Ricci Flow and Einstein Metrics in Low Dimensions

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1. Introduction

The purpose of this essay is to give an expository account of Hamilton's work during the 1980's on the Ricci flow on surfaces [**21**], 3-manifolds [**19**], and 4-manifolds [**20**]. We have restricted our attention to these papers both because of our personal familiarity with them, and because they deal directly with constructing Einstein metrics — in fact, constant sectional curvature metrics.

The study of the Ricci flow began with Hamilton's seminal 1982 paper 'Three-manifolds with positive Ricci curvature.' In this paper he not only introduced the notion of the Ricci flow, but applied it to classify closed 3-manifolds with positive Ricci curvature. Later, in another very important 1986 paper 'Four-manifolds with positive curvature operator,' Hamilton extended his methods to show that closed 4-manifolds with positive curvature operator are topologically either S^4 or $\mathbb{RP}^4$. An important development in this paper is the use of the 'weak and strong maximum principles for systems,' which enabled Hamilton to also classify both 3-manifolds with nonnegative Ricci curvature and 4-manifolds with nonnegative curvature operator. Furthermore, Hamilton also greatly simplified the computations in his original paper. The last of Hamilton's papers in the 80's on the Ricci flow appeared in 1988 and was entitled 'The Ricci flow on surfaces.' Here Hamilton proved for any initial metric on a surface the convergence of the Ricci flow to a constant curvature metric (except for the case of a metric on the 2-sphere with variable signed curvature, a condition which was later removed by similar methods.) The real jewel of this paper is the techniques that Hamilton introduced in this paper. In particular, he obtained both a 'Harnack estimate' and an 'entropy estimate.' The Harnack estimate is especially important in the analysis of singularities (see [**27**].)

The rest of this paper is organized as follows. In section 2, we describe the basic facts about the Ricci flow. Then in section 3 we quickly review the maximum principle, which is the main tool used in the study of the Ricci flow. In sections 4 through 6, we devote one section each to Hamilton's results on surfaces, 3-manifolds, and 4-manifolds. Finally, in the last section we reference some of the other important works on the Ricci flow.

2. Basic facts about the Ricci flow

2.1. The equation. The Ricci flow is a nonlinear heat equation which deforms metrics in the direction of minus the Ricci tensor. Let M be a differentiable manifold. A family of Riemannian metrics $g(t)$, $t \in [0,T)$, where $T \in (0,\infty]$, is called a solution to the *Ricci flow* if

$$\frac{\partial g}{\partial t}(x,t) = -2\, Rc\,(x,t)\,,$$

at all points $x \in M$ and times $t \in [0,T)$. In other words, for any tangent vectors X and Y at x we have:

$$\frac{\partial g}{\partial t}(X,Y)(x,t) = -2\, Rc(X,Y)(x,t),$$

for all $x \in M$ and $t \in [0,T)$. Taking $X = \partial/\partial x^i$ and $Y = \partial/\partial x^j$, we obtain the component form of the Ricci flow equation:

$$\frac{\partial}{\partial t} g_{ij} = -2\, R_{ij}\,,$$

which is the usual way we shall write the equation.

2.2. Short-time existence. The first question is that of short-time existence. On a compact manifold a solution exists for short time for any smooth initial metric:

THEOREM 2.1. (Hamilton 1982, DeTurck 1982) *Given any smooth, compact Riemannian manifold* (M, g_o), *there exists a unique smooth solution* $g(t)$ *to the Ricci flow with initial condition* $g(0) = g_o$ *on some time interval* $[0,\epsilon)$.

The [**19**] original proof of this result uses the Nash-Moser implicit function theorem and is rather involved. We suggest that the reader consult [**18**] for a vastly simplified proof.

2.3. Fundamental evolution equations. Once we are given the equation for the time evolution of the metric, in order to understand how the geometry of the metric evolves, we need to first derive the equations for the geometric quantities associated to the metric, such as the Christoffel symbols and the Riemann curvature tensor. In particular, we have (see [**19**] for proofs of all of the formulas in this subsection:)

LEMMA 2.2. *Under the Ricci flow, the Christoffel symbols evolve by*

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right).$$

Using the definition of the Riemann curvature tensor in terms of the Christoffel symbols, and applications of the Bianchi identities, one derives:

LEMMA 2.3. *Under the Ricci flow, the Riemann curvature (4,0)-tensor satisfies the following reaction-diffusion equation*

$$\frac{\partial}{\partial t} R_{ijkl} = \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})$$

where $B_{ijkl} = R_{ipjq} R_{kplq}$.

Although B is quadratic in Rm, it is not exactly the square of Rm. To define the square, consider Rm as a self-adjoint operator on 2-forms

$$Rm : \wedge^2 M \to \wedge^2 M$$

defined by

$$Rm\,(U)_{ij} = \sum_{p,q,r,s} g^{pr} g^{qs} R_{ijpq} U_{rs} = R_{ijpq} U_{pq}$$

By one of the symmetries of the Riemann curvature tensor, we have

$$\langle Rm\,(U)\,, V\rangle = \langle U, Rm\,(V)\rangle\,,$$

that is, Rm is self-adjoint. Here the inner product on 2-forms is defined by

$$\langle U, V\rangle = g^{ik} g^{jl} U_{ij} V_{kl}.$$

Now we can square Rm as an operator to obtain

$$Rm^2 : \wedge^2 TM^* \to \wedge^2 TM^*$$

which is given by

$$Rm^2\,(U)_{ij} = Rm\,[Rm\,(U)]_{ij} = R_{ijpq} R_{pqrs} U_{rs}.$$

Hence we write

$$\left(Rm^2\right)_{ijkl} = R_{ijpq} R_{pqkl}.$$

Although this is obviously the most natural definition of the square of Riemann curvature operator, there is another concept of square which will be useful. This definition applies whenever one has a self-adjoint operator on a Lie algebra. The reason this is relevant is that $\wedge^2 M$ has a Lie algebra structure which makes it isomorphic to $so\,(n)$. In particular, we define the Lie bracket of two 2-forms by

$$[U, V]_{ij} = U_{ip} g^{pq} V_{qj} - V_{ip} g^{pq} U_{qj}.$$

Noting that the matrix of components of a 2-form is antisymmetric and that in coordinates where $g^{ij} = \delta_{ij}$ we have

$$[U, V]_{ij} = U_{ip} V_{pj} - V_{ip} U_{pj} = (UV - VU)_{ij}\,.$$

This gives the isomorphism between $\left(\wedge^2 M_x, [\ ,\ ]\right)$ and $so\,(n)$ for any given point $x \in M$. Choose any basis $\{\phi^a\}$ of $\mathfrak{g}$ and let c_c^{ab} denote the structure constants:

$$\left[\phi^a, \phi^b\right] = \sum_c c_{abc} \phi^c.$$

Now we can define the square using the Lie bracket. Given a Lie algebra $\mathfrak{g}$ with a Lie bracket $[\ ,\ ]$ and an inner product $\langle\ ,\ \rangle$, the **Lie square** $L^\# : \mathfrak{g} \to \mathfrak{g}$ of an operator $L : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$L^\#_{ab} = c_{ace} c_{bdf} L_{cd} L_{ef}.$$

Computations yield

$$\left(Rm^2\right)_{ijkl} = 2(B_{ijkl} - B_{ijlk})$$

and

$$\left(Rm^\#\right)_{ijkl} = 2\left(B_{ikjl} - B_{iljk}\right).$$

Hence

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + \left(Rm^2\right)_{ijkl} + \left(Rm^{\#}\right)_{ijkl}. \tag{1}$$

Now we go on to the evolution equations for the Ricci tensor and scalar curvature. Since the Ricci tensor is the trace of the Riemann curvature tensor, one easily obtains from Lemma 2.3:

COROLLARY 2.4. *Under the Ricci flow, the Ricci tensor satisfies*

$$\frac{\partial}{\partial t}R_{jk} = \Delta R_{jk} + 2g^{pr}g^{qs}R_{pjqk}R_{rs} - 2g^{pq}R_{pj}R_{qk}.$$

Taking a second trace, one has:

COROLLARY 2.5. *Under the Ricci flow, the scalar curvature function evolves by*

$$\frac{\partial}{\partial t}R = \Delta R + 2\left|Rc\right|^2.$$

In order to make use of these nice equations, one needs the maximum principle, which in the parabolic case yields bounds for solutions to reaction-diffusion equations such as the ones above.

3. Maximum principles

In this section we recall the various versions of the maximum principle that are required for the study of the Ricci flow. We start with the scalar maximum principle and work our way up to the maximum principle for systems where the solution is a section of a vector bundle. We consider both the weak and the strong maximum principles. An excellent basic reference is [**47**]. See [**28**], p.99, for the scalar heat equation on a manifold, and [**19**], [**20**] for the parabolic maximum principle for tensors and systems.

3.1. Weak maximum principle.

3.1.1. *Scalar equations.* The heat equation is the prototype for parabolic equations. One of the most important properties it satisfies is the maximum principle, which says that for any smooth solution to the heat equation, whatever pointwise bounds hold at $t = 0$ also hold for $t > 0$.

THEOREM 3.1. (Scalar Maximum Principle I: pointwise bounds are preserved) *Let $u : M^n \times [0,T) \to \mathbb{R}$ be a C^2 solution to the heat equation*

$$\frac{\partial u}{\partial t} = \Delta u$$

on a complete Riemannian manifold. If $C_1 \leq u(x,0) \leq C_2$ for all $x \in M$, for some constants $C_1, C_2 \in \mathbb{R}$, then $C_1 \leq u(x,t) \leq C_2$ for all $x \in M$ and $t \in [0,T)$.

More generally, one may allow the metric to depend on time and also add in gradient and reaction terms. Namely, consider the semi-linear heat equation

$$\frac{\partial u}{\partial t} = \Delta u + \langle X, \nabla u\rangle + F(u) \tag{2}$$

where $\Delta = \Delta_{g(t)}$ is the laplacian with respect to a time-dependent metric $g(t)$, $X = X(t)$ is a time-dependent vector field, and $F : \mathbb{R} \to \mathbb{R}$ is a smooth function.

PROPOSITION 3.2. (Scalar Maximum Principle II: ODE gives pointwise bounds for PDE) *Let $u : M^n \times [0,T) \to \mathbb{R}$ be a C^2 solution to (2). If $C_1 \leq u(x,0) \leq C_2$ for all $x \in M$, for some constants $C_1, C_2 \in \mathbb{R}$, then $\phi_1(t) \leq u(x,t) \leq \phi_2(t)$ for all $x \in M$ and $t \in [0,T)$, where $\phi_i(t)$, $i = 1,2$, are the solutions to the associated ordinary differential equation*

$$\frac{d\phi_i}{dt} = F(\phi_i)$$
$$\phi_i(0) = C_i.$$

3.1.2. *Systems.* The maximum principle is quite robust, it applies to general classes of second-order parabolic equations and even to some systems, such as the Ricci flow. A simple example of the maximum principle for systems is the following.

