

New curvature flows in complex geometry

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ABSTRACT. The Anomaly flow is a flow of $(2, 2)$ -forms on a 3-fold which was originally motivated by string theory and the need to preserve the conformally balanced property of a Hermitian metric in the absence of a $\partial\bar{\partial}$ -Lemma. It has revealed itself since to be a remarkable higher order extension of the Ricci flow. It has also led to several other curvature flows which may be interesting from the point of view of both non-Kähler geometry and the theory of non-linear partial differential equations. This is a survey of what is known at the present time about these new curvature flows.

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1. Introduction

The search for canonical metrics is at the interface of several particularly active fields in mathematics. The theory of non-linear partial differential equations certainly plays a major role, since a canonical metric is typically defined by a curvature condition, and hence is a non-linear system of second

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order equations in the metric. The curvature condition should optimize the metric in some sense, which suggests a variational principle, or a link with the field equation of a physics theory. The existence of a canonical metric, or of a minimizer for the variational problem, is usually tied to some deep geometric properties of the underlying space, such as stability in algebraic geometry. This interaction between the theory of partial differential equations, theoretical physics, and algebraic geometry has been particularly fertile for complex, and especially Kähler geometry. There, the complex and the Riemannian structures fit seamlessly, and the search for Hermitian-Yang-Mills connections [29, 97], Kähler-Einstein metrics [7, 101, 22], and Kähler metrics with constant scalar curvature (see [30, 19] for recent advances and [84] for a survey) have motivated many exciting developments in the last 50 years.

However, conditions such as Yang-Mills connections, or metrics of constant scalar or Ricci curvature are all linear in the curvature form. Recent advances in theoretical physics, especially string theory, suggest that the notion of canonical metrics should be widened to include conditions which involve its higher powers. This is in view of the Green-Schwarz anomaly cancellation mechanism [50], which involves the square of the curvature, and also of string-theoretic corrections to the Einstein-Hilbert and the Yang-Mills actions. Since the curvature form with respect to the Chern connection of the Hermitian metric is a $(1, 1)$ -form (valued in the space of endomorphisms of the tangent space or a vector bundle), equations involving its p -th power for $p > 1$ will be equations for (p, p) -forms. But the Kähler condition is a condition on $(1, 1)$ -forms, so we expect to have to replace it by conditions on (p, p) -forms, such as Michelsohn's balanced condition [68], which will usually be weaker. Because of these accumulated difficulties, systems involving powers of the curvature may seem inaccessible but for the breakthrough by Fu and Yau [43, 44], who found some special solutions by PDE methods to one such system, namely the Hull-Strominger system [56, 57, 90]. The Fu-Yau solution also revealed that the system has a lot of structure and may perhaps be amenable to a more systematic approach.

If we have to implement a condition which is weaker than the Kähler condition, we face a very serious obstacle, namely the lack of a $\partial\bar{\partial}$ -Lemma in non-Kähler geometry. Perhaps surprisingly, in some cases such as the Hull-Strominger system, this can be circumvented by introducing a suitable geometric flow, which is a flow of $(2, 2)$ -forms on a 3-dimensional complex manifold [75]. Even more surprising is that the resulting flow, called the Anomaly flow, turns out to be a generalization of the Ricci flow, albeit with corrections due to the metric not being Kähler and to the higher powers of the curvature tensor [78]. Besides its original motivation as a parabolic approach to solving the Hull-Strominger system, the Anomaly flow admits many variants and generalizations, which can potentially be useful in addressing some classic questions of non-Kähler geometry. While many extensions of the Ricci and related flows have been introduced over the years in

the literature [8, 10, 11, 47, 61, 64, 65, 88, 89, 94], the ones arising from the Anomaly flow seem all new. In fact, while for many of the existing generalizations, only short-time existence theorems and/or derivative estimates are available at the present time, the long-time behavior and convergence of the Anomaly-related flows can be established in a number of highly non-trivial cases. This can perhaps be attributed to their natural handling of the conformally balanced condition, and also argues for their interest from the point of view of the theory of non-linear partial differential equations.

The purpose of the present paper is to present an introduction to the Anomaly and related flows, and to survey some of what is known. Their study has just begun, and many basic questions are as yet unanswered. We hope that the paper will motivate many researchers to help answer these questions.

2. The reference flow: the Kähler-Ricci flow

When discussing the new flows, it will be instructive to compare them to a reference flow in complex geometry, namely the Kähler-Ricci flow. We begin by recalling some basic facts about the Kähler-Ricci flow. The literature on the subject is immense, and we shall only touch upon those aspects where a comparison with the new flows would be particularly helpful.

2.1. The Ricci flow. We start with the Ricci flow. Let X be a Riemannian manifold. For simplicity, we shall assume throughout this paper that X is compact, unless indicated specifically otherwise. The Ricci flow is the flow of metrics $t \rightarrow g_{ij}(t)$ given by

$$(2.1) \quad \partial_t g_{ij}(t) = -2 R_{ij}(t), \quad g_{ij}(0) = g_{ij}^0$$

where g_{ij}^0 is an initial metric, and $R_{ij}(t)$ is the Ricci curvature of $g_{ij}(t)$. In normal coordinates near any given point, the Ricci curvature is given by

$$(2.2) \quad R_{ij} = \frac{1}{2}(-g^{pq}\partial_p\partial_q g_{ij} - g^{pq}\partial_i\partial_j g_{pq} + g^{pq}\partial_p\partial_j g_{iq} + g^{pq}\partial_i\partial_q g_{pj}).$$

Using De Turck's trick [27], the last three terms can be removed by gauge fixing, and the first term $-g^{pq}\partial_p\partial_q g_{ij}$ is used to obtain a unique short time solution to the Ricci flow.

Initially developed by Hamilton, the early successes of the Ricci flow include the proof of its convergence to a space form for 3-folds with positive Ricci curvature [53] and, with appropriate surgeries, for 4-folds with positive curvature operator [54]. As everyone knows, these successes culminated in Perelman's spectacular proof of the Geometrization Conjecture [72, 73]. It has been a very active research direction ever since, and many deep and powerful results continue to be obtained.

2.2. The Kähler-Ricci flow. Let X be now a compact Kähler manifold. The Kähler-Ricci flow is just the Ricci flow

$$(2.3) \quad \partial_t g_{\bar{k}j} = -R_{\bar{k}j}(t)$$

where $g_{\bar{k}j}(0)$ (and hence $g_{\bar{k}j}(t)$ for all t) is Kähler.

We recall the notion of Chern unitary connection defined by a Hermitian metric $g_{\bar{k}j}$. To a Hermitian metric $g_{\bar{k}j}$, we can associate its form $\omega = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$. The metric is said to be Kähler if $d\omega = 0$. The Kähler class of ω is then the de Rham cohomology class $[\omega]$.

Given a Hermitian metric ω , the Chern unitary connection ∇ is defined by the requirement that $\nabla_{\bar{k}}V^j = \partial_{\bar{k}}V^j$ and unitarity, i.e., $\nabla_{\bar{k}}g_{\bar{m}j} = 0$. In general, the Chern unitary connection differs from the Levi-Civita connection. But the two are the same if ω is Kähler.

For Kähler metrics, the Ricci curvature is given by the very simple formula

$$(2.4) \quad R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log \omega^n,$$

where n is the complex dimension of X . The corresponding Ricci form $Ric(\omega)$ is defined by $Ric(\omega) = iR_{\bar{k}j}dz^j \wedge d\bar{z}^k$, or equivalently,

$$(2.5) \quad Ric(\omega) = -i\partial\bar{\partial} \log \omega^n.$$

Clearly $Ric(\omega)$ is a closed form, and defines itself a cohomology class. Since for two different metrics ω and $\tilde{\omega}$, the ratio $\omega^n/\tilde{\omega}^n$ is a well-defined smooth function, $[Ric(\omega)]$ is independent of ω . It is called the first Chern class $c_1(X)$. More geometrically, let K_X be the canonical bundle of X , that is, the bundle of $(n, 0)$ -forms on X . We can think of ω^n as a positive section of $K_X \otimes \bar{K}_X$, and hence as a metric on the anti-canonical bundle K_X^{-1} . The Ricci curvature defined by (2.4) can then be viewed as the curvature of the line bundle K_X^{-1} , and its de Rham cohomology class $c_1(X)$ can be identified as the first Chern class of K_X^{-1} .

2.3. Immediate advantages of the Kähler condition. Since $\partial_t \omega = -Ric(\omega)$, the closedness of $Ric(\omega)$ implies that ω evolves only by closed forms, and hence it remains closed if it is closed at time $t = 0$. In other words, the Kähler property is preserved. The resulting cohomological information accounts for some marked differences between the Kähler-Ricci flow and the general Ricci flow on just Riemannian manifolds.

(a) In the Kähler case, the Kähler-Ricci flow implies $\partial_t[\omega(t)] = -c_1(X)$, and hence, even though $\omega(t)$ may still be difficult to find, we know at least its cohomology class

$$(2.6) \quad [\omega(t)] = [\omega(0)] - t c_1(X).$$

(b) Thus the flow can only exist for t such that $[\omega(t)] > 0$, in the sense that $[\omega(t)]$ contains a positive representative. This suggests that the maximum time of existence for the flow should be given by

$$(2.7) \quad T = \sup\{t : [\omega(0)] - t c_1(X) > 0\}$$

which is determined by the cohomology class of $[\omega(0)]$, and does not depend on $\omega(0)$ itself. This was confirmed by Tian and Z. Zhang [91].

(c) On a compact Kähler manifold, the $\partial\bar{\partial}$ -Lemma says that if ω and $\tilde{\omega}$ are any two $(1,1)$ -forms satisfying $[\tilde{\omega}] = [\omega]$, then we can write $\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi$ for some smooth function φ unique up to an additive constant, which is called the potential of $\tilde{\omega}$ with respect to ω . In particular, a Kähler metric ω is determined by its cohomology class $[\omega]$ and a scalar function. We can apply the $\partial\bar{\partial}$ -Lemma to reduce the Kähler-Ricci flow to a scalar flow as follows. First we choose a known and evolving $(1,1)$ -form $\hat{\omega}(t)$ with $\partial_t\hat{\omega}(t) = i\partial\bar{\partial}\log\Omega = -c_1(X)$, $\hat{\omega}(0) = \omega(0)$, where Ω is a fixed smooth, strictly positive section of K_X . Then $[\hat{\omega}(t)] = [\omega(t)]$, and we can set

$$(2.8) \quad \omega(t) = \hat{\omega}(t) + i\partial\bar{\partial}\varphi(t)$$

with $\varphi(t)$ now the unknown scalar function. The Kähler-Ricci flow is equivalent to

$$(2.9) \quad i\partial\bar{\partial}\partial_t\varphi(t) = i\partial\bar{\partial}\log\omega^n(t) - i\partial\bar{\partial}\log\Omega = i\partial\bar{\partial}\log\frac{\omega^n(t)}{\Omega}$$

Changing if necessary Ω by a constant multiple, this equation is equivalent to the following parabolic complex Monge-Ampère type equation for the scalar function φ ,

$$(2.10) \quad \partial_t\varphi = \log\frac{(\hat{\omega}(t) + i\partial\bar{\partial}\varphi)^n}{\Omega}, \quad \hat{\omega}(t) + i\partial\bar{\partial}\varphi > 0.$$

This equivalence between a flow of metrics and a much simpler flow of scalar functions is arguably the most useful consequence of the Kähler property, and one which will be sorely missing when we consider the new curvature flows.

