Recent results on k-th Yau algebras over simple elliptic singularities $\tilde{E}_6$

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Dedicated to Professor Shing-Tung Yau on the occasion of his 70th birthday

Abstract. Recently, we have introduced a series of finite dimensional solvable Lie algebras (i.e., k-th Yau algebras) associated to an isolated hypersurface singularity. These Lie algebras are subtle invariants of singularities. The purpose of this paper is to summarize the results that we have obtained recently on k-th moduli algebras and k-th Yau algebras associated to isolated hypersurface singularities.

1. Introduction

The moduli algebra $\mathcal{A}(\mathcal{V}_p)$ of an isolated hypersurface singularity $(\mathcal{V}_p, 0) \subset (\mathbb{C}^n, 0)$ defined by $p(x_1, \ldots, x_n) = 0$ is a finite dimensional $\mathbb{C}$-algebra defined by

$$\mathbb{C}[[x_1, \ldots, x_n]]/\langle p, J(p)\rangle,$$

where $J(p) := \langle \partial_1(p), \ldots, \partial_n(p)\rangle$ denotes the Jacobi ideal of p. Clearly, it is an Artinian local algebra with maximal ideal m. Its dimension denoted by τ, called Tjurina number, is an important invariant of an isolated hypersurface singularity.

Let V_1 and V_2 be two isolated hypersurface singularities; $A(V_1)$ and $A(V_2)$ be the moduli algebras. The well-known Mather-Yau theorem [1] states that $(V_1, 0) \cong (V_2, 0) \iff A(V_1) \cong A(V_2)$. Motivated from the Mather-Yau theorem, Yau considered the Lie algebra of $\mathbb{C}$-derivations of moduli algebra $A(V)$, i.e., $L(V) = \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. The finite dimensional Lie algebra

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$L(V)$ was called Yau algebra and its dimension $\lambda(V)$ was called Yau number. The Yau algebra plays an important role in singularity theory [2]. Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities begin from eighties ([3, 4, 2, 5, 6], [7, 8, 9, 10, 11, 12, 13, 14, 15]). In particular, Yau algebras of simple singularities and simple elliptic singularities were computed and a number of elaborate applications to deformation theory were presented in [16] and [2]. However, the Yau algebra can not characterize the simple singularties completely. In [17], it was shown that if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic. Therefore, a natural question is to find new Lie algebras which can be used to distinguish singularities (at least for the simple singularities) completely. The is the motivation for us to introduce the series of new k-th Yau algebra, which will be used to characterize isolated hypersurface singularities.

Recall that we have the following theorem.

Theorem 1 ([18], Theorem 2.26). *Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*

1) $(V(f), 0) \cong (V(g), 0)$;
2) *For all* $k \geq 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra;*
3) *There is some* $k \geq 0$ *such that* $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebras, where* $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ as above, then the claim of the equivalence of 1) and 3) is exactly the same as the Mather-Yau theorem.

Based on Theorem 1, it is natural for us to introduce the new series of k-th Yau algebras $L^k(V)$ which are defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$. Its dimension is denoted as $\lambda^k(V)$. This number $\lambda^k(V)$ is a new numerical analytic invariant of a singularity. We call it k-th Yau number. In particular, $L^0(V)$ is exactly the Yau algebra, thus $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. Therefore, we have reasons to believe that these new Lie algebras $L^k(V)$ and numerical invariants $\lambda^k(V)$ will also play an important role in the study of singularities.

In this paper, we announce the recent results, obtained in [12], on k-th moduli algebras and k-th Yau algebras associated to isolated hypersurface singularities.

Given a family of complex projective hypersurfaces in $\mathbb{CP}^n$, the Torelli problem studied by P. Griffiths and his school asks whether the period map is injective on that family. Precisely, it is equivalent to saying whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. From the perspective of singularity theory, a complex projective hypersurface can be viewed as an isolated hypersurface singularity. The corresponding question arises: whether the family of isolated complex hypersurface singularities can be distinguished by means of their Yau algebras.

We require that the Milnor number μ is constant along this family in view of the theorem of Lê and Ramanujan [22]. Knowing that Tjurina number τ is also a complex analytic invariant, we may restrict ourselves to consider only a (μ, τ)-constant family of isolated complex hypersurface singularities.

As shown in [17], simple hypersurface singularities except A_6 and D_5 are completely determined by their Yau algebras. Khimshiashvili obtained similar results in this spirit for certain types of fewnomial singularities in [20].

In [23], Saito defined a simple elliptic singularity to be a normal surface singularity such that the exceptional set of the minimal resolution is a smooth elliptic curve, and classified those which are hypersurface singularities into the following three types:

$$\begin{aligned}
\tilde{E}_6 &: x^3 + y^3 + z^3 + txyz = 0, \quad t^3 \neq 27;\\
\tilde{E}_7 &: x^4 + y^4 + z^2 + tx^2y^2 = 0, \quad t^2 \neq 4;\\
\tilde{E}_8 &: x^6 + y^3 + z^2 + tx^4y = 0, \quad 4t^3 \neq 27.
\end{aligned}$$

It turns out that all these three families are (μ, τ)-constant. Saito determined the holomorphic classes of these families by computing the j-invariants within each type. The isomorphism problem for the moduli algebras of simple elliptic singularities has been extensively studied in purely algebraic terms and is now well-understood. It was shown by Eastwood [24] in a very explicit form how one can recover Saito's j-invariants directly from the corresponding moduli algebra. In [6], Chen, Seeley and Yau gave a detailed characterization of linear isomorphisms of moduli algebras arising from $\tilde{E}_6$.

Seeley and Yau proved the Torelli type theorems on $\tilde{E}_7$ and $\tilde{E}_8$ stating that Yau algebras are isomorphic if and only if their moduli algebras are isomorphic [2]. However, the case of $\tilde{E}_6$ looks quite different. It is pointed out in [5] that Yau algebra does not depend on the parameter t of $\tilde{E}_6$. In fact, it provides an instance of continuous family of eight-dimensional representations of a solvable Lie algebra.

Since the Yau algebra is not enough to determine the analytic structure of simple elliptic singularity family $\tilde{E}_6$, we have to extend the definition of Yau algebra. In [13], we introduced a series of new Lie algebras, i.e., the k-th Yau algebra which is the derivation Lie algebra of the k-th moduli algebra. The dimension of the k-th Yau algebra, called k-th Yau number, becomes an important invariant for isolated hypersurface singularities. The classical Hilbert polynomial for projective varieties measures the growth of the dimension of twisted modules. It is interesting to find that similar polynomial can be deduced from k-th Yau numbers, see Theorem C, and an explicit formula concerning isolated homogeneous hypersurface singularities is stated in Corollary 4. Detailed descriptions of k-th Yau algebras for k sufficiently large are given by Theorems 2, 10 and Corollary 3. We also have obtained a concise proof for theorems in [6] concerning the automorphism group of $\tilde{E}_6$ by using some properties of the 1-st Yau algebra, see Theorem D and Remark 36.

We have already shown that the complex structures of simple elliptic singularities can be totally distinguished by means of k-th Yau algebras for sufficiently large k. We have also investigated the isomorphism group of such Lie algebras explicitly. This problem is essentially linked with the classification problem that aims to classify all finite dimensional nilpotent Lie algebras. So far, the classification of all Lie algebras over $\mathbb{C}$ is obtained in dimension up to 6, and nilpotent complex Lie algebras are classified only in dimension up to 7. Two-step nilpotent, or metabelian, Lie algebras form the first non-trivial subclass of nilpotent algebras. Even in the case of metabelian algebras, the classification is a rather complicated problem and is far from being solved. The solution in dimension up to 9 is given in [**25**, **26**]. In greater dimensions, only partial results are obtained.

Isomorphisms coming from Lie algebras are reduced to the solutions of certain types of algebraic equation systems in essence. However, it seems impossible to obtain isomorphisms directly in this way as usually these systems contain thousands of variables and relations when the dimension we considering is large. The basic idea to identify the isomorphic classes is to extract invariants from the algebraic structures. Actually, the crucial step in the proof of Torelli theorems of 0-th Yau algebras over both $\tilde{E}_7$ and $\tilde{E}_8$ is to find four distinct invariant lines, and compute their cross-ratio. This approach works perfectly for any integer $k \geqslant 1$, in our forthcoming publications the details could be found. But it is a great pity that the case for $\tilde{E}_6$ looks completely different. Because of the coordinate symmetry, we can hardly obtain invariant lines. This makes the problem over $\tilde{E}_6$ be the most intractable one among these three families. Nevertheless, invariant-theoretic approach is still available. Technically, we split the problem into three cases: (1) $k = 0$ or 1; (2) $k = 2$; (3) $k \geqslant 3$.

In the first case, we show that the 1-st Yau algebra on $\tilde{E}_6$ does not depend on t which is highly similar to the case of the 0-th Yau algebra. It is formally given by the following theorem.

Theorem A ([**12**]). *The* 1*-st Yau algebra of* $\tilde{E}_6$ *is independent on the parameter* t *and it gives rise to a continuous family of* 11*-dimensional representations of a solvable Lie algebra.*

For $k \geqslant 2$, we develop an algebraic method to understand the isomorphisms of moduli algebras and Yau algebras respectively. Roughly speaking, explicit bases for all k-th Yau algebras can be entirely described with the help of a generalized Koszul sequence, see Theorems 9 and 10. It enables us to construct a family $\{M_t\}$ of 45-dimensional metabelian Lie algebras for case (2) and a similar family $\{N_t\}$ of 15-dimensional for case (3). It takes us a lot of effort to show that a symmetric three-dimensional subspace Z remains unchanged under isomorphisms of $\{M_t\}$ through a subtle calculation on some derivations of Lie algebras. Along this way, we discover that any such isomorphism gives rise to a linear isomorphic map between the correspondent moduli algebras. The computation for $\{N_t\}$ is not exactly the same. As the derivations become trivial in this situation, we introduce the so-called quasi-derivations to prove the invariant property of a three-dimensional subspace, say $\Delta_{1,t}$. These invariant-theoretic approaches bring novel ideas in the study of classification problem of Lie algebras. It turns out that the analytic structure of family $\tilde{E}_6$ is completely dependent on $\{M_t\}$ or $\{N_t\}$. Conclusively, we achieve a Torelli type theorem for any $k \geqslant 2$ which can be precisely expressed in the following Theorem B. As a consequence, we are able to construct infinitely many one parameter family of nilpotent Lie algebras of arbitrary dimensions under certain condition (see Theorem C).

THEOREM B. *Let $\{V_t\}$ represent a family of simple elliptic singularities of type $\tilde{E}_6$. If integer $k \geqslant 2$, then L_t^k and L_s^k are isomorphic as Lie algebras if and only if V_t is biholomorphic to V_s.*

Encouraged by the above results, we formulate the following conjecture.

CONJECTURE. *Suppose that $\{\mathcal{V}_{\mathbf{t}}\}$ is a (μ, τ)-constant family of isolated hypersurface singularities with parameters $\mathbf{t} = (t_1, \ldots, t_m)$. Then the Torelli type theorem holds for the k-th Yau algebra of $\{\mathcal{V}_{\mathbf{t}}\}$, which states that $\mathcal{V}_{\mathbf{t}}$ and $\mathcal{V}_{\mathbf{s}}$ are biholomorphically equivalent if and only if their k-th Yau algebras are equivalent for sufficiently large k.*

We also emphasize that our novel results including Theorems 2, 9, 10 etc. are of interest for isolated hypersurface singularities, not merely in the study of $\tilde{E}_6$.