PROPOSITION 3.3. (preserving nonnegativity of a 2-tensor) *Let $(M, g(t))$ be a time-dependent Riemannian manifold, where the metric depends smoothly on time (e.g., $g(t)$ is a solution to the Ricci flow,) and $\alpha(t) \in \Gamma(TM^* \otimes_S TM^*)$ a symmetric 2-tensor satisfying the semi-linear heat equation*

$$\frac{\partial \alpha}{\partial t} = \Delta_{g(t)}\alpha + \beta,$$

where $\beta(t) = f(\alpha, g(t))$ is a symmetric 2-tensor, which is a smooth (Lipschitz should be enough) function of α and $g(t)$, satisfying the condition that

$$\beta_{ij} V^i V^j (x,t) \geq 0$$

whenever $V(x,t)$ is a null eigenvector of $\alpha(t)$:

$$\alpha_{ij} V^i (x,t) = 0.$$

If $\alpha(0) \geq 0$, then $\alpha(t) \geq 0$ for all $t \geq 0$.

Idea of the proof. Suppose $\alpha > 0$ at $t = 0$ and (x_0, t_0) is a point and time such that

$$\alpha_{ij} V^i (x_0, t_0) = 0$$

for the first time for some tangent vector V at (x_0, t_0). Then $\alpha_{ij} W^i W^j (x,t) \geq 0$ for all $x \in M$, $t \in [0, t_0]$, and tangent vectors W. Extend V to a neighborhood of (x_0, t_0) in space and time so that

$$\frac{\partial V}{\partial t}(x_0, t_0) = 0$$
$$\nabla V (x_0, t_0) = 0 \tag{3}$$
$$\Delta V (x_0, t_0) = 0.$$

In particular, this may be accomplished by parallel translating V in space along geodesic rays emanating from x_0, and taking V to be independent of time. Then at any point in the neighborhood of (x_0, t_0),

$$\frac{\partial}{\partial t}\left(\alpha_{ij} V^i V^j\right) = \left(\frac{\partial}{\partial t}\alpha_{ij}\right) V^i V^j$$
$$= \left(\Delta \alpha_{ij} + \beta_{ij}\right) V^i V^j.$$

On the other hand, by (3), at (x_0, t_0) we have

$$\left(\Delta \alpha_{ij}\right) V^i V^j = \Delta\left(\alpha_{ij} V^i V^j\right) \geq 0.$$

Combining this with our assumption

$$\beta_{ij}V^iV^j\,(x_0,t_0)\geq 0,$$

we conclude

$$\frac{\partial}{\partial t}\left(\alpha_{ij}V^iV^j\right)=\Delta\left(\alpha_{ij}V^iV^j\right)+\beta_{ij}V^iV^j\geq 0$$

at (x_0,t_0). Hence, if $\alpha_{ij}V^iV^j$ becomes zero, it wants to increase. Note that in the heuristic arguments above, we have not assumed that β is a smooth function of α and $g(t)$.

A fancier version of the maximum principle for systems, which is used for the Ricci flow, is as follows. Let $V \overset{\pi}{\to} M$ be a vector bundle over a manifold M with a time-dependent Riemannian metric $g(t)$ on M, a fixed metric h on the fibers of V, and a time-dependent connection (covariant derivative) $\nabla(t)$ on V compatible with h. The time-dependent laplacian $\Delta(t)$ acting on sections of V is defined by

$$\Delta=\mathrm{trace}_g\left(\nabla^2\right)=g^{ij}\nabla_i\nabla_j$$

where $\nabla:\Gamma(V)\to\Gamma(V\otimes TM^*)$ is the connection on V and $\nabla:\Gamma(V\otimes TM^*)\to\Gamma(V\otimes TM^*\otimes TM^*)$ is defined using the connection on V and the Levi-Civita connection on TM^*. Suppose that a time-dependent section $\sigma(t)\in\Gamma(V)$ satisfies the parabolic equation

$$\frac{\partial\sigma}{\partial t}=\Delta\sigma+F(\sigma), \tag{4}$$

where $F:V\to V$ is fiber preserving, i.e., F is a vertical vector field on V. Analogous, to the case of the semi-linear heat equation, we consider the corresponding ODEs to (4) obtained by dropping the laplacian term

$$\frac{ds}{dt}=F(s)$$

which are ordinary differential equations on the fibers $V_x=\pi^{-1}(x)$ for each $x\in M$. The analogue for systems to the initial pointwise bounds $C_1\leq u(x,0)\leq C_2$, which we assumed in the maximum principle for the scalar heat equation, is to assume that the initial data $\sigma(0)$ lies in a subset $\mathcal{K}\subset V$ which is invariant under parallel translation in V and fiberwise convex, i.e., $\mathcal{K}_x=\mathcal{K}\cap V_x$ is a convex subset of V_x for all $x\in M$. The invariance under parallel translation corresponds to the interval $[C_1,C_2]$ being independent of $x\in M$, and the fiberwise convexity corresponds to the interval $[C_1,C_2]$ being convex in $\mathbb{R}$ (here $V=M\times\mathbb{R}$.)

The maximum principle for systems says that the associated ODE can give bounds for the PDE in the following sense

THEOREM 3.4. (maximum principle for systems) *Let $\sigma(t)\in\Gamma(V)$ be a solution to*

$$\frac{\partial\sigma}{\partial t}=\Delta\sigma+F(\sigma).$$

Suppose that $\mathcal{K}(t)\subset V$ is a time-dependent subset invariant under parallel translation and fiberwise convex such that for any solution $s(t)\in V_x$ to the ODE

$$\frac{ds}{dt}=F(s)$$

with $s(0) \in \mathcal{K}(0)_x$ *stays in* $\mathcal{K}_x$, *i.e.,* $s(t) \in \mathcal{K}(t)_x$ *for all* $x \in M$ *and* $t \geq 0$. *Then any solution* $\sigma(t)$ *to the PDE with* $\sigma(0) \in \mathcal{K}(0)$ *stays in* $\mathcal{K}$, *i.e.,* $\sigma(t) \in \mathcal{K}(t)$ *for all* $t \geq 0$.

Proof in a special case. We first consider the case of a flat trivial bundle, that is, where the solution is a function on M with values in $\mathbb{R}^k$. To understand why we need the fibers $\mathcal{K}_x$ to be convex, it is enough to consider the case of vector valued functions on the unit interval $[0,1]$

$$u : [0,1] \to \mathbb{R}^k,$$

where the values at the endpoints are fixed

$$u(0) = \vec{a}$$
$$u(1) = \vec{b}.$$

The heat equation smooths out the function u (in infinite time) to the linear function $u_\infty(s) := (1-s)\,\vec{a} + s\vec{b}$. That is, for the theorem to be true, we need that if $\vec{a}, \vec{b} \in \mathcal{K}$, then $(1-s)\,\vec{a} + s\vec{b} \in \mathcal{K}$. That is, $\mathcal{K}$ is convex.

The statement of the theorem in the special case of a flat trivial bundle is:

PROPOSITION 3.5. *Let* $u : M \to \mathbb{R}^k$ *be a solution to*

$$\frac{\partial u}{\partial t} = \Delta u + F(u) \tag{5}$$

where $F : \mathbb{R}^k \to \mathbb{R}^k$. *Let* $\mathcal{K} \subset \mathbb{R}^k$ *be a closed convex set such that any solution* U *to the ODE corresponding to (5)*

$$\frac{dU}{dt} = F(U)$$

which starts in $\mathcal{K}$ *stays in* $\mathcal{K}$, *i.e., if* $U(0) \in \mathcal{K}$, *than* $U(t) \in \mathcal{K}$ *for all* $t \geq 0$. *Then any solution to the PDE (5) which starts in* $\mathcal{K}$ *also stays in* $\mathcal{K}$.

Proof. Given $x \in \mathbb{R}^k$, let $d(x,\mathcal{K})$ denote the distance from x to $\mathcal{K}$ (with $d(x,\mathcal{K}) = 0$ for $x \in \mathcal{K}$.) Associated to the solution $u(x,t)$ to (5), let

$$s(t) := \sup_{x \in M} d(u(x,t),\mathcal{K})$$

be the maximum distance of u from the set $\mathcal{K}$ at time t. We shall show that $s(t)$ grows at most exponentially:

$$\frac{ds}{dt} \leq Cs$$

for some constant $C < \infty$. Since $s(0) = 0$, we conclude that $s(t) = 0$ for all $t \geq 0$, from which the proposition follows. For computational purposes, we describe the distance function $d(x,\mathcal{K})$ in terms of support functions for the convex set $\mathcal{K}$.

DEFINITION 1. We say that a linear function $l : \mathbb{R}^k \to \mathbb{R}$ is a *support function* for $\mathcal{K}$ at a point $v \in \partial\mathcal{K}$ if

1. $l(w - v) \leq 0$ for all $w \in \mathcal{K}$
2. $|l| = 1$.

We write $l \in \operatorname{supp}_v \mathcal{K}$. The distance function may be rewritten as

$$d(x,\mathcal{K}) = \left(\sup_{\substack{l \in \operatorname{supp}_v \mathcal{K} \\ v \in \partial\mathcal{K}}} l(x-v) \right)^+ . \tag{6}$$

Now the maximum distance of u to $\mathcal{K}$ may be written as

$$s(t) = \sup_{x\in M} \left\{ \sup_{\substack{l \in \operatorname{supp}_v \mathcal{K} \\ v \in \partial\mathcal{K}}} l(u(x,t)-v) \right\}^+ = \sup_{\substack{l \in \operatorname{supp}_v \mathcal{K} \\ v \in \partial\mathcal{K}}} \left\{ \sup_{x\in M} l(u(x,t)-v) \right\}^+ .$$

Since M, $\operatorname{supp}_v\mathcal{K}$ and $\partial\mathcal{K}$ are compact sets (we may assume $\mathcal{K}$ is a compact set by using a cutoff function,) we have

$$\begin{aligned} \frac{d}{dt}s(t) &\le \sup \left\{ \left[\frac{\partial}{\partial t} m(u-w)\right](y,t) : \begin{array}{l} m \in \operatorname{supp}_w \mathcal{K},\ w \in \partial\mathcal{K},\ y \in M \\ m(u(y,t)-w) = \sup_{l,v,x} l(u(x,t)-v) \end{array} \right\} \\ &= \sup \left\{ \left[m(\Delta u) + m(F(u))\right](y,t) : \begin{array}{l} m \in \operatorname{supp}_w \mathcal{K},\ w \in \partial\mathcal{K},\ y \in M \\ m(u(y,t)-w) = \sup_{l,v,x} l(u(x,t)-v) \end{array} \right\}, \end{aligned}$$

where we used

$$\frac{\partial}{\partial t} m(u-w) = m\left(\frac{\partial u}{\partial t}\right) = m(\Delta u + F(u)).$$