2.4. The Kähler-Ricci flow in analytic/algebraic geometry.

Early on, the Kähler-Ricci flow provided a parabolic approach to the problem of finding Kähler-Einstein metrics, that is Kähler metrics satisfying the condition

$$(2.11) \quad Ric(\omega) = \mu\omega$$

for μ constant. This problem was solved using elliptic methods by Yau [101] for $\mu = 0$ and by Yau [101] and Aubin [7] for $\mu < 0$. Such metrics can also be viewed as stationary points of the (normalized) Kähler-Ricci flow, and an alternative proof using the Kähler-Ricci flow was given by H.D. Cao [17]. The case $\mu > 0$ was solved only relatively recently in 2013 by Chen, Donaldson, and Sun [20, 21, 22]. Subsequent progress on the Kähler-Ricci flow approach in this case can be found in [92, 23] and references therein.

In the previous application, the initial assumption, which is necessary for the existence of Kähler-Einstein metrics, was that $c_1(X)$ was either zero, or definite. More recently, in an important development, the existence of Kähler-Einstein metrics was also established by Wu and Yau [98, 99, 100], starting instead from an assumption of negative holomorphic sectional curvature. Wu and Yau used elliptic methods, but Nomura [70], Tong [93], and Huang et al [55] have been able to recover and extend the results of Wu and Yau in several ways, using the Kähler-Ricci flow.

The above applications of the Kähler-Ricci flow concern the cases when a canonical metric does exist, and the Kähler-Ricci flow converges to it. Several longer range and presently ongoing programs are based on the expectation that the Kähler-Ricci flow can still help find a canonical metric, or a canonical complex structure, even when there is none on the original manifold. These include the analytic minimal model program of Song and Tian [87], with early contributions by Tsuji [96], which requires the continuation of the Kähler-Ricci flow beyond singularities, possibly on a different manifold; and flows on unstable manifolds, where one would have to address the phenomena of jumps to a different complex structure and moduli theory. A major difficulty in general is to identify the limiting complex manifold. So far only the case of S^2 with an arbitrary configuration of marked points, including semistable and unstable ones, has been fully worked out in [83]. The convergence in the stable case had been obtained in [67].

3. Motivation for the new flows

While the study of the Kähler-Ricci flow remains a very active research area, with many hard and deep questions left open, a number of new problems arising from different areas seem to require new flows. Remarkably, while their origin is a priori unrelated, these new flows turn out to be in many ways just higher order corrections of the Ricci flow, in a natural setting which is somewhere intermediate between the Riemannian and the Kähler settings. We begin with an informal discussion of the motivation and qualitative features of these flows, leaving the precise formulation to the next section.

3.1. Anomaly cancellation and powers of the curvature. In string theory, a crucial requirement discovered by M. Green and J. Schwarz [50] for the cancellation of anomalies is the equation

$$(3.1) \quad dH = \frac{\alpha'}{4} (\text{Tr}(Rm \wedge Rm) - \text{Tr}(F \wedge F)).$$

Here space-time is a 10-dimensional manifold, and the fields in the theory include a metric g_{ij} , a Hermitian metric h on a vector bundle E , and a 3-form H . The expressions Rm and F are the curvatures of ω and h , viewed as 2-forms valued in the endomorphisms of the tangent space and the bundle

E respectively. The coefficient α' is the string tension, which is a positive constant.

Note that this is an equation of 4-forms involving the square of the curvature. By contrast, more familiar equations such as the Kähler-Einstein equation or the Yang-Mills equation are linear in the curvature. The right hand side of the equation (3.1) arises rather from Pontryagin or Bott-Chern characteristic classes, which arise themselves from gauge and gravitational anomalies. To derive it from an action, one would need to expand in the string tension α' , and keep both the leading terms as well as the next order expansion terms.

As shown by Candelas, Horowitz, Strominger, and Witten [16], in compactifications of the heterotic string from 10-dimensional to 4-dimensional space-time, supersymmetry requires the existence of a covariantly constant spinor from which an integrable complex structure with nowhere vanishing holomorphic $(3, 0)$ -form Ω can be constructed on the internal 6-dimensional manifold. Restricted to the internal space, the metric g_{ij} can then be identified with a $(1, 1)$ -form ω , and H becomes the torsion of ω , $H = i(\bar{\partial} - \partial)\omega/2$. The bundle E turns out to be a holomorphic vector bundle, and the curvature F is that of the Chern unitary connection of h on E . In the equation (3.1), it is natural to view Rm as the curvature of the Chern unitary connection of ω . However, it has been shown by Hull [57] that other unitary connections are also admissible. Altogether, the equation (3.1) is then an equation for $(2, 2)$ -forms, which implies that $\text{Tr}(Rm \wedge Rm)$ is a $(2, 2)$ -form, although Rm itself may not be a $(1, 1)$ -form.

We observe that, except in the special case when the right hand side of (3.1) vanishes identically (which can only happen if the vector bundles $T^{1,0}(X)$ and E have the same second Chern class), the equation (3.1) implies that the Hermitian metric ω is not Kähler. Thus, besides introducing an equation of $(2, 2)$ -forms and the square of the curvature tensor, the anomaly cancellation mechanism of Green and Schwarz also suggests a widening of metrics under consideration to non-Kähler metrics.

3.2. Weakenings of the Kähler condition. Many weakenings of the Kähler condition on Hermitian metrics are known, and we shall actually discuss some of them in section §6 below. But for our purposes, the most important condition is the notion of balanced metric introduced by Michelsohn [68]. A metric ω on an n -dimensional complex manifold is said to be *balanced* if

$$(3.2) \quad d(\omega^{n-1}) = 0$$

and a manifold is said to be balanced if it admits a balanced metric. Depending on the circumstances, this condition may well prove to be preferable to the Kähler condition, because it is invariant under birational transformations, as was shown by Alessandrini and Bassanelli [2].

Perhaps unexpectedly, the notion of balanced metric arose relatively recently in a completely different context, which is the one discussed in the previous section of compactifications of the heterotic string to 4-dimensional Minkowski space preserving supersymmetry. In the original set-up considered in [16], the compactification was achieved by a product of Minkowski space time with the internal space. In the generalization considered by Hull [56, 57] and Strominger [90], the product is replaced by a warped product. Supersymmetry still leads to the internal space being a Calabi-Yau 3-fold, with a holomorphic non-vanishing $(3, 0)$ -form Ω , but the Hermitian metric ω must satisfy the condition

$$(3.3) \quad d(\|\Omega\|_{\omega} \omega^2) = 0$$

(as reformulated by Li and Yau [63]). This condition just means that the metric $\|\Omega\|_{\omega}^{\frac{1}{2}} \omega$ is balanced, and we shall just refer to ω as being *conformally balanced*. For more recent developments on balanced metrics, please see the survey paper [40] and references therein.

3.3. Detection of instability. Even when the solution of an elliptic equation can already be found by a particular flow under e.g. a stability condition, other flows can be potentially useful too. This is because, in the absence of a stability condition, they may fail to converge in different ways, which would provide then different ways of detecting instability. A well-known example is the method of continuity for the Kähler-Einstein problem for Fano manifolds. The failure of convergence is detected there by the emergence of a non-trivial multiplier ideal sheaf [69, 86]. By contrast, the behavior of the Kähler-Ricci flow in the unstable case is still obscure. On the other hand, a third flow again with Kähler-Einstein metrics as stationary point is the inverse Monge-Ampère flow recently introduced by Collins et al [26]. This flow appears to provide more information than the Kähler-Ricci flow when the underlying manifold is an unstable toric Fano manifold, as it produces then an optimal destabilizing test configuration.

3.4. New types of partial differential equations. The appearance of powers of the curvature, in contrast with the equations of constant scalar or constant Ricci curvature with which we are more familiar, can be expected to result also in new types of PDE's that have not been encountered before.

Two new difficulties will also have to be addressed:

(a) The first is the lack of a $\partial\bar{\partial}$ -lemma in non-Kähler geometry. So the equations will usually not reduce to scalar equations. In this respect, the non-Kähler flows that we shall encounter may be closer to the Ricci flow than the Kähler-Ricci flow.

(b) But even when they happen to be reducible to a scalar equation, more often than not, these equations will not be concave in the eigenvalues of the Hessian. This will prevent the use of classical and powerful PDE tools

such as those of Caffarelli-Nirenberg-Spruck [12, 13] and the Evans-Krylov theorem, and other methods will have to be developed.

Thus the new geometric flows may be interesting in their own right from the point of view of the theory of non-linear partial differential equations.

4. The Anomaly flow

We now formulate precisely the first of the new flows which appeared recently in complex geometry. This is the Anomaly flow introduced by the authors in [75], and given this name in recognition of the anomaly cancellation mechanism introduced in [50]. There are actually several versions of the Anomaly flow, and we begin with the simplest version.

Let X be a compact complex 3-fold, equipped with a nowhere vanishing holomorphic $(3, 0)$ -form Ω . Let $t \rightarrow \Phi(t)$ be a given path of closed $(2, 2)$ -forms, with $[\Phi(t)] = [\Phi_2(0)]$ for each t . Let ω_0 be an initial metric which is conformally balanced, i.e., $\|\Omega\|_{\omega_0}\omega_0^2$ is a closed $(2, 2)$ -form. Then the Anomaly flow is the flow of $(2, 2)$ -forms defined by

$$(4.1) \quad \partial_t(\|\Omega\|_{\omega}\omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\text{Tr}(Rm \wedge Rm) - \Phi),$$

where $\|\Omega\| = (i\Omega \wedge \bar{\Omega}\omega^{-3})^{\frac{1}{2}}$ is the norm of Ω with respect to ω . Here α' is the string tension introduced earlier in (3.1). Mathematically, the equation (4.1) is well-defined for any real number α' and we fix $\alpha' \in \mathbf{R}$ unless specified otherwise.

The expression Rm is the curvature of the Chern unitary connection defined by the Hermitian metric ω , viewed as a section of $\Lambda^{1,1} \otimes \text{End}(T^{1,0}(X))$. We shall on occasion consider other connections too, and say explicitly so when this is the case.

Note that the stationary points of the flow are precisely given by the Green-Schwarz anomaly cancellation equation (3.1), and that the flow preserves the conformally balanced condition. Indeed, by Chern-Weil theory, its right hand side is a closed $(2, 2)$ -form and hence for all t ,

$$d(\|\Omega\|_{\omega}\omega^2) = 0$$

at all times t , if $d(\|\Omega\|_{\omega}\omega^2) = 0$ at time $t = 0$.

