Einstein Metrics from Symmetry and Bundle Constructions

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Introduction.

In this article we will primarily discuss the construction of Einstein metrics whose holonomy group is generic, i.e., the restricted holonomy is $SO(n)$, where n is the dimension of the manifold. Unfortunately, such Einstein metrics are not at all well-understood. There is no known obstruction for Einstein metrics in dimensions greater than 4, nor is there a general existence theorem for Einstein metrics with generic holonomy. For a discussion of obstructions in dimension 4, see the essay by LeBrun in this volume.

Recall that the Einstein equation $Ric(g) = \Lambda g$ is a non-linear second order system of partial differential equations which is invariant under the action of the diffeomorphism group of the manifold. (We will call the constant Λ the *Einstein constant*, while physicists call it the *cosmological constant.*) In the absence of any general understanding of the solutions of this system, the current strategy for constructing examples is to employ either symmetry or bundle structures to reduce the Einstein equation to more manageable systems of equations.

By the use of symmetry we mean constructing Einstein metrics having a finite-dimensional Lie group of isometries. Generally speaking, progress has been made only when the Lie group acts transitively on the manifold or acts with hypersurface principal orbits. Under these assumptions, the Einstein equation becomes respectively a system of algebraic or ordinary differential equations.

By the use of bundle structures we mean constructing Einstein metrics on the total spaces of bundles which are put together from special families of metrics on the fibres and base using suitable connections. In this situation the Einstein condition translates into a coupled system of equations involving the Ricci curvatures of the fibres and base, as well as the curvature of the connection. Since bundles have structural groups which play a role in the construction, we may, in the spirit of physicists, regard bundle constructions as exploiting the "internal" symmetry of the manifolds. Indeed, these bundle constructions originated from Kaluza-Klein theories of supergravity.

Where the methods surveyed here also produce Einstein metrics with special holonomy, a brief account of the results will be given. The reader is referred to the relevant chapters in this volume for further information.

We do not claim to give a complete survey of all work done in the above topics. Rather, this article only surveys those developments which the author knows how to link into a coherent whole. Also, in view of the excellent book of A. Besse [**17**], we will concentrate only on developments in the last decade.

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1. Kaluza-Klein Constructions on Principal and Fibre Bundles.

Let $\pi : P \to M$ be a smooth principal G-bundle, where G is a compact Lie group, and n and d denote respectively the dimensions of P and M. Let ϕ be a connection on P with curvature form $\Omega = d\phi + [\phi, \phi]$. Given a left-invariant metric $\langle\ ,\ \rangle$ on G and a metric g^* on M, we may use ϕ to construct a metric g on P given by the formula

$$g(X,Y) = g^*(\pi_*(X), \pi_*(Y)) + \langle \phi(X), \phi(Y) \rangle. \tag{1.1}$$

Then $\pi : (P, g) \to (M, g^*)$ becomes a Riemannian submersion with totally geodesic fibres. We refer readers to Chapter 9 of [**17**] for the basic theory of Riemannian submersions. The connection ϕ is said to be *Yang-Mills* if Ω is coclosed as an $ad(\mathfrak{g})$-valued 2-form on M. Clearly, this notion depends on the choice of g^*. The *Kaluza-Klein ansatz* is the construction of Einstein metrics g on P of the type described. In physical theories, (M, g^*) is space-time and matter fields are sections of vector bundles associated to the principal G-bundles P. A good reference for Kaluza-Klein theory from the physical viewpoint is [**50**]. For a good mathematical account, see [**23**].

The Einstein condition for g is equivalent to the system

(1.2)
$$Ric_G(\phi(U), \phi(V)) + \frac{1}{4}\sum_{i,j} \langle \Omega(e_i, e_j), \phi(U)\rangle \langle \Omega(e_i, e_j), \phi(V)\rangle = \Lambda \langle \phi(U), \phi(V) \rangle,$$

$$Ric(g^*)(\pi_*(X), \pi_*(Y)) - \frac{1}{2}\sum_{i} \langle \Omega(X, e_i), \Omega(Y, e_i)\rangle = \Lambda g^*(\pi_*(X), \pi_*(Y), \tag{1.3}$$

together with the Yang-Mills condition for ϕ. (In the above, Λ is the Einstein constant, U, V are vertical tangent vectors, X, Y are horizontal tangent vectors, and $\{e_1, \cdots, e_d\}$ is a g-orthonormal basis of horizontal tangent vectors.) This Einstein condition follows immediately from (9.61) in [**17**], and the "unknowns" in the above system are ϕ, $\langle\ ,\ \rangle$, and g^*. The Yang-Mills condition is equivalent to the condition that the horizontal and vertical distributions of ϕ are orthogonal with respect to $Ric(g)$.

As was pointed out in [**17**, (9.62)], the Einstein condition implies that g^* has constant scalar curvature and the pointwise norm of Ω must be constant. The latter condition obviously holds if Ω is parallel or if M is homogeneous and ϕ is an invariant connection. However, it is not clear to the author how strong this condition is, especially when G is non-abelian.

The equations (1.2) and (1.3) are in general coupled equations on P. If $\langle\ ,\ \rangle$ is a bi-invariant metric, then (1.3) becomes an equation on M and (1.2) is invariant under the right action of G. However, it is a non-vacuous condition for the second term of the left-hand side of (1.2) to be invariant under the left action of G, contrary to the claims in (9.63) of [**17**] and the ensuing corollary.

As in the situation of Kaluza-Klein theory, the case of an abelian G is more approachable and we will discuss this case first.

Let G be an r-torus T^r. Notice that all left-invariant metrics on T^r are bi-invariant. We will think of T^r as an r-fold product of circles $S^1 = \sqrt{-1}\mathbb{R}/2\pi\sqrt{-1}\mathbb{Z}$. Then a principal torus bundle P is classified by r cohomology classes $\chi_1, \cdots, \chi_r$ in $H^2(M;\mathbb{Z})$, which can be thought of as the Euler classes of the circle bundles P/T^{r-1}, where T^{r-1} ranges over the r codimension 1 subtori obtained by omitting one of the circle factors. Given a connection ϕ on P, the $\mathbb{R}^r$-valued 2-form $\frac{\sqrt{-1}}{2\pi}\Omega$ is the pull-back of an $\mathbb{R}^r$-valued 2-form $\eta = \eta_1 + \cdots + \eta_r$ on M whose components η_i represent χ_i. If a metric g^* is chosen on M, then there is a connection on P such that the corresponding 2-forms η_i are harmonic. If in addition $H^1(M;\mathbb{R}) = 0$, then the choice of ϕ is unique up to gauge equivalence. Thus when G is abelian, the Yang-Mills condition is easily satisfied. Recall, however, that the pointwise norm of η_i must also be constant.

In order to increase the chances of solving (1.2) and (1.3), we need to be able to vary g^* in a family of metrics whose Ricci tensors are simple and whose scalar curvature functions are constant. In general, the harmonic forms η_i will vary with g^*, so at least some information about this variation is required in solving the Einstein equation.

With these considerations in mind, let (M_j, J_j^*), $j = 1, \cdots, m$, be Fano manifolds, i.e., Kähler manifolds with positive first Chern class. By [**129**] they admit a Kähler metric with positive definite Ricci tensor, so by [**73**] they are simply connected. The cohomology group $H^2(M_j;\mathbb{Z})$ is torsion free and so the first Chern class $c_1(M_j)$ can be written as $p_j\alpha_j$ where p_j is a positive integer and α_j is an indivisible class in $H^2(M_j;\mathbb{Z})$.

We assume further that these Fano manifolds are equipped with a Kähler-Einstein metric g_j^* normalized so that $Ric(g_j^*) = p_j g_j^*$. This assumption is nontrivial and we refer the reader to Tian's article in this volume for up-to-date information. We will denote the Kähler form of g_j^* by ω_j^* and its Ricci form by ρ_j^*.

Now let $M = M_1 \times \cdots \times M_m$ and π_j be the projection map onto M_j. We will consider principal T^r bundles P_χ over M which are classified by cohomology classes χ_i of the form

$$\chi_i = \sum_{j=1}^{m} b_{ij}\pi_j^*\alpha_j, \qquad 1 \le i \le r,$$

where b_{ij} are integers. On M we let g^* denote a general product metric of the form $x_1 g_1^* + \cdots + x_m g_m^*$ with $x_j > 0$. Every such metric is Kähler with respect to the product complex structure on M. Furthermore, the 2-forms $\eta_i = \frac{1}{2\pi}\sum_j b_{ij}\omega_j$ are harmonic with respect to any of the product metrics g^*. We equip P_χ with a connection ϕ such that $\frac{\sqrt{-1}}{2\pi}d\phi = \pi^*(\eta_1 + \cdots + \eta_r)$.

THEOREM 1.1. [**124**] *Let $\pi : P_\chi \longrightarrow M$ be a principal r-torus bundle with characteristic classes $\chi = (\chi_1, \cdots, \chi_r)$ as described above. If the matrix $B = (b_{ij})$ has maximal rank, then there is an Einstein metric g with positive scalar curvature* *on P_χ of the form (1.1) where $\langle\ ,\ \rangle$ is a certain left-invariant metric on T^r and g^* is a certain product metric.*

Because the connection form has been fixed and the metrics g_i^* are Einstein, the Einstein condition in the situation of the theorem becomes a system of algebraic equations in the scaling parameters $x_1, \cdots, x_m$ and in the components of the left-invariant metric $\langle\ ,\ \rangle$. It turns out that the latter are determined by the former, and so we are reduced to a system involving only the x_j. This is then solved by a degree argument. Notice that the rank assumption on B is necessary in view of Bonnet-Myers, as the fundamental group of P is finite iff the rank of B is maximal. Note that the submersed product metric g^* is generally not Einstein.

When $r = 1$ and $m = 1$ in the above theorem, we recover the well-known theorem of S. Kobayashi [**74**]. The Einstein metrics on circle bundles over $\mathbb{CP}^1 \times \mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ were independently found by the physicists D'Auria, Castellani, Fré, and van Nieuwenhuizen [**35**], [**46**] in their quest for 11-dimensional supergravity theories. Circle bundles over an m-fold product of $\mathbb{CP}^1$ was studied by Rodionov [**103**] in the context of homogeneous Einstein metrics.

The Einstein manifolds constructed in Theorem 1.1 display many interesting geometrical and topological properties. Especially noteworthy are the following, whose details can be found in [**124**].

1. There are compact simply connected manifolds in all odd dimensions greater than 4 which admit infinitely many pairwise non-isometric Einstein metrics (with positive scalar curvature) belonging to different path components of the moduli space of Einstein structures. If the volumes of these Einstein metrics are normalized to be 1, then the Einstein constants have 0 as an accumulation point. For example, for each $k \geq 1, S^2 \times S^{2k+1}$ and certain non-trivial $\mathbb{RP}^{2k+1}$ or S^{4k+1} bundles over S^2 exhibit this property. Furthermore, the infinitely many Einstein metrics on any of these manifolds all have isomorphic transitive isometry groups which are not conjugate in the diffeomorphism group, and hence represent inequivalent actions by the same abstract group.

   In §2D we will describe some recent examples of C. Böhm [**19**] which include even-dimensional manifolds, e.g., S^6, S^8, admitting infinitely many inhomogeneous Einstein metrics of volume 1 such that the sequence of Einstein constants converge to a *positive* value.
2. In dimension 7, among the circle bundles over $\mathbb{CP}^1 \times \mathbb{CP}^2$, there are certain bundles P_χ such that for each homotopy 7-sphere Σ, the manifold $P_\chi \sharp \Sigma$ (connected sum) exhibits the phenomena described in (1) above. For different homotopy spheres, the spaces are homeomorphic but not diffeomorphic. These results follow from Theorem 1.1 and the classification theorem of Kreck and Stolz [**79**]. Thus it would appear that Einstein metrics with positive scalar curvature do not always show a preference for one differential structure over another.
3. Condition C of Palais-Smale consequently fails in general for the total scalar curvature functional on the space of Riemannian structures of volume 1.
4. There are Einstein metrics of positive scalar curvature (in odd dimensions) whose (connected) isometry group acts with arbitrarily large cohomogeneity. (Recall that the *cohomogeneity* of a compact Lie group action is the codimension of any principal (generic) orbit.) Indeed, provided that the

characteristic class χ is complicated enough in a suitable sense, the cohomogeneity of P_χ is the sum of the cohomogeneities of the factors of M. Hence the above fact follows from the existence of cohomogeneity 1 Kähler-Einstein Fano manifolds [**75**]. (In [**76**] Kähler-Einstein Fano manifolds of arbitrary large cohomogeneity are constructed by a blowing-down process.)

5. There are odd-dimensional Einstein manifolds with positive scalar curvature which have Einstein moduli spaces of positive dimension. Indeed these are circle bundles of sufficiently complicated topology over Kähler-Einstein Fano manifolds with positive-dimensional Kähler-Einstein moduli.

One may be tempted to extend Theorem 1.1 by letting the base (M, g^*) vary over all Kähler manifolds with constant scalar curvature, or by choosing more general elements of $H^2(M;\mathbb{Z})$ to be the characteristic classes of P_χ. However, at least in the case of circle bundles, we have the following converse.

THEOREM 1.2. [**116**] *Let $\pi : (P, g) \to (M, g^*)$ be a principal circle bundle such that g is an Einstein metric making π into a Riemannian submersion with totally geodesic fibres onto a compact Kähler manifold. Suppose further that the Euler class of P is a cohomology class of type $(1,1)$ with respect to the complex structure of M. Then (M, g^*) is isometric to a Kählerian product $\prod_j (M_j, g_j^*)$ where g_j^* is a Kähler-Einstein metric on a Fano manifold M_j and the Euler class of P is a linear combination of the first Chern classes of M_j.*

We shall give the principal ideas in the proof of the above theorem. First, since the scalar curvature of g^* must be constant, the contracted second Bianchi identity implies that the Ricci form of g^* is harmonic. Therefore, the 2-form corresponding to the second term of the left-hand side of (1.3) is also harmonic. Using these facts, one shows that the eigenvalues of the symmetric operator S given by $g^*(S(X), Y) = -\Omega(J^*(X), Y)$ are constant over M and the eigenspaces corresponding to distinct eigenvalues have constant dimension. The eigenbundles $\mathcal{E}_j$ are therefore well-defined. They are actually J^*-invariant and satisfy a strong integrability condition: $[\mathcal{E}_i, \mathcal{E}_i] \subset \mathcal{E}_i$ for all i and $[\mathcal{E}_i \oplus \mathcal{E}_j, \mathcal{E}_i \oplus \mathcal{E}_j] \subset \mathcal{E}_i \oplus \mathcal{E}_j$ for all $i \neq j$. We consider next the leaves of eigenspace foliation $\mathcal{E}_j$, which are complex submanifolds. Using the Riemannian submersion structure, one checks that in the induced metric the leaves all have Ricci curvature bounded below by that of g^*. The compactness of M then implies that all the leaves are compact simply connected regular submanifolds of M. Finally, using a Bochner argument, one shows that all the leaves are totally geodesic and give a de Rham decomposition of (M, g^*). Equation (1.3) then implies that each de Rham factor is Kähler-Einstein and that the curvature form of the circle bundle is a linear combination of the Kähler classes of the factors.