Since m is linear, we have

$$m(\Delta u)(y,t) = \Delta(m(u))(y,t) \le 0$$

by our assumptions on m and y. Given m and y as before, the point $w \in \partial\mathcal{K}$ is the unique point in $\mathcal{K}$ closest to $u(y,t)$

$$d(w, u(y,t)) = d(\mathcal{K}, u(y,t)) = s(t).$$

We have

$$m(F(w)) \le 0,$$

so that

$$\begin{aligned} m(F[u(y,t)]) &= m(F(w)) + m(F[u(y,t)] - F(w)) \\ &\le |m|\,|F[u(y,t)] - F(w)| \\ &\le C\,|u(y,t) - w| = Cs(t). \end{aligned}$$

Hence we conclude

$$\frac{d}{dt}s(t) \le Cs(t). \quad \text{q.e.d.}$$

Proof of the general case. Since the proof is similar, we only highlight the differences. Define

$$\begin{aligned} s(t) &:= \sup_{x\in M} d(\sigma(x,t), \mathcal{K}_x) \\ &= \sup_{x\in M} \left\{ \sup_{\substack{l \in \operatorname{supp}_v \mathcal{K}_x \\ v \in \partial\mathcal{K}_x}} l(\sigma(x,t)-v) \right\}^+ . \end{aligned}$$

As before, the time-derivative of the maximal distance function satisfies

$$\frac{d}{dt}s(t) \leq \sup\left\{ [m(\Delta\sigma) + m(F(\sigma))](y,t) : \begin{array}{c} m \in \mathrm{supp}_w \mathcal{K}_y,\ w \in \partial\mathcal{K}_y,\ y \in M \\ m(\sigma(y,t) - w) = \sup_{l,v,x} l(\sigma(x,t) - v) \end{array} \right\}.$$

We have $m(F(\sigma))(y,t) \leq 0$ as before. To show that $m(\Delta\sigma)(y,t) \leq 0$, we extend $m \in V_y^*$ to a neighborhood U of y by parallel translation along geodesics emanating from y. That is $m \in \Gamma(V^*|_U)$ and

$$\nabla m(y) = 0$$
$$\Delta m(y) = 0.$$

CLAIM 1. $m(x)$ is a support function for $\mathcal{K}_x$ for x in a neighborhood of y.

The theorem now follows from

$$m(\Delta\sigma)(y,t) = \Delta(m(\sigma))(y,t) \leq 0. \text{ q.e.d.}$$

Remark (how the convexity of $\mathcal{K}$ is used in the proof.) Only for convex sets do we have the property that

$$l(x - v) \leq 0 \text{ for all } x \in \mathcal{K}$$

for any support function l at v. In particular, if $\mathcal{K}$ is not convex, then (6) does not hold.

3.2. Strong maximum principle.

3.2.1. *Scalar heat equation.* The strong maximum principle says that if a supersolution to the heat equation is bounded from below by a constant C and if the bound is preserved in time, then for positive time, either the solution is strictly greater than C or identically equal to C. In other words, if we have a supersolution which initially is nonnegative everywhere and positive at one point, then for positive time, the supersolution is positive everywhere.

PROPOSITION 3.6. (Strong Maximum Principle, I.: scalar equation) *Let (M,g) be a complete Riemannian manifold and $u : M \times [0,T) \to \mathbb{R}$ be a solution to the equation*

$$\frac{\partial u}{\partial t} \geq \Delta u + \langle X, \nabla u \rangle$$

with $u \geq 0$ everywhere. If $u(x_0, t_0) = 0$ at some point $x_0 \in M$ and time $t_0 \in (0,T)$, then $u \equiv 0$ on $M \times [0,T)$.

Proof. See Protter-Weinberger [**47**], Theorem 4 of Ch. 3, sect. 2.

3.2.2. *Systems.* To apply the strong maximum principle to the reaction-diffusion equation satisfied by the curvature tensor under the Ricci flow, we need a version for systems of functions (i.e., tensors or sections of vector bundles.) Let $(M^n, g(t))$ be a manifold with a time-dependent Riemannian metric (e.g., a solution to the Ricci flow.) Let $\pi : V \to M$ be a vector bundle with a fixed metric h on the fibers and a time-dependent connection $A(t)$ compatible with h. The maximum principle for systems is (for the proof of the results in this section, see section 8 of [**20**]:)

PROPOSITION 3.7. *Let $\sigma(t) \in \Gamma(V)$ be a time-dependent section of V satisfying*

$$\frac{\partial \sigma}{\partial t} = \Delta\sigma + F(\sigma), \tag{7}$$

where $F : V \to V$ *is a fiber preserving map. Suppose that*

$$k : V \to \mathbb{R}$$

is a function such that

1. *it is invariant under parallel translation: for every path in* M *and horizontal lift* α *in* V*, we have* k *is constant along* α*,*
2. $k|_{V_x}$ *is convex for all* $x \in M$*.*
   Then

A. (weak maximum principle) *If for all* $x \in M$*, the sets*

$$\{v \in V_x : k(v) \leq c\}$$

are preserved by the ODE

$$\frac{ds}{dt} = F(s)$$

for some $c \in \mathbb{R}$*, then the inequality*

$$k(\sigma) \leq c$$

is also preserved by the PDE (7).

B. (strong maximum principle) *If* $k(\sigma(x, 0)) > c$ *for some* $x \in M$*, then*

$$k(\sigma(x, t)) > c, \text{ for all } x \in M \text{ and } t > 0.$$

This is the general vector bundle statement of the strong maximum principle for systems. Since we are interested in obtaining bounds for the Riemann curvature tensor under the Ricci flow, and the Riemann curvature tensor is a bilinear form, the version of the strong maximum principle we shall use is.

COROLLARY 3.8. (Strong Maximum Principle, II.: bundle formulation) *Let* $\beta(t) \in \Gamma(V^* \otimes_S V^*)$ *be a time-dependent symmetric bilinear form on* V *satisfying the semi-linear parabolic equation*

$$\frac{\partial \beta}{\partial t} = \Delta \beta + F(\beta),$$

where $F : V^* \otimes_S V^* \to V^* \otimes_S V^*$ *is a fiber preserving map with* $F(\beta) \geq 0$ *whenever* $\beta \geq 0$*. If* $\beta \geq 0$ *at* $t = 0$*, then* $\beta \geq 0$ *for* $t > 0$ *(by the weak maximum principle) and there exists a* $\delta > 0$ *such that on the time interval* $[0, \delta]$

PROPOSITION 3.9.

1. *the rank of* β *is constant,*
2. *the null space*

   $$null(\beta(t)) := \{v \in V : \beta(t)(v, w) = 0 \text{ for all } w \in V \text{ with } \pi(w) = \pi(v)\}$$

   is independent of time and invariant under parallel translation in space (in other words, invariant under parallel translation in space and time,) and furthermore
3. $$null(\beta(t)) \subset null(F[\beta(t)]).$$

Applying the above result to the curvature operator $Rm : \wedge^2 M \to \wedge^2 M$ of a solution to the Ricci flow yields:

THEOREM 3.10. (Strong Maximum Principle, III.: application to curvature) *Let* $(M, g(t))$ *be a solution to the Ricci flow with nonnegative curvature operator* $Rm \geq 0$ *on* $U \times [0, \varepsilon)$ *where* $U \subset M$ *is an open set and* $\varepsilon > 0$. *Then either*

1. (positive curvature) $Rm > 0$ *on* $U \times (0, \varepsilon)$, *or*
2. (holonomy group reduces) *for* $t > 0$, *the image of the Riemann curvature operator is invariant under parallel translation in space and independent of time. Furthermore, for each point* $x \in M$ *and* $t \in (0, \varepsilon)$, *the image of* $Rm(x, t)$ *is a proper Lie subalgebra of* $\wedge^2 M_x \cong so(n)$ *which is isomorphic to the holonomy algebra of* $g(t)$.

4. Surfaces

In this section we describe Hamilton's results on the Ricci flow on surfaces. The first thing to notice about the Ricci flow on surfaces is that the equation simplifies. Since when $n = 2$ we have $R_{ij} = \frac{1}{2} R g_{ij}$, the Ricci flow equation becomes

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij}.$$

This is a conformal flow, that is, the solutions $g(t)$ satisfy $g(t) = e^{u(t)} g(0)$ for some time-dependent function u on M^2.

4.1. Uniformization theorem and main result. The classical uniformization theorem in complex analysis is equivalent to the following differential geometric statement:

PROPOSITION 4.1. *Given an oriented Riemannian surface* (M, h), *there exists a function* $u : M \to \mathbb{R}$ *such that the metric* $g = e^u h$ *has constant Gaussian curvature* $K_g \equiv 1$, 0, *or* -1.

The Ricci flow provides a constructive way of proving the above result. Namely, we have

THEOREM 4.2. *(Hamilton 1988) If* M^2 *is a closed surface, then for any initial metric* g_0 *on* M, *the solution to the area normalized Ricci flow*

$$\begin{aligned} \frac{\partial}{\partial t} g &= (r - R)\, g \\ g(0) &= g_0, \end{aligned}$$

(where $r = \int R dA / \int dA$ *is the average scalar curvature) exists for all time and has constant area. Moreover,*

1. *If the Euler characteristic of* M *is non-positive, then the solution metric* $g(t)$ *converges to a smooth constant curvature metric as* $t \to \infty$.
2. *If the scalar curvature* R *of the initial metric* g_0 *is positive, then the solution metric* $g(t)$ *converges to a smooth positive constant curvature metric as* $t \to \infty$.

Since given a surface with positive Euler characteristic, it is easy to find a metric with positive scalar curvature, the uniformization theorem follows from the above result. In [**14**] the condition that the initial metric have positive scalar curvature was removed. In [**3**] a new proof of this result was given using the Aleksandrov reflection method analogous to the work of R. Schoen on the Yamabe problem (which in dimension 2 is the elliptic version of the Ricci flow.) In [**22**], a proof via an isoperimetric estimate and a singularity analysis was given.

