Statistics and Its Interface Volume 4 (2011) 129–130

Discussion of 'Threshold models in time series analysis — 30 years on'

Peter Brockwell*

The paper, *Threshold models in time series analysis — 30 years on*, of Howell Tong is discussed and some remarks on the impact and possible future development of threshold models and their continuous-time analogues are made.

AMS 2000 subject classifications: Primary 37M10, 62M10; secondary 91B84.
Keywords and phrases: Threshold model, Non-linear time series, Continuous time, Ergodicity.

Professor Tong is to be congratulated for his masterly overview of the history, key ideas, successes and current developments in the field of threshold models in time series analysis. This area has proved to be a particularly fertile area for both theoretical and applied studies in the thirty years since the appearance of the seminal paper by Tong and Lim (1980).

Of course there is no end to the classes of nonlinear models which could conceivably be used to represent observed time series data. The problem is to formulate nonlinear models which are susceptible to mathematical and statistical analysis and which can reproduce the most striking and commonly occurring features of observed time series which are impossible to reproduce using linear models. These include phenomena such as persistence of volatility, jump resonance, amplitude-frequency dependency and limit cycles. In the late nineteen seventies a variety of useful and relatively tractable families began to appear, all motivated by essentially the same goals. Besides threshold models, other examples were the bilinear model of Granger and Anderson (1978) (see also Subba-Rao and Gabr (1984)), the exponential autoregressive model of Ozaki and Oda (1978), the state-dependent model of Priestley (1980), the random-coefficient autoregression model of Nicholls and Quinn (1982), and the ARCH and GARCH models of Engle (1982) and Bollerslev (1986) respectively. An excellent discussion of these models can be found in Chapter 3 of Tong (1990). Of these models, the ARCH and GARCH models, specifically designed to capture the volatility clustering observed widely in financial data, have had the greatest impact in econometrics, but threshold models have been highly influential in a broader range of applications and have also found financial applications (see for example Tsay, 2005).

An attractive unifying notion which is emphasized in Section 3 of the paper under discussion is the *Threshold Principle*, i.e. the notion of piecewise linearization with the particular regime determined by the value of an indicator time series whose properties may be specified in various ways and which may be observed, unobserved or partially observed. This allows the incorporation of a very wide class of processes into the family of threshold models, special cases of which are given in Section 3.

In the thirty years since the publication of Tong and Lim (1980), interest in threshold (and other nonlinear) models has flourished, with the development of a substantial body of research into questions of ergodicity and the existence of stationary distributions as discussed in Section 4 of the paper. For the self-exciting threshold AR(1) process with delay 1, necessary and sufficient conditions for the existence of a stationary version and properties of the least squares estimators of the coefficients when the thresholds are assumed known were established by Chan et al. (1985). For the same process with a single threshold and arbitrary delay, necessary and sufficient conditions for ergodicity were obtained by Chen and Tsay (1991). In general however such questions are far from fully resolved and remain active areas of research. The selection and estimation of threshold models for given data sets also remains a challenging problem. Systematic approaches are proposed in the papers of Tsay (1989, 1998) and an overview of more recent developments is contained in the book of Fan and Yao (2003).

The continuous-time threshold models touched upon at the end of Section 3 have some appealing properties arising from their specification via stochastic differential rather than stochastic difference equations. Existence and stability of continuous-time SETARMA models have been studied by Stramer et al. (1996) and applied to the analysis of financial data in Brockwell and Williams (1997). For Gaussian SETAR processes observed continuously in time, exact maximum likelihood estimation can be carried out as in Brockwell et al. (2007) and the results applied to estimation for such models on the basis of observed high-frequency data.

Threshold models have played an important role in drawing the attention of probabilists and statisticians to the need for and the potential benefits to be derived from nonlinear models. It also led to the two books Tong (1983, 1990), the

*The author gratefully acknowledges the support of the National Science Foundation Grant DMS 1107031.

first of which contains computer programs for implementing model-fitting and the second of which provides an overview of nonlinear time series modelling, highlighting the use of the dynamical systems approach to the analysis of these models and the use of Markovian methods and Lyapunov functions for their analysis. The dynamical system framework has now become fundamental in tackling many of the problems associated with non-Gaussian and non-linear models in time series analysis. In conjunction with Markov chain Monte Carlo and particle filtering methods, it has permitted in principle the analysis of extremely complex non-linear problems. The development of computationally efficient and rapidly convergent algorithms however is still a challenging area. The theoretical analysis of ergodicity for many important classes of widely used non-linear models is also an area in which much remains to be done.

Received 7 May 2011

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Peter Brockwell
Colorado State University
Fort Collins, Colorado
USA

Columbia University
New York, New York
USA

E-mail address: pb2141@columbia.edu