There are, however, Einstein metrics of type (1.1) on circle bundles over Kähler base manifolds. Of course, the submersed metric on the base is not Kähler.

THEOREM 1.3. *There are Einstein metrics on the total spaces of the following principal S^1 bundles over the specified coadjoint orbits:*

(i) [**119, 36, 92, 56, 77, 26**] *any non-trivial S^1 bundle over $SU(3)/T^2$,*

(ii) [**110**] *any non-trivial S^1 bundle over $SU(p+q+r)/S(U(p)U(q)U(r))$ and $SO(2n)/U(n-1)U(1), n \geq 3$,*

(iii) [**111**] *Let G/L be a coadjoint orbit where G is semisimple and the Lie algebra of L is obtained by deleting a simple root from the Dynkin diagram of $\mathfrak{g}$ which has a coefficient of 2 in the expression of the maximal root of G as a* *linear combination of the simple roots. Then on the principal circle bundle corresponding to the $U(1)$ factor in L there is an Einstein metric other than that from Kobayashi's theorem* [**74**].

In the above theorem, note that since all base manifolds are coadjoint orbits, for any homogeneous complex structure one chooses, the first Chern class is positive. Thus all 2-forms are of type $(1,1)$. In (i) the principal circle bundles P_χ can be indexed by 2 integers k, l, where in order to eliminate covering manifolds one assumes that they are relatively prime. Furthermore, notice that the Weyl group $N(T)/T$ acts on the right on $SU(3)/T$, and so there are some obvious diffeomorphisms among the bundles, e.g., $P_{1,1} \cong P_{0,1}$ and $P_{1,2} \cong P_{1,-1}$. (We caution the reader that our notation is such that $P_{2,2}$ corresponds to the first Chern class of $SU(3)/T$.) Similar remarks apply to the other cases where the subgroup has a non-trivial normalizer.

The existence of an Einstein metric in (i) was first obtained in [**119**] in order to show that in a fixed dimension there can *already* be infinitely many homotopy types among *homogeneous* Einstein manifolds. These metrics were rediscovered in [**36**] in a more explicit form. A second Einstein metric was constructed by Page and Pope [**92**]. These physicists also showed that the Einstein metrics have Killing spinors, a fact later rediscovered by Friedrich and Kath [**56**]. Finally, Kowalski and Vlášek [**77**], in a very careful study of these examples, discovered that for large k, one of the Einstein metrics on $P_{-k-1,k}$ also has positive sectional curvature. A third Einstein metric was discovered on $P_{-1,1}$ in [**26**]. All the Einstein metrics in (i) are also related to G_2 structures, as was discovered in [**30**].

If we put $p = q = r = 1$ in (ii), we recover (i). The second Einstein metric in (iii) lies in the canonical variation [**17**, 9.70] of the Kobayashi metric.

As for examples with non-abelian G, the following framework unifies many known Einstein metrics. Suppose that M is an irreducible Riemannian manifold such that the structural group G of its holonomy bundle $\tilde{P}$ is non-simple. Let $G = H \cdot K$ where $\cdot$ means the quotient of the product by a finite normal subgroup. Then $P = \tilde{P}/H$ is a principal $\overline{K}$ bundle over M for a certain quotient $\overline{K}$ of K. We can ask for an Einstein metric of type (1.1) on P.

EXAMPLE 1.1. If (M, g^*) is Kähler-Einstein Fano, then $G = U(n)$ and we can let $H = SU(n)$. One is then precisely in the situation of Kobayashi's theorem [**74**].

EXAMPLE 1.2. If (M, g^*) is quaternionic-Kähler with positive scalar curvature, then either $G = Sp(n) \cdot Sp(1)$ and we can let $H = Sp(n)$, or M is quaternionic symmetric and $G = H \cdot Sp(1)$ with $H \subset Sp(n)$. Then $\overline{K} = SO(3)$ unless $M = \mathbb{HP}^n$, in which case $K = \overline{K} = Sp(1)$. The connection on P induced by the Levi-Civita connection is Yang-Mills with constant norm, as was observed by independently in [**32**] and [**90**]. Using this connection, one can construct two non-isometric Einstein metrics of type (1.1) on P [**17**, 14.85]. When $M = \mathbb{HP}^n$, $P = S^{4n+3}$, and the two Einstein metrics are the constant curvature metric and the Jensen metric [**68**]. Alternatively, since P is a principal circle bundle over the quaternionic-Kähler twistor space of M, we can also appeal to Kobayashi's theorem and the canonical variation [**17**, 9.70] to obtain the two Einstein metrics. These metrics also occur among those in Theorem 1.3(iii). Still another viewpoint is that the Einstein metrics come from 3-Sasakian structures [**25, 26**].

EXAMPLE 1.3. If M is a compact irreducible hermitian symmetric space, $G = K \cdot U(1)$. Then if we let $H = U(1)$, we obtain two non-isometric Einstein metrics on P. Except in the case $K = SU(p)SU(q), p \neq q$, one of these Einstein metrics was found by Jensen [**68**]. For the remaining case and the second Einstein metric (which comes from the canonical variation), see [**120**, Theorem 4].

EXAMPLE 1.4. If M is a compact quaternionic symmetric space, $G = K \cdot Sp(1)$, and if we let $H = Sp(1)$, there are again two non-isometric Einstein metrics on P. When K is simple, one of the Einstein metrics was again found in [**68**]. For the rest, see [**120**, Theorem 2].

EXAMPLE 1.5. If M is a compact irreducible symmetric space whose isotropy group is non-simple, and H is not one of the choices already discussed, then Einstein metrics on the bundle P for such a choice of H were again obtained in [**68**].

Instead of principal bundles, we can also consider Kaluza-Klein constructions on associated fibre bundles of principal bundles. As before, let $\pi : P \to M$ be a principal G-bundle with connection ϕ whose curvature form is Ω. Let G act almost effectively on a manifold F and let $W = P \times_G F$. If g^* is a metric on M and $\langle\ ,\ \rangle$ now denotes a G-invariant metric on F, then (1.1) defines a metric g so that the projection $\pi : (W, g) \to (M, g^*)$ is a Riemannian submersion with totally geodesic fibres. The Einstein condition for g is again equivalent to the Yang-Mills condition on ϕ and equations similar to (1.2) and (1.3).

In order to describe these equations precisely, recall that a point in W is an equivalent class $[p, x]$ where $p \in P, x \in F$ and $(p, x) \sim (pg, g^{-1}x)$. Having chosen a representative (p, x), there is an inclusion $i_p : F \to W$ given by $i_p(x) = [p, x]$. Because $i_{pg} = i_p \circ g$, i_p is an isometry between $(F, \langle\ ,\ \rangle)$ and the fibre through $[p, x]$ with the metric induced from g. To take care of horizontal directions, we make use of $j_x : P \to W$ given by $j_x(p) = [p, x]$, which satisfies $j_{gx} = j_x \circ R_g$. Then the equations analogous to (1.2) and (1.3) are respectively

$$(1.4)\qquad Ric_F(i_{p*}^{-1}(U), i_{p*}^{-1}(V)) + \frac{1}{4}\sum_{i,j} \langle \overline{\Omega(\tilde{e}_i, \tilde{e}_j)}_x, i_{p*}^{-1}(U)\rangle \, \langle \overline{\Omega(\tilde{e}_i, \tilde{e}_j)}_x, i_{p*}^{-1}(V)\rangle$$

$$= \Lambda \langle i_{p*}^{-1}(U), i_{p*}^{-1}(V)\rangle,$$

$$(1.5)\ Ric(g^*)(\pi_*(X), \pi_*(Y)) - \frac{1}{2}\sum_i \langle \overline{\Omega(\tilde{X}, \tilde{e}_i)}_x, \overline{\Omega(\tilde{Y}, \tilde{e}_i)}_x \rangle = \Lambda g^*(\pi_*(X), \pi_*(Y)),$$

where $\sim$ denotes horizontal lifts and for $Z \in \mathfrak{g}$, $\overline{Z}_x$ denotes the value of the Killing field induced by Z on F at x. Unlike the principal bundle case, it is possible for $\overline{Z}$ to vanish at some points.

We now describe some Einstein metrics on bundles for which G acts transitively on F. The first family gives quaternionic analogues of Einstein metrics given by Theorem 1.1.

THEOREM 1.4. [**120**] *Let $(M_j, g_j^*), 1 \leq j \leq m$, be quaternionic Kähler manifolds with positive scalar curvature and P_j be the canonical $SO(3)$-bundle over M_j associated with the quaternionic-Kähler structure. Let $P = P_1 \times \cdots \times P_m$, $G = SO(3) \times \cdots \times SO(3)$, (m factors), and $F = G/\Delta SO(3)$ where $\Delta SO(3)$ denotes the diagonally embedded subgroup. Then $W = P \times_G F$ admits an Einstein* *metric with positive scalar curvature of type (1.1) submersing onto a product of the metrics* g_j^* *and having a normal homogeneous fibre metric.*

(A *normal homogeneous* metric on G/K is a G-invariant Riemannian metric induced by some bi-invariant metric on G, not necessarily positive definite.)

This theorem is proved in a similar way as Theorem 1.1. On the other hand, the Einstein metric can also be deduced as a special case of 3-Sasakian reduction discovered by Boyer, Galicki and Mann [**25, 26**]. See the article by the first two authors in this volume for details and up-to-date information.

Constructions similar to those in Theorem 1.4 can be performed with the bundles P in Examples 1.3 and 1.4. Namely, let $M_1 \times \cdots \times M_m$ be the m-fold product of the same compact quaternionic (resp. irreducible hermitian) symmetric space M, and let $\tilde{P}_j$ be the holonomy bundle of M_j with group $G = H \cdot K$ where $H = Sp(1)$ (resp. $U(1)$). Denote by P_j the quotient $\tilde{P}_j/H$, which is a principal $\overline{K}$-bundle. Then under certain conditions there are Einstein metrics of type (1.1) on $W = (P_1 \times \cdots \times P_m)/\Delta\overline{K}$. We refer the reader to Theorems 3 and 5 in [**120**] for details. Here we only mention two examples to indicate the possibilities.

EXAMPLE 1.6. For $M = \mathbb{HP}^n, n \geq 1$, there is an Einstein metric of type (1.1) on W if the number of factors m satisfies

$$2n^2(m-2) \leq n(3m^2 - 7m + 6) + 5m^2 - 5m + 2.$$

EXAMPLE 1.7. For $M = \mathbb{CP}^n$, there is an Einstein metric on W of type (1.1) provided

$$n^2(2-m) + n(m^2 - 2m) + 2m^2 + m - 2 \geq 0.$$

2. Einstein Metrics of Cohomogeneity One.

A. Generalities. Let G be a compact Lie group. A connected G-manifold is said to be of *cohomogeneity* 1 if the principal orbits are hypersurfaces. In this section we will be concerned with G-invariant Einstein metrics on such manifolds whose full isometry groups do not act transitively. The orbit space of a cohomogeneity 1 manifold is either an interval $\hat{I}$ whose boundary points represent singular orbits, or it is a circle. We will only concern ourselves with the former situation. For cohomogeneity 1 metrics, the Einstein condition reduces to a system of nonlinear ordinary differential equations on $\hat{I}$ together with appropriate boundary conditions to ensure that we have a smooth metric. The first systematic study of cohomogeneity 1 Einstein metrics was carried out in [**16**]. Some recent works about manifolds of cohomogeneity 1 which contain useful information include [**1, 8, 87, 97, 113**].

We will give first a geometric description of the Einstein condition for a cohomogeneity 1 metric following [**55**].

Let $(\hat{M}, \hat{g})$ be a cohomogeneity 1 G-manifold of dimension $n+1$ with a G-invariant metric. Let $P = G/K$ be the principal orbit type and $Q_i = G/H_i$ be the singular orbit types. There are at most 2 singular orbits, and when we are concentrating on one of them, we will use Q and H respectively to denote the orbit and its corresponding isotropy group. We can easily arrange for $K \subset H_i$. For example, we can choose a unit speed geodesic that starts from a singular orbit and intersects each principal orbit orthogonally. Then the points in the geodesic belonging to principal orbits all have the same isotropy group K, which then lies in the isotropy groups of the points on the geodesic belonging to the singular orbits. It follows from the cohomogeneity 1 condition that H_i must act transitively on the unit sphere in the normal slice to Q_i. So $H_i/K \cong S^{k_i}$, and P may be viewed as the unit sphere bundle of the normal bundle of Q_i in $\hat{M}$, which has the form $\nu(Q_i) = G \times_{H_i} V_i$, where H_i acts orthogonally on the slice representation $V_i \cong \mathbb{R}^{k_i+1}$. (This last identification is given by the normal exponential map.)

Let $\hat{M}_0$ denote the union of the principal orbits in $\hat{M}$. The geodesic chosen above gives a diffeomorphism $\hat{M}_0 \cong I \times P$, where $I = int(\hat{I})$. The pull-back of $\hat{g}$ via this diffeomorphism takes the form

$$dt^2 + g_t, \quad t \in I,$$

where g_t is a 1-parameter family of G-invariant metrics on P. It is occasionally useful to fix a background metric g_b on P of type (1.1) where g^* is a G-invariant metric on Q, ϕ is a connection for the principal bundle $H \to G \to G/H$, and $\langle \ , \ \rangle$ is the constant curvature 1 metric on $H/K \cong S^k$. In terms of g_b, we can think of g_t as a g_b-symmetric endomorphism of TP. The Ricci tensor of g_t can be thought of as an endomorphism r_t of TP, symmetric with respect to g_t but not in general so with respect to g_b.

If we can construct a *smooth* metric $\hat{g}$ on $\hat{M}$ such that on $\hat{M}_0$ the Einstein equation is satisfied, then by continuity we have an Einstein metric on $\hat{M}$. In order to write down the Einstein equation on $\hat{M}_0$, we introduce the shape operator $\mathcal{L}_t$ of the principal orbits $\{t\} \times P$. This is the endomorphism of TP given by $\mathcal{L}_t(X) = \hat{\nabla}_X N$, where N is the unit vector field $\partial/\partial t$.

By using the Gauss and Codazzi equations, we easily obtain the Einstein equation for $\hat{g}$ on $\hat{M}_0$ as a system on P. This is the system below corresponding to the choice $\epsilon = 1$.

$$g' = 2g\mathcal{L}, \tag{2.1}$$

$$\mathcal{L}' + tr(\mathcal{L})\mathcal{L} - \epsilon r_t = -\epsilon\Lambda \cdot \mathbf{I}, \tag{2.2}$$

$$tr(\mathcal{L}') + tr(\mathcal{L}^2) = -\epsilon\Lambda, \tag{2.3}$$

$$tr(X \lrcorner\, d^{\nabla}\mathcal{L}) = 0, \tag{2.4}$$

for all $X \in TP$, where Λ is the Einstein constant, $\lrcorner$ denotes interior multiplication, and d^{∇} is the exterior covariant derivative $T^*P \otimes TP \to \Lambda^2(T^*P) \otimes TP$ formed using the Levi Civita connection ∇^t of g_t. If we take $\epsilon = -1$ instead, we obtain the Einstein condition for the Lorentz metric $-dt^2 + g_t$.

