Remarks on G_2-manifolds with boundary

Simon Donaldson

ABSTRACT. This article is based on the author's lecture at the Journal of Differential Geometry Conference, Harvard 2017. We discuss closed and torsion-free G_2-structures on a 7-manifold with boundary, with prescribed 3-form on the boundary. Much of the article is based on an observation that there is an intrinsic notion of "mean-convexity" for such boundary data. When the boundary data is mean-convex, classical arguments from Riemannian geometry can be applied. Another theme in the article is a connection with the maximal submanifold equation, in spaces of indefinite signature.

CONTENTS

1. Introduction

In this paper we discuss special geometric structures on manifolds of dimensions 6 and 7, and the connections between these arising in the case of a 7-manifold with boundary. Our approach is largely based on a seminal paper of Hitchin, published in the *Journal of Differential Geometry* [**6**] which emphasises differential forms and volume functionals (see also [**7**]). In dimension 6 the primary structure of interest is a Calabi-Yau structure, i.e. a Riemannian 6-manifold with holonomy contained in $SU(3)$ or, equivalently, a complex 3-manifold with a Kähler metric and a holomorphic 3-form of constant non-zero norm. In dimension 7, the primary structure of interest is a torsion free G_2-structure *i.e.* a Riemannian 7-manifold with holonomy contained in G_2. But in each case there are useful variants of these where we relax the holonomy condition and consider "closed" G_2 and $SL(3,\mathbf{C})$ structures. In Section 2 we review the basic theory and then discuss the elementary differential geometry of a hypersurface in a G_2 manifold. The main observation is a connection between the mean curvature of the hypersurface and the intrinsic geometry of the submanifold. In Section 3 we begin with simple remarks about the question of deforming a closed G_2-structure on a manifold with boundary to a torsion-free structure. Then we apply some classical Riemannian geometry to the case when the boundary is constrained to have positive mean curvature. In Section 4 we consider G_2-cobordisms and discuss connections with deformations of $SL(3,\mathbf{C})$-structures "tamed" by a symplectic form. We explain the possible relevance of G_2-cobordisms to questions of Torelli type for Calabi-Yau 3-folds. In Section 5 we consider related questions for maximal submanifolds, which arise as dimensional reductions and adiabatic limits of the special holonomy theory.

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2. Basics

2.1. Algebraic structures. We review the special features of 3-forms in 6 and 7 dimensions. First let V be a 6-dimensional real vector space.

DEFINITION 1. *A 3-form $\rho \in \Lambda^3 V^*$ is called definite if for each non-zero $v \in V$ the contraction $i_v(\rho) \in \Lambda^2 V^*$ has rank 4.*

The first basic fact is that if V has an orientation a definite 3-form ρ defines a complex structure I_ρ on V (and reversing the orientation changes I_ρ to $-I_\rho$). To see this, for $v \in V$ let N_v be the null space of $i_v(\rho)$ *i.e.*

$$N_v = \{v' \in V : i_{v'} i_v \rho = 0\}.$$

Clearly v is in N_v and v' is in N_v if and only if v is in $N_{v'}$. The condition that ρ is definite is that N_v has dimension 2, for all non-zero v. For each non-zero v' in N_v the form $i_{v'}(\rho)$ induces a non-degenerate symplectic form $\Omega_{v'}$ on V/N_v. We fix the orientation on V/N_v so that $\Omega_{v'}^2 > 0$; then the map $v' \mapsto \Omega_{v'}^2$ defines a conformal structure on the two-dimensional vector space N_v. If we are given an orientation of V we get an induced orientation on N_v, so we get a complex structure on N_v in the usual way. Then we define $I_\rho(v) \in N_v$ using this complex structure on N_v and it is clear that $I_\rho^2 = -1$ since $N_{I_\rho v} = N_v$.

The second basic fact is that the definite 3-forms form a single open orbit under the action of $GL(V)$. Any such form is equivalent to the standard model

$$\rho_0 = \mathrm{Re}(dz_1 dz_2 dz_3) \tag{1}$$

on $\mathbf{C}^3$, and the complex structure defined by ρ_0 is the standard one (for the standard orientation on $\mathbf{C}^3$). In other words, giving an oriented real 6-dimensional vector space V with a definite form ρ is equivalent to giving a 3-dimensional complex space with a non-zero complex form of type $(3,0)$ and the stabiliser in $GL^+(V)$ of ρ is isomorphic to $SL(3,\mathbf{C})$. To see this choose any non-zero v in V, let $v' = I_\rho(V)$ and set $\Omega = \Omega_v, \Omega' = \Omega_{v'}$. By definition these form an orthonormal pair of 2 forms on the 4-dimensional vector space V/N_v with respect to the wedge-product and it is well-known that such a pair defines a complex structure on V/N_v such that $\Omega - i\Omega'$ is a non-zero element of the complex exterior square. Thus there are complex co-ordinates z_2, z_3 on V/N_v such that

$$\Omega - i\Omega' = dz_2 dz_3.$$

Let $z_1 = x_1 + iy_1$ be a standard complex co-ordinate on N_v corresponding to the basis element v. Choose a complementary sub-space Q to N_v in V such that $\rho|_Q = 0$—it is easy to check from the definitions that these exist. Then z_1, z_2, z_3 become co-ordinates on V and it follows from the definitions that

$$\rho = dx_1 \wedge \mathrm{Re}(dz_2 dz_3) - dy_1 \wedge \mathrm{Im}(dz_2 dz_3) = \mathrm{Re}(dz_1 dz_2 dz_3).$$

Given the complex structure I_ρ defined by a definite form ρ and orientation we see that there is another definite form $\tilde{\rho}$ characterised by the fact that $\rho + i\tilde{\rho}$ is of type $(3,0)$. The volume element defined by ρ is

$$\mathrm{vol}_\rho = \frac{1}{4}\rho \wedge \tilde{\rho}, \tag{2}$$

and the first variation of the volume form, with respect to a variation $\delta\rho$ is

$$\delta\mathrm{vol}_\rho = \tfrac{1}{2}(\delta\rho) \wedge \tilde{\rho}. \tag{3}$$

We also have the variation in $\tilde{\rho}$. This is given by

$$\delta\tilde{\rho} = -i(\delta\rho)_{3,0} - i(\delta\rho)_{2,1} + i(\delta\rho)_{1,2} + i(\delta\rho)_{0,3}, \tag{4}$$

in terms of the decomposition of $\delta\rho$ into bi-type determined by the complex structure I_ρ.

Now let U be a 7-dimensional real vector space.

Definition 2. *A 3-form $\phi \in \Lambda^3 U^*$ is called definite if for each non-zero $u \in U$ the contraction $i_u(\phi) \in \Lambda^2 U^*$ has rank* 6.

Recall that a real 2-form ω on a complex vector space is called a *taming form* if $\omega(\xi, I\xi) > 0$ for all non-zero vectors ξ. This is equivalent to saying that the (1,1) component of ω is a positive $(1,1)$ form in the standard sense. For U as above fix a non-zero vector $\nu \in U$ and complementary subspace $V \subset U$, so we have a fixed isomorphism $U = \mathbf{R}\nu \oplus V$. We can write any $\phi \in \Lambda^3 U^*$ as

$$\phi = \omega \wedge dt + \rho \tag{5}$$

where $\omega \in \Lambda^2 V^*, \rho \in \Lambda^3 V^*$ and $dt \in U^*$ is dual to ν. It is easy to check from the definitions that ϕ is definite if and only if ρ is a definite 3-form on V and ω is a taming form for the complex structure induced on V by ρ and one of the orientations on V. Now suppose that U has a fixed orientation. We say that a definite form ϕ is *positive* if, in the description above, ω is a taming form for the complex structure defined by ρ and the induced complex structure on V. By continuity this is the independent of the choice of vector ν and complementary subspace V. It is also equivalent to saying that

$$(i_u\phi)^2 \wedge \phi > 0 \tag{6}$$

for all non-zero $u \in U$. Then the expression on the left hand side of (6) defines a conformal structure on U and a Euclidean structure g_ϕ in this conformal class can be fixed by requiring that $|\phi|^2_{g_\phi} = 7$. The condition that ν is orthogonal to V in terms of the representation (5) is that $\omega \wedge \rho = 0$, or equivalently that ω has type $(1,1)$ with respect to the complex structure on V. The condition that ν has length 1 is that $\frac{1}{6}\omega^3 = \mathrm{vol}_\rho$. It follows that any positive form is equivalent to the model on $\mathbf{R} \oplus \mathbf{C}^3$

$$\phi_0 = \omega_0 dt + \rho_0,$$

where $\omega_0 = dx_1dy_1 + dx_2dy_2 + dx_3dy_3$ is the standard symplectic form on $\mathbf{C}^3$ and $\rho_0 = \mathrm{Re}(dz_1, dz_2dz_3)$, as above. Thus the positive forms make up a single orbit for the action of $GL^+(U)$ on $\Lambda^3 U^*$.