4.1. The Hull-Strominger system. The main motivation for the Anomaly flow is to solve the following system of equations for supersymmetric compactifications of the heterotic string, proposed independently by Hull [56, 57] and Strominger [90], which generalizes an earlier proposal by Candelas, Horowitz, Strominger, and Witten [16]. Let X be a 3-fold equipped with a holomorphic non-vanishing $(3, 0)$ -form Ω and a holomorphic vector bundle $E \rightarrow X$ with $c_1(E) = 0$. Then we look for a Hermitian

metric ω on X and a Hermitian metric h on E on X satisfying

$$\begin{aligned}
 i\partial\bar{\partial}\omega - \frac{\alpha'}{4} (\text{Tr}(Rm \wedge Rm) - \text{Tr}(F \wedge F)) &= 0 \\
 d(\|\Omega\|_\omega \omega^2) &= 0 \\
 \omega^2 \wedge F &= 0
 \end{aligned}
 \tag{4.2}$$

The third condition is the familiar Hermitian-Yang-Mills equation, so the novelty of the Hull-Strominger system resides essentially in the first two equations. Actually, the second equation in the above system was originally written in a different way in [56, 57] and [90]. The formulation given above, which brings to light the conformally balanced condition, is due to Li and Yau [63].

The solutions of the Hull-Strominger system can be viewed as generalizations of Ricci-flat Kähler metrics in the following sense. Assume that

$$\frac{\alpha'}{4} (\text{Tr}(Rm \wedge Rm) - \text{Tr}(F \wedge F)) = 0.
 \tag{4.3}$$

Then the first equation reduces to $i\partial\bar{\partial}\omega = 0$. Combined with the second equation $d(\|\Omega\|_\omega \omega^2) = 0$, it implies that ω is both Kähler and Ricci-flat (a detailed proof of this fact is given in §6.1 below). One simple way of guaranteeing the condition (4.3), as suggested in [16], is to take $E = T^{1,0}(X)$, and $\omega = h$. If ω is Kähler, then the first equation in the Hull-Strominger system is satisfied. The third condition on F coincides then with the Ricci-flat condition on ω . This implies that $\|\Omega\|_\omega$ is constant, and hence the second equation is a consequence of ω being Kähler. Thus Ricci-flat Kähler metrics provide a consistent solution to the Hull-Strominger system.

An immediate difficulty when trying to solve the Hull-Strominger system in general is how to implement the second condition in the absence of an analogue of a $\partial\bar{\partial}$ -Lemma. While there are many Ansätze that can produce a balanced metric starting from a given one, none of them is particularly special, and they all lead to very unwieldy expressions for their curvatures. Anomaly flows are designed to circumvent this difficulty, and produce solutions of the Hull-Strominger system without appealing to any particular ansatz. The prototype of an Anomaly flow was introduced in (4.1). For the purpose of solving the Hull-Strominger system, we fix a conformally balanced metric $\omega(0)$ on X and a metric $h(0)$ on E and introduce the following coupled flow $t \rightarrow (\omega(t), h(t))$,

$$\begin{aligned}
 \partial_t(\|\Omega\|_\omega \omega^2) &= i\partial\bar{\partial}\omega - \frac{\alpha'}{4} (\text{Tr}(Rm \wedge Rm) - \text{Tr}(F \wedge F)) \\
 h(t)^{-1}\partial_t h(t) &= -\frac{\omega^2 \wedge F}{\omega^3}
 \end{aligned}
 \tag{4.4}$$

As noted before, the flow preserves the conformally balanced condition for ω , and hence its stationary points are automatically balanced if they exist. Since they will manifestly also satisfy the first and third equations in the

Hull-Strominger system, they satisfy the complete Hull-Strominger system. Thus we shall have circumvented the problem of the absence of a $\partial\bar{\partial}$ -Lemma for conformally balanced metrics.

We observe that, while the flow (4.4) is a coupled flow for $\omega(t)$ and $h(t)$, there are cases when F can be viewed as known. For example, in the case of Calabi-Eckmann-Goldstein-Prokushkin fibrations over a $K3$ surface, the Hermitian-Yang-Mills condition on F does not change as ω evolves, and h can be taken as fixed, given by the Hermitian-Yang-Mills metric corresponding to the metric $\omega(0)$ on X . Setting $\Phi = \text{Tr}(F \wedge F)$, we see that the flow (4.4) reduces to the flow (4.1).

4.2. Formulation as a flow of metrics instead of (2, 2) forms.

While the formulation of the Anomaly flow as a flow of (2, 2)-forms is ideally suited to the manifest preservation of the conformally balanced condition of Hermitian metrics, for an analytic study of the flow, we need another, more conventional formulation as a flow of (1, 1)-forms. This is provided by the following theorem [75, 78]:

THEOREM 1. *Consider the anomaly flow (4.1) with a conformally balanced initial metric.*

(a) *The flow can also be expressed as*

$$(4.5) \quad \partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega} \left\{ -\tilde{R}_{\bar{k}j} + g^{s\bar{r}} g^{p\bar{q}} T_{\bar{q}s j} \bar{T}_{p\bar{r}\bar{k}} - \frac{\alpha'}{4} g^{s\bar{r}} (R_{[\bar{k}s}{}^\alpha{}_\beta R_{\bar{r}j]}{}^\beta{}_\alpha - \Phi_{\bar{k}s\bar{r}j}) \right\}$$

Here $\tilde{R}_{\bar{k}j} = g^{p\bar{q}} R_{\bar{q}p\bar{k}j}$ is the Chern-Ricci tensor, and $T = i\partial\omega = \frac{1}{2} T_{\bar{k}j m} dz^m \wedge dz^j \wedge d\bar{z}^k$ is the torsion tensor. The bracket $[\cdot, \cdot]$ denote anti-symmetrization in each of the two sets of barred and unbarred indices.

(b) *The flow is parabolic and exists at least for a short time when $\alpha' Rm$ satisfied the following positivity condition*

$$(4.6) \quad |\xi|^2 |A|^2 + \frac{\alpha'}{2} g^{q\bar{b}} g^{a\bar{p}} g^{s\bar{r}} g^{\alpha\bar{\gamma}} R_{[\bar{r}q}{}^\beta{}_\alpha \xi_{\bar{p}} \xi_s] A_{\bar{\gamma}\beta} A_{\bar{b}a} > 0$$

for all Hermitian tensors A and vectors ξ . In particular, it is parabolic and exists at least for a short time if $|\alpha' Rm(0)| < 1/2$.

The above formulas for the Anomaly flow should be viewed in the context of conformally balanced metrics. In general, we define the curvature tensor of the Chern unitary connection of a Hermitian metric $g_{\bar{k}j}$ by

$$(4.7) \quad [\nabla_j, \nabla_{\bar{k}}]V^p = R_{\bar{k}j}{}^p{}_q V^q$$

for any vector field V^p . Explicitly,

$$(4.8) \quad R_{\bar{k}j}{}^p{}_q = -\partial_{\bar{k}}(g^{p\bar{m}} \partial_j g_{\bar{m}q}).$$

When $g_{\bar{k}j}$ is not Kähler, there are several candidates for the notion of Ricci curvature,

$$(4.9) \quad R_{\bar{k}j} = R_{\bar{k}j}{}^p{}_p, \quad \tilde{R}_{\bar{k}j} = R^p{}_{p\bar{k}j}, \quad R'_{\bar{k}j} = R_{\bar{k}}{}^p{}_{pj}, \quad R''_{\bar{k}j} = R^p{}_{j\bar{k}p}$$

with corresponding notions of scalar curvature R, \tilde{R}, R' and R'' . We always have $R' = R''$ and $R = \tilde{R}$. However, for conformally balanced metrics, we also have the following relations

$$(4.10) \quad R'_{\bar{k}j} = R''_{\bar{k}j} = \frac{1}{2}R_{\bar{k}j} = \tilde{R}_{\bar{k}j} - \nabla^m T_{\bar{k}jm}.$$

For parabolicity purposes, it is crucial that it is the Chern-Ricci curvature $\tilde{R}_{\bar{k}j}$ that appears in the right hand side of (4.5) rather than the other notions of Ricci curvature. Indeed, explicitly,

$$\tilde{R}_{\bar{k}j} = -g^{p\bar{q}}g_{\bar{k}m}\partial_{\bar{q}}(g^{m\bar{\ell}}\partial_p g_{\bar{\ell}j}) = -\Delta g_{\bar{k}j} + \dots$$

which is the analogue of the equation (2.2) for the Ricci flow and shows that the flow is parabolic. Note that the other Ricci tensor

$$R_{\bar{k}j} = -\partial_j\partial_{\bar{k}} \log \omega^n = -g^{p\bar{q}}\partial_j\partial_{\bar{k}}g_{\bar{q}p} + \dots$$

has a natural interpretation as the first Chern class for X , but would not have been appropriate in the right hand side of (4.5) for parabolicity purposes.

4.3. Comparison with the Kähler-Ricci flow. With the formulation (4.5) for the Anomaly flow, it is easier now to compare it with the Kähler-Ricci flow. Perhaps surprisingly, since their original motivations are rather different, the two flows appear rather similar, with the Anomaly flow a higher order version of the Kähler-Ricci flow and the additional complications of $\|\Omega\|_{\omega}^{-1}$, of torsion, and of quadratic terms in the curvature tensor.

(a) In the Kähler-Ricci flow, the $(1,1)$ -cohomology class is determined $[\omega(t)] = [\omega(0)] - tc_1(X)$. In the Anomaly flow (4.1), we have rather

$$[\|\Omega\|_{\omega(t)}\omega^2(t)] = [\|\Omega\|_{\omega(0)}\omega^2(0)] - t\frac{\alpha'}{4}(c_2(X) - [\Phi_2(0)])$$

(b) However, the $(2,2)$ cohomology class $[\|\Omega\|_{\omega}\omega^2]$ provides much less information than the $(1,1)$ -cohomology class $[\omega]$. For example, the volume is an invariant of an $(1,1)$ cohomology class, but not of an $(2,2)$ -cohomology class.

(c) As an indirect consequence, the maximum time of the Anomaly flow is not determined by cohomology alone and depends on the initial data. In this respect, the Anomaly flow is closer to the Ricci flow than the Kähler-Ricci flow.

