})) = \begin{cases} 0, & 1\le r\le n-p, \\ (-1)^p h^{p,p}, & r=0. \end{cases}$

The index of a Fano manifold is by definition the largest root of κ that can be extracted, and it follows from a well-known characterization of Kobayashi-Ochiai [**69**] that if the index $2n+2$, the Fano manifold is biholomorphically equivalent to $\mathbb{CP}^{2n+1}$. The index of a twistor space is $n+1$ unless L itself has a square root which occurs if and only if $\varepsilon = 0$. Two simply-connected complex contact manifolds are contact-isomorphic if and only if they are biholomorphic [**80**]. A fuller discussion of automorphism groups will be postponed until §7, but part (i) of the next theorem now follows.

THEOREM 5.5. [**102, 80**] *Let M be a positive QK manifold of dimension $4n$.*
(i) If $\varepsilon=0$ then M is isometric to $\mathbb{HP}^n$.
(ii) If $b_2(M)>0$ then M is isometric to $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$.

As first pointed out by LeBrun, the characterization of QK manifolds with $b_2 \ge 1$ is a spin-off of results of Wiśniewski [**118**] within the context of Mori's programme. The crucial property of the twistor space Z of such a manifold is the existence of a rational curve C with $C\cdot L = 1$ whose homology class is not proportional to that of a fibre of π. Through each point the family of such rational curves actually spans out a projective space and Z can be identified with the total space of the projectivization of a vector bundle over a variety X. The mapping $f\colon Z\to X$ is a so-called Fano contraction, and its fibres are tangent to the contact distribution. The key point here is that if C is any rational curve satisfying $L\cdot C = 1$, then the pullback of θ to C is zero since $H^1(\mathbb{CP}^1,\mathcal{O}(\Omega^1(1))) = 0$. It turns out that the existence of a contact structure on Z allows one to deduce that X is isomorphic to $\mathbb{CP}^{n+1}$, and $Z\cong\mathbb{P}(T^*\mathbb{CP}^{n+1})$.

A study of Fano manifolds Z with $b_2(Z)=1$ (corresponding to $b_2(M)=0$) is accomplished in the papers [**70, 87**]. A key theorem asserts that in each fixed dimension, the top power of $c_1(Z)$ is bounded, and this implies that there are only finitely many deformation types. On the other hand, under appropriate hypotheses, a Fano contact structure is rigid under defomation, whence

THEOREM 5.6. [**80**] *Up to homothety, there are only finitely many positive QK manifolds of dimension $4n$.*

The general theory of polarized varieties, as described by Fujita [**45**], is especially relevant to the study of low-dimensional Fano manifolds. We set

$$R_k = H^0(Z,\mathcal{O}(L^k)),\quad r_k = \dim R_k, \tag{5.3}$$

and omit the subscripts when $k=1$. The fact that L is ample implies that the natural map

$$\nu_k\colon Z\to\mathbb{P}(R_k^*)$$

is an embedding for k sufficiently large. However, we shall be more concerned with $\nu=\nu_1$, which is a well-defined mapping only if the base locus B of the linear system $|L|$ is empty. Relative to the 'polarization' defined by L, the Δ-genus is defined by

$$\Delta(Z) = \deg(Z) - r + 2n + 1,$$

where $\deg(Z) = \langle \ell^{2n+1}, [Z] \rangle$ and $\ell = c_1(L)$. Then

$$\Delta(Z) \geq \dim B + 1 \tag{5.4}$$

(with the convention that $\dim \emptyset = -1$), and this equation limits the size of B.

Given that $c_1(TZ) = (n+1)\ell$, the Riemann-Roch theorem implies that

$$r_k = \langle e^{k\ell} \mathrm{td}(Z), [Z] \rangle = \left\langle e^{\ell(n+1+2k)/2} \hat{A}(Z), [Z] \right\rangle, \tag{5.5}$$

where $\hat{A}(Z)$ is defined by (8.4). It follows that there exists a polynomial

$$P(k) = \deg(Z) \frac{k^{2n+1}}{(2n+1)!} + \text{lower powers of } k,$$

such that $r_k = P(k)$ for $k \geq 0$. This is the so-called Hilbert polynomial of the polarized variety (Z, L). Geometrical properties of μ are encapsulated in the natural homomorphism

$$\bigoplus_{k=0}^{\infty} S^k R \to \bigoplus_{k=0}^{\infty} R_k \tag{5.6}$$

of coordinate rings. The space R_k is spanned by the pullbacks of homogeneous polynomials of degree k to Z. We shall see in §6 that, for a twistor space, $R = R_1$ is isomorphic to the complexification $\mathfrak{g}_c$ of the Lie algebra of the isometry group G of M. Indeed, the individual linear mappings $s_k \colon S^k R \to R_k$ of (5.6) are G-equivariant, and an understanding of the resulting representations leads to models for twistor spaces.

The dimension of the space of polynomials of degree k in $N+1$ variables equals $\binom{k+N}{N}$, and the Hilbert polynomial of $(\mathbb{CP}^N, \mathcal{O}(1))$ is

$$\binom{t+N}{N} = \frac{1}{N!}(t+N)(t+N-1)\cdots(t+1).$$

If X is an embedded hypersurface of $\mathbb{CP}^N$ of degree h, then the kernel of (5.6) is generated by the element of $S^k R$ whose zero set defines Z. It follows that the Hilbert polynomial of X is

$$\binom{t+N}{N} - \binom{t+N-h}{N}.$$

A dual situation occurs when Y is a covering of $\mathbb{CP}^N$ of degree d branched over a hypersurface of degree dh. In this case, the cokernel of (5.6) is generated by an element of R_h, and the Hilbert polynomial of Y is

$$\binom{t+N}{N} + \binom{t+N-h}{N}.$$

Example. These situations are combined when Z is a branched covering of a hypersurface of $\mathbb{CP}^N$. An analysis of the representations R_k for $n = 4$ shows that one of the many potential twistor spaces Z of a real 16-dimensional QK manifold has

$$P(t) = \binom{t+10}{10} - \binom{t+8}{10} + \binom{t+6}{10} - \binom{t+4}{10} = \frac{4t^9}{9!} + \text{lower terms},$$

and the values of $r_k = P(k)$ for $k \geq 1$ are

$$11,\ 65,\ 275,\ 936,\ 2728,\ 7072,\ 16720,\ 36685, \dots . \tag{5.7}$$

This is consistent with Z being the double-covering of a hyperquadric H in $\mathbb{CP}^{10}$, branched over the intersection of H with an octic, although a positive identification of this sort requires more explicit knowledge of (5.6).