Given a positive 3-form ϕ as above we get a 4-form $*_\phi\phi \in \Lambda^4 U^*$, where $*_\phi$ is the $*$-operator defined by the Euclidean structure g_ϕ.

2.2. Hypersurface geometry. Now we turn to differential geometry and we recall two basic facts.

1. If N is an oriented 6-manifold and ρ is a 3-form on N which is definite at every point then the induced almost-complex structure I_ρ is integrable if and only if $d\rho = 0$ and $d\tilde{\rho} = 0$.
2. If M is an oriented 7-manifold and ϕ is a 3-form on M which is positive at each point then ϕ is covariant constant with respect to the Levi-Civita connection of the Riemannian metric g_ϕ induced by ϕ if and only if $d\phi = 0$ and $d *_\phi \phi = 0$. In this case we say that ϕ is a torsion-free G_2-structure on M.

In this paper we want to consider relaxing these conditions so we say that:

- a *closed* $SL(3,\mathbf{C})$*-structure* on an oriented 6-manifold is given by a definite 3-form ρ with $d\rho = 0$;
- a *closed* G_2*-structure* on an oriented 7-manifold is given by a positive 3-form ϕ with $d\phi = 0$.

LEMMA 1. *If ρ is a closed $SL(3,\mathbf{C})$-structure on an oriented 6-manifold N then the 4-form $d\tilde{\rho}$ has type $(2,2)$ with respect to the almost-complex structure I_ρ.*

This is clear from Hitchin's variational point of view. Any definite form ρ defines a volume form vol_ρ as above. By (3) the variation of the volume with respect to a compactly supported variation $\delta\rho$ of ρ is

$$\int_N \delta\rho \wedge \tilde{\rho}.$$

Let v be a compactly-supported vector field on N and $\delta\rho$ be the variation given by the Lie derivative $\mathcal{L}_v\rho$. Since ρ is closed this is $d(i_v\rho)$. Diffeomorphism invariance of the volume implies that

$$\int_N d(i_v\rho)\tilde{\rho} = \int_N i_v\rho \wedge d\tilde{\rho} = 0.$$

Since this is true for all v we must have $i_v\rho\wedge d\tilde{\rho} = 0$, pointwise for all tangent vectors v, which is just the condition that $d\tilde{\rho}$ has type $(2,2)$.

One of the main observations in this paper is that, as a form of type $(2,2)$, there is a notion of *positivity* of the tensor $d\tilde{\rho}$. In fact, using the volume form, the 4-forms of type $(2,2)$ can be identified with the Hermitian forms on T^*N, for which we have the standard notion of positivity. Any $(2,2)$-form σ can be written in suitable co-ordinates at a point as

$$-\tfrac{1}{4}(\lambda_1 dz_2 d\overline{z}_2 dz_3 d\overline{z}_3 + \lambda_2 dz_3 d\overline{z}_3 dz_1 d\overline{z}_1 + \lambda_3 dz_1 d\overline{z}_1 dz_2 d\overline{z}_2). \tag{7}$$

The form is positive if each $\lambda_i > 0$. We also consider the weaker notion of *semipositivity*, by which we mean that all λ_i are non-negative and at least one is strictly positive. Equivalently, a $(2,2)$ form σ is semi-positive if $\omega\wedge\sigma > 0$ for all positive $(1,1)$-forms ω. We will say that a closed $SL(3,\mathbf{C})$ structure ρ is *mean-convex* if $d\tilde{\rho}$ is semi-positive at each point, and *strictly mean-convex* if it is positive at each point. Likewise for *mean-concave*. Changing the sign of the almost-complex structure interchanges the two conditions.

There is a scalar invariant $\det(\sigma)$ of a $(2,2)$-form σ on a manifold N with $SL(3,\mathbf{C})$ structure. To define this we use the volume form to identify σ with an element $\underline{\sigma}$ of $\Lambda^2 TN$, then take $\underline{\sigma}^3 \in \Lambda^6 TN$ and multiply by the $\frac{1}{6}$ of the volume form, to get a scalar (the factor being a convenient normalisation). In terms of the explicit representation (7) in standard co-ordinates at a point we have $\det\sigma = \lambda_1\lambda_2\lambda_3$. Thus for a strictly mean-convex structure ρ the function $\det(d\tilde{\rho})$ is strictly positive.

Now consider an oriented 7-manifold M with torsion-free G_2-structure ϕ and a 6-dimensional submanifold $N \subset M$. By the discussion in 2.1 above the restriction of ϕ to N is a closed definite 3-form ρ. One first observation is that the induced Riemannian measure on N coincides with $|\mathrm{vol}_\rho|$, where the choice of sign of the volume form vol_ρ depends on an orientation of N, or equivalently a co-orientation of $N \subset M$.

Fix a choice of unit normal vector field ν. From standard Riemannian theory we have at each point $p \in N$ the second fundamental form $B \in s^2 T_p^* N$. On the other hand we have an induced $SL(3,\mathbf{C})$ structure ρ on N and a 2-form ω on N given by the contraction $i_\nu(\phi)$. The 2-form ω is a positive $(1,1)$-form with respect to the almost-complex structure defined by ρ. Using this almost-complex structure we write B as a sum $B = B_{1,1} + B_{\mathbf{C}}$ where $B_{1,1}$ is the real part of a Hermitian form and $B_{\mathbf{C}}$ is the real part of a complex quadratic form. The component $B_{1,1}$ is somewhat analogous to the Levi form of a real hypersurface in a complex Kähler manifold. Using the standard identification we let $\beta_{1,1}$ be the $(1,1)$ form on N corresponding to the Hermitian form $B_{1,1}$. The mean curvature μ of N in M is the trace of B which can be written as

$$\mu = \beta_{1,1} \wedge \omega^2 \ (\mathrm{vol})^{-1}. \tag{8}$$

PROPOSITION 1. *In this situation we have* $\beta_{1,1} \wedge \omega = \frac{1}{2} d\tilde{\rho}$, *so*

$$\mu = \tfrac{1}{2}(d\tilde{\rho} \wedge \omega) \ (\mathrm{vol})^{-1}.$$

In particular, if the induced $SL(3, \mathbf{C})$*-structure* ρ *is mean convex then the mean curvature* μ *is positive with respect to the normal direction* ν.

REMARKS.

- The point of the Proposition is that it relates $d\tilde{\rho}$, which is an intrinsic invariant of the structure on N, to the mean curvature which is an extrinsic invariant of the submanifold $N \subset M$.
- If ρ is strictly mean convex there is a stronger statement

$$\mu \geq \tfrac{3}{2} \det(d\tilde{\rho})^{1/3}. \tag{9}$$

  In terms of a representation (7) in standard co-ordinates at a point this is the arithmetic-geometric mean inequality for the λ_i.

