The Stability of Minkowski Space-Time

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1. Introduction

The *general theory of relativity*, discovered by Einstein in 1915 [**9, 10**], is a unified theory of space, time and gravitation. According to general relativity, the *space-time manifold* is a four-dimensional oriented differentiable manifold $\mathcal{M}$ which is endowed with a *Lorentzian metric* g, that is, a continuous assignment of g_p, a symmetric bilinear form of index 1 in $T_p\mathcal{M}$, at each $p \in \mathcal{M}$. The Lorentzian metric divides $T_p\mathcal{M} \setminus 0_p$ into three subsets, I_p, N_p, S_p, the set of *time-like, null, space-like* vectors at p, according as to whether the quadratic form g_p is respectively negative, zero, or positive. The subset N_p is a double cone $N_p^+ \bigcup N_p^-$, the *null cone* at p. The subset I_p is the interior of this cone, an open set consisting of two components I_p^+ and I_p^-, the *future* and *past* components respectively. The boundaries of these components are the corresponding components of N_p. The subset S_p is the exterior of the null cone, a connected open set.

A curve in $\mathcal{M}$ is called *causal* if its tangent vector at each point belongs to the set $I \bigcup N$ correponding to that point. We assume that $(\mathcal{M}, g)$ is *time oriented*, that is a continuous choice of future component of I_p at each $p \in \mathcal{M}$ can and has been made. A causal curve is then *future directed* or *past directed* according as to whether its tangent vector at a point belongs to the subset $I^+ \bigcup N^+$ or $I^- \bigcup N^-$ corresponding to that point. The *causal future* $\mathcal{J}^+(\mathcal{K})$ of a set $\mathcal{K} \subset \mathcal{M}$ is the set of points which can be reached by a future directed causal curve initiating at $\mathcal{K}$. Similarly $\mathcal{J}^-(\mathcal{K})$, the *causal past* of $\mathcal{K}$, is the set of points which can be reached by a past directed causal curve initiating at $\mathcal{K}$. The boundaries $\partial\mathcal{J}^+(\mathcal{K}) \setminus \mathcal{K}$ and $\partial\mathcal{J}^-(\mathcal{K}) \setminus \mathcal{K}$ are hypersurfaces generated by null geodesics, *null hypersurfaces*, with the past end points of the null geodesics generating $\partial\mathcal{J}^+(\mathcal{K}) \setminus \mathcal{K}$ and the future end points of those generating $\partial\mathcal{J}^-(\mathcal{K}) \setminus \mathcal{K}$ all lying in $\mathcal{K}$. The specification of $\mathcal{J}^+(p)$ and $\mathcal{J}^-(p)$ for every $p \in \mathcal{M}$ defines the *causal structure*, which is equivalent to the conformal geometry of $\mathcal{M}$.

A hypersurface $\mathcal{H}$ in $\mathcal{M}$ is called *space-like* if at each $x \in \mathcal{H}$ the restriction of g_x to $T_x\mathcal{H}$ is positive definite. We denote by $\overline{g}$ the induced metric or first fundamental form of $\mathcal{H}$:

$$\overline{g}_x = g_x\Big|_{T_x\mathcal{H}}$$

The pair $(\mathcal{H}, \overline{g})$ is then a Riemannian manifold. The orthogonal complement of $T_x\mathcal{H}$ in $T_x\mathcal{M}$ is a one dimensional linear subspace of $T_x\mathcal{M}$ contained in I_x. There is therefore a unique future directed unit time-like vector N_x whose span is this orthogonal complement, the *unit normal* to $\mathcal{H}$ at x. We denote by k the second fundamental form of $\mathcal{H}$. Its components in an arbitrary frame e_i, $i = 1, 2, 3$ in $\mathcal{H}$ are given by:

$$k_{ij} = g(\nabla_{e_i} N, e_j)$$

where we denote by ∇ the covariant derivative operator on $\mathcal{M}$ associated to g. A space-like hypersurface $\mathcal{H}$ in $\mathcal{M}$ is called a *Cauchy hypersurface* if $(\mathcal{H}, \overline{g})$ is complete and each causal curve in $\mathcal{M}$ intersects $\mathcal{H}_0$ at one and only one point. We assume that $(\mathcal{M}, g)$ posesses such a Cauchy hypersurface. This assumption essentially means that we consider only space-times arising from the evolution of initial data. Under this assumption we can define on $\mathcal{M}$ a *time function*, that is a differentiable function t such that at each $p \in \mathcal{M}$, $dt \cdot X > 0$ whenever $X \in I_p^+$. The level sets $\mathcal{H}_t$ of a time function constitute a foliation of $\mathcal{M}$ into space-like hypersurfaces. The *lapse function* of the foliation is defined by:

$$\phi = (-g^{\mu\nu}\partial_\mu t \partial_\nu t)^{-1/2}$$

It measures the normal separation of the leaves of the foliation. We also have the time-like future directed vectorfield whose components in an arbitrary frame are given by:

$$T^\mu = -\phi^2 g^{\mu\nu}\partial_\nu t$$

It is characterized by the fact that its integral curves are orthogonal to the foliation and are parametrized by t. The one parameter group of diffeomorphisms generated by T maps the hypersurfaces $\mathcal{H}_t$ onto each other. We call T the *time translation* vectorfield corresponding to the time function t. The space-time manifold $\mathcal{M}$ is represented by the product $\Re \times \mathcal{H}_0$, where we identify $p \in \mathcal{M}$ with the pair (t, x) and the integral curve of T through p intersects $\mathcal{H}_0$ at x. In this representation we have:

$$T = \frac{\partial}{\partial t}$$

and the space-time metric is given by:

$$g = -\phi^2 dt^2 + \overline{g}$$

If e_i, $i = 1, 2, 3$ is a local frame in $\mathcal{H}_0$ we propagate it to a local frame in each $\mathcal{H}_t$ according to:

$$[T, e_i] = 0$$

The components of the first fundamental form of $\mathcal{H}_t$ then satisfy the first variation equations:

$$\frac{\partial \overline{g}_{ij}}{\partial t} = 2\phi k_{ij} \tag{1.1}$$

2. The Einstein Vacuum Equations

In general relativity the connection of the Lorentzian metric g is identified with the gravitational force, while its curvature, which produces geodesic deviation, is identified with the tidal force. Einstein's basic physical insight in discovering the theory was the fact that the gravitational force can be locally elliminated by going to a freely falling frame, just as the connection coefficients can be made to vanish along a geodesic by going to cylindrical normal coordinates (*equivalence principle*).

The laws of general relativity are the *Einstein equations* [10] linking the space-time curvature to the matter content:

$$G_{\mu\nu} = 2T_{\mu\nu} \tag{2.1}$$

Here $G_{\mu\nu}$ is the *Einstein tensor*, given by:

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu} \tag{2.2}$$

with $R_{\mu\nu}$ the Ricci tensor and R the scalar curvature of the metric $g_{\mu\nu}$, while $T_{\mu\nu}$ is the *energy-momentum tensor* of matter. The twice contracted Bianchi identities,

$$\nabla^{\nu} G_{\mu\nu} = 0, \tag{2.3}$$

then imply the *energy-momentum conservation laws*:

$$\nabla^{\nu} T_{\mu\nu} = 0, \tag{2.4}$$

Thus general relativity incorporates the equations of motion of classical mechanics.

In the absense of matter equations (2.1) reduce to the *Einstein vacuum equations* for the space-time manifold:

$$R_{\mu\nu} = 0 \tag{2.5}$$

In the present article we shall confine our attention to this case.