(d) The diffusion operator for the Anomaly flow is more complicated. It includes the Laplacian as for the Kähler-Ricci flow, but also the Riemann

curvature, e.g.

$$(4.11) \quad \delta R_{\bar{k}j}{}^\rho{}_\lambda \rightarrow \frac{1}{2\|\Omega\|} (\Delta(\delta R_{\bar{k}j}{}^\rho{}_\lambda) + 2\alpha' g^{\rho\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^\beta{}_\alpha \nabla_s \nabla_{\bar{\mu}}] \delta R_{\bar{k}j}{}^\alpha{}_\beta)$$

Another complication is that the torsion stays identically 0 in the Kähler-Ricci flow, but it evolves in the Anomaly flow. Explicitly, its evolution is given by

$$(4.12) \quad \begin{aligned} \partial_t T_{\bar{p}jq} &= \frac{1}{2\|\Omega\|_\omega} \left[\Delta T_{\bar{p}jq} - \alpha' g^{s\bar{r}} (\nabla_j (R_{[\bar{p}s}{}^\alpha{}_\beta R_{\bar{r}q]}{}^\beta{}_\alpha - \Phi_{\bar{p}s\bar{r}q}) \right. \\ &\quad \left. + \alpha' g^{s\bar{r}} \nabla_q (R_{[\bar{p}s}{}^\alpha{}_\beta R_{\bar{r}j]}{}^\beta{}_\alpha - \Phi_{\bar{p}s\bar{r}j}) \right] \\ &\quad + \frac{1}{2\|\Omega\|_\omega} (T^m{}_{jq} \Psi_{\bar{p}m} - T_j \Psi_{\bar{p}q} + T_q \Psi_{\bar{p}j} + \nabla_j (T\bar{T})_{\bar{p}q} - \nabla_q (T\bar{T})_{\bar{p}j}) \\ &\quad - \frac{1}{2\|\Omega\|_\omega} (T^r{}_{q\lambda} R^\lambda{}_{r\bar{p}j} - T^r{}_{j\lambda} R^\lambda{}_{r\bar{p}q}) \end{aligned}$$

where $\Delta = g^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$ is the Laplacian, and Ψ is the right hand side of (4.1).

4.4. The Anomaly flow for the Hull-Strominger-Ivanov system.

In the original formulation of Strominger [90] of the Hull-Strominger system (4.2), it was suggested that Rm be the curvature of the Chern unitary connection of ω , as Rm would then be a $(1, 1)$ -form, which would be consistent with the first equation in (4.2) being an equation of $(2, 2)$ -forms. On the other hand, in [58], Hull argued that actually Rm can be the curvature of any connection, at the only cost of adding finite counterterms to the effective action.

More recently, in [59], Ivanov argued that, in order to obtain the field equations of the heterotic string, the Hull-Strominger system ought to be supplemented by the condition that Rm be the curvature of an $SU(3)$ instanton. By definition, an $SU(3)$ instanton with respect to a metric ω is given by another Hermitian metric $\tilde{\omega}$ on X whose Chern unitary connection would satisfy

$$(4.13) \quad \omega^2 \wedge \tilde{F} = 0.$$

Here \tilde{F} is the curvature of the Chern unitary connection of $\tilde{\omega}$, and the right hand side is 0 because $c_1(X) = 0$. Thus a complete system of equations for a supersymmetric vacuum of the heterotic string would consist of three unknowns $(\omega, \tilde{\omega}, h)$ satisfying both (4.2) and (4.13). We shall refer to the resulting system for $(\omega, \tilde{\omega}, h)$ as the Hull-Strominger-Ivanov system. Kähler Ricci-flat metrics can still be viewed as special solutions of the Hull-Strominger-Ivanov system, in the sense that if ω is such a metric and we set $E = T^{1,0}(X)$, $h = \tilde{\omega} = \omega$, then we would obtain a solution of this system.

In [46], Garcia-Fernandez et al. introduced an elegant way of reformulating the Hull-Strominger-Ivanov system. For this, let (X, Ω) be as before,

and let \mathcal{E} be a holomorphic vector bundle over X . Let c be also a bilinear form

$$(4.14) \quad c : \Lambda^{1,1}(\mathcal{E}) \times \Lambda^{1,1}(\mathcal{E}) \rightarrow \Lambda^{2,2}(X) \cap \text{Ker } d.$$

(for some applications, it may be necessary to impose the stronger condition that in the right hand side, $\text{Ker } d$ should be replaced by the smaller space $\text{Im } i\partial\bar{\partial}$.) For each Hermitian metric \mathcal{H} on \mathcal{E} , let $F_{\mathcal{H}}$ be the curvature of the corresponding Chern unitary connection. Then the following system of equations for (ω, \mathcal{H})

$$(4.15) \quad \begin{aligned} i\partial\bar{\partial}\omega - \frac{\alpha'}{4}c(F_{\mathcal{H}}, F_{\mathcal{H}}) &= 0 \\ d(\|\Omega\|_{\omega}\omega^2) &= 0 \\ F_{\mathcal{H}}^{0,2} = F_{\mathcal{H}}^{2,0} = 0, \quad \omega^2 \wedge F_{\mathcal{H}} &= 0 \end{aligned}$$

is a generalization of the Hull-Strominger-Ivanov system. Indeed, if we let $\mathcal{E} = T^{1,0}(X) \otimes E$, $\mathcal{H} = (\tilde{\omega}, h)$, and

$$(4.16) \quad c((A_1, B_1), (A_2, B_2)) = \text{Tr}(A_1 \wedge A_2) - \text{Tr}(B_1 \wedge B_2)$$

where (A_i, B_i) is the corresponding decomposition of $\Lambda^{1,1}(\mathcal{E}) = \Lambda^{1,1}(T^{1,0}(X)) \oplus \Lambda^{1,1}(E)$, we get back the Hull-Strominger-Ivanov system. In their paper [46] Garcia-Fernandez et al. also found a variational principle for the Hull-Strominger-Ivanov system, in other words, a space of (ω, \mathcal{H}) on which a functional $M(\omega, \mathcal{H})$ can be defined, whose critical points are exactly the solutions of the system. We won't reproduce the formula for $M(\omega, \mathcal{H})$ here, but note only that a simple version in the special case with $\alpha' = 0$ and no \mathcal{H} appears below in (6.12).

It is straightforward to formulate a version of the Anomaly flow for the Hull-Strominger-Ivanov system. In view of the previous reformulation (4.15) of the system, it suffices to consider the following version $t \rightarrow (\omega(t), \mathcal{H}(t))$ of the Anomaly flow,

$$(4.17) \quad \begin{aligned} \partial_t(\|\Omega\|_{\omega}\omega^2) &= i\partial\bar{\partial}\omega - \frac{\alpha'}{4}c(F_{\mathcal{H}}, F_{\mathcal{H}}) \\ \mathcal{H}^{-1}\partial_t\mathcal{H} &= -\frac{\omega^2 \wedge F_{\mathcal{H}}}{\omega^3}. \end{aligned}$$

for initial data $(\omega(0), \mathcal{H}(0))$ with $\omega(0)$ conformally balanced. Clearly this system again preserves the conformally balanced property of $\omega(t)$, and its stationary points satisfy the Hull-Strominger-Ivanov system (4.15). In fact, this system is even simpler than the original Anomaly flow defined in (4.4), because it has no quadratic term in $Rm(\omega)$. There are quadratic terms in $F_{\mathcal{H}}$, but these are simpler than $Rm(\omega)$.

The parabolicity of the flow (4.17), and hence its short-time existence is an immediate consequence of the results of [75] and [78], and more specifically of formulas of the type (4.5).

5. Illustrative cases of the Anomaly flow

The Anomaly flow is still largely unexplored, and the many additional difficulties compared to the Kähler-Ricci flow which were described in section §4.3, suggest that a lot of work will have to be done before we can achieve a level of understanding anywhere close to our present one for the Kähler-Ricci flow. However, many special cases of the original Anomaly flow (4.4) have now been worked out, each of which sheds a different light on the flow, and each of which may ultimately be useful in developing the general theory.

5.1. The flow on toric fibrations over Calabi-Yau surfaces. Let $(Y, \hat{\omega})$ be a Calabi-Yau surface, equipped with a nowhere vanishing holomorphic $(2, 0)$ -form Ω_Y . Let $\omega_1, \omega_2 \in H^2(Y, \mathbf{Z})$ satisfy $\omega_1 \wedge \hat{\omega} = \omega_2 \wedge \hat{\omega} = 0$.

From this data, building on ideas going back to Calabi and Eckmann [15], Goldstein and Prokushkin [48] showed how to construct a toric fibration $\pi : X \rightarrow Y$, equipped with a $(1, 0)$ -form θ on X satisfying $\partial\theta = 0$, $\bar{\partial}\theta = \pi^*(\omega_1 + i\omega_2)$. Furthermore, the form $\Omega = \sqrt{3}\Omega_Y \wedge \theta$ is a holomorphic nowhere vanishing $(3, 0)$ -form on X , and for any scalar function u on M , the $(1, 1)$ -form

$$(5.1) \quad \omega_u = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}$$

is a conformally balanced metric on X .

The first non-Kähler solution of the Hull-Strominger was actually found by Fu and Yau [43, 44] in 2006, using these fibrations and elliptic partial differential equations methods. Besides the solutions themselves, the Fu-Yau work also revealed the rich and deep structure of the Hull-Strominger system. Since then many other non-Kähler solutions have been found, see e.g. [3, 4, 34, 36, 37, 38, 43, 41, 71].

In general, for a given elliptic system, there is an infinite number of flows that will admit it as stationary points, but most of them will not behave well for large time. Usually one has the freedom of selecting a well-behaved flow. But in the present case, the Anomaly flow was dictated by the requirement that the balanced condition is preserved, and there does not appear any other flow that meets this requirement. Thus it is particularly important to determine whether the Anomaly flow is a viable tool for solving the Hull-Strominger system. This is confirmed by the following theorem in the case of toric fibrations [79]:

THEOREM 2. *Consider the Anomaly flow on the fibration $X \rightarrow Y$ constructed above, with initial data $\omega(0) = \pi^*(M\hat{\omega}) + i\theta \wedge \bar{\theta}$, where M is a positive constant. Fix h on the bundle $E \rightarrow Y$ satisfying the Hermitian-Yang-Mills equation $\hat{\omega} \wedge F = 0$. Then $\omega(t)$ is of the form $\pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}$ and, assuming an integrability condition on the data (which is necessary), there exists $M_0 > 0$, so that for all $M \geq M_0$, the flow exists for all time,*

and converges to a metric ω_∞ with $(X, \pi^*E, \omega_\infty, \pi^*(h))$ satisfying the Hull-Strominger system.

The reason why the Anomaly flow on Calabi-Eckmann-Goldstein-Prokushkin fibrations is simpler than the general case is because it descends to a flow on the base Y of the form.

(5.2)

$$\partial_t \omega = -\frac{1}{2\|\Omega\|_\omega} \left[\frac{R}{2} - |T|^2 - \frac{\alpha'}{4} \sigma_2(iRic_\omega) + 2\alpha' \frac{i\partial\bar{\partial}(\|\Omega\|_\omega \rho)}{\omega^2} - 2\frac{\mu}{\omega^2} \right] \omega$$

with an initial metric of the form $\omega(0) = M\hat{\omega}$. Here ω is now a Hermitian metric on the base Y , and $\sigma_2(\Phi) = \Phi \wedge \Phi \omega^{-2}$ is the usual determinant of a real $(1, 1)$ -form Φ , relative to the metric ω . ρ and μ are given smooth $(1, 1)$ -form and $(2, 2)$ -form, respectively, both are determined explicitly by $\hat{\omega}$, F and θ .

