Ramanujan Rediscovered

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Ramanujan Mathematical Society

Lecture Notes Series

Volume 14

Ramanujan Rediscovered

Proceedings of a Conference on Elliptic
Functions, Partitions, and q-Series in
Memory of K. Venkatachaliengar, Bangalore,
June 2009

Volume editors

Bruce C. Berndt (University of Illinois at Urbana)
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Preface

On 1–5 June, 2009, a conference in number theory was held on the beautiful campus of Infosys in Bangalore. The impetus for organizing this meeting was to recognize and commemorate K. Venkatachaliengar, an outstanding, well-known mathematician, who taught primarily at universities in Bangalore and Mysore for most of his career. He was born on 8 December, 1908, and so the meeting marked the centenary of Venkatachaliengar's birth. In the last several decades of his long life of 95 years, KV, as he was affectionately known to most of his friends, had become keenly interested in the life and work of India's greatest mathematician, Srinivasa Ramanujan, and so it was natural for Ramanujan's first loves of theta functions, partitions, and q-series to be the focus of the conference. Accordingly, over 50 mathematicians gathered for the presentation of 32 lectures in memory of both Ramanujan and KV. This volume comprises 13 papers by mathematicians who lectured at the meeting. In addition, three papers on the life and work of KV, along with a complete list of his publications, are offered.

All the participants, especially the organizers, are extremely grateful to Infosys Corporation for making available its exceptional facilities, staff, and financial support for the conference. The International Institute of Information Technology, Bangalore, and especially its Director, Professor Sadagopan, deserve our warm thanks for their generous support. Lastly, it is a pleasure to acknowledge the munificent financial support of the Indian Mathematical Society.

Foreword

To commemorate the birth centenary of Prof. K. Venkatachaliengar, an International Conference "Ramanujan Rediscovered" was held at Infosys, Bangalore on June 1–5, 2009. It was jointly organized by the International Institute of Information Technology, Bangalore and the Indian Mathematical Society. The Conference was inaugurated with the lighting of the traditional lamp by Professors S. Sadagopan (Director, IIIT, Bangalore), B. C. Berndt (co-chair, University of Illinois at Urbana-Champaign, USA), A. K. Agarwal (co-chair, Panjab University, Chandigarh), Ravichandran (Indian Institute of Management, Indore), Mr. Dinesh (Infosys) and Mr. Srikantan Moorthy (VP, Infosys). In their inaugural speeches, the speakers highlighted the mathematical contributions of S. Ramanujan and K. Venkatachaliengar. The inauguration function was followed by three keynote addresses delivered by Professors B. C. Berndt, A. K. Agarwal and Ravichandran. After the keynote addresses Professor A. K. Agarwal gave a talk on the history, objectives and activities of the Indian Mathematical Society.

In all there were about 250 participants. There was a large number of overseas participants. Prof. G. N. Srinivasa Prasanna, IIIT, B was the convener of the Conference. There were two parallel sessions-one for mathematics and the other for Information Technology (IT). In mathematics, the main topics discussed were: elliptic functions, q-series, partitions and related number theory and in IT session the main topics covered were: discrete mathematics, game theory, bioinformatives, optimization techniques and formal methods in software engineering. There were 18 invited speakers in mathematics session and 7 in IT session. There were about 10 contributed talks and 4 poster presentations in mathematics session.

On the first day of the Conference a film "God, Zero and Infinity" on Ramanujan's story was screened in the evening. The film directed by Santosh Dhavala and narrated by Tom Alter was liked by all participants. The other highlights of the Conference were: (1) a skype session on June 3 in which Prof. Zhi-Guo Liu presented his paper as he could not attend the Conference physically, (2) a cultural program and a banquet at Manipal County in the evening of June 3, and (3) on June 4, Prof. S. Sadagopan met all the participants over a cup of tea and presented them momentos. The Conference was concluded on June 5 with a popular talk on Ramanujan by Prof. B. C. Berndt.

In my opinion the Conference was very stimulating and indeed achieved its goals. The proceedings of the conference will be published by the Ramanujan Mathematical Society in its Lecture Notes Series.

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K. Venkatachaliengar (KVI) was born on 8 December 1908 in a small village called Kadaba about sixty kilometers northwest of Bangalore. He was the last of four brothers and two sisters all born to K. Venkatarama Iyengar (father) and Venkatalaxamma (mother). He lost his mother when he was still a toddler and thus missed motherly love. His father was a farmer by profession and was well read.

From early childhood, KVI showed great aptitude at studies, and his only limitation was lack of books and learned teachers in the small village. When he was around ten years his father also passed away and the burden of raising the family was thrust on the young shoulders of his elder brother, K. Anantha Char. After completing primary schooling in Kadaba, he continued his higher schooling in Tumkur and Bangalore. He was always at the top of his class and, apart from brilliance at mathematics, had great liking for the classics and Kannada language.

After the school final, he entered the Central College, and all his student colleagues and teachers alike immediately took note of his extraordinary talents. At this time, it is pertinent to note that generally the standards of mathematics among the Indian universities were very low, and teaching was like school teaching and research unknown.

Luckily for KVI, he had as teachers two very bright mathematicians, Prof. K. S. K. Iyengar and Prof. B. S. Madhava Rao, who were his seniors by a few years. Prof. KSK probably had studied in Cambridge and was very proficient in pure mathematics. Prof. B. S. Madhava Rao had completed his Masters from the Calcutta University and was more inclined towards applied mathematics and theoretical physics. Along with these teachers, he also had a senior fellow student, also very talented in mathematics, V. R. Tiruvenkatachar. It was very lucky for university mathematics in India that a very talented quartet was in the Central College as contemporaries. Abundant talents they had in mathematics but soon they learnt that the books in modern pure mathematics were not in English but mostly in French and German. India, being a British colony, usually had books only from England and in English. This quartet immediately went about repairing this sad state of affairs and started getting books to learn French and German from the continent. They self-acquired reading ability in these languages and subsequently started getting from the continent works of the great mathematicians of the French and German schools.