6. Isometry Groups and Moment Mappings

Let M be a manifold with a symplectic 2-form ω, and a vector field X which is an infinitesimal automorphism of ω. Thus,

$$0 = \mathcal{L}_X\omega = X \lrcorner\ d\omega + d(X \lrcorner\ \omega) = d(X \lrcorner\ \omega),$$

and there exists a real-valued function f (defined on at least an open set of M) such that $X \lrcorner\ \omega = df$. This basic observation underlies much of this section, though we shall see in due course that analogues of f can be constructed on manifolds with geometrical structures that are not obviously 'symplectic'.

Next, suppose that M is a hyper-Kähler manifold, so that we can choose symplectic 2-forms $\omega_1, \omega_2, \omega_3$ associated to a standard triple of complex structures. If X is a Killing vector field on M whose corresponding 1-parameter group of isometries preserves the HK structure, then the above observation shows that, locally, there exist functions f_1, f_2, f_3 such that $df_i = X \lrcorner\ \omega_i$. These functions constitute the 'hyper-Kähler moment mapping' for the 1-dimensional group action, but it is convenient to represent them by means of the 2-form $\zeta = \sum_{i=1}^{3} f_i\omega_i$, so that

$$d\zeta = \sum_{i=1}^{3} df_i \wedge \omega_i = \tfrac{1}{2} X \lrcorner\ \Omega$$

in terms of (1.4).

This process generalizes to the QK case as follows. First we identify V with the subbundle of $\bigwedge^2 T^*M$ with fibre isomorphic to $\mathfrak{sp}(1)$. A section ζ of the bundle V is called a 'twistor function' if

$$d\zeta = \tfrac{1}{2} X_\zeta \lrcorner\ \Omega, \tag{6.1}$$

for some vector field X_ζ. The terminology is taken from **[60]**, and the equation (6.1) is equivalent to the assertion that $\overline{D}_2\zeta = 0$, where $\overline{D}_2$ is the operator described in (3.6) using the Levi-Civita connection.

LEMMA 6.1. **[102]** *Let M be a QK manifold of dimension $4n \geq 8$ with non-zero scalar curvature. The mapping $\zeta \longmapsto X_\zeta$ establishes a bijective correspondence between the space of twistor functions and the space of Killing vector fields.*

The inverse mapping is obtained as follows. If X is a Killing vector field then at each point ∇X belongs to the subspace of $\operatorname{End} T$ determined by the holonomy algebra $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ **[71]**. Then, up to a universal constant, $(\nabla X)^V = s\zeta$, where the left-hand side is the component of ∇X in V. This works because the relevant component of the derivative of $(\nabla X)^V$ is proportional to X, thanks to the Ricci identity and Einstein condition.

Remark. It follows that, given a Killing vector field X on a QK 8-manifold,

$$(\sqrt{s}\,\zeta, X) \in \Sigma^2 \oplus \Lambda^1_0 \subset \Delta$$

is an eigenvector for the Dirac operator. Other eigensections of Δ are generated by vector fields X for which ∇X belongs to the subspace of $\operatorname{End} T$ isomorphic to $\bigwedge^2 E \subset \mathfrak{gl}(2,\mathbb{H})$. Such vector fields are non-isometric automorphisms of the quaternionic structure, and in the compact case exist only on $\mathbb{HP}^2$ [**4, 79**].

The fibre of V at $x \in M$ is naturally isomorphic to the space $H^0(Z_x, \mathcal{O}(2))$ of holomorphic sections of the restriction of the holomorphic line bundle L to the twistor line Z_x. In this way we obtain a mapping f from sections of V to sections of L over Z, and the following is a well-known example of the 'twistor transform':

LEMMA 6.2. [**102**] *The mapping f induces an isomorphism between the space of twistor functions and the space $H^0(Z, \mathcal{O}(L))^\sigma$ of σ-invariant holomorphic sections of L over Z.*

Consider the beginning of the long exact sequence

$$0 \to H^0(Z, \mathcal{O}(D)) \to H^0(Z, \mathcal{O}(TZ)) \leftrightarrows H^0(Z, \mathcal{O}(L)) \to \dots \tag{6.2}$$

associated to (5.1). It is known that map $Y \longmapsto Y \lrcorner\, \theta$ induces an isomorphism between the space of infinitesimal automorphisms of the contact structure and $H^0(Z, \mathcal{O}(L))$, and it follows that any σ-invariant automorphism of the contact structure arises from an isometry of M [**91**]. Such an isometry will, in turn, induce a holomorphic vector field on Z, and this process provides the indicated splitting of the sequence (6.2).

Let Y be a holomorphic vector field on Z preserving the contact structure, and let $s = Y \lrcorner\, \theta$ be the corresponding section of L. The latter defines a genuine function on U, and the equation

$$0 = \mathcal{L}_Y \theta = d(Y \lrcorner\, \theta) + Y \lrcorner\, d\theta = ds + Y \lrcorner\, \omega$$

allows us to interpret $-s$ as a holomorphic moment mapping on U. More invariantly, given $s \in \mathfrak{g}_c$, a moment mapping

$$\mu \colon U \to \mathfrak{g}_c^*$$

is defined by $\mu(u)(s) = s(u)$ for $u \in U$ and $s \in \mathfrak{g}_c$.

The mapping

$$\nu \colon Z \to \mathbb{P}(H^0(Z, \mathcal{O}(L))^*) \cong \mathbb{P}(\mathfrak{g}_c^*)$$

discussed in §5 may now be regarded as the projectivization of μ. The image $\nu(Z_x)$ of each fibre is determined by the restriction

$$i_* \colon H^0(Z, \mathcal{O}(L)) \to H^0(Z_x, \mathcal{O}(2)),$$

which corresponds (after complexification) to the mapping

$$Y \lrcorner\, \theta \mapsto (\nabla Y)_x^V \in V.$$

Suppose that the image of i_* is 3-dimensional for every $x \in M$, so that ν maps the conic Z_x to a conic in $\mathbb{P}(\mathfrak{g}_c^*)$. After dualizing, one then obtains a mapping $M \to \mathbb{G}\mathrm{r}_3(\mathfrak{g})$, and the situation may be summarized schematically:

$$
\begin{array}{ccc}
U & \xrightarrow{\mu} & \mathfrak{g}_c^* \\
\downarrow & & \downarrow \\
Z & \stackrel{\nu}{\dashrightarrow} & \mathbb{P}(\mathfrak{g}_c^*) \\
\downarrow & & \\
M & \dashrightarrow & \mathbb{G}\mathrm{r}_3(\mathfrak{g})
\end{array} \tag{6.3}
$$

Dotted arrows indicate mappings whose domain of definition may be a subset of that indicated.