Note that (2.1) is essentially the definition of $\mathcal{L}_t$, which must also be symmetric with respect to g_t. Equation (2.4) is just $\widehat{Ric}(X, N) = 0$, and equations (2.2–2.3) represent the Einstein condition in the direction of the principal orbit and N respectively. Let s_t denote the scalar curvature of r_t. Then if we take the trace of (2.2) and use (2.3), we immediately obtain the equation

$$\epsilon s - (tr(\mathcal{L}))^2 + tr(\mathcal{L}^2) = (n-1)\epsilon\Lambda. \tag{2.5}$$

It is possible to interpret this equation as a first integral of a suitable Hamiltonian system.

By using the contracted second Bianchi identity, A. Back has deduced the following useful lemma [**13**].

LEMMA 2.1. *Let $\hat{g} = dt^2 + g_t$ be an equidistant family of hypersurfaces $I \times P$ satisfying (2.1) and (2.2) for some constant Λ. Let the scalar curvature s_t of g_t be constant for each $t \in I$ and v_t be the volume distortion of g_t with respect to some background metric on P. Then $\widehat{Ric}(X, N)v$ is constant in t for any $X \in TP$. Furthermore, if (2.4) is also satisfied, then $(\widehat{Ric}(N, N) - \Lambda)v^2$ is constant in t.*

Applying this lemma together with Theorem 5.2 in [**48**] gives

PROPOSITION 2.2. *Let $\hat{M}$ be a cohomogeneity 1 G-manifold with at least one singular orbit of dimension strictly smaller than that of the principal orbits. If $\hat{g}$ is a G-invariant metric of class C^3 such that (2.1) and (2.2) are satisfied on $I \times P$, then $\hat{g}$ is actually a smooth Einstein metric and hence real analytic.*

Proofs of the above statements can be found in [**55**]. Proposition 2.2 implies that we can focus on equation (2.2), provided we can ensure that the solution represents a smooth enough metric. Here, a C^3 metric is needed because the contracted second Bianchi identity is used in the proof. In special cases, the smoothness requirement can sometimes be weakened. We describe now a practical criterion for smoothness for the metrics $\hat{g}$, following [**55**], and then give an example illustrating how one applies this criterion in practice.

Let $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) denote the subspace of the tangent space of G/K at the coset (K) corresponding to H/K (resp. G/H). For example, we could choose an $Ad(K)$-invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

such that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_+$ and $\mathfrak{p}_-$ are $Ad(H)$-invariant. A smooth G-invariant metric $\hat{g}$ on $G \times_H V$ is equivalent to an H-equivariant smooth map

$$\psi : V \longrightarrow S^2(V \oplus \mathfrak{p}_-),$$

where H acts on V via the slice representation and on $\mathfrak{p}_-$ by the isotropy representation of G/H. We can approximate ψ near the origin by Taylor polynomials whose homogeneous parts are H-equivariant polynomials of degree p on V with coefficients in $A := S^2(V \oplus \mathfrak{p}_-)$, i.e., elements of $Hom_H(S^p(V), A)$.

On the other hand, in writing $\hat{g}$ in the form $dt^2 + g_t$, we are really restricting ψ to a ray in V emanating from the origin. We then obtain a smooth curve $a(t)$ in A^K, the K-invariant elements in A. Conversely, given such a smooth curve $a : \mathbb{R}_+ \to A^K$, we obtain a smooth map $V \setminus 0 \to A$ by using the H-action. The smoothness question is when such a map extends smoothly to an H-equivariant map $\psi : V \to A$.

LEMMA 2.3. [**55**] *A smooth map $a : \mathbb{R}_+ \to A^K$ extends to a smooth map $\psi : V \to A$ as above iff each Taylor coefficient a_p of $a(t)$ is the restriction of an element of $Hom_H(S^p(V), A)$ to the unit sphere $S^k \subset V$.*

Clearly, entirely analogous criteria exist for smoothness of G-invariant tensors of other types on $G \times_H V$. One just has to replace A above by the relevant H-representation.

We now make some observations regarding the lowest degree Taylor coefficients. First, note that smoothness implies that $a_0 \in A^H$. Now

$$A^H = S^2(V \oplus \mathfrak{p}_-)^H = S^2(V)^H \oplus (V \otimes \mathfrak{p}_-)^H \oplus S^2(\mathfrak{p}_-)^H,$$

and V is an irreducible H-representation since H acts transitively on the unit sphere in V. The component of a_0 in $S^2(V)^H = \mathbf{1}$ is the Euclidean metric because in the exponential coordinate system, spheres with decreasing radii must become round to first order. The component of a_0 in $S^2(\mathfrak{p}_-)^H$ is just the G-invariant metric on Q induced by $\hat{g}$. a_0 has no component in $(V \otimes \mathfrak{p}_-)^H$ because V is the normal slice to Q at the coset $(H) \in G/H$. Thus a_0 is just the identity map relative to a suitable background metric.

Next we consider the first order Taylor coefficient a_1. Smoothness implies that it is an H-equivariant linear map $V \to A$. It is not difficult to see that there are no non-zero H-equivariant linear maps $V \to S^2(V)$. Hence, $tr(a_1)$ comes only from $V \to S^2(\mathfrak{p}_-)$. This part of a_1 is just the shape operator of Q by (2.1). Since $tr(a_1)$ is an H-invariant linear function on V, it must be zero. Hence we have deduced the following corollary using only local smoothness considerations.

COROLLARY 2.4. **[65]** *If $(\hat{M}, \hat{g})$ is a smooth Riemannian manifold of cohomogeneity* 1 *with a singular orbit Q, then Q is a minimal submanifold.*

In **[65]**, the above corollary followed from an equivariant variational principle.

EXAMPLE 2.1. Let $\hat{M} = S^4$ be the unit sphere in $\mathbb{R}^5$, viewed as the space of 3×3 symmetric matrices with real entries and trace 0. Let $G = O(3)$ act by conjugation on these symmetric matrices. Then the principal orbits consist of matrices in S^4 with distinct eigenvalues and the principal isotropy group is $K = O(1)^3$. The two singular orbits comprise matrices in S^4 with 2 distinct eigenvalues. The isotropy group H is, up to conjugation, $O(2)\times O(1)$, and Q is the projective plane, minimally embedded as the Veronese surface. The isotropy representation of G/K is

$$(-1 \otimes -1 \otimes 1) \oplus (-1 \otimes 1 \otimes -1) \oplus (1 \otimes -1 \otimes -1),$$

where ± 1 denote respectively the trivial/non-trivial representation of $O(1) \approx \mathbb{Z}/2$. With the above choice of H, $\mathfrak{p}_+ = -1 \otimes -1 \otimes 1$. The slice representation at the singular orbit Q is $\rho^2 \otimes \mathbf{1}$, where ρ^m is the irreducible 2-dimensional representation of $O(2)$ lying in the mth symmetric power of the usual representation ρ^1 which does not already lie in the $(m-2)$nd symmetric power. Then we have H-module decompositions

$$S^2(\mathfrak{p}_-) = (\rho^2 \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1}),$$

$$S^m(V) = (\rho^{2m} \otimes \mathbf{1}) \oplus S^{m-2}(V).$$

Hence for $m \geq 1$, $Hom_H(S^{2m}(V), S^2(\mathfrak{p}_-)) \approx Hom_H(S^2(V), S^2(\mathfrak{p}_-))$, which is 1-dimensional and is generated by t^2 times the identity matrix. Likewise, we have $Hom_H(S^{2m-1}(V), S^2(\mathfrak{p}_-)) \approx Hom_H(V, S^2(\mathfrak{p}_-))$, which is again 1-dimensional, generated by

$$\begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix}$$

where (t_1, t_2) are Euclidean coordinates in $V \approx \mathbb{R}^2$ and $t^2 = t_1^2 + t_2^2$. Up to a constant, this is the shape operator of the Veronese surface in S^4. On the other hand, it is a general fact (see **[55**, §1, Lemma 2**]**) that for a compact linear group H acting transitively on the unit sphere in V, one has $Hom_H(S^{2m-1}(V), S^2(V)) = 0$ and $Hom_H(S^{2m}(V), S^2(V)) \approx Hom_H(S^2(V), S^2(V))$. In the present example, this last space has dimension 2 and for the generators one can take t^2 times the identity matrix and

$$\begin{pmatrix} t_2^2 & -t_1 t_2 \\ -t_1 t_2 & t_1^2 \end{pmatrix}.$$

However, only multiples of the second generator are candidates for the second order Taylor coefficient of a smooth metric.

Let B denote the bi-invariant metric on $O(3)$ given by $-tr(XY)$. We express g_t as

$$h(t)^2 \ 2B|_{\mathfrak{p}_+} \oplus f_1(t)^2 \ \frac{3}{2}B|_{\mathfrak{p}'_-} \oplus f_2(t)^2 \ \frac{3}{2}B|_{\mathfrak{p}''_-}.$$

(The coefficients in front of B are chosen so that $dt^2 + 2B|_{\mathfrak{p}_+}$ is the Euclidean metric on $V = \mathbb{R}^2$.) It follows that smoothness of $\hat{g}$ means that

$$\begin{pmatrix} f_1(t)^2 \\ f_2(t)^2 \end{pmatrix} = \sum_{j=0}^{\infty} \left\{ a_{2j+1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^{2j+1} + a_{2j} \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{2j} \right\}.$$

As for $h(t)$, smoothness is equivalent to it being odd with $h'(0) = 1$. Note that for the usual metric on S^4, in terms of B above, $h(t) = \sin t$, $f_1(t) = \cos t - \frac{1}{\sqrt{3}} \sin t$, and $f_2(t) = \cos t + \frac{1}{\sqrt{3}} \sin t$ over the interval $[0, \frac{\pi}{3}]$.

B. Initial Value Problem. A basic analytical question about the Einstein system (2.1–2.4) is the initial value problem. The easier case is the initial value problem at a principal orbit. Considerably subtler is the initial value problem at a singular orbit. We begin with the easier case.

THEOREM 2.5. [**55**] *Let G be a compact Lie group and K be a closed subgroup such that G/K is connected. Let h be a given G-invariant metric on G/K and $\mathcal{L}_0$ be an h-symmetric endomorphism of $T(G/K)$ such that for all $X \in T(G/K)$ we have $tr(X \neg \ d^{\nabla^h} \mathcal{L}_0) = 0$. Then there is a unique Einstein metric $\hat{g} = dt^2 + g_t$ defined on $(-\epsilon, \epsilon) \times G/K$, for some $\epsilon > 0$, with $g_0 = h$ and $\mathcal{L}_0$ equal to the shape operator of $\{0\} \times G/K$. Furthermore, $\hat{g}$ depends continuously on the initial values h and $\mathcal{L}_0$.*

Let us now assume that there is a singular orbit of strictly smaller dimension than the principal orbits. By Proposition 2.2, for the initial value problem, we need only consider the equations (2.1) and (2.2). In a neighbourhood around Q, the term $\mathcal{L}_t$ has t^{-1} dependence while r_t has t^{-2} dependence. So the differential equations have a singularity at $t = 0$. Of course, these equations are very nonlinear, especially because of the Ricci term, whose dependence on the metric g_t cannot be very explicitly written down if we want to leave G/K general. (r_t is a rational function of the components of g_t, but the constants in the expression depend on the specific G/K.) The linearization of (2.2) has the form $z' = t^{-2}\mathcal{A}(t)z$ where $\mathcal{A}(0)$ is a lower triangular matrix. The initial value problem for the linear case, though well-understood, is not completely trivial. In particular, a formal power series solution cannot be expected in all cases.

The singular initial value problem has been solved under an additional assumption.

THEOREM 2.6. [**55**] *Assume that as K-representations, V and $\mathfrak{p}_-$ have no irreducible sub-representations in common. Then, given any G-invariant metric g^* on Q and any G-equivariant homomorphism $\mathcal{L}_1 : \nu(Q) \longrightarrow S^2(T^*Q)$, there exists* *a smooth G-invariant Einstein metric on some open disk bundle of $\nu(Q)$ with any prescribed sign (positive, zero, or negative) of the Einstein constant Λ and having g^* and $\mathcal{L}_1$ as initial metric and shape operator on Q.*

The theorem is proved by the classical method of asymptotic series. The key step is to show that there is a formal power series solution any finite truncation of which defines a smooth metric on $\nu(Q)$. This involves input from geometry and representation theory since from a purely analytic point of view there is no reason to expect *power series* solutions at all. (From the smoothness discussion above, asymptotic series which are not power series do not give rise to a smooth metric.) One then applies a Picard iteration scheme to sufficiently high order truncations of the formal power series solution to get a smooth metric defined in a tube around Q. Alternatively, for this last step, one may quote a theorem of Malgrange [**83**].

Uniqueness is not true for the above singular initial value problem. It turns out that in general one needs to prescribe a finite number of additional Taylor coefficients in order to obtain a unique solution. These parameters can be calculated explicitly using representation theory once the triple $K \subset H \subset G$ is given. Non-uniqueness can be explained as follows. In constructing the formal power series solution, as is customary, one has to solve for Taylor coefficients recursively in terms of Taylor coefficients of lower degrees. The linear operators involved in this process are only injective above a certain critical degree which varies from situation to situation. Non-uniqueness comes from the kernels of these operators in lower degrees. In fact, there are sequences of examples for which the critical degrees tend to infinity (see example 3, §5 of [**55**]).

When the assumption on V and $\mathfrak{p}_-$ as K-representations does not hold, the initial value problem has been solved in the special case of the Kervaire spheres in [**13**]. The statement of the result is the same as in Theorem 2.6. It is conceivable that Theorem 2.6 holds without the technical assumption on V and $\mathfrak{p}_-$.

C. Examples With Special Holonomy. Under the further assumption of special holonomy, classification theorems are often available in addition to the construction of examples. We shall begin with cohomogeneity 1 hyperkähler metrics, which are metrics on $4n$-dimensional manifolds whose holonomy lies in $Sp(n)$. Alternatively, these are Riemannian manifolds which are Kähler with respect to 3 complex structures satisfying the multiplicative relations between the quaternions i, j, and k. See the article by A. Dancer in this volume for further information.

If we assume that the hyperkähler metric is irreducible, then since the Ricci tensor is zero, a cohomogeneity 1 metric exists only on a noncompact manifold. Calabi constructed [**31**] a complete hyperkähler metric on $T^*\mathbb{CP}^n$ of cohomogeneity 1 under $PSU(n+1)$. When $n = 1$, this metric was discovered earlier by Eguchi-Hanson [**54**]. In dimensions greater than 4, one has the following classification theorem.

THEOREM 2.7. [**43**] *Let $(\hat{M}, \hat{g})$ be an irreducible hyperkähler manifold of dimension greater than 4 which is of cohomogeneity 1 with respect to a compact simple Lie group G. Then, up to coverings, $\hat{M}$ is an open subset of either $T^*\mathbb{CP}^n$ with the Calabi metric or the $\mathbb{H}^*$ or $\mathbb{H}^*/\mathbb{Z}_2$ bundle over a quaternionic symmetric space of compact type with the Swann metric. If g is in addition complete, then it is isometric to the Calabi metric.*

R. Bielawski [**18**] independently obtained the classification theorem under the additional assumption of completeness.

To describe the Swann metric, recall from Example 1.2 that every quaternionic Kähler manifold has a canonical $SO(3)$ bundle over it. Therefore there is an associated $\mathbb{H}^*/\mathbb{Z}_2$ bundle, which is an $\mathbb{H}^*$ bundle in the case of the quaternionic projective space. A. Swann constructed an incomplete hyperkähler metric on this bundle in [**114**].