The proof of Proposition 1 is a straightforward calculation which can be done in various ways. For one approach we first observe that, since a torsion-free structure agrees with the flat model to order two at a point, it suffices to consider the case when M is the flat model $\mathbf{R} \times \mathbf{C}^3$ as in (2.1). We take N to be the graph of a function $f : \mathbf{C}^3 \to \mathbf{R}$ with f and df vanishing at the origin. The induced 3-form (pulled back to $\mathbf{C}^3$) is given by

$$\rho = \rho_0 + df \wedge \omega_0.$$

Then the formula (4) for the variation of $\tilde{\rho}$ shows that

$$\tilde{\rho} = \tilde{\rho}_0 - I(df \wedge \omega_0) + O(|z|^2),$$

where I acts as $+i$ on $\Lambda^{2,1}$ and $-i$ on $\Lambda^{1,2}$. So, at the origin

$$d\tilde{\rho} = -d(I df \wedge \omega_0) = (-2i\overline{\partial}\partial f) \wedge \omega_0.$$

On the other hand, in the familiar way the second fundamental form B at the origin is given by the Hessian of f and the $(1,1)$ component $\beta_{1,1}$ is $-i\overline{\partial}\partial f$, hence the desired formula.

We give an another approach to the calculation for the mean curvature. Let g be a function of compact support on N. This gives a variation vector field $g\nu$ and the variation of the induced Riemannian volume of N is

$$\delta \mathrm{Vol} = \int_N g\ \mu\ \mathrm{vol}_N$$

As we observed at the beginning of this subsection, for any hypersurface in M the Riemannian volume coincides with the volume computed from the induced $SL(3,\mathbf{C})$ structure. The variation in the induced 3-form ρ is the Lie derivative $-\mathcal{L}_{g\nu}(\phi)$ and since ϕ is closed this is

$$\delta\rho = -d(g i_\nu \phi) = -d(g\omega),$$

so by (3) the variation in the volume is

$$\delta \mathrm{Vol} = -\tfrac{1}{2}\int_N d(g\omega)\wedge\tilde{\rho} = \tfrac{1}{2}\int_N (g\omega)\wedge d\tilde{\rho},$$

and since this true for all g we must have $\mu \mathrm{vol}_N = \frac{1}{2}\omega\wedge d\tilde{\rho}$. This derivation has the advantage that it shows the formula for the mean curvature applies for hypersurfaces in 7-manifolds with *closed* G_2-structures (but the formula for $\beta_{1,1}$ in Proposition 1 then acquires an extra, trace-free, term).

For completeness we also give a formula for the component $B_{\mathbf{C}}$ of the second fundamental form. Contraction of vectors with $\rho - i\tilde{\rho}$ defines an isomorphism $\Lambda^{0,2} = \overline{TN}$, hence $\Lambda^{1,2} = TN^* \otimes \overline{TN}$. Using the Hermitian metric we identify $\overline{TN}$ with T^*N so $\Lambda^{1,2} = T^*N\otimes T^*N$. In particular we have an embedding of the symmetric tensors $s^2(T^*N)\subset\Lambda^{1,2}$. (These are the primitive (1,2)-forms.) The component $B_{\mathbf{C}}$ of the second fundamental form is the real part of an element of $s^2(T^*N)$, so using these identifications it corresponds to a form $\beta_{1,2}\in\Lambda^{1,2}$. Then we have

$$d\omega = \tfrac{1}{2}\mu\rho - \tfrac{1}{2}\beta_{1,2} - \tfrac{1}{2}\overline{\beta_{1,2}}. \tag{10}$$

We leave the verification to the reader.

2.3. Examples.

1. Take M to be $\mathbf{R}^7$ with the standard flat G_2-structure and N to be the 6-sphere with outward-pointing normal. The induced $SL(3,\mathbf{C})$ structure is strictly mean convex and the corresponding almost-complex structure on S^6 is the standard one. More generally, recall that a *nearly Kähler* structure on an oriented 6-manifold N is given by a closed definite 3-form ρ and a 2-form ω which is a positive $(1,1)$-form with respect to the almost complex structure and satisfying

   $$\tfrac{1}{6}\omega^3 = \mathrm{vol}_\rho \quad d\tilde{\rho} = 2\omega^2 \quad d\omega = 3\rho.$$

   So the $SL(3,\mathbf{C})$-structure is mean-convex. A nearly Kähler-structure defines a conical torsion-free G_2-structure on $(0,\infty)\times N$ with

   $$\phi = r^3\rho + r^2 dr\wedge\omega.$$

   In particular, we recover the flat structure on $\mathbf{R}^7$ from the 6-sphere in this way.

2. We say that an $SL(3,\mathbf{C})$ structure ρ on N is *tamed* by a symplectic form Ω if Ω has positive $(1,1)$ component (as in 2.1 above). Then on a compact manifold N,
$$\int_N d\tilde{\rho}\wedge\Omega = \int_N \tilde{\rho}\wedge d\Omega = 0.$$
It follows that a mean-convex $SL(3,\mathbf{C})$ structure on N does not admit a taming form.
3. We consider a dimensional-reduction related to a construction of Baraglia [1]. Let $T^4 = \mathbf{R}^4/\mathbf{Z}^4$ be the 4-torus and identify $H^2(T^4)$ with $\mathbf{R}^{3,3}$, the indefinite quadratic form defined by cup-product. We also regard $\mathbf{R}^{3,3}$ as the space of constant 2-forms on T^4. Let $\Sigma\subset\mathbf{R}^{3,3}$ be a "space-like" surface (i.e. a 2-dimensional submanifold on which the quadratic form restricts to a Riemannian metric). Then we have a canonical 3-form ρ_Σ on $\Sigma\times T^4$. If we write $f:\Sigma\to\mathbf{R}^{p,q}$ for the inclusion map then, in terms of local co-ordinates s_1, s_2 on Σ,
$$\rho_\Sigma = -\left(\frac{\partial f}{\partial s_1}ds_1 + \frac{\partial f}{\partial s_2}ds_2\right), \tag{11}$$
where the partial derivatives are interpreted as constant co-efficent 2-forms on T^4. In other words $\rho_\Sigma = -df$, where the right hand side is interpreted as a 3-form on $\Sigma\times T^4$. The submanifold $\Sigma\subset\mathbf{R}^{3,3}$ has a mean curvature vector μ_Σ which is normal to $T\Sigma$. Suppose that, at a point, μ_Σ is a spacelike vector in $\mathbf{R}^{3,3}$. Then $T\Sigma+\mathbf{R}\mu_\sigma$ is a maximal positive subspace in $\mathbf{R}^{3,3}$ and as such has a canonical orientation. Thus if we are given an orientation of Σ it makes sense to say that μ_Σ is "outward pointing".

LEMMA 2. *In this situation*
$$d\tilde{\rho}_\Sigma = \mathrm{vol}_\Sigma\otimes\mu_\Sigma,$$
where vol_Σ *is the induced area 2-form on* Σ *and* μ_Σ *is viewed as a constant coefficient 2-form on* T^4*. The 3-form* ρ_Σ *is mean-convex if and only if* μ_Σ *is space-like and outward pointing.*

We leave the proof as an exercise for the reader. Note that in this situation $d\rho_\Sigma$ has rank at most 1, so ρ_Σ is never strictly mean-convex.

Let $\Xi\subset\mathbf{R}^{3,3}$ be a 3-dimensional space-like submanifold, with inclusion map $F:\Xi\to\mathbf{R}^{3,3}$. For the same reason as above it inherits an orientation, so we have an induced volume form χ. We define a 3-form on $\Xi\times T^4$:
$$\phi_\Xi = -dF + \chi,$$
using notation as above. Then Baraglia shows that this defines a torsion-free G_2-structure if and only if Ξ is a *maximal submanifold*

(that is, a stationary point for the induced volume function, with respect to compactly supported variations in Ξ). For a surface $\Sigma \subset \Xi$, Proposition 1 amounts to the elementary statement that if μ_Σ is spacelike and outward pointing then the mean curvature of $\Sigma \subset \Xi$ is positive.

