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ON RICCI-FLAT TWISTOR THEORY

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1. Background

In the autumn of 1952, I had the honour to be taken on as a research student at the University of Cambridge, to work in algebraic geometry under the supervision of the renowned mathematician William V.D. Hodge. As I recall it, there were four of us, starting under Hodge at the same time. The research that then interested him was broadly divided into that which was centred on algebraic geometry and a more topological line arising from his work on harmonic integrals. I had specifically started on the algebraic geometry side, but I was finding things rather too strictly "algebraic", for my tastes, with not much of a realization of this algebra into what I thought of as "geometry". Noticing that I was not entirely happy with spending my time dealing with questions in ideal theory, local rings, and so on, Hodge suggested that I might like to sit in on a supervision session, the supervisee being the only one of the four of us who was working on the harmonic integrals side of things. The idea intrigued me because that work seemed to be rather more geometrical in nature than the problems that I had been looking at, so with considerable expectations I turned up. The student was a "Mr Attia"—or, at least, that is how Hodge used to refer to him—and I remember being totally snowed under by Mr Attia's breadth of knowledge and comprehension; indeed, I recall not understanding a single word of what was going on. Of course "Attia" was really "Atiyah"–and one of the difficulties about being a research student, especially at a place like Cambridge, is that one never knows who one's co-research students really are (or will be)! "Not

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understanding a single word" may perhaps be nothing to be ashamed of under such circumstances.

Over my research-student period, there was much interest in what was then referred to as "the theory of stacks". I remember trying to struggle with stacks, for a little, but I then made life easier for myself by deciding that my interests lay largely elsewhere, so I spent a good deal of my time learning about general relativity, quantum mechanics, mathematical logic, and various other matters purely mathematical. After I left Cambridge, in 1959, my interests had moved more and more in the direction of theoretical physics, mainly general relativity, but also quantum mechanics. Later I developed my interest in what I referred to as the theory of "twistors", which took advantage of many algebraic/geometrical notions that I had learned about during my student days—most particularly the Klein representation of lines in projective 3-space as points of a 4-quadric.

The basic idea of twistor theory (for flat Minkowski space-time M) was, in effect, to take the Klein representation "in reverse", where the conformally compactified space-time $M^{\#}$ is, roughly speaking, taken as the "Klein Quadric" of another space PN (Penrose 1967). More precisely, we regard the natural complexification $CM^{\#}$, of $M^{\#}$, as the Klein representation of complex straight lines in a certain CP^3 called projective twistor space PT. The space PN is a real 5-submanifold (given by the vanishing of a Hermitian quadratic form of signature $++--$) of the real 6-manifold PT. The projective lines which lie in PN are "Klein-represented" by the points of of the real 4-manifold $M^{\#}$. Then, using this description, the basic physical notions of space-time, particles, fields, etc. would be interpreted in terms of the projective geometry of PT or PN, or of the geometry/analysis of the underlying vector space T, simply called *twistor space.* It later turned out (Penrose 1969) that *massless* fields, in particular, find an elegant description in terms of contour integrals in twistor space. In particular, linearized gravitational fields (massless fields of spin 2) can be neatly accommodated within this scheme. (See Penrose 1987, for an account of the curious history of all this.)

Yet, this approach did not directly cope with the space-time curvature which would be needed in order that the gravitational field proper could be incorporated into twistor theory, in accordance with Einstein's general relativity. However, through a roundabout route, originating with an idea due to E.T. Newman (Newman 1976, cf. Penrose 1992 for the relevant history), I had come to the conclusion that "half" of the gravitational field—the "left-handed" half that is described by an anti-self-dual (ASD) Weyl curvature—can indeed be incorporated into twistor geometry, where the notion of twistor space has to be generalized away from the *flat* twistor model PT (or T) to a "curved" one $P\mathcal{T}$ (or $\mathcal{T}$), this "curvature" not being anything that shows up at the *local* level (T and $\mathcal{T}$ being locally identical), but arising from the *global* structure of $P\mathcal{T}$. I had realized that I needed to understand how to describe *defomations* of complex manifolds (particularly non-compact ones) and that this could indeed accommodate genuine ASD Weyl curvature into the (complex) "space-time". Moreover, the condition of *Ricci-flatness* for such ASD complex-Riemannian 4-manifolds can be easily incorporated. I consulted a few people about how to describe such deformations and under what circumstances the needed *4-parameter* family of "Klein" lines would persist in $P\mathcal{T}$, but it was not until Michael Atiyah explained Kodaira's various theorems on this question to me—and more importantly, how to *use* these theorems in the context that I needed, that I began to see what all those "stacks"—now called *sheaves*—had really been about, all the time. This provided the necessary background for the construction that I referred to as "the non-linear graviton" (Penrose 1976) in which complex ASD Ricci-flat 4-manifolds can be described in terms of a kind of "Klein representation" of lines in appropriate complex 3-manifolds.