Combined with their innate brilliance and studying the great French and German works, they entered eagerly to pursue research and teaching of mathematics. They were even conducting courses in French and German for mathematics students!

After completing his B.Sc. (Hons) from the Central college, KVI went to Calcutta University for his Masters. Here his mathematical brilliance was immediately noticed by his teachers. During his time in Calcutta, an incident is worth recording. A British gentleman, who was good at mathematics, proposed a hard problem regarding some property of a triangle (the famous Morley Trisector Theorem) and remarked that it was beyond the ability of Indian students to solve. When KVI heard of this in the evening, he solved the same before going to bed and gave two different solutions the next morning!

KVI completed his Masters and as usual stood first. Some years later, he also obtained his D.Sc. from the Calcutta University. The thesis examiners for his D.Sc. thesis were Hermann Weyl, Garrett Birkhoff and F. W. Levi. Weyl went out of his way to praise the thesis and predicted a bright future for the candidate.

When he returned to Bangalore after his Masters, the immediate problem was getting a job. Since in those days the number of positions was rigidly fixed, one had to wait for the retirement of a senior member before one could get a job! For some time, KVI worked as a lecturer in a private college in Belgaum.

KVI had known Sir C. V. Raman probably even from Calcutta days and when Sir C. V. Raman was the director of the Indian Institute of Science, he finally offered some position to KVI at the Indian Institute of Science, with a salary around Rs. 80/- per month, a princely sum in those days. The first task Sir C. V. Raman assigned KVI was to conduct a course on integral equations at the Institute. During this time, KVI also worked out the mathematical theory of some problem connected with a vibrating string which had attracted the interest of Sir C. V. Raman, and a detailed paper resulted from this. Most of this was during World War II, during which time Max Born was also at the Indian Institute of Science.

KVI finally got a proper teaching job at the Central College after a vacancy arose. Here he had the excellent company of the other three mathematicians (KSK, BSMR, and VRT), and they started courses of high standard in modern mathematics and built up an excellent library. This was also the golden age of their research work. Since, fortunately, four of them were there, they collaborated on many problems and a number of papers of the highest standard followed. KSK passed away early around the 1940's, but the other three carried on their work vigorously and went on publishing results. One such was on particle spins and was published in the Proceedings of the Royal Society in 1946 [1].

Unfortunately, due to a variety of reasons, this collaborative work was put to an end, due to KVI being transferred out of Bangalore to a nearby small town called Tumkur, where only junior classes were held. In spite of lacking good students of senior classes and a library, KVI went about his research work ceaselessly. In the Tumkur College where he had been posted, he was actively looking for some talented students to teach some advanced mathematics. Luckily for him he found a fellow physics lecturer, K. N. Srinivasa Rao, who showed some mathematical ability. KVI immediately caught on to him and started a course for his colleague. Finding that he had not learnt group theory, he first taught him group theory and then complex variables, etc. After finding that Srinivasa Rao was very sincere and learnt all his teaching well, around 1948–50, he managed to get him a fellowship from the Council of Scientific and Industrial Research. This allowed KNS to devote all his time for mathematical studies under KVI without the burden of teaching students elementary physics at the college. This happened when both were posted to a college in Mysore.

In 1953, KVI was transferred back to Bangalore and this time to the Engineering college. Since there were no postgraduate courses here, to satisfy his intellectual prowess, he introduced advanced mathematics for those students who showed an above average interest in mathematics. This was besides the normal curriculum.

Around 1962, the University of Mysore created a postgraduate centre in Mysore called Manasa Gangothri, and KVI moved there as the head of the department. Here again he started building a proper library and introduced many advanced subjects such as abstract algebra, etc. After reaching the age of 58, he retired in 1966 from the University of Mysore and settled down in Bangalore. Here he had the company of his earlier collaborators Prof. B. S. Madhava Rao and Prof. V. R. Tiruvenkatachar. For two years, he served as an emeritus professor at Madurai University, teaching advanced courses to the faculty there. The following is worth mentioning here. One of the faculty members told him that he was stuck on some problem connected with Krull algebra. KVI on the spot proved the necessary result, and this was submitted as a paper in a special volume of a journal in honour of Prof. Krull. Prof. Krull personally wrote a letter of appreciation.

During this time, Prof. KVI was also associated with the National Council for Educational Research and Training, and heading the geometry study group. He made geometry exciting for youngsters, by having plastic stick models of polyhedra made. These, and the wooden models of complex topological surfaces, made during his Central College days (1940–50's) by a master carpenter Giriappa, can be used for teaching at all levels. For example, counting can be taught to a child, trigonometry to middle and high school students, and graph colouring and group theory to Ph.D. level candidates. The topological surfaces reflect some of the deepest aspects of topology, including non-Euclidean relativistic geometries. This material has been donated to the International Institute of Information Technology, Bangalore, and is on display there.

When he was past 50, he noticed that mathematical books could be procured from the USSR through some specialized book stores specializing in USSR books. Since these could be obtained at rock bottom prices which he could afford, he started teaching himself enough Russian to understand mathematics in Russian.

At this time Prof. B. S. Madhava Rao was in his house in South Bangalore, and Prof. V. R. Tiruvenkatachar lived near KVI's house. KVI and Prof. V. R. Tiruvenkatachar used to go for walks in the mornings and evenings discussing mathematics as usual. They used to visit Prof. B. S. Madhava Rao around once every few months.

After his retirement, KVI decided to devote most of his energies on Ramanujan's life and works. This culminated in his beautiful book, *Development of Elliptic Functions According to Ramanujan*, which has been described in detail elsewhere in this volume.

In 1977, Bruce Berndt began to devote all of his research efforts toward proving the claims left behind by Ramanujan in notebooks that he compiled before heading to Cambridge. After the publication of Berndt's first volumes on Ramanujan, KVI was very happy, and told Berndt that this appeared to be the only example where all the work, including unpublished work, of a great mathematician is being critically examined.

Being a born-mathematician, KVI always breathed and lived mathematics. Whenever he sat down to dinner, he never failed to scribble mathematics on his dinner plate. His wife had to disturb him from this so that he could be served. This trait was with him throughout his life. More often than not, the scribbling included a triangle.