We are now in a position to explain the fundamental link between quaternion-Kähler geometry and complex nilpotent orbits. Let G_c denote a complex semisimple Lie group and let G be a compact subgroup of G_c corresponding to a real form $\mathfrak{g}$ of its Lie algebra $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$. An element of $\mathfrak{g}_c$ is nilpotent if and only if it lies in the intersection of all invariant polynomials on $\mathfrak{g}_c$, and the set of such elements forms the 'nilpotent variety' $\mathcal{N}$. Suppose that M admits a group of isometries G that does not preserve any almost-complex structure that arises as a local section of Z. Roughly speaking this is a 'fullness' assumption for the way the group interacts with the quaternionic structure. In this case, L^k cannot have any G-invariant divisors for any k, and so $H^0(Z, \mathcal{O}(L^k))$ cannot contain any G-invariant elements. If follows that $\nu(Z) \subset \mathcal{N}/\mathbb{C}^*$.

The fullness condition is certainly satisfied if ν is an embedding (i.e. L is very ample), or more generally if $\dim \nu(Z) = 2n+1$. In these cases, it follows that $\nu(Z)$ is a nilpotent coadjoint orbit, and this leads to

THEOREM 6.3. **[16]** *If Z is a compact Fano twistor space and $\nu(Z)$ has the same dimension as Z, then M must be a Wolf space.*

Let us now consider the general nilpotent orbit. The algebra of invariant polynomials on $\mathfrak{g}$ is generated by a finite set $p_i \in S^{k_i}\mathfrak{g}^*$ with degrees $k_1, \dots, k_r$, and we may take $k_1 = 2$ and p_1 to be the Killing form. Thus, any nilpotent element of $\mathfrak{g}_c$ must be isotropic or 'null' relative to the Killing form, and if $\mathfrak{g} = \mathfrak{su}(2)$ this condition is of course sufficient. More generally, fix a non-zero homomorphism $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$. Then for any $V \in L_\rho$, it is easy to see that isotropic elements of V_c are nilpotent not just in $V_c \cong \mathfrak{sl}(2, \mathbb{C})$, but also in $\mathfrak{g}_c$. Thus, isotropic elements of V_c belong to the G_c-orbit

$$U_\rho = \{\mathrm{Ad}(g)(\xi) : g \in G_c\}, \tag{6.4}$$

where $\xi = \rho(A_1 + iA_2)$ in the notation of (2.2). Conversely, any nilpotent orbit in $\mathfrak{g}_c$ may be written in the form (6.4) for some element ξ arising from a real homomorphism $\mathfrak{su}(2) \to \mathfrak{g}$ **[72]**.

The fact that $\xi \in U_\rho$ if and only if $\lambda\xi \in U_\rho$ for any non-zero complex number λ (infinitesimally, it is actually true that $\xi \in (\mathrm{ad}(e))^2\mathfrak{g}_c)$ ensures that the projectivized nilpotent orbit

$$Z_\rho = U_\rho/\mathbb{C}^*$$

is defined as a submanifold of $\mathbb{P}(\mathfrak{g}_c)$. Now, U_ρ is equipped with the Kostant-Kirillov (holomorphic) symplectic form ω, and has even complex dimension. A contact 1-form θ is induced on Z_ρ for which the quotient line bundle L over Z is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}(\mathfrak{g}_c)$.

Consider the quadric

$$Q = \{[v] \in \mathbb{P}(\mathfrak{g}_c) : \langle v, v\rangle = 0\}$$

of isotropic elements, itself isomorphic to the Grassmannian $\mathbb{G}\mathrm{r}_2(\mathfrak{g})$. Given $V \in \mathbb{G}\mathrm{r}_3(\mathfrak{g})$, we let C_V denote the conic $\mathbb{P}(V_c) \cap Q$. Recall Theorem 2.4. We have already shown that if $V \in L_\rho$ then $C_V \subset Z_\rho$, and this is also valid for $V \in M_\rho$ for the following reasons. Let $\xi = v_1 + iv_2$ be an isotropic vector in $V \in M_\rho \setminus L_\rho$, where $\{v_1, v_2, v_3\}$ is an oriented orthonormal basis of V. It follows that

$$\begin{aligned} \nabla\psi\big|_V(\xi) &= (-[v_2, v_3] - \psi(V)v_1) + i(-[v_3, v_1] - \psi(V)A_v) \\ &= i[v_1 + iv_2, v_3] - \psi(V)(v_1 + iv_2), \end{aligned}$$

belongs to $T_\xi U_\rho$. This means that tangent vectors to the flow lines of ψ preserve the nilpotency property to first order at all points of $M_\rho \setminus L_\rho$. It is also easy to check that for all $\alpha \in T_V\mathbb{G}\mathrm{r}_3(\mathfrak{g})$, we have $\alpha \in T_0 \oplus T_+$ if and only if $\alpha(\xi) \in \mathrm{ad}(\xi)\mathfrak{g}_c$ for all isotropic elements $\xi \in V_c$.

Let $\tilde{Z}_\rho$ denote the tautological bundle over M_ρ whose fibre at a point $V \in M_\rho$ is the conic C_V, so that there is a natural mapping $f\colon \tilde{Z}_\rho \to Z_\rho$. Since $C = C_V$ satisfies (4.1), it belongs to a complex $4n$-dimensional family of projective lines all of the form C_V with $V \in \mathbb{G}\mathrm{r}_3(\mathfrak{g})$. We may restrict to a family of real dimension $4n$ by considering only those lines which are invariant by the real structure σ of $\mathfrak{g}_c$ which preserves Z_ρ. It follows that $f(Z_\rho)$ is an open subset of Z_ρ. On the other hand, the Morse theory implies that $f(\tilde{Z}_\rho)$ is closed, since if $\{V_n\}$ is a sequence in M_ρ converging to $V_\infty \in \mathbb{G}\mathrm{r}_3(\mathfrak{g})$, then the flow line containing V_∞ emanates from a critical manifold L'_ρ with $\psi|L'_\rho \geq \psi|L$. It is now easy to show that f is bijective, and this analysis leads to the following characterization of Z_ρ as a twistor space.