The principal part of the Ricci tensor is:

$$(1/2)g^{\alpha\beta}(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\mu\nu})$$

For a given metric g the symbol σ_ξ at a point $p \in \mathcal{M}$ and a covector $\xi \in T^*_p\mathcal{M}$ is the linear operator on $S_2(T_p\mathcal{M})$, the space of symmetric bilinear forms $\dot{g}$ in $T_p\mathcal{M}$ (variations of g), obtained by the replacement:

$$\partial_\mu g_{\alpha\beta} \mapsto \xi_\mu \dot{g}_{\alpha\beta}$$

This gives:

$$(\sigma_\xi \cdot \dot{g})_{\mu\nu} = (1/2)(\xi_\mu\xi^\alpha\dot{g}_{\alpha\nu} + \xi_\nu\xi^\alpha\dot{g}_{\alpha\mu} - g^{\alpha\beta}\dot{g}_{\mu\nu} - \xi_\mu\xi_\nu g^{\alpha\beta}\dot{g}_{\alpha\beta})$$

which we can write as:

$$\sigma_\xi \cdot \dot{g} = (1/2)(\xi \otimes i_\xi\dot{g} + i_\xi\dot{g} \otimes \xi - (\xi,\xi)\dot{g} - \xi \otimes \xi \mathrm{tr}\dot{g})$$

Here $i_\xi\dot{g}$ is the covector obtained by contracting $\dot{g}$ with the vector corresponding to ξ:

$$(i_\xi\dot{g})_\nu = \xi^\mu \dot{g}_{\mu\nu}$$

and we denote $g^{\alpha\beta}\xi_\alpha\xi_\beta = (\xi,\xi)$. We see that for any given ξ and any other covector ζ, the variation

$$\dot{g} = \xi \otimes \zeta + \zeta \otimes \xi$$

belongs to the null space $N(\sigma_\xi)$ of the symbol at ξ. Thus the Einstein equations seem at first sight to be a degenerate differential system, $N(\sigma_\xi)$ being non-zero for any $\xi \in T^*_p\mathcal{M}$. This is due to the fact that the equations are generally covariant; proper account must be taken of the geometric equivalence of metrics related by a diffeomorphism. Since any one parameter group of diffeomorphisms is generated by a vectorfield ζ and the infinitesimal action of the group on the space of metrics is the Lie derivative

$$(\mathcal{L}_\zeta g)_{\mu\nu} = \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$$

the symbol of which is $\xi \otimes \zeta + \zeta \otimes \xi$, we consider the quotient Q_ξ of $S_2(T_p\mathcal{M})$ by the following equivalence relation: $\dot{g}_1 \sim \dot{g}_2$ if and only if $\dot{g}_1 - \dot{g}_2 = \xi \otimes \zeta + \zeta \otimes \xi$ for some covector ζ at p. The symbol σ_ξ when reduced to Q_ξ is then seen to have zero null space when ξ is not null. Moreover, when ξ is a non-zero null covector, choosing a null conjugate to $\underline{\xi}$ to ξ, i.e. another null covector in the same component of the dual null cone at p such that $(\xi, \underline{\xi}) = -2$, we can identify Q_ξ with the space of all $\dot{g} \in S_2(T_p\mathcal{M})$ such that $i_\xi \dot{g} = \underline{0}$. Then $N(\sigma_\xi)$ is seen to be the subspace of Q_ξ consisting of those $\dot{g}$ which also verify $i_{\underline{\xi}} \dot{g} = 0$ and $\mathrm{tr}\dot{g} = 0$. Therefore, when ξ is null $N(\sigma_\xi)$ can be identified with $\hat{S}_2(\Pi)$, the space of trace-free symmetric bilinear forms on Π, the space-like plane which is the intersection of the null spaces of ξ and $\underline{\xi}$. Thus, the two dimensional space $\hat{S}_2(\Pi)$ represents the space of *dynamical degrees of freedom of the gravitational field at a point* (gravitational waves).

In terms of the foliation induced by a time function t the Einstein vacuum equations become:

$$\overline{R} - |k|^2 + (\mathrm{tr}k)^2 = 0 \tag{2.6}$$

$$\overline{\nabla}^j k_{ij} - \overline{\nabla}_i \mathrm{tr}k = 0 \tag{2.7}$$

$$\frac{\partial k_{ij}}{\partial t} = \overline{\nabla}_i \overline{\nabla}_j \phi - (\overline{R}_{ij} + k_{ij}\mathrm{tr}k - 2k_{im}k^m_j)\phi \tag{2.8}$$

Equations (2.6) and (2.7) correspond to the Gauss and Codazzi equations respectively, while equations (2.8) represent the second variation equations and must be considered in conjunction with equations (1.1). Here $\overline{\nabla}$ is the covariant derivative operator, $\overline{R}_{ij}$ the Ricci tensor and $\overline{R}$ the scalar curvature on $\mathcal{H}_t$, defined by $\overline{g}$. Note that ϕ is left completely undetermined by the above equations, a freedom which corresponds to the complete arbitrariness in chosing the time function. Remark that by virtue of the identities (2.3), if $\overline{g}, k$ satisfy equations (1.1,2.8), then equations (2.6,2.7) are satisfied on any $\mathcal{H}_t$ provided that they are satisfied on $\mathcal{H}_0$. Therefore they can be regarded as constraints on given initial conditions for $\overline{g}, k$. Accordingly, an *initial data set* for the Einstein vacuum equations is defined to be a triplet $(\mathcal{H}_0, \overline{g}_0, k_0)$ consisting of a complete three dimensional Riemannian manifold $(\mathcal{H}_0, \overline{g}_0)$ equipped with a 2-covariant symmetric tensorfield k_0, satisfying the constraint equations (2.6,2.7). By a *development* of such an initial data set we mean a Lorentzian manifold $(\mathcal{M}, g)$ satisfying the Einstein vacuum equations (2.5) and an embedding of $\mathcal{H}_0$ as a Cauchy hypersurface in $\mathcal{M}$ such that $\overline{g}_0$ and k_0 are the induced first and second fundamental forms respectively.

3. Asymptotic Flatness

The central mathematical problem of the theory is the study of the developments of general asymptotically flat initial data sets. These represent isolated gravitating physical systems. By an *asymptotically flat initial data set* we mean an initial data set $(\mathcal{H}, \overline{g}, k)$ such that the complement of a compact set in $\mathcal{H}$ is diffeomorphic to the complement of a closed ball in $\Re^3$ and there exists a coordinate system in this complement relative to which the metric components $\overline{g}_{ij}$ approach δ_{ij} and those of k approach zero, sufficiently rapidly for the notions of *total energy*, *linear momentum* and *angular momentum* to be well defined and finite. The Arnowitt , Deser and Misner [**1**] definitions of these notions are, respectively,

$$E = \frac{1}{4}\lim_{r\to\infty}\int_{S_r}\sum_{i,j}(\partial_i\overline{g}_{ij} - \partial_j\overline{g}_{ii})dS_j \tag{3.1}$$

(3.2)
$$P^i = -\frac{1}{2}\lim_{r\to\infty}\int_{\mathcal{S}_r}\sum_j (k_{ij} - \overline{g}_{ij}\mathrm{tr}k)dS_j$$

(3.3)
$$J^i = -\frac{1}{2}\lim_{r\to\infty}\int_{\mathcal{S}_r}\sum_{j,m,n} \in_{ijm} x^j(k_{mn} - \overline{g}_{mn}\mathrm{tr}k)dS_n$$

where $\mathcal{S}_r$ is the coordinate sphere of radius r and dS_i are the components of its oriented area element.

The notions of total energy, linear and angular momentum are in particular well defined and finite if there is a coordinate system in a neighborhood of infinity in which

(3.4)
$$g_{ij} = (1 + M_0/2\pi r)\delta_{ij} + o_2(r^{-3/2}),\ k_{ij} = o_1(r^{-5/2}) \text{ as } r \to \infty$$

Here, a function f is said to be $o_n(r^{-\alpha})$ as $r \to \infty$ if f is C^n and $\partial^m f = o(r^{-m-\alpha})$ as $r \to \infty$, for any $m = 0, ..., n$ where ∂^m denotes all partial derivatives of order m. Initial data sets verifying (3.4) we call *strongly asymptotically flat.* The leading term

$$(1 + M_0/2\pi r)\delta_{ij}$$

in the expansion of the metric of a strongly asymptotically flat initial data set we call the *Schwarzschild part* of the metric. The conditions (3.4) imply:

$$E = M_0,\ P^i = 0$$

Thus a strongly asymptotically flat initial data set defines a *center of mass frame.* The *posivite mass theorem* first proved by R. Schoen and S.T. Yau [**14**] and later, by a different method, by E. Witten [**15**], states that $M_0 \geq 0$ with equality if and only if the initial data set is embedded in the flat Minkowski space-time.

The total energy, the linear momentum and the angular momentum are *conserved quantities.* That is, given a time function t whose lapse function ϕ tends to 1 at infinity each level set, then if the zero level set $\mathcal{H}_0$ defines an asymptotically flat initial data set, so do all the level sets $\mathcal{H}_t$ and the values of each of these quantities are the same for all the $\mathcal{H}_t$.