Moreover, We have also obtained some other results. In [**13**], we have studied the k-th Yau number which is the dimension of k-th Yau algebra $L^k(\mathcal{V}_p)$ which we denote as $\lambda^k(\mathcal{V}_p)$. These numbers $\lambda^k(\mathcal{V}_p)$ are new numerical analytic invariants of singularities. We have formulated a conjecture $\lambda^{(k+1)}(\mathcal{V}_p) > \lambda^k(\mathcal{V}_p), k \geq 0$ and have proven this conjecture for large class of singularities. In [**14**], we have computed the Lie algebras $L^1(\mathcal{V}_p)$ for fewnomial isolated singularities. We have also formulated a sharp upper estimate conjecture for the $\lambda^k(\mathcal{V}_p)$ of weighted homogeneous isolated hypersurface singularities and we have proven this conjecture in case of $k = 1$ for large class of singularities.

Due to the space limit, this paper is to summarize mainly the results that we have obtained in [**12**]. The structure of the paper is as follows. In Section 2, we present some basic definitions and recent results about the k-th Tjurina numbers and the k-th Yau numbers for isolated hypersurface singularities. In Section 3, we present a connection between local function algebras and k-th moduli algebras. In Section 4, the bases of k-th moduli algebras and their associated k-th Yau algebras are computed. We also describe the k-th Yau number of $\tilde{E}_6$ explicitly. Theorem B is reduced to compute the set of isomorphisms of two families of metabelian Lie algebras in Section 5. Section 6 and 7 are devoted to present isomorphism groups of metabelian

Lie algebras which is a key step in the proof of Theorem B. In the final section, we characterize the set of isomorphisms of the 1-st Yau algebras which yields a similar result revolving around the 0-th Yau algebras.

2. General results

Let $\mathcal{O}_n := \mathbb{C}[[x_1, \dots, x_n]]$ be a formal power series ring over $\mathbb{C}$ with maximal ideal m. Let $\mathcal{V}_p$ be a germ of isolated hypersurface singularity at the origin in $\mathbb{C}^n$ represented as the zero locus of polynomial $p = p(x_1, \dots, x_n)$. Denote by $\mathcal{R}(\mathcal{V}_p)$ the local function algebra (or coordinate ring) of $\mathcal{V}_p$, i.e.,

$$\mathcal{R}(\mathcal{V}_p) = \mathcal{O}_n/\langle p\rangle.$$

The k-th moduli algebra of $\mathcal{V}_p$ is defined by

$$\mathcal{A}^k(\mathcal{V}_p) = \mathcal{O}_n/\langle p, m^k J(p)\rangle,$$

where $J(p) = \langle \partial_1(p), \dots, \partial_n(p)\rangle$ denotes the Jacobi ideal of p.

For an isolated hypersurface singularity $\mathcal{V}_p$ determined by a polynomial $p = p(x_1, \dots, x_n)$, the k-th Yau algebra of $\mathcal{V}_p$ is defined by

$$L^k(\mathcal{V}_p) := \mathrm{Der}(\mathcal{A}^k(\mathcal{V}_p), \mathcal{A}^k(\mathcal{V}_p)).$$

It is a natural generalization of Yau algebra [**4**]. The dimension of k-th Yau algebra shall be called k-th Yau number.

When we restrict ourselves to consider the homogeneous isolated hypersurface singularities, the natural embedding from $L^k(\mathcal{V}_p)$ to a free $\mathcal{A}^k(\mathcal{V}_p)$-module generated by $\partial_1, \dots, \partial_n$ endows $L^k(\mathcal{V}_p)$ with a grading structure by setting $\deg(\partial_1) = \dots = \deg(\partial_n) = -1$. A homogeneous derivation is a derivation whose nonzero terms all have the same degree.

We obtain the following results. The details and proofs can be found in [**12**].

THEOREM 2 ([**12**]). *Suppose that $\mathcal{V}_p$ is an isolated singularity determined by a polynomial $p = p(x_1, \dots, x_n)$ with $n \geqslant 2$. For k sufficiently large, there exists an exact sequence of $\mathcal{A}^k(\mathcal{V}_p)$-modules:*

(1)
$$0 \longrightarrow L^k(\mathcal{V}_p) \longrightarrow \mathcal{A}^k(\mathcal{V}_p)\langle \partial_1, \dots, \partial_n\rangle \xrightarrow{\psi_1} \mathcal{A}^k(\mathcal{V}_p) \xrightarrow{\psi_0} \mathcal{A}^0(\mathcal{V}_p) \longrightarrow 0,$$

where ψ_1 maps ∂_i to $\partial_i(p)$, and ψ_0 denotes the quotient map.

According to the proof of Theorem 2 and Proposition 1.2 in [**28**], one can conclude that when $\mathcal{V}_p$ is weight-homogeneous and k is large enough, as an $\mathcal{A}^k(\mathcal{V}_p)$-module, $L^k(\mathcal{V}_p)$ is generated by $m^k\partial_i$, $\mathcal{H}$ and $\mathcal{E}_{i,j}$ for $1 \leqslant i < j \leqslant n$, where $\mathcal{H}$ represents the Euler derivation, and $\mathcal{E}_{i,j}$ is given by

$$\mathcal{E}_{i,j} := \partial_i(p)\partial_j - \partial_j(p)\partial_i.$$

In particular, we achieve the following corollary for homogeneous singularities.

COROLLARY 3 ([**12**]). *Suppose that $\mathcal{V}_p$ is homogeneous. Let k be an integer verifying $m^k\partial_{i,j}(p) \subseteq J(p)$ for all $i, j = 1, \dots, n$. Then*

$$L^k(\mathcal{V}_p) \cong m^k\mathcal{A}^k(\mathcal{V}_p)\langle\partial_1, \dots, \partial_n\rangle \oplus \mathrm{Der}_L^k(\mathcal{V}_p),$$

where $\mathrm{Der}_L^k(\mathcal{V}_p)$ consists of derivations of degree strictly less than $k-1$. Moreover, the subspace $\mathrm{Der}_L^k(\mathcal{V}_p)$ can be viewed as an $\mathcal{A}^k(\mathcal{V}_p)$-module generated by derivations $\mathcal{H} = x_1\partial_1 + \cdots + x_n\partial_n$ and $\mathcal{E}_{i,j}$ for $1 \leqslant i < j \leqslant n$.

The following theorem appears to be the main result in this section.

THEOREM C ([**12**]). *Let $\mathcal{V}_p$ be an isolated hypersurface singularity with multiplicity r. Denote by τ its Tjurina number. Then there exists a polynomial $P(k)$ of degree $n-1$ such that*

$$\dim \mathcal{A}^k(\mathcal{V}_p) = P(k) \text{ and } \dim L^k(\mathcal{V}_p) = (n-1)P(k) + \tau,$$

for k sufficiently large. Furthermore, the leading coefficient of $P(k)$ equals $r/(n-1)!$.

Hereafter, we define the binomial coefficient to be zero when its numerator is negative. We obtain the consequent result for homogeneous singularities as follows.

COROLLARY 4 ([**12**]). *For homogeneous singularity $\mathcal{V}_p$ with multiplicity r, we have*

$$\dim \mathcal{A}^k(\mathcal{V}_p) = (r-1)^n + \binom{k+r-2+n}{n} - \binom{k-2+n}{n} - \theta_{r+k-2}, \tag{2}$$

where θ_i denotes the coefficient of t^i in the series $(1+t+t^2+\cdots+t^{r-2})^n/(1-t)$.

Hence, for sufficiently large k, we have

$$\dim \mathcal{A}^k(\mathcal{V}_p) = \binom{k+r-2+n}{n} - \binom{k-2+n}{n}, \tag{3}$$

and

$$\dim L^k(\mathcal{V}_p) = (n-1)\binom{k+r-2+n}{n} - (n-1)\binom{k-2+n}{n} + (r-1)^n. \tag{4}$$

We define the Hilbert-Poincaré series of homogeneous singularity $\mathcal{V}_p$ by

$$HS(t) := \sum_{k=0}^{\infty} \dim \mathcal{A}^k(\mathcal{V}_p) \cdot t^k.$$

As a consequence of the previous Corollary, we find that

$$HS(t) = \frac{(r-1)^n}{1-t} + \frac{1-t^r-(1-t^{r-1})^n}{t^{r-2}(1-t)^{n+1}}.$$

3. Isomorphisms of moduli algebras over $\tilde{E}_6$

Let us consider a family $\{V_t\}$ of elliptic singularity of type $\tilde{E}_6$ defined by

$$V_t := \{(x,y,z) : f_t(x,y,z) = x^3 + y^3 + z^3 + txyz = 0\},$$

where $t^3 \neq -27$. As in Section 2, we denote the local function algebra of V_t by $\mathcal{R}(V_t)$ and the k-th moduli space of V_t by $\mathcal{A}^k(V_t)$.

Let I_A and I_B be two ideals of $\mathcal{O}_n$ generated by homogeneous polynomials. Define $\mathfrak{A} := \mathcal{O}_n/I_A$ and $\mathfrak{B} := \mathcal{O}_n/I_B$ to be the correspondent local algebras. Denote the set of algebraic isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ by $\mathrm{Iso}(\mathfrak{A},\mathfrak{B})$. By the Lifting Lemma (Lemma 1.23 in [**18**]),

$$\mathrm{Iso}(\mathfrak{A},\mathfrak{B}) = \{\phi \in \mathrm{Aut}(\mathcal{O}_n) : \phi I_A \subseteq I_B \text{ and } \phi^{-1} I_B \subseteq I_A\}.$$

Define the set of $\mathbb{C}$-linear isomorphisms by

$$\mathrm{GL}(\mathfrak{A},\mathfrak{B}) := \{\phi \in \mathrm{GL}(n) : \phi I_A \subseteq I_B \text{ and } \phi^{-1} I_B \subseteq I_A\}.$$

THEOREM 5 ([**12**]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be local algebras defined above. Let ϕ be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then the linear part $[\phi]$ of ϕ is also an isomorphism. Furthermore,*

$$\mathrm{Iso}(\mathfrak{A},\mathfrak{B}) \cong \mathrm{Aut}_{id}(\mathfrak{B}) \rtimes \mathrm{GL}(\mathfrak{A},\mathfrak{B}) \cong \mathrm{GL}(\mathfrak{A},\mathfrak{B}) \ltimes \mathrm{Aut}_{id}(\mathfrak{A}),$$

where $\mathrm{Aut}_{id}(\cdot)$ denotes an automorphism group

$$\mathrm{Aut}_{id}(\mathfrak{A}) := \{\phi \in \mathrm{Iso}(\mathfrak{A},\mathfrak{A}) : [\phi] = id\}.$$

Therefore, $\mathfrak{A}$ is algebraic isomorphic to $\mathfrak{B}$ if and only if there exists a linear isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

We encounter a natural problem. On elliptic singularities $\tilde{E}_6$, what is the relation between isomorphisms of local function algebras and of k-th moduli algebras? To answer this we establish the following interesting theorem.