THEOREM 6.4. [**114**] *Let $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ be a non-zero homomorphism. There is a fibration $\pi\colon Z_\rho \to M_\rho$ such that*
(i) $\pi^{-1}(V) = C_V$;
(ii) C_V has normal bundle $N_V \cong 2n\mathcal{O}(1)$ where $2n$ is its rank;
(iii) C_V is transverse to the contact distribution.

The only known proof of (ii) relies on Kronheimer's analysis of the nilpotent orbit U_ρ [**75**], though it is instructive to indicate its validity for $V \in L_\rho$. Suppose that Z_ρ has complex dimension $2n+1$. The fibre of N_V at $[\xi] \in C_V$ is obtained by applying ξ to the right-hand side of (2.6). Decomposing each space $\mathrm{ad}(\xi)\Sigma^k$ in (2.6) into weights, it follows that

$$\deg(N_V) = \sum_{k\geq 1} k\mu_k = 2n,$$

and we may write $N_V \cong \bigoplus_{i=1}^{2n} \mathcal{O}(k_i)$ with $\sum_{i=1}^{2n} k_i = 2n$. The existence of holomorphic vector fields pointing in all directions at each point of Z_ρ implies that $k_i \geq 0$. Since C_V is a conic,

$$\left.\frac{T\mathbb{P}(\mathfrak{g}\mathbb{C})}{T\mathbb{P}(V\mathbb{C})}\right|_Z \cong V^\perp \otimes \mathcal{O}(2),$$

and this vector bundle contains N_V as a subbundle. Because $\mathcal{O}(-k_i+2)$ contains no non-zero holomorphic sections if $-k_i+2<0$, it follows that $k_i\le 2$. Moreover $k_i=2$ for some i if and only if N_V has a constant section determined by a non-zero element in $\bigcap_{[\xi]\in C_V} \mathrm{ad}(\xi)\mathfrak{g}_c$, but this intersection is easily seen to be zero.

Remark. Given ρ, there exists a complex flag manifold $F_\rho = G_c/P$ and a 'canonical fibration'

$$f_\rho\colon Z_\rho = U_\rho/\mathbb{C}^* \to F_\rho$$

with the property that the fibres of f are tangent to the contact distribution of Z_ρ. This mapping gives rise to a contact map $U_\rho/\mathbb{C}^* \to \mathbb{P}(T^*F)$ that was exploited by Kobak [**67**] in the study of harmonic maps. This construction is closely related to the Springer resolution of the nilpotent variety of $\mathfrak{g}_c$.

7. Divisors and Quotients

To begin this section, let M be a (not necessarily compact) 4-dimensional manifold with an anti-self-dual conformal structure, so $W_+=0$. This is equivalent to saying that M is a 1-dimensional quaternionic manifold, as the assumption imples that the twistor space Z is a complex manifold with a holomorphic line bundle L satisfying $L^2\cong\kappa^*$.

Suppose that M has a complex structure I, and let D_I denote the divisor of Z formed from the disjoint union of the sections I and $-I$.

THEOREM 7.1. [**97**] *$D_I\in|L|$ if and only if the conformal class contains a Kähler metric.*

Such a metric has zero scalar curvature by standard curvature properties, and is therefore 'scalar-flat Kähler' (SFK). This theorem leads to a characterization of anti-self-dual Hermitian surfaces [**24**].

Now suppose that M is a QK manifold of dimension $4n\ge 4$. From the point of view of §6, the relevance of Theorem 7.1 is that a non-zero Killing vector field X gives rise to a such a divisor of L. The corresponding complex structure I_ζ is the one whose 2-form is $\zeta/|\zeta|$, and is defined away from the zero set

$$M_0=\{x\in M:\zeta(x)=0\}.$$

If $x\in M\setminus M_0$ then D intersects Z_x in two points corresponding to $\pm I_\zeta$. To check explicitly that I_ζ is integrable, extend $I_\zeta=I_1$ to an orthonormal basis $\{I_1,I_2,I_3\}$ of V. In terms of (1.2), the condition (6.1) tells us that $\alpha_i=I_i\alpha$ for $i=1,2$ and some 1-form α. This implies that

$$\nabla_{I_\zeta X}=I_\zeta\nabla_X I_\zeta,$$

which is indeed the condition that I_ζ be a complex structure.

Suppose that M is a QK manifold with an action by S^1 preserving the quaternionic structure. Let $X=X_\zeta$ denote the corresponding Killing vector field, and let f denote the S^1-equivariant function $\|\zeta\|^2$. It follows from (6.1) that $df=2\,X\lrcorner\,\zeta$, or

$$\frac{1}{\|\zeta\|}\mathrm{grad}\, f = I_\zeta X. \tag{7.1}$$

The various moment equations can be neatly encapsulated in the statement that, if t is a formal variable that behaves like a closed 2-form with $X \lrcorner\ t = 0$, then

$$\hat{\Omega} = \Omega - 2\zeta t + f t^2$$

satisfies $d\hat{\Omega} + (X \lrcorner\ \hat{\Omega})t = 0$. This means that $\hat{\Omega}$ is an equivariantly closed extension of Ω, and the situation is analogous to that of Kähler geometry [**14, 7**].

From (7.1), the set of critical points of f consists of the union of the zero set M_0 and the set

$$M^{S^1} = \{x \in M : X(x) = 0\}$$

of fixed points of the S^1-action. If $M_0 \cap M^{S^1} = \emptyset$, so that S^1 acts freely on M_0, then the result of Galicki-Lawson (Theorem 7.3 below) implies that $Q = M_0/S^1$ is a QK manifold. The gradient flow therefore determines a diagram

$$Q \leftarrow M_c/S^1 \rightarrow M^{S^1}, \tag{7.2}$$

where $M_c = f^{-1}(c)$, c is less than the first critical value of f, and every connected component of the fixed point set M^{S^1} is a Kähler submanifold of M. The Hermitian manifold $(M \setminus M_0, g, I_z)$ has been studied in [**5**], and M_c/S^1 may be regarded as a Kähler quotient of it. The correspondence (7.2) realizes M as a 'vehicle' for the abstract geometry associated to the quotient Q, and can also be used to compute the topology of Q using the methods of [**66**]. This and other applications of Morse theory to quaternion-Kähler geometry discussed in this section, are due to Battaglia [**14**].