4. The Maximal Time Function

In a space-time arising from asymptotically flat initial conditions we can define a unique *maximal time function t.* This is defined by the condition that its level sets $\mathcal{H}_t$ are complete space-like hypersurfaces of maximal volume on which ϕ tends to 1 at infinity and $P^i = 0$. The maximality condition is expressed by:

(4.1)
$$\mathrm{tr}k = 0$$

Relative to the maximal time function the constraint equations (2.6,2.7) reduce to:

(4.2)
$$\overline{R} = |k|^2$$

(4.3)
$$\overline{\nabla}^j k_{ij} = 0$$

while the evolution equations (1.1,2.8) reduce to:

(4.4)
$$\frac{\partial \overline{g}_{ij}}{\partial t} = 2\phi k_{ij}$$

(4.5)
$$\frac{\partial k_{ij}}{\partial t} = \overline{\nabla}_i\overline{\nabla}_j\phi - (\overline{R}_{ij} - 2k_{im}k^m_j)\phi$$

Furthermore, taking the trace of (4.4) and imposing (4.1) we obtain the following elliptic equation for the lapse function:

$$\overline{\triangle}\phi = |k|^2\phi \tag{4.6}$$

A complete maximal space-like hypersurface in Minkowski space-time is necessarily a hyperplane. Thus if the initial data set $(\mathcal{H}_0, \overline{g}_0, k_0)$ satisfies the maximality condition $\mathrm{tr}k_0 = 0$, it has trivial development if and only if $(\mathcal{H}_0, \overline{g}_0)$ is the Euclidean space and $k_0 = 0$. In the following we shall restrict ourselves to strongly asymptotically flat initial data sets satisfying the maximality condition. An appropriate version of the local existence theorem gives us a development $\mathcal{M}$ represented by the product $\mathcal{I} \times \mathcal{H}_0$, where $\mathcal{I}$ is an interval containing 0 and the projection to the first factor is the maximal time function. We remark here that $\mathcal{I} = \Re$ does not imply that the development is geodesically complete, for we may have $\inf_{\mathcal{M}} \phi = 0$.

5. Statement of The Problem

The simplest solution of the Einstein vacuum equations is of course the flat Minkowski space-time of special relativity, introduced by Minkowski in 1908 [**13**] as the geometric framework of that theory, in a work which was instrumental in the transition from Einstein's formulation of special relativity of 1905 [**8**] to his discovery of the general theory in 1915 [**9**]. Minkowski space-time is the manifold $\Re^4$ together with the metric η whose components form the diagonal matrix with entries -1, 1, 1, 1.

The problem which we shall discuss in the present article is the problem of the global stability of Minkowski space-time in the framework of general relativity. That is, whether any asymptotically flat initial data set which is sufficiently close to a trivial one has a development which is a geodesically complete space-time approaching the Minkowski space-time at infinity along any geodesic. This question has been answered in the affirmative in my joint work with Sergiu Klainerman [**7**] when asymptotic flatness of the initial data set is meant in the strong sense defined above and an appropriate notion of closeness is required.

In the following we shall discuss the main ideas and methods of the proof, after a brief exposition of general methods of treating problems of global stability of the trivial solution for field theories in Minkowski space-time and a discussion of the peculiar difficulties present in the problem at hand and the obstacles that had to be overcome.

6. Field Theories in a Given Spacetime

Consider a field theory in a given space-time $(\mathcal{M}, g)$ whose field equations are derivable from an *action* $\mathcal{A}$. For any domain $\mathcal{D}$ with compact closure in $\mathcal{M}$ the action in $\mathcal{D}$ is the integral:

$$\mathcal{A}[\mathcal{D}] = \int_{\mathcal{D}} L d\mu_g \tag{6.1}$$

where L is the *Lagrangian*. The field equations of the theory express the condition that for any such domain $\mathcal{D}$ the action is stationary with respect to variations of the field with support in $\mathcal{D}$. On the other hand, variations of the action, supported in $\mathcal{D}$, with respect to the underlying metric, give rise to the energy-momentum tensor through the formula:

$$\dot{\mathcal{A}}[\mathcal{D}] = -\frac{1}{2}\int_{\mathcal{D}} T^{\mu\nu}\dot{g}_{\mu\nu}d\mu_g \tag{6.2}$$

By its definition $T^{\mu\nu}$ is symmetric. If $\mathcal{A}$ is invariant under diffeomorphisms of $\mathcal{M}$ reducing to the identity outside $\mathcal{D}$, then the field equations imply that $T^{\mu\nu}$ is divergence-free:

$$\nabla^\nu T_{\mu\nu} = 0 \tag{6.3}$$

This is in accordance with (2.4), so the theory is compatible with general relativity.

Now suppose that X is a vectorfield generating a one parameter group of isometries of $(\mathcal{M}, g)$ (Killing vectorfield). Then the 1-form

$$P_\mu = -T_{\mu\nu}X^\nu \tag{6.4}$$

is divergence-free

$$\nabla^\mu P_\mu = 0 \tag{6.5}$$

or, equivalently, the dual 3-form ${}^\star P$ is closed:

$$d{}^\star P = 0 \tag{6.6}$$

It follows that the integral of ${}^\star P$ on two homologous hypersurfaces is the same and the integral

$$\int_{\mathcal{H}} {}^\star P$$

on a Cauchy hypersurface $\mathcal{H}$ is a conserved quantity, that is, its value is the same for all Cauchy hypersurfaces. This is essentially what is called *Noether's Principle.* Moreover if the action is invariant under conformal transformations of the metric then the energy-momentum tensor is trace-free and these considerations extend to the case where X generates a one parameter group of conformal isometries of $(\mathcal{M}, g)$ (conformal Killing vectorfield). An important requirement on a physical theory is that the energy-momentum tensor should satisfy the positivity condition:

$$T(X_1, X_2) \geq 0$$

for any pair X_1, X_2 of time-like future directed vectors at a point. Then, provided that the vector multiplier X above is time-like future directed, the quantity

$$\int_{\mathcal{H}} {}^\star P = \int_{\mathcal{H}} T(X, N)d\mu_{\overline{g}}$$

is non-negative, N being the unit normal to $\mathcal{H}$. As its value is the same as that on the Cauchy hypersurface on which the initial data is given, it provides an estimate for the solution in terms of the initial data.

Furthermore, if we suppose, as is natural, that the Lagrangian posesses the symmetries of the underlying metric, the pullback by an isometry of a solution is also a solution of the field equations. Moreover, if the field equations are linear then the difference of two solutions is also a solution. It follows that given a vectorfield which generates a one parameter group of isometries of the space-time, the Lie derivative of a solution with respect to this vectorfield is also a solution of the same equations, being the limit of a difference quotient of solutions. In the case of a conformally invariant action, the same is true for the Lie derivative with respect to a vectorfield generating conformal space-time isometries. Thus in the linear case the previous construction applies to Lie derivatives as well, in fact to iterated

Lie derivatives of arbitrary order, giving a series of positive conserved quantities controlling the solutions. In fact, once enough such quantities of sufficiently high order are obtained, the Sobolev inequalities imply uniform decay estimates of the solutions at infinity.

In the non-linear case, the Lie derivative of a solution is no longer a solution of the same field equations. An analogous construction does give energy tensors corresponding to the Lie derivatives, but their divergence no longer vanishes. The positive quantities obtained using suitable vector multipliers as before, are consequently not conserved. The difference of the values corresponding to two Cauchy hypersurfaces is the integral of error terms over the space-time region bounded by the hypersurfaces. Nevertheless, if we have enough quantities of sufficiently high order at our disposal then the integral of the error terms may be estimated, using Sobolev-type inequalities, in terms of the quantities themselves. Thus one arrives at a closed system of ordinary differential inequalities which controls the growth of these quantities in time and implies that they remain bounded for all time provided that their initial values are sufficiently small. This yields a global existence theorem for small initial data.

In the case that the underlying space-time is the Minkowski space-time, there is a large group of conformal isometries available, consisting of the space-time translations, the space-time rotations (Lorentz group), the scaling, and the inverted space-time translations, generated by the vectorfields:

$$T_\mu = \partial_\mu; \ \mu = 0,1,2,3 \tag{6.7}$$

$$\Omega_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu; \ \mu, \nu = 0,1,2,3 \tag{6.8}$$

$$S = x^\mu \partial_\mu \tag{6.9}$$

$$K_\mu = -2x_\mu S + (x,x)\partial_\mu; \ \mu = 0,1,2,3 \tag{6.10}$$

respectively. Here,

$$x_\mu = \eta_{\mu\nu} x^\nu, \ (x,x) = \eta_{\mu\nu} x^\mu x^\nu$$

Of these only the time translations and the inverted time translations are generated by everywhere time-like future directed vectorfields, T_0 and K_0 respectively, and are thus suitable for use as multipliers. Lie derivatives can be taken with respect to all generating vectorfields.

The general method outlined above grew as a synthesis of the *conformal method* which I introduced in the case of the Yang-Mills equations [**4**] and later applied it to quasilinear hyperbolic systems of scalar equations [**5**], and the *commutator method* introduced by Klainerman [**12**] in the study of non-linear perturbations of the wave equation. The conformal method corresponded to a special case of the method just outlined, namely the case where Lie derivatives are taken only with respect to inverted space-time translations and only the inverted time translation is used as a multiplier, the integrations being carried over space-like hyperboloids, while Klainerman's commutator method corresponded to the case where Lie derivatives are taken only with respect to the Lorentz group and scaling and only the usual time translation is used as a multiplier, the integrations being carried over space-like hyperplanes.