Around the 1960's, he was requested by the then pontiff of the Parkala Mutt (Hindu religious body) at Mysore, who was a very great scholar, to become the president of the Mutt for some administrative duties. Since the request came from a great seer, KVI accepted it, even though he had until then never taken part in any such organized cultural activity. Once he accepted this task, he devoted his energies to it and started learning certain portions of philosophy required by his duties. Due to his intellectual acumen, he succeeded eminently and served the Swamiji (seer) and the Mutt with utmost devotion.

The triangle remained his most endearing object spanning his entire life. He had a great liking to imbibe mathematics into any willing child or young person. The elliptic functions and lots of other mathematics were first taught to his grandson – of course it went well above the young child's (between 7–15 years of age) head. He also taught his grandson the "Morley-Trisector Theorem." His grandson was inspired enough and learnt enough about the construction to do it during one evening when he had gone for a walk, and showed it to him when he returned with great pride. KVI taught all this, plus trigonometry, Gauss polynomials, complex variables, especially using Hurwitz's double series approach, etc. Some of this material is with his grandson in India and the USA. He was especially happy when his great-granddaughter Anagha gave him a lecture on polyhedra!

Even though he walked the high roads of mathematics, he never in his whole life felt any 'pride'. He was always eager to teach any willing soul less endowed than himself in mathematics.

In closing, we include below views of Prof. KVI from some of his former students: Prof. Mahabala, a prominent Indian Computer Scientist; Prof. Prabhu, a control theorist and former Head of Electrical Engineering at IIT-Kanpur; and Dr. Ramamani, his Ph.D. student.

KV Fondly Remembered by Prof. H. N. Mahabala, IIITb

Prof. K. Venkatachala Iyengar was a young researcher's delight. He would often recall results of great mathematicians of the past. He did not like teaching routine math courses to undergraduate students, but was transferred to an engineering college due to departmental politics. His talent was wasted on disinterested engineering students. At any opportunity, he would talk about great mathematicians and their work. Students were perplexed. He would walk across to talk to graduate students where I was a student studying for B.Sc. (Hons.) in mathematics. We were fascinated by his obsession with mathematics. He would advise us to learn Russian, because Russian books on advanced mathematics were sold very cheap! We heard from him about Bourbaki. We were attracted to read Hardy's book on Ramanujan.

At that time (1950's) a scientific exposition was held, and who should organize the maths exhibits, but KV. There were wooden models of interesting solids, made by a carpenter under his close direction, which we volunteers had to explain. We found out to our great surprise that the circle was not the only curve with constant width. He got constructed a model where one could roll the curve to check that it had constant width. We had wire frames to demonstrate minimal surfaces using soap films. We had to prepare a soap solution using oxalic acid and some other ingredients. We enjoyed confounding the visitors with minimal surfaces, in particular using the cube frame. We found out how interesting mathematics can be. We wrote in big letters the digits of pi accurate to some 600 digits and fixed it along the wall for visitors to check that pi was a non-repeating irrational number. The lattice components were developed later. He would train us on how to explain mathematics and would frequently call us "Hopeless Fellows," all with a smirk on his face. He breathed mathematics and explained in an excited way about great results in mathematics. We set up a 5-inch telescope and showed the moon and Saturn to long queues of people. It is surprising that he was the only faculty member who helped us put up a good show. His complete disregard for every day etiquette and lack of interest in things other than mathematics made him a poor conversationalist though.

I am proud that IIITb celebrated KV's centenary, with many Ramanujan enthusiasts around the world participating.

Professor K. Venkatachaliengar by Prof. Prabhu

It was the academic year 1958–1959. I was a third year Electrical (Power) Engineering student in the Bachelor of Engineering course in the Government College of Engineering, Bangalore. It was a traditional course. It dealt mainly with engineering practice of those days. Much of it was descriptive. Engineering education was poised to undergo radical changes. It was already happening in the advanced countries. Our batch of students were lucky. The university had started taking a few tentative steps to usher in refreshing changes. We were offered a new curriculum which had incorporated some small, but significant, changes to the old curriculum. One aspect of it was to offer us two elective courses, one in electronics and radio engineering, and the other in mathematics. We were extraordinarily lucky in getting Professor Venkatachaliengar as our mathematics teacher. He taught us the theory of complex variables.

Mathematics was not given much importance earlier in engineering curricula. Now, of course, everybody knows the importance of a good mathematical foundation to build the superstructure of engineering knowledge and ability.

For Professor Venkatachaliengar, who was an outstanding mathematician, spending time teaching undergraduate engineering students, rather than guiding research work of gifted mathematics students, must have been a difficult experience. He was a mathematics teacher in a government run system, and had been posted to an engineering college. It must have been highly demoralizing and depressing for him. But his behavior did not indicate anything of this. He was one of the best teachers I have ever had. Always full of energy, he had a child-like ability to get thrilled with ideas and insights, and was ever ready to share his knowledge. It was just a blackboard and a piece of chalk; but what an experience they wrought through the hands of Professor Venkatachaliengar!

It was only later, much later, that I realized the value of what we had gained as students of Professor Venkatachaliengar. He made a big difference to us. He taught us deep things. He taught us that no matter what the circumstances in your life may be, as a teacher you should always be ever enthusiastic and dedicated towards your students.

A Brief Reminiscence by Dr. V. Ramamani

Scientist, Defence Research and Development Organization (Retd.)

My association with Prof. KV was a long one – for over 40 years. I was a postgraduate student of Prof. KV during 1961–63. Thereafter, I continued as a research scholar and did a Ph.D. under his able guidance. However, I continued to meet him. Whatever knowledge I gained in his association is a valuable one. Prof. KV was a versatile scholar both in pure and applied mathematics. But his love for S. Ramanujan was unique. He was devoting a lot his time in solving problems found in the note books of S. Ramanujan. His other favorite subjects were ancient Indian mathematics and astronomy. Prof. KV was a simple and a straightforward person and never hesitated to express his opinion. He expected hard work from his students and encouraged them whenever they did good work.