Example. The only complete example of this construction arises from the action of S^1 on $\mathbb{H}^{n+1}$ by left multiplication by e^{it} which we now examine in some detail. The commutator of S^1 in $Sp(n+1)$ is $U(n+1)$, and we may regard $\mathbb{H}^{n+1}$ is the real vector space underlying $\mathbb{C}^{n+1} \otimes \mathbb{C}^2$ by writing a quaternionic vector as $a + bj$ with $a, b \in \mathbb{C}^{n+1}$. The action of $\mathbb{H}^*$ on $\mathbb{C}^2$ extends to an action of $GL(2,\mathbb{C})$ commuting with $U(n+1)$, and this action is free outside the set S of simple elements (those with rank less than 2) in the tensor product. The mapping $a + bj \longmapsto \langle a, b\rangle$ defines an isomorphism

$$(\mathbb{H}^{n+1} \setminus S)/GL(2,\mathbb{C}) \cong \mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+1}),$$

and it follows that $\mathbb{HP}^n \setminus \mathbb{CP}^n$ is the total space of a bundle over $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+1})$ with fibres isomorphic to $GL(2,\mathbb{C})/\mathbb{H}^* \cong \mathbb{HP}^1 \setminus \mathbb{CP}^1$. The 2-form ζ on $\mathbb{HP}^n$ can be identified with the mapping

$$a + bj \longmapsto \overline{(a+bj)}i(a+bj) = (\|a\|^2 - \|b\|^2)i + 2\overline{a}b\,k,$$

and the fibre over a 2-plane $\Pi \in \mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+1})$ intersects M_0 in the circle $U(2)/Sp(1)$ each point of which is represented by a unitary basis of Π. Thus M_0/S^1 is also isomorphic to $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+1})$.

This example illustrates the way in which a quotient by a non-compact group may be replaced by a compact quotient. If X is the Killing vector field corresponding to the S^1 action on $\mathbb{HP}^n$, then the tangent vectors I_1X, I_2X, I_3X defined at any given point arise from a basis of $\mathfrak{sl}(2,\mathbb{C})/\mathfrak{su}(2)$, and are always orthogonal to M_0. In general one does not have the luxury of an $SL(2,\mathbb{C})$ action, though lifting to the twistor space does enable one to complexify an S^1 action. In the example above, $M^{S^1} = \mathbb{CP}^n$ and M_c/S^1 can be identified with the flag manifold

$U(n+1)/(U(1)\times U(1)\times U(n))$. In general one might conjecture that M_c/S^1 is isomorphic to the twistor space of Q.

THEOREM 7.2. **[14]** *If S^1 acts on a positive QK manifold M and the action is free on M_0, then M is isometric to $\mathbb{HP}^n$.*

Moreover, it is effectively a consequence of Theorem 5.5(ii) that $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ is the only positive QK manifold that can be obtained as a QK quotient by a circle action **[13]**.

Passing to the more general case of a group action, suppose that G is a connected Lie group acting on a QK manifold M as a group of isometries. For each point $x\in M$ we obtain a mapping $\mathfrak{g}\to V$. This gives rise to a 'moment section' $\gamma\in\Gamma(M,\mathfrak{g}^*\otimes V)$ which is G-equivariant.

THEOREM 7.3. **[47]** *If G acts freely on the zero set M_0 of γ then $Q=M_0/G$ is a QK manifold.*

This quotient construction commutes with the hyper-Kähler quotient construction on U, in the sense that the bundle associated to Q with fibre $\mathbb{H}^*/\mathbb{Z}_2$ can be realized as an HK quotient of U **[113]**. There is also a corresponding notion of 'contact' quotient construction on Z, enabling one to extend the commutativity of (6.3) to quotients as well as embeddings. Incidentally, the paper **[52]** describes an example of a holomorphic contact orbifold obtained as a Kähler S^1-quotient, and used to compactify a moduli space of Higgs bundles.

Remark. The quaternionic quotient construction arises naturally in relation to the decomposition of a tensor product into irreducible components. Let G be a compact Lie group with complexification G_c. Suppose that W is a complex $(2n+2)$-dimensional vector space upon which G acts linearly, commuting with an antilinear involution $j\colon W\to W$ and an identification $W^*\cong W$. The action $\mathfrak{g}\otimes W\to W$ induces a G-equivariant mapping

$$\mathfrak{p}\colon W\otimes W\to\mathfrak{g}^*,$$

which is best regarded as projection to a summand of the tensor product. Let

$$W_0=\{w\in W:\mathfrak{p}(w\otimes w)=0=\mathfrak{p}(w\otimes jw)\}$$

(the notation is taken from **[103]**). Then the quotient by G of the flat hyper-Kähler structure on W is W_0/G, and the QK quotient of $\mathbb{P}(W)\cong\mathbb{HP}^n$ is $W_0/(G\times\mathbb{H}^*)$. In this set-up, the function $f(w)=i\mathfrak{p}(w\otimes jw)$ is a real-valued moment mapping for the action of G on the Kähler submanifold $W_0'=\{w\in W:\mathfrak{p}(w\otimes w)=0\}$, and the curves $\exp(itA)$, with $A\in\mathfrak{g}$, are trajèctories the gradient flow of $\|f\|^2$. It follows from the theory of **[66]** that $W_0/G=f^{-1}(0)/G$ can be identified with the geometrical invariant theory quotient of W_0'.

Less standard actions of S^1 give rise to a host of quaternion-Kähler orbifolds. Starting from $\mathbb{HP}^2$, an analogous but more general procedure produces self-dual Einstein metrics, for example on the weighted projective planes

$$\mathbb{CP}^2_{1,1,k}=\frac{\mathbb{C}^3\setminus\{0\}}{(x,y,z)\longmapsto(\lambda x,\lambda y,\lambda^k z)},\quad\lambda\in\mathbb{C}^*.$$

This orbifold has $(0,0,1)$ as a unique singular point, and may be viewed as a 1-point compactification of the total space of the line bundle L^{-k} over $\mathbb{CP}^1$. In **[60]**, Joyce shows that it admits a $U(2)$-invariant twistor function characterizing the SFK metric constructed by LeBrun in [**76**]. He also shows that formal combinations of such metrics can be then deformed to self-dual metrics on connected sums $n\mathbb{CP}^2$.

Although complete metrics with negative scalar curvature exist [**78**], singularities seem inevitable in the positive case. From the quotient point of view, these arise when G does not act freely on M_0. An exhaustive study of the resulting singularities has been given in [**37**].