4.2. The energy. In dimension 2, given a fixed metric h in the conformal class, the normalized Ricci flow is actually the gradient flow for a *relative energy functional* E_h on the space Met of all metrics which are in the conformal class of h and have the same area as h. Let

$$Met = \left\{ g = e^u h : \int_M e^u dA_h = \int_M dA_h \right\}.$$

The relative energy functional E_h is given by:

$$E_h : Met \to \mathbb{R},$$

where

$$E_h(g) = \int_M \ln(g/h) \left(R_g dA_g + R_h dA_h \right).$$

Clearly

$$E_h(g) = -E_g(h).$$

The reason that E_h is called a relative energy functional is:

LEMMA 4.3. *If g, h, and k are any 3 metrics in a conformal class, then*

$$E_h(g) = E_k(g) + E_h(k).$$

To define the gradient of the energy functional, we need to define a metric on the infinite dimensional space of metrics in a conformal class with fixed area. We consider the L^2-metric:

$$\langle \ , \ \rangle_{L^2} : T\left(Met\right)_g \to T\left(Met\right)_g$$

defined by

$$\langle a \cdot g, b \cdot g \rangle_{L^2} = 2 \int_M ab \cdot dA_g,$$

where a and b are smooth functions on M with $\int_M a \cdot dA_g = 0$ and $\int_M b \cdot dA_g = 0$. We compute that

$$\nabla E_h(g) = (R - r) \cdot g.$$

Thus the normalized Ricci flow is the same as the gradient flow of the energy:

$$\frac{d}{dt} E_h(g) = -\nabla E_h(g).$$

The time-derivative of the energy is given by:

LEMMA 4.4. *Under the normalized Ricci flow, the time-derivative of the energy is*

$$\frac{d}{dt} E_h(g) = -2 \int_M (R - r)^2 dA \leq 0.$$

An important fact is that the energy functional is bounded from below:

PROPOSITION 4.5. (Onofri 1982) *If $M \cong S^2$, then the energy E_h is bounded below on Met and the minimum of E_h is obtained on the real 5-dimensional family of constant curvature metrics with fixed area in the conformal class of h.*

The reader may find it surprising that the energy functional does not play an important role in the study of the Ricci flow on surfaces. In fact, most proofs of convergence results do not use the energy functional. However, the definition of the energy functional does extend to the Ricci flow on Kähler manifolds (see [**42**] and [**52**].)

4.3. Evolution of curvature. Similar to Lemma 2.5, we have (note that when $n = 2$, $Rc = \frac{1}{2}Rg$)

LEMMA 4.6. *Under the normalized Ricci flow,*

$$\frac{\partial}{\partial t}R = \triangle R + R(R - r). \tag{8}$$

This type of evolution is known as a reaction-diffusion equation. The Laplacian term is causing the diffusion of R, whereas the quadratic terms in R represent the reaction terms. If the right-hand-side only contained the Laplacian term, then the equation would be the heat equation (albeit the Laplacian is respect to a time-dependent metric) and R would tend to a constant as t approaches ∞. On the other hand, if the right-hand-side only contained the $R(R - r)$ term, then the equation would be an ODE and the solution would blow up in finite time for any initial data satisfying $R(0) > \max\{r, 0\}$. The answer to the question of how the scalar curvature behaves under the normalized Ricci flow depends on whether the diffusion or the reaction term dominates. It turns out, as we shall show later, that it is the diffusion term which dominates. Dropping the Laplacian term yields the following ordinary differential equation:

$$\frac{d}{dt}s = s(s - r), \tag{9}$$

where the function $s = s(t)$ plays the role of R. The solution to the ODE above with initial data $s(0) = s_o$ is:

$$s(t) = \frac{r}{1 - \left(1 - \frac{r}{s_o}\right)e^{rt}},$$

when $r \neq 0$ and $s_o \neq 0$. When $r = 0$, we have:

$$s(t) = \frac{s_o}{1 - s_o t},$$

and when $s_o = 0$,

$$s(t) \equiv 0.$$

We conclude that for all values of r, the solution blows up in finite time when $s_o > \max\{r, 0\}$:

$$\lim_{t \to T} s(t) = \infty,$$

where if $r = 0$, then $T = 1/s_o$, and if $r \neq 0$, then $T = -\frac{1}{r}\ln\left(1 - \frac{r}{s_o}\right) \in (0, \infty)$. Hence we cannot obtain an upper bound for the curvature under the normalized Ricci flow by directly applying the maximum principle to equation (8). On the other hand, the ODE behaves much better when $s_o < \min\{r, 0\}$, in which case we have:

$$s(t) - r \geq s_o - r.$$

In view of the solution $s(t)$ of equation (9), we may apply the maximum principle to the equation for R to conclude:

LEMMA 4.7. *Let $\{g(t)\}$ be a solution to the normalized Ricci flow on a surface M.*

1. *If $r < 0$, then*
$$R - r \geq \frac{r}{1 - \left(1 - \frac{r}{R_{\min}(0)}\right) e^{rt}} - r \geq (R_{\min}(0) - r)\, e^{rt}.$$
Note that in this case, $\lim_{t\to\infty} e^{rt} = 0$.
2. *If $r = 0$, then*
$$R \geq \frac{R_{\min}(0)}{1 - R_{\min}(0)\, t} > -\frac{1}{t}.$$
3. *If $r > 0$ and $R_{\min}(0) < 0$, then*
$$R - r \geq \frac{r}{1 - \left(1 - \frac{r}{R_{\min}(0)}\right) e^{rt}} \geq R_{\min}(0)\, e^{-rt}.$$

Hence we have uniform lower bounds for the curvature under the normalized Ricci flow, whereas our upper bounds for the curvature blow up in finite time. In the next section we shall obtain a uniform upper bound for the curvature when $r \leq 0$ and an exponential upper bound for the curvature when $r > 0$.

4.4. Ricci solitons and more estimates for curvature. In this subsection we consider Ricci solitons, which are fixed points of the normalized Ricci flow in the space of metrics (in a conformal class with fixed area) modulo the action of the group of conformal diffeomorphisms. These special solutions to the Ricci flow motivate certain quantities we consider in estimating the curvature. It turns out that the good quantities to estimate are the ones which are constant on Ricci solitons. In particular, by estimating such a quantity, we shall obtain upper bounds for the scalar curvature. These upper bounds are uniform, except in the case where $r > 0$, in which case they are exponential. We shall also show that the only Ricci solitons on a closed surface are the constant curvature metrics.

DEFINITION 2. A solution $\{g(t)\}$ to the normalized Ricci flow is a *Ricci soliton* if there exists a one-parameter family of conformal diffeomorphisms $\{\varphi(t)\}$ such that
$$g(t) = \varphi(t)^* g(0).$$

Differentiating this equation with respect to time implies:

$$\frac{\partial}{\partial t} g = L_X g, \tag{10}$$

where $\{X(t)\}$ is the one-parameter family of vector fields generated by $\{\varphi(t)\}$. Substituting the normalized Ricci flow into (10) yields:
$$(r - R) g_{ij} = \nabla_i X_j + \nabla_j X_i.$$
If $X = -\nabla f$ is minus the gradient of some time-dependent function f, then we obtain the equation:
$$(R - r) g_{ij} = 2 \nabla_i \nabla_j f.$$

In this case we say that $\{g(t)\}$ is a *gradient Ricci soliton*. Tracing implies:

$$\Delta f = R - r. \tag{11}$$

This equation is solvable for f since $\int_M (R-r)\, dA = 0$. The solution f is referred to as the *potential of the curvature*, and it is unique up to an additive constant since harmonic functions are constants. The reader should note that the potential f can be defined by equation (11) for any metric g, not just for Ricci solitons. Defining

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2} \Delta f \cdot g_{ij}$$

to be the trace-free part of the Hessian of f, we see that the gradient Ricci soliton equation is equivalent to:

$$M_{ij} = 0.$$

Taking the divergence of M, we obtain:

$$\begin{aligned} div(M)_i &= \nabla_j M_{ij} = \nabla_j \nabla_i \nabla_j f - \frac{1}{2} \nabla_i \Delta f \\ &= \frac{1}{2}(\nabla_i R + R \nabla_i f). \end{aligned}$$

Note that dividing by R, we find that the gradient Ricci soliton equation implies:

$$\nabla(\ln R + f) = 0,$$

that is,

$$\ln R + f = C,$$

where C is a (time-dependent) constant.

More importantly, the gradient Ricci soliton equation implies:

$$\begin{aligned} 0 &= \nabla_i R + R \nabla_i f = \nabla_i R + (R-r)\nabla_i f + r \nabla_i f \\ &= \nabla_i R + 2\nabla_i \nabla_j f \cdot \nabla_j f + r \nabla_i f = \nabla_i \left(R + |\nabla f|^2 + r f \right). \end{aligned}$$

That is, on a gradient Ricci soliton,

$$R + |\nabla f|^2 + r f = C,$$

where C is a (time-dependent) constant. Since $R + |\nabla f|^2 + r f$ is constant on Ricci solitons, we expect that it will satisfy a nice evolution equation in general. The potential f itself satisfies a nice evolution equation:

LEMMA 4.8. *Under the normalized Ricci flow, the potential of the curvature satisfies (provided we suitably adjust the additive constant in the definition of f):*

$$\frac{\partial}{\partial t} f = \Delta f + r f. \tag{12}$$

Applying the maximum principle to equation (12) yields:

COROLLARY 4.9. *Under the normalized Ricci flow, there exists a constant C such that*

$$|f| \le C e^{rt}.$$

A consequence of this estimate is that when $r \le 0$, as long as the solution exists, the metrics $g(t)$ are uniformly equivalent.

PROPOSITION 4.10. *If $r \leq 0$, then under the normalized Ricci flow, there exists a constant $C \geq 1$ depending only on the initial metric g_o such that:*

$$\frac{1}{C} g(0) \leq g(t) \leq C\, g(0).$$

Define $H = R - r + |\nabla f|^2$. Since $H + rf$ is constant on Ricci solitons and f satisfies a nice evolution equation in general, it is natural that H should satisfy a nice evolution equation.

LEMMA 4.11. *Under the normalized Ricci flow:*

$$\frac{\partial}{\partial t} H = \triangle H - 2\,|M_{ij}|^2 + rH. \tag{13}$$

Applying the maximum principle to equation (13) implies:

COROLLARY 4.12. *Under the normalized Ricci flow, there exists a constant C depending only on the initial metric g_o such that:*

$$H \leq C\, e^{rt},$$

in particular,

$$R - r \leq C\, e^{rt}.$$

Combining our previous estimates for R, we have:

PROPOSITION 4.13. *Under the normalized Ricci flow, there exists a constant C depending only on the initial metric g_o such that:*

1. (a) If $r < 0$, then

$$-C\, e^{rt} \leq R - r \leq C\, e^{rt}.$$

(b) If $r = 0$, then

$$-\frac{C}{1 + Ct} \leq R \leq C.$$

(c) If $r > 0$, then

$$-C\, e^{-rt} \leq R - r \leq C\, e^{rt}.$$

Before going to the long-time existence theorem in the next subsection, which follows from the above estimates for the curvature, we conclude this subsection with a proof of the fact that the only Ricci solitons on a closed surface are the constant curvature metrics.

PROPOSITION 4.14. *If $(M, g(t))$ is a family of Ricci solitons, then $g(t) = g(0)$ is a constant curvature metric.*

PROOF. Multiplying the Ricci soliton equation

$$(r - R)g_{ij} = \nabla_i X_j + \nabla_j X_i$$

by $R\, g_{ij}$ yields:

$$-\int_M (R - r)^2\, dA = \int_M (r - R) R\, dA = \int_M R \operatorname{div} X\, dA.$$

Since X is a conformal Killing vector field, integrating by parts and applying the Kazdan-Warner identity implies:

$$\int_M (R-r)^2\, dA = \int_M \nabla R \cdot X\, dA = 0.$$

Hence $R \equiv r$ and the proposition is proved.