7. Weyl Fields and Bianchi Equations

If one tries to apply the general method just outlined to the problem of the global stability of the Minkowski space-time in general relativity, one quickly reaches an impasse for the following two reasons. First, the energy-momentum tensor in the case of gravitation, defined as in (6.2) above, but relative to the Einstein-Hilbert action:

$$\mathcal{A}[\mathcal{D}] = -\frac{1}{4}\int_{\mathcal{D}} R d\mu_g$$

vanishes, as this expresses the field equations of gravitation, namely the Einstein vacuum equations. And, second, space-time in general relativity posesses in general no symmetries, hence the conformal isometry group is trivial and the vectorfields required in the construction do not exist. At this point two main ideas were introduced which overcame these obstacles.

The first idea was that instead of the Einstein equations we should concentrate our attention on the Bianchi identities

$$\nabla_{[\alpha} R_{\beta\gamma]\delta\epsilon} = 0 \tag{7.1}$$

(here [ ] stands for cyclic permutation), considering them as equations for the curvature. This leads us to introduce the concept of a *Weyl field* $W_{\alpha\beta\gamma\delta}$, in a given space-time, a 4-covariant tensorfield posessing the algebraic properties of the Weyl or conformal curvature tensor. The natural field equations for a Weyl field are the *Bianchi equations*, identical in form to the Bianchi identities:

$$\nabla_{[\alpha} W_{\beta\gamma]\delta\epsilon} = 0 \tag{7.2}$$

We can write these simply as:

$$DW = 0 \tag{7.3}$$

A particular case of a Weyl field is, of course, the Riemann curvature tensor of a metric satisfying the Einstein vacuum equations, but the situation considered here is more general as there need be no connection between a Weyl field and the underlying space-time metric. In a four dimensional space-time the dual ${}^*W$ of a Weyl field W is also a Weyl field and if W satisfies the Bianchi equations so does ${}^*W$. The operator D although formally identical to the exterior derivative, is not an exterior differential operator and $D^2 \neq 0$. As a consequence, the Bianchi equations imply an algebraic condition:

$$R_\mu{}^{\alpha\beta\gamma}\,{}^*W_{\nu\alpha\beta\gamma} - R_\nu{}^{\alpha\beta\gamma}\,{}^*W_{\mu\alpha\beta\gamma} = 0$$

The Bianchi equations are conformally covariant. If f is a conformal isometry of $(\mathcal{M}, g)$, that is $f^*g = \Omega^2 g$ for some positive function Ω, and W is a solution of the Bianchi equations then so is $\Omega^{-1} f^* W$.

To a Weyl field we can associate a tensorial quadratic form, a 4-covariant tensorfield which is fully symmetric and trace-free. This tensorfield is a generalization of one found previously by Bel and Robinson **[3]** so we call it the *Bel-Robinson tensor*. It is given by:

$$Q_{\alpha\beta\gamma\delta} = (1/2)(W_{\alpha\rho\gamma\sigma} W_\beta{}^\rho{}_\delta{}^\sigma + {}^*W_{\alpha\rho\gamma\sigma}\,{}^*W_\beta{}^\rho{}_\delta{}^\sigma) \tag{7.4}$$

and satisfies the following positivity condition:

$$Q(X_1, X_2, X_3, X_4) \geq 0$$

for any tetrad of time-like future directed vectors at a point, with equality if and only if W vanishes at that point. Furthermore, if W satisfies the Bianchi equations then Q is divergence-free:

$$\nabla^\alpha Q_{\alpha\beta\gamma\delta} = 0 \tag{7.5}$$

It follows that given three vector fields X_1, X_2, X_3, each generating a one parameter group of conformal isometries of $(\mathcal{M}, g)$, some or all of which possibly coincident, then the 1-form

$$P = -Q(\cdot, X_1, X_2, X_3) \tag{7.6}$$

is divergence-free, consequently the integral

$$\int_{\mathcal{H}} {}^\star P$$

on a Cauchy hypersurface $\mathcal{H}$ is a conserved quantity, which is positive definite in the case that the X_1, X_2, X_3 are all time-like future directed.

Given a Weyl field W and a vector field X the usual Lie derivative $\mathcal{L}_X W$ of W with respect to X is not in general a Weyl field. However we can define a modified Lie derivative $\hat{\mathcal{L}}_X W$ which is a Weyl field:

$$\begin{aligned} \hat{\mathcal{L}}_X W_{\alpha\beta\gamma\delta} &= \mathcal{L}_X W_{\alpha\beta\gamma\delta} - (1/8)\mathrm{tr}\pi W_{\alpha\beta\gamma\delta} \\ &\quad -(1/2)(\hat{\pi}_\alpha{}^\mu W_{\mu\beta\gamma\delta} + \hat{\pi}_\beta{}^\mu W_{\alpha\mu\gamma\delta} + \hat{\pi}_\gamma{}^\mu W_{\alpha\beta\mu\delta} + \hat{\pi}_\delta{}^\mu W_{\alpha\beta\gamma\mu}) \end{aligned} \tag{7.7}$$

Here $\pi_{\mu\nu} = \mathcal{L}_X g_{\mu\nu}$ and $\hat{\pi}$ is the *deformation tensor* of X, namely the trace-free part of π. The modified Lie derivative commutes with the Hodge dual:

$$\hat{\mathcal{L}}_X {}^\star W = {}^\star\hat{\mathcal{L}}_X W \tag{7.8}$$

As a consequence of the linearity and the conformal invariance of the Bianchi equations, if W is a solution of these equations and X is a vector field generating a one parameter group of conformal isometries f_t, then

$$\hat{\mathcal{L}}_X W = \frac{d}{dt}(\Omega_t^{-1} f_t^* W)\Big|_{t=0}$$

is also a solution of the same equations. Therefore the considerations regarding conserved quantities can be applied to the Weyl field $\hat{\mathcal{L}}_X W$ as well.

8. The Optical Function

The second main idea of the proof of the global stability of Minkowski space-time was in overcoming the obstacle that a general metric in fact posesses only a trivial conformal isometry group. The idea originates in the observation that a space-time which arises from asymptotically flat initial conditions should itself be asymptotically flat, approaching the Minkowski space-time at infinity. Thus we have a group acting at infinity as a conformal isometry. The problem is how to extend this action to the whole space-time in such a way that the deviation from conformal isometry is globally small and approaching zero at infinity sufficiently rapidly. The crux of the idea was the solution of this problem by means of a geometric construction. It turns out that we can only define the action of the subgroup of the Minkowskian conformal group consisting of the time translations, the scaling, the inverted time translations and the spatial rotation group $O(3)$ leaving the total energy-momentum vector invariant, however this subgroup suffices to derive a complete system of estimates. First, the action of the group of time translations is the simplest to define, for, we have a unique maximal time function. The corresponding time translation vector field T generates the action, mapping the maximal hypersurfaces of vanishing linear momentum $\mathcal{H}_t$ onto each other. The action of the other groups is defined with the help of an *optical function* u.

This is a function whose level sets $\mathcal{C}_u$ are null hypersurfaces, defined as follows. We start with a surface $\mathcal{S}_{0,0}$ diffeomorphic to S^2 on $\mathcal{H}_0$ and we define the level set $\mathcal{C}_0$ to be the outer component of $\partial\mathcal{J}^+(\mathcal{S}_{0,0})$, an outgoing null hypersurface. The surface $\mathcal{S}_{0,0}$ must be chosen so that the null geodesics generating the latter have no future end points. We would like then to define the level sets $\mathcal{C}_u$, $u \neq 0$, to be other outgoing null hypersurfaces such that, if we consider the surfaces $\mathcal{S}_{t,u} = \mathcal{H}_t \bigcap \mathcal{C}_u$, the restriction to $\mathcal{S}_{t,u}$ of minus the signed distance function along $\mathcal{H}_t$ from $\mathcal{S}_{t,0}$ tends to u as $t \to \infty$. However, this definition can be implemented only after global existence has already been proven.