The above classification is also valid for a compact semisimple cohomogeneity one group action provided that any $\mathfrak{su}(2)$ ideal in $\mathfrak{g}$ acts trivially on the three complex structures on $\hat{M}$.

THEOREM 2.8. *A non-flat hyperkähler* 4*-manifold of cohomogeneity* 1 *with respect to a compact connected simple group is one of the following.*

(i) [**15**] *a member of a* 2*-parameter family of $SU(2)$-invariant incomplete examples or the Eguchi-Hanson metric on $T^*\mathbb{CP}^1$,*
(ii) [**60**] *the $U(2)$-invariant Taub-NUT metric on $\mathbb{R}^4$,*
(iii) [**11**] *up to a double covering, the* 2*-monopole space M_2^0, which is the unique complete hyperkähler* 4*-manifold with cohomogeneity* 1 *under $G = SO(3)$ and such that G rotates the complex structures,*
(iv) [**58**] *a member of a family of incomplete examples with $G = SU(2)$, which also acts transitively on the complex structures.*

Cohomogeneity 1 Kähler-Einstein metrics of non-positive scalar curvature on holomorphic line bundles over Kähler manifolds can be found among the bundle constructions of Calabi [**31**], Bérard Bergery [**16**], Page and Pope [**93**]. For these authors, the Euler class of the line bundle is proportional to the first Chern class of the base. Theorem 3.2 generalizes these examples in the bundle context to line bundles over a product of Fano manifolds such that the Euler class is a linear combination of the first Chern classes of the de Rham factors of the base. Furthermore, certain blow-downs of the zero section are also allowed, as was anticipated by Calabi [**31**, p. 277]. In the cohomogeneity 1 context, the choices for the Euler class of the line bundles are even more numerous. We have the following classification/existence theorem.

THEOREM 2.9. [**45**] *Let G be a compact connected semisimple Lie group acting with cohomogeneity* 1 *via isometries on a Kähler-Einstein manifold $(\hat{M}, \hat{g})$ which is irreducible and not hyperkähler. Suppose further that the isotropy representation of the principal orbit G/K splits into pairwise inequivalent irreducible subrepresentations.*

(i) *There is a coadjoint orbit G/L with a fixed invariant complex structure J^* so that $K \subset L, L/K \cong S^1$ and the induced metric on each principal orbit gives $G/K \to G/L$ the structure of a Riemannian submersion with totally geodesic fibres onto an invariant Kähler metric on G/L.*
(ii) *The complex structure on $\hat{M}$ is induced by J^*, the underlying connection of the Riemannian submersions, and the metric on the fibres. On $\hat{M}_0$, the union of all the principal orbits, the Kähler-Einstein metric can be expressed explicitly in terms of rational functions which depend on* $\dim H^2(G/L; \mathbb{R})$ *continuous parameters in the Ricci flat case and on a single constant of integration otherwise.*

(iii) *When there is a singular orbit G/H, then it is also a coadjoint orbit with an invariant complex structure induced from J^*. Moreover, it is a totally geodesic Kähler submanifold of $\hat{M}$ and H/L is analytically isomorphic to a complex projective space $\mathbb{CP}^{l-1}$.*

(iv) *Let χ denote the Euler class of the circle bundle $L/K \to G/K \to G/L$. Then the cohomology class*

$$c_1(G/L, J^*) + l\chi$$

is 0 when restricted to H/L, and, as an element of $H^2(G/H; \mathbb{R})$, is positive, zero, or negative depending on the sign of the Einstein constant.

(v) *The geometric data in (i), (iii), (iv) are sufficient for the construction of a smooth G-invariant Kähler-Einstein metric on a neighborhood of the zero section of the bundle $G \times_H \mathbb{C}^l$, and this metric extends to a complete metric on the underlying smooth vector bundle when the Einstein constant is non-positive.*

(vi) *If $(\hat{M}, \hat{g})$ is complete, then either there is a singular orbit G/H as above and $\hat{M} \cong G \times_H \mathbb{C}^l$, or else M is compact and the Einstein constant is positive.*

Of course, the condition on the isotropy representation of G/K is not always satisfied, but since it is satisfied for all coadjoint orbits G/L (L has maximal rank in G) a generic choice of K with $L/K \cong S^1$ will result in a G/K with the same property. In any event, the existence part of the theorem (i.e., part (v)) remains valid without this condition on the isotropy representation.

For the above theorem, the semisimplicity of G provides us with a moment map which takes orbits in $\hat{M}$ to coadjoint orbits in $\mathfrak{g}^*$. Under the assumption on the isotropy representation of G/K we obtain (i). The Einstein condition is then seen to be the same as (3.2-3.4) in the bundle situation discussed in the next section. One therefore gets explicit local solutions in the same manner. Note that the analysis of the singular orbits shows that the admissible quadruples (G, H, L, K) can be enumerated in terms of combinatorial data. Also, moduli of the Ricci-flat Kähler metrics come from the choice of an invariant Kähler metric on G/H. When the Einstein constant Λ is non-zero, the cohomology class in (iv) is really Λ times the Kähler class of the metric on G/H.

While the condition on the isotropy representation of G/K is generically satisfied, interesting Kähler-Einstein metrics nevertheless exist in situations where the condition does not hold. The Calabi metric on $T^*\mathbb{CP}^n$ is one example. We also have

THEOREM 2.10. [**112**] *There exists a complete Ricci-flat Kähler metric of cohomogeneity* 1 *on the cotangent bundle of a compact symmetric space of rank* 1.

The complex structures on the above spaces are special cases of adapted complex structures on tubes of zero sections of tangent bundles of real analytic manifolds constructed by Lempert, Szöke, [**82, 115**] and Guillemin and Stenzel [**59**].

Cohomogeneity 1 Kähler-Einstein metrics of positive scalar curvature were first constructed by Sakane [**110**] on certain $\mathbb{CP}^1$ bundles over a product of two compact Hermitian symmetric spaces. Later, Koiso and Sakane [**75, 76**] generalized this construction to the bundle (rather than the strictly cohomogeneity 1) situation and discovered the *sufficiency* of the vanishing of the Futaki invariant for existence in this set-up. (This is not true for the general existence problem in the Fano case, cf Tian's article.) These constructions will be discussed further in §3 below. As in the non-positive case there is the following classification/existence theorem.

THEOREM 2.11. [**75, 76, 45, 102**] *Let G and $(\hat{M}, \hat{g})$ be as in Theorem 2.9 and suppose that the Einstein constant is positive. In addition to (i) and (ii), we have the following analogues of (iii) and (iv):*

(iii)* *Each singular orbit G/H_i, $i = 1, 2$, is a coadjoint orbit with an invariant complex structure induced from J^*. They are totally geodesic Kähler submanifolds of $\hat{M}$. Furthermore, $H_i/L \cong \mathbb{CP}^{l_i - 1}$ and their isotropy representations have no common root spaces.*

(iv)* *Let χ be as in Theorem 2.9. Then for $i = 1, 2$, the class*

$$c_1(G/L, J^*) + (-1)^{i+1} l_i \chi$$

restricts to 0 on H_i/L and lies in the Kähler cone in $H^2(G/H_i; \mathbb{R})$.

The geometric data in (i), (iii) and (iv)* together with the vanishing of the Futaki integral*

$$\int_{-l_1}^{l_2} \prod_j (\lambda_j x - 1)^{d_j/2} x dx$$

are sufficient for the existence of a G-invariant Kähler-Einstein metric with positive constant on $\hat{M}$ having the stated orbit types.

In the above, d_j is the (real) dimension of the jth irreducible summand in the isotropy representation of G/L and λ_j is the corresponding eigenvalue of the curvature form of the circle bundle $L/K \to G/K \to G/L$, which can be expressed in terms of the first Chern class of G/L and the Euler class of the circle bundle. As in Theorem 2.9 the existence part does not require the condition on the isotropy representation of the principal orbit.

The special case of 4-dimensional Kähler-Einstein manifolds with cohomogeneity 1 has also been analysed. Here, the Ricci flat case is precisely the hyperkähler case, which has already been mentioned.

When $G = SU(2)$, Dancer and Strachan [**42**] proved that the complete cohomogeneity 1 Kähler-Einstein metrics with negative Einstein constant form two families. One of the families consists of $U(2)$-invariant metrics on complex line bundles over $\mathbb{CP}^1$ with Chern class < -2. These are just the noncompact Kähler examples discovered independently in [**16**], [**31**], and [**58**], and can be viewed as special cases of Theorem 3.2(ii) below. The second family consists of triaxial metrics, i.e., the metric components in the 3 independent directions in the principal orbits (S^3) are unequal. On the other hand, compact solutions must be the canonical Einstein metrics on $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Quaternionic-Kähler manifolds with positive scalar curvature and of cohomogeneity 1 with respect to a compact connected isometry group have been investigated in [**8**], resulting in a partial classification. Recently, Dancer and Swann [**44**] proved that a complete quaternionic-Kähler manifold with positive scalar curvature which has a semisimple compact group of isometries with cohomogeneity 1 must be quaternionic symmetric. The methods in [**44**] involve the associated twistor space of the quaternionic-Kähler manifold and its complex contact geometry. As a result, they also obtain information in the incomplete as well as non-compact cases.

For metrics of cohomogeneity 1 with holonomy G_2 or $Spin(7)$, see Theorem 3.7.

In closing this subsection on cohomogeneity one Einstein metrics with special holonomy, we would like to mention Hitchin's classification [**64**] of the cohomogeneity one $SU(2)$-invariant anti-self-dual Einstein metrics on 4-manifolds. Recall (cf LeBrun's article) that an oriented Riemannian 4-manifold is anti-self-dual (ASD) if the self-dual part of its Weyl tensor vanishes identically. While ASD Einstein metrics do not have special holonomy in general, the anti-self-duality gives an extra structure which can be used to analyse the Einstein condition via twistor theory. Furthermore, an ASD Einstein metric with zero scalar curvature is locally hyperkählerian, so Hitchin's classification includes the Einstein manifolds in Theorem 2.8.

THEOREM 2.12. [**64**] *Suppose that* $(\hat{M}, \hat{g})$ *is a complete ASD Einstein manifold with an isometric* $SU(2)$ *action with cohomogeneity* 1.

(i) *If the scalar curvature is positive,* M *is either* S^4 *or* $\mathbb{CP}^2$ *with the canonical metric.*
(ii) *If the scalar curvature is zero, then* M *is isometric to flat* $\mathbb{R}^4$, $\mathbb{R}^4$ *with the Taub-NUT metric,* T^*S^2 *with the Eguchi-Hanson metric, or the Atiyah-Hitchin* 2*-monopole space.*
(iii) *If the scalar curvature is negative,* M *is either the unit* 4*-ball with the flat metric, the Bergmann metric, Pedersen's metric* [**99**], *or a member of a family of metrics arising from solutions of Painlevé VI, or else* M *is the complex line bundle over* S^2 *with Euler class* < -2 *equipped with the Bérard Bergery metric.*

Part(i) of the above result recovers a well-known earlier theorem of Hitchin [**63**]. The conformal structure of the Bérard Bergery metric in (iii) was studied by Pedersen [**99**] and LeBrun [**81**]. In [**64**], Hitchin actually gives a local classification, from which the above global classification follows by examining completeness issues. The proof of the local classification is twistorial in nature. The $SU(2)$ action can be lifted to a Lie algebra of holomorphic vector fields on the twistor space Z. Generically, one obtains from this a section of the anti-canonical line bundle over Z and a flat connection on the trivial $SU(2)^{\mathbb{C}}$ bundle over the complement of the zero set of the above-mentioned section. Restricting the connection to a connected family of twistor lines intersecting the zero set transversally, one obtains an isomonodromic deformation of connections over $\mathbb{CP}^1$, whose residues can be associated to a solution of Painlevé's sixth. The Einstein condition then gives strong restrictions on the above data, and the local classification results from a detailed analysis of the possibilities. The non-generic situation corresponds to the locally hypercomplex case.

D. Examples with Generic Holonomy. Solutions of (2.1–2.4) with generic holonomy include some of the very first examples of cohomogeneity 1 Einstein metrics, e.g., the Page metric on $\mathbb{CP}^2 \sharp (-\mathbb{CP}^2)$ [**91**] and its generalizations [**16, 93**]. These however will be dealt with in the broader context of §3, where the bundle structure plays a more important role and allows examples with little or no symmetry to be constructed. We would like to mention, however, that 4-dimensional Einstein orbifolds with $U(2)$-actions of cohomogeneity 1 have been studied in detail in [**100**]. Both Kähler and non-Kähler Einstein orbifolds with positive Einstein constants were found, but not with zero or negative constants. It is interesting to compare this study with Theorems 2.9, 2.11, and Theorems 3.1–3.5 below since these results show that blow-downs of the singular orbits can be realized on manifolds when the base of the bundle is more complicated. On the other hand, when there are no manifold solutions in these situations, it might in turn be possible to find many orbifold solutions as in **[100]**.

We turn now to the work of C. Böhm, who studied the cohomogeneity 1 Einstein equations (2.1–2.4) in the situation where the isotropy representation of the principal orbit G/K splits into two inequivalent sub-representations and H/K is a sphere of dimension greater than 1. Because of the first integral (2.5), the Einstein equation can be thought of as a vector field on the 3-dimensional constant energy hypersurface defined by it.

THEOREM 2.13. **[19]** *There exists infinitely many pairwise non-isometric Einstein metrics of cohomogeneity* 1 *with positive scalar curvature on* $S^{n+1}, 4 \leq n \leq 8$.

This theorem provides for the first time infinitely many inhomogeneous Einstein metrics on standard spheres as well as the existence of more than one Einstein metric on even-dimensional spheres. The group G in these examples is $SO(p+1)\times SO(q+1)$ where $p+q=n$, $p,q \geq 2$ and the principal isotropy group $K = SO(p) \times SO(q)$. The two singular orbits are $S^p \times \{*\}$ and $\{*\} \times S^q$. Using this range of values of p and q, Böhm obtains one infinite sequence of pairwise non-isometric Einstein metrics on S^5 and S^6, *two* infinite sequences of non-isometric Einstein metrics on S^7 and S^8, and *three* such infinite sequences on S^9.

A new phenomenon is exhibited by these sequences of Einstein metrics. Let the Einstein constants be normalized to be equal to 1. For fixed p, q, the sequence of Einstein metrics converges in the Gromov-Hausdorff distance to the singular *Einstein* metric

$$dt^2 + \frac{p-1}{n-1}\sin(t)^2 g_{S^p} + \frac{q-1}{n-1}\sin(t)^2 g_{S^q}.$$

Away from the singular orbit, the sequence actually converges in the C^∞ topology. Notice that the volume of the limiting space is positive and the sectional curvatures blow up at the singularities. Also, the group action survives in the limit with the exception that the singular orbits are blown down to points. By contrast, a sequence of similarly normalized examples from Theorem 1.1 have bounded sectional curvatures and volumes tending to 0. Furthermore, if the diameter of the fibres tends to 0, which is automatic in the case of circle bundles, then the sequence of Einstein manifolds collapse (in the sense of Gromov) to the base with some product metric, not necessarily Einstein.