THEOREM 6 ([**12**]). *For $k \geqslant 2$, we have the following equivalence*

$$\mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s)) \cong \mathrm{GL}(\mathcal{A}^k(V_t), \mathcal{A}^k(V_s)).$$

Generically, the theorem is still true for $k = 0$ and 1 though the above proof is not available. The only exceptional case is when $t^3, s^3 = 0$ or 216. To see this, we set $t = s = 0$. Let us consider linear maps of type

$$\phi_{\alpha,\beta,\gamma} : (x,y,z) \mapsto (\alpha x, \beta y, \gamma z). \tag{5}$$

One may check that when $\alpha, \beta, \gamma \neq 0$, each $\phi_{\alpha,\beta,\gamma}$ is a linear automorphism of $\mathcal{A}^0(V_0)$. While $\phi_{\alpha,\beta,\gamma}$ represents an automorphism of $\mathcal{R}(V_0)$ only in the case when $\alpha^3 = \beta^3 = \gamma^3 \neq 0$. It follows that

$$\mathrm{GL}(\mathcal{R}(V_0), \mathcal{R}(V_0)) \subsetneq \mathrm{GL}(\mathcal{A}^0(V_0), \mathcal{A}^0(V_0)).$$

In [6], Chen, Seeley and Yau found that the linear isomorphisms of the 0-th moduli algebras of $\tilde{E}_6$ can be expressed as matrices

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

whose entries satisfy the following system of quadratic equations:

$$\begin{cases} 3sa_1^2 + tsa_2a_3 - 18b_1c_1 - 3tb_2c_3 - 3tb_3c_2 = 0, \\ 3sb_1^2 + tsb_2b_3 - 18a_1c_1 - 3ta_2c_3 - 3ta_3c_2 = 0, \\ 3sc_1^2 + tsc_2c_3 - 18a_1b_1 - 3ta_2b_3 - 3ta_3b_2 = 0, \\ 3sa_2^2 + tsa_1a_3 - 18b_2c_2 - 3tb_1c_3 - 3tb_3c_1 = 0, \\ 3sb_2^2 + tsb_1b_3 - 18a_2c_2 - 3ta_1c_3 - 3ta_3c_1 = 0, \\ 3sc_2^2 + tsc_1c_3 - 18a_2b_2 - 3ta_1b_3 - 3ta_3b_1 = 0, \\ 3sa_3^2 + tsa_1a_2 - 18b_3c_3 - 3tb_2c_1 - 3tb_1c_2 = 0, \\ 3sb_3^2 + tsb_1b_2 - 18a_3c_3 - 3ta_2c_1 - 3ta_1c_2 = 0, \\ 3sc_3^2 + tsc_1c_2 - 18a_3b_3 - 3ta_2b_1 - 3ta_1b_2 = 0. \end{cases} \tag{6}$$

By solving this equation system, they obtained the following theorem.

THEOREM 7 ([6]). *Let ρ be a primitive cube root of unity. Assume that $t^3, s^3 \neq 0, 216, -27$. Up to a scale multiplication, any linear isomorphism from $\mathcal{A}^0(V_t)$ to $\mathcal{A}^0(V_s)$ is given by one of the 216 matrices generated by*

$$A_1 := \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 := \begin{pmatrix} \rho & \rho^2 & 1 \\ \rho^2 & \rho & 1 \\ 1 & 1 & 1 \end{pmatrix}, A_3 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Moreover, for the case $t, s \in \{0, 6, 6\rho, 6\rho^2\}$, we see that moduli algebras $\mathcal{A}^0(V_0)$, $\mathcal{A}^0(V_6)$, $\mathcal{A}^0(V_{6\rho})$, $\mathcal{A}^0(V_{6\rho^2})$ are isomorphic to each other with isomorphism group generated by A_1, A_2, A_3, A_4 and nonsingular diagonal matrices.

One can check that each matrix A_i is an isomorphism between the local function algebras $\mathcal{R}(V_t)$ and $\mathcal{R}(V_s)$. Moreover, diagonal matrices represent linear maps of type (5). They become isomorphisms belonging to $\mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s))$ only in the case $\alpha^3 = \beta^3 = \gamma^3 \neq 0$. Hence, they are generated by A_1, A_3, A_4 and scale matrices. Therefore, as a consequence of Theorem 7 we achieve the following corollary.

COROLLARY 8 ([12]). *The set $\mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s))$ of isomorphisms forms a subgroup of $\mathrm{GL}(3)$ generated by A_1, A_2, A_3, A_4 and scale matrices. In particular, for $t^3, s^3 \neq 0, 216, -27$, we have*

$$\mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s)) \cong \mathrm{GL}(\mathcal{A}^0(V_t), \mathcal{A}^0(V_s)). \tag{7}$$

Using this result, it is easy to compute the j-invariant for $\tilde{E}_6$. Following [6], we set

$$j(t) := \frac{t^3(t^3-216)^3}{(t^3+27)^3}.$$

Then $\mathcal{R}(V_t)$ is isomorphic to $\mathcal{R}(V_s)$ if and only if $j(t)=j(s)$.

It remains to understand the set of isomorphism from $\mathcal{A}^1(V_t)$ to $\mathcal{A}^1(V_s)$, which will be studied in the final section.

4. Bases for k-th Yau algebras

Our aim in this section is to describe bases for k-th moduli algebra $\mathcal{A}^k(V_t)$ and the correspondent k-th Yau algebra $L_t^k := \mathrm{Der}(\mathcal{A}^k(V_t), \mathcal{A}^k(V_t))$.

Note that the Jacobi ideal is given by

$$J(f_t) = \langle 3x^2+tyz, 3y^2+txz, 3z^2+txy\rangle.$$

We obtain from definition that

$$\mathcal{A}^0(V_t) = \mathbb{C}[[x,y,z]]/\langle 3x^2+tyz, 3y^2+txz, 3z^2+txy\rangle,$$

and

$$\begin{aligned}&\mathcal{A}^1(V_t)\\ &=\mathbb{C}[[x,y,z]]/\langle 3x^3+txyz, 3y^3+txyz, 3z^3+txyz, x^2y, x^2z, y^2x, y^2z, z^2x, z^2y\rangle.\end{aligned}$$

For $k \geqslant 2$, it is known from the proof of Theorem 6 that

$$\mathcal{A}^k(V_t) = \mathbb{C}[[x,y,z]]/\langle f_t, m^k J(f_t)\rangle = \mathbb{C}[[x,y,z]]/\langle f_t, m^{k+2}\rangle.$$

One may describe the bases for the k-th moduli algebra of V_t as follows.

Case 1: $k=0$. There exists a monomial basis of $\mathcal{A}^0(V_t)$, namely

$$\mathfrak{A}_0 = \{1, x, y, z, xy, xz, yz, xyz\}.$$

Case 2: $k=1$. The set of monomials

$$\mathfrak{A}_1 = \{1, x, y, z, x^2, y^2, z^2, xy, xz, yz, xyz\}$$

forms a basis of $\mathcal{A}^1(V_t)$.

Case 3: $k \geqslant 2$. Denote by $\mathfrak{S}_k$ the set of monomials with degree less or equal k. In particular, $\mathfrak{S}_k$ equals the empty set for $k<0$. The set of monomials

$$\begin{aligned}\mathfrak{A}_2 &= \{1, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^2y, x^2z, y^3, y^2x, y^2z, z^3, z^2x, z^2y, xyz\}\\ &= \mathfrak{S}_3 \setminus \{x^3\}\end{aligned}$$

forms a basis for $\mathcal{A}^2(V_t)$. In general, for $k \geqslant 2$, the k-th moduli algebra $\mathcal{A}^k(V_t)$ has a basis

$$\mathfrak{A}_k = \mathfrak{S}_{k+1} \setminus \{x^3\mathfrak{S}_{k-2}\}.$$

So the maximal degree of elements contained in $\mathcal{A}^k(V_t)$ is $k+1$. By counting the cardinalities of bases in each case, we have

$$\dim \mathcal{A}^k(V_t) = \begin{cases} 8 & \text{for } k = 0, \\ 11 & \text{for } k = 1, \\ \frac{3}{2}k^2 + \frac{9}{2}k + 4 & \text{for } k \geqslant 2. \end{cases}$$

By a proper shifting, the polynomial with respect to k for the case $k \geqslant 2$ coincides with the Hilbert polynomial of V_t as a projective variety.

We turn to investigate the bases for the k-th Yau algebra.

Case 1: $k = 0$. For $t^3 = 0$ or 216, $\dim L_t^0 = 12$. For $t^3 \neq 0, 216$, we find that L_t^0 is of dimension 10 with basis

$$\{x\partial_x + y\partial_y + z\partial_z, 6xy\partial_x - txz\partial_y, 6xz\partial_x - txy\partial_z, 6xy\partial_y - tyz\partial_x,$$
$$6zy\partial_y - txy\partial_z, 6xz\partial_z - tyz\partial_x, 6yz\partial_z - txz\partial_y, xyz\partial_x, xyz\partial_y, xyz\partial_z\}.$$

Case 2: $k = 1$. For $t^3 = 0$ or 216, $\dim L_t^1 = 24$. For $t^3 \neq 0, 216$, we get $\dim L_t^1 = 22$. A basis for L_t^1 in the latter case is given by

$$\{x\partial_x + y\partial_y + z\partial_z\} \cup \{\eta\partial_x, \eta\partial_y, \eta\partial_z : \eta \in \mathfrak{A}_1 \text{ and } \deg \eta = 2 \text{ or } 3\}.$$

Case 3: $k = 2$. In this case, we obtain $\dim L_t^2 = 46$ for arbitrary t and a basis for L_t^2 given by

$$\{x\partial_x + y\partial_y + z\partial_z\} \cup \{\eta\partial_x, \eta\partial_y, \eta\partial_z : \eta \in \mathfrak{A}_2 \text{ and } \deg \eta = 2 \text{ or } 3\}.$$

Case 4: $k \geqslant 3$. According to Corollary 3, we already know that L_t^k is generated by $\mathcal{H}$, $\mathcal{E}_{i,j}, m^k\partial_x, m^k\partial_y, m^k\partial_z$. The problem is that they are linearly dependent, so we need to investigate their relations. The general situation, namely isolated homogeneous hypersurface singularities will be considered. The following theorem is viewed as a generation of Koszul sequence.