4. We consider a different dimension reduction, as in [4]. This time we take a flat 3-torus $\mathbf{R}^3/\mathbf{Z}^3$ and an oriented 3-manifold Y. Let $\sigma_1, \sigma_2, \sigma_3$ be closed 2-forms on Y which are linearly independent at each point and let θ_i be co-ordinates on $\mathbf{R}^3$. Then we have a definite 3-form

$$\rho = d\theta_1 d\theta_2 d\theta_3 - \sum \sigma_i d\theta_i$$

on the 6-manifold $Y \times T^3$. By elementary linear algebra there is a unique basis of 1-forms ϵ_i such that

$$\sigma_i = \epsilon_j \wedge \epsilon_k,$$

for cyclic permutations (ijk) of (123). One finds that

$$\tilde{\rho} = -\epsilon_1 \epsilon_2 \epsilon_3 + \sum_{\text{cyclic}} \epsilon_i d\theta_j d\theta_k. \tag{12}$$

The condition that σ_i are closed means that

$$d\epsilon_i = \sum S_{ij} \sigma_j,$$

where (S_{ij}) is a symmetric matrix and we have

$$d\tilde{\rho} = \sum S_{ai} \sigma_a \wedge d\theta_j d\theta_k$$

(where a ranges over $1, 2, 3$ and (ijk) over cyclic permutations). The condition that ϕ is mean-convex is that S is a nonnegative matrix (and not 0).

3. Manifolds with boundary

3.1. Gluing closed forms. The main focus of this paper is a compact oriented 7-manifold M with boundary N and a given closed $SL(3, \mathbf{C})$-structure ρ on N. In addition we consider an "enhancement" of ρ which is an equivalence class of closed 3-forms over M equal to ρ on N under the equivalence relation $\psi \sim \psi + d\alpha$ where α is a 2-form vanishing on N. Thus the existence of an enhancement is the condition that the cohomology class $[\rho]$ extends to $H^3(M)$ and the difference of two enhancements is naturally an element of $H^3(M, N)$. We write $\hat{\rho}$ for an enhancement class. Then we have two existence questions.

- Is there a closed G_2-structure with boundary value $\hat{\rho}$?
- Is there a torsion-free G_2-structure with boundary value $\hat{\rho}$?

There are also corresponding uniqueness questions. In [5] we showed that the second question, modulo diffeomorphisms fixing the boundary pointwise, corresponds to an elliptic boundary value problem of index 0.

The next Proposition is a reflection of the fact that closed G_2-structures form a more flexible class than the torsion-free structures. Let M_1, M_2 be oriented 7-manifolds with boundary (not necessarily compact) and suppose that N_1, N_2 are compact components of the boundary. Write $\iota_i : N_i \to M_i$ for the inclusion maps. Suppose that there is a diffeomorphism $\gamma : N_1 \to N_2$ which is orientation reversing (for the orientations induced from M_i). Then in the standard way we can form a manifold $M_1 \sharp_\gamma M_2$ by gluing the boundary components N_1, N_2 using γ.

Proposition 2. *Suppose that ϕ_1, ϕ_2 are closed G_2-structures on M_1, M_2 and that $\iota_1^*(\phi_1) = \gamma^* \iota_2^*(\phi_2)$. Then there is closed G_2 structure ϕ on $M_1 \sharp_\gamma M_2$. Moreover ϕ can be chosen arbitrarily close to ϕ_i outside an arbitrarily small neighbourhood of N_i.*

We sketch a proof. Assume for simplicity that M_i are compact with just one boundary component N_i, so $M = M_1 \sharp_\gamma M_2$ is a closed manifold. We have an L^∞ 3-form Φ_0 on M, defined to be equal to ϕ_i on int $M_i \subset M$. It follows from the hypothesis and Stokes' formula that $d\Phi_0 = 0$ in the weak sense. Let K_ϵ be the operators defining the 1-parameter heat semigroup for the Hodge Laplacian on 3-forms on M (with some choice of Riemannian metric) and for $\epsilon > 0$ set $\Phi_\epsilon = K_\epsilon \Psi_0$. Then Φ_ϵ is a smooth, closed 3-form and the only point to check is that Φ_ϵ is positive. Near the boundary $N_1 \subset M_1$ we choose a collar neighbourhood with normal co-ordinate $t \in [0, \delta)$ so that the 3-form is

$$\rho_t + \omega_t dt, \tag{13}$$

where ρ_t, ω_t are t-dependent forms on N_1. The positivity condition is that ρ_t is definite and that ω_t has positive $(1,1)$-component for the almost-complex structure defined by ρ_t. Gluing the corresponding representation of ϕ_2, we can write Φ_0 in the same form (13), but now with t in an interval $(-\delta, \delta)$ and piecewise-smooth forms ρ_t, ω_t. The 3-forms ρ_t are continuous across $t = 0$ but ω_t has a jump discontinuity. The key point now is that the set of 2-forms with positive $(1,1)$ part (with respect to a fixed complex structure) forms a convex cone. In this description, the action of K_ϵ is given, to a very good approximation, by a positively weighted average of the forms at nearby points. It follows easily from this that $K_\epsilon \Phi_0$ is a positive 3-form for small ϵ.

Alternatively, we can construct suitable smoothing operators like K_ϵ by explicit local formulae, thus avoiding use of any analytical theory of the heat equation. Such a construction also applies when M_i are not compact, or have additional boundary components. Moreover we can arrange that ϕ is exactly equal to ϕ_i outside arbitrarily small neighbourhoods of N_i.

3.2. A fundamental difficulty. Going back to the general questions at the beginning of Section 3.1: the most optimistic, naive, hope would be that any closed G_2-structure can be deformed, through closed G_2-structures with fixed boundary data, to a torsion-free structure. There are various reasons why this cannot be true and we will discuss one such difficulty in this subsection.

For small $\lambda > 0$ let Ω_λ be a bounded domain with smooth boundary in $\mathbf{R}^7 = \mathbf{C}^3 \times \mathbf{R}$ which is diffeomorphic to a ball and which near the origin is given by

$$\{(z,t) \in \mathbf{C}^3 \times \mathbf{R} : 0 < t < |z|^2 + \lambda\}. \tag{14}$$

The boundary of Ω_λ is an embedded sphere $\iota_\lambda : S^7 \to \mathbf{R}^7$ and the embeddings can clearly be taken to have a smooth limit ι_0 which is an immersion. Let ϕ be the standard flat G_2-structure on $\mathbf{R}^7$ and $\rho^{(\lambda)} = \iota_\lambda^*(\phi)$ for $\lambda \geq 0$. Then for $\lambda > 0$ the 3-form $\rho^{(\lambda)}$ is the boundary value of a torsion-free G_2-structure on B^7 but these have no smooth limit as λ tends to 0. On the other hand we will show that $\rho^{(0)}$ is the boundary value of a closed G_2 structure on the ball.

Let $\pi : \mathbf{C}^3 \times \mathbf{R} \to \mathbf{C}^3 \times \mathbf{R}$ be the map $\pi(z,t) = (z, |z|^2 t)$ and let Φ be the 3-form $\Phi = \pi^*\phi$ on a region

$$U_\kappa = \{(z,t) \in \mathbf{C}^3 \times \mathbf{R} : |z| \leq \kappa, 0 \leq t \leq 1\}.$$

Thus

$$\Phi = \rho_0 + t(d|z|^2)\omega_0) + |z|^2\omega_0 dt.$$

We can fix a small $\kappa > 0$ so that, for this range of t and $|z|$, the 3-form $\rho_0 + t(d|z|^2)\omega_0$ is definite on $\mathbf{C}^3$ and ω_0 has positive $(1,1)$ part with respect to this form. Then Φ is a positive 3-form on U_κ except for the points where $z = 0$. Let η be a 1-form on $\mathbf{C}^3$ with $d\eta = \omega_0$ and let χ be a standard cut-off function on $\mathbf{C}^3$, vanishing for $|z| \geq \kappa/2$ and equal to 1 for $|z| \leq \kappa/4$. For small $\epsilon > 0$ let

$$\Phi_\epsilon = \Phi + \epsilon d(\chi \eta dt).$$

So

$$\Phi_\epsilon = \omega_0(|z|^2 + \epsilon\chi)dt + \epsilon d\chi \wedge \eta dt + (\rho_0 + t(d|z|^2).$$

A moments thought shows that Φ_ϵ is a closed positive 3-form on U_κ for small ϵ. By construction, Φ_ϵ has the same boundary values as Φ on the boundaries $t = 0, 1$ and agrees with Φ for $|z| > \kappa/4$. Now it is clear that we can choose a smooth map $I_0 : B^7 \to \mathbf{R}^7$ extending the immersion ι_0 and choose a region $\tilde{U} \subset B^7$ such that there is a diffeomorphism $h : \tilde{U} \to U_\kappa$ with $I_0 = \pi \circ h$ on $\tilde{U}$ and such that I_0 is an immersion outside $\tilde{U}$. Then the 3-form ϕ which is equal to $h^*\Phi_\epsilon$ on $\tilde{U}$ and to $I^*(\phi_0)$ outside $\tilde{U}$ is a closed G_2-structure on B^7 with boundary value $\rho^{(0)}$.