By then I had been in Oxford for several years, where Michael now was, and he made a special point of providing me and my research group with illuminating expository sessions, in which he explained to us, in his characteristically revealing way, the beauty, the essential simplicity, and the relevant uses of sheaf cohomology. One point, in particular, that I found valuable was Michael's deliberate use of *Čech* cohomology in his expositions, rather than the more frequently used Dolbeault approach. In my opinion, the Čech approach provided a much greater clarity, in the context of the problems of relevance to us, and it was certainly sufficient for our immediate needs. It soon emerged, on the basis of Michael's encouraging insights, that the contour integral expressions that I had previously adopted for the description of (linear) massless fields really were themselves expressions of (Čech) sheaf cohomology. Accordingly, an (analytic) massless field of helicity $n/2$ in M would be interpreted as an element of $H^1(Q, \mathcal{O}(-n-2))$, where Q is some suitable open subregion of PT, related to the domain of definition (assumed appropriate) of the massless field in M, and where $\mathcal{O}(-n-2)$ is the sheaf of twisted holomorphic functions on PT, locally given by holomorphic functions on T of homogeneity degree $-n-2$ (cf. Eastwood, Penrose, and Wells 1981; here n is an integer, the *spin* of the field being $|n|/2$, where the *sign* of n tells us the "handedness" of the field). This clarified numerous points of previous confusion.

These insights also led to a direct interpretation of the linear massless fields of helicity-2 (linearized ASD gravity) as providing infinitesimal deformations of (regions of) projective twistor space PT (cf. Penrose and Rindler 1986), this being a *weak-field* version of the above "non-linear graviton". A point to note is that we are here concerned with transition functions that are constructed from holomorphic twistor functions $f_{ij}(Z^\alpha)$ that are homogeneous of degree $+2$ (corresponding to helicity $n/2 = -2$). (Here, I am beginning to use the standard 2-spinor/twistor index-notation of Penrose and Rindler 1986. The *twistor* Z^α is an element of flat twistor space $\underset{\smile}{T}$, "α" being a 4-dimensional *abstract index.*) Thus, the family of Čech representative functions $\{f_{ij}\}$, for the $H^1(Q, \mathcal{O}(-n-2))$ element, defined on the overlaps $U_i \cap U_j$ of a suitable Čech cover $\{U_i\}$ of Q (with $f_{ij} = -f_{ji}$, and $f_{ij} - f_{ik} + f_{jk} = 0$ on triple overlaps), directly provides the family of infinitesimal transition functions for piecing together the infinitesimally curved twistor space $\mathcal{T}$. These infinitesimal transition functions are provided by "sliding infinitesimally along" the vector field

$$\varepsilon^{AB}\frac{\partial f_{ij}}{\partial \omega^A}\frac{\partial}{\partial \omega^B},$$

where I now adopt the 2-spinor/twistor index-notation $(\omega^A, \pi_{A'})$ for the spinor parts of the twistor Z^α, taken with respect to some origin O in M. Note that the homogeneity degree $+2$ of f_{ij} exactly balances the two $\partial/\partial\omega$ contributions, each of degree -1.