4.5. Bernstein-Shi estimates and long-time existence. The long time existence of the solution follows from Proposition 4.13 and estimates for the higher derivatives of the curvature. For the Ricci flow, such higher derivative estimates first appeared in [**19**] using interpolation inequalities. However, new higher derivative estimates were obtained by W. Shi [**49**] using the method of Bernstein (see [**39**].) These estimates are have also proved useful for the analysis of singularities.

PROPOSITION 4.15. *Suppose that $(M, g(t))$ is a solution to the normalized Ricci flow. There exists a universal constant C (depending on r if $r < 0$) such that if $R \le M$ at $t = 0$ for some constant $M > 0$, then*

$$|\nabla R|\,(x,t) \le \frac{C \cdot M}{t^{1/2}},$$

for all $x \in M$ and $t \in \left[0, \frac{1}{C \cdot M}\right]$.

Similarly, one can prove estimates for the higher covariant derivatives of the scalar curvature.

PROPOSITION 4.16. *Suppose that $(M, g(t))$ is a solution to the normalized Ricci flow. There exists a constant C_m depending only on m (and r if $r < 0$) such that if $R \le M$ at $t = 0$ for some constant $M > 0$, then*

$$|\nabla^m R|\,(x,t) \le \frac{C_m \cdot M}{t^{m/2}},$$

for all $x \in M$ and $t \in \left[0, \frac{1}{C_m \cdot M}\right]$.

Based on the curvature and higher derivative estimates, we have the following long-time existence result

THEOREM 4.17. *If (M, g_o) is a closed Riemannian surface, then a solution $g(t)$ to the normalized Ricci flow exists for all time.*

Concerning the convergence, we have

THEOREM 4.18. *If $r \le 0$, then the metrics $g(t)$ converge uniformly in any C^k-norm to a smooth metric g_∞ as $t \to \infty$, and the metric has g_∞ has constant curvature.*

In the following subsections we shall consider the more difficult case where $R > 0$ initially.

4.6. Entropy. In this subsection we assume $R > 0$ initially. Since we have not been able to apply the maximum principle to obtain a uniform upper bound for the curvature, we consider integral quantities. Perhaps the most important such quantity for the Ricci flow on surfaces is the *entropy* N defined for a metric with positive curvature by:

$$N(g) = \int_M \ln R \cdot R\, dA.$$

The only reason this quantity is called the entropy is because it resembles other quantities called entropy which are the integral of a positive function times its logarithm. Our sign convention is the opposite of the usual one and we shall show that the entropy is decreasing (instead of increasing) under the normalized Ricci flow. The time-derivative of the entropy is given by (see [**21**]:)

LEMMA 4.19. *If $R(g_o) > 0$, then under the normalized Ricci flow*

$$\frac{dN}{dt} = -\int_M \frac{|\nabla R|^2}{R}\,dA + \int_M (R-r)^2\,dA$$

Hamilton originally proved that the entropy is non-increasing: $dN/dt \leq 0$ by showing

$$\frac{d}{dt}\left(\frac{dN}{dt}\right) \geq c_1\left(\frac{dN}{dt}\right)^2 + c_2\frac{dN}{dt}$$

and concluding that if dN/dt were ever positive, then it would tend to infinity in finite time, contradicting the long-time existence of the solution established in the previous subsection. A direct proof of this fact was given in [**15**]:

PROPOSITION 4.20. *If $R(g_o) > 0$, then under the normalized Ricci flow, then*

$$\frac{dN}{dt} = -\int_M \frac{|\nabla R + R\nabla f|^2}{R}\,dA - 2\int_M |M_{ij}|^2\,dA \leq 0.$$

As a consequence, we have

COROLLARY 4.21. *If $R(g_o) > 0$, then under the normalized Ricci flow, then the entropy is a strictly decreasing function of time unless $R(g_o) \equiv r$ in which case it is constant in time.*

4.7. Harnack estimate. In this section we describe Hamilton's Harnack inequality for the scalar curvature function under the normalized Ricci flow, which is modelled on the Li-Yau Harnack inequality [**41**] for the heat equation on a Riemannian manifold. Recall that for a Ricci soliton,

$$\nabla R + R\nabla f = 0. \tag{14}$$

Let

$$L = \ln R.$$

Dividing equation (14) by R and taking the divergence implies:

$$Q := \Delta L + R - r = 0.$$

The quantity Q is known as the *Harnack quantity.* We shall obtain a lower bound for Q depending only on the initial metric g_o by applying the maximum principle to the evolution equation which it satisfies. To compute its evolution equation, we first need to compute the evolution equation for L.

LEMMA 4.22. *Under the normalized Ricci flow,*

$$\frac{\partial}{\partial t}L = \Delta L + |\nabla L|^2 + R - r.$$

From the lemma we may rewrite the Harnack quantity as:

$$Q = \frac{\partial}{\partial t} L - |\nabla L|^2 .$$

The evolution for Q is given by:

LEMMA 4.23. *Under the normalized Ricci flow,*

$$\frac{\partial}{\partial t} Q = \Delta Q + 2\langle \nabla L, \nabla Q \rangle + 2 \left| \nabla \nabla L + \frac{1}{2}(R - r) g \right|^2 + rQ.$$

Using the inequality $|a_{ij}|^2 \geq \frac{1}{2}(\operatorname{trace}_g a)^2$ for any symmetric 2-tensor on a surface, we have

COROLLARY 4.24.

$$\frac{\partial}{\partial t} Q \geq \Delta Q + 2\langle \nabla L, \nabla Q \rangle + Q^2 + rQ. \tag{15}$$

Applying the maximum principle to equation (15) implies:

PROPOSITION 4.25. *There exists a constant $C > 1$ depending only on g_o such that:*

$$Q = \frac{\partial}{\partial t} \ln R - |\nabla \ln R|^2 \geq -\frac{Cr\, e^{rt}}{C\, e^{rt} - 1}.$$

This estimate for Q is known as a *differential Harnack inequality.* Integrating it along paths in space and time yields a classical Harnack inequality which gives a lower bound for the curvature at some point and time in terms of the curvature at an earlier time and another point. In particular, let $x_1, x_2 \in M$ be any two points and $0 \leq t_1 < t_2$ be two times. Define

$$A = A(x_1, t_1, x_2, t_2) = \inf_\gamma \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt} \right|^2 dt,$$

where the infimum is taken over all C^1-paths $\gamma : [t_1, t_2] \to M$ joining x_1 and x_2. Then we have

PROPOSITION 4.26. *Let $(M, g(t))$ be a solution to the normalized Ricci flow. If $x_1, x_2 \in M$ and $0 \leq t_1 < t_2$, then there exists a constant $C > 1$ depending only on g_0 such that:*

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq \frac{C\, e^{rt_1} - 1}{C\, e^{rt_2} - 1} \cdot e^{-\frac{1}{4}A} \geq e^{-C(t_2 - t_1)} \cdot e^{-\frac{1}{4}A}.$$

4.8. Uniform bounds on R. In this subsection we show how the entropy estimate and the Harnack inequality may be used to obtain uniform positive upper and lower bounds for the scalar curvature in the case where $R(g_o) > 0$. The upper bound may also be obtained using the gradient estimate for the scalar curvature of subsection 4.5.

PROPOSITION 4.27. *If the initial metric g_o has positive scalar curvature $R(g_o) > 0$, then there exists a constant $C \in [1, \infty)$ such that*

$$R(x, t) \leq C$$

for all $x \in M$ and $t \in [0, \infty)$.

We divide the outline of the proof into several steps below. Our goal is to show that for any t_o, $R_{\max}(t_o)$ is bounded above by some constant depending only on g_o. First we show that $R_{\max}$ at most doubles on a small time-interval.

LEMMA 4.28. *Given any $t_o \in [0,\infty)$, we have*

$$R(x,t) \le 2R_{\max}(t_o)$$

for all $t \in [t_o, t_o + \frac{1}{2R_{\max}(t_o)}]$ and $x \in M$.

This implies that the metrics are uniformly equivalent in that same time-interval.

COROLLARY 4.29. *Given any $t_o \in [0,\infty)$, we have*

$$\frac{1}{e} g(x,t_o) \le g(x,t) \le \sqrt{e} \cdot g(x,t_o)$$

for all $t \in [t_o, t_o + \frac{1}{2R_{\max}(t_o)}]$ and $x \in M$.

Let $p \in M$ be a point such that

$$R(p,t_o) = R_{\max}(t_o).$$

We next show that at time $t_o + \frac{1}{2R_{\max}(t_o)}$ in a small ball about p, R is comparable to $R(p,t_o)$.

LEMMA 4.30. *Assume that $R_{\max}(t_o) \ge 1$. Given any constant C', there exists a constant C depending only on g_o and C' such that if*

$$\rho(x,p) \le \frac{C'}{\sqrt{R_{\max}(t_o)}},$$

then

$$R(x, t_o + \frac{1}{2R_{\max}(t_o)}) \ge C \cdot R_{\max}(t_o).$$

Next we show that the diameter of (M,g) is uniformly bounded from above.

LEMMA 4.31. *There exists a constant $C > 0$ depending only on g_0 such that*

$$\operatorname{diam}(M, g(t)) \le C.$$

The lemma above enables us to apply the Harnack inequality again to obtain a positive lower bound for R.

LEMMA 4.32. *There exists a constant $c > 0$ depending only on g_0 such that*

$$R(x,t) \ge c > 0,$$

for all $x \in M$ and $t \in [0,\infty)$.

4.9. Asymptotic approach to soliton. In this subsection we show that when $R > 0$ the metric approaches a Ricci soliton as $t \to \infty$ under the normalized Ricci flow. Recall from subsection 4.4 that on a Ricci soliton

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2}(R - r)\, g_{ij} \equiv 0.$$

LEMMA 4.33. *Under the normalized Ricci flow,*

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} - 2RM_{ij} + rM_{ij}.$$

Next we compute the evolution equation for the square of the norm of M_{ij} :

LEMMA 4.34.

$$\frac{\partial}{\partial t}|M_{ij}|^2 = \triangle |M_{ij}|^2 - 2|\nabla_k M_{ij}|^2 - 2R|M_{ij}|^2.$$

Using the estimate $R > C > 0$ and the maximum principle yields:

PROPOSITION 4.35. *Under the normalized Ricci flow, there exists a constants $C_1, C_2 > 0$ such that*

$$|M_{ij}|^2 \leq C_1 \cdot e^{-C_2 t}.$$

4.10. Convergence of the normalized flow. We now consider the modified flow

$$\frac{\partial}{\partial t} g_{ij} = -Rg_{ij} + (L_{\nabla f} g)_{ij} = 2M_{ij}.$$

The solution $\bar{g}(t)$ of this flow with $\bar{g}(0) = g_0$ is equivalent to the solution $g(t)$ of the original normalized flow with $g(0) = g_0$ in that there exists a one-parameter family of diffeomorphisms ϕ_t of M^2 such that

$$\bar{g}(t) = \phi_t^* g(t).$$

Hence for the modified flow, we also have the estimate

$$|\bar{M}_{ij}|^2 \leq C_1 \cdot e^{-C_2 t}.$$

This implies that the modified flow converges exponentially to a limit metric $\bar{g}_\infty$ (one can obtain the necessary higher derivative estimates.) The limit metric for the modified flow satisfies

$$(\bar{M}_\infty)_{ij} \equiv 0.$$

Furthermore, the curvature and its derivatives converge to their limit exponentially fast. Now by Proposition 4.14, we conclude that $\bar{g}_\infty$ has constant curvature:

$$\bar{R}_\infty \equiv r.$$

This implies that $\bar{R}(t)$ tends to a constant exponentially fast, which in turn implies that the solution $g(t)$ to the original normalized flow converges exponentially fast to a constant curvature metric g_∞. This completes the outline of the proof of part 2 of Theorem 4.2.