This example does not completely rule out the possibility that $\rho^{(0)}$ is the boundary value of a torsion-free G_2-structure (because there could be some other solution which is not the limit of the flat solutions for $\lambda > 0$), but it seems unlikely that this happens. In any case this phenomenon—of different parts of the boundary coming together—will be a serious problem in any kind of existence theory.

3.3. Some Riemannian geometry. In this subsection we consider a compact Riemannian manifold X of dimension $(n+1)$ with smooth boundary Y such that

- The Ricci curvature of X is non-negative;
- The mean curvature μ of the boundary (with respect to the outward pointing normal) is bounded below by a positive constant μ_0.

We recall four results, each of a standard nature, which hold in this situation. Let $\mathcal{P}$ be the set of smooth maps $\gamma : [0,1] \to X$ with $\gamma(0), \gamma(1) \in Y$ but with $\gamma(t)$ in the interior of X for $0 < t < 1$. For $\delta > 0$ we write $\mathcal{P}_\delta$ for the subset of $\mathcal{P}$ given by paths of length at most δ.

Proposition 3. *For any path γ in $\mathcal{P}$ there is a small variation in $\mathcal{P}$ which decreases the length.*

By considering the first variation it suffices to consider the case when γ is a geodesic which is normal to the boundary at the end points. Take an orthonormal frame of $TY_{\gamma(0)}$ and parallel transport these along γ to get variation vector fields v_i. The second variation of arc length under the variation v_i (adapted to lie in $\mathcal{P}$ in the obvious way) is

$$-\int_\gamma K(v_i, \gamma') - B_{\gamma(0)}(v_i(0)) - B_{\gamma(1)}(v_i(1)),$$

where B is the second fundamental form of the boundary and $K(\ )$ is the sectional curvature. Summing over i, the sum of the second variations is

$$-\int_\gamma \mathrm{Ric}(\gamma') - \mu(\gamma(0)) - \mu(\gamma(1)) < 0,$$

so at least one of the variations decreases length.

Proposition 4. *Let δ be the minimum length of a geodesic segment in $\mathcal{P}$ which is orthogonal to Y at $\gamma(0)$. Then any path in $\mathcal{P}_\delta$ can be contracted to a point through paths in $\mathcal{P}_\delta$.*

This follows from the previous result and an argument of Morse-theory type.

Proposition 5. *The distance of any point of X to the boundary Y is at most $n\mu_0^{-1}$.*

Let x_0 be a point in the interior of X and γ minimise length among paths from x_0 to the boundary. Let v_i be a parallel orthonormal frame along γ as before and consider the variation vector fields tv_i (where we assume that γ is parametrised by arc-length). The second variation formula shows that if the sum of the second variations is positive then the length of γ is at most $n\mu_0^{-1}$. (Equality holds when X is a ball in $\mathbf{R}^{n+1}$ with centre x_0).

Proposition 6. $\mathrm{Vol}(X) \leq \frac{n}{(n+1)\mu_0}\mathrm{Vol}(Y)$

This follows from a variant of the Bishop comparison inequality, see [8].

The relevance of these results for our purposes is that the hypotheses are satisfied for $(X, Y) = (M, N)$ where M has a torsion-free G_2-structure and the boundary $SL(3, \mathbf{C})$ structure on N is strictly mean-convex. We define

$$m(\rho) = \min_N (\det\, d\tilde{\rho})^{1/3}$$

so $\mu(\rho) > 0$ and by (9) we can take $\mu_0 = \frac{3}{2} m(\rho)$

- Proposition 3 shows, roughly speaking, that the phenomenon discussed in 3.1 cannot occur for mean-convex boundary data. So one can be more optimistic about an existence theory for torsion-free G_2-structures with prescribed boundary data in the case when this boundary data is mean-convex.
- Proposition 6 gives an upper bound

$$\mathrm{Vol}(M) \leq \frac{4}{7m(\rho)} \mathrm{Vol}(N). \tag{15}$$

The point here is that the right hand side is entirely determined by the $SL(3, \mathbf{C})$ structure on N. Note that equality holds when M is a ball in $\mathbf{R}^7$. There are reasons to expect that a torsion-free G_2-structure maximises the volume among all closed G_2 structures with given boundary data (see the discussion in [5]). This raises the question whether the inequality (15) is true for *closed* G_2-structures on M, with strictly mean-convex boundary.

There is a variant of this discussion for submanifolds, related to Example 3 in 2.3. Let $\Sigma \subset \mathbf{R}^{p,q}$ be an oriented space-like $(p-1)$-dimensional submanifold with spacelike, outward-pointing, mean curvature μ_Σ. Suppose that Σ is the oriented boundary of a spacelike submanifold Ξ. Let ν be the outward pointing unit normal to Σ in Ξ. Then the mean curvature μ of Σ in Ξ is $\langle \mu_\Sigma, \nu \rangle$. Now we have an elementary inequality

$$\langle \mu_\Sigma, \nu \rangle \geq \|\mu_\Sigma\| = \sqrt{\langle \mu_\Sigma, \mu_\Sigma \rangle}.$$

If Ξ is a maximal submanifold the Ricci curvature of the induced metric is non-negative and we deduce from Proposition 6 that

$$\mathrm{Vol}(\Xi) \leq \frac{p-1}{p} \mathrm{Vol}(\Sigma) \left(\min_\Sigma \|\mu_\Sigma\| \right)^{-1}. \tag{16}$$

(With equality for a standard ball in $\mathbf{R}^p \subset \mathbf{R}^{p,q}$.) The question that arises is whether this holds for *any* spacelike Ξ with boundary Σ.

4. G_2-cobordisms

In this section we consider a pair of compact, connected, 6-manifolds with closed $SL(3,\mathbf{C})$ structures $(N_0,\rho_0),(N_1,\rho 1)$ and a cobordism M from N_0 to N_1 with closed or torsion-free G_2 structure ϕ restricting to ρ_i on the boundary. Proposition 2 shows that the existence of a closed G_2-cobordism defines a transitive relation on $SL(3,\mathbf{C})$ structures, but the orientations in the set-up mean that this relation is not symmetric (or at least, not in an obvious way).

Our main focus is on the case when N_0, N_1 are diffeomorphic (so we just write N) and M is a product, as a smooth manifold. Choosing such a product structure we can express a 3-form ϕ in the usual way as $\rho_t + \omega_t dt$. The existence of a closed G_2-cobordism is equivalent to the existence of a path ρ_t from ρ_0 to ρ_1 through closed $SL(3,\mathbf{C})$-structures on N such that

$$\frac{d\rho_t}{dt} = d\omega_t \tag{17}$$

where ω_t has positive $(1,1)$ part with respect to ρ_t. Of course we need to assume that ρ_0,ρ_1 define the same cohomology class in $H^3(N)$. If we ignore the positivity condition it is known that we can find some path ρ_t of $SL(3,\mathbf{C})$-structures. In [2], Crowley and Nordström show that "co-closed" G_2-structures on 7-manifolds obey an h-principle and the same arguments apply to closed $SL(3,\mathbf{C})$-structures [9]. (The main point is that any hypersurface in $\mathbf{R}^7$ acquires a closed $SL(3,\mathbf{C})$-structure, just as any hypersurface in $\mathbf{R}^8$ acquires a co-closed G_2-structure.) Easy bundle theory considerations show that ρ_0,ρ_1 are homotopic as definite 3-forms and the h-principle shows that these forms can be taken to be closed, in a fixed cohomology class.