Theorem 9 ([12]). *Suppose that $\mathcal{V}_p$ is an isolated homogeneous hypersurface singularity associated with a homogeneous polynomial $p = p(x_1, x_2, \ldots, x_n)$ of degree r in $n \geqslant 2$ variables. Set $\mathcal{O} := \mathcal{O}_n := \mathbb{C}[[x_1, \ldots, x_n]]$ and $\mathcal{R} := \mathcal{R}(\mathcal{V}_p) := \mathcal{O}/\langle p \rangle$. Let $\mathcal{C}^1$ be a free $\mathcal{O}$-module of rank n with generators $\mathfrak{b}^0_{[1]}, \ldots, \mathfrak{b}^0_{[n]}$. For $s \geqslant 0$, define $\mathcal{C}^s$ to be a free $\mathcal{O}$-module of the form*

$$\mathcal{C}^s := \bigoplus_{j=0}^{\kappa} \bigwedge^{s-2j} \mathcal{C}^1, \text{ where } \kappa := \left\lfloor \frac{s}{2} \right\rfloor.$$

Denote by $\mathfrak{b}^0_{[i_1,\ldots,i_s]}, \mathfrak{b}^1_{[i_1,\ldots,i_{s-2}]}, \ldots, \mathfrak{b}^\kappa_{[i_1,\ldots,i_{s-2\kappa}]}$ (or $\mathfrak{b}^\kappa_\emptyset$ when $s = 2\kappa$) the generators of $\mathcal{C}^s$. In particular, $\mathfrak{b}^0_\emptyset = 1 \in \mathcal{C}^0 = \mathcal{O}$. Endow $\mathcal{C}^s$ with a grading structure by defining the degree of $\mathfrak{b}^j_{[i_1,\ldots,i_{s-2j}]}$ to be $(r-1)(s-2j) + r(j-1)$. For $s \geqslant 2$, we define ϕ_s to be a morphism from $\mathcal{C}^s$ to $\mathcal{C}^{s-1}$ preserving both grading structures:

$$\mathfrak{b}^j_{[i_1,i_2,\ldots,i_{s-2j}]} \mapsto \sum_{l=1}^{n} x_l \mathfrak{b}^{j-1}_{[l,i_1,i_2,\ldots,i_{s-2j}]} + \sum_{\mu=1}^{s-2j} (-1)^{\mu+1} \partial_{i_\mu}(p) \mathfrak{b}^j_{[i_1,\ldots,\widehat{i_\mu},\ldots,i_{s-2j}]}.$$

Hereafter, superscript $\widehat{\cdot}$ *means the term is omitted. In addition, we define* ϕ_1 *the morphism from* $\mathcal{C}^1$ *to* $\mathcal{C}^0$ *by* $\mathfrak{b}^0_{[i]} \mapsto \partial_i(p)$.

Then there exists an exact sequence of graded $\mathcal{R}$*-modules:*

(8)
$$\longrightarrow \mathcal{C}^3 \otimes \mathcal{R} \xrightarrow{\psi_3} \mathcal{C}^2 \otimes \mathcal{R} \xrightarrow{\psi_2} \mathcal{C}^1 \otimes \mathcal{R} \xrightarrow{\psi_1} \mathcal{R}[r] \xrightarrow{\psi_0} \mathcal{R}[r]/J(p) \longrightarrow 0,$$

where ψ_i *is induced by* ϕ_i *for* $i \geqslant 1$ *and* ψ_0 *denotes the quotient map.*

As a consequence of the previous theorem, we obtain a resolution for $\mathrm{Der}^k_L(\mathcal{V}_p)$. See the definition in Corollary 3.

THEOREM 10 ([**12**]). *Fix an isolated homogeneous singularity* $\mathcal{V}_p$ *associated with a homogeneous polynomial* $p = p(x_1, x_2, \dots, x_n)$ *with* $n \geqslant 2$. *Assume that* k *is large enough such that* $m^k \partial_{i,j}(p)$ *is contained in* $J(p)$ *for* $i, j = 1, \dots, n$. *Denote by* $\mathcal{C}^s_L$ *the quotient module of* $\mathcal{C}^s$ *modulo elements of degree* $\geqslant k-1$. *Then there exists an exact sequence (of finite length) of graded* $\mathcal{R}$*-modules:*

(9)
$$\longrightarrow \mathcal{C}^4_L \otimes \mathcal{R} \xrightarrow{\psi_4} \mathcal{C}^3_L \otimes \mathcal{R} \xrightarrow{\psi_3} \mathcal{C}^2_L \otimes \mathcal{R} \xrightarrow{\psi_2} \mathrm{Der}^k_L(\mathcal{V}_p) \longrightarrow 0.$$

As a torsion $\mathcal{R}$*-module, the generators of* $\mathrm{Der}^k_L(\mathcal{V}_p)$ *can be chosen as*

$$\mathcal{H} := x_1\partial_1 + x_2\partial_2 + \cdots + x_n\partial_n,$$
$$\mathcal{E}_{i,j} := \partial_i(p)\partial_j - \partial_j(p)\partial_i \quad \text{for } 1 \leqslant i < j \leqslant n,$$

with relations

$$m^{k-1}\mathcal{H} = m^{k-r+1}\mathcal{E}_{i,j} = 0, \tag{10a}$$

$$\partial_i(p)\mathcal{H} + \sum_{l=1}^{n} x_l \mathcal{E}_{l,i} = 0, \tag{10b}$$

and

$$\partial_l(p)\mathcal{E}_{i,j} + \partial_i(p)\mathcal{E}_{j,l} + \partial_j(p)\mathcal{E}_{l,i} = 0. \tag{10c}$$

An alternative proof is given in [**12**] for polynomial (4) as an application of sequence (9).

We return to the case of simple elliptic singularities $\tilde{E}_6$. According to the second assertion of Theorem 10, we define

$$\mathcal{H} := x\partial_x + y\partial_y + z\partial_z,$$
$$\mathcal{X}_t := \partial_z(f_t)\partial_y - \partial_y(f_t)\partial_z = (3z^2 + txy)\partial_y - (3y^2 + txz)\partial_z,$$
$$\mathcal{Y}_t := \partial_x(f_t)\partial_z - \partial_z(f_t)\partial_x = (3x^2 + tyz)\partial_z - (3z^2 + txy)\partial_x,$$
$$\mathcal{Z}_t := \partial_y(f_t)\partial_x - \partial_x(f_t)\partial_y = (3y^2 + txz)\partial_x - (3x^2 + tyz)\partial_y.$$

Then $L_t^k/m^k\langle\partial_x,\partial_y,\partial_z\rangle$ is isomorphic to an $\mathcal{R}(V_t)$-module generated by $\mathcal{X}_t,\mathcal{Y}_t,\mathcal{Z}_t,\mathcal{H}$ with relations:

$$z\mathcal{Y}_t - y\mathcal{Z}_t = (3x^2+tyz)\mathcal{H}, \tag{11a}$$

$$x\mathcal{Z}_t - z\mathcal{X}_t = (3y^2+txz)\mathcal{H}, \tag{11b}$$

$$y\mathcal{X}_t - x\mathcal{Y}_t = (3z^2+txy)\mathcal{H}, \tag{11c}$$

$$(3x^2+tyz)\mathcal{X}_t + (3y^2+txz)\mathcal{Y}_t + (3z^2+txy)\mathcal{Z}_t = 0, \tag{11d}$$

and

$$m^{k-1}\mathcal{H} = m^{k-2}\mathcal{X}_t = m^{k-2}\mathcal{Y}_t = m^{k-2}\mathcal{Z}_t = 0. \tag{11e}$$

Hence, one may choose a basis of L_t^k of the form

$$\begin{aligned}&\{\xi\mathcal{X}_t : \xi\in\mathfrak{S}_{k-3}\setminus x^2\mathfrak{S}_{k-5}\}\\ &\cup\{\xi\mathcal{Y}_t,\xi\mathcal{Z}_t : \xi\in\mathfrak{S}_{k-3}\setminus x\mathfrak{S}_{k-4}\}\\ &\cup\{\lambda\mathcal{H} : \lambda\in\mathfrak{S}_{k-2}\setminus x^2\mathfrak{S}_{k-4}\}\\ &\cup\{\eta\partial_x,\eta\partial_y,\eta\partial_z : \eta\in\mathfrak{S}_{k+1}\setminus x^3\mathfrak{S}_{k-2} \text{ and } \deg\eta = k \text{ or } k+1\}.\end{aligned}$$

A direct calculation shows that the k-th Yau number satisfies

$$\dim L_t^k = 3k^2+9k+16,$$

which coincides with polynomial (4) in Corollary 4.

5. Torelli theorem for the k-th Yau algebra

In this section, we aim to prove Theorem B. The basic idea is to compute the set of isomorphisms from L_t^k to L_s^k. The grading structure of L_t^k is not invariant under isomorphisms which creates many difficulties. To overcome this issue we introduce a new family of graded Lie algebras with the same isomorphic classes instead.

Throughout this section, we assume $k\geqslant 2$. First we wish to define a series of Lie subalgebras of L_t^k:

$$\mathfrak{g}_0\supseteq\mathfrak{g}_1\supseteq\mathfrak{g}_2\supseteq\cdots\supseteq\mathfrak{g}_k\supseteq\mathfrak{g}_{k+1}. \tag{12}$$

Observing from the basis of L_t^k studied in Section 4 that the degree of derivations ranges from 0 to k, we shall set $\mathfrak{g}_{k+1}:=\{0\}$. Let us define $\mathfrak{g}_1:=[L_t^k,L_t^k]$ and iteratively,

$$\mathfrak{g}_i:=\{u\in L_t^k : [u,\mathfrak{g}_1]\in\mathfrak{g}_{i+1}\},$$

where i ranges from k to 2. From the construction above, we see that each subspace $\mathfrak{g}_i$ is generated by all derivations of degree $\geqslant i$. Denote by $D_{i,t}^k$ the quotient space $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ which is represented by homogeneous derivations of degree i. We define a graded Lie algebra called $\bar{L}_t^k$ by

$$\bar{L}_t^k:=\oplus_{i=0}^k D_{i,t}^k,$$

with Lie bracket operations inherited from L_t^k. From definition, we have $\bar{L}_t^k \cong L_t^k$ as Lie algebras. Let us denote by $\mathfrak{M}_d$ the set of monomials of degree d. According to the basis of $\bar{L}_t^k$, we find that, for $i = k$ or $k-1$,

$$D_{i,t}^k \cong \frac{\mathbb{C}\langle \eta\partial_x, \eta\partial_y, \eta\partial_z : \eta \in \mathfrak{M}_{i+1}\rangle}{\mathbb{C}\langle \xi f_t\partial_x, \xi f_t\partial_y, \xi f_t\partial_z : \xi \in \mathfrak{M}_{i-2}\rangle}, \tag{13}$$

and for $0 \leqslant i < k-1$,

$$D_{i,t}^k \cong \frac{\mathbb{C}\langle \lambda\mathcal{H}, \xi\mathcal{X}_t, \xi\mathcal{Y}_t, \xi\mathcal{Z}_t : \lambda \in \mathfrak{M}_i, \xi \in \mathfrak{M}_{i-1}\rangle}{\mathbb{C}\langle \text{relations generated by (11a)-(11d) and } f_t\rangle}, \tag{14}$$

as vector spaces. Hence, for a fixed subscript i, the subspace $D_{i,t}^k$ remains unchanged when k ranks from $i+2$ to an arbitrary large integer. This enables us to define the limit of sequence L_t^k with $k \in \mathbb{N}$ as

$$\bar{L}_t^\infty := \oplus_{i=0}^\infty D_{i,t}^{i+2}.$$

It is natural to wonder whether there exists a canonical definition for the limit of a general sequence of k-th Yau algebras of isolated singularities.