Example. The action of S^1 on the Wolf space $M = \mathbb{G}\mathrm{r}_4(\mathbb{R}^7)$ arising from inclusions $S^1 \subset U(3) \subset SO(7)$ has been described in [**68**]. The fixed point set M^{S^1} consists of two copies of $\mathbb{CP}^2$, one of which lies in M_0. In fact, the quotient M_0/S^1 can be identified with the locally-symmetric singular space $\mathbb{Z}_3\backslash G_2/SO(4)/\mathbb{Z}_3$ which is itself an $\mathbb{H}^*$-quotient of the nilpotent variety $\{A \in \mathfrak{sl}(3,\mathbb{C}) : A^3 = 0\}$.

Let M be an arbitrary QK manifold. The subgroup S^1 of $SO(3)$ fixing the complex structure I_1 on the associated bundle U gives rise to a moment mapping $\mu_1 : U \to \mathbb{R}$ which equals the square of radial distance measured on the fibres of U over M. Then

$$Z \cong \mu_1^{-1}(1)/S^1$$

may be regarded as a Kähler quotient of U. The Kähler form of Z can be used to construct a Kähler quotient of Z, which provides an unexplored link between quaternion-Kähler and Kähler geometry. For example the action of S^1 on an open set of a QK manifold M gives rise to a Kähler metric on a space of the same real dimension.

One can always define a 'quadratic moment mapping'

$$F : M \to S^2\mathfrak{g}^*$$

by $F(x)(A,B) = \langle \zeta_A(x), \zeta_B(x)\rangle$, for $A,B \in \mathfrak{g}$. This is the generalization of the function f defined for an S^1 action above, and can also be regarded as the $Sp(1)$-invariant component of the hyper-Kähler moment mapping on the associated bundle U. A computation of the derivative of F shows that it will be an immersion if, at each point x, γ has rank 3 and the quaternionic span of the tangent space to the orbit $G(x)$ equals T_xM. In general, $S^2\mathfrak{g}^*$ will decompose into a number of irreducible G-modules U_i, and we may write $F(x) = \sum F_i(x)$, with $F_i(x) \in U_i$. One of the U_i is the 1-dimensional space spanned by the Killing form of $\mathfrak{g}$, and there will be other trivial summands if and only if G is not simple. The orbit $G(x)$ will then fibre over each of the orbits $G(F_i(x))$ with $F_i(x) \neq 0$. For example, if G is not simple, a component of F determines a G-equivariant mapping $M \to \mathfrak{g}^*$, and the orbits of G will all fibre over coadjoint ones.

Example. Let $M = \mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4})$ and $G = SO(n+4)$. If $\mathfrak{g} = \mathfrak{so}(n+4)$ then the decomposition of $S^2\mathfrak{g}$ is well known from the theory of curvature tensors; we may write

$$S^2\mathfrak{g} \cong W \oplus \Lambda^4 \oplus S^2_0 \oplus \mathbb{R}.$$

The component of F in $\mathbb{R}$ is the norm of the moment mapping, and must be constant given that G acts transitively on M. Since the stabilizer of $F(m)$ must contain the stabilizer of m, we may deduce that $F(M) \subset \Lambda^4 \oplus \mathbb{R}$, and ignoring the $\mathbb{R}$-component, F is a Plücker embedding.

We remarked in §2 that the Wolf space $\mathbb{G}\mathrm{r}_4(\mathbb{R}^8)$ has some special properties. Each point of the twistor space Z of its non-compact dual defines a null line in the complex quadric Q^4 (in $\mathbb{CP}^5$), which is a compactification and complexification of Minkowski space. This is the basic example of a phenomenon discovered in **[77]** that associates to any real analytic conformal manifold N of signature $(3, n-1)$ a QK $4n$-manifold M with $s < 0$. The twistor space of M is an open set of the space of null geodesics of a complexification of N.

8. Characteristic Classes and Constraints

Suppose that M is a QK manifold for which (partly for convenience of exposition) $\varepsilon = 0$, so that M has a distinguished principal G-bundle P with $G = Sp(n) \times Sp(1)$. Then P may be regarded as the pullback of the universal bundle E_G by a suitable map f to the classifying space $B_G = E_G/G$. Characteristic classes on M arise by pulling back elements of the cohomology ring $H^*(B_G)$, which is identified with the subspace of W-invariant elements of $H^*(B_T)$ where T is a maximal torus of G and $W = N(T)/T$ is the Weyl group.

If $x_1, \ldots, x_n$ are coordinates on the Lie algebra $\mathfrak{sp}(n)$ and y is a coordinate on $\mathfrak{sp}(1)$, then W permutes the former and changes an arbitrary number of signs, so that $|W| = 2^{n+1}n!$. We may interpret the symbols $x_1, \ldots, x_n, y$ as cohomology classes on B_T by means of the natural isomorphism $H^1(T) \cong H^2(B_T)$, and $H^*(B_G)$ is generated by the elementary polynomials

$$c_{2k} = \sum_{r_1 < r_2 < \cdots < r_k} {x_{r_1}}^2 {x_{r_2}}^2 \cdots {x_{r_k}}^2, \qquad k = 1, \ldots, n,$$
$$u = -y^2.$$

Then $f^*c_2, \ldots, f^*c_{2n}$ are the Chern classes of the vector bundle E, and f^*u is minus the 2nd Chern class of H. Moreover, these classes are well defined as elements of $H^*(M, \mathbb{Q})$ even when the structure of M does not lift to G. For example, $4f^*u$ is always integral and represented in de Rham cohomology by a constant multiple of the 4-form Ω. In practice it is often convenient to replace the Chern classes of E by the Pontrjagin classes of M. The kth such class equals f^*p_k (for $1 \le k \le n$) where p_k is the polynomial of degree $2k$ defined by

$$\sum_{k=1}^{2n} (-1)^k p_k = \prod_{r=1}^{2n} (1 - {X_r}^2).$$

An irreducible G-module V decomposes under T as the direct sum of 1-dimensional spaces V_λ, where the weight $\lambda \in \mathfrak{t}^*$ describes the eigenvalues of $(e^{ix_1}, \ldots, e^{ix_1}, e^{iy})$. The representation ring $R[T]$ is thus the polynomial algebra generated by $e^{\pm x_1}, \ldots, e^{\pm x_n}$, $e^{\pm y}$, and restriction to T defines an injective ring homomorphism

$$\mathrm{ch}\colon R[G] \to \mathbb{C}[[x_1, \ldots, x_n, y]],$$

whose image is contained in the space of W-invariant formal power series. The usual Chern character of the vector bundle $P \times_G V$ is simply $f^*\mathrm{ch}(V)$.