Example. Consider the standard closed definite form ρ on S^6. Then $-\rho$ is also a closed definite form and there is an obvious homotopy

$$\rho_t = \cos(\pi t)\rho + \sin(\pi t)\tilde{\rho},$$

through definite forms, but these are not closed. The Crowley-Nordström theory shows that there is some homotopy through closed definite forms. Note that such a homotopy cannot be invariant under G_2, acting on S^6. It is interesting to ask whether there is a closed G_2-cobordism from ρ to $-\rho$ (or from $-\rho$ to ρ).

4.1. Taming forms and cobordisms. There is a further connection between homotopy of definite forms and G_2-cobordism in the presence of a taming form.

Lemma 3. *Let Ω be a symplectic form on N. If ρ_0,ρ_1 can be joined by a path ρ_t of closed definite 3-forms in a fixed cohomology class such that ρ_t is tamed by Ω for each t then there is a closed G_2-cobordism from ρ_0 to ρ.*

The proof is easy: the hypotheses mean that we can find closed, Ω-tamed, $SL(3,\mathbf{C})$ structures ρ_t and 2-forms $\tilde{\omega}_t$ such that $d\tilde{\omega}_t$ is the t-derivative of ρ_t. Then we set $\omega_t = \tilde{\omega}_t + A\Omega$ for large A.

We say that a closed G_2-cobordism is tamed by Ω if there is a product structure $M = N \times [0,1]$ with respect to which ρ_t is tamed by Ω for all t.

There is a more precise statement of this Lemma, involving the "enhancement" of the boundary data. For simplicity we consider the case when $H^2(N) = \mathbf{R}$ and fix a pair of 2-cycles S_0, S_1 representing a generator of $H_2(N)$. Also fix a 3-chain $W \subset M = N \times [0,1]$ with boundary $-S_0$ in one end and S_1 in the other. So for any closed form ϕ with boundary values ρ_0, ρ_1 we have a real number

$$I_W(\phi) = \int_W \phi,$$

which depends only on the relative homology class of W. The more refined question is to ask for which values of I_W (if any) is there a closed G_2-cobordism from ρ_0 to ρ_1. In the presence of a symplectic form Ω as above, fix the sign of S_1 so that $\langle \Omega, S_1 \rangle > 0$. Then

$$\int_W \Omega dt = \int_W d(t\Omega) = \int_{S_1} \Omega > 0.$$

The more precise statement of Lemma 3 is that, under the hypotheses of the Lemma, there is a κ_0 such that for all $\kappa \geq \kappa_0$ there is a closed G_2-cobordism ϕ from ρ_0 to ρ_1 with $I_W(\phi) = \kappa$. Motivated by this we can define an invariant $D_W(\rho_0, \rho_1)$ of a pair ρ_0, ρ_1 to be the infimum of the values of $I_W(\phi)$ such that there is a closed G_2-cobordism ϕ from ρ_0 to ρ_1 (and $+\infty$ if this set is empty). Of course this depends on the choice of W, but the invariants given by different choices are related by the addition of a constant determined by homological considerations.

There is a potential connection between these ideas and the enumerative geometry of holomorphic curves in N. The appropriate theory would probably be an extension of the "Donaldson-Thomas" invariants to the symplectic case and, since such a theory has not so far been set up rigourously, we only sketch the idea. Suppose, in the simplest situation, that there is a single holomorphic curve for the almost complex structure defined by ρ_0 in the homology class $[S_0]$ so we take S_0 to be this holomorphic curve. Similarly suppose that there is single holomorphic curve in this homology class for the almost complex structure defined by ρ_1 and take S_1 to be that curve. Suppose further that throughout the 1-parameter family ρ_t there is just a single holomorphic curve S_t, giving a smoothly varying family from S_0 to S_1. This family defines a cycle W in $N \times [0,1]$ and we have

$$I_W(\phi) = \int_0^1 \left(\int_{S_t} \omega_t \right) dt. \tag{18}$$

The derivation of this equation uses the fact that for any tangent vector v to N at a point of S_t the contraction $i_v(\rho_t)$ vanishes on the tangent space of S_t, which is a characterisation of holomorphic curves in this setting. The point now is that $I_W(\phi) > 0$ since ω_t has positive $(1,1)$ component and its integral over a curve is positive.

Of course the situation above cannot be expected to hold in general. However we do have compactness results for holomorphic curves in the case of tamed structures and the discussion can be extended. For example we might have a finite number of holomorphic curves, with respect to ρ_i, in the given homology class and we then take the cycles S_i to be sum of these, with suitable signs. But we will not try to go into further details here. The general point is that we can hope that there are preferred chains W which impose a constraint $I_W(\phi) > 0$, at least for tamed, closed, G_2-cobordisms.

4.2. More Riemannian geometry and questions of Torelli type. Now consider a compact Riemannian manifold with boundary which gives a cobordism from Y_0 to Y_1. Then we have another result of a standard nature.

PROPOSITION 7. *If the Ricci curvature of X is non-negative and the mean curvature of both boundary components (with respect to the outward normals) is non-negative then X is isometric to a product $Y \times [0, L]$ and in particular Y_1, Y_2 are isometric.*

To prove this we consider the harmonic function h on X, equal to 0 on Y_0 and to 1 on Y_1. Then we have

$$\Delta(|\nabla h|^2) = |\nabla\nabla h|^2 + \mathrm{Ric}(\nabla h) \tag{19}$$

and integrating we obtain

$$\int_{Y_0} \nabla_\nu |\nabla h|^2 + \int_{Y_1} \nabla_\nu |\nabla h|^2 + \int_X |\nabla\nabla h|^2 + \mathrm{Ric}(\nabla h) = 0, \tag{20}$$

where ∇_ν denotes the normal derivative. The second fundamental form of the boundary is the quadratic form defined by $B(\xi) = \langle \nabla_\xi \nu, \xi\rangle$ for vectors ξ tangent to the boundary. So if ξ_i is an orthornomal frame the mean curvature is

$$\mu = \sum_i \langle \nabla_{\xi_i} \nu, \xi_i\rangle.$$

On the boundary, write $\nabla h = f\nu$. Then

$$\sum_i \nabla_{\xi_i} \langle \nabla h, \xi_i\rangle = \sum \langle \nabla_{\xi_i} f\nu, \xi_i f\rangle = f^2 \mu.$$

On the other hand

$$\Delta h = \sum_i \langle \nabla_{\xi_i} \nabla h, \xi_i\rangle + \langle \nabla_\nu \nabla h, \nu\rangle,$$

so $f\mu = -\langle \nabla_\nu \nabla h, \nu\rangle$, since h is harmonic. Thus

$$\nabla_\nu(|\nabla h|^2) = -f^2\mu = -\mu|\nabla h|^2$$

and (20) becomes

$$\int_{Y_0} \mu|\nabla h|^2 + \int_{Y_1} \mu|\nabla h|^2 + \int_X |\nabla\nabla h|^2 + \mathrm{Ric}(\nabla h) = 0.$$

Under our hypotheses all terms are non-negative so must vanish identically. In particular $\nabla\nabla h = 0$ and this leads easily the product decomposition.

We apply this to the case of a torsion-free G_2-cobordism.

COROLLARY 1. *Let* $(N_0, \rho_0), (N_1, \rho_1)$ *be a pair of compact 6-manifolds with integrable* $SL(3, \mathbf{C})$ *structures. If there is a torsion-free* G_2*-cobordism* M *from* (N_0, ρ_0) *to* (N_1, ρ_1) *then they are isomorphic.*

This follows from Proposition 7 since the Ricci curvature of M and mean curvature of the boundary vanish (the latter by Proposition 1).

This corollary potentially has some bearing on questions of Torelli type for Calabi-Yau 3-folds. That is, the question whether a Calabi-Yau structure is uniquely determined by the cohomology class $[\rho] \in H^3(N; \mathbf{R})$. In fact the usual algebraic geometry formulation is in terms of the larger data $[\rho + i\tilde{\rho}] \in H^3(N, \mathbf{C})$. There are examples showing that "global Torelli" fails, in the standard algebraic geometry formulation [**10**]. But it is possible that there could be alternative formulations with positive answers.