Because of the inexplicit nature of Böhm's solutions, one cannot yet decide whether or not the infinitely many Einstein metrics belong to different components of the Einstein moduli space. However, from the convergence to the singular Einstein space, it is possible to check that the Einstein metrics with different G's belong to different components of the moduli space and also belong to different components than the homogeneous Einstein metrics on spheres.

Besides low-dimensional spheres, Böhm has also constructed cohomogeneity 1 Einstein metrics on certain low-dimensional product manifolds.

THEOREM 2.14. **[19]** *There exists infinitely many non-isometric Einstein metrics of cohomogeneity* 1 *on* $\hat{M} = S^{p+1} \times Q^q$, *where* $5 \leq p+q+1 \leq 9, p > 1, q > 1$, *and* Q *is a non-flat compact isotropy irreducible homogeneous space* $\overline{G}/\overline{H}$.

In the above, the group G is $SO(p+1) \times \overline{G}$ and the principal isotropy group is $K = SO(p) \times \overline{H}$. The two singular orbits are both $Q = G/H$, with $H = SO(p+1) \times \overline{H}$. Note that $\overline{G}/\overline{H}$ is just the effective version of G/H.

Furthermore, Böhm was able to construct analytically an Einstein metric on $\mathbb{HP}^2 \sharp (-\mathbb{HP}^2)$. Numerical solutions were obtained in [**94**] on the connected sum of two $\mathbb{HP}^n$ for a range of n.

We will now give a sketch of the methods employed by Böhm to obtain the above existence theorems.

In the situations of Theorems 2.13 and 2.14, the principal orbit is a product manifold whose isotropy representation consists of two inequivalent irreducible summands $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Hence the metric $\hat{g}$ can be written as $dt^2 + f_1(t)^2 g_b|_{\mathfrak{p}_1} + f_2(t)^2 g_b|_{\mathfrak{p}_2}$, where g_b is an appropriately normalised background product Einstein metric on P. For Theorem 2.14, Böhm looks for solutions on an interval $[0, T]$ with boundary conditions $f_1(0) = 0 = f_1(T)$, $f_1'(0) = 1 = -f_1'(T)$, and $f_2(0) = f_2(T) = a > 0$, $f_2'(0) = 0 = f_2'(T)$. Geometrically, this means that reflection about the midpoint of the interval $[0, T]$ is an isometry and the principal orbit at the midpoint is totally geodesic. In Theorem 2.13, the boundary conditions used are instead $f_1(0) = 0 = f_2(T)$, $f_1'(0) = 1 = -f_2'(T)$, and $f_2(0) = a > 0, f_1(T) = b > 0$, $f_1'(T) = 0 = f_2'(0)$. In either case, the boundary conditions are precisely the smoothness conditions for the particular singular orbit type, and the initial value problem for Einstein metrics has a unique solution depending continuously on the single initial value a or b.

For any (local) solution emanating from a singular orbit it is first shown that the trace of the shape operator of the principal orbits is strictly decreasing and reaches zero before the maximal time of existence of the solution. Such a zero is called a *turning point*. Furthermore, all the critical points of the function $w = f_1/f_2$ of t are non-degenerate. Let N_a denote the number of critical points of w occurring before the turning point of the solution $f_a = (f_1, f_2)$ with initial value a. N_a is finite and remains constant as a is varied in an interval $[a_1, a_2] \subset \mathbb{R}_+$ provided that no a in the interval corresponds to a reflection symmetric solution, i.e., one which reaches a totally geodesic principal orbit ($f_1'(t^*) = f_2'(t^*) = 0$), which can therefore be extended to a global solution by reflection. On the other hand, if f_a passes through a reflection symmetric solution, then N_a jumps up or down by at most 1. Consequently, the change in N_a as a is varied can be used to detect and give a lower bound for reflection symmetric solutions.

In order to exploit this fact, Böhm shows that in the examples of the theorems above, N_a tends to $+\infty$ as $a \to 0$. It is here that the dimension restrictions in Theorems 2.13 and 2.14 enter crucially, together with the special properties of two-dimensional vector fields.

Recall that the first integral (2.5) implies that the Einstein equations can be viewed as a vector field on the three-dimensional constant energy manifold $E(w, w', f_2, f_2') = 0$. Böhm first shows that the spherical cone $(dt^2 + \sin(t)^2 g_b)$ of the product Einstein metric of the principal orbit is a local attractor for the integral curves of the Einstein vector field. Next, he uses special charts to study this vector field in detail and establishes certain rotational behaviour of the solutions. In a chart parametrized by w, f_2, f_2', it is shown that after a suitable blow-up, the Einstein vector field extends to a vector field V defined on a rectangular region in the boundary $f_2 \equiv 0$. V has two zeros: $(0,0)$, and z which corresponds to the spherical cone. Now z is a focal point (in the sense that the linearization of V at z has non-real eigenvalues) only if $n \leq 8$. In this case, the integral curve starting from $(0,0)$ eventually spirals around z. Here, one has to use Poincaré-Bendixson and also rule out the possibility of a limit cycle. From this behaviour of the integral curve, one can then deduce the limiting behaviour of N_a as $a \to 0^+$.

Theorem 2.14 now follows immediately from the above properties of N_a.

In order to prove Theorem 2.13, Böhm again uses the attracting property of the Einstein spherical cone. This time, he constructs a 2-dimensional slice whose origin is a point on the integral curve of the spherical cone. He shows that solutions emanating from the two singular orbits with small enough initial values a or b intersect this slice at a unique point. As a (resp. b) tends to 0, the locus of the intersection point is a clockwise (resp. anti-clockwise) spiral, both with the origin as limit point. The two spirals intersect in infinitely many points (in the slice), and each intersection represents an integral curve emanating from one singular orbit which continues to the other singular orbit. In this way, one obtains infinitely many Einstein metrics.

Finally, simple geometric arguments show that the metrics constructed cannot be homogeneous and cannot be isometric to each other.

In the case of $\mathbb{HP}^2\sharp(-\mathbb{HP}^2)$, the zero z of the vector field V is a node and one only has $N_a \geq 1$ as $a \to 0^+$.

Readers who are familiar with the many constructions of minimal submanifolds in spheres and other symmetric spaces using equivariant geometry will recognize that Böhm uses many of the same techniques. Of course, the Einstein equation is somewhat more complicated because one is dealing with a system rather than a single ODE.

A large family of complete, non-compact Einstein metrics of cohomogeneity 1 has very recently been found by Böhm [**21**] as a result of further study of the dynamic properties of the cohomogeneity 1 Einstein equations (2.1–2.4).

THEOREM 2.15. [**21**] *Let $m \geq 1$ and $k \geq 3$ be integers and $G_i/K_i, 1 \leq i \leq m$, be non-flat, compact isotropy irreducible spaces. Then $\mathbb{R}^k \times G_1/K_1 \times \cdots \times G_m/K_m$ has an m-dimensional family of complete Einstein metrics with negative scalar curvature as well as an $(m-1)$-dimensional family of complete Ricci flat metrics. All these metrics are of cohomogeneity* 1 *under the group $SO(k) \times G_1 \times \cdots \times G_m$.*

In certain cases, Böhm also finds finite subgroups of $SO(k) \times G_1 \times \cdots \times G_m$ which act freely on the product manifold, and in this way obtain families of Einstein metrics on the corresponding quotient manifolds.(Compare Theorem 4.1(i) below.)

E. Non-Existence. It follows from the analyses in [**16**] and [**93**] that the cohomogeneity 1 Einstein equations, specifically (3.2–3.4), can fail to have global smooth solutions, and hence there are closed simply connected manifolds of cohomogeneity 1 with respect to a fixed G-action which do not admit any G-invariant Einstein metrics. We present here one rather intriquing example, which was already mentioned in [**17**, p. 275].

EXAMPLE 2.2. As in §1, let P_b be the principal $U(1)$ bundle over $S^2 = \mathbb{CP}^1$ with Euler class $b \cdot \alpha$, where α is the generator of $H^2(S^2;\mathbb{Z})$ corresponding to the hyperplane bundle. P_b is really the lens space $U(2)/(U(1)\cdot\mathbb{Z}_b)$. The associated $\mathbb{CP}^1$

bundles are closed manifolds with an almost effective cohomogeneity 1 $U(2)$-action. (In fact, with the natural induced complex structure, these are the Hirzebruch surfaces.) For $|b| \geq 2$, it follows from **[16]** or **[93]** that there are no $U(2)$-invariant Einstein metrics. However, there are only 2 diffeomorphism types among the $\mathbb{CP}^1$ bundles: $S^2 \times S^2$ when b is even and $\mathbb{CP}^2 \sharp (-\mathbb{CP}^2)$ when b is odd. These two smooth manifolds admit respectively a homogeneous (the product metric) and a cohomogeneity 1 Einstein metric (the Page metric). This shows that the same manifold can have infinitely many cohomogeneity 1 actions by the same abstract group, but only some actions support invariant Einstein metrics.

Recently, Böhm has obtained a non-existence criterion for cohomogeneity 1 Einstein metrics on closed manifolds in terms of the orbit structure and the geometry of the principal orbit.

THEOREM 2.16. **[20]** *Let $\hat{M}$ be a closed G-manifold with cohomogeneity 1 and two singular orbits $Q_i = G/H_i$, $i = 1, 2$. Let G/K be the principal orbit type, with $K \subset H_i$. Suppose that $\mathfrak{h}_i = \mathfrak{k} \oplus \mathfrak{p}_i$ are $Ad(K)$ invariant decompositions, and $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_l$ is the decomposition of the isotropy representation of G/K into $Ad(K)$ invariant isotypic components. (Isotypic means a direct sum of equivalent irreducible representations.) If for some j, $\mathfrak{m}_j$ is $Ad(K)$-irreducible, $\mathfrak{m}_j \cap (\mathfrak{p}_1 \cup \mathfrak{p}_2) = \{0\}$, and the restriction of the trace-free part of the Ricci tensor of* **any** *G-invariant metric on G/K to the summand $\mathfrak{m}_j$ is negative definite, then there cannot be any smooth G-invariant Einstein metrics on $\hat{M}$.*

A large number of examples satisfying the hypotheses of the above theorem can be constructed **[20]** using compact homogeneous manifolds which do not admit any homogeneous Einstein metrics (see §4A). A simple example is the following. Let $G = SO(k+1) \times \overline{G}$, $H = SO(k+1) \times \overline{H}$, and $K = SO(k) \times \overline{H}$, where $\overline{G}/\overline{H}$ is $SO(2l)/(SO(l) \times U(1)), l \geq 32$. The G-manifold is $S^{k+1} \times (\overline{G}/\overline{H})$, and has no G-invariant Einstein metrics if $1 \leq k \leq l/3$.

3. Modified Kaluza-Klein Ansatz on Fibre Bundles.

In this section we consider a useful modification of the Kaluza-Klein ansatz. Let $\tilde{\pi} : \tilde{P} \longrightarrow M$ be a principal G-bundle and F a manifold on which G acts almost effectively with cohomogeneity 1. Let K denote a principal isotropy group of this action and let H (resp. H_1, H_2) denote the isotropy group(s) of the singular orbit(s). We will only refer to the situation having one singular orbit since analogous statements hold for the other singular orbit, if it is present. As in §2 we may assume that $K \subset H$. Recall also that H/K is diffeomorphic to a sphere S^k.

Let W be the manifold $\tilde{P} \times_G F$. Then W is the union of a 1-parameter family of hypersurfaces diffeomorphic to $P = \tilde{P} \times_G (G/K) = \tilde{P}/K$ which collapse onto $Q = \tilde{P} \times_G (G/H) = \tilde{P}/H$. P is an $H/K \cong S^k$ bundle over Q and this sphere bundle may be identified with the unit sphere bundle of the normal bundle of Q in W.

We can construct a metric $\hat{g}$ on W as follows. We choose a connection ϕ on the principal bundle $\tilde{P}$. This induces connections on $W \rightarrow M$ as well as on P and Q. Recall that a G-invariant metric on F can be written as $dt^2 + q_t$ where q_t is a 1-parameter family of homogeneous metrics on G/K defined on an interval $I = (0, T)$ where T is allowed to be $+\infty$. Let g_t^* be a 1-parameter family of metrics on M defined on the same interval. Using the connection ϕ, we let

$$\hat{g} = dt^2 + q_t + \tilde{\pi}^* g_t^*. \tag{3.1}$$

Of course there has to be boundary conditions at $t = 0$ and, in the compact case, at $t = T$ to ensure that $\hat{g}$ is smooth, just as in the cohomogeneity 1 case. Notice that for a fixed t, $g_t = q_t + \tilde{\pi}^* g_t^*$ makes $(P, g_t) \longrightarrow (M, g_t^*)$ into a Riemannian submersion with totally geodesic fibres. The modified Kaluza-Klein ansatz asks for $\hat{g}$ to be Einstein.

In order to write down the Einstein condition for $\hat{g}$, we need only observe that $W \setminus Q \cong I \times P$ is an equidistant family of hypersurfaces with unit normal field $N = \partial/\partial t$. Introducing the shape operators $\mathcal{L}_t$ of the hypersurfaces as in the cohomogeneity 1 case, we see that the Einstein condition for $\hat{g}$ is again given by (2.1–2.4) where the Ricci operator r_t of g_t can be computed using the theory of Riemannian submersions.

If we examine the arguments in Lemma 2.1 and Proposition 2.2 we find that Proposition 2.2 also holds in the present situation provided that for each t the scalar curvature of g_t^* is constant and the pointwise norm of the curvature form Ω is a constant function on $\tilde{P}$.

We will first discuss a special case that gives rise to large families of Einstein hermitian metrics and also unifies and generalizes many known examples. In particular, we obtain Einstein metrics on certain Fano manifolds when Kähler-Einstein metrics are obstructed (see Theorems 3.3 and 3.4).

Let (M_j, J_j, g_j^*) be Kähler-Einstein Fano manifolds as in §1 and let P_χ be a principal $U(1)$ bundle over $M = M_1 \times \cdots \times M_m$, where its Euler class $\chi = \sum_j b_j \pi_j^* \alpha_j \in H^2(M; \mathbb{Z})$. For F we take $\mathbb{C}$ or $S^2 = \mathbb{CP}^1$ or $\mathbb{RP}^2$, on which S^1 acts by complex multiplication in the first two cases. In the last case, the circle acts by the induced action on the $\mathbb{Z}/2$ quotient, so that the singular orbits are a point and a circle. Then W is a complex line bundle (resp. $\mathbb{CP}^1$, $\mathbb{RP}^2$ bundle) over M. As before, the connection ϕ on P_χ will be chosen so that $\frac{\sqrt{-1}}{2\pi} d\phi$ is the pull-back of a 2-form harmonic with respect to the product metric on M. We let

$$\hat{g} = dt^2 + h(t)^2 \langle\ ,\ \rangle + \sum_{j=1}^m f_j(t)^2 \pi_j^* g_j^*,$$

where $h, f_1, \cdots, f_m$ are smooth positive functions on I and $\langle\ ,\ \rangle$ is the metric on S^1 so that it has length 2π. We shall denote the real dimension of M_j by $2n_j$.