Prof. KV was a voracious reader and knowledgeable in varied fields. I had the opportunity and privilege to meet him on numerous occasions and during every visit I learned something from him. It was a treat to listen to him.

References

[1] B. S. Madhava Rao, V. R. Thiruvenkatachar, and K. Venkatachaliengar, Algebra related to elementary particles of spin $\frac{3}{2}$, *Proc. Roy. Soc. London, Ser. A.*, **187** (1946) 385–397.

K. Venkatachaliengar

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During the period from 1985 to 1994, the author received at least nine lengthy letters from Venkatachaliengar. The first, written on 15 July 1985, came in response to receiving a copy of the author's first book on Ramanujan's notebooks [4], which the author had earlier sent to Venkatachaliengar. In order to set the context of a long passage that we are going to quote from this letter, we need to recall a brief history of Ramanujan's *Quarterly Reports*.

On 26 February 1913, Sir Gilbert Walker, Head of the Meteorological Observatory in Madras, sent a letter to Francis Dewsbury, Registrar at the University of Madras, exhorting the University to provide Ramanujan with a scholarship. On 1 May 1913, Ramanujan was granted such a scholarship with the stipulation that he write *Quarterly Reports* on his research activity. Before departing for England in March of 1914, Ramanujan wrote three reports, dated 5 August 1913, 7 November 1913, and 9 March 1914. An account of these reports may be found in the author's book [4] or his papers [2,3]. In his first letter to the author, Venkatachaliengar provided an account of the deliberations that took place before the scholarship was approved. In the quote below, S. R. is an abbreviation for Srinivasa Ramanujan.

> The English professors in Madras headed by Littlehailes stoutly opposed the grant of a research scholarship to S. R. although the Board of Studies in Mathematics in Madras University consisting of a majority of Indian professors had recommend[ed] the award at the suggestion of the competent distinguished mathematician G. Walker. The Indian professors were apprehensive; the vice-chancellor was the chief justice of the Madras High Court who was appointed directly by the Secretary of State of India (India office, London). One of the members of the Syndicate of the Madras Univ. was Justice Sundaram Iyer of the Madras High Court who had been briefed by the Indian professors; at the meeting of the Syndicate all the English Profs. opposed the proposal with all their vehement speeches. After all of them had talked out their breath, Justice Sundaram Iyer stood up and read out the preamble to Madras University act passed by the British Parliament in London: The Madras University is founded to *promote learning* and *encourage research* .... The English Chief Justice was told that nothing had been

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> done by the Univ. till that time to encourage research and he immediately stifled the opposition and made the recommendation to the Governor of Madras Lord Pentland, who was also a direct appointee of the British Govt. Even with this there was a hitch; and Sir Francis Spring wrote a letter to the private secretary of the Governor strongly recommending the case. After S. R. became famous, Prof. Littlehailes used to take S. R. in the sidecar of his motorbike. His Indian colleagues were having a hearty laugh. This was narrated to me by late Prof. S. R. Ranganathan, who had written a book on S. R.

Descriptions of events leading to Ramanujan's scholarship vary somewhat; see for example, [17] and [4, p. 295]. Venkatachaliengar's account highlights the objections of English mathematicians, in particular Littlehailes. In one of the two Presidential addresses [20] that Venkatachliengar gave before the Indian Mathematical Society, he pointed out that this was the first research scholarship awarded by the University some 50 years after the British Parliament had passed the Act. Sir Francis Spring was the Chairman of the Madras Port Trust Office where Ramanujan served as a clerk, and was one of Ramanujan's earliest and most devoted supporters. Ranganathan was Librarian at the University of Madras and wrote the first thorough biography of Ramanujan [17]. He is also famous in the field of library science and is considered to be one of the founders of modern library science. Littlehailes was Professor of Mathematics at Presidency College in Madras and later became Vice-Chancellor of the University of Madras; more information about him can be found in [6].

In his visits to India, the author can recall three long conversations with Venkatachaliengar. Although our conversations were chiefly about mathematics, in particular, about modular equations, he would often relate interesting facts about Indian mathematics and mathematicians, including Ramanujan, over a broad expanse of the twentieth century. It is unfortunate that portions of these conversations were not recorded for posterity.

An examination of the bibliography of Venkatachaliengar shows a broad stretch of interests. For roughly the first half of his career, Venkatachaliengar's interests were algebraic, covering a wide variety of topics. In the second half of his career, his interests took on a more analytic bent, and he became fascinated with the work of Ramanujan. Although he published few papers on Ramanujan's work, his passion for Ramanujan was evident in his two Presidential addresses to the Indian Mathematical Society [20], his highly original monograph on Ramanujan's discoveries in elliptic functions [21], and his unpublished, handwritten monograph on selected results from Ramanujan's work that he coauthored with his close friend V. R. Thiruvenkatachar [19]. We offer a few remarks about some of Venkatachaliengar's publications.

Venkatachaliengar's monograph [21] is thoroughly reviewed by S. Cooper elsewhere in this volume. A corrected, edited, and more widely distributed edition of this work is greatly desired.

A typed version of [19] has been made by M. D. Hirschhorn. This will be the starting point of an edited version that is being undertaken by the present author and several of his graduate students, and which will be published by the Ramanujan Mathematical Society.

Readers might be interested in the origin of the author's paper [7] with Venkatachaliengar. In early 1999, the author visited the University of Mysore, and one of his hosts, Professor S. Bhargava, accompanied him to the home of retired University of Mysore Professor T. S. Nanjundiah, who presented the author with a handwritten partial manuscript of a paper that he had once started to write with Venkatachaliengar. Although the Dedekind eta-function $\eta(\tau)$ was not mentioned in this partial manuscript, the intended authors had very cleverly derived the transformation formula for $\eta(\tau)$. Nanjundiah demurred about his own contributions to the paper, and so eventually with the gracious accedence of both Nanjundiah and Venkatachaliengar, the present author added a few minor contributions of his own and coauthored [7] with the latter. Readers might turn to the paper by the author, C. Gugg, S. Kongsiriwong, and J. Thiel [5] in this volume, where they will find Venkatachaliengar's ingenious idea generalized.