Since ω stays conformal to $\hat{\omega}$, we can set $\omega = e^u \hat{\omega}$, and rewrite also this flow as a parabolic flow in u ,

$$(5.3) \quad \partial_t u = \frac{1}{2} e^{-u} \left(\Delta_{\hat{\omega}} e^u + \alpha' \hat{\sigma}_2(i\partial\bar{\partial}u) - 2\alpha' \frac{i\partial\bar{\partial}(e^{-u}\rho)}{\hat{\omega}^2} + 2\frac{\mu}{\hat{\omega}^2} \right)$$

where both the Laplacian $\Delta_{\hat{\omega}}$ and the determinant $\hat{\sigma}_2$ are written with respect to the fixed metric $\hat{\omega}$. This flow is not concave in the Hessian of u , and this makes it inaccessible to standard non-linear PDE techniques. In practice, we actually solve the problem by using the previous, Ricci-flow like formulation.

Some key ingredients of the proof are the following estimates. We illustrate only features which are peculiar to the flow (5.2), the details can be found in the original paper [79].

5.1.1. *A priori estimate for $\|\Omega\|_\omega$.* The first estimate is a C^0 estimate for $\|\Omega\|_\omega = e^{-u}$, or equivalently an upper and a lower bound for e^u , but which has to be sharper than the usual C^0 estimates. This is because it does not suffice to show that the upper and lower bounds are independent of time, but also that they remain comparable to the size of the initial data.

LEMMA 1. *Assume that the flow (5.2) exists and is parabolic for $t \in [0, T)$ and starts from $u_0 = \log M$. Then there exists M_0 such that, for all $M \geq M_0$, we have*

$$(5.4) \quad \sup_{X \times [0, T)} e^u \leq C_1 M, \quad \sup_{X \times [0, T)} e^{-u} \leq C_2 M^{-1}$$

with constants C_1, C_2 depending only on $(Y, \hat{\omega})$, ρ , μ , and α' .

This estimate is proved by parabolic Moser iteration. The first step is to isolate from the flow a term comparable to the L^2 norm of the gradient of $e^{(k+1)u/2}$, for arbitrary k . For example, to establish the sup norm bound,

we use

$$\begin{aligned}
 & \frac{k}{2} \int_X e^{(k+1)u} \{ \hat{\omega} + \alpha' e^{-2u} \rho \} \wedge i\partial u \wedge \bar{\partial} u + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \frac{\hat{\omega}^2}{2!} \\
 &= -\frac{k}{2} \int_X e^{ku} i\partial u \wedge \bar{\partial} u \wedge \omega' \\
 (5.5) \quad &+ \int_X e^{ku} \mu - \alpha' \left(1 - \frac{1}{1-k} \right) \int_X e^{(k-1)u} i\partial \bar{\partial} \rho
 \end{aligned}$$

with a form ω' which is positive as long as the flow exists and is parabolic. To prove the lemma, we assume first that $\sup_X e^{-u(\cdot,t)} < \delta$ for all $t \in [0, T]$, where δ is a fixed small positive constant depending only on the data $(Y, \hat{\omega})$, ρ , μ , and α' . We later show that $\sup_X e^{-u(\cdot,t)} < \delta$ is preserved. The above estimate implies the bound

$$\begin{aligned}
 & \frac{k}{2} \int_X e^{(k+1)u} |\nabla u|^2 + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \\
 (5.6) \quad & \leq (\|\mu\|_{L^\infty} + 2\|\alpha'\rho\|_{C^2}) \left(\int_X e^{ku} + \int_X e^{(k-1)u} \right).
 \end{aligned}$$

We can then apply the Sobolev inequality and iterate the bound. But the need for obtaining precise bounds in terms of M requires that we need to consider not just large time, but also small time. This is because *some* finite bound is automatic for small time from parabolicity, but this is not the case for *the specific* bound of the type stated in the lemma.

For the lower bound, we need to use the upper bound to establish the preliminary integral estimate

$$(5.7) \quad \int_X e^{-u} \leq 2C_0 M^{-1}$$

again assuming to begin with that $\sup_X e^{-u(\cdot,t)} < \delta$ for all $t \in [0, T]$. Isolating the gradient, applying the Sobolev inequality, and iterating as before starting from the above integral estimate, separately for the cases of small time and large time, we obtain

$$(5.8) \quad \sup_{X \times [0, T]} e^{-u} \leq C_2 M^{-1}$$

assuming that $\sup_X e^{-u(\cdot,t)} < \delta$ for all $t \in [0, T]$. The proof of Lemma 1 is then completed by the following argument. Start from $u(0) = \log M$ with M large enough so that the condition $\sup_X e^{-u(\cdot,0)} < \delta$ is satisfied. The estimates which have already been obtained imply that this condition is preserved as long as the flow exists, and actually the better estimate (5.8) holds. Q.E.D.

5.1.2. *A priori estimate for the torsion.* The previous C^0 estimate is the only one which relies on the formulation of the Anomaly flow as a scalar parabolic equation. From now on, the other estimates make use rather of the formulation (5.2) as a geometric flow of metrics.

LEMMA 2. *There exists $M_0 > 1$ with the following property. Let the flow (4.5) start from a constant function $u(0) = \log M$ with $M \geq M_0$. If*

$$(5.9) \quad |\alpha' Ric_\omega| \leq 10^{-6}$$

along the flow, then there exists $C_3 > 0$ depending only on $(Y, \hat{\omega})$, ρ , μ , and α' , such that

$$(5.10) \quad |T|^2 \leq C_3 M^{-1}.$$

The proof is by the maximum principle applied to the function

$$(5.11) \quad G = \log |T|^2 - \Lambda \log \|\Omega\|_\omega$$

with $\Lambda = 1 + 2^{-3}$. Set

$$(5.12) \quad F^{p\bar{q}} = g^{p\bar{q}} + \alpha' \|\Omega\|_\omega^3 \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2} (Rg^{p\bar{q}} - R^{p\bar{q}})$$

where the $(1, 1)$ -form $\tilde{\rho}^{p\bar{q}}$ is defined by picking up the coefficient of $u_{\bar{q}p}$ in the expansion

$$(5.13) \quad -\alpha' i\partial\bar{\partial}(e^{-u}\rho) = (\alpha' e^{-u} \tilde{\rho}^{p\bar{q}} u_{\bar{q}p} + \dots) \frac{\hat{\omega}^2}{2}.$$

We can then show that, at a maximum point of G , we must have

$$(5.14) \quad 0 \leq F^{p\bar{q}} \partial_p \partial_{\bar{q}} G - \frac{1}{200} |T|^2 + C \|\Omega\|_\omega (1 + \frac{\|\Omega\|_\omega^{\frac{1}{2}}}{|T|}).$$

If $|T| \leq \|\Omega\|_\omega^{\frac{1}{2}}$, the desired estimate follows is a trivial consequence of Lemma 1. Otherwise, we obtain at a maximum point

$$(5.15) \quad |T|^2 \leq 200 C \|\Omega\|_\omega (1 + \frac{\|\Omega\|_\omega^{\frac{1}{2}}}{|T|})$$

which again implies the desired estimate.

5.1.3. *A priori estimate for the curvature.* Perhaps the most technically elaborate estimate is the following C^2 estimate:

LEMMA 3. *Start the flow with a constant function $u_0 = \log M$. There exists $M_0 > 1$ such that for every $M \geq M_0$, if*

$$(5.16) \quad \|\Omega\|_\omega \leq C_2 M, \quad |T|^2 \leq C_3 M^{-1}$$

along the flow, then

$$(5.17) \quad |\alpha' Ric_\omega| \leq C_5 M^{-\frac{1}{2}}$$

where C_5 only depends on $(Y, \hat{\omega})$, ρ , μ , and α' . Here, C_2 and C_3 are the constants given in Lemmas 1 and 2 respectively.

This lemma is proved by considering the following quantity

$$(5.18) \quad |\alpha' Ric_\omega|^2 + \Theta|T|^2$$

where Θ is a suitable constant. This quantity starts from the value 0 at time $t = 0$. If we choose Θ to be

$$(5.19) \quad \Theta = \max\{4|\alpha'|^{-1}, 8(\alpha')^2(C_4 + 1)\}$$

then we can show that

$$(5.20) \quad \partial_t(|\alpha' Ric_\omega|^2 + \Theta|T|^2) \leq 0$$

the first time that it reaches the value $(2\Theta C_3 + 1)M^{-1}$. This implies that it never exceeds this value, and Lemma 3 is proved.

5.1.4. *The long-time existence of the flow.* A priori estimates for the derivatives of the curvature and of the torsion, in terms of suitable powers of M and constants depending only on the geometric data can be formulated and established in the same way, when $|\alpha' Ric_\omega| \leq \delta_1$ and $|T| \leq \delta_2$ for some fixed small positive constants δ_1 and δ_2 .

Once the a priori estimates are available, the long-time existence of the flow can be established as follows. If we start from $u(0) = \log M$, we have $|\alpha' Ric_\omega| = |T| = 0$ at time $t = 0$. So the set of times t for which $|\alpha' Ric_\omega| \leq \delta_1$, $|T| \leq \delta_2$ on $X \times [0, t)$ is closed and not empty. By the a priori estimates, we obtain even better estimates $|\alpha' Ric_0| < C_5 M^{-\frac{1}{2}}$, $|T|^2 \leq C_3 M^{-1}$, which show that it is also open. Thus we obtain estimates for all derivatives and all time, which implies the existence of the flow for all time.

5.1.5. *The convergence of the flow.* The preceding bounds imply the existence of a sequence of times for which the flow is convergent. To get the full convergence, we introduce

$$(5.21) \quad J(t) = \int_X v^2 \frac{\hat{\omega}^2}{2!}$$

with $v = \partial_t e^u$. Using the a priori estimates and the fact that v has average 0 at all times, we can show that

$$(5.22) \quad \frac{dJ(t)}{dt} \leq -\eta J$$

for some strictly positive constant η if M is large enough. Thus v tends to 0 in L^2 norm, and it is not difficult, using the a priori estimates, to deduce then that it tends to 0 in C^∞ norm. The proof of Theorem 2 is complete.

5.2. The Anomaly flow on hyperkähler fibrations over Riemann surfaces. Next we describe a very recent work of T. Fei, Z. Huang, and S. Picard [34, 35]. Their set-up is certain hyperkähler fibrations over Riemann surfaces originating from works of Fei [31, 32, 33], which are themselves built on earlier constructions of Calabi [14] and Gray [49].

Let Σ be a Riemann surface and a holomorphic map $\varphi : \Sigma \rightarrow \mathbf{CP}^1$ with $\varphi^*O(2) = K_\Sigma$. By pulling back sections of $O(2)$, we obtain three holomorphic

(1, 0)-forms μ_1, μ_2, μ_3 . Let $\hat{\omega}$ be the metric proportional to $i \sum_{j=1}^3 \mu_j \wedge \bar{\mu}_j$ normalized so that $\int_{\Sigma} \hat{\omega} = 1$.

Next, take a flat 4-torus (T^4, g) which is viewed as a hyperkähler manifold with complex structures I, J, K satisfying $I^2 = J^2 = K^2 = -1$ and $IJ = K$. Let $\omega_I, \omega_J, \omega_K$ be the corresponding Kähler forms. For any point $(\alpha, \beta, \gamma) \in S^2$, there is a compatible complex structure $\alpha I + \beta J + \gamma K$ on T^4 . Let now $X = \Sigma \times T^4$, with the complex structure $j_{\Sigma} \oplus (\alpha I + \beta J + \gamma K)$, where we have let $\varphi = (\alpha, \beta, \gamma)$.