The Einstein equation is the following system.

$$-\frac{h''}{h} - \sum_{j=1}^m 2n_j \frac{f_j''}{f_j} = \Lambda \tag{3.2}$$

$$-\frac{h''}{h} - \sum_{j=1}^m 2n_j \frac{h' f_j'}{h f_j} + \sum_{j=1}^m n_j \frac{b_j^2}{2} \frac{h^2}{f_j^4} = \Lambda \tag{3.3}$$

$$-\frac{f_i''}{f_i} - \frac{h' f_i'}{h f_i} - \sum_j 2n_j \frac{f_i' f_j'}{f_i f_j} + \left(\frac{f_i'}{f_i}\right)^2 - \frac{b_i^2}{2} \frac{h^2}{f_i^4} + \frac{r_i^*}{f_i^2} = \Lambda \tag{3.4}$$

where r_i^* is the Ricci endomorphism of g_i^*, which is $p_i \cdot \mathbf{I}$ in our situation.

For the detailed analysis of these equations we refer the reader to [**45, 118**]. We will, however, make a few remarks about the special features of this system. First, equating the first two equations gives the relation

$$\sum_{j=1}^{m} 2n_j \left(\frac{f_j''}{f_j} - \frac{h'f_j'}{hf_j} + \frac{b_j^2}{4}\frac{h^2}{f_j^4} \right) = 0. \tag{3.5}$$

Let

$$\mu_i = \frac{f_i''}{f_i} - \frac{h'f_i'}{hf_i} + \frac{b_i^2}{4}\frac{h^2}{f_i^4}. \tag{3.6}$$

It turns out that explicit solutions can be obtained by setting all μ_i to be identically zero. To see this, we change variables by letting $dr = h(t)dt$ and defining $A_0(r) = h(t)^2$ and $A_i(r) = f_i(t)^2$. Then the Einstein equations become

$$\frac{1}{2}A_0'' + \frac{1}{2}A_0'(\log v)' + A_0 \sum_{j=1}^{m} n_j \left(\frac{A_j''}{A_j} - \frac{1}{2}\left(\frac{A_j'}{A_j}\right)^2 \right) = -\Lambda, \tag{3.7}$$

$$\frac{1}{2}A_0'' + \frac{1}{2}A_0'(\log v)' - \frac{A_0}{2}\sum_{j=1}^{m} n_j \frac{b_j^2}{A_j^2} = -\Lambda, \tag{3.8}$$

$$\frac{A_0'A_i'}{2A_i} + \frac{A_0}{2}\left(\frac{b_i^2}{A_i^2} + \frac{A_i''}{A_i} - \left(\frac{A_i'}{A_i}\right)^2 + (\log v)'\frac{A_i'}{A_i} \right) - \frac{r_i^*}{A_i} = -\Lambda, \tag{3.9}$$

where

$$v = \prod_{j=1}^{m} f_j^{2n_j} = \prod_j A_j^{n_j}.$$

There are two types of solutions of the equation $\mu_i = 0$. Either

$$A_i(r) = \pm(b_i r + a_i),$$

or

$$A_i(r) = \lambda_i(r + c_i)^2 - \frac{1}{4}\frac{b_i^2}{\lambda_i},$$

where a_i and λ_i are constants of integration. We shall refer to these two cases respectively as the linear and quadratic cases. Observe that (3.9) is a *linear* equation in A_0, so it can be integrated, and A_0 can be expressed in terms of the functions A_i.

In order to have explicit local solutions of (3.7–3.9), there will be consistency conditions to satisfy so that the m equations in (3.9) determine the same function A_0. One can then see that all of (3.7–3.9) hold. Using these local solutions, we can analyse the boundary conditions that will ensure that the solutions extend to a smooth metric on W. When W is non-compact, there are additional conditions which guarantee a complete metric.

In the cases where $F = \mathbb{C}$ or S^2, there is a complex structure J_h on W obtained by lifting the product complex structure of the base to the horizontal tangent spaces using the connection ϕ and defining $J_h(N)$ to be $h^{-1}U$, where U is the infinitesimal generator of the S^1 action, i.e., $\phi(U) = \sqrt{-1}$. Then $\hat{g}$ is hermitian with respect to J_h. In the compact and Ricci-flat cases, J_h is equivalent to the natural complex structure induced from the base and fibres. Furthermore, the Kähler condition for

$\hat{g}$ becomes simply $A_i' = -b_i$, $1 \le i \le m$. Therefore, the linear case of the explicit solutions corresponds to Kähler metrics modulo orientation.

If $\hat{g}$ is Kähler, then the right S^1 action on $I \times P$ has a momentum map and it is easy to see that it corresponds to an anti-derivative of h. Thus the change of variable used above to simplify the equations (3.2–3.4) is not at all ad hoc. It gives an explicit representation of the Einstein metric $\hat{g}$ in terms of rational functions. Also, there is a geometrical interpretation of the condition $\mu_i = 0, 1 \le i \le m$. It can be shown (see [**118**, §7]) that it is equivalent to

$$-(J_h \cdot \hat{R})(X,Y,Z,V) := \hat{R}(J_hX, J_hY, J_hZ, J_hV) = \hat{R}(X,Y,Z,V),$$

for all tangent vectors X, Y, Z, V of W, where $\hat{R}$ in the above is the Riemann curvature tensor of $\hat{g}$.

Thus far in the case under consideration, we have $\tilde{P} = P_\chi$ and $Q = M$, and we will refer to the situation as an S^1 collapse. On the other hand, it is possible to have S^k collapses with $k > 1$. In that case, one of the factors of M must be $\mathbb{CP}^{l-1}$ with $k = 2l-1$. Our convention is then that the S^1 collapse case is identified with the case when $l = 1$ and one of the factors of M reduces to a point.

We will now give precise statements of the existence theorems and describe the special cases which were previously known. We begin with the linear (Kähler) case.

THEOREM 3.1. [**75, 76**] *Let $m \ge 3$, $(M_1, J_1^*) = (\mathbb{CP}^{l_1-1}, can)$ and $(M_m, J_m^*) = (\mathbb{CP}^{l_m-1}, can)$. Suppose that $b_1 = -b_m = -1$ and $b_2, \cdots, b_{m-1}$ are non-zero integers. Suppose that further that $l_1 b_j > -p_j$ and $p_j > l_m b_j$ for all j, $2 \le j \le m-1$. Then there is a Kähler Einstein metric with positive scalar curvature on $[P_\chi \times_{U(1)} \mathbb{CP}^1]/\sim$, (where $\sim$ means collapsing $M \times \{0\}$ onto $M_2 \times \cdots \times M_m$ and/or $M \times \{\infty\}$ onto $M_1 \times \cdots \times M_{m-1}$) iff the Futaki integral*

$$\int_{-l_1}^{l_m} \prod_{j=1}^{m} \left(\frac{p_j}{b_j} - x\right)^{n_j} x\,dx = 0.$$

(Note that $n_1 = l_1 - 1$ and $n_m = l_m - 1$. Hence when $l_1 = 1$ or $l_m = 1$, then the corresponding factors are identically 1 in the above integral.) Actually, Theorem 3.1 is a version of the existence theorem of Koiso-Sakane adapted to the present framework in order to facilitate comparison with Theorem 3.4 below. The general form of their theorem [**75**, Theorem 4.2] deals with compactifications of hermitian line bundles over a Kähler-Einstein Fano manifold such that the eigenvalues of the curvature form of the line bundles are constant with respect to the Ricci form of the base.

THEOREM 3.2. [**117, 118, 45**] *Let $m \ge 2$ and $(M_1, J_1^*) = (\mathbb{CP}^{l_1-1}, can)$, $l_1 \ge 1$. Assume that χ is determined by integers $b_1 = -1$ and $b_2, \cdots, b_m$.*

(i) *If $b_j l_1 = -p_j$ for all $j \ge 2$, then there is an $(m-1)$-parameter family of complete Ricci-flat Kähler metrics on $[P_\chi \times_{U(1)} \mathbb{C}]/\sim$, where $\sim$ means collapsing $M \times \{0\}$ onto $M_2 \times \cdots \times M_m$.*

(ii) *If $-b_j l_1 > p_j$ for all $j \ge 2$, then there exists a complete Kähler-Einstein metric with negative scalar curvature and infinite volume on $[P_\chi \times_{U(1)} D]/\sim$, where D is an open disk containing 0 in $\mathbb{C}$ and $\sim$ is as in (i).*

The $m = 2, l_1 = 1$ case is due to Bérard Bergery. We turn next to the quadratic case and begin with compact examples.

THEOREM 3.3. [**117, 118, 45**] *Let $b_1, \cdots, b_m$, $m \geq 2$, be integers satisfying*

$$|b_1| = 1, \quad 0 < l_1|b_j| < p_j, \quad j > 1$$

where $(M_1, J_1^*) = (\mathbb{CP}^{l_1-1}, can)$, $l_1 \geq 1$. Then there is an Einstein metric with positive scalar curvature on $[P_\chi \times_{U(1)} \mathbb{CP}^1]/\sim$ where $\sim$ means collapsing $M \times \{0\}$ and $M \times \{\infty\}$ down to $M_2 \times \cdots \times M_m$. The Einstein metric is hermitian with respect to J_h and has $\mathbb{Z}/2$ symmetry about the equator of $\mathbb{CP}^1$.

When $m = 2$ and $l_1 = 1$, we obtain the well-known Einstein metrics of Bérard Bergery [**16**] and Page and Pope [**93**]. Taking M_2 further to be $\mathbb{CP}^1$ we recover the Page metric [**91**] on $\mathbb{CP}^2\sharp(-\mathbb{CP}^2)$, which has the distinction of being the first compact *inhomogeneous* Einstein metric of positive scalar curvature discovered. Note that in these cases $\mu_i = 0$ is not an additional condition.

THEOREM 3.4. [**117, 118, 45**] *Let $m \geq 3$ and $(M_1, J_1^*) = (\mathbb{CP}^{l_1-1}, can)$, $(M_m, J_m^*) = (\mathbb{CP}^{l_m-1}, can)$, with $l_1 \geq 1, l_m \geq 1$. Suppose that there are integer $b_1, \cdots, b_m$, and $\varepsilon_j = \pm 1, 1 \leq j \leq m$, with the following properties:*

(i) *$|b_1| = 1 = |b_m|$.*
(ii) *For $2 \leq i \leq m-1$, if $\varepsilon_i = 1$, then $0 < l_1|b_i| < p_i$ and if $\varepsilon_i = -1$, then $0 < l_m|b_i| < p_i$.*
(iii) *$\varepsilon_1 = 1$, $\varepsilon_m = -1$, and when $l_1 = l_m = 1$, then at least one of the ε_i, $2 \leq i \leq m-1$, is positive.*
(iv) *The integral*

$$\int_{-l_1}^{l_m} \prod_{j=1}^{m} \left(\frac{p_j}{|b_j|} + \varepsilon_j x \right)^{n_j} x dx < 0.$$

Then there exists an Einstein metric with positive scalar curvature on $[P_\chi \times_{U(1)} \mathbb{CP}^1]/\sim$, where $\sim$ means collapsing $M \times \{0\}$ onto $M_2 \times \cdots \times M_m$ and/or collapsing $M \times \{\infty\}$ onto $M_1 \times \cdots \times M_{m-1}$. The Einstein metric is hermitian with respect to the complex structure J_h and does not have $\mathbb{Z}/2$ symmetry with respect to the equator of S^2.

The Einstein metrics in this theorem should perhaps be grouped with the Kähler-Einstein metrics in Theorem 3.1. This is because the integral condition in (iv) complements that in Theorem 3.1, which expresses the vanishing of the Futaki invariant evaluated on the (real) holomorphic vector field $h\partial/\partial t$. For example, over $\mathbb{CP}^1 \times \mathbb{CP}^n, n > 1$, the $\mathbb{CP}^1$ bundles with $(b_1, b_2) = (-1, k), 0 < k < n+1$, do not admit any Kähler-Einstein metric, but they all admit an Einstein metric by the above theorem.

As for non-Kähler Einstein metrics with non-positive Einstein constant, we have

THEOREM 3.5. [**117, 118, 45**] *Let $(M_1, J_1^*) = (\mathbb{CP}^{l_1-1}, can)$ with $l_1 \geq 1$ and $b_1, \cdots, b_m$, $m \geq 2$, be non-zero integers.*

(i) *If $l_1|b_j| < p_j$ for $j > 1$ and $|b_1| = 1$ whenever $l_1 > 1$, then there exists a complete Ricci flat J_h-hermitian metric on $[P_\chi \times_{U(1)} \mathbb{C}]/\sim$, where $\sim$ denotes collapsing $M \times \{0\}$ onto $M_2 \times \cdots \times M_m$.*
(ii) *If $|b_1| = 1$ when $l_1 > 1$, then there exists a 1-parameter family of non-homothetic complete Einstein metrics with negative constant on $[P_\chi \times_{U(1)} D]/\sim$, where $D \subset \mathbb{C}$ is an open disk about 0 and $\sim$ denotes collapsing*

$M \times \{0\}$ onto $M_2 \times \cdots \times M_m$. These Einstein metrics are J_h hermitian and have infinite volume.

In all the above results (3.2)–(3.5), the versions without blow-downs were first obtained in [**117**]. Complete proofs together with a study of the associated hermitian geometry and analogues in Einstein-Weyl geometry can be found in [**118**]. The present versions were announced in the preprint version of [**45**]. The proofs consist of combining the analysis in [**118**] with the study of the blow-down situation in the cohomogeneity 1 Kähler case in [DaWa].

Just as in [**31**] and [**16**], it is also possible to construct both Kähler and non-Kähler Einstein metrics with negative constant on the total spaces of complex line bundles over a product of compact Kähler-Einstein manifolds with negative first Chern class and/or compact Ricci-flat Hodge manifolds. For a factor M_j of the first type, as in §1, we shall write the first Chern class as $-p_j \cdot \alpha_j$, where α_j is indivisible, p_j is positive, and we normalize the Kähler-Einstein metric to have constant $-p_j$. For a Ricci-flat factor, we shall assume that the Kähler class is of the form $2\pi\alpha_j$, i.e., the Kähler-Einstein metric is Hodge.

THEOREM 3.6. *Let $M = M_1 \times \cdots \times M_m$ be a product of compact Kähler-Einstein manifolds of the above types and P_χ be the principal $U(1)$-bundle over M with Euler class $\chi = \sum_j b_j \pi_j^* \alpha_j$, $b_j \neq 0$. Let W_χ denote the associated complex line bundle with the induced complex structure.*

(i) *If $b_j < 0$ for all j, then there exists a complete Kähler-Einstein metric with negative constant on a disk subbundle of W_χ.*
(ii) *For any choice of the b_j, there exists a 1-parameter family of non-homothetic hermitian but non-Kähler Einstein metrics with negative constant on a disk subbundle of W_χ.*

Note that in the above theorem, the base M can contain factors of both types.

The modified Kaluza-Klein ansatz can also be used to construct Einstein metrics with special holonomy.