Venkatachaliengar supervised the doctoral dissertation of only one student, V. Ramamani [12]. In her beautiful thesis, she extended some of the ideas from Ramanujan's epic paper [14]. Two of the author's doctoral students, Heekyoung Hahn and Tim Huber, in turn used her work in their own research, [8,9] and [11], respectively.

Ramamani and Venkatachaliengar coauthored [13], which, for the present author, has been the latter's most influential paper, for he has lectured on its content in his classes and seminars. Recall first a famous theorem of Euler on partitions: The number of partitions of the positive integer n into odd parts is equal to the number of partitions of n into distinct parts. J. J. Sylvester [18] gave a beautiful extension of Euler's theorem in 1884, and Ramamani and Venkatachaliengar provided an elegant proof in [13], which one can find in the text by G. E. Andrews [1, pp. 24–25] and which motivated a further proof by M .D. Hirschhorn [10]. We state Sylvester's theorem.

Theorem 1. *Let $A_k(n)$ denote the number of partitions of the positive integer n into odd parts such that exactly k different parts occur. Let $B_k(n)$ denote the number of partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, where $\lambda_i \geq \lambda_{i+1}$ and $1 \leq i \leq r-1$, such that the sequence $(\lambda_1, \dots, \lambda_r)$ is composed of exactly k noncontiguous sequences of one or more consecutive integers. Then $A_k(n) = B_k(n)$, for all k and n.*

As intimated above, Venkatachaliengar served as President of the Indian Mathematical Society for two years. In his two addresses to the Society [20], he urged Indian mathematicians to study Ramanujan's (earlier) notebooks [15] and lost notebook [16]. In particular, he focused on particular examples from the lost notebook and his proofs of them. He also discussed the manuscripts of Ramanujan that were found in the library at Oxford University and published with Ramanujan's lost notebook [16].

References

[1] G. E. Andrews, The Theory of Partitions, Addison–Wesley, Reading, MA, 1976; reissued: Cambridge University Press, Cambridge (1998).

[2] B. C. Berndt, The quarterly reports of S. Ramanujan, *Amer. Math. Monthly*, **90** (1983) 505–516.

[3] B. C. Berndt, Ramanujan's quarterly reports, *Bull. London Math. Soc.*, **16** (1984) 449–489.

[4] B. C. Berndt, Ramanujan's Notebooks, Part I, Springer-Verlag, New York (1985).

[5] B. C. Berndt C. Gugg, S. Kongsiriwong and J. Thiel, A proof of the general theta transformation formula, this volume, pp. 53–62.

[6] B. C. Berndt and R. A. Rankin, Ramanujan: Letters and Commentary, American Mathematical Society, Providence, RI, 1995; London Mathematical Society, London (1995).

[7] B. C. Berndt and K. Venkatachaliengar, On the transformation formula for the Dedekind eta-function, in Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, F. G. Garvan and M. E. H. Ismail, eds., Kluwer, Dordrecht (2001) pp. 73–77.

[8] H. Hahn, Convolution sums of some functions on divisors, *Rocky Mt. J. Math.*, **37** (2007) 315–344.

[9] H. Hahn, Eisenstein series associated with $\Gamma_0(2)$, *Ramanujan J.*, **15** (2008) 235–257.

[10] M. D. Hirschhorn, Sylvester's partition theorem and a related result, *Michigan Math. J.*, **21** (1975) 133–136.

[11] T. Huber, Zeros of Generalized Rogers–Ramanujan Series and Topics from Ramanujan's Theory of Elliptic Functions, Ph.D. thesis, University of Illinois at Urbana–Champaign, Urbana (2007).

[12] V. Ramamani, Some Identities Conjectured by Srinivasa Ramanujan Found in His Lithographed Notes Connected with Partition Theory and Elliptic Modular Functions – Their Proofs – Interconnection with Various Other Topics in the Theory of Numbers and Some Generalisations Thereon, Ph.D. thesis, University of Mysore, Mysore (1970).

[13] V. Ramamani and K. Venkatachaliengar, On a partition theorem of Sylvester, *Michigan Math. J.*, **19** (1972) 137–140.

[14] S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.*, **22** (1916) 159–184.

[15] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay (1957).

[16] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi (1988).

[17] S. R. Ranganathan, Ramanujan: The Man and the Mathematician, Asia Publishing House, Bombay (1967).

[18] J. J. Sylvester, A constructive theory of partitions arranged in three acts, an interact, and an exodion, *Amer. J. Math.*, **5** (1884) 251–330; **6** (1886) 334–336.

[19] V. R. Thiruvenkatachar and K. Venkatachaliengar, Ramanujan at Elementary Levels: Glimpses, Unpublished handwritten notes.

[20] K. Venkatachaliengar, Ramanujan Manuscripts, I, II, Presidential Addresses given in December 1979, 1980 to the Indian Mathematical Society, *Math. Student*, **52** (1984) 1–4; 215–233.

[21] K. Venkatachaliengar, Development of Elliptic Functions According to Ramanujan, Tech. Rep. 2, Madurai Kamaraj University, Madurai (1988).

A Review of Venkatachaliengar's Book on Elliptic Functions

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1. Introduction

One of the things Professor K. Venkatachaliengar is best remembered for is his book: "Development of Elliptic Functions according to Ramanujan", published by Madurai Kamaraj University in 1988[1]. In this review we shall survey several of the topics covered in the book and discuss some of the methods and ideas.

We will deliberately be brief and not go too heavily into the details. The exception is in Section 4, where a complete proof of Ramanujan's ${}_1\psi_1$ summation formula, due to Venkatachaliengar, is given.

Throughout this review, τ will always be a complex number with positive imaginary part, and $q = \exp(2\pi i\tau)$, so that $|q| < 1$. Occasionally, reference will be made to the Weierstrassian functions $\wp$, ζ, and σ, and the Weierstrassian invariants g_2 and g_3. Definitions and all of the main properties of the Weierstrassian functions can be found in Chapter 20 of the book by Whittaker and Watson [42]. For this review, it should be sufficient to refer to the equations (13)–(16), (49) and (50), below.