QUESTION 1. *Suppose* (N, ρ_0) *and* (N, ρ_2) *are integrable* $SL(3, \mathbf{C})$ *structures.*

- *If there is a closed* G_2*-cobordism between the structures are they isomorphic?*
- *If* ρ_0, ρ_1 *are homotopic through tamed, closed,* $SL(3, \mathbf{C})$*-structures in a fixed cohomology class are they isomorphic?*

In other words, it is possible that examples where the Torelli property fails come from Calabi-Yau structures in different connected components under tamed deformations (although by the Crowley-Nordström theory discussed above they lie in the same connected component of closed $SL(3, \mathbf{C})$ structures). To explain the relevance of Corollary 1, suppose that ρ_s is a path of tamed, closed, structures from ρ_0 to ρ_1. We showed in [**5**] that for small s there is a torsion-free G_2-cobordism from ρ_0 to ρ_s. If this can be continued all the way to $s = 1$ we would deduce from Corollary 1 that ρ_0 and ρ_1 are isomorphic.

Continuing in a speculative vein, similar ideas could possibly be relevant to proving *existence* of Calabi-Yau structures. Suppose that ρ_0 is a real-analytic, closed $SL(3, \mathbf{C})$ structure on N. Then it is straightforward to show that there a torsion-free G_2-cobordism from ρ_0 to some ρ_1 close to ρ_0. Fix these boundary values ρ_0, ρ_1 and attempt to vary the enhancement data, i.e. seek torsion-free G_2-cobordisms ϕ_L with $I_W(\phi) = L$ and with $L \to \infty$. The simplest picture of what could happen is that, for a suitable family of base points in M, the based Gromov-Hausdorff limit as $L \to \infty$ is a product $N \times \mathbf{R}$, with a Calabi-Yau structure on N.

For another question, let ρ_0 be mean-concave and ρ_1 be mean-convex. Then Proposition 7 shows that there is no torsion-free G_2-cobordism for ρ_0 to ρ_1. (The signs are confusing here—the condition that ρ_0 is mean concave says that the boundary has positive mean curvature with respect to the outward normal, due to the switch in orientation). This can also be seen using geodesics and the second variation formula, as in Proposition 3. The question arises whether there can be a *closed* G_2-cobordism from ρ_0 to ρ_1.

5. Variants for maximal submanifolds

We can develop the same ideas in the direction of existence and uniqueness questions for maximal submanifolds. This is related to the G_2-discussion via the dimension reduction procedure described in Example 3 of 2.3, but can also be pursued independently. Consider a space-like $(p-1)$-dimensional submanifold $\Sigma \subset \mathbf{R}^{p,q}$ as in 3.2. Suppose that Ξ_0, Ξ_1 are two p-dimensional maximal spacelike submanifolds with boundary Σ. For $L > 0$ we consider $[0, L] \times \mathbf{R}^{p,q} \subset \mathbf{R}^{p+1,q}$ and the set

$$T = [0, L] \times \Sigma \cup \{0\} \times \Xi_0 \cup \{L\} \times \Xi_1.$$

PROPOSITION 8. *If there is a compact $(p+1)$-dimensional maximal submanifold Z in $\mathbf{R}^{p+1,q}$ with boundary T then $\Xi_0 = \Xi_1$.*

(More precisely, Z should be a "manifold with corners".) To see this we follows the proof of Proposition 7. The linear projection to the first factor is a harmonic function h on Z. The maximal condition implies that the Ricci curvature of Z is nonnegative and the fact that Ξ_i are maximal implies that the mean curvature of Ξ_i in Z vanishes. The new feature is that Z has an extra boundary component $[0, L] \times \Sigma$. Let $e \in \mathbf{R}^{p+1,q}$ be the coordinate vector corresponding to the $[0, L]$ factor. Then $|\nabla h|$ at a point of Z is the length of the orthogonal projection of e to the tangent space of Z (with respect to the indefinite form). At points of the boundary component $[0, L] \times \Sigma$ the first vector e lies inside this tangent space so $|\nabla h| = 1$. A moments thought shows that the normal derivative of $|\nabla h|^2$ vanishes, thus we do not get any contribution to the boundary term and the same argument applies to show that $\nabla\nabla h = 0$. This means that $|\nabla h| = 1$ everywhere which can only happen if e is tangent to Z at each point and we deduce that $\Xi_0 = \Xi_1$ and $Z = [0, L] \times \Xi_0$.

Finally we consider a more complicated geometric set-up, following [**3**] (to which we refer for more details). Let P be a p-dimensional manifold and $Q \subset P$ a co-oriented submanifold of co-dimension 2. We regard (P, L) as an orbifold, so we have orbifold charts around points of Q modelled on $\mathbf{R}^{p-2} \times \mathbf{C}$, with the involution $z \mapsto -z$ on the $\mathbf{C}$ factor. We consider a flat affine orbifold bundle $V \to P$ with structure group the affine extension Γ of $O(p, q)$. Thus over $P \setminus L$ we have a flat Γ-bundle in the usual sense and the orbifold structure over a point x of Q is given by an element r_x of order 2 in Γ. We suppose that the r_x are reflections in "timelike" vectors. Given this data, we have a notion of a *branched section* u of V. By definition this is given over $P \setminus Q$ by a section of the flat bundle. Locally, over small open sets $\Pi \subset P \setminus Q$ this is represented by a map $u_\Pi : \Pi \to \mathbf{R}^{p,q}$ and we require that this be an embedding with image a space-like submanifold. Around a point x of Q the behaviour of u can be described as follows. There is an orthogonal decomposition

$$\mathbf{R}^{p,q} = \mathbf{C} \times \mathbf{R}^{p-2} \times \mathbf{R} \times \mathbf{R}^{q-1}$$

in which the reflection r_x acts as -1 on the $\mathbf{R}$ factor and $+1$ on the other factors. The factor $\mathbf{C} \times \mathbf{R}^{p-2}$ is a positive subspace for the indefinite form and the factor $\mathbf{R} \times \mathbf{R}^{q-1}$ is a negative subspace. We can choose local co-ordinates $(w, \tau) \in \mathbf{C} \times \mathbf{R}^{p-2}$ on P such that Q is defined by $w = 0$ and the section is give by a multi-valued function

$$u(w, \tau) = (w, \tau, f(w^{1/2}, \tau), g(w, \tau)), \tag{21}$$

where f is an odd function in the $w^{1/2}$ variable. In other words, the orbifold co-ordinate z is $w^{1/2}$ and f is a genuine function $f(z, \tau)$ with $f(-z, \tau) = -f(z, \tau)$. We require that

$$g = O(|w|^2), \nabla g = O(|w|), \nabla^2 g = O(1), \tag{22}$$

$$f = O(|w|^{3/2}), \nabla f = O(|w|^{1/2}), \nabla^2 f = 0(|w|^{-1/2}). \tag{23}$$

Finally we can define *maximal branched sections* of V to be branched sections which away from Q are locally given by parametrised maximal submanifolds of $\mathbf{R}^{p,q}$. Around points of Q they correspond to branched maximal subvarities, with co-dimension 2 singularities.

Maximal branched sections are certainly not unique. We define an equivalence relation on branched sections of V as follows. If $f : (P, Q) \to (P, Q)$ is a diffeomorphism and if there is an isomorphism $\tilde{f} : f^*(V) \to V$ then $u \sim \tilde{f}(f^*(u))$. (In particular, if f is isotopic to the identity the flat structure defines a lift $\tilde{f}$.) Then if u_0 is a maximal branched section and if $u_1 \sim u_0$ then so also is u_1. Locally, this just corresponds to different choices of parametrisation of the same maximal submanifold. Another simple way in which uniqueness can fail occurs when there is a *covariant constant* section s of the flat orbifold vector bundle $\underline{V}$ associated to the affine bundle V. In that case we can change a maximal branched section u to another $u + s$. Locally this just corresponds to translation of the maximal subvariety. In most cases of interest there will be no such covariant constant sections.