Let $\mathrm{Iso}(L_t^k, L_s^k)$ be the set of all isomorphisms from L_t^k to L_s^k. Denote by $\mathrm{Iso}(\bar{L}_t^k, \bar{L}_s^k)$ the set of isomorphisms from $\bar{L}_t^k$ to $\bar{L}_s^k$ preserving both grading structures. Since the Lie bracket of L_t^k coincides with the one of $\bar{L}_t^k$, there exists a natural embedding

$$\mathrm{Iso}(\bar{L}_t^k, \bar{L}_s^k) \hookrightarrow \mathrm{Iso}(L_t^k, L_s^k).$$

Conversely, any isomorphism from L_t^k to L_s^k maps the series (12) of L_t^k to the correspondent one of L_s^k analogously which induces an isomorphism that preserves the grading structures of $\bar{L}_t^k$ and $\bar{L}_s^k$. So we obtain a natural map

$$\mathrm{Iso}(L_t^k, L_s^k) \to \mathrm{Iso}(\bar{L}_t^k, \bar{L}_s^k).$$

More essential relations between these two sets are described by the following theorem.

THEOREM 11 ([**12**]). *Let L_t^k be the k-th Yau algebra of $\tilde{E}_6$. Let $\bar{L}_t^k$ be the graded Lie algebra induced by L_t^k. Denote by $\mathrm{Aut}_{id}(L_t^k)$ the group of automorphisms of L_t^k which induce the identity map on $\bar{L}_t^k$. Then*

$$\mathrm{Iso}(L_t^k, L_s^k) \cong \mathrm{Aut}_{id}(L_s^k) \rtimes \mathrm{Iso}(\bar{L}_t^k, \bar{L}_s^k) \cong \mathrm{Iso}(\bar{L}_t^k, \bar{L}_s^k) \ltimes \mathrm{Aut}_{id}(L_t^k).$$

Therefore, L_t^k is isomorphic to L_s^k if and only if $\bar{L}_t^k$ is isomorphic to $\bar{L}_s^k$.

DEFINITION 12. A Lie algebra L is called a metabelian Lie algebra if $[L, [L, L]] = 0$. Its signature is a pair (m, n), where

$$m = \dim L/[L, L] \text{ and } n = \dim[L, L].$$

We define two families of metabelian Lie algebras $\{M_t\}$ and $\{N_t\}$ as

$$M_t := D_{1,t}^2 \oplus [D_{1,t}^2, D_{1,t}^2],$$

and

$$N_t := D_{1,t}^3 \oplus [D_{1,t}^3, D_{1,t}^3].$$

For simplicity we write $\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z$ instead of $x\mathcal{H}, y\mathcal{H}, z\mathcal{H}$ respectively throughout this paper. Denote by K_t a linear subspace generated by $f_t\partial_x, f_t\partial_y, f_t\partial_z$. It can be easily deduced from Equations (13) and (14) that

$$D^2_{1,t} \cong \mathbb{C}\langle \eta\partial_x, \eta\partial_y, \eta\partial_z : \eta \in \mathfrak{M}_2\rangle,$$

and

$$D^3_{1,t} \cong \mathbb{C}\langle \mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z, \mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t\rangle.$$

Then a direct calculation shows that

$$D^2_{2,t} = [D^2_{1,t}, D^2_{1,t}] \cong \mathbb{C}\langle \eta\partial_x, \eta\partial_y, \eta\partial_z : \eta \in \mathfrak{M}_3\rangle / K_t,$$

and

$$D^3_{2,t} \supsetneq [D^3_{1,t}, D^3_{1,t}] \cong \mathbb{C}\langle x\mathcal{X}_t, y\mathcal{X}_t, z\mathcal{X}_t, x\mathcal{Y}_t, y\mathcal{Y}_t, z\mathcal{Y}_t, x\mathcal{Z}_t, y\mathcal{Z}_t, z\mathcal{Z}_t\rangle.$$

This yields that the signatures of M_t and N_t are $(18, 27)$ and $(6, 9)$ respectively.

The following theorem will play a significant role in the proof of Theorem B.

THEOREM 13 ([**12**]). *Denote by* $\operatorname{Iso}(M_t, M_s)$ *and* $\operatorname{Iso}(N_t, N_s)$ *the sets of isomorphisms preserving grading structures. Then we have the equivalences*

$$\operatorname{Iso}(M_t, M_s) \cong \operatorname{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s)), \tag{15}$$

and

$$\operatorname{Iso}(N_t, N_s) \cong \operatorname{GL}(\mathcal{A}^0(V_t), \mathcal{A}^0(V_s)). \tag{16}$$

Therefore, the Lie algebra M_t (resp. N_t) is isomorphic to M_s (resp. N_s) if and only if their j-invariants $j(t)$ and $j(s)$ are equal.

We finish this section with a proof of Theorem B.

PROOF OF THEOREM B. We first assume that V_t is biholomorphic to V_s. By Mather-Yau Theorem (or Theorem 6) there exists an isomorphism from $\mathcal{A}^k(V_t)$ to $\mathcal{A}^k(V_s)$. It gives rise to an isomorphism from L^k_t to L^k_s since k-th Yau algebra is constructed from the k-th moduli algebra.

Conversely, suppose that ϕ is an isomorphism from L^k_t to L^k_s. Then it induces an isomorphism that preserves the grading structures of $\bar{L}^k_t$ and $\bar{L}^k_s$ from Theorem 11. So ϕ maps isomorphically from $D^k_{i,t}$ to $D^k_{i,s}$ for $i = 1, 2$. Theorem 13 states that the j-invariants $j(t)$ and $j(s)$ are equal. Therefore, we find that V_t is biholomorphic to V_s. □

REMARK 14. *Suppose that $t^3, s^3 \neq 0, 216, -27$. The proof actually shows that* $\operatorname{Iso}(\bar{L}^k_t, \bar{L}^k_s) \cong \operatorname{GL}(\mathcal{A}^0(V_t), \mathcal{A}^0(V_s)) \cong \operatorname{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s))$ *for $k \geqslant 2$ or even $k = \infty$. It remains to prove Theorem 13 which is explained in the next two sections.*

6. Isomorphisms of $\{M_t\}$

We prove the equivalence between $\mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s))$ and $\mathrm{Iso}(M_t, M_s)$ in this section. Recall that M_t admits a direct sum decomposition

$$M_t = D_1 \oplus (D_2/K_t),$$

where D_1, D_2 are generated by all derivations $\eta\partial_x, \eta\partial_y, \eta\partial_z$ for $\eta \in \mathfrak{M}_2, \mathfrak{M}_3$ respectively. It is closely related to a graded metabelian Lie algebra M_* with decomposition

$$M_* := D_1 \oplus D_2.$$

We shall mention that M_* actually is a Lie subalgebra of the Yau algebra of $\mathbb{C}[[x, y, z]]/m^4$. We employ the symbols $[\cdot,\cdot]_t$, $[\cdot,\cdot]_s$ and $[\cdot,\cdot]$ to distinguish Lie brackets on M_t, M_s and M_*. Knowing $[u, v]_t = [u, v] \mod K_t$, we observe that both M_t and M_s are quotient Lie algebras of M_*.

If ϕ is an isomorphism from M_t to M_s preserving both grading structures, then for $u, v \in M_t$,

$$\phi[u, v]_t = [\phi u, \phi v]_s.$$

Let $A := \phi|_{D_1}$ and $B := \phi|_{D_2/K_t}$. Then for $u, v \in D_1$,

$$[Au, Av] = B[u, v] \mod K_s.$$

The linear map B can be extended to $\mathrm{GL}(D_2)$ whose image on the subspace K_t is K_s. And then there exists a map

$$C : D_1 \times D_1 \to K_s$$

such that

$$[Au, Av] = B[u, v] + C(u, v).$$

Obviously, $C(u, v)$ shall be an anti-commutative bilinear map. Note that the Lie bracket

$$[\cdot,\cdot] : D_1 \times D_1 \to D_2$$

is surjective and both $[D_1, D_2]$ and $[D_2, D_2]$ vanish. We realize that the isomorphism ϕ is completely determined by the linear map A, though maps B and C may not be unique. In fact, the bilinear map C can be cancelled by applying the following theorem.

THEOREM 15 ([**12**]). *Suppose that ϕ is a degree-preserving isomorphism from M_t to M_s. Let $A \in \mathrm{GL}(D_1)$ be the restriction of ϕ to D_1. Then there exists a unique linear automorphism $B \in \mathrm{GL}(D_2)$, such that*

$$[Au, Av] = B[u, v].$$

In addition, ϕ induces an automorphism of M_* which preserves the grading structure of M_*, i.e.,

$$\mathrm{Iso}(M_t, M_s) \subseteq \mathrm{Aut}(M_*).$$

Before we give a proof, we introduce three derivations of Lie algebra M_* and deduce a lemma. For $u \in M_*$, we define

$$\delta_x(u) := [x\partial_x, u],$$
$$\delta_y(u) := [y\partial_y, u],$$
$$\delta_z(u) := [z\partial_z, u].$$

Though $x\partial_x$ is not contained in M_*, derivation δ_x is well-defined as an endomorphism of M_*. Notice that by the Jacobi identity of the Yau algebra of $\mathbb{C}[[x, y, z]]/m^4$, we have

$$\delta_x[u, v] = [\delta_x(u), v] + [u, \delta_x(v)].$$

This implies δ_x is indeed a derivation of M_*. All generators $\eta\partial_x, \eta\partial_y, \eta\partial_z$ with $\eta \in \mathfrak{M}_2 \cup \mathfrak{M}_3$ of basis for M_* are eigenvectors of δ_x. Given u one of such generators, we define $\deg_x(u)$ to be the degree of u with respect to x. Then

$$\delta_x(u) = \deg_x(u) \cdot u.$$

So the eigenvalues of $\delta_x|_{D_1}$ are $\{-1, 0, 1, 2\}$ and those of $\delta_x|_{D_2}$ are $\{-1, 0, 1, 2, 3\}$. In the same manner, similar properties hold for δ_y and δ_z.

LEMMA 16 ([**12**]). *Assume that s is a nonzero complex number. Then there exist decompositions of linear maps on D_2:*

$$id|_{D_2} = P_x + Q_x = P_y + Q_y = P_z + Q_z$$

such that $\operatorname{im} Q_\alpha = K_s$ *and* $\delta_\alpha D_2 \subseteq \operatorname{im} P_\alpha$ *for* $\alpha = x, y, z$.