The page numbers that appear in the section headings refer to Venkatachaliengar's book [39].

2. A generalization of Ramanujan's identity [pp. 1–13]

The first part of Venkatachaliengar's book takes much of its inspiration from Ramanujan's paper "On Certain Arithmetical Functions" [32]. Two of the fundamental formulas in Ramanujan's paper are

[1]There is no publication date in Venkatachaliengar's book [39]. Dr. G. N. Srinivasa Prasanna has told me that according to Professor Soundararajan of Madurai University, the book went to press in Dec. 1987 and was released in Feb. 1988. This is consistent with information I have received from Professors R. A. Askey and B. C. Berndt. The article [38], published in vol. 52 of The Mathematics Student, reports that Venkatachaliengar's book was published in 1980. This is most likely an error. For, although this volume of The Mathematics Student has a publication year of 1984, MathSciNet indicates that it was not actually published until 1990. We conclude that Venkatachaliengar's book [39] must have been published in 1988.

$$\left(\frac{1}{4}\cot\frac{\theta}{2}+\sum_{j=1}^{\infty}\frac{q^j}{1-q^j}\sin j\theta\right)^2$$
$$=\left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2+\sum_{j=1}^{\infty}\frac{q^j}{(1-q^j)^2}\cos j\theta+\frac{1}{2}\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}(1-\cos j\theta) \qquad (1)$$

and

$$\left(\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12}+\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}(1-\cos j\theta)\right)^2$$
$$=\left(\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12}\right)^2+\frac{1}{12}\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}(5+\cos j\theta). \qquad (2)$$

These identities are each valid in the horizontal strip $|\text{Im}\,\theta| < 2\pi\ \text{Im}\,\tau$, although all of the functions involved have analytic continuations to functions that are meromorphic on the complex plane.

The identities (1) and (2) can be expressed in terms of theta and elliptic functions, as follows. It is easy to check that the theta function

$$u(\theta,t)=-i\sum_{j=-\infty}^{\infty}q^{(2j+1)^2/8}e^{i(2j+1)\theta/2}, \quad \text{where} \quad q=e^{-t},$$

is a solution of the heat equation

$$2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial\theta^2}.$$

Therefore, $\log u$ satisfies the non-linear partial differential equation

$$\left(\frac{\partial\log u}{\partial\theta}\right)^2=2\frac{\partial\log u}{\partial t}-\frac{\partial^2\log u}{\partial\theta^2}. \qquad (3)$$

By Jacobi's triple product identity [5, p. 10] we have

$$u(\theta,t)=2q^{1/8}\sin\frac{\theta}{2}\prod_{j=1}^{\infty}(1-q^je^{i\theta})(1-q^je^{-i\theta})(1-q^j). \qquad (4)$$

In 1951, B. van der Pol [36] observed that if (4) is substituted into (3), the result is Ramanujan's identity (1). Moreover, if we apply the differential operator $\partial/\partial\theta$ to (3) and put $w=-\partial\log u/\partial\theta$, the result simplifies to Burgers' equation [14]

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial\theta}=\frac{1}{2}\frac{\partial^2 w}{\partial\theta^2}$$

which has applications in fluid mechanics.

Ramanujan's identity (2) may be shown to be equivalent to the identity

$$\wp^2(\theta) = \frac{1}{6}\wp''(\theta) + \frac{g_2}{12}$$

satisfied by the Weierstrass elliptic $\wp$ function, [42, p. 450].

Venkatachaliengar's book begins with a generalization of Ramanujan's identity (1). Let ρ_1 and ρ_2 be the functions defined by

$$\rho_1(x) = \frac{1}{2} + \sum{}' \frac{x^n}{1-q^n} \quad \text{and} \quad \rho_2(x) = -\frac{1}{12} + \sum{}' \frac{q^n x^n}{(1-q^n)^2}, \tag{5}$$

where the primes denote that the summations are over all non-zero integers n. These series converge in the annuli $|q| < |x| < 1$ and $|q| < |x| < |q|^{-1}$, respectively. Venkatachaliengar's identity is:

$$\rho_1(x)\rho_1(y) - \rho_1(xy)\{\rho_1(x) + \rho_1(y)\} = \rho_2(x) + \rho_2(y) + \rho_2(xy). \tag{6}$$

It may be proved by computing the coefficient of $x^m y^n$ on the left hand side and noting that it simplifies to zero unless $n = 0$, $m = 0$ or $m = n$. The non-zero terms give rise to the three functions that occur on the right hand side of the identity.

The analytic continuation for ρ_1 may be obtained as follows. We have

$$\begin{aligned} \rho_1(x) &= \frac{1}{2} + \sum_{j=1}^{\infty} \left(\frac{x^j}{1-q^j} + \frac{x^{-j}}{1-q^{-j}} \right) \\ &= \frac{1}{2} + \frac{x}{1-x} + \sum_{j=1}^{\infty} \left(\frac{q^j x^j}{1-q^j} - \frac{q^j x^{-j}}{1-q^j} \right). \end{aligned} \tag{7}$$

Expanding each term as a geometric series and interchanging the order of summation in the resulting double series gives

$$\begin{aligned} \rho_1(x) &= \frac{1+x}{2(1-x)} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (q^{jk}x^j - q^{jk}x^{-j}) \\ &= \frac{1+x}{2(1-x)} + \sum_{k=1}^{\infty} \left(\frac{q^k x}{1-q^k x} - \frac{q^k x^{-1}}{1-q^k x^{-1}} \right). \end{aligned} \tag{8}$$

Thus $\rho_1(x)$ is analytic for $0 < |x| < \infty$ except for simple poles at $x = q^k$ for every integer k. In a similar way, we find that

$$\rho_2(x) = -\frac{1}{12} + \sum_{k=1}^{\infty} \left(\frac{kq^k x}{1-q^k x} + \frac{kq^k x^{-1}}{1-q^k x^{-1}} \right), \tag{9}$$

and therefore $\rho_2(x)$ is analytic for $0 < |x| < \infty$ except for simple poles at $x = q^k$ for every *non-zero* integer k.