As an illustration, first consider

$$\begin{aligned} \mathrm{ch}(E \otimes H) &= (2n - c_2 + \tfrac{1}{12}(c_2{}^2 - 2c_4) + \cdots)(2 + u + \tfrac{1}{12}u^2 + \cdots) \\ &= 4n + 2(nu - c_2) + \tfrac{1}{6}(c_2{}^2 - 2c_4 - 6c_2u + nu^2) + \cdots \end{aligned}$$

and it follows that

$$\begin{aligned} p_1 &= 2(nu - c_2), \\ p_2 &= {c_2}^2 + 2c_4 + (6-4n)c_2 u + (2n-1)nu^2. \end{aligned}$$

The Euler class of the oriented real vector bundle TM is the pullback of the class

$$e = \prod_{r=1}^{2n} X_r = \prod_{r=1}^{n} ({x_r}^2 - y^2) = \sum_{r=0}^{n} c_{2r} u^{n-r} \tag{8.1}$$

on B_G. The last term coincides with the Chern class c_{2n} of the virtual vector bundle $E - H$. In this sense, the lemma is analogous to the fact that the Euler class e underlying a complex vector bundle V is its top Chern class.

Example. Over the projective space $\mathbb{HP}^n$, the direct sum $E \oplus H$ can be identified with the trivial bundle with fibre $\mathbb{H}^{n+1}$ with H the tautological line bundle and E its complement. Hence

$$(1 + c_2 + c_4 + \cdots + c_{2n})(1-u) = 1,$$

$c_{2k} = u^k$ and $e = (n+1)u^n$. Since $\chi(\mathbb{HP}^n) = n+1$, we deduce that $u^n = 1$. On an arbitrary compact QK manifold, with our choice of orientation, all one knows is that the 'quaternionic volume'

$$v(M) = \langle (4u)^n, [M] \rangle$$

is a positive integer, and $\deg(Z) = 2v(M)$. A known estimate for the Chern number c_1^{2n+1} on the twistor space Z [**80**] implies that $v(M) \le 2v(\mathbb{HP}^n) = 2 \cdot 4^n$. It is also known that the integer

$$v'(M) = (n-1)v(M) - 2\left\langle p_1 (4u)^{n-1}, [M] \right\rangle \tag{8.2}$$

is non-negative, with equality if and only if M is $\mathbb{HP}^n$ [**102**].

Let

$$D \colon \Gamma(M, \Delta_+) \to \Gamma(M, \Delta_-) \tag{8.3}$$

denote the Dirac operator described in §3. If $V \in R[G]$, the Dirac operator may be coupled to the corresponding virtual vector bundle in order to define the index

$$\operatorname{ind}(V) = \dim \ker D - \dim \operatorname{coker} D.$$

The Atiyah-Singer index theorem implies that

$$\operatorname{ind}(V) = \left\langle \operatorname{ch}(V) \hat{A}(TM) \right\rangle,$$

where angular brackets indicate the evaluation or 'integration' of a cohomology class on the fundamental cycle $[M]$. The $\hat{A}$ class is defined by

$$\begin{aligned} \hat{A} = \prod_{r=1}^{2n} \frac{\frac{1}{2}X_r}{\sinh(\frac{1}{2}X_r)} &= 1 + \hat{A}_1 + \hat{A}_2 + \cdots \\ &= 1 - \tfrac{1}{24}p_1 + \tfrac{1}{45\Delta 2^{\,}}(7{p_1}^2 - 4p_2) + \cdots \end{aligned} \tag{8.4}$$

To present the results of this section in a more general context, we shall first consider the case of a compact oriented 8-dimensional Riemannian 8-manifold M with a holonomy group K of rank less than 4. If M is neither locally reducible nor symmetric, then K has to be one of the groups

K	$Sp(2)Sp(1)$	$Spin(7)$	$SU(4)$	$Sp(2)$
k	0	1	2	3

In all cases M is a spin manifold. The last three groups correspond to Ricci-flat geometries, and the integer k is the dimension of the space of parallel spinors [**117**], which we may arrange to equal $\dim\ker D$ in (8.3).

The existence of the $*$ operator on 4-forms allows one to decompose $b_4 = b^+ - b^-$, where

$$b^{\pm} = \dim\{\sigma \in \gamma(M, \bigwedge^4 T^*M) : *\sigma = \pm\sigma,\ d\sigma = 0\}.$$

The Hirzebruch signature theorem implies that

$$b^+ - b^- = \tfrac{1}{45}(7p_2 - {p_1}^2),$$

and the Atiyah-Singer index theorem asserts that k is equal to the $\hat{A}$ genus $\hat{A}_2$ given in (8.4). Using the extra equation

$$4p_2 - {p_1}^2 = 8\chi$$

that follows merely from the topological reduction (see for example (8.1)), we obtain

PROPOSITION 8.1. [**62**] $k = \frac{1}{24}(b^+ - 2b^- + b_3 - b_2 + b_1 - 1)$.

It follows that the Betti numbers are 'relatively large' in the Ricci-flat cases. Indeed, if M is an irreducible HK manifold so that $H = Sp(2)$ then $b_1 = 0$ and Hodge theory implies that $b^- = 3h_0^{1,1} = 3(b_2 - 3)$. It follows that

$$b_3 + b_4 = 10b_2 + 46, \tag{8.5}$$

and the right-hand side is at least 76. By contrast, if $H = Sp(2)Sp(1)$ and $s > 0$, then b_1, b_3, b^- all vanish, and

$$b_4 = 1 + b_2. \tag{8.6}$$

This equation plays a crucial role in proofs that M must in fact be one of the symmetric spaces $\mathbb{Gr}_2(\mathbb{C}^4)$, $G_2/SO(4)$, $\mathbb{HP}^2$ (recall Theorem 2.1). If M is not the complex Grassmannian, then $b_4 = 1$ which gives an extra relation between characteristic classes.

It is rather surprising that the equations (8.5),(8.6) have non-trivial generalizations to higher dimensions:

THEOREM 8.2. [**108, 80**] *Let M be a compact Riemannian manifold of dimension $4n$ with holonomy group K and scalar curvature s.*

(i) If $K \subseteq Sp(n)$ (so that $s = 0$) then $n\chi = 6\sum_{i=1}^{2n}(-1)^i i^2 b_{2n-i}$.