In the course of the proof, a continuity argument, we have a final maximal hypersurface $\mathcal{H}_{t_*}$. We would like then to define u on $\mathcal{H}_{t_*}$ to be minus the signed distance function along $\mathcal{H}_{t_*}$ from $\mathcal{S}_{t_*,0}$. However, the definition is inappropriate because this is only as smooth as the metric, two orders of differentiability smoother than the curvature, even though $\mathcal{S}_{t_*,0}$ itself is of the maximal smoothness allowed, one order smoother than the metric. With such a loss of smoothness we would not arrive at a closed system of estimates. We instead define u on $\mathcal{H}_{t_*}$ by imposing certain equation for the lapse function a of the foliation of $\mathcal{H}_{t_*}$ generated by u:

$$a = (\overline{g}^{ij}\partial_i u \partial_j u)^{-1/2} \tag{8.1}$$

As the lapse function measures the normal separation of the leaves of the foliation, the equation for a, to be given below, can be thought of as an equation of motion for a surface on a the three dimensional Riemannian manifold. The given surface $\mathcal{S}_{t_*,0}$, which is to be the zero level set of u on $\mathcal{H}_{t_*}$, plays the role of an initial condition. To write the equation for a in a form which is as simple as possible we shall neglect the terms contributed by the second fundamental form of $\mathcal{H}_{t_*}$. Then a satisfies on each surface $\mathcal{S}_{t_*,u}$, level set of u on $\mathcal{H}_{t_*}$ the equation:

$$\not{\Delta} \log a = f - \overline{f}, \ \overline{\log a} = 0 \tag{8.2}$$

where f is the function:

$$f = K - \frac{1}{4}(\mathrm{tr}\theta)^2 \tag{8.3}$$

Here K is the Gauss curvature of $\mathcal{S}_{t_*,u}$ and θ is the second fundamental form of $\mathcal{S}_{t_*,u}$ relative to $\mathcal{H}_{t_*}$. Also, $\not{\nabla}$ is the covariant derivative operator on $\mathcal{S}_{t_*,u}$ associated to the induced metric γ. Finally, we denote by an overline the mean value of a function on $\mathcal{S}_{t_*,u}$. To see why the function u constructed by solving (8.2,8.3) has the required smoothness properties, recall the trace of the second variation equations of the foliation of a three dimensional Riemannian manifold induced by a function u:

$$\frac{\partial \mathrm{tr}\theta}{\partial u} = \not{\Delta} a + \frac{1}{2}a(\overline{R} + |\theta|^2 + (\mathrm{tr}\theta)^2 - 2K)$$

Since we are neglecting the second fundamental form of $\mathcal{H}_{t_*}$ we have, in accordance with (4.3), $\overline{R} = 0$; therefore, by virtue of (8.2) this reduces to:

$$\frac{1}{a}\frac{\partial \mathrm{tr}\theta}{\partial u} = \frac{1}{2}|\hat{\theta}|^2 + \frac{1}{2}(\mathrm{tr}\theta)^2 + |\not{\nabla} \log a|^2 - \overline{f} \tag{8.4}$$

Here we denote by $\hat{\theta}$ the trace-free part of θ. The gain in smoothness is evident from the fact that the curvature terms have been elliminated in favor of terms which are one order smoother. The propagation equation (8.4) is considered in conjunction with the Codazzi equations:

$$\nabla^B \hat{\theta}_{AB} - \frac{1}{2}\nabla_A \mathrm{tr}\theta = \overline{R}_{A3} \tag{8.5}$$

an elliptic equation for $\hat{\theta}$ on each $S_{t_*,u}$, and with the Gauss equation:

$$K - \frac{1}{4}(\mathrm{tr}\theta)^2 + \frac{1}{2}|\hat{\theta}|^2 = -\overline{R}_{33} \tag{8.6}$$

to complete the smoothness argument. Here e_A, $A = 1, 2$ is an arbitrary local frame in $\mathcal{S}_{t_*,u}$, complemented by e_3, the unit outward normal to $\mathcal{S}_{t_*,u}$ in $\mathcal{H}_{t_*}$.

Once the surfaces $\mathcal{S}_{t_*,u}$ have been constructed, the null hypersurfaces $\mathcal{C}_u$ are defined to be the inner components of $\partial\mathcal{J}^-(\mathcal{S}_{t_*,u})$ and the construction of the optical function is complete.

9. Vector Fields and the Controlling Quantity

The surfaces $\mathcal{S}_{t,u}$ define a two parameter foliation of the space-time slab bounded by $\mathcal{H}_0$ and $\mathcal{H}_{t_*}$. Let $r(t, u)$ be the *area radius* of $\mathcal{S}_{t,u}$, defined by:

$$r(t,u) = \sqrt{\frac{\mathrm{Area}(\mathcal{S}_{t,u})}{4\pi}} \tag{9.1}$$

We then define the function

$$\underline{u} = u + 2r \tag{9.2}$$

Let L and $\underline{L}$ be respectivelly the outgoing and incoming null normals to $\mathcal{S}_{t,u}$ whose component along T is equal to T. We then have:

$$T = \frac{1}{2}(L + \underline{L}) \tag{9.3}$$

and we define the generator of scalings by:

$$S = \frac{1}{2}(\underline{u}L + u\underline{L}) \tag{9.4}$$

and the generator of inverted time translations by:

$$K = \frac{1}{2}(\underline{u}^2 L + u^2 \underline{L}) \tag{9.5}$$

To define the action of the rotation group $O(3)$ on $\mathcal{H}_{t_*}$, we consider the vector field on $\mathcal{H}_{t_*}$ whose components in an arbitrary frame in $\mathcal{H}_{t_*}$ are given by:

$$U^i = a^2 \overline{g}^{ij} \partial_j u \tag{9.6}$$

The integral curves of U are orthogonal to the foliation induced by u on $\mathcal{H}_{t_*}$ and are parametrized by u. The one parameter group of diffeomorphisms generated by U maps the surfaces $\mathcal{S}_{t_*,u}$ onto each other. The induced metric γ on $\mathcal{S}_{t_*,u}$ rescaled by the factor r^{-2} tends along the flow of U to a metric of Gauss curvature equal to 1 as $u \to -\infty$. We can thus attach the standard sphere S^2 at infinity on $\mathcal{H}_{t_*}$. We have the standard action of $O(3)$ on S^2 by isometries. The action is then extended to $\mathcal{H}_{t_*}$ by conjugation: Given an element $O \in O(3)$ and a point $p_* \in \mathcal{S}_{t_*,u}$, there is a point $q \in S^2$, the ideal point at parameter value $-\infty$ along the integral curve of U through p_* at parameter value u. The action of $O(3)$ on S^2 gives us the point

$Oq \in S^2$. The point $Op_* \in \mathcal{S}_{t_*,u}$ is then defined to be the point at parameter value u along the integral curve of U leading to the ideal point Oq at parameter value $-\infty$.

The action of $O(3)$ is then extended to the space-time slab using the vector field L. The integral curves of L are the null geodesic generators of the hypersurfaces $\mathcal{C}_u$ and are parametrized by t. The one parameter group of diffeomorphisms generated by L maps the surfaces $\mathcal{S}_{t,u}$ corresponding to the same value of u but different values of t onto each other. Given an element $O \in O(3)$ and a point $p \in \mathcal{S}_{t,u}$, to obtain the point Op we follow the integral curve of L through p at parameter value t to the point $p_* \in \mathcal{S}_{t_*,u}$ at parameter value t_*. The action of $O(3)$ on $\mathcal{H}_{t_*}$ just defined gives us the point $Op_* \in \mathcal{S}_{t_*,u}$. The point $Op \in \mathcal{S}_{t,u}$ is then defined to be the point at parameter value t along the integral curve of L through Op_* at parameter value t_*.

The three rotation vector fields ${}^{(a)}\Omega$, $a = 1,2,3$, generating the above action satisfy:

$$[{}^{(a)}\Omega, L] = 0 \tag{9.7}$$

$$g({}^{(a)}\Omega, L) = g({}^{(a)}\Omega, T) = 0 \tag{9.8}$$

and, of course, the commutation relations of the Lie algebra of $O(3)$:

$$[{}^{(a)}\Omega, {}^{(b)}\Omega] = \in_{abc} {}^{(c)}\Omega \tag{9.9}$$

The group orbits are the surfaces $S_{t,u}$.

By the above construction the deformation tensors of the generating vector fields depend entirely on the geometric properties of the hypersurfaces $\mathcal{C}_u$ and $\mathcal{H}_t$.

Once the vector fields are defined we consider the 1-form P, given by

$$P = P_0 + P_1 + P_2 \tag{9.10}$$

where:

$$\begin{aligned}
P_0 &= -Q(R)(\cdot, \overline{K}, T, T) \\
P_1 &= -Q(\hat{\mathcal{L}}_O R)(\cdot, \overline{K}, \overline{K}, T) - Q(\hat{\mathcal{L}}_T R)(\cdot, \overline{K}, \overline{K}, \overline{K}) \\
P_2 &= -Q(\hat{\mathcal{L}}_O^2 R)(\cdot, \overline{K}, \overline{K}, T) - Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T R)(\cdot, \overline{K}, \overline{K}, \overline{K}) \\
&- Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T R)(\cdot, \overline{K}, \overline{K}, \overline{K}) - Q(\hat{\mathcal{L}}_T^2 R)(\cdot, \overline{K}, \overline{K}, \overline{K})
\end{aligned} \tag{9.11}$$

and

$$\overline{K} = K + T$$

while O stands for the collection ${}^{(a)}\Omega$, $a = 1,2,3$. Here $Q(W)$ is the Bel-Robinson quadratic form associated to the Weyl field W and R stands for the space-time curvature, the original Weyl field. We then define the *controlling quantity*:

$$E = \max\{E_1, E_2\} \tag{9.12}$$

where

$$E_1 = \sup_t \int_{\mathcal{H}_t} {}^\star P, \; E_2 = \sup_u \int_{\mathcal{C}_u} {}^\star P \tag{9.13}$$

and everything is restricted to the space-time slab ${}^{(t_*)}\mathcal{M} = \bigcup_{t \in [0,t_*]} \mathcal{H}_t$ under consideration.