From (8) and (9) we deduce the properties

$$\rho_1(x) = -\rho_1(x^{-1}) \quad \text{and} \quad \rho_2(x) = \rho_2(x^{-1}). \tag{10}$$

Combining (6) and (10) we obtain the symmetric form

$$\rho_1(x)\rho_1(y) + \rho_1(y)\rho_1(z) + \rho_1(z)\rho_1(x) = \rho_2(x) + \rho_2(y) + \rho_2(z), \tag{11}$$

which holds for all complex numbers x, y and z that satisfy $xyz = 1$.

Venkatachaliengar shows how (11) can be used to deduce two forms of the addition formula for the Weierstrass elliptic $\wp$ function [42, pp. 440–441]:

$$\wp(\alpha+\beta) = \frac{1}{4}\left(\frac{\wp'(\alpha) - \wp'(\beta)}{\wp(\alpha) - \wp(\beta)}\right)^2 - \wp(\alpha) - \wp(\beta) \tag{12}$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ \wp(\alpha) & \wp(\beta) & \wp(\gamma) \\ \wp'(\alpha) & \wp'(\beta) & \wp'(\gamma) \end{vmatrix} = 0, \quad \text{where } \alpha + \beta + \gamma = 0,$$

as well as the differential equation [42, p. 437]

$$(\wp'(\theta))^2 = 4\wp^3(\theta) - g_2\,\wp(\theta) - g_3.$$

Moreover, taking the limit as $y \to 1$ in (6) gives

$$-\rho_1^2(x) + x\rho_1'(x) = 2\rho_2(x) + \rho_2(1).$$

Putting $x = \exp(i\theta)$ in this and appealing to (7), we obtain Ramanujan's identity (1); that is, Venkatachaliengar's identities (6) and (11) are generalizations of Ramanujan's identity (1).

3. Ramanujan's Eisenstein series [pp. 15–18, 31–32]

Ramanujan's Eisenstein series P, Q and R are defined by

$$P = P(q) = 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \tag{13}$$

$$Q = Q(q) = 1 + 240\sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^j}, \tag{14}$$

$$R = R(q) = 1 - 504\sum_{j=1}^{\infty} \frac{j^5q^j}{1-q^j}. \tag{15}$$

In terms of Weierstrassian parameters [42, pp. 437, 446], it turns out that

$$P(q) = \frac{12}{\pi}\eta_1, \quad Q(q) = 12g_2 \quad \text{and} \quad R(q) = 216g_3, \tag{16}$$

where the corresponding Weierstrass elliptic function has periods 2π and $2\pi\tau$.

The series P, Q and R were favorites of Ramanujan. They were studied extensively by him in the paper [32], in his second notebook [34] and in the lost notebook [35]. A survey of the Eisenstein series occurring in the lost notebook has been given by B. C. Berndt and A. J. Yee [8].

The Eisenstein series (13)–(15) arise as coefficients when the identities (1) and (2) are expanded in powers of θ. Ramanujan [32, p. 165] showed that by equating the coefficients of θ^2, θ^4 and θ^6 in the power series expansions of (1), and the coefficients of θ^4 in (2), the following differential equations may be deduced:

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \quad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \quad \text{and} \quad q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{17}$$

These have come to be known as Ramanujan's differential equations, although they were known earlier to G.-H. Halphen in 1886 [23, p. 450]. Detailed proofs of Ramanujan's differential equations, following Ramanujan's method, are given at the beginning of Chapter 2 of Venkatachaliengar's book, and also in Chapter 4 of the introductory text by B. C. Berndt [5]. The reader is encouraged to work through the proof in Ramanujan's original paper [32], as well.

As noted by Ramanujan, the differential equations in (17) imply, by logarithmic differentiation, the result

$$Q^3 - R^2 = 1728q\prod_{j=1}^{\infty}(1-q^j)^{24}.$$

Let U_n and V_n be defined by

$$U_n = \frac{1^{n+1} - 3^{n+1}q + 5^{n+1}q^3 - 7^{n+1}q^6 + \cdots}{1 - 3q + 5q^3 - 7q^6 + \cdots},$$

$$V_n = \frac{1^n - 5^n q - 7^n q^2 + 11^n q^5 + 13^n q^7 + \cdots}{1 - q - q^2 + q^5 + q^7 + \cdots},$$

or equivalently,

$$U_n = \frac{\sum j^{n+1} q^{j^2/8}}{\sum j q^{j^2/8}} \quad \text{and} \quad V_n = \frac{\sum (-1)^{(k-1)/6} k^n q^{k^2/24}}{\sum (-1)^{(k-1)/6} q^{k^2/24}}, \tag{18}$$

where the sums range over all integers j and k (positive as well as negative) that satisfy $j \equiv 1 \pmod 4$ and $k \equiv 1 \pmod 6$, respectively. Clearly, $U_0 = V_0 = 1$. In the lost notebook [35, p. 369], Ramanujan derived the recurrence relations

$$U_{n+2} = PU_n + 8q\frac{dU_n}{dq}, \tag{19}$$

$$V_{n+2} = PV_n + 24q\frac{dV_n}{dq}. \tag{20}$$

To prove these, recall that the denominators in (18) have factorizations given by [5, pp. 12, 14]

$$\sum_{j\equiv 1 \pmod 4} j q^{j^2/8} = q^{1/8}\prod_{\ell=1}^{\infty}(1-q^{\ell})^3 \tag{21}$$

and

$$\sum_{k\equiv 1 \pmod 6} (-1)^{(k-1)/6} q^{k^2/24} = q^{1/24}\prod_{\ell=1}^{\infty}(1-q^{\ell}). \tag{22}$$