PROOF OF THEOREM 15. Let us assume that ϕ is an isomorphism from M_t to M_s represented by (A, B, C). That is for $u, v \in D_1$,

$$[Au, Av] = B[u, v] + C(u, v).$$

Since the matrix A_2 defined in Theorem 7 induces an isomorphism from M_0 to M_6, we may assume that both parameters t and s are nonzero. Let P_x, Q_x be the linear maps constructed in Lemma 16. Then

$$\begin{aligned}(17)\qquad [Au, Av] &= B[u, v] + C(u, v) = P_xB[u, v] + Q_xB[u, v] + C(u, v)\\ &= P_xB[u, v] + C_x(u, v),\end{aligned}$$

where

$$C_x(u, v) := C(u, v) + Q_xB[u, v].$$

Set $\delta_x^A := A^{-1}\delta_x A$. Applying δ_x on both sides of Equation (17), we obtain

$$\begin{aligned}&\delta_x P_x B[u, v] + \delta_x C_x(u, v)\\ &\quad= \delta_x[Au, Av]\\ &\quad= [\delta_x Au, Av] + [Au, \delta_x Av]\\ &\quad= [A\delta_x^A u, Av] + [Au, A\delta_x^A v]\\ &\quad= P_xB[\delta_x^A u, v] + P_xB[u, \delta_x^A v] + C_x(\delta_x^A u, v) + C_x(u, \delta_x^A v).\end{aligned}$$

As a consequence of Lemma 16, we obtain $\delta_x D_2 \subseteq \operatorname{im} P_x$ and $\operatorname{im} C_x \subseteq K_s = \operatorname{im} Q_x$. It follows that the sum $C_x(\delta_x^A u, v) + C_x(u, \delta_x^A v)$ is contained in the intersection of $\operatorname{im} P_x$ and $\operatorname{im} Q_x$. Thus,

$$C_x(\delta_x^A u, v) + C_x(u, \delta_x^A v) = 0. \tag{18}$$

Since δ_x^A is a similarity transformation of δ_x, the eigenvalues of $\delta_x^A|_{D_1}$ shall be $-1, 0, 1, 2$ and the eigenvectors are of the form $A^{-1}\eta\partial_x, A^{-1}\eta\partial_y, A^{-1}\eta\partial_z$ with $\eta \in \mathfrak{M}_2$. Hence, one may suppose that both u, v are eigenvectors of δ_x^A with eigenvalues $\deg_x(Au), \deg_x(Av)$ respectively. Then it follows from Equation (18) that

$$(\deg_x(Au) + \deg_x(Av))C_x(u, v) = 0.$$

Thus, for $\deg_x(Au) + \deg_x(Av) \neq 0$,

$$C(u, v) = -Q_x B[u, v].$$

Similarly, we can conclude that for $\deg_y(Au) + \deg_y(Av) \neq 0$,

$$C(u, v) = -Q_y B[u, v],$$

and for $\deg_z(Au) + \deg_z(Av) \neq 0$,

$$C(u, v) = -Q_z B[u, v].$$

Notice that

$$\begin{aligned}&(\deg_x Au + \deg_x Av) + (\deg_y Au + \deg_y Av) + (\deg_z Au + \deg_z Av)\\&= (\deg_x + \deg_y + \deg_z)(Au) + (\deg_x + \deg_y + \deg_z)(Av) = 2.\end{aligned}$$

So at least one of the previous three conditions holds for arbitrary eigenvectors u and v. It yields that the value of $C(u, v)$ depends only on the Lie bracket $[u, v]$. Since C is a bilinear map, there exists a linear map Q from D_2 to K_s such that for arbitrary vectors u and v,

$$C(u, v) = Q[u, v].$$

Let $B_Q := B + Q$. Then

$$[Au, Av] = B[u, v] + C(u, v) = (B + Q)[u, v] = B_Q[u, v].$$

Applying the same arguments for ϕ^{-1}, we get

$$[A^{-1}u, A^{-1}v] = B_Q^*[u, v].$$

Hence,

$$[u, v] = [AA^{-1}u, AA^{-1}v] = B_Q[A^{-1}u, A^{-1}u] = B_Q B_Q^*[u, v].$$

This implies $B_Q B_Q^* = id$. Thus, $B_Q \in \mathrm{GL}(D_2)$ and

$$[A^{-1}u, A^{-1}v] = B_Q^{-1}[u, v].$$

This implies that (A, B_Q) represents an automorphism of M_* whose inverse is given by (A^{-1}, B_Q^{-1}). ☐

REMARK 17. *Theorem 15 tells us that every isomorphism from M_t to M_s gives rise to an automorphism of M_*. So our next task is to describe the automorphism group of M_*. Once we prove that* $\mathrm{Aut}(M_*) \cong \mathrm{GL}(3)$*, then the first equivalence of Theorem 13 follows obviously. In fact, if* $\phi \in \mathrm{GL}(3)$ *induces an isomorphism from M_t to M_s, then it maps isomorphically from K_t to K_s. This yields $\phi(f_t) = \lambda f_s$ for some constant λ. So ϕ is indeed a linear isomorphism from $\mathcal{R}(V_t)$ to $\mathcal{R}(V_s)$ and vice versa.*

The inclusion $\mathrm{GL}(3) \subseteq \mathrm{Aut}(M_*)$ is trivial since all non-degenerate linear maps are automorphisms of $\mathbb{C}[[x, y, z]]/m^4$.

The idea for the converse is as follows. Define Z to be a linear subspace of D_1 generated by $\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z$. Observe that all automorphisms induced by $\mathrm{GL}(3)$ leave the subspace Z invariant. To see this, let A be a matrix contained in $\mathrm{GL}(3)$. Write

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} a_1' & b_1' & c_1' \\ a_2' & b_2' & c_2' \\ a_3' & b_3' & c_3' \end{pmatrix}.$$

The induced map $\phi_A : \mathbb{C}[[x, y, z]]/m^4 \to \mathbb{C}[[x, y, z]]/m^4$ is expressed as

$$\begin{cases} \phi_A(x) = a_1 x + b_1 y + c_1 z, \\ \phi_A(y) = a_2 x + b_2 y + c_2 z, \\ \phi_A(z) = a_3 x + b_3 y + c_3 z. \end{cases}$$

Then the explicit expression for $\phi_A(\mathcal{H})$ is given by

$$\begin{aligned} \phi_A(x\partial_x + y\partial_y + z\partial_z) = \quad & (a_1 x + b_1 y + c_1 z)(a_1'\partial_x + a_2'\partial_y + a_3'\partial_z) \\ & + (a_2 x + b_2 y + c_2 z)(b_1'\partial_x + b_2'\partial_y + b_3'\partial_z) \\ & + (a_3 x + b_3 y + c_3 z)(c_1'\partial_x + c_2'\partial_y + c_3'\partial_z) \\ = & \; x\partial_x + y\partial_y + z\partial_z. \end{aligned}$$

Hence, we get

$$\begin{cases} \phi_A(\mathcal{H}_x) = a_1 \mathcal{H}_x + b_1 \mathcal{H}_y + c_1 \mathcal{H}_z, \\ \phi_A(\mathcal{H}_y) = a_2 \mathcal{H}_x + b_2 \mathcal{H}_y + c_2 \mathcal{H}_z, \\ \phi_A(\mathcal{H}_z) = a_3 \mathcal{H}_x + b_3 \mathcal{H}_y + c_3 \mathcal{H}_z, \end{cases} \tag{19}$$

which implies that the matrix representation of $\phi|_Z$ is the same as A. Hence, we shall necessarily prove that any algebraic automorphism of M_* maps Z to itself. Taking advantage of this claim, one may construct various invariants from Z which eventually characterize all algebraic automorphisms.

We begin with a useful lemma.

LEMMA 18 ([**12**]).

(1) *The intersection I of linear spaces $[\mathcal{H}_x, M_*]$ and $[\mathcal{H}_y, M_*]$ is generated by $x^2y\partial_z, xy^2\partial_z, xyz\partial_z$. Furthermore, we have*

$$\{u : [\mathcal{H}_x, u] \subseteq I\} = Z \oplus \langle xy\partial_z, y^2\partial_z, yz\partial_z \rangle,$$

and

$$\{u : [\mathcal{H}_y, u] \subseteq I\} = Z \oplus \langle x^2\partial_z, xy\partial_z, xz\partial_z\rangle.$$

(2) *Fix $\lambda \in \mathbb{C}$. If Ω is an endomorphism of D_1 such that for $u, v \in Z$,*

$$[u, \Omega v] + \lambda[\Omega u, v] = 0. \tag{20}$$

Then Ω maps Z to itself.

For a vector $v \in D_1$, we define the kernel of v by

$$\ker(v) = \{u \in D_1 : [u, v] = 0\}.$$

For a subspace $V \subseteq D_1$, we define

$$\ker(V) = \{u \in D_1 : [u, v] = 0 \text{ for all } v \in V\}.$$

Lemma 19 ([12]). *Assume that ϕ is an automorphism of M_* preserving grading structure. Let A and B be the restrictions of ϕ to D_1 and D_2 respectively. Then*

$$[Z, Z] = [A(Z), A(Z)] = 0.$$

Moreover, for any non-zero vector u contained in Z, we have

$$\ker(u) = Z \text{ and } \ker(Au) = A(Z).$$

Lemma 20 ([12]). *With the same notations in Lemma 19, the following assertions hold.*

(1) *Fix $u, v \in Z$ and express Au and Av as*

$$Au = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2,$$
$$Av = \beta_{-1} + \beta_0 + \beta_1 + \beta_2,$$

according to the degree of x (resp. y, z). Then $[\alpha_i, \beta_j] = 0$ for $i, j = -1, 0, 1, 2$.

(2) *For any vector $h \in A(Z)$, we write*

$$h = \sum_{k,l,m=-1}^{2} \alpha_{k,l,m}$$

according to the relative degrees of x, y, z. Then each component $\alpha_{k,l,m}$ of the expression is also contained in $A(Z)$.

Lemma 21 ([12]). *The dimensions of kernels of vectors contained in the monomial basis of D_1 are given by*

$$\dim\ker x^2\partial_x = \dim\ker y^2\partial_y = \dim\ker z^2\partial_z = 7;$$
$$\dim\ker y^2\partial_x = \dim\ker z^2\partial_x = \dim\ker x^2\partial_y = \dim\ker z^2\partial_y = \dim\ker x^2\partial_z$$
$$= \dim\ker y^2\partial_z = 8;$$
$$\dim\ker yz\partial_x = \dim\ker xz\partial_y = \dim\ker xy\partial_z = 7;$$
$$\dim\ker xy\partial_x = \dim\ker xz\partial_x = \dim\ker xy\partial_y = \dim\ker yz\partial_y = \dim\ker xz\partial_z$$
$$= \dim\ker yz\partial_z = 5.$$

LEMMA 22 ([12]). *Let $\phi = (A, B)$ represent an isomorphism of Lie algebra M_* preserving grading structure. Then A maps Z to itself.*

Based on above Lemmas, we can prove the following result.

THEOREM 23 ([12]). *Let ϕ be an automorphism of Lie algebra M_* preserving grading structure. Then ϕ is induced by some linear map $P \in \mathrm{GL}(3)$, or equivalently*

$$\mathrm{Aut}(M_*) \cong \mathrm{GL}(3).$$

Now we conclude that the first equivalence of Theorem 13 follows from Remark 17 and Theorem 23.