(ii) If $K \subseteq Sp(n)Sp(1)$ and $s > 0$ then $\sum_{i=0}^{n-1}[6i(n-1-i) - (n-1)(n-3)]b_{2i} = \frac{1}{2}n(n-1)b_{2n}$.

The more attractive form of the first equation reflects the fact that products of HK manifolds remain HK. Quaternion-kähler manifolds enjoy no such functorial properties, though the second equation can be expressed in the more memorable form

$$\sum_{i=1}^{\lfloor m/2 \rfloor} i(m-i)(m-2i)\gamma_i = 0, \tag{8.7}$$

by setting $m = n+1$, $\gamma_i = \beta_{2i} - \beta_{2m-2i}$ and $\beta_{2i} = b_{2i} - b_{2i-4}$ (see (1.6) and [**49**]).

Remark. Fix k with $1 \le k \le n$. There is a non-trivial action of $\mathbb{Z}_2$ on $\mathbb{HP}^n$ induced by changing signs of k of the coordinates of $\mathbb{H}^{n+1}$. Work on other manifolds with exceptional holonomy suggest that in assessing the relevance of topological constraints to orbifolds like $N = \mathbb{HP}^n/\mathbb{Z}_2$ it is more relevant to consider the 'string-theoretic' Poincaré polynomial $P_{st}(t)$ formed by adding contributions for the fixed point set. The latter is a disjoint union $\mathbb{HP}^{k-1} \sqcup \mathbb{HP}^{n+1-k}$, so

$$P_{st}(t) = Q_n + t^{2k}Q_{n-k} + t^{2n-2k+2}Q_{k-1},$$

where $Q_n = \sum_{i=0}^{n} t^{4i}$. It follows that, in the above notation, N has

$$\beta_{2i} = \begin{cases} 1, & i = k \text{ or } n+1-k, \\ 0, & \text{otherwise.} \end{cases}$$

and $\gamma_{2i} = 0$ for all i. This provides some justification for the symmetry in (8.7).

The complicated nature of $\hat{A}_k$ for $k \ge 2$ leads one to seek virtual representations V for which $\mathrm{ch}(V)$ has only terms of near top dimension. To make this precise, let $\mathcal{F}_k$ denote the space spanned by virtual representations V of $Sp(n)Sp(1)$ such that $\mathrm{ch}(V)$ has no terms of degree less than $2k$. The sequence of ideals $\cdots \supset \mathcal{F}_k \supset \mathcal{F}_{k+1} \supset \cdots$ is an example of the γ-filtration that is used to derive the graded structure of cohomology from K-theory [**46**]. It is easy to describe the restriction of this filtration to $R[Sp(1)]$. Since ch is a ring homomorphism and $\mathcal{F}_j\mathcal{F}_k \subset \mathcal{F}_{j+k}$, we have $(\Sigma^2 - 3)^k \in \mathcal{F}_k$ for any k, and in fact $(\Sigma^2 - 3)^k$ generates $\mathcal{F}_k \cap R[Sp(1)]$. Expanding both sides of the equation

$$\mathrm{ch}((\Sigma^2 - 3)^{n+k}) = (4u + \tfrac{4}{3}u^2 + \cdots)^{n+k}$$

and recalling (5.5) and (8.2) yields

PROPOSITION 8.3. [**53**] *Let M be a compact QK manifold of dimension $8m$. Then*

$$\sum_{i=0}^{m+k} (-1)^i \binom{4m+2k+1}{i} r_{m+k-i} = \begin{cases} \frac{1}{48}(v' + 3(n-1)v), & k = -1, \\ v, & k = 0, \\ 0, & k \ge 1. \end{cases}$$

An analogous result holds when M has odd quaternionic dimension.

Example. Let $n = 2m = 4$. The Hilbert polynomial is completely determined by the dimension $r = r_1$ of the isometry group and the quaternionic volume v. Combining the formula for $k = -1$ with (5.4) gives

$$7 + \tfrac{3}{16}v \le r \le 2v + 9.$$

For example, $r = 11$ and $v = 2$ gives the values (5.7) in §5, though the resulting polarized variety with $\Delta = 2$ must be smooth and has topology inconsistent with that of a twistor space.

A deeper analysis of the filtration $\mathcal{F}_k$ is used in the proof of Theorem 8.2. This relies on the existence of $W_n \in R[G]$ such that

$$\mathrm{ch}(W_n) = y(1 + \tfrac{1}{24}p_1),$$

where y has degree $4n-4$, ensuring that $\mathrm{ind}(W_n)$ vanishes [**107**]. The element W_n is generated from representations of the form $R^{p,q} = \Lambda_0^p \otimes \Sigma^q$, and the associated vector bundles may be coupled to the Dirac operator provided $p+q+n$ is even. The associated indices

$$i^{p,q} = \mathrm{ind} D(R^{p,q})$$

play a fundamental role.

The index of the Dirac operator on M coupled to Σ^q is equal to that of the Dolbeault complex on Z coupled to the line bundle $\mathcal{O}(q-n)$, whence

$$i^{p,q} = \chi(Z, \mathcal{O}(\pi^* \Lambda_0^p(q-n)).$$

Using $h^{p,p}(Z) = b_{2p-2} + b_{2p}$ (with $b_{-2} = 0$), Lemma 5.5 implies that on a positive QK manifold of dimension $4n$,

$$i^{p,q} = \begin{cases} 0, & n = p+q+2r,\ r > 0, \\ (-1)^p(b_{2p-2} + b_{2p}), & n = p+q, \\ r_{(q-n)/2}, & p = 0,\ q \geq n. \end{cases}$$

Observe that $i^{0,n+2} = r_1$ is the dimension of the isometry group. The next result is the generalization of (2.1) to the case of an abstract QK manifold, and is proved in the same way as Theorem 8.3:

PROPOSITION 8.4. **[41]** $\sum\limits_{p=0}^{n} (-1)^p i^{p,n+2-p} = 2\chi + b_{2n-2} + b_{2n}$.

It follows from **[79]** that $i^{1,n+1} = 0$ unless $M = \mathbb{HP}^n$. It is an interesting problem to understand the indices $i^{p,n-p+1}$ for $p \geq 2$, and identify the cohomology spaces that might contribute to their non-vanishing.

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