5. Three-manifolds

In this section we present Hamilton's seminal result concerning the Ricci flow on closed 3-manifolds with positive Ricci curvature. In this exposition we closely follow Hamilton's paper [**19**], while omitting most of the detailed computations. We suggest that the reader consult [**20**] for a proof of the curvature estimates which simplified his earlier computations, but which also requires more machinery.

Let (M^n, g) be a closed Riemannian n-manifold with positive Ricci curvature. By Myers' Theorem, the fundamental group of M is finite (one shows that a positive lower bound for the Ricci curvature gives an upper bound for the diameter of the manifold, and then applies this result to the universal cover $\tilde{M}$ with the lifted metric $\tilde{g}$ to conclude that $\tilde{M}$ is compact.) When $n = 3$, $\tilde{M}^3$ is compact and simply-connected, and hence a homotopy 3-sphere. We have the well-known

CONJECTURE 1. *(Poincaré)* Any closed simply-connected 3-manifold is diffeomorphic to S^3.

Hence, if the Poincaré Conjecture is true, one concludes that $\tilde{M}^3$ is diffeomorphic to S^3. Furthermore, we also have

CONJECTURE 2. *(Spherical Space Form)* Any discrete group of diffeomorphisms acting freely on S^3 is conjugate to a group of isometrics.

Hence, if the Spherical Space Form Conjecture is also true, we have M^3 is diffeomorphic to S^3/Γ, where Γ is a discrete subgroup of $O(4)$. In particular, M^3 admits a metric with constant positive sectional curvature. It is this last statement that Hamilton proved, the existence of a constant positive sectional curvature metric on a positively Ricci curved closed 3 -manifold - which would be a consequence of Myers' Theorem and the Poincaré and Spherical Space Form Conjectures.

THEOREM 5.1. (Hamilton 1982) *Given any smooth, compact 3-dimensional Riemannian manifold (M, g_o) with positive Ricci curvature, there exists a unique smooth solution $g(t)$ to the normalized Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{3} r \cdot g_{ij},$$

with initial condition $g(0) = g_o$ on the time interval $[0, \infty)$. Moreover, the solution $g(t)$ converges exponentially fast to a constant sectional curvature metric. In particular, M^3 is diffeomorphic to a spherical space form.

We outline the proof in the following subsections.

5.1. Positivity of the Ricci tensor is preserved. We shall first study the (unnormalized) Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij},$$

and prove estimates for the curvature and its derivatives. These estimates will imply that the solution to the normalized Ricci flow converges to a constant curvature metric. We first recall the evolution equation for the scalar curvature function under the Ricci flow (Corollary 2.5 :)

$$\frac{\partial}{\partial t} R = \triangle R + 2 |Rc|^2 .$$

By the maximum principle, if the initial metric g_o has positive scalar curvature: $R_o > 0$, then the solution $g(t)$ has positive scalar curvature: $R(t) > 0$ as long as it exists.

In dimension 3, the Riemann curvature tensor is completely determined by the Ricci tensor:

$$R_{ijkl} = R_{il} g_{jk} + R_{jk} g_{il} - R_{ik} g_{jl} - R_{jl} g_{ik} - \frac{1}{2} R \left(g_{il} \, g_{jk} - g_{ik} \, g_{jl} \right). \tag{16}$$

Recall that Corollary 2.4 says that under the Ricci flow, the evolution equation for the Ricci tensor is:

$$\frac{\partial}{\partial t} R_{jk} = \triangle R_{jk} + 2 g^{pr} g^{qs} R_{pjqk} R_{rs} - 2 g^{pq} R_{pj} R_{qk}.$$

Applying (16) to this yields

LEMMA 5.2. *Under the Ricci flow, the Ricci tensor satisfies the following reaction-diffusion equation:*

$$\frac{\partial}{\partial t} R_{jk} = \triangle R_{jk} - 6 g^{pq} R_{jp} R_{qk} + 3R\, R_{jk} + \left(2 |Rc|^2 - R^2\right) g_{jk},$$

where $|Rc|^2 = g^{pq} g^{rs} R_{pr} R_{qs}$ *is the square of the norm of the Ricci tensor.*

Applying the maximum principle for tensors Proposition 3.3, we have

COROLLARY 5.3. *If the initial metric* g_o *has semi-positive Ricci curvature:* $Rc_o \geq 0$, *then as long as the solution* $g(t)$ *to the Ricci flow exists,* $g(t)$ *has semi-positive Ricci curvature:* $Rc(t) \geq 0$.

PROOF. Let

$$S_{jk} = -6 g^{pq} R_{jp} R_{qk} + 3R\, R_{jk} + \left(2 |Rc|^2 - R^2\right) g_{jk}.$$

We need to show that if V is a null-eigenvector of R_{jk}, i.e., $R_{jk} V^j = 0$, then

$$S_{jk} V^j V^k \geq 0.$$

Diagonalizing the Ricci tensor with respect to the metric

$$R_{jk} = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}$$

we find that the tensor S_{jk} is also diagonal and is given by:

$$S_{jk} = \begin{pmatrix} \begin{matrix} -2\lambda^2 + \mu^2 + \nu^2 \\ +\lambda\mu + \lambda\nu - 2\mu\nu \end{matrix} & & \\ & \begin{matrix} -2\mu^2 + \lambda^2 + \nu^2 \\ +\mu\lambda + \mu\nu - 2\lambda\nu \end{matrix} & \\ & & \begin{matrix} -2\nu^2 + \lambda^2 + \mu^2 \\ +\nu\lambda + \nu\mu - 2\lambda\mu \end{matrix} \end{pmatrix}.$$

If $\lambda = 0$ with corresponding (unit) null-eigenvector V, then

$$S_{jk} V^j V^k = \mu^2 + \nu^2 - 2\mu\nu = (\mu - \nu)^2 \geq 0.$$

hence the null-eigenvector condition is satisfied and the proposition follows.

5.2. Pinching of the Ricci tensor is preserved. Let $\lambda \leq \mu \leq \nu$ denote the eigenvalues (in increasing order) of the Ricci tensor with respect to the metric. The corollary says that if $\lambda \geq 0$ at $t = 0$, then $\lambda \geq 0$ for all $t \geq 0$. Next we show that any positive pinching of the Ricci tensor is preserved. That is, if there exists an $\epsilon > 0$ such that

$$\lambda \geq \epsilon\, (\lambda + \mu + \nu)$$

at $t = 0$, then $\lambda \geq \epsilon\, (\lambda + \mu + \nu)$ for all $t \geq 0$. A computation yields

LEMMA 5.4. *Under the Ricci flow,*

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{R_{jk}}{R}\right) &= \triangle\left(\frac{R_{jk}}{R}\right) + \frac{2}{R} g^{pq} \nabla_p R\, \nabla_q \left(\frac{R_{jk}}{R}\right) \\ &\quad + \frac{-6 g^{pq} R_{jp} R_{qk} + 3R\, R_{jk} + \left(2 |Rc|^2 - R^2\right) g_{jk}}{R} - \frac{R_{jk}}{R^2} \cdot 2 |Rc|^2 \end{aligned}$$

Again, applying the maximum principle for tensors, we obtain

COROLLARY 5.5. *If the initial metric g_o has positive scalar curvature $R_o > 0$ and satisfies the pinching condition: $Rc_o \geq \epsilon R_o g_o$, for some $\epsilon > 0$, then as long as the solution $g(t)$ to the Ricci flow exists, $g(t)$ also has positive scalar curvature $R(t) > 0$ and satisfies the pinching condition: $Rc(t) \geq \epsilon R(t) g(t)$.*

PROOF. We compute that

$$\frac{\partial}{\partial t}\left(\frac{R_{jk}}{R} - \epsilon\, g_{jk}\right) = \Delta\left(\frac{R_{jk}}{R} - \epsilon\, g_{jk}\right) + \frac{2}{R} g^{pq} \nabla_p R \nabla_q \left(\frac{R_{jk}}{R}\right) + T_{jk},$$

where

$$T_{jk} = \frac{-6 g^{pq} R_{jp} R_{qk} + 3R\, R_{jk} + \left(2\,|Rc|^2 - R^2\right) g_{jk}}{R} - \frac{R_{jk}}{R^2} \cdot 2\,|Rc|^2 + 2\epsilon\, R_{jk}.$$

Since the positivity of the scalar curvature is preserved under the Ricci flow, it is sufficient to show that if V is a null-eigenvector of $\frac{R_{jk}}{R} - \epsilon\, g_{jk}$, i.e., $\left(\frac{R_{jk}}{R} - \epsilon\, g_{jk}\right) V^j = 0$, then

$$R^2 T_{jk} V^j V^k \geq 0.$$

Diagonalizing the Ricci tensor as before and assuming

$$\lambda - \epsilon\,(\lambda + \mu + \nu) = 0$$

with corresponding (unit) null-eigenvector V, we find that the tensor $R^2 T_{jk}$ is also diagonal and:

$$R^2 T_{jk} V^j V^k = \lambda^2\,(-2\lambda + \mu + \nu) + (\mu + \nu)\,(\mu - \nu)^2.$$

Since $0 < \lambda \leq \mu \leq \nu$, we have

$$R^2 T_{jk} V^j V^k \geq 0,$$

and the null-eigenvector condition is satisfied.