10. The Continuity Argument

The values of the integral of ${}^*P$ on two homologous hypersurfaces are not the same, for the vector fields T, S, K and O are not exact conformal Killing vector fields. The difference of these values is the integral of error terms, linear in the deformation tensors of the vector fields and quadratic in the Weyl fields, over the space-time region bounded by the hypersurfaces. The crucial point and success of the geometric construction is the fact that these error integrals can be bounded in terms of the controlling quantity itself.

The proof of the stability theorem is by the method of continuity and it involves a complex bootstrap argument. Starting with a strongly asymptotically flat initial data set satisfying the maximality condition, and using an appropriate version of the local existence theorem we can assume that the space-time is maximally extended up to a value t_* of the maximal time function. This value is defined to be the maximal one such that certain geometric quantities defined by the hypersurfaces $\mathcal{H}_t$ and $\mathcal{C}_u$ remain bounded by a small positive number ε_0. These quantities include, in particular,

$$\sup_{t,u} \sup_{\mathcal{S}_{t,u}} \left|r^2 K - 1\right|$$

which controlls the isoperimetric constant of the surfaces $\mathcal{S}_{t,u}$, on which the Sobolev inequalities depend. They also include:

$$\sup_{t} \sup_{\mathcal{H}_t} (1 - \phi)$$

(note that by the maximum principle applied to (4.6): $\phi \leq 1$). It then follows that a cetrain norm of the deformation tensors of the vector fields T, S, K and O in the space-time slab bounded by $\mathcal{H}_0$ and $\mathcal{H}_{t_*}$ is less than another small positive constant ε_1. Using this bound for the deformation tensors, as well as the Sobolev inequalities, we are able to estimate the integral of the error terms over the space-time slab by $c\varepsilon_1 E$ and thus arrive at an inequality of the form:

$$E \leq c(D + \varepsilon_1 E)$$

where D stands for initial data. When ε_1 is chosen sufficiently small, which is achieved by choosing ε_0 suitably small, this implies $E \leq cD$. On the other hand we are able to show that the aforementioned geometric quantities associated to the hypersurfaces $\mathcal{H}_t$ and $\mathcal{C}_u$ are bounded by cE. Thus if D is suitably small this bound does not exceed $\varepsilon_0/2$, which by contituity contradicts the maximality of t_*, unless of course $t_* = \infty$, in which case, in view of the fact that ϕ has a positive lower bound, we have geodesic completeness and the theorem is proved.

We remark that the estimate of the error terms would fail if it were not for the fact that the worst error terms vanish due to a simple algebraic identity: if A, B, C are any three symmetric trace-free two dimensional matrices then $\mathrm{tr}(ABC) = 0$. The reason why such matrices appear can be traced back to the symbol of the Einstein equations; they represent the dynamical degrees of freedom of the gravitational field.

The smallness condition on the initial data which is required in the proof of the theorem is the following. Take a point $p \in \mathcal{H}_0 = \mathcal{H}$ and a positive real number

λ. Let d_p be the distance function on $\mathcal{H}$ from p. Set:

$$
\begin{aligned}
D(p,\lambda) &= \sup_{\mathcal{H}}\{\lambda^{-2}(d_p^2+\lambda^2)^3|\overline{Ric}|^2\} \\
&+ \lambda^{-3}\left\{\int_{\mathcal{H}}\sum_{l=0}^{3}(d_p^2+\lambda^2)^{l+1}|\overline{\nabla}^l k|^2 d\mu_{\overline{g}}\right. \\
&+ \left.\int_{\mathcal{H}}\sum_{l=0}^{1}(d_p^2+\lambda^2)^{l+3}|\overline{\nabla}^l B|^2 d\mu_{\overline{g}}\right\}
\end{aligned} \tag{10.1}
$$

Here, $|\overline{Ric}|^2 = \overline{R}^{ij}\overline{R}_{ij}$, $\overline{\nabla}^l$ denotes the covariant derivative of order l, and B is the Bach tensor or conformal curvature of $(\mathcal{H},\overline{g})$, a symmetric trace-free 2-covariant tensorfield given by:

$$
B_{ij} = \frac{1}{2}(\in_i{}^{ab}\,\overline{\nabla}_a\hat{\overline{R}}_{jb} + \in_j{}^{ab}\,\overline{\nabla}_a\hat{\overline{R}}_{ib}) \tag{10.2}
$$

with $\hat{\overline{R}}_{ij}$ the traceless part of $\overline{R}_{ij}$. Then it is the dimensionless invariant

$$
\inf_{p\in\mathcal{H},\lambda>0} D(p,\lambda)
$$

which must be sufficiently small.

11. The Geometry of Maximal and Null Hypersurfaces

The most difficult and complex step in the proof of the stability theorem is the step demonstrating that if the geometric quantities defined by the hypersurfaces $\mathcal{H}_t$ and $\mathcal{C}_u$ do not exceed ε_0 they are in fact bounded by cE.

The instrinsic and extrinsic geometry of a maximal hypersurface $\mathcal{H}_t$ is is controlled by the elliptic system:

$$
\overline{R}_{ij} - k_{im}k^m_j = E_{ij} \tag{11.1}
$$

$$
\overline{\nabla}_i k_{jm} - \overline{\nabla}_j k_{im} = \in_{ij}^{n} H_{mn},\ \overline{\nabla}^j k_{ij} = 0,\ \mathrm{tr}k = 0 \tag{11.2}
$$

Here E_{ij} and H_{ij} stand for the *electric* and *magnetic* parts of the space-time curvature respectively, symmetric trace-free 2-covariant tensorfiels on $\mathcal{H}_t$, defined in terms of an arbitray frame e_i, $i = 1,2,3$ in $\mathcal{H}_t$ by:

$$
E_{ij} = R(e_i,\hat{T},e_j,\hat{T}),\ H_{ij} = -{}^{\star}R(e_i,\hat{T},e_j,\hat{T}) \tag{11.3}
$$

where $\hat{T}$ is the unit normal to $\mathcal{H}_t$. These are directly controlled by the quantity E. The estimates however involve the foliation of $\mathcal{H}_t$ given by the surfaces $\mathcal{S}_{t,u}$, the level sets of the restriction to $\mathcal{H}_t$ of the optical function u, and some control of the properties of this foliation, provided by the a priori assumption that the geometric quantities do not exceed ε_0, is needed in order to proceed.

The intrinsic geometry of a null hypersurface $\mathcal{C}_u$ is described in terms of the foliation of $\mathcal{C}_u$ given by the surfaces $\mathcal{S}_{t,u}$. If we denote by $e_0 = \phi^{-1}T$ the unit normal to $\mathcal{H}_t$, then e_+ and e_-, respectively the outgoing and incoming null normals to to $\mathcal{S}_{t,u}$, whose component along e_0 is equal to e_0, are given by:

$$
e_+ = e_0 + e_3,\ e_- = e_0 - e_3 \tag{11.4}
$$

where $e_3 = -a^{-1}U$ is the unit outward normal to $\mathcal{S}_{t,u}$ in $\mathcal{H}_t$. We have:

$$
e_+ = \phi^{-1}L \tag{11.5}
$$

As e_+ is tangent to $\mathcal{C}_u$, χ, the second fundamental form of $\mathcal{S}_{t,u}$ relative to e_+ is an aspect of the intrinsic geometry of $\mathcal{H}_u$. Its components in an arbitrary local frame e_A, $A = 1, 2$ in $\mathcal{S}_{t,u}$ are given by:

$$\chi_{AB} = g(\nabla_{e_A} e_+, e_B) \tag{11.6}$$

The second fundamental form of $\mathcal{S}_{t,u}$ relative to e_-, which is transverse to $\mathcal{H}_u$, we denote by $\underline{\chi}$:

$$\underline{\chi}_{AB} = g(\nabla_{e_A} e_-, e_B) \tag{11.7}$$

We have:

$$\chi = \theta + \eta, \ \underline{\chi} = -\theta + \eta \tag{11.8}$$

where θ is the second fundamental form of $\mathcal{S}_{t,u}$ relative to $\mathcal{H}_t$ and η is the restriction of k to $\mathcal{S}_{t,u}$. As we have already discussed how k is estimated we shall describe below how estimates for χ are obtained; the intrinsic geometry of $\mathcal{S}_{t,u}$ is controlled by the Gauss equation:

$$K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}\hat{\chi} \cdot \hat{\underline{\chi}} = -\rho \tag{11.9}$$

where

$$\rho = \frac{1}{4} R(e_-, e_+, e_-, e_+) \tag{11.10}$$

and we denote by $\hat{\chi}, \hat{\underline{\chi}}$ the traceless parts of $\chi, \underline{\chi}$, respectively. The function $\mathrm{tr}\chi$ satisfies along the integral curves of L (which are parametrized by t) the propagation equation:

$$\frac{1}{\phi}\frac{\partial \mathrm{tr}\chi}{\partial t} = \nu\mathrm{tr}\chi - \frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 \tag{11.11}$$