Then X is a non-kähler complex manifold with trivial canonical bundle, and non-vanishing holomorphic (3, 0)-form given by

$$\Omega = \mu_1 \wedge \omega_I + \mu_2 \wedge \omega_J + \mu_3 \wedge \omega_K$$

Furthermore, for any real function $f \in C^{\infty}(\Sigma)$, as shown in [31, 32], the (1, 1)-form

$$(5.23) \quad \omega_f = e^{2f} \hat{\omega} + e^f \omega', \quad \text{with } \omega' = \alpha \omega_I + \beta \omega_J + \gamma \omega_K$$

is a conformally balanced metric on X . Furthermore, $\|\Omega\|_{\omega_f} = e^{-2f}$ and

$$(5.24) \quad \|\Omega\|_{\omega_f} \omega_f^2 = 2 \text{vol}_{T^4} + 2e^f \hat{\omega} \wedge \omega',$$

The Anomaly flow on X preserves the above ansatz and reduces then to the following conformal flow on Σ for a metric $\omega(t)$

$$(5.25) \quad \partial_t e^f = \frac{1}{2} [\hat{g}^{z\bar{z}} \partial_z \partial_{\bar{z}} (e^f + \frac{\alpha'}{2} \kappa e^{-f}) - \kappa (e^f + \frac{\alpha'}{2} \kappa e^{-f})]$$

where κ is the Gauss curvature of $\hat{\omega}$. Explicitly, $\kappa = -\varphi^* \omega_{FS} / \hat{\omega}$ and is always < 0 , except at the branch points of φ , where it vanishes. Despite the extent to which Riemann surfaces have been studied, the flow (5.25) does not seem to have appeared before. It may be useful for future investigations to have the following more intrinsic formulation of it,

$$(5.26) \quad \partial_t \omega = \frac{\omega}{|\mu|^2} (-R + |\nabla \log |\mu|^2|^2) + \frac{\alpha'}{2} (\Delta |\nabla \varphi|^2 - |\nabla \varphi|^4)$$

where all norms and operators are taken with respect to the evolving metric $\omega(t)$. The following criterion for the long-time existence of the flow in this setting was proved in [35]:

THEOREM 3. *Assume that $\alpha' > 0$. Suppose that a solution to (5.25) exists on $[0, T)$. If*

$$(5.27) \quad \sup_{\Sigma \times [0, T)} \|\Omega\|_{\omega_f} = \sup_{\Sigma \times [0, T)} e^{-2f} < \infty,$$

then the flow can be extended to $[0, T + \epsilon)$ for some $\epsilon > 0$.

The proof is short and can be given here. Let the function u be defined by

$$(5.28) \quad u = e^f + \frac{\alpha'}{2} \kappa e^{-f}.$$

Then u evolves by

$$(5.29) \quad \partial_t u = \frac{1}{2} \left(1 - \frac{\alpha'}{2} \kappa e^{-f} \right) (\Delta_{\hat{\omega}} u - \kappa u).$$

Since $\alpha' > 0$, $\kappa \leq 0$, and $e^{-f} \leq C$, this is a parabolic equation for u with bounded coefficients. We may therefore apply the theorem of Krylov-Safonov [62] to obtain parabolic Hölder estimates for u . Since $2e^f = (u + \sqrt{u^2 - 2\alpha'\kappa})$, we may deduce Hölder estimates for e^f and e^{-f} . Applying parabolic Schauder theory and a bootstrap argument, we obtain higher order estimates for e^f which allow us to extend the flow past T . Q.E.D.

However, it may happen that $e^{-f} \rightarrow \infty$ as $t \rightarrow T$ for some finite time T , and the flow will terminate in finite time. Indeed, integrating (5.25) gives

$$(5.30) \quad \frac{d}{dt} \int_{\Sigma} e^f \hat{\omega} = \frac{1}{2} \int_{\Sigma} (-\kappa) e^f \hat{\omega} - \frac{\alpha'}{4} \int_{\Sigma} \kappa^2 e^{-f} \hat{\omega}.$$

Applying the Cauchy-Schwarz inequality,

$$(5.31) \quad \frac{d}{dt} \int_{\Sigma} e^f \hat{\omega} \leq \frac{\|\kappa\|_{L^\infty(\Sigma)}}{2} \int_{\Sigma} e^f \hat{\omega} - \frac{\alpha'}{4} \left(\int_{\Sigma} (-\kappa) \hat{\omega} \right)^2 \left(\int_{\Sigma} e^f \hat{\omega} \right)^{-1}.$$

Since the genus $g \geq 3$ for this construction, the Gauss-Bonnet theorem gives $\int_{\Sigma} (-\kappa) \hat{\omega} > 0$. By studying this ODE for $\int_{\Sigma} e^f \hat{\omega}$, we see that if $\int_{\Sigma} e^f \hat{\omega}$ is too small initially then it will collapse to zero in finite time. This provides an example of the Anomaly flow on $[0, T)$ with $T < \infty$ where $\|\Omega\|_{\omega_f} = e^{-2f} \rightarrow \infty$ as $t \rightarrow T$.

At the other end, it was proved in [35] that for initial data with e^f large, then the flow will exist for all time, and collapse the fibers in the limit:

THEOREM 4. *Assume that $e^f \gg 1$ at the initial time. Then the flow exists for all time, and*

$$\frac{\omega_f}{\int_X \|\Omega\|_{\omega_f} \frac{\omega_f^3}{3!}} \rightarrow p^*(q_1^2 \hat{\omega})$$

where q_1 is the first eigenfunction of the operator $-\Delta_{\hat{\omega}} - |\nabla \varphi|_{\hat{\omega}}^2$. Furthermore

$$\left(X, \frac{\omega_f}{\int_X \|\Omega\|_{\omega_f} \frac{\omega_f^3}{3!}} \right) \rightarrow (\Sigma, q_1^2 \hat{\omega}) \quad \text{in Gromov - Hausdorff topology.}$$

The two regions of initial data e^f small or large leave out the particularly interesting region of intermediate e^f . It is actually in this intermediate region that Fei, Huang, and Picard [34] found infinitely many topologically distinct examples, which is remarkable, as the analogous statement has been conjectured for Calabi-Yau manifolds, but has not been proved as yet.

When comparing the Anomaly flow to the Kähler-Ricci flow in section §4.3, we had noted that the Anomaly flow preserves the $(2, 2)$ de Rham cohomology class $[\|\Omega\|_{\omega}\omega^2]$ (when $c_2(X) = c_2(E)$), but that this condition is much weaker than the preservation of the $(1, 1)$ -Kähler class as in the case of the Kähler-Ricci flow. Nevertheless, it can give important information, as we shall now discuss in the present case.

Taking the de Rham class of (5.24) yields

$$(5.32) \quad \frac{1}{2}[\|\Omega\|_{\omega_f}\omega_f^2] = [\text{vol}_{T^4}] + [e^f\alpha\hat{\omega}][\omega_I] + [e^f\beta\hat{\omega}][\omega_J] + [e^f\gamma\hat{\omega}][\omega_K].$$

Therefore $[\|\Omega\|_{\omega_f}\omega_f^2] \in H^4(X, \mathbf{R})$ is parametrized by the vector

$$(5.33) \quad V = \left(\int_{\Sigma} e^f \alpha \hat{\omega}, \int_{\Sigma} e^f \beta \hat{\omega}, \int_{\Sigma} e^f \gamma \hat{\omega} \right).$$

Since the class $[\|\Omega\|_{\omega_f}\omega_f^2]$ is preserved along the Anomaly flow, the vector V is constant along the flow, and since $(\alpha, \beta, \gamma) \in S^2$, we have

$$(5.34) \quad \int_{\Sigma} e^f \hat{\omega} \geq |V|.$$

This is a bound from below for the flow in terms of cohomological data. It gives more refined information about the finite time singularity obtained when $\int_{\Sigma} e^f \hat{\omega} \ll 1$. Since the ODE (5.31) forces $\int_{\Sigma} e^f \hat{\omega} \rightarrow 0$ in finite time, we see that the long-time existence criterion must fail first, and $e^{-2f} = \|\Omega_f\|_{\omega_f} \rightarrow \infty$ as $t \rightarrow T$ on a set of measure zero.

Finally, we note that the flow admits an energy functional

$$(5.35) \quad I(u) = \int_{\Sigma} |\nabla u|_{\hat{\omega}}^2 \hat{\omega} + \int_{\Sigma} \kappa u^2 \hat{\omega},$$

which is monotone decreasing for arbitrary initial data. This functional was used in [35] to provide convergence of the normalized solution in the regime $u \geq 0$.

It is possible that the existence of solutions to the Hull-Strominger system may be tied to some stability condition. In this case, the functional $I(u)$ may turn out to be useful. We note that [34] introduced a condition which they call the ‘‘hemisphere condition’’, which may also be related to some notion of stability.

5.3. The case of unimodular Lie groups. Next, we consider the Anomaly flow on unimodular Lie groups. This is a natural setting to consider, since any left-invariant metric will be conformally balanced. The left-invariance property will also reduce the Anomaly flow to a system of ordinary differential equations. This will allow us to tackle the first case when the Anomaly flow is a genuine system instead of just a scalar equation. It will be then easier to trace the effects of a non-zero parameter α' . An interesting feature of the Hull-Strominger equations will emerge, which is that solutions may require a different connection on the Gauduchon line of metrics

associated to a Hermitian metric than the Chern unitary connection. This had been anticipated in the physics literature [5, 9, 37, 56], and worked out explicitly by Fei and Yau [36] for stationary points. The only caveat is that we shall be dealing with non-compact settings, which may not behave exactly in the same way as in the compact case.

Consider left-invariant metrics on a 3-dimensional complex Lie group X . Let $\{e_a\}$ be a basis of left-invariant holomorphic vector fields on X , with structure constants c^d_{ab} ,

$$[e_a, e_b] = \sum_d e_d c^d_{ab}$$

Then $\Omega = e^1 \wedge e^2 \wedge e^3$ is a holomorphic nowhere vanishing $(3, 0)$ -form on X , and if X is unimodular in the sense that

$$\sum_d c^d_{db} = 0, \quad \text{for any } b,$$

then any metric $\omega = ig_{\bar{a}b}e^b \wedge \bar{e}^a$ is balanced [1]. Unimodular Lie groups in 3-d are given by \mathbf{C}^3 , the Heisenberg group, the rigid motions of \mathbf{R}^2 , and $SL(2, \mathbf{C})$.