5.3. Pinching improves. The corollary says that the pinching constant of the Ricci tensor is preserved. Now we will show that the pinching constant improves. We consider the scalar quantity

$$|Rc|^2 - \frac{1}{3} R^2 = \left| Rc - \frac{1}{3} R\, g \right|^2.$$

Since $n = 3$, a metric is Einstein: $Rc - \frac{1}{3} R\, g = 0$ if and only if it has constant sectional curvature. Thus this quantity measures the difference of the metric from having constant sectional curvature. For a quantity to have geometric meaning independent of the size of the metric, it is necessary for it to be scale-invariant, i.e., if the metric is multiplied by a constant, then the quantity remains unchanged. A scale-invariant quantity measuring the difference of the metric from having constant sectional curvature is:

$$\frac{\left|Rc - \frac{1}{3} R\, g\right|^2}{R^2} = \frac{|Rc|^2}{R^2} - \frac{1}{3}.$$

From Corollary 5.5, we expect that the maximum of this quantity decreases in time, which is actually the case. However, more is true; namely the maximum of the quantity

$$\frac{\left|Rc - \frac{1}{3} R\, g\right|^2}{R^{2-\delta}}$$

is decreasing in time provided $\delta > 0$ is sufficiently small. Since we are assuming the initial metric has positive Ricci curvature, which is preserved under the Ricci flow, the metric is always shrinking under the Ricci flow. Furthermore, from the evolution equation for R, the minimum of the scalar curvature is increasing in time under the Ricci flow. Thus one would hope that the minimum of the scalar curvature increases to infinity as t approaches the final time. If this is the case then we can conclude that the scale-invariant quantity measuring the difference of the metric from having constant sectional curvature decreases to zero since:

$$\frac{\left|Rc - \frac{1}{3} R\, g\right|^2}{R^2} \leq C\, R^{-\delta}.$$

One would then expect that under the normalized Ricci flow, the metric converges to a constant curvature metric. We now proceed to prove the estimate for:

$$f := \frac{|Rc|^2 - \frac{1}{3} R^2}{R^{2-\delta}}.$$

The evolution equation for f is given by

LEMMA 5.6. *For* $\delta \in [0,1]$,

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1-\delta)}{R} \langle \nabla R, \nabla f \rangle + \frac{2}{R^{3-\delta}} \left(2U + \delta\, |Rc|^2 \left(|Rc|^2 - \frac{1}{3} R^2 \right) \right),$$

where

$$\begin{aligned} U &= -|Rc|^4 - 2R \cdot tr_g\left(Rc^3\right) + \frac{5}{2} R^2 |Rc|^2 - \frac{1}{2} R^4 \\ &= -\lambda^2 (\lambda - \mu)(\lambda - \nu) - \left[\nu^2 (\nu - \lambda) - \mu^2 (\mu - \lambda)\right](\nu - \mu) \leq 0 \end{aligned}$$

By applying the maximum principle to the lemma, we have

COROLLARY 5.7. *If* $Rc(g_o) > 0$, *then there exists a* $\bar{\delta} > 0$ *depending only on* g_o *such that if* $\delta \in [0, \bar{\delta}]$ *and* $R^{\delta-2}\left(|Rc|^2 - \frac{1}{3}R^2\right) \leq C$ *at* $t = 0$, *then*

$$\frac{|Rc|^2 - \frac{1}{3} R^2}{R^2} \leq C \cdot R^{-\delta}$$

as long as the solution exists.

PROOF. It suffices show that

$$2U + \delta\, |Rc|^2 \left(|Rc|^2 - \frac{1}{3} R^2 \right) \leq 0 \tag{17}$$

for $\delta > 0$ sufficiently small depending on g_o. We compute:

$$U \leq -\lambda^2 (\lambda - \mu)^2 - \nu^2 (\nu - \mu)^2 .$$

Since $Rc(g_o) > 0$ and M is compact, there exists a constant $\epsilon > 0$ such that $Rc(g_o) \geq \epsilon R_o g_o$. By lemma 5.5 , we have $Rc \geq \epsilon R g$ as long as the solution exists. Hence

$$U \leq -\epsilon^2 R^2 \left[(\lambda - \mu)^2 + (\nu - \mu)^2 \right].$$

On the other hand,

$$|Rc|^2 - \frac{1}{3} R^2 = \frac{1}{3} \left[(\lambda - \mu)^2 + (\lambda - \nu)^2 + (\mu - \nu)^2 \right] \leq \left[(\lambda - \mu)^2 + (\nu - \mu)^2 \right].$$

Thus

$$U \leq -\epsilon^2 R^2 \left(|Rc|^2 - \frac{1}{3} R^2 \right),$$

and inequality (17) holds for all

$$\delta \leq 2\epsilon^2.$$

5.4. The gradient estimate for the scalar curvature. In this subsection we obtain a gradient estimate for the scalar curvature. This estimate is important because it enables us to compare curvatures at different points, whereas the pinching estimate of the previous section is a pointwise estimate for the curvatures. Note that the contracted second Bianchi identity implies that an Einstein metric (which is a solution to the degenerate elliptic equation $0 = -R_{ij} + \frac{1}{n} r g_{ij}$) in dimension at least 3 has constant scalar curvature, i.e., the gradient of the scalar curvature is zero. In the case of the normalized Ricci flow on a closed 3-manifold, which is the degenerate parabolic equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij},$$

we have that the metric approaches an Einstein metric where the scalar curvature becomes large. In particular, we have the following estimate (where the left-hand-side is a scale invariant quantity measuring the difference of the metric from being Einstein:)

$$\frac{|Rc|^2 - \frac{1}{3} R^2}{R^2} \leq C \cdot R^{-\delta}.$$

Hence it is natural to expect that the gradient of the scalar curvature approaches zero in some sense. Moreover, one might also expect that in order to prove this, one needs to use the contracted second Bianchi identity. Both are in fact the case, using the contracted second Bianchi identity we shall show that given any β sufficiently small, there exists a constant C depending only on β and g_o such that

$$|\nabla R|^2 \leq \beta R^{3-\bar{\delta}/2} + C.$$

This estimate may written as

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\bar{\delta}/2} + C\, R^{-3},$$

where the left-hand-side is a scale invariant quantity and the right-hand-side is small for R large. To prove this estimate, we need to compute several evolution equations, the first of which is for the square of the norm of the gradient of the scalar curvature. A computation gives

LEMMA 5.8.

$$\frac{\partial}{\partial t}|\nabla R|^2 = \triangle |\nabla R|^2 - 2|\nabla\nabla R|^2 + 4\langle \nabla R, \nabla |Rc|^2\rangle.$$

Next we divide $|\nabla R|^2$ by R (the reader may wonder why we divide by R and not some other power of R, the computations bear this out) and compute its evolution equation.

LEMMA 5.9.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) &= \triangle\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 \\ &\quad +\frac{4}{R}\langle \nabla R, \nabla |Rc|^2\rangle - \frac{2|Rc|^2}{R^2}|\nabla R|^2.\end{aligned}$$

The evolution equation for $\frac{|\nabla R|^2}{R}$ has only one bad (positive) term on the right-hand-side, which is

$$\frac{4}{R}\langle \nabla R, \nabla |Rc|^2\rangle.$$

We remedy this by adding to $\frac{|\nabla R|^2}{R}$ the quantity $|Rc|^2 - \frac{1}{3}R^2$, which introduces a good (negative) term which cancels out the bad term. The evolution equation for $|Rc|^2 - \frac{1}{3}R^2$ is:

LEMMA 5.10.

$$\begin{aligned}\frac{\partial}{\partial t}\left(|Rc|^2 - \frac{1}{3}R^2\right) &= \triangle\left(|Rc|^2 - \frac{1}{3}R^2\right) - 2\left(|\nabla Rc|^2 - \frac{1}{3}|\nabla R|^2\right) \\ &\quad -8tr_g(Rc^3) + \frac{26}{3}R|Rc|^2 - 2R^3.\end{aligned}$$

The good term in the evolution equation for $|Rc|^2 - \frac{1}{3}R^2$ is

$$-2\left(|\nabla Rc|^2 - \frac{1}{3}|\nabla R|^2\right).$$

That this term dominates the bad term $\frac{4}{R}\langle \nabla R, \nabla |Rc|^2\rangle$ follows from:

LEMMA 5.11.

$$|\nabla Rc|^2 - \frac{1}{3}|\nabla R|^2 \geq \frac{1}{37}|\nabla Rc|^2.$$

PROOF. Using the fact that for any 2-tensor a_{ij} (not necessarily symmetric:)

$$|a_{ij}|^2 \geq \frac{1}{3}\left(g^{ij}a_{ij}\right)^2,$$

we have by the second contracted Bianchi identity

$$\begin{aligned}|\nabla Rc|^2 - \frac{1}{3}|\nabla R|^2 &= \left|\nabla_i R_{jk} - \frac{1}{3}\nabla_i R\, g_{jk}\right|^2 \geq \frac{1}{3}\left|g^{ij}\left(\nabla_i R_{jk} - \frac{1}{3}\nabla_i R\, g_{jk}\right)\right|^2 \\ &= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3}\right)^2|\nabla R|^2 = \frac{1}{108}|\nabla R|^2,\end{aligned}$$

and the lemma follows easily.

COROLLARY 5.12.

$$\begin{aligned}\frac{\partial}{\partial t}\left(|Rc|^2-\frac{1}{3}R^2\right) &= \triangle\left(|Rc|^2-\frac{1}{3}R^2\right)-\frac{2}{37}|\nabla Rc|^2 \\ &\quad -8tr_g(Rc^3)+\frac{26}{3}R\,|Rc|^2-2R^3.\end{aligned}$$

Since we have (using the estimates $|Rc|\leq R$ and $|\nabla R|\leq\sqrt{3}\,|\nabla Rc|$ to obtain the last inequality:)

$$\frac{4}{R}\left|\langle\nabla R,\nabla|Rc|^2\rangle\right|\leq\frac{8\,|Rc|}{R}|\nabla R|\cdot|\nabla Rc|\leq 8\sqrt{3}\,|\nabla Rc|^2,$$

we consider the quantity

$$V=\frac{|\nabla R|^2}{R}+\frac{37}{2}\left(8\sqrt{3}+1\right)\left(|Rc|^2-\frac{1}{3}R^2\right)$$

which satisfies the evolution equation:

$$\begin{aligned}\frac{\partial}{\partial t}V &= \Delta V-2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2-\frac{2\,|Rc|^2}{R^2}|\nabla R|^2-|\nabla Rc|^2 \\ &\quad +\frac{37}{2}\left(8\sqrt{3}+1\right)\left(-8\mathrm{tr}_g(Rc^3)+\frac{26}{3}R\,|Rc|^2-2R^3\right).\end{aligned}$$

The only bad term on the right-hand-side is

$$W=-8\mathrm{tr}_g(Rc^3)+\frac{26}{3}R\,|Rc|^2-2R^3.$$

Some algebra yields:

LEMMA 5.13.

$$W\leq\frac{50}{3}R\left(|Rc|^2-\frac{1}{3}R^2\right).$$

PROOF. We may rewrite W as:

$$W=-8\left\langle R_{ij}-\frac{1}{3}Rg_{ij},g^{kl}R_{ik}R_{jl}\right\rangle+6R\left(|Rc|^2-\frac{1}{3}R^2\right).$$

The first term on the right-hand-side may be estimated as follows:

$$-8\left\langle R_{ij}-\frac{1}{3}Rg_{ij},g^{kl}R_{ik}R_{jl}\right\rangle\leq\frac{32}{3}R\left|R_{ij}-\frac{1}{3}Rg_{ij}\right|^2$$

and the lemma follows.