Now consider the product $P \times [0, L]$ with projection $\pi : P \times [0, L] \to P$. The pull-back $\pi^*(V)$ is a flat affine orbifold bundle over $(P \times [0, L], Q \times [0, L])$. We consider the bundle $\pi^*(V) \times \mathbf{R}$ over $(P \times [0, L], Q \times [0, L])$ with the obvious structure of an affine orbifold bundle with fibre $\mathbf{R}^{p+1,q}$. Let e be the covariant constant section of the vector bundle $\underline{\pi^*(V) \times \mathbf{R}}$ corresponding to the unit vector in the $\mathbf{R}$ factor. If we have two branched sections u_0, u_1 of V we can consider branched sections U of $\pi^*(V) \oplus \mathbf{R}$ with boundary conditions that $U = u_0$ over $P \times \{0\}$ and $U = u_1 + Le$ over $P \times \{L\}$.

Proposition 9. *If u_0 and u_1 are two branched maximal section of V and if there is a branched maximal section U of $\pi^*(V)\times \mathbf{R}$ with these boundary values then $u_1 \sim u_0 + s$ for a covariant constant section s of V_0.*

The section U induces a Riemannian metric Γ on $P\times[0,L]$ with a singularity along $Q\times[0,L]$. In the local-co-ordinates given by (21) this metric is uniformly equivalent to the Euclidean metric, with Lipschitz metric tensor. As before the metric has non-negative Ricci curvature away from the singular set. We write h for the function on $P\times[0,L]$ given by projection of U to the $\mathbf{R}$ factor in $\pi^V\times\mathbf{R}$. Thus $h=0,L$ on the two boundary components. The local geometry away from the singular set is just as before but we need to check that the singularity does not affect the argument. Let N_ϵ be a tubular neighbourhood of Q of radius ϵ and consider

$$\int_{(P\setminus N_\epsilon)\times[0,L]} \Delta|\nabla h|^2$$

There is a new boundary term

$$\int_{\partial N_\epsilon\times[0,L]} \nabla_\nu |\nabla h|^2. \tag{24}$$

The integrand is locally $\langle B(\nabla h,\nu),e\rangle$ where $B(\ )$ is the second fundamental form of the image of U, regarded now as a bilinear form on the tangent space. It follows from (21) that B is $O(\epsilon^{-1/2})$ and $|\nabla h|$ is $O(1)$. The local bi-Lipschitz property implies that the volume of $\partial N_\epsilon\times[0,L]$ is $O(\epsilon)$ so the integral in (24) is $O(\epsilon^{1/2})$ and taking $\epsilon\to 0$ we deduce that $\nabla\nabla h=0$, as before. In particular the length $|\nabla h|$ is a constant c and $c\geq 1$ (since it is the projection of a unit vector to a maximal positive subspace in $\mathbf{R}^{p+1,q}$). The local representation (21) shows that ∇h, regarded as the gradient vector field, is Lipschitz on $P\times[0,L]$ and this implies that the integral curves run from one boundary component to the other (as in the smooth case). The same argument shows that the (singular) Riemannian manifold $(P\times[0,L],\Gamma)$ is isometric to a Riemannian product, say $(P,g)\times[0,L/c]$, where the function h on $P\times[0,L]$ goes over to the function $\tilde{h}(x,t)=ct$ on $(P,g)\times[0,L/c]$.

Suppose first that $c=1$. This implies that at each point the gradient vector ∇h in the tangent space of $P\times[0,L]$ maps under the derivative of U to the fixed vector e. Let $F:(P,g)\times[0,L]\to P\times[0,L]$ be the diffeomorphism given by the Riemannian product structure, equal to the identity on $P\times\{0\}$. The pull back by F of ∇h is the unit vector field ∂_t in the $[0,L]$ factor. Thus the derivative $\partial_t(F^*(U))$ is equal to e. The flat structure, and the fact that F is the identity on $P\times\{0\}$, gives a canonical isomorphism $\tilde{F}:F^*(\pi^*(V)\times\mathbf{R})\to\pi^*V\times\mathbf{R}$. So we can regard $F^*(U)$ as a 1-parameter family $(F^*(U))_t$ of sections of the bundle $V\times\mathbf{R}\to P$ and our identification of the t-derivative shows that

$$(F^*(U))_t = u_0 + te. \tag{25}$$

Now let F be given on the other boundary component by $F(x,L) = (f(x), L)$ for a diffeomorphism $f : P \to P$ and let $\tilde{f}$ be the restriction of $\tilde{F}$. Then (25) specialises to $\tilde{f}(f^*(u_1)) = u_0$, which shows that $u_1 \sim u_0$. The argument above is essentially the same as that in the proof of Proposition 8, once we know that $|\nabla h| = 1$. The extra difficulty that arises now is to analyse the case when $c > 1$. To handle this we need a lemma from local differential geometry.

Lemma 4. *Suppose X is a connected p-dimensional Riemannian manifold (not necessarily complete) and suppose that $f_s : X \to \mathbf{R}^{p,q}$ is a smooth family of spacelike embeddings for $s \in (-\delta, \delta)$. Fix $c > 1$ and let $\mathbf{R}^{p,q} \times \mathbf{R}$ have the standard indefinite form, positive on the $\mathbf{R}$ factor. Let*

$$\Phi : X \times (-\delta, \delta) \to \mathbf{R}^{p+1,q}$$

be the map $\Phi(x, s) = (f_s(x), cs)$. Suppose that

1. *Φ is an isometric embedding, with space-like image, for the Riemannian product metric on $X \times (-\delta, \delta)$;*
2. *the image of Φ is a maximal space-like submanifold in $\mathbf{R}^{p,q} \times \mathbf{R}$;*

Then there is a vector $\nu \in \mathbf{R}^{p,q}$ with $|\nu|^2 = 1 - c^2 < 0$ such that $f_s(x) = f_0(x) + s\nu$ and the image $f_0(X)$ lies in a hyperplane normal to ν.

Write $\frac{\partial f_s}{\partial s} = \nu_{s,x}$ so ν takes values in $\mathbf{R}^{p,q}$. The isometric embedding condition in item (1) is equivalent to

- $|\nu_{s,x}|^2 = 1 - c^2$;
- $\nu_{s,x}$ is orthogonal to the tangent space of $f_s(X)$ at $f_s(x)$;
- each f_s is an isometric embedding of X in $\mathbf{R}^{p,q}$.

When the codimension, q, is large these conditions admit many solutions so we have to bring in the second hypothesis, that the image of Φ is a maximal submanifold. Let Γ be the Gauss map of the image of Φ. The maximal submanifold condition implies that

$$|\frac{\partial \Gamma}{\partial s}|^2 = \mathrm{Ric}_{X \times (-\delta,\delta)}(\partial s)$$

where on the left hand side we use the standard Riemannian metric on the Grassmann manifold of maximal positive subspaces. Since, for the product manifold, this component of the Ricci curvature is zero we deduce that Γ is constant in s. By simple linear algebra and the orthogonality condition this implies that $\nu_{s,x}$ is independent of s so we can write ν_xv and $f_s = f_0 + s\nu_x$. From this one deduces easily that ν_x is independent of x, and the orthogonality shows that X_0 lies in a hyperplane normal to ν.

Given this Lemma it is easy to extend the proof that we gave for the case $c = 1$ to the general case.

There are two situations in which this result interacts with G_2-geometry.

1. Take $p = 2, q = 19$ and $P = S^2$. Consider a polarised Calabi-Yau threefold N which admits a holomorphic Lefschetz fibration

$N \to S^2$ with $K3$ fibres. The cohomology of the fibres orthogonal to the Kähler class defines a flat orbifold vector bundle (with Q the finite set of critical values) and a class in $H^3(N)$ yields a lift to an affine bundle V. The period map of the complex structure defines a branched maximal section. The uniqueness question is a version of the Torelli problem for $K3$-fibred Calabi-Yau 3-folds. In the argument, the maximal section U over $S^2 \times [0, L]$ corresponds to the "adiabatic limit" of a G_2-cobordism with a Kovalev-Lefschetz fibration, as discussed in [**3**].

2. Take $p = 3, q = 19$. Then P is a 3-manifold and $Q \subset P$ is a link. The uniqueness question for maximal sections is the adiabatic limit of a "Torelli problem" for closed G_2-manifolds with Kovalev-Lefschetz fibrations.

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Department of Mathematics, Imperial College, London, UK; and Simons Center for Geometry and Physics, Stony Brook, New York, USA

E-mail address: `s.donaldson@imperial.ac.uk`