Because ω is not necessarily Kähler, there are many natural unitary connections associated to it. The most familiar one is the Chern unitary connection, defined earlier. But other unitary connections $\nabla^{(\kappa)}$, called the Gauduchon line, can be defined by

$$\nabla_j^{(\kappa)} V^k = \nabla_j V^k - \kappa T^k_{jm} V^m$$

Here $T = i\partial\omega$ is the torsion. The connections with $\kappa = 1$ and $\kappa = 1/2$ are known respectively as the Bismut connection and the Lichnerowicz connection. The following theorem was proved in [80]:

THEOREM 5. *Set $\tau = 2\kappa^2(2\kappa - 1)$, and assume that $\alpha'\tau > 0$. Then*

- (a) *When $X = \mathbf{C}^3$, any metric, and hence the flow is stationary.*
- (b) *When X is nilpotent, there is no stationary point, and hence the flow can never converge. If the initial metric is diagonal, the metric remains diagonal along the flow, the lowest eigenvalue remains constant, while the other two eigenvalues tend to ∞ at a constant rate.*
- (c) *When X is solvable, the stationary points of the flow are precisely the metrics with*

$$g_{\bar{1}2} = g_{\bar{2}1} = 0, \quad \alpha'\tau g^{3\bar{3}} = 1.$$

The flow is asymptotically unstable near any stationary point. However, the condition $g_{\bar{1}2} = g_{\bar{2}1} = 0$ is preserved along the flow, and the flow with any initial metric satisfying this condition converges to a stationary point.

- (d) *When $X = SL(2, \mathbf{C})$, there is a unique stationary point, given by the diagonal metric*

$$g_{\bar{a}b} = \frac{\alpha'\tau}{2} \delta_{ab}$$

The linearization of the flow at the fixed point admits both positive and negative eigenvalues, and hence the flow is asymptotically unstable.

The details can be found in [80]. Here we shall stress only that, underlying all this is the remarkable fact, worked out by Fei and Yau, that while Rm is not a $(1, 1)$ -form for general $\nabla^{(\kappa)}$ connections, $\text{Tr}(Rm \wedge Rm)$ is a $(2, 2)$ -form. More precisely,

$$\text{Tr}(Rm \wedge Rm)_{\bar{k}\bar{\ell}ij} = \overline{\tau c^r_{kl} c^s_{rp} c^q_{ij} c^s_{qp}}$$

We should also say that the equations appear complicated, for example, they are given by

$$\partial_t g_{11} = \frac{1}{2\|\Omega\|} g^{3\bar{3}} g_{11} \left(1 - \frac{\alpha'\tau}{4} g^{3\bar{3}}\right), \quad \partial_t g_{22} = \frac{1}{2\|\Omega\|} g^{3\bar{3}} g_{22} \left(1 - \frac{\alpha'\tau}{4} g^{3\bar{3}}\right), \text{ etc.}$$

in the case of the group of rigid motions, and in the case of $SL(2, \mathbf{C})$ by

$$\partial_t \lambda_1 = \frac{(\lambda_1 \lambda_2 \lambda_3)^{\frac{1}{2}}}{2} \left\{ \frac{\alpha'\tau}{4} \left(\frac{2}{\lambda_1} + \frac{\lambda_1}{\lambda_2^2} + \frac{\lambda_1}{\lambda_3^2} \right) - \frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2} \right\}$$

with similar equations for λ_2 and λ_3 . But they reveal a lot of structure upon closer inspection, and this leads to the above detailed information on the phase diagram of the corresponding dynamical systems.

6. A flow of balanced metrics with Kähler fixed points

The special case $\alpha' = 0$ of the Anomaly flow may actually be of independent geometric interest as well. It is given by $\partial_t(\|\Omega\|_{\omega} \omega^2) = i\partial\bar{\partial}\omega$ for 3-folds X , and a natural generalization to arbitrary n dimensions for $n \geq 3$ can be defined by [82]

$$(6.1) \quad \partial_t(\|\Omega\|_{\omega} \omega^{n-1}) = i\partial\bar{\partial}\omega^{n-2}.$$

Here X is assumed to be a compact complex manifold of dimension n , which admits a nowhere vanishing holomorphic $(n, 0)$ -form Ω , and the initial metric $\omega(0)$ is assumed to be conformally balanced, $d(\|\Omega\|_{\omega(0)} \omega^{n-1}(0)) = 0$.

6.1. Formulation as a flow of metrics. As in the case of 3 dimensions, the above flow of $(n - 1, n - 1)$ -forms admits a useful reformulation as a flow of metrics, as it was given in [82]:

THEOREM 6. *Consider the flow (6.1) with X , Ω , and $\omega(0)$ as described above. Set $\omega = ig_{\bar{k}j} dz^j \wedge d\bar{z}^k$. Then the evolution of $g_{\bar{k}j}$ is given by*

(a) *When $n = 3$,*

$$(6.2) \quad \partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_{\omega}} \left[-\tilde{R}_{\bar{k}j} + g^{m\bar{\ell}} g^{s\bar{r}} T_{\bar{r}mj} \bar{T}_{s\bar{\ell}k} \right]$$

(b) When $n \geq 4$,

$$(6.3) \quad \begin{aligned} \partial_t g_{\bar{k}j} &= \frac{1}{(n-1)\|\Omega\|_\omega} \left[-\tilde{R}_{\bar{k}j} + \frac{1}{2(n-2)}(|T|^2 - 2|\tau|^2) g_{\bar{k}j} \right. \\ &\quad \left. - \frac{1}{2} g^{q\bar{p}} g^{s\bar{r}} T_{\bar{k}qs} \bar{T}_{j\bar{p}\bar{r}} + g^{s\bar{r}} (T_{\bar{k}js} \bar{T}_{\bar{r}} + T_s \bar{T}_{j\bar{k}\bar{r}}) + T_j \bar{T}_{\bar{k}} \right]. \end{aligned}$$

Recall that the torsion terms are defined by $T = i\partial\omega$, $T_\ell = g^{j\bar{k}} T_{\bar{k}j\ell}$, and we have introduced the $(1, 0)$ -form $\tau = T_\ell dz^\ell$.

The reason that the flow (6.1) may be interesting is because its stationary points are Ricci-flat Kähler metrics. In fact, the stationary points of the flow satisfy the so-called *astheno-Kähler* condition $i\partial\bar{\partial}\omega^{n-2} = 0$ introduced in [60]. The fact that conformally balanced and astheno-Kähler metrics must be Kähler had been shown earlier by Matsuo-Takahashi [66] and Fino-Tomassini [39]. With the above formulas (6.2) and (6.3) for the flow (6.1), it is easy to give an independent proof of this fact. We give the proof when $n \geq 4$, the same argument works for $n = 3$ and is even simpler. At a stationary point $\partial_t g_{\bar{k}j} = 0$, and taking the trace of the right hand side of (6.3) gives

$$0 = -\tilde{R} + \frac{n}{2(n-2)}(|T|^2 - 2|\tau|^2) - \frac{1}{2}|T|^2 + 3|\tau|^2$$

But for conformally balanced metrics, $\tilde{R} = R = \Delta \log \|\Omega\|_\omega^2$. Simplifying yields

$$(n-2)\Delta \log \|\Omega\|_\omega^2 = |T|^2 + 2(n-3)|\tau|^2 \geq 0$$

Thus $\log \|\Omega\|_\omega^2$ is constant by maximum principle, $|T|^2 = 0$, and ω is Kähler and Ricci-flat.

6.2. A convergence theorem. It is a well-known and still open question in complex geometry to determine when a balanced or conformally balanced complex manifold X is actually Kähler. The flow (6.1) appears particularly well-suited to address this question when $c_1(X) = 0$. In particular, from the above discussion, the Kähler property is a necessary condition for the convergence of the flow. At this moment, we can prove a partial converse [82]:

THEOREM 7. *Assume that the initial data $\omega(0)$ satisfies $\|\Omega\|_{\omega(0)}\omega(0)^{n-1} = \hat{\chi}^{n-1}$, where $\hat{\chi}$ is a Kähler metric. Then the flow exists for all time $t > 0$, and as $t \rightarrow \infty$, the solution $\omega(t)$ converges smoothly to a Kähler, Ricci-flat, metric ω_∞ satisfying*

$$\omega_\infty = \|\Omega\|_{\chi_\infty}^{-2/(n-2)} \chi_\infty$$

where χ_∞ is the unique Kähler Ricci-flat metric in the cohomology class $[\hat{\chi}]$, and $\|\Omega\|_{\chi_\infty}$ is an explicit constant.

It is instructive to examine the corresponding PDE in greater detail.

The key starting point is that, in this special case, we can actually identify an evolution of $(1, 1)$ -cohomology classes. Define the smooth function $f \in C^\infty(X, \mathbf{R})$ by

$$e^{-f} = \frac{1}{(n-1)} \|\Omega\|_{\hat{\chi}}^{-2}$$

Let $t \rightarrow \varphi(t)$ be the following Monge-Ampère type flow

$$\partial_t \varphi = e^{-f} \frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n}, \quad \text{with } \varphi(x, 0) = 0,$$

subject to the plurisubharmonicity condition $\hat{\chi} + i\partial\bar{\partial}\varphi > 0$. Then, $\omega(t)$ defined by

$$(6.4) \quad \|\Omega\|_{\omega(t)} \omega^{n-1}(t) = \chi^{n-1}(t) \quad \text{with } \chi(t) = \hat{\chi} + i\partial\bar{\partial}\varphi(t) > 0$$

is precisely the solution of the Anomaly flow.

We note the similarity with the Kähler-Ricci flow and with the flow recently introduced by Collins et al [26], except we have here the Monge-Ampère determinant itself, instead of its log or its inverse. In this respect, the Anomaly flow with $\alpha' = 0$ can be viewed more like the complex analogue of the inverse Gauss curvature flow, which for convex bodies can be expressed as

$$\partial_t u = \frac{\det(ug_{ij} + \nabla_i \nabla_j u)}{\det g_{ij}}, \quad u(x, 0) = u_0(x) > 0$$

where u is the support function $u : \mathbf{S}^n \times [0, T]$ defined by $u(N, t) = \langle P, N \rangle$ where $P \in M_t$ is the point on M_t with normal N , and (\mathbf{S}^n, g_{ij}) is the standard sphere.

But unlike the Kähler-Ricci flow, the equation is not concave. Thus many well-known methods for the theory of non-linear parabolic partial differential equations, including those recently developed in [85]. Rather, the equation has to be studied on its own. We describe now some of the main steps.

6.2.1. *Estimates for $\partial_t \varphi$.* Despite the difficulties we have noted, the flow has a very important property, which is that $\partial_t \varphi \equiv \dot{\varphi}$ as well as the oscillation of φ can be controlled at all times. To see this, let L be the linear elliptic second order operator defined by

$$(6.5) \quad L = e^{-f} \frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n} \chi^{j\bar{k}} \hat{\nabla}_j \hat{\nabla}_{\bar{k}}$$

where we have denoted by $\hat{\nabla}$ the connection with respect to the reference metric $\hat{\chi}$. Then differentiating the flow gives

$$(6.6) \quad (\partial_t - L)\dot{\varphi} = 0$$

and hence, by the maximum principle

$$(6.7) \quad \inf_X \dot{\varphi}(0) \leq \dot{\varphi} \leq \sup_X \dot{\varphi}(0).$$

which shows that $\dot{\varphi}$ is uniformly bounded for all times. Next, we can rewrite the flow as a Monge-Ampère equation with bounded right hand side,

$$(6.8) \quad (\hat{\chi} + i\partial\bar{\partial}\varphi)^n = e^f \hat{\chi}^n$$

Yau’s C^0 estimate gives then a uniform bound on the oscillation $\sup_X \varphi - \inf_X \varphi$.