THEOREM 3.7. [**29, 57**] *There are complete Einstein metrics of type (3.1) with the indicated holonomy on the following bundles:*

(i) *the bundle of anti-self-dual 2-forms over S^4 or $\mathbb{CP}^2$; holonomy type G_2*
(ii) *the bundle of real spinors over S^3; holonomy type G_2*
(iii) *the negative spin bundle over S^4 regarded as a self-dual manifold; holonomy type $Spin(7)$.*

In (i) above, $\tilde{P}$ is an $SO(3)$-bundle and $SO(3)$ acts on $F = \Lambda^2_-(\mathbb{R}^4) \cong \mathbb{R}^3$ in the usual manner. Likewise in (iii), $\tilde{P}$ is an $SU(2)$-bundle with $SU(2)$ acting in the usual way on $F = \mathbb{C}^2$, which is also a cohomogeneity 1 action. The fact that the base manifolds are self-dual and Einstein is significant because these properties provide $\tilde{P}$ with the Yang-Mills connection ϕ required in the construction. In (ii), $\tilde{P}$ is the trivial $SU(2)$ bundle over S^3 and $SU(2) = Spin(3)$ acts on $\mathbb{H}$ by quaternion multiplication. The bundles in (ii) and (iii) are both topologically trivial. Furthermore, all the metrics in the above theorem are explicit and, since the base manifolds are homogeneous, are of cohomogeneity 1 as well. Many incomplete metrics with G_2 or $Spin(7)$ holonomy were also found in [**29**]. We refer the reader to the article by Joyce for further information about the search for Einstein metrics with these holonomy groups.

Finally, we mention some numerical solutions of the Einstein system of the modified Kaluza-Klein ansatz.

In [**94**], the authors considered the situation where $G = Sp(1)$ and $\tilde{P} = S^{4n+3}$ is the total space of the Hopf bundle over $\mathbb{HP}^n$. The fibre F is $S^4 = \mathbb{H} \cup \{\infty\}$ or $\mathbb{H}$, on which $Sp(1)$ acts as the unit quaternions. In the first case, W is $\mathbb{HP}^{n+1} \sharp (-\mathbb{HP}^{n+1})$. Numerical evidence for an Einstein metric of type (3.1) was given. As was pointed out in §3, the case $n = 1$ has now been analytically established by Böhm [**19**]. He also produced numerical evidence for a second solution when $n \geq 2$. In the second case, W is either $\mathbb{H}^{n+1}$ or a non-trivial quaternionic line bundle over $\mathbb{HP}^n$. Page and Pope produced numerical solutions with negative and zero Einstein constants. Recently, Böhm [**21**] gave a proof of these numerical results as well as the corresponding result for the Hopf bundle over the Cayley plane.

A second situation was studied by Gibbons, Page, and Pope in [**57**]. Here, $G = SO(3)$ and $\tilde{P} \to M$ is the canonical $SO(3)$ bundle of a quaternionic Kähler manifold with positive scalar curvature. The fibre F is either $S^3 = \mathbb{R}^3 \cup \{\infty\}$ or $\mathbb{R}^3$ with G acting in the usual way as rotations. The hypersurfaces P are the quaternionic-Kähler twistor spaces of M. In the first case, numerical solutions exist when the dimension of M is small, e.g., when $M = S^4$ or $\mathbb{CP}^2$. Again, Böhm found a second numerical solution for $M = \mathbb{HP}^n$ for a certain n. In the second case, numerical solutions were also found, which have again recently been proved in [**21**] for all values of $n \geq 3$.

4. Homogeneous Einstein Metrics.

The Einstein condition for a homogeneous metric is a system of algebraic equations for which one seeks a real solution satisfying some positivity condition reflecting the positive definiteness of the metric which the solution represents. By the theorem of Alekseevsky-Kimel'fel'd [**6**], Ricci-flat homogeneous spaces are flat. A homogeneous Einstein manifold with positive Einstein constant must be compact with finite fundamental group by Bonnet-Myers, while an Einstein manifold with negative Einstein constant must be noncompact by Bochner's theorem.

A. Positive Einstein Constant. We begin with a qualitative picture. Homogeneous Einstein metrics on a compact homogeneous space G/K are precisely the critical points of the scalar curvature function on the space of G-invariant metrics with a fixed value for the volume. This fact was first exploited in [**67**] for left-invariant metrics on Lie groups.

THEOREM 4.1. [**123**] *Let G/K be an effective, compact homogeneous space with G and K compact and connected. Let $\mathcal{S}$ denote the function that assigns to each G-invariant metric of volume 1 its scalar curvature.*

(i) *$\mathcal{S}$ is bounded from below iff the universal cover of G/K has the form $\mathbb{R}^k \times (G_1/K_1) \times \cdots \times (G_l/K_l)$ where G_j/K_j are isotropy irreducible. In this case, $\mathcal{S}$ is proper iff $k = 0$. When $k = 0$, there is a unique critical point and $\mathcal{S}$ is bounded below by a positive constant. The critical point corresponds to the product Einstein metric. If $k \geq 1$, then $\mathcal{S}$ has a critical point iff G/K is a torus.*

(ii) *$\mathcal{S}$ is bounded from above and proper iff $\mathfrak{k}$ is a maximal subalgebra (by inclusion) in $\mathfrak{g}$. In this case, $\mathcal{S}$ has a global maximum which is therefore an Einstein metric.*

((iii) *$\mathcal{S}$ is bounded from above but is not proper iff $K \cdot U(1)$ is a subgroup of G and $G/(K \cdot U(1))$ is a compact irreducible hermitian symmetric space other than $SO(n+2)/(SO(n)SO(2))$.*

Isotropy irreducible spaces will be defined and described presently. Compact homogeneous spaces with $\mathfrak{k}$ maximal in $\mathfrak{g}$ are quite numerous and Theorem 4.1 therefore gives rise to many homogeneous Einstein manifolds. Roughly speaking, there are as many maximal subalgebras as there are irreducible representations. To explain this we recall some classification theorems of Dynkin in [**52**]. Let K be a compact simple Lie group, $\rho : K \longrightarrow SU(N)$ be an irreducible finite-dimensional unitary representation of K, and τ be the usual complex 1-dimensional representation of the circle. What Dynkin proved in the unitary case can be formulated (see [**121**]) more succintly as follows. $\rho(K)$ is maximal in $SU(N)$ unless $\rho(K) \subset SU(n) \subset SU(N)$ in which (a) the realification of $\rho \otimes \tau : K \times U(1) \longrightarrow U(n)$ is the isotropy representation of an irreducible hermitian symmetric space, and (b) the second inclusion belongs to a distinguished subset of the (irreducible) exterior powers of the vector representation of $SU(n)$. For the symplectic and orthogonal cases, see [**121**].

When K is closed but not necessarily connected, and $\mathfrak{k}$ is maximal in $\mathfrak{g}$, then 4.1(ii) still implies that G/K admits a G-invariant Einstein metric. This is because the $Ad(K)$-invariant inner products on the tangent space at the coset (K) form a closed subset of the $Ad(K_0)$-invariant inner products, where K_0 is the identity component of K.

Theorem 4.1 implies that when $\mathfrak{k}$ is not maximal in $\mathfrak{g}$ then generically the scalar curvature function $\mathcal{S}$ would be unbounded from above and from below. In this case, $\mathcal{S}$ can fail to have any critical points. For example, let $G = SU(4)$ and $SU(2)$ be the subgroup given by the irreducible 4-dimensional (symplectic) representation. Then $M = SU(4)/SU(2)$ does not have any homogeneous Einstein metric. Other families of similar examples can be found in [**123**]. A more complicated family was constructed recently by Park and Sakane [**96**]. The existence of compact homogeneous spaces which do not admit homogeneous Einstein metrics shows that in higher dimensions, Hamilton's Ricci flow with an initial metric of positive Ricci tensor need not converge to an Einstein metric, even if it exists for all time. (Hamilton's flow preserves the symmetries of the initial metric.)

From the point of view of Riemannian geometry, isometric Einstein metrics are always identified. In the homogeneous situation, the normalizer $N(K)$ of K in G acts on G/K via the adjoint action and hence induces an action of $N(K)/K$ on the space of invariant metrics. The moduli space of homogeneous Einstein structures can therefore be regarded as the quotient of the set of G-invariant Einstein metrics of volume 1 by this action. Since the space of volume 1 G-invariant Einstein metrics is diffeomorphic to Euclidean space, its quotient by the $N(K)/K$ action is contractible. (This is a special case of a general theorem of R. Oliver that the orbit space of a topological action of a compact Lie group on a contractible space is contractible.) Put another way, the domain of the scalar curvature function $\mathcal{S}$ does not acquire topology in passing to the quotient by $N(K)/K$.

An open general question for homogeneous Einstein manifolds with positive constant is the following. Suppose that G/K is a compact homogeneous manifold with finite fundamental group. Is it true that in the quotient of the space of G-invariant metrics with volume 1 by the above action of $N(K)/K$, there are only finitely many Einstein structures ?

Moving now to specific examples, we begin with *isotropy irreducible* spaces. These are connected homogeneous spaces G/K where K is compact and acts irreducibly on the tangent space. As was observed in [**127**], these spaces are Einstein since any $Ad(K)$-invariant inner product on the tangent space at the coset (K) must have Ricci tensor (also $Ad(K)$-invariant) proportional to itself by Schur's lemma. The main problem is therefore the classification of such spaces.

Among the isotropy irreducible spaces are the irreducible symmetric spaces, which were classified by Cartan in [**33, 34**]. In view of the importance of holonomy in the study of Einstein metrics, we might suggest [**126**, §2] as a modern version of Cartan's first method of classification, which is based on classifying the holonomy representation. The classification of the remaining isotropy irreducible spaces involves two steps. First, one classifies the *strongly isotropy irreducible* spaces, which are those for which the identity component of K already acts irreducibly on the tangent space. This classification is due independently to Manturov [**84, 85, 86**] and J. Wolf [**127**]. It should be noted that Wolf in addition made an extensive study of the geometry of these spaces. Another classification of these spaces appeared in [**78**] much later. There is a conceptual relation between the strongly isotropy irreducible quotients of the classical groups and irreducible symmetric spaces which allows one to deduce the classification of the former from that of the latter. This relationship was noticed by C. T. C. Wall [**127**, pp. 147-148], and is proved in two different ways in [**126**] and [**62**].

The isotropy irreducible spaces which are not strongly isotropy irreducible were classified in [**125**] using the classification of normal homogeneous Einstein quotients of compact connected simple groups in [**122**]. It was shown in [**122**] that if G/K is a homogeneous space with G compact, connected, and simple, then the Killing form metric is Einstein iff the Casimir operator of the isotropy representation is a multiple of the identity. This fact formed the basis of the classification in [**122**], which is relevant to the isotropy irreducible case because the key case in this latter classification is the situation of a compact and simple G, with the Killing form inducing an Einstein metric on G_0/K_0, where G_0 and K_0 are the identity components of G and K respectively. Instead of describing the classification in [**125**] in detail, we will illustrate the phenomenon using an example.

EXAMPLE 4.1. Let $G = SO(nk), K = SO(k) \times \cdots \times SO(k)$ (n times). Then the isotropy representation of G/K is easily seen to be

$$\sum_{1\leq i<j\leq n} \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \rho_k \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \rho_k \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},$$

where the ρ_k belongs to the ith and jth factor in each summand. Hence G/K is not strongly isotropy irreducible. Although the irreducible summands are inequivalent, they become equivalent under outer automorphisms of K. Because the Casimir constants are equal, the Killing form metric is Einstein. On the other hand, the outer automorphisms which interchange the $SO(k)$ factors actually extend to automorphisms of G. Therefore, if we enlarge G to include these automorphisms, we obtain an isotropy irreducible space. Note that depending on which automorphisms are included there could be different ways of enlarging G to give an isotropy irreducible space.

In general, therefore, one has to select from the spaces classified in [**122**] those whose isotropy representations are permuted transitively by automorphisms of K and carry out the corresponding analysis of the extensions of the necessary automorphisms to G.

The Einstein condition for the Killing form metric on G/K where G is semisimple, compact, connected but *non-simple* has been studied recently in [**104, 105, 106, 108, 109, 95, 89**].

In [**104, 108**], Rodionov proved that if G is compact, connected, semisimple but nonsimple, and K is a closed *simple* subgroup such that G/K is simply connected and effective, then $G = K \times \cdots \times K$ and K is embedded diagonally. So G/K is actually isotropy irreducible but not strongly isotropy irreducible unless $G = K \times K$, in which case we get a symmetric space. The automorphisms which one has to add to get isotropy irreducibility could be any subgroup of the permutation group which permutes the K factors in G transitively.

Several new infinite families of homogeneous spaces G/K whose Killing form metric is Einstein were produced in [**89**] and [**109**]. We mention here one of the families in these references.

EXAMPLE 4.2. [**109, 89**] Let $G = Sp(m+n) \times Sp(n)^p$ and $K = Sp(m) \times Sp(n)$, where $m, n \geq 1$. Suppose also that the embedding of K in G is given by composition of

$$id \times \Delta : Sp(m) \times Sp(n) \subset Sp(m) \times Sp(n)^{p+1},$$

where Δ denotes the diagonal map, and the obvious embedding

$$(Sp(m) \times Sp(n)) \times Sp(n)^p \subset Sp(m+n) \times Sp(n)^p.$$

Then it was shown in [**109**] that the Einstein condition is satisfied for the Killing form metric on G iff m, n, p satisfy the diophantine equation $2n^2 + (4-p)n + 2 - p = 2m(p(n+1)+m)$. This equation was completely solved in [**89**] and there are infinitely many solutions.

In the study of homogeneous manifolds it sometimes happens that several Lie groups act transitively on the same manifold. If a subgroup $H \subset G$ acts transitively on G/K, then the isotropy group of the H-action is $K \cap H$. Therefore the dimension of the space of invariant metrics could very well increase. When this happens, there may be more Einstein metrics. This phenomenon is of special interest if the underlying manifold is important in many geometrical situations. The irreducible symmetric spaces fit this criterion. In this regard, Ziller determined all the homogeneous Einstein metrics on the compact rank 1 symmetric spaces [**130**]. The non-symmetric Einstein metrics on these manifolds all come from Hopf fibrations by scaling the fibres differently. They are

1. the Jensen metric on S^{4n+3} from the fibration $Sp(n+1)/Sp(n) \longrightarrow Sp(n+1)/(Sp(n) \times Sp(1))$,
2. the metric found in [**24**] on S^{15} from the fibration $Spin(9)/Spin(7) \longrightarrow Spin(9)/Spin(8)$,
3. the Ziller metric on $\mathbb{CP}^{2n+1}$ from the twistor fibration $Sp(n+1)/(Sp(n) \times U(1)) \longrightarrow Sp(n+1)/(Sp(n) \times Sp(1))$.

THEOREM 4.2. [**70**] *Let G/K be a compact irreducible symmetric space of rank > 1 such that G is the identity component of the isometry group of the symmetric metric. Assume that $G/K \neq (K \times K)/K$. Then there is a non-symmetric homogeneous Einstein metric with respect to a transitive subgroup $H \subset G$ precisely in the following cases:*

(i) $G = SO(2n), K = U(n), H = SO(2n-1),\ H \cap K = U(n-1),\ n \geq 4,$

(ii) $G = SO(7), K = SO(2) \times SO(5), H = G_2,\ H \cap K = U(2),$

(iii) $G = SO(8), K = SO(3) \times SO(5), H = Spin(7),\ H \cap K = SO(4).$

There are respectively 1, 2, and 2 non-symmetric Einstein metrics.