2. The Googly problem

Although all this was remarkably satisfying, a definite problem began to loom large. For if twistor theory is to be taken to be a *physical* theory, the gravitational field as it is *actually* understood, must be described by a (Weyl) curvature for a space-time which possesses *both* an SD (self-dual) and an ASD part. In the case of *weak-field* gravity, regarded as a massless field of spin 2, this is neatly accommodated because the $\mathcal{O}(-6)$ Čech cohomology handles the right-handed (SD) part of the gravitational field in a closely analogous way to the $\mathcal{O}(+2)$ Čech cohomology description of the left-handed (ASD) part of the gravitational field. Moreover, if we regard these as referring to the non-projective twistor space T rather than to the projective PT, then there is an easy way of expressing the *sum* of the SD and ASD parts, to obtain a twistor-cohomological description of *full* (neither SD nor ASD) weak-field gravity. Yet, for this to provide an actual deformation of twistor space, we need an active role for the $\mathcal{O}(-6)$-cohomology, analogous, in some appropriate way, to the way in which the

$\mathcal{O}(+2)$-cohomology infinitesimally *deforms* twistor space, thus leading to the "non-linear graviton" construction referred to above. The problem of introducing SD Weyl curvature into the geometry of twistor space has been referred to as the (gravitational) *googly* problem of twistor theory—in reference to the cricketing term "googly" for a ball that spins in a right-handed sense even though the bowling action suggests a left-handed spin. Taking the cricketing analogy further, I now refer to the original "non-linear graviton" (mentioned above; as given in Penrose 1976) as the *leg-break construction.*

Somewhat over a year ago, a new approach to the relevant googly geometry has come about (see Penrose 1999), in which the googly (SD) information is encoded in the way that the twistor space $\mathcal{T}$ sits above its "projective" version $P\mathcal{T}$, where the leg-break (ASD) information resides in the structure of $P\mathcal{T}$, essentially just as before. In 1978 Michael and his colleagues showed (Atiyah, Hitchin, and Singer 1978) how my original leg-break construction could be adapted to the case of an ordinary (positive-definite) ASD Riemannian Ricci-flat 4-space (the ASD condition being non-trivial in the positive-definite case, unlike the situation with the Lorentzian signature of general relativity). The purpose of this article is to point out that there is also a Riemannian version of the new googly geometry, although I have not worked out all the requirements for this. It is my hope that these ideas will be taken up seriously by someone, and that there may be some interesting new things to say about general (neither SD nor ASD) Ricci-flat Riemannian 4-spaces in accordance with these *twistorial* ideas.

I can give only a very brief account of the new googly geometry here; otherwise there is danger of things getting unhelpfully bogged down in the notation. In any case, it is probable that any Riemannian geometric approach would rely upon some different concepts which might be better expressed in ways other than those that naturally suit Lorentzian space-time geometry. It should be made clear, also, that there are still major unresolved issues with regard to the googly geometry, parts of the programme being still in a conjectural state. Moreover, there are some aspects of the construction that rely upon conditions of asymptotic flatness that are appropriate in the Lorentzian case, whereas I do not know to what extent these Lorentzian ideas can be taken over to the case of a Riemannian Ricci-flat 4-space.

The Riemannian case does have one clear advantage over the Lorentzian case, in relation to the ideas of twistor theory. Since the condition of Ricci-flatness becomes a set of *elliptic* equations, we must expect that the solutions are analytic in the interior regions. Indeed, this is the case (see

Kazdan 1983). Thus, for any Riemannian Ricci-flat 4-manifold $\mathcal{M}$, there exists a (local) complexification $C\mathcal{M}$, which need be merely a "thickening" of the real 4-manifold $\mathcal{M}$ into a (non-compact) real 8-manifold which is a complex 4-manifold $C\mathcal{M}$ of topology $\mathcal{M} \times R^4$.