7. Isomorphisms of $\{N_t\}$

We wish to prove equivalence (16) of Theorem 13 in this section. Recall from Section 5 that N_t is a metabelian Lie algebra with decomposition

$$N_t = D_{1,t} \oplus [D_{1,t}, D_{1,t}].$$

Throughout this section, we adopt $[\cdot,\cdot]_s$ to distinguish Lie bracket operations of N_s from those of N_t. The basis of $D_{1,t}$ stated in Section 5 is given by

$$\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z, \mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t, \tag{21}$$

and the one of $[D_{1,t}, D_{1,t}]$ is

$$x\mathcal{X}_t, y\mathcal{X}_t, z\mathcal{X}_t, x\mathcal{Y}_t, y\mathcal{Y}_t, z\mathcal{Y}_t, x\mathcal{Z}_t, y\mathcal{Z}_t, z\mathcal{Z}_t.$$

A direct calculation shows that

$$\begin{aligned}
&[\mathcal{H}_x, \mathcal{X}_t] = x\mathcal{X}_t, [\mathcal{H}_x, \mathcal{Y}_t] = y\mathcal{X}_t, [\mathcal{H}_x, \mathcal{Z}_t] = z\mathcal{X}_t,\\
&[\mathcal{H}_y, \mathcal{X}_t] = x\mathcal{Y}_t, [\mathcal{H}_y, \mathcal{Y}_t] = y\mathcal{Y}_t, [\mathcal{H}_y, \mathcal{Z}_t] = z\mathcal{Y}_t,\\
&[\mathcal{H}_z, \mathcal{X}_t] = x\mathcal{Z}_t, [\mathcal{H}_z, \mathcal{Y}_t] = y\mathcal{Z}_t, [\mathcal{H}_z, \mathcal{Z}_t] = z\mathcal{Z}_t,
\end{aligned}$$

and then

$$\begin{aligned}
[\mathcal{Z}_t, \mathcal{X}_t] &= [(3y^2 + txz)\partial_x - (3x^2 + tyz)\partial_y, (3z^2 + txy)\partial_y - (3y^2 + txz)\partial_z]\\
&= (t^2x^2z - 18z^2y - 3txy^2)\partial_x + (3tz^3 - 3tx^3)\partial_y\\
&\quad + (18x^2y + 3ty^2z - t^2xz^2)\partial_z\\
&= 6y\mathcal{Y}_t + t((tx^2z + 3xy^2)\partial_x + (3z^3 - 3x^3)\partial_y + (-3y^2z - txz^2)\partial_z)\\
&= tz\mathcal{X}_t + 6y\mathcal{Y}_t + tx\mathcal{Z}_t.
\end{aligned}$$

Similar computation shows that

$$[\mathcal{Z}_t, \mathcal{X}_t] = tz\mathcal{X}_t + 6y\mathcal{Y}_t + tx\mathcal{Z}_t, \tag{22a}$$

$$[\mathcal{X}_t, \mathcal{Y}_t] = ty\mathcal{X}_t + tx\mathcal{Y}_t + 6z\mathcal{Z}_t, \tag{22b}$$

$$[\mathcal{Y}_t, \mathcal{Z}_t] = 6x\mathcal{X}_t + tz\mathcal{Y}_t + ty\mathcal{Z}_t. \tag{22c}$$

Thus, we deduce the equalities

$$6[\mathcal{H}_x,\mathcal{X}_t]+t[\mathcal{H}_y,\mathcal{Z}_t]+t[\mathcal{H}_z,\mathcal{Y}_t]=[\mathcal{Y}_t,\mathcal{Z}_t], \tag{23a}$$

$$t[\mathcal{H}_x,\mathcal{Z}_t]+6[\mathcal{H}_y,\mathcal{Y}_t]+t[\mathcal{H}_z,\mathcal{X}_t]=[\mathcal{Z}_t,\mathcal{X}_t], \tag{23b}$$

$$t[\mathcal{H}_x,\mathcal{Y}_t]+t[\mathcal{H}_y,\mathcal{X}_t]+6[\mathcal{H}_z,\mathcal{Z}_t]=[\mathcal{X}_t,\mathcal{Y}_t]. \tag{23c}$$

As in Section 6, we put $Z:=\langle\mathcal{H}_x,\mathcal{H}_y,\mathcal{H}_z\rangle$ and obtain

$$[Z,Z]=0. \tag{23d}$$

One can conclude that N_t is actually isomorphic to a free metabelian Lie algebra generated by elements of (21) modulo relations (23a)-(23d).

We make a few observation regarding how matrices A_1 and A_2 denoted in Theorem 7 act on $\{N_t\}$. Denote by ϕ_{A_1} and ϕ_{A_2} the induced maps of A_1 and A_2 on $D_{1,t}$ respectively; then

$$\begin{cases}\phi_{A_1}(\mathcal{H}_x)=\rho\mathcal{H}_x,\\ \phi_{A_1}(\mathcal{H}_y)=\mathcal{H}_y,\\ \phi_{A_1}(\mathcal{H}_z)=\mathcal{H}_z,\\ \phi_{A_1}(\mathcal{X}_t)=\mathcal{X}_s,\\ \phi_{A_1}(\mathcal{Y}_t)=\rho^2\mathcal{Y}_s,\\ \phi_{A_1}(\mathcal{Z}_t)=\rho^2\mathcal{Z}_s,\end{cases}$$

and

$$\begin{cases}\phi_{A_2}(\mathcal{H}_x)=\rho\mathcal{H}_x+\rho^2\mathcal{H}_y+\mathcal{H}_z,\\ \phi_{A_2}(\mathcal{H}_y)=\rho^2\mathcal{H}_x+\rho\mathcal{H}_y+\mathcal{H}_z,\\ \phi_{A_2}(\mathcal{H}_z)=\mathcal{H}_x+\mathcal{H}_y+\mathcal{H}_z,\\ \phi_{A_2}(\mathcal{X}_t)=(2/9\rho+1/9)(t+3)(\rho\mathcal{X}_s+\rho^2\mathcal{Y}_s+\mathcal{Z}_s),\\ \phi_{A_2}(\mathcal{Y}_t)=(2/9\rho+1/9)(t+3)(\rho^2\mathcal{X}_s+\rho\mathcal{Y}_s+\mathcal{Z}_s),\\ \phi_{A_2}(\mathcal{Z}_t)=(2/9\rho+1/9)(t+3)(\mathcal{X}_s+\mathcal{Y}_s+\mathcal{Z}_s).\end{cases}$$

Put $\Delta_{1,t}:=\langle\mathcal{X}_t,\mathcal{Y}_t,\mathcal{Z}_t\rangle$. We discover that both subspaces Z and $\Delta_{1,t}$ are invariant under ϕ_{A_1} and ϕ_{A_2}. Furthermore, the matrix representations restricted to Z and $\Delta_{1,t}$ are proportional. They are of vital importance for the proof of Theorem 13. The following lemma is devoted to proving the first discovery for arbitrary isomorphism from N_t to N_s.

LEMMA 24 ([**12**]). *Let ϕ be a degree-preserving isomorphism from N_t to N_s. Then $\phi(Z)=Z$.*

We have seen in Section 6 that derivations of Lie algebras play an important role in the study of isomorphisms of $\{M_t\}$. However, in the case of $\{N_t\}$, it turns out that derivations are trivial and hence provide no information. So we need to extend the definition of derivations.

DEFINITION 25. Let W be a one-dimensional subspace of $[D_{1,t},D_{1,t}]$. We call δ a quasi-derivation of N_t associated to W, if δ is a degree-preserving derivation of quotient Lie algebra N_t/W. Alternatively, a quasi-derivation

$$\delta : D_{1,t} \to D_{1,t}, [D_{1,t}, D_{1,t}] \to [D_{1,t}, D_{1,t}]$$

is an endomorphism of N_t such that

$$[\delta u, v] + [u, \delta v] = \delta[u, v] \mod W. \tag{24}$$

Set $\delta^* := \phi\delta\phi^{-1}$, $u^* := \phi u$ and $v^* := \phi v$. Applying ϕ on both sides of Equation (24), we obtain

$$\phi[\delta u, v] + \phi[u, \delta v] = \phi\delta[u, v] \mod \phi W.$$

Hence,

$$\begin{aligned} [\delta^* u^*, v^*]_s + [u^*, \delta^* v^*]_s &= \phi[\delta u, v] + \phi[u, \delta v] \\ &= \phi\delta[u, v] \\ &= \delta^*[u^*, v^*]_s \mod \phi W. \end{aligned}$$

This means that δ^* is a quasi-derivation of N_s associated to ϕW. So the induced map $\phi_* : \delta \mapsto \delta^*$ gives a bijection from quasi-derivations of N_t to those of N_s. A quasi-derivation δ is said to be degenerate if for $u \in Z, v \in D_{1,t}$,

$$\delta u = 0 \text{ and } [u, \delta v] = \delta[u, v].$$

The set of degenerate quasi-derivations can be characterized by the following lemma.

LEMMA 26 ([**12**]). *Let δ be a linear map from $\Delta_{1,t}$ to $D_{1,t}$ and let W be a one-dimensional subspace of $[D_{1,t}, D_{1,t}]$. Then δ can be extended to be a degenerate quasi-derivation associated to W if and only if $\sigma_x(\delta), \sigma_y(\delta), \sigma_z(\delta)$ are contained in W, where*

$$\sigma_x(\delta) := 6[\mathcal{H}_x, \delta\mathcal{X}_t] + t[\mathcal{H}_y, \delta\mathcal{Z}_t] + t[\mathcal{H}_z, \delta\mathcal{Y}_t] - [\delta\mathcal{Y}_t, \mathcal{Z}_t] - [\mathcal{Y}_t, \delta\mathcal{Z}_t], \tag{25a}$$

$$\sigma_y(\delta) := t[\mathcal{H}_x, \delta\mathcal{Z}_t] + 6[\mathcal{H}_y, \delta\mathcal{Y}_t] + t[\mathcal{H}_z, \delta\mathcal{X}_t] - [\delta\mathcal{Z}_t, \mathcal{X}_t] - [\mathcal{Z}_t, \delta\mathcal{X}_t], \tag{25b}$$

$$\sigma_z(\delta) := t[\mathcal{H}_x, \delta\mathcal{Y}_t] + t[\mathcal{H}_y, \delta\mathcal{X}_t] + 6[\mathcal{H}_z, \delta\mathcal{Z}_t] - [\delta\mathcal{X}_t, \mathcal{Y}_t] - [\mathcal{X}_t, \delta\mathcal{Y}_t]. \tag{25c}$$

Denote by $\Delta_{2,t}$ the subspace of $[D_{1,t}, D_{1,t}]$ spanned by

$$[\mathcal{Y}_t, \mathcal{Z}_t], [\mathcal{Z}_t, \mathcal{X}_t], [\mathcal{X}_t, \mathcal{Y}_t].$$

Next, we wish to prove that $\phi(\Delta_{2,t}) = \Delta_{2,s}$ as an application of quasi-derivations.