The non-symmetric Einstein metric in Theorem 4.2(i) was first found in [**123**]. The case of group manifolds was studied earlier in [**67, 68**] and more extensively in [**40**]. While a complete classification has not been achieved, there are numerous examples of left-invariant Einstein metrics. For example, every compact simple Lie group of dimension greater than 3 has a left-invariant Einstein metric other than the Killing form metric. $SO(2n)$ and $SO(2n+1)$ have at least $3n-2$ distinct left-invariant Einstein metrics.

Another important family of homogeneous spaces consists of the coadjoint orbits of compact connected semisimple Lie groups G, also known as the generalized flag manifolds. Each coadjoint orbit is of the form $G/C(T)$ where $C(T)$ is the centralizer of a torus in G. It has a natural invariant complex structure and the first Chern class is positive. The existence of a homogeneous Kähler-Einstein metric is due to Koszul. Both [**17**, chapter 8] and [**7**] are excellent references for these and other classical facts about coadjoint orbits. The question therefore arises if there are any other homogeneous Einstein metrics on the generalized flag manifolds.

The classification of [**122**] shows that the Killing form metric is Einstein for the generalised flag manifolds $SU(nk)/S(U(k)\times\cdots\times U(k))$, $Sp(3n-1)/(Sp(n)\times U(2n-1))$, $SO(3n+2)/(SO(n)\times U(n+1))$, $E_6/(Spin(8)\cdot U(1)\cdot U(1))$, and G/T where T is a maximal torus in a compact semisimple connected Lie group whose local factors are of type A_l, D_l, E_6, E_7 or E_8. In [**9**] and [**72**] all the homogeneous Einstein metrics on certain families of coadjoint orbits were determined by explicitly solving the Einstein equation, e.g., $SU(p+q+r)/S(U(p)U(q)U(r))$ and $SO(2n)/(U(n-1)U(1))$. In addition, Arvanitoyeorgos found at least $\frac{n!}{2} + n + 1$ *solutions* of the Einstein equation on $SU(n)/T$ when $n \geq 4$, of which $n+1$ are non-Kähler. It should be pointed out that in [**9**] and [**72**], the distinct solutions sometimes represent isometric metrics. For example, the $\frac{n!}{2}$ solutions in the $SU(n)/T$ case are just the Kähler-Einstein metrics for the $n!$ distinct invariant complex structures (which correspond to the natural complex structures induced by different embeddings of $SU(n)/T$ as coadjoint orbits). The action of the Weyl group (the symmetric group) identifies them as Riemannian metrics. Similarly, n of the other solutions are also isometric. The remaining solution is the Killing form metric.

Homogeneous Einstein metrics with positive sectional curvature have been studied. One takes the classification of homogeneous spaces admitting positive sectional curvature (due to Berger, Bérard Bergery, Wallach, and B. Wilking) and examines which of the positively curved homogeneous metrics are Einstein as well. For the compact symmetric spaces of rank 1, Ziller [**130**] found that all homogeneous Einstein metrics have positive sectional curvature. In [**122**, 5.4] it is shown that in Berger's classification of normal homogeneous manifolds with positive sectional curvature, $Sp(2)/SU(2)$ is the only space other than the rank 1 symmetric spaces that has a positively curved normal homogeneous Einstein metric. The situation with the Aloff-Wallach spaces was studied in [**77**] (cf remarks about Theorem 1.3(i)). Although homogeneous Einstein metrics with positive curvature were found on an infinite family, a complete classification has not been achieved. The remaining cases of $F_4/Spin(8), SU(5)/(Sp(2)\cdot U(1)), Sp(3)/(Sp(1)Sp(1)Sp(1))$, and $SU(3)/T^2$ were studied by Rodionov in [**104**].

The classification of compact homogeneous Einstein manifolds of low dimension has also been attempted. A classical result of G. Jensen [**66**] is that all four-dimensional homogeneous Einstein manifolds are symmetric. Compact five-dimensional homogeneous Einstein manifolds with positive scalar curvature were classified in [**5, 107**]. Besides symmetric metrics and product metrics, there is an infinite family of homogeneous Einstein metrics on $S^2 \times S^3$. These are precisely the circle bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$ in Theorem 1.1. While they are all diffeomorphic, their homogeneous structures are all distinct. See remark (1) after Theorem (1.1).

While a complete classification has not been obtained, the dimension 7 case has been extensively studied by mathematicians as well as by theoretical physicists in connection with 11-dimensional supergravity theory. See [**50**, pp. 63-64] for more references. The circles bundles over $\mathbb{CP}^1 \times \mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, and $SU(3)/T^2$ encountered in Theorems 1.1 and 1.3 are homogeneous Einstein manifolds of dimension 7. Indeed, every simply connected compact homogeneous 7-manifold admits a homogeneous Einstein metric with positive scalar curvature, by [**37**].

Finally, a well-known technical difficulty in studying homogeneous Einstein metrics is the presence of multiplicities in the decomposition of the isotropy representation into irreducible representations. In [**71**] examples of this phenomena are studied. They include $Spin(8)/G_2 \cong S^7 \times S^7$, $Spin(7)/SU(3) \cong S^7 \times S^6$, $Spin(8)/U(3) \cong S^7 \times G_2^+(\mathbb{R}^8)$, and the Stiefel manifolds $SO(n+1)/SO(n-1)$.

Kerr classified all the G-homogeneous Einstein metrics on these spaces, although some of the Einstein metrics were already known. In the first three cases, modulo the action of $N(K)/K$, there is at least one Einstein metric other than the product Einstein metric. In the case of the Stiefel manifolds, there is a unique invariant Einstein metric, a fact already proved in [**14**] and later reproved in [**10**]. The existence of this Einstein metric, however, already follows from Kobayashi's theorem [**74**], since the Stiefel manifold under consideration is a circle bundle over the corresponding oriented Grassmann of 2-planes. In all of the above cases except $Spin(8)/G_2$, the group $N(K)/K$ is a circle and Kerr uses this action to eliminate one of the off-diagonal components of the invariant metrics. It turns out that apart from $Spin(8)/U(3)$ all the non-product Einstein metrics, up to isometry, can be found among G-invariant metrics which are diagonal with respect to a fixed decomposition of the isotropy representation.

B. Negative Einstein Constant. Let $(M = G/K, g)$ be a homogeneous Einstein manifold with negative scalar curvature. Then it follows from Bochner's theorem that both M and G are non-compact. The well-known unsolved conjecture of D. V. Alekseevsky asserts that K must be a maximal compact subgroup of G. If this conjecture is true, then using the Levi and Iwasawa decompositions, it follows that (M, g) is isometric to a left-invariant metric on some solvable Lie group S. Indeed, all known examples are isometric to left-invariant metrics on simply connected solvable Lie groups. If S is unimodular, then a theorem of Doti-Miatello [**49**] shows that there are no left-invariant Einstein metrics, generalizing an earlier result of Milnor [**88**] for the nilpotent case.

The most classical examples of homogeneous Einstein manifolds with negative scalar curvature are the symmetric spaces of non-compact type and the non-compact homogeneous Kähler-Einstein manifolds, which are shown to be precisely the bounded homogeneous domains with the Bergmann metric in [**39**] and in unpublished work of Koszul. For the latter class of examples, Piatetskii-Shapiro [**101**] has shown that there exists continuous Einstein moduli on the bounded homogeneous domains. The remaining case for which the holonomy is not generic is that of homogeneous quaternionic-Kähler manifolds, whose classification was begun by Alekseevsky [**2**] and completed by Cortés [**38**].

We will therefore consider below the non-unimodular case with generic holonomy. Also, we will confine ourselves to the simply connected case and hence will describe results in terms of Lie algebras. We begin with several definitions, following [**128**] and [**61**].

A *metric solvable Lie algebra* is a pair $(\mathfrak{s}, g)$, where $\mathfrak{s}$ is a finite-dimensional solvable Lie algebra and g is a left-invariant metric. A solvable Lie algebra is *completely solvable* if for all $X \in \mathfrak{s}$ the eigenvalues of ad_X are real. A metric solvable Lie algebra $(\mathfrak{s}, g)$ is called *standard* if the g-orthogonal complement of the derived algebra $[\mathfrak{s}, \mathfrak{s}]$ is an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{s}$. The left-invariant metric g is then referred to as a standard left-invariant metric. Finally, a standard metric solvable Lie algebra is of *Iwasawa type* if in addition (a) for all $0 \neq X \in \mathfrak{a}$, $ad_X \neq 0$ and is a symmetric linear operator with respect to g, and (b) there exists some $X_0 \in \mathfrak{a}$ such that $ad_{X_0} | \, [\mathfrak{s}, \mathfrak{s}]$ has positive eigenvalues.

There are many examples of simply connected Einstein solvmanifolds with non-positive sectional curvature. See [**3, 47, 41, 22**] for earlier results. Recently, Wolter [**128**] constructed two infinite families of such examples using Heisenberg groups. These examples include the examples of Deloff. Wolter's construction uses a general sufficient condition for a left-invariant metric on a solvable Lie algebra of Iwasawa type to be Einstein. This condition is then verified in special cases to give Einstein solvmanifolds with non-positive sectional curvature. In [**80**], Lanzendorf classified all the left-invariant Einstein metrics (with non-positive sectional curvature) that can be obtained by Wolter's construction, and obtained some new examples.

In [**53**], an infinite family of quite explicit examples was constructed as follows. Let K be a compact connected Lie group with an almost faithful irreducible representation ρ on $\mathbb{R}^n$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathbb{R}^n \oplus \mathbb{R}A$ be equipped with an inner product g such that the three summands are orthogonal, $g(A, A) = 1$, on $\mathbb{R}^n$ it is an arbitrary K-invariant inner product, and on $\mathfrak{k}$ it is $-\frac{1}{4n}$ times the trace form of ρ. Define the Lie bracket on $\mathfrak{g}$ by using the Lie bracket of $\mathfrak{k}$ and by declaring that $ad(A)$ acts as the identity on $\mathbb{R}^n$ and twice the identity on $\mathfrak{k}$, that $[\mathfrak{k}, \mathbb{R}^n] = 0$, and that $[\mathbb{R}^n, \mathbb{R}^n] \subset \mathfrak{k}$ so that for $X, Y \in \mathbb{R}^n$, $Z \in \mathfrak{k}$, one has $g([X, Y], Z) = g(\rho(Z)X, Y)$. Then g gives a left-invariant Einstein metric on the associated simply connected solvable Lie group.

The above examples as well as all other known examples are of standard type.

Recently, J. Heber, building on the work in [**128**], has made a very systematic study of Einstein solvmanifolds of standard type. We will only describe some of the results in [**61**], referring the reader to that paper for details as well as other results.

Heber has also constructed in [**61**] many examples of solvable Lie algebras which do not admit any left-invariant Einstein metrics. While nonstandard left-invariant Einstein metrics are not known to exist, there is a dichotomy and a uniqueness theorem for standard Einstein metrics.

THEOREM 4.3. [**61**] *Let $\mathfrak{s}$ be a solvable Lie algebra. If $\mathfrak{s}$ admits a standard left-invariant Einstein metric, then it cannot admit a non-standard left-invariant Einstein metric. Furthermore, if $\mathfrak{s}$ admits two standard left-invariant Einstein metrics g_1 and g_2, then there is a positive constant c and an isometry ϕ such that $g_1 = c\phi^* g_2$. If $\mathfrak{s}$ is in addition completely solvable, then ϕ is an automorphism of $\mathfrak{s}$.*

Heber has also characterized when an Einstein left-invariant metric must be of standard type. The characterization is an open algebraic condition, which is satisfied, for example, if $\mathfrak{s}$ is completely solvable, or if the Killing form is either non-negative semidefinite or have signature at most 1. According to Azencott and Wilson [**12**], $(\mathfrak{s}, g)$ is standard as well if g has non-positive sectional curvature.

The first step towards classifying standard left-invariant Einstein metrics is the theorem in [**61**] that such a metric solvable Lie algebra $(\mathfrak{s}, g)$ is isometric to one of Iwasawa type after possibly altering the Lie bracket. Combining [**4**, Corollary 1.10] with this theorem, one obtains the following interesting fact.

THEOREM 4.4. [**4, 61**] *A simply connected standard Einstein solvmanifold with negative scalar curvature has a quotient of finite volume iff it is symmetric.*

Returning to the situation of the associated (solvable) algebra of Iwasawa type of a standard, Einstein, metric solvable Lie algebra, a canonical element $X_0 \in \mathfrak{a}$ can now be chosen such that ad_{X_0} acting on $[\mathfrak{s}, \mathfrak{s}]$ has eigenvalues which are positive integers with no common divisors. The ordered list of eigenvalues together with their multiplicities is called the *eigenvalue type* of $(\mathfrak{s}, g)$. For a fixed dimension, only finitely many eigenvalue types can occur. Heber then gives an outline of the steps towards a complete classification of standard Einstein metrics. In particular, he gives a reduction theorem which allows the description of Einstein solvable Lie algebras of Iwasawa type for which $\dim \mathfrak{a} > 1$ in terms of those with $\dim \mathfrak{a} = 1$.

Now let n be a positive integer and define $\mathcal{M}^n$ to be space of all Einstein metric solvable Lie algebras of dimension n and scalar curvature -1 modulo the action of the diffeomorphism group of the underlying $\mathbb{R}^n$. Let $\mathcal{M}^n_{st}$ be the subspace of Einstein algebras of standard type. Both spaces are regarded as subspaces of the space of all metric solvable Lie algebras of dimension n modulo diffeomorphisms, equipped with the C^∞ topology.

THEOREM 4.5. [**61**] *Let $\mathcal{M}^n_\lambda \subset \mathcal{M}^n_{st}$ be the subset of standard Einstein metric solvable Lie algebras with a fixed eigenvalue type λ. Then*

(i) *$\mathcal{M}^n_{st}$ is a finite disjoint union of spaces $\mathcal{M}^n_\lambda$ which are homeomorphic to compact, semi-algebraic analytic subsets of some auxiliary Euclidean space,*
(ii) *each $\mathcal{M}^n_\lambda$ is open in $\mathcal{M}^n$ in the C^∞ topology.*

Finally, Heber has computed the dimensions of those spaces $\mathcal{M}^n_\lambda$ which contain an irreducible non-compact symmetric space of rank 1.

THEOREM 4.6. [**61**] *Let $\mathcal{M}^n_\lambda$ contain a rank 1 symmetric space of non-compact type. In the cases of real or complex hyperbolic space, $\mathcal{M}^n_\lambda$ consists of only one point.*

For quaternionic hyperbolic space of real dimension $n = 4(m+1)$, *if* $m = 1$ *then the symmetric metric is an isolated point in* $\mathcal{M}_\lambda^n$, *while for* $m \geq 2$, *a neighborhood of the symmetric metric in* $\mathcal{M}_\lambda^n$ *has dimension* $8m^2 - 6m - 8$. *For the hyperbolic Cayley projective plane, the corresponding dimension is 84.*

There is actually a geometrical description of a neighborhood of the symmetric space in $\mathcal{M}_\lambda^n$ in terms of spaces of orbits in certain representations. See **[61]** for further details.

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