Here,

$$\nu = g(\nabla_{e_+} e_0, e_3) = \nabla_3 \log\phi + \delta, \ \delta = k_{33} \tag{11.12}$$

Note that by virue of the Einstein vacuum equations no curvature term appears on the right hand side of (11.11). The propagation equation (11.11) is considered in conjunction with the Codazzi equation:

$$\nabla\!\!\!/^B \hat{\chi}_{AB} - \frac{1}{2}\nabla\!\!\!/_A \mathrm{tr}\chi = \epsilon^B \hat{\chi}_{AB} - \frac{1}{2}\epsilon\mathrm{tr}\chi - \beta_A \tag{11.13}$$

$$\epsilon_A = k_{A3}, \ \beta_A = \frac{1}{2} R(e_A, e_+, e_-, e_+) \tag{11.14}$$

an elliptic equation for $\hat{\chi}$ on each $\mathcal{S}_{t,u}$, to obtain the required optimal estimates for χ, one order of differentiability smoother than the space-time curvature.

The foliation of space-time given by the null hypersurfaces $\mathcal{C}_u$ are described in terms of the foliation of each $\mathcal{H}_t$ given by the surfaces $\mathcal{S}_{t,u}$. The properties of the latter include, besides what we have already discussed, the lapse function a given, on each $\mathcal{H}_t$, by (8.1). The estimation of $\log a$ is the most subtle part of the argument. It is accomplished by introducing the *mass aspect* function:

$$\mu = -\nabla\!\!\!/ \cdot \zeta + K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} \tag{11.15}$$

where

$$\zeta = \nabla\!\!\!/ \log a - \epsilon \tag{11.16}$$

The function μ turns out to satisfy along the integral curves of L the propagation equation:

$$\begin{aligned}\frac{1}{\phi}\frac{\partial\mu}{\partial t}+\mu\mathrm{tr}\chi &= 2\hat{\chi}\cdot(\nabla\!\!\!/\hat{\otimes}\zeta)-2\zeta\cdot\beta\\ &-\frac{1}{2}\mathrm{tr}\chi(\nabla\!\!\!/\cdot\lambda+|\lambda|^2+\frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}-\rho)\\ &+(\zeta-\lambda)\cdot(\nabla\!\!\!/\mathrm{tr}\chi-\epsilon\mathrm{tr}\chi)\\ &-\frac{1}{4}\mathrm{tr}\underline{\chi}|\hat{\chi}|^2+\zeta\cdot\hat{\chi}\cdot\lambda\end{aligned}\tag{11.17}$$

Here,

$$\lambda_A=g(\nabla_{e_+}e_0,e_A)=\nabla\!\!\!/_A\log\phi+\epsilon_A\tag{11.18}$$

and we denote by $\nabla\!\!\!/\hat{\otimes}\zeta$ the 2-covariant symmetric trace-free tensorfield on $\mathcal{S}_{t,u}$ given by:

$$(\nabla\!\!\!/\hat{\otimes}\zeta)_{AB}=\frac{1}{2}(\nabla\!\!\!/_A\zeta_B+\nabla\!\!\!/_B\zeta_A-\gamma_{AB}\nabla\!\!\!/\cdot\zeta)$$

What is remarkable here is that, by virtue of the Einstein vacuum equations, the right hand side of (11.17) does not contain terms involving the first derivatives of the curvature. This fact allows us to consider the propagation equation (11.17) in conjuction with the definition (11.15), which is equivalent to:

$$\Delta\!\!\!/\log a=-\mu+\nabla\!\!\!/\cdot\epsilon+K+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\tag{11.19}$$

an elliptic equation for $\log a$ on each $\mathcal{S}_{t,u}$, to obtain the required optimal estimates for $\log a$, two orders of differentiability smoother than the space-time curvature.

We remark that equation (8.2) on $\mathcal{H}_{t_*}$, when the terms contributed by the second fundamental form of $\mathcal{H}_{t_*}$ are no longer neglected, takes in terms of the function μ the form, simply:

$$\mu=\overline{\mu}\tag{11.20}$$

where $\overline{\mu}$ denotes the mean value of μ on each $\mathcal{S}_{t,u}$.

12. Asymptotic Behaviour

Once the proof of the stability theorem is completed we show that the optical function ${}^{(t_*)}u$ defined during the course of the proof in the slab ${}^{(t_*)}\mathcal{M}$, converges as $t_*\to\infty$ to a global optical function u. For each $t\geq 0$, the 0-level set of ${}^{(t_*)}u$ is the part of $\mathcal{C}_0$, the 0-level set of u, contained in the slab ${}^{(t_*)}\mathcal{M}$. Thus the restrictions of ${}^{(t_*)}L$, L to $\mathcal{C}_0\bigcap{}^{(t_*)}\mathcal{M}$ coincide. We shall describe in the remainder of this article the asymptotic behaviour of the solutions. The derivation of these results is found in the last chapter of "".

Let us denote by ${}^{(t_*)}\omega_t$ and ω_t the one parameter groups of transformations generated by ${}^{(t_*)}L$ and L respectively. Let us also denote by ${}^{(t_*)}\psi_s$ and ψ_s the one parameter groups of transformations generated by ${}^{(t_*)}U$ and U respectively. Given a diffeomorphism χ of S^2 onto the surface $\mathcal{S}_{0,0}$ we define the one parameter family $\varphi_{t,0}$ of diffeomorphisms of S^2 onto $\mathcal{S}_{t,0}$ by:

$$\varphi_{t,0}=\omega_t\circ\chi$$

We then define the one parameter family ${}^{(t_*)}\varphi_{t_*,s}$ of diffeomorphisms of S^2 onto ${}^{(t_*)}\mathcal{S}_{t_*,s}$ by:

$$^{(t_*)}\varphi_{t_*,s} = \psi_s \circ \varphi_{t_*,0}$$

Finally we define the two parameter family ${}^{(t_*)}\varphi_{t,s}$, $t \in [0,t_*]$, of diffeomorphisms of S^2 onto ${}^{(t_*)}\mathcal{S}_{t,s}$ by:

$$^{(t_*)}\varphi_{t,s} = \omega_{t-t_*} \circ {}^{(t_*)}\varphi_{t_*,s}$$

We then show that as $t_* \to \infty$, ${}^{(t_*)}\varphi_{t,s}$ converges for each t and s to a diffeomorphism of S^2 onto $\mathcal{S}_{t,s}$.

We call a n-covariant tensorfield w on $\mathcal{M}$ $\mathcal{S}_{t,u}$-tangent if at each $p \in \mathcal{M}$ and for any n-tuplet $X_1, ..., X_n$ of vectors at $p \in \mathcal{S}_{t,u}$ we have:

$$w(X_1, ..., X_n) = w(\Pi X_1, ..., \Pi X_n)$$

where Π is the orthogonal projection to $T_p\mathcal{S}_{t,u}$. Given any such tensorfield we define:

$$\tilde{w}_{t,u} = \varphi^*_{t,u}(r^{-n}w)$$

Then $\tilde{w}_{t,u}$ is a n-covariant tensorfield on S^2, for each t and u. We say that on $\mathcal{C}_u$ w tends to a limit $W(u)$ as $t \to \infty$, and we write:

$$\lim_{\mathcal{C}_u, t\to\infty} w = W(u)$$

if:

$$\lim_{t\to\infty} \tilde{w}_{t,u} = W(u)$$

on S^2. It then follows that:

$$\frac{\partial W}{\partial u} = \lim_{\mathcal{C}_u, t\to\infty} \Pi \mathcal{L}_U w$$