6.2.2. *The C^2 estimate.* The key estimate is the following C^2 estimate: let φ be a solution of (6.1) on $X \times [0, T]$ satisfying the positivity condition (6.4). Then there exists constants $C, A > 0$ depending only on $X, \hat{\chi}$, and f such that

$$(6.9) \quad \Delta_{\hat{\chi}}\varphi(x, t) \leq C e^{A(\tilde{\varphi}(x,t) - \inf_{X \times [0,T]} \tilde{\varphi})}$$

for all $(x, t) \in X \times [0, T]$. Here $\tilde{\varphi}$ is defined by

$$(6.10) \quad \tilde{\varphi}(x, t) = \varphi(x, t) - \frac{1}{V} \int_X \varphi(\cdot, t) \hat{\chi}^n, \quad V = \int_X \hat{\chi}^n.$$

The proof is by the maximum principle applied to the following test function

$$(6.11) \quad G(x, t) = \log \text{Tr } h - A\tilde{\varphi} + \frac{B}{2} \left[\frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n} \right]^2$$

where $h^j_k = \hat{\chi}^{j\bar{p}} \chi_{\bar{p}k}$, and $A, B > 0$ are suitable constants. We note the extra term $B[(\hat{\chi} + i\partial\bar{\partial}\varphi)^n / \hat{\chi}^n]^2 / 2$, which does not appear in the standard test function used in the study of the complex Monge-Ampère type equations or the Kähler-Ricci flow. Indeed, this is the main innovation of this test function, and it is required in order to overcome the difficult terms caused by the lack of concavity of the flow as well as cross terms resulting from the conformal factor e^{-f} .

6.2.3. *The $C^{2,\alpha}$ estimate.* The lack of concavity is again a source of difficulties in getting $C^{2,\alpha}$ estimates and the absence of an Evans-Krylov theorem. Here we adapt instead techniques for $C^{2,\alpha}$ estimates developed by B. Chow and D. Tsai [18, 95] (see also Andrews [6]) in the study of Gauss curvature type flows in convex geometry. This implies the long-time existence of the flow.

Finally the convergence of the flow can be established in several ways. One elegant way is to use the dilaton functional

$$(6.12) \quad M(\omega) = \int_X \|\Omega\|_{\omega} \omega^n$$

introduced by Garcia-Fernandez et al [46]. This functional can be shown to be monotone decreasing for the flow (6.1) with conformally Kähler initial data, with $dM(t)/dt \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof of Theorem 7.

6.3. Singularities and divergence. Since the convergence of the flow (6.1) is intimately related to the question of whether the underlying conformally balanced manifold X is actually Kähler, it would be very valuable to find geometric criteria for when it develops singularities and/or diverges. An elementary example is provided by the case of Calabi-Eckmann-Goldstein-Prokushkin fibrations with $\alpha' = 0$. If ω_1 and ω_2 are the two harmonic forms used in the construction of the toric fibration $\pi : X \rightarrow Y$, and ω_u is the metric (5.1), then

$$i\partial\bar{\partial}\omega_u = (\Delta_{\hat{\omega}}e^u + \|\omega_1\|_{\hat{\omega}}^2 + \|\omega_2\|_{\hat{\omega}}^2)\hat{\omega}^2/2,$$

so that the Anomaly flow $\partial_t(\|\Omega\|_{\omega_u}\omega_u^2) = i\partial\bar{\partial}\omega_u$ becomes

$$(6.13) \quad \partial_t e^u = \frac{1}{2}(\Delta_{\hat{\omega}}e^u + \|\omega_1\|_{\hat{\omega}}^2 + \|\omega_2\|_{\hat{\omega}}^2).$$

It follows immediately that

$$(6.14) \quad \partial_t \left(\int_Y e^u \hat{\omega}^2 \right) = \int_Y (\|\omega_1\|_{\hat{\omega}}^2 + \|\omega_2\|_{\hat{\omega}}^2) \frac{\hat{\omega}^2}{2} > 0.$$

Thus the flow cannot converge unless $\omega_1 = \omega_2 = 0$, which is exactly when the fibration π becomes the trivial product $Y \times T$, which is indeed Kähler.

The study of the formation of singularities in a flow begins with the identification of which geometric quantity must blow up in order for the flow not to extend further. For the original Anomaly flow in dimension 3, the following theorem was established in [78]:

THEOREM 8. *Assume that $n = 3$ and we have a solution of the Anomaly flow (4.5) with $\alpha' = 0$ (or (6.2)) on the time interval $[0, \frac{1}{A}]$. Then for all non-negative integers k , there exists a constant C_k , depending on a uniform lower bound for $\|\Omega\|_{\omega}$, such that, if*

$$(6.15) \quad |Rm|_{\omega} + |DT|_{\omega} + |T|_{\omega}^2 \leq A, \quad (z, t) \in X \times [0, \frac{1}{A}]$$

then

$$(6.16) \quad |D^k Rm(z, t)|_{\omega} + |D^{k+1}T(z, t)|_{\omega} \leq \frac{C_k A}{t^{\frac{k}{2}}}, \quad (z, t) \in X \times (0, \frac{1}{A}].$$

In particular, the flow will exist for all time, unless there is a time $T > 0$ and a sequence (z_j, t_j) with $t_j \rightarrow T$ with either $\|\Omega(z_j, t_j)\|_{\omega(t_j)} \rightarrow 0$, or

$$(6.17) \quad (|Rm|_{\omega}^2 + |DT|_{\omega}^2 + |T|_{\omega}^2)(z_j, t_j) \rightarrow \infty.$$

Indeed, the above theorem holds for the flow (6.2) in any dimension $n \geq 3$. We expect this theorem to generalize to the flow (6.1) for arbitrary $n \geq 3$.

7. More speculative flows

We conclude with a brief discussion of some more speculative flows.

7.1. Flow of balanced metrics with source term. In §6, we investigated the flow (6.1) for conformally balanced metrics, which is just the higher dimensional generalization of the Anomaly flow (4.1) with zero slope parameter α' . We can also consider a similar flow with source term

$$(7.1) \quad \partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega - \Upsilon,$$

where Υ is a given $i\partial\bar{\partial}$ -exact $(2,2)$ -form on the compact complex 3-fold X . Indeed, this flow provides a natural approach to a question raised by Garcia-Fernandez [45], namely, whether for any given real $i\partial\bar{\partial}$ -exact $(2,2)$ -form Υ on a compact conformally balanced 3-fold (X, Ω, ω_0) , there exists another conformally balanced metric ω with $i\partial\bar{\partial}\omega = \Upsilon$ and $[\|\Omega\|_\omega \omega^2] = [\|\Omega\|_{\omega_0} \omega_0^2]$? This question is itself closely related to a conjecture on the solvability of the Hull-Strominger system, raised by Yau. Thus the long time behavior and convergence of the flow (7.1) is of particular interest.

7.2. The Anomaly flow in arbitrary dimension. The original motivation for the Hull-Strominger system, and hence for the Anomaly flow was for compactifications of the 10-dimensional heterotic string to 4-dimensional space-times, and thus the internal complex manifold X must have complex dimension 3. The Green-Schwarz anomaly cancellation mechanism made also use specifically of the second Chern classes of $T^{1,0}(X)$ and of the bundle $E \rightarrow X$, which appear in (3.1). However, from the pure mathematical standpoint, the Anomaly flow seems to fit naturally in a family of flows which can be defined in arbitrary dimension n , and are given by

$$(7.2) \quad \partial_t(\|\Omega\|_\omega \omega^{n-1}) = i\partial\bar{\partial}\omega^{n-2} - \alpha'(c_{n-1}(Rm) - \Phi_{n-1}(t))$$

where $c_{n-1}(Rm) = \text{Tr}(Rm \wedge \cdots \wedge Rm)$ is the $(n - 1)$ -th Chern class of $T^{1,0}(X)$, and $\Phi_{n-1}(t)$ is a given flow of closed $(n - 1, n - 1)$ -forms, for example $\Phi_{n-1}(t) = c_{n-1}(F(t))$ where $F(t)$ is the curvature of a flow of Hermitian metrics $H(t)$.

Just as Hermitian-Yang-Mills equations originally appeared in dimension 4 but proved to be mathematically important in arbitrary dimensions, we expect that these flows will be interesting for $n > 3$ as well. We note that other generalizations of the Hull-Strominger system to arbitrary dimensions have been proposed earlier in [43, 44]. Restricted to Calabi-Eckmann-Golstein-Prokushkin fibrations, they lead to the following interesting n -dimensional generalizations of the equation studied by Fu and Yau [43, 44], called the Fu-Yau equation,

$$(7.3) \quad i\partial\bar{\partial}(e^u \hat{\omega} - \alpha' e^{-u} \rho) \wedge \hat{\omega}^{n-2} + \alpha' i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u \wedge \hat{\omega}^{n-2} + \mu = 0,$$

where μ is a given smooth (n, n) form. When $n = 2$, the above equation reduces to the right hand side of equation (5.3) and was studied by Fu and Yau using elliptic methods in [43, 44]. The general dimensional case of (7.3) has been subsequently studied in [74, 77, 81, 24, 25]. They are of Hessian type, and suggest interesting questions about a priori estimates for such equations, some of which have been addressed in [28, 51, 52, 76, 77].

7.3. Flows without the form Ω . For some questions, e.g., the relation between balanced cone and Kähler cone [42], it would be desirable to eliminate the assumption of the existence of a non-vanishing holomorphic form Ω . Thus it is tempting to consider the flow

$$\partial_t \omega^{n-1} = i\partial\bar{\partial}\omega^{n-2}$$

on an n -dimensional complex manifold X , for initial data $\omega(0)$ which are balanced, i.e. $d\omega(0)^{n-1} = 0$. The stationary points are still exactly the Kähler metrics, although they are not necessarily Ricci-flat. Explicitly, we can still write it in terms of metrics,

$$(7.4) \quad \begin{aligned} \partial_t g_{\bar{k}j} &= -\frac{1}{n-1} \nabla^m T_{\bar{k}jm} \\ &+ \frac{1}{2(n-1)} \left[-g^{q\bar{p}} g^{s\bar{r}} T_{\bar{k}qs} \bar{T}_{j\bar{p}\bar{r}} + \frac{|T|^2}{n-1} g_{\bar{k}j} \right] \end{aligned}$$

Compared to the expression (6.3) for the Anomaly flow, we see that the key term $\tilde{R}_{\bar{k}j}$ has cancelled out, and the flow is not parabolic. The lack of parabolicity poses severe problems, and it is unclear whether this flow can be used in any way. It would be very interesting if one can find a flow of balanced metrics with Kähler metrics as fixed points, but which is parabolic. In this context, we note that another flow of balanced metrics has been introduced by Bedulli and Vezzoni in [8], based on a generalization of the Calabi flow, and its short-time existence established there.

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