THEOREM 27 ([**12**]). *Assume that $t^3 \neq -27$. Degenerate quasi-derivations of N_t can be described as follows.*

Case (1): For $t^3 \neq 0, 216, -27$, any degenerate quasi-derivation δ_t^0 of N_t is of the form

$$\begin{cases} \delta_t^0(\mathcal{X}_t) = (t\beta\mu - t\gamma\nu)\mathcal{H}_x + (t\alpha\mu - 6\beta\nu)\mathcal{H}_y + (6\gamma\mu - t\alpha\nu)\mathcal{H}_z \\ \qquad\qquad - (\beta\mu + \gamma\nu)\mathcal{X}_t + \alpha\mu\mathcal{Y}_t + \alpha\nu\mathcal{Z}_t, \\ \delta_t^0(\mathcal{Y}_t) = (6\alpha\nu - t\beta\lambda)\mathcal{H}_x + (t\gamma\nu - t\alpha\lambda)\mathcal{H}_y + (t\beta\nu - 6\gamma\lambda)\mathcal{H}_z + \beta\lambda\mathcal{X}_t \\ \qquad\qquad - (\alpha\lambda + \gamma\nu)\mathcal{Y}_t + \beta\nu\mathcal{Z}_t, \\ \delta_t^0(\mathcal{Z}_t) = (t\gamma\lambda - 6\alpha\mu)\mathcal{H}_x + (6\beta\lambda - t\gamma\mu)\mathcal{H}_y + (t\alpha\lambda - t\beta\mu)\mathcal{H}_z + \gamma\lambda\mathcal{X}_t \\ \qquad\qquad + \gamma\mu\mathcal{Y}_t - (\alpha\lambda + \beta\mu)\mathcal{Z}_t, \end{cases}$$

for some complex numbers $\alpha, \beta, \gamma, \lambda, \mu, \nu$ and the correspondent vector space W_0 is spanned by the vector

$$\alpha[\mathcal{Y}_t, \mathcal{Z}_t] + \beta[\mathcal{Z}_t, \mathcal{X}_t] + \gamma[\mathcal{X}_t, \mathcal{Y}_t].$$

In addition, there exists a nonzero quasi-derivation associated to a one-dimensional linear subspace W if and only if W is contained in the linear subspace $\Delta_{2,t}$.

Case (2): For $t = 0$, the degenerate quasi-derivation of N_t is given by $\delta_0^1, \delta_0^2, \delta_0^3$ or of the form described in Case (1), where δ_0^1 is defined by

$$\begin{cases} \delta_0^1(\mathcal{X}_t) = (\zeta/6)\mathcal{Y}_t + (\eta/6)\mathcal{Z}_t, \\ \delta_0^1(\mathcal{Y}_t) = -\eta\mathcal{H}_x - 6\gamma\mathcal{H}_z + \beta\mathcal{X}_t - \alpha\mathcal{Y}_t, \\ \delta_0^1(\mathcal{Z}_t) = \zeta\mathcal{H}_x + 6\beta\mathcal{H}_y + \gamma\mathcal{X}_t - \alpha\mathcal{Z}_t, \end{cases}$$

and δ_0^2, δ_0^3 are the image of δ_0^1 under isomorphisms $\phi_{A_4A_3,*}, \phi_{A_3A_4,*}$ respectively. Explicitly, δ_0^2 and δ_0^3 can be expressed as

$$\begin{cases} \delta_0^2(\mathcal{X}_t) = \zeta\mathcal{H}_y + 6\gamma\mathcal{H}_z - \beta\mathcal{X}_t + \alpha\mathcal{Y}_t, \\ \delta_0^2(\mathcal{Y}_t) = (\zeta/6)\mathcal{Z}_t + (\eta/6)\mathcal{X}_t, \\ \delta_0^2(\mathcal{Z}_t) = -6\alpha\mathcal{H}_x - \eta\mathcal{H}_y + \gamma\mathcal{Y}_t - \beta\mathcal{Z}_t, \end{cases}$$

and

$$\begin{cases} \delta_0^3(\mathcal{X}_t) = -6\beta\mathcal{H}_y - \eta\mathcal{H}_z - \gamma\mathcal{X}_t + \alpha\mathcal{Z}_t, \\ \delta_0^3(\mathcal{Y}_t) = 6\alpha\mathcal{H}_x + \zeta\mathcal{H}_z - \gamma\mathcal{Y}_t + \beta\mathcal{Z}_t, \\ \delta_0^3(\mathcal{Z}_t) = (\zeta/6)\mathcal{X}_t + (\eta/6)\mathcal{Y}_t. \end{cases}$$

The correspond one-dimensional linear subspaces W_1, W_2, W_3 of $\delta_0^1, \delta_0^2, \delta_0^3$ are respectively spanned by

$$\alpha[\mathcal{Y}_t, \mathcal{Z}_t] + \beta[\mathcal{Z}_t, \mathcal{X}_t] + \gamma[\mathcal{X}_t, \mathcal{Y}_t] + \zeta[\mathcal{H}_x, \mathcal{Y}_t] + \eta[\mathcal{H}_x, \mathcal{Z}_t],$$
$$\alpha[\mathcal{Y}_t, \mathcal{Z}_t] + \beta[\mathcal{Z}_t, \mathcal{X}_t] + \gamma[\mathcal{X}_t, \mathcal{Y}_t] + \zeta[\mathcal{H}_y, \mathcal{Z}_t] + \eta[\mathcal{H}_y, \mathcal{X}_t],$$
$$\alpha[\mathcal{Y}_t, \mathcal{Z}_t] + \beta[\mathcal{Z}_t, \mathcal{X}_t] + \gamma[\mathcal{X}_t, \mathcal{Y}_t] + \zeta[\mathcal{H}_z, \mathcal{X}_t] + \eta[\mathcal{H}_z, \mathcal{Y}_t].$$

Case (3): For $t = 6\rho^j$ with $j = 0, 1, 2$, the degenerate quasi-derivation of N_t is given by the image of $\delta_0^1, \delta_0^2, \delta_0^3$ under $\phi_j := \phi_{A_1^jA_2,*}$ with associated linear spaces $\phi_j(W_1), \phi_j(W_2), \phi_j(W_3)$ respectively except those described in Case (1).

As an immediate consequence of the previous theorem, we find that N_0 can only be equivalent to $N_6, N_{6\rho}, N_{6\rho^2}$. Moreover, we obtain the following corollary.

COROLLARY 28 ([12]). *Let ϕ be an isomorphism contained in* $\mathrm{Iso}(N_t, N_s)$*. Then $\phi(\Delta_{2,t}) = \Delta_{2,s}$.*

With the help of this result, we can show that $\phi(\Delta_{1,t}) = \Delta_{1,s}$.

LEMMA 29. *Let $\Delta_{1,t}$ be a subspace of $D_{1,t}$ generated by $\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t$. Let $\Delta_{2,t}$ be a subspace of $[D_{1,t}, D_{1,t}]$ generated by $[\mathcal{X}_t, \mathcal{Y}_t]$, $[\mathcal{Y}_t, \mathcal{Z}_t]$, $[\mathcal{Z}_t, \mathcal{X}_t]$. Then $\Delta_{1,t}$ is the unique three-dimensional subspace of D_1 such that*

$$[\Delta_{1,t}, \Delta_{1,t}] = \Delta_{2,t}.$$

Hence, the decomposition

$$D_{1,t} = Z \oplus \Delta_{1,t}$$

remains unchanged under isomorphisms.

THEOREM 30. *The isomorphism group of $N_0, N_6, N_{6\rho}, N_{6\rho^2}$ is equivalent to a subgroup of* $\mathrm{GL}(3)$ *generated by matrices A_1, A_2, A_3, A_4 and diagonal matrices.*

THEOREM 31 ([12]). *Assume that $t^3 \neq 0, 216, -27$. Suppose that $A : Z \to \Delta_{1,t}$ and $B : [D_{1,t}, D_{1,t}] \to [D_{1,t}, D_{1,t}]$ are linear maps satisfying*

$$B[u, v] = [Au, v] \text{ for } u \in Z, v \in \Delta_{1,t} \text{ and } B(\Delta_{2,t}) = 0.$$

Then A is of the form

$$\begin{cases} A(\mathcal{H}_x) = \lambda \mathcal{X}_t, \\ A(\mathcal{H}_y) = \lambda \mathcal{Y}_t, \\ A(\mathcal{H}_z) = \lambda \mathcal{Z}_t, \end{cases}$$

for some constant λ.

We have shown that the matrix representations of $\phi|_Z$ and $\phi|_{\Delta_{1,t}}$ are proportional for arbitrary isomorphism $\phi \in \mathrm{Iso}(N_t, N_s)$.

LEMMA 32 ([12]). *Suppose that ϕ is a degree-preserving isomorphism form N_t to N_s whose restriction to Z is given by*

$$\begin{cases} \phi(\mathcal{H}_x) = a_1 \mathcal{H}_x + b_1 \mathcal{H}_y + c_1 \mathcal{H}_z, \\ \phi(\mathcal{H}_y) = a_2 \mathcal{H}_x + b_2 \mathcal{H}_y + c_2 \mathcal{H}_z, \\ \phi(\mathcal{H}_z) = a_3 \mathcal{H}_x + b_3 \mathcal{H}_y + c_3 \mathcal{H}_z. \end{cases}$$

Then the restriction of ϕ to $\Delta_{1,t}$ can be expressed as

$$\begin{cases} \phi(\mathcal{X}_t) = \theta a_1 \mathcal{X}_s + \theta b_1 \mathcal{Y}_s + \theta c_1 \mathcal{Z}_s, \\ \phi(\mathcal{Y}_t) = \theta a_2 \mathcal{X}_s + \theta b_2 \mathcal{Y}_s + \theta c_2 \mathcal{Z}_s, \\ \phi(\mathcal{Z}_t) = \theta a_3 \mathcal{X}_s + \theta b_3 \mathcal{Y}_s + \theta c_3 \mathcal{Z}_s, \end{cases}$$

for some constant θ.

8. Isomorphisms of 1-st moduli algebras

In this section, we determine the set of linear isomorphisms from $\mathcal{A}^1(V_t)$ to $\mathcal{A}^1(V_s)$. Clearly, any such isomorphism induces an isomorphism belonging to $\mathrm{GL}(\bar{L}_t^1, \bar{L}_s^1)$. This fact plays an important role in the proof of the following theorem which has been mentioned in Section 3.

Theorem 33. *For $t^3, s^3 \neq 0, 216, -27$, the following holds:*

$$\mathrm{GL}(\mathcal{A}^1(V_t), \mathcal{A}^1(V_s)) \cong \mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s)).$$

With the help of Theorem 33, one can easily obtain the following corollary without knowledge of Theorem 7.

Corollary 34 ([**12**]). *If $t^3, s^3 \neq 0, 216, -27$, then we find*

$$\mathrm{GL}(\mathcal{R}(V_t), \mathcal{R}(V_s)) \cong \mathrm{GL}(\mathcal{A}^0(V_t), \mathcal{A}^0(V_s)) \cong \mathrm{GL}(\mathcal{A}^1(V_t), \mathcal{A}^1(V_s)).$$

Lemma 35. *Assume that $t^3, s^3 \neq 0, 216, -27$. Let ϕ be a linear isomorphism from $\mathcal{A}^1(V_t)$ to $\mathcal{A}^1(V_s)$ represented by a square matrix. Let a_1, a_2, a_3 be the entries in the first row of ϕ. There are only two cases:*

(1) *If all elements a_1, a_2, a_3 are nonzero, then $a_1^3 = a_2^3 = a_3^3$;*
(2) *At least two elements of $\{a_1, a_2, a_3\}$ equal zero.*

Finally we give a complete description for $\mathrm{GL}(\mathcal{A}^1(V_t), \mathcal{A}^1(V_s))$.

Theorem D ([**12**]). *If $t^3, s^3 \neq 0, 216, -27$, then the set of linear isomorphisms from $\mathcal{A}^1(V_t)$ to $\mathcal{A}^1(V_s)$ forms a group of 216 matrices generated by A_1, A_2, A_3 and A_4 (defined in Theorem 7) up to a scale multiplication.*

Remark 36. *As a consequence of Corollary 34 and Theorem D, we obtain a new proof of Theorem 7.*

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