The first step in the proposed construction of a "twistor space" $\mathcal{T} = \mathcal{T}(\mathcal{M})$, for $\mathcal{M}$ is to produce the *relative* twistor space $\mathcal{T}_p$, where p is any point of $\mathcal{M}$. This is a perfectly rigorous procedure, which I shall outline shortly. The second step would be to attempt to provide a local identification between what I shall call a "comprehensive" (open) region of $\mathcal{T}_p$, and an analogous comprehensive region of $\mathcal{T}_q$, for different points $p, q \in \mathcal{M}$ for which p and q are close enough to each other for this to be achieved. I shall describe the idea behind the notion of "comprehensive" in a moment. In the absence of a more satisfactory procedure, this identification could be via some "ideal" twistor space $\mathcal{T}_\infty$, which we try to think of as being defined as a limit of $\mathcal{T}_p$, as $p \to \infty$, there being identifications of compehensive regions of each of $\mathcal{T}_p$ and $\mathcal{T}_q$ with one and the same compehensive region of $\mathcal{T}_\infty$.

The idea behind this "compehensive" notion is that such a comprehensive region contains the essential global structure that is to be carried from $\mathcal{T}_p$ to $\mathcal{T}_q$ (perhaps via $\mathcal{T}_\infty$). This is to be analogous to what happens in the procedure of *analytic continuation*, as applied to $C\mathcal{M}$. In fact, something of this very nature is already part of the original leg-break construction, although this point does not seem to have been particularly emphasized before. In that case (now taking $C\mathcal{M}$ to be ASD), we can construct the standard leg-break twistor spaces $\mathcal{T}_a$, $\mathcal{T}_b$ of intersecting open neighbourhoods of points $a, b \in C\mathcal{M}$. The twistor space of the *intersection* of these neighbourhoods, provides an identification between open regions of $\mathcal{T}_a$ and $\mathcal{T}_b$ that is sufficiently "comprehensive" that the essential analytic geometry of $C\mathcal{M}$ is carried from $\mathcal{T}_a$ to $\mathcal{T}_b$ via this region. In simple enough ASD situations, it is possible to "glue" all the $\mathcal{T}_x$-spaces together so as to obtain *one* all-inclusive (Hausdorff) twistor space $\mathcal{T}$, but there are other situations when this is not possible, at least if one requires a Hausdorff geometry. When $C\mathcal{M}$ is not ASD, the situation appears to be like this, but essentially more complicated, and some appropriate attitude towards this geometry (not yet fully formulated) seems to be required.

It is not yet clear to me how all this is to work, for general Ricci-flat Riemannian 4-spaces, but there is a "generic" family of Lorentzian space-times for which it can indeed be carried out. These are the space-times that I refer to as "strongly asymptotically flat" radiative analytic vacuums. Think of a sourceless (analytic) gravitational wave that comes in from infinity and then finally disperses out to infinity again, leaving no remnant in the form of a black hole or ay other kind of undispersed localized curvature. In fact, it is only the final dispersing of the wave out to infinity that is needed here, and the work of Friedrich (1986, 1998) is sufficient to establish the "generic" nature of solutions of the Einstein vaccum equations satisfying the needed conditions. What is required is an analytic future-null conformal infinity $\mathcal{I}^+$, with a regular future vertex i^+ (see Penrose and Rindler 1986, Chapter 9). In this case, the required twistor space "$\mathcal{T}^\infty$" actually does exist, this being the space $\mathcal{T}_{i^+}$, and for points $a, b, \cdots$ of $C\mathcal{M}$, "close enough" to i^+, there will indeed be comprehensive regions of $\mathcal{T}_a, \mathcal{T}_b, \cdots$ that can be identified with comprehensive regions of this $\mathcal{T}^\infty$. It is probably not appropriate to go into the details, here, of why this appears to work in the Lorentzian case, but in any case I do not see any reason to expect that this should directly carry over to the Reimannian situation.

Let me leave this issue aside as largely unresolved. However, I should try to explain, briefly, how the relative twistor spaces $\mathcal{T}_a$, are to be constructed. Here, there is no real difference between the Lorentzian and Riemannian cases. In $C\mathcal{M}$, each point $a \in C\mathcal{M}$ has its light cone $\mathcal{C}_a$, consisting of all the points of $C\mathcal{M}$ that lie on null geodesics through a. On $\mathcal{C}_a$, there are curves known as α-lines, which are the curves that "appear intrinsically" to be the intersections of $\mathcal{C}_a$ with α-planes in $C\mathcal{M}$ (SD totally null complex 2-sufaces), even though there may be no actual α-planes in $C\mathcal{M}$. The equation of an α-line, with tangent vector $o^A\pi^{A'}$ can be expressed as