Combining the lemmas above, we have

$$\begin{aligned}\frac{\partial}{\partial t}V &\leq \Delta V-|\nabla Rc|^2+\frac{37}{2}\left(8\sqrt{3}+1\right)\frac{50}{3}R\left(|Rc|^2-\frac{1}{3}R^2\right) \\ &\leq \Delta V-|\nabla Rc|^2+CR^{3-\bar{\delta}}.\end{aligned}$$

On the other hand, we compute

$$\frac{\partial}{\partial t}R^{2-\bar{\delta}/2}=\triangle\left(R^{2-\bar{\delta}/2}\right)-\left(2-\bar{\delta}/2\right)\left(1-\bar{\delta}/2\right)R^{-\bar{\delta}/2}|\nabla R|^2+2(2-\bar{\delta}/2)R^{1-\bar{\delta}/2}|Rc|^2.$$

If we choose $\bar{\beta}$ depending only on g_o such that

$$\bar{\beta}\cdot\left(2-\bar{\delta}/2\right)\left(1-\bar{\delta}/2\right)R_{\min}(0)^{-\bar{\delta}/2}\leq\frac{1}{3},$$

then using the estimate

$$|\nabla R|^2\leq 3\,|\nabla Rc|^2,$$

we have for all $\beta\in[0,\bar{\beta}]$:

$$\begin{aligned}\frac{\partial}{\partial t}\left[V-\beta R^{2-\bar{\delta}/2}\right] &\leq \triangle\left[V-\beta R^{2-\bar{\delta}/2}\right]+CR^{3-\bar{\delta}}-\beta 2(2-\bar{\delta}/2)R^{1-\bar{\delta}/2}\,|Rc|^2\\ &\leq \triangle\left[V-\beta R^{2-\bar{\delta}/2}\right]+C(\beta,g_o).\end{aligned}$$

By the maximum principle, we conclude that

$$V-\beta R^{2-\bar{\delta}/2}\leq C,$$

for some constant C depending only on β and g_o. Hence we have the following:

PROPOSITION 5.14. *There exists a constants $\bar{\delta}$, $\bar{\beta}$ and C depending only on g_o such that for all $\beta\in[0,\bar{\beta}]$:*

$$\frac{|\nabla R|^2}{R}\leq\beta R^{2-\bar{\delta}/2}+C(\beta,g_o).$$

5.5. Convergence. Let $[0,T)$ denote the maximum time interval of existence of the solution $g(t)$ to the Ricci flow. As a consequence of the pinching improves estimate (Corollary 5.7) and the gradient estimate (Proposition 5.14,) one can show that

LEMMA 5.15.

$$\lim_{t\to T}\frac{R_{\max}(t)}{R_{\min}(t)}=1.$$

We now consider the solution $\tilde{g}\left(\tilde{t}\right)$ to the volume normalized equation with the same initial data

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}_{ij} &= \frac{2}{3}\tilde{r}\tilde{g}_{ij}-2\tilde{R}_{ij}\\ \tilde{g}(0) &= g_o,\end{aligned}$$

which exists on a maximal time interval $[0,\tilde{T})$. Now combining the pinching estimate (all scale-invariant estimates for solutions of the unnormalized flow also hold for solutions of the normalized flow)

$$\tilde{R}_{ij}\geq\varepsilon\tilde{R}\tilde{g}_{ij}$$

and Myers' Theorem with the fact that the volume is preserved under the flow, yields a uniform upper bound for the minimum scalar curvature

$$\tilde{R}_{\min}\left(\tilde{t}\right)\leq C<\infty$$

for all $\tilde{t}\in[0,\tilde{T})$. Thus, by the lemma above, we also have

$$\tilde{R}_{\max}\left(\tilde{t}\right)\leq C<\infty.$$

Using Corollary 5.7 and the above estimates for the scalar curvature, Hamilton [**19**], section 17, now proves that the solution exists for all time, exponentially converges to a limit metric which has constant positive sectional curvature.

6. Four-manifolds

In this section we outline the proof of Hamilton's classification 4 -manifolds with positive curvature operator.

THEOREM 6.1. (Hamilton 1986) *Given any smooth, compact 4-dimensional Riemannian manifold* (M, g_o) *with positive curvature operator, there exists a unique smooth solution* $g(t)$ *to the normalized Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{1}{2} r \cdot g_{ij},$$

with initial condition $g(0) = g_o$ *on the time interval* $[0, \infty)$. *Moreover, the solution* $g(t)$ *converges to a constant sectional curvature metric. In particular,* M^4 *is diffeomorphic to either* S^4 *or* $\mathbb{R}P^4$.

The general strategy of the proof is much along the lines of Hamilton's 3 - manifold result, except that the analysis of the curvature operator is significantly more complicated. We start by recalling that

$$Rm : \wedge^2 M \to \wedge^2 M$$

is a self-adjoint linear map at each point in the manifold, and satisfies the equation (1)

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + \left(Rm^2\right)_{ijkl} + \left(Rm^{\#}\right)_{ijkl},$$

where $Rm^{\#}$ is the square using the Lie algebra structure constants of $\wedge^2 M$. Now since $n = 4$, we have the decomposition of 2-forms

$$\wedge^2 M = \wedge^2_+ M \oplus \wedge^2_- M,$$

where

$$\begin{aligned} \wedge^2_+ &= \left\{\alpha \in \wedge^2 : *\alpha = \alpha\right\} \\ \wedge^2_- &= \left\{\alpha \in \wedge^2 : *\alpha = -\alpha\right\}, \end{aligned}$$

into self-dual and anti-self-dual 2-forms corresponding to the isomorphism

$$so(4) \cong so(3) \oplus so(3).$$

The Lie algebra bracket restricted to each of the factors $\wedge^2_+$ and $\wedge^2_-$ is the cross-product (since they are isomorphic to $so(3)$.) Hence if we decompose the Riemann curvature operator as

$$Rm = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where $A : \wedge^2_+ \to \wedge^2_+$, $B : \wedge^2_- \to \wedge^2_+$, and $C : \wedge^2_- \to \wedge^2_-$, then

$$Rm^{\#} = \begin{pmatrix} A^{\#} & B^{\#} \\ \left(B^{\#}\right)^t & C^{\#} \end{pmatrix}.$$

The ODE corresponding the PDE (1)

$$\frac{dS}{dt} = S^2 + S^{\#}.$$

may be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}A &= \Delta A + A^2 + 2A^{\#} + B^tB\\ \frac{\partial}{\partial t}B &= \Delta B + AB + BC + 2B^{\#}\\ \frac{\partial}{\partial t}C &= \Delta C + C^2 + 2C^{\#} + {}^tBB.\end{aligned}$$

Let $a_1 \le a_2 \le a_3$, $b_1 \le b_2 \le b_3$, $c_1 \le c_2 \le c_3$ denote the ordered eigenvalues of the symmetric matrices A, $\sqrt{B^tB} = \sqrt{BB^t}$, C, respectively. Under the above system of ODEs, we find that the eigenvalues evolve by

$$\begin{aligned}\frac{d}{dt}a_1 &\ge a_1^2 + a_2a_3 + b_1^2\\ \frac{d}{dt}a_3 &\le a_3^2 + a_1a_2 + b_3^2\\ \frac{d}{dt}c_1 &\ge c_1^2 + c_2c_3 + b_1^2\\ \frac{d}{dt}c_3 &\le c_3^2 + c_1c_2 + b_3^2\\ \frac{d}{dt}(b_2+b_3) &\le a_2b_2 + a_3b_3 + b_2c_2 + b_3c_3 + 2b_1b_2 + 2b_1b_3.\end{aligned}$$

The maximum principle for systems (Theorem 3.4,) reduces the problem of obtaining bounds for Rm to finding suitable convex sets in $so(4) \otimes_S so(4)$ which are preserved by the system of ordinary differential equations above. Using this method and the above inequalities for the evolution of the eigenvalues, Hamilton proved the following:

PROPOSITION 6.2. *Depending on the initial metric g_o, there exist constants $C < \infty$ and $\varepsilon > 0$ such that the following inequalities remain true under the flow as long as the solution exists:*

1. $a_3 \le C \cdot a_1$
2. $c_3 \le C \cdot c_1$
3. $(b_2 + b_3)^2 \le C \cdot a_1c_1$
4. $(b_2 + b_3)^{2+\varepsilon} \le C \cdot a_1c_1(a - 2b + c)^{\varepsilon}$
5. $(b_2 + b_3)^{2+\varepsilon} \le C \cdot a_1c_1$
6. $a_3 \le a_1 + C \cdot a_1^{1-\varepsilon}$
7. $c_3 \le c_1 + C \cdot c_1^{1-\varepsilon}$.

The above estimates should be viewed as a 4-dimensional analogue to Corollary 5.5 and Corollary 5.7 in the 3-dimensional case. In particular, these estimates enable one to prove the same gradient estimate for the scalar curvature as in the 3-dimensional case (Proposition 5.14,) which in turn leads to the long-time existence and convergence of the solution to a constant curvature metric.

7. Concluding remarks

We have not included in this account the important recent work of Hamilton on the analysis of singularities which develop under the flow and the possibility of performing geometric surgeries to avoid these singularities **[27]**, **[24]**, **[23]** (see **[12]** for a brief survey of this material,) nor the important works of others, such as (we refer to [**27**] for a more complete bibliography:)

1. Positive curvature pinched manifolds - H. Chen [**13**], Gerhard Huisken [**31**], Christophe Margerin [**43**], Seiki Nishikawa [**46**]
2. Kähler manifolds - S. Bando [**1**], Huai-Dong Cao [**7**], [**8**], [**9**], [**10**], [**11**], Ngaiming Mok [**45**], Wan-Xiong Shi [**49**], [**50**], [**51**]
3. Ricci solitons - Robert Bryant [**5**], Tom Ivey [**33**], [**35**], [**36**], Koiso [**38**]
4. Negative curvature pinched manifolds - Maung Min-Oo [**44**], Rugang Ye [**55**]
5. Surfaces - Bartz-Struwe-Ye [**3**], Chow [**14**], Daskalopoulos-del Pino [**17**], Hamilton [**22**], Lang-Fang Wu [**53**], [**54**]
6. Special metrics - Cafora-Isenberg-Jackson [**6**], Hamilton-Isenberg [**30**], Isenberg-Jackson [**32**], Dan Knopf [**37**], Leviton-Rubinstein [**40**]
7. Smoothing properties - S. Bando [**2**], Bemelmans-Min-Oo-Ruh [**4**]
8. Harnack inequalities and applications - Hamilton [**25**], [**26**], Sun-Chin Chu [**16**]
9. Manifolds with boundary - Ying Shen [**48**]
10. Compactness - Hamilton [**29**]

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