The induced metric γ on $\mathcal{S}_{t,u}$ tends in this sense to a metric $\overset{\circ}{\gamma}$ on S^2, which is independent of u and of Gauss curvature equal to 1. Therefore $(S^2, \overset{\circ}{\gamma})$ can be identified with the unit sphere in $\Re^3$. Also,

$$\lim_{\mathcal{C}_u, t\to\infty} \phi = \lim_{\mathcal{C}_u, t\to\infty} a = 1 \tag{12.1}$$

and:

$$\lim_{\mathcal{C}_u, t\to\infty} r\mathrm{tr}\chi = - \lim_{\mathcal{C}_u, t\to\infty} r\mathrm{tr}\underline{\chi} = 2 \tag{12.2}$$

Moreover,

$$\lim_{\mathcal{C}_u, t\to\infty} r^2\hat{\chi} = \Sigma(u), \quad \lim_{\mathcal{C}_u, t\to\infty} r\hat{\underline{\chi}} = \Xi(u) \tag{12.3}$$

where Σ and Ξ are symmetric trace-free 2-covariant tensorfields on S^2 depending on u and related by:

$$\frac{\partial \Sigma}{\partial u} = -\frac{1}{2}\Xi \tag{12.4}$$

Also,

$$\Xi = o(|u|^{-3/2}) \text{ as } |u| \to \infty \tag{12.5}$$

Note that according to (12.2,12.3),

$$\lim_{\mathcal{C}_u, t\to\infty} \frac{\hat{\theta}}{\mathrm{tr}\theta} = -\frac{1}{4}\Xi(u) \tag{12.6}$$

so the surface $\mathcal{S}_{t,u}$ for fixed u does not become umbilical relative to $\mathcal{H}_t$ as $t \to \infty$.

The space-time curvature decomposes relative to the surfaces $\mathcal{S}_{t,u}$ into the $\mathcal{S}_{t,u}$-tangent 2-covariant symmetric trace-free tensorfields α, $\underline{\alpha}$, whose components in an arbitrary local frame e_A, $A = 1,2$ in $\mathcal{S}_{t,u}$ are given by:

$$\alpha_{AB} = R(e_A, e_+, e_B, e_+), \ \underline{\alpha}_{AB} = R(e_A, e_-, e_B, e_-) \tag{12.7}$$

the $\mathcal{S}_{t,u}$-tangent 1-forms β, $\underline{\beta}$, with components:

$$\beta_A = \frac{1}{2}R(e_A, e_+, e_-, e_+), \ \underline{\beta}_A = \frac{1}{2}R(e_A, e_-, e_-, e_+) \tag{12.8}$$

and the functions ρ and σ, defined by:

$$\rho = \frac{1}{4}R(e_-, e_+, e_-, e_+), \ \sigma\varepsilon(e_A, e_B) = \frac{1}{2}R(e_A, e_B, e_-, e_+) \tag{12.9}$$

where ε is here the area 2-form of $\mathcal{S}_{t,u}$. We have:

$$\begin{aligned}
&\lim_{\mathcal{C}_u, t\to\infty} r^{7/2}\alpha = 0, &&\lim_{\mathcal{C}_u, t\to\infty} r\underline{\alpha} = A(u)\\
&\lim_{\mathcal{C}_u, t\to\infty} r^{7/2}\beta = 0, &&\lim_{\mathcal{C}_u, t\to\infty} r^2\underline{\beta} = B(u)\\
&\lim_{\mathcal{C}_u, t\to\infty} r^3\rho = P(u), &&\lim_{\mathcal{C}_u, t\to\infty} r^3\sigma = Q(u)
\end{aligned} \tag{12.10}$$

where A is a symmetric trace-free 2-covariant tensorfield, B is a 1-form and P and Q are functions on S^2, all depending on u and having the decay properties:

$$\begin{aligned}
A = o(|u|^{-5/2}), \qquad & B = o(|u|^{-3/2})\\
P - \overline{P} = o(|u|^{-1/2})), \qquad & Q = o(|u|^{-1/2}))\\
& \text{as } |u| \to \infty
\end{aligned} \tag{12.11}$$

while:

$$\begin{aligned}
\overline{P} = o(|u|^{-1/2}) \qquad & \text{as } u \to \infty\\
\overline{P} + \frac{M_0}{2\pi} = o(|u|^{-1/2}) \qquad & \text{as } u \to -\infty
\end{aligned} \tag{12.12}$$

Here M_0 is the ADM mass. Moreover A and B are related to Ξ according to:

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{2}A \tag{12.13}$$

and (relative to an arbitrary local frame in S^2)

$$\overset{\circ}{\nabla}{}^B \ \Xi_{AB} = B_A \tag{12.14}$$

The following result shows that the ADM mass enters the asymptotic expansion of the area radius of the sections $\mathcal{S}_{t,u}$ of a null hypersurface $\mathcal{C}_u$ as $t \to \infty$:

$$r(t,u) = t - \frac{M_0}{2\pi}\log t + O(1) \text{ :at constant } u \text{ as } t \to \infty \tag{12.15}$$

The *Hawking mass* $m(t,u)$ contained by a surface $\mathcal{S}_{t,u}$ is defined by [**11**]:

$$m(t,u) = 2\pi r\left(1 + \frac{1}{16\pi}\int_{\mathcal{S}_{t,u}} \mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right) \tag{12.16}$$

Note that:

$$\overline{\mu} = \frac{m}{2\pi r^3} \tag{12.17}$$

The *Bondi mass* $M(u)$ contained in C_u is defined by:

$$M(u) = \lim_{t\to\infty} m(t,u) \tag{12.18}$$

One of the achievements of our work was the rigorous derivation of the formula:

$$\frac{\partial M}{\partial u} = -\frac{1}{8}\int_{S^2} |\Xi|^2 d\mu_{\overset{\circ}{\gamma}} \tag{12.19}$$

due to Bondi [**2**]. Morever, we obtain:

$$\lim_{u\to-\infty} M(u) = M_0, \ \lim_{u\to\infty} M(u) = 0 \tag{12.20}$$

Our final result has to do with the difference of the limits Σ^+, Σ^-, of Σ as $u \to \infty$, $u \to -\infty$, respectively. This difference is determined by the equation:

$$\overset{\circ}{\nabla}{}^B (\Sigma^+_{AB} - \Sigma^-_{AB}) = \overset{\circ}{\nabla}_A \Phi \tag{12.21}$$

where Φ is the solution of:

$$\overset{\circ}{\Delta} \Phi = -2(F - \overline{F}), \ \overline{\Phi} = 0 \tag{12.22}$$

and F is the function on S^2 defined by:

$$F = \frac{1}{8}\int_{-\infty}^{\infty} |\Xi(u)|^2 du \tag{12.23}$$

In view of (12.19), $F/4\pi$ is the total energy radiated to infinity in a given direction, per unit solid angle. The integrability condition of (12.21,12.22), is that F is L^2-orthogonal to the 1st eigenspace of $\overset{\circ}{\Delta}$:

$$F_{(1)} = 0 \tag{12.24}$$

Now the L^2-inner products of F with the three Cartesian coordinate functions x^i, i=1,2,3, on $S^2 \subset \Re^3$, which form an orthogonal basis for the 1st eigenspace of $\overset{\circ}{\Delta}$ represent the components of the total linear momentum radiated to infinity. Since the initial and final states both have zero linear momentum, (12.24) expresses here the law of conservation of linear momentum.

The solution of (12.21,12.22), evaluated at an arbitrary pair X, Y of vectors in $\Re^3$, tangent to S^2 at an arbitrary point ξ, is given by:

$$(\Sigma^+ - \Sigma^-)(X,Y) = -\frac{1}{2\pi}\int_{|\xi'|=1} (F - F_{[1]})(\xi')\frac{(X,\xi')(Y,\xi') - (1/2)(X,Y)|\Pi\xi'|^2}{1-(\xi,\xi')} d\mu_{\overset{\circ}{\gamma}}(\xi') \tag{12.25}$$

Here the subscript [1] denotes the projection on the sum of the 0th and 1st eigenspaces of $\overset{\circ}{\Delta}$, the projection on the 0th eigenspace being the mean value, $(,)$ denotes inner product in $\Re^3$ and Π denotes projection to the plane orthogonal to ξ. Now, by (12.4) we have:

$$\Sigma^+ - \Sigma^- = -\frac{1}{2}\int_{-\infty}^{\infty} \Xi(u) du \tag{12.26}$$

In view of (12.26,12.23), equation (12.25) constitutes a non-linear relationship satisfied by Ξ on $\Re \times S^2$. The non-linearity of Einstein's equations is therefore partially retained even at infinity!

It turns out that the difference $\Sigma(u) - \Sigma^-$ is directly related to the instantaneous displacements of faraway test masses with respect to a reference test mass, relative to which they are initially at rest. The difference $\Sigma^+ - \Sigma^-$, thus yields a permanent displacement of the test masses, a non-linear effect, which is observable in principle (see [**6**] for the details).

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