$$\pi^{B'}\nabla_{0B'}\pi_{A'} \ \propto \ \pi_{A'}$$

on $\mathcal{C}_a$. Here *suffixes* 0 and 0′ are to denote components obtained by contraction with spinors o^A and with $\tilde{o}^{A'}$, respectively, where the tangents to the null geodesics through a (i.e. generators of $\mathcal{C}_a$) are the null vectors $o^A\tilde{o}^{A'}$. (When $\mathcal{M}$ is SD the twistor lines are null geodesics on $\mathcal{C}_a$, but in the general case they are not.) It should be remarked that the definition of a twistor line is *conformally invariant.*

The *points of the projective* relative twistor space $P\mathcal{T}_a$ are just the α-lines on $\mathcal{C}_a$. We define the *non*-projective relative twistor space $\mathcal{T}_a$ by fixing the proportionality scale in the above equation according to the conformally invariant equation

$$\pi^{B'}\nabla_{0B'}\pi_{A'} \ = \ K\ \pi_{A'} \times (\pi_{0'})^{-5}\text{Þ}_c\tilde{\psi}_{0'0'0'0'} \ ,$$

along the α-lines on $\mathcal{C}_a$. Here Þ_c is the conformally invariant "thorn" operator defined in *Spinors and Space-Time*, Vol. 1 (Penrose and Rindler

1984) p. 395, which is a modified version of the covariant derivative operator $\nabla_{00'}$, and $\tilde{\psi}_{A'B'C'D'}$ is the (conformally invariant) helicity $+2$ massless field related to the SD Weyl spinor $\tilde{\Psi}_{A'B'C'D'}$ by

$$\tilde{\psi}_{A'B'C'D'} \;=\; \Omega^{-1}\tilde{\Psi}_{A'B'C'D'} \;,$$

where Ω is a *conformal factor* which is needed when we go to a new metric $\Omega^2 g$ which is regular on $\mathcal{I}^+$, where g is the given metric of $\mathcal{M}$. We shall require this for $\mathcal{T}^\infty$, though for $\mathcal{T}_a$ we can take $\Omega = 1$. The quantity K is a particular numerical constant whose value has not yet been determined, at the time of writing.

For the detailed meaning of all these quantities, see Penrose and Rindler (1986), Penrose (1999). Apart from the precise (as yet undetermined) value of K, the form of this equation is dictated by requirements of conformal invariance. The space $\mathcal{T}^\infty$ (and hence, each $\mathcal{T}_a$) has a structure determined from a 1-form ι and a 3-form θ (just given up to proportionality), subject to

$$\iota \wedge d\iota = 0 \;,\; \iota \wedge \theta = 0$$

and a further condition that can be given as

$$d\theta \otimes \iota \;=\; -2\theta \oslash d\iota$$

where the bilinear operator $\oslash$, acting between an n-form and a 2-form, is defined by

$$\eta \oslash (dp \wedge dq) \;=\; \eta \wedge dp \otimes dq - \eta \wedge dq \otimes dp \;.$$

In the original leg-break construction, the forms θ and ι provide the essential local structure of $\mathcal{T}$. In flat space we have

$$\begin{aligned} \iota &= \varepsilon_{A'B'}\pi^{A'} d\pi^{B'} \;, \\ \theta &= 1/6\; \varepsilon_{\alpha\beta\gamma\delta}\; Z^\alpha \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \;. \end{aligned}$$

Here, we merely have

$$\Pi \;=\; d\theta \otimes \iota \text{ and } \Sigma = d\theta \otimes d\theta \otimes \theta$$

(or something equivalent) as being specified as local structure assigned to $\mathcal{T}^+$. We also retain the condition $d\theta \otimes \iota = -2\theta \oslash d\iota$. For any particular choice of ι and θ, consistent with these relations, we can provide a definition of the "Euler vector field" $\Upsilon = \theta \div \phi$, and the projective space

$P\mathcal{T}^+$ is the factor space of $\mathcal{T}^+$ by the integral curves of Υ. For further details, see Penrose (1999).

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