Therefore,

$$\begin{aligned}
8q\frac{d}{dq}\log U_n
&= 8q\frac{d}{dq}\log\left(\sum_{j\equiv 1 \pmod 4} j^{n+1} q^{j^2/8}\right) - 8q\frac{d}{dq}\log\left(q^{1/8}\prod_{\ell=1}^{\infty}(1-q^{\ell})^3\right)\\
&= \frac{\sum_{j\equiv 1 \pmod 4} j^{n+3} q^{j^2/8}}{\sum_{j\equiv 1 \pmod 4} j^{n+1} q^{j^2/8}} - \left(1-24\sum_{\ell=1}^{\infty}\frac{\ell q^{\ell}}{1-q^{\ell}}\right)\\
&= \frac{U_{n+2}}{U_n} - P.
\end{aligned}$$

The recurrence for U_n now follows, and the recurrence for V_n may be proved by the same method. Using the recurrence relations (19), (20) and Ramanujan's differential equations (17), we obtain

$$U_0 = 1,\quad U_2 = P,\quad U_4 = \frac{1}{3}(5P^2-2Q),\quad U_6 = \frac{1}{9}(35P^3-42PQ+16R),$$

$$V_0 = 1,\quad V_2 = P,\quad V_4 = 3P^2-2Q,\qquad V_6 = 15P^3-30PQ+16R.$$

By induction, it follows that for any positive integer n,

$$U_{2n} = \sum_{\substack{i,j,k\ge 0\\ i+2j+3k=n}} \mu_{ijk} P^i Q^j R^k \quad\text{and}\quad V_{2n} = \sum_{\substack{i,j,k\ge 0\\ i+2j+3k=n}} \nu_{ijk} P^i Q^j R^k, \tag{23}$$

where μ_{ijk} and ν_{ijk} are rational and integer constants, respectively.

In his presidential address delivered to the Indian Mathematical Society [38], Venkatachaliengar described the proof of (23) outlined by Ramanujan in the lost notebook [35, p. 369] as "disarmingly simple". The identities in (23) are discussed further in [2, pp. 355–364] and [39, pp. 31–32]. Analogues of (23) for which the products in (21) and (22) are replaced with $q^{r/24}\prod_{\ell=1}^{\infty}(1-q^{\ell})^r$ for $r \in \{2, 4, 6, 8, 10, 14, 26\}$ have been given in [16].

4. Ramanujan's ${}_1\psi_1$ summation formula [pp. 22–30]

One of Ramanujan's most important and well-known results is the ${}_1\psi_1$ summation formula. Venkatachaliengar gave two proofs, and we reproduce one of them here. This elegant proof makes use of symmetry and functional equations. First, recall Jacobi's triple product identity, which in one form is [5, p. 10]

$$\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1+q^{2n-1}z^{-1})(1-q^{2n}) = \sum_{n=-\infty}^{\infty} q^{n^2}z^n. \tag{24}$$

It may be regarded as the Laurent series expansion of the product

$$\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}) \tag{25}$$

in the annulus $0 < |z| < \infty$. Ramanujan's ${}_1\psi_1$ summation formula involves a generalization of this product to include a denominator which consists of two infinite products. Specifically, let

$$\phi(z,\alpha,\beta) = \prod_{n=1}^{\infty}\frac{(1+q^{2n-1}z)(1+q^{2n-1}z^{-1})}{(1+\alpha q^{2n-1}z)(1+\beta q^{2n-1}z^{-1})} \tag{26}$$

and suppose in addition that

$$|\beta q| < |z| < |\alpha q|^{-1}. \tag{27}$$

The poles of ϕ are given by

$$z \in \{-\beta q, -\beta q^3, -\beta q^5, \ldots\} \cup \{-(\alpha q)^{-1}, -(\alpha q^3)^{-1}, -(\alpha q^5)^{-1}, \ldots\},$$

and the purpose of the condition (27) is to ensure that ϕ is analytic in an annulus that separates the two families of poles. By Laurent's theorem, ϕ has an expansion of the form

$$\phi(z,\alpha,\beta) = \sum_{n=-\infty}^{\infty} c_n(\alpha,\beta)z^n \tag{28}$$

in the annulus (27). It is easy to check that

$$(1+\alpha qz)\phi(z,\alpha,\beta) = (\beta+qz)\phi(q^2z,\alpha,\beta). \tag{29}$$

If we put

$$\Psi(z) = (1+\alpha qz)\phi(z,\alpha,\beta),$$

we see that $\Psi(z)$ has a Laurent expansion valid in the larger annulus

$$|\beta q| < |z| < |\alpha q^3|^{-1}, \tag{30}$$

because the pole of $\phi(z,\alpha,\beta)$ at $z = -(\alpha q)^{-1}$ has been eliminated. Now

$$\Psi(z) = (1+\alpha qz)\sum_{n=-\infty}^{\infty} c_n(\alpha,\beta)z^n = \sum_{n=-\infty}^{\infty}(c_n(\alpha,\beta)+\alpha q c_{n-1}(\alpha,\beta))z^n$$

and by (29),

$$\Psi(z) = (\beta + qz) \sum_{n=-\infty}^{\infty} c_n(\alpha, \beta) q^{2n} z^n$$

$$= \sum_{n=-\infty}^{\infty} (\beta c_n(\alpha, \beta) q^{2n} + c_{n-1}(\alpha, \beta) q^{2n-1}) z^n.$$

Both of these expansions for Ψ are valid in the larger annulus given by (30). Comparing coefficients of z^n gives

$$c_n(\alpha, \beta) + \alpha q c_{n-1}(\alpha, \beta) = \beta c_n(\alpha, \beta) q^{2n} + c_{n-1}(\alpha, \beta) q^{2n-1},$$

hence, we obtain the recurrence relation

$$c_n(\alpha, \beta) = \frac{q^{2n-2} - \alpha}{1 - q^{2n}\beta} q c_{n-1}(\alpha, \beta).$$

Iterating, gives

$$c_n(\alpha, \beta) = \frac{(1-\alpha)(q^2-\alpha)\cdots(q^{2n-2}-\alpha)}{(1-\beta q^2)(1-\beta q^4)\cdots(1-\beta q^{2n})} q^n c_0(\alpha, \beta), \tag{31}$$

for any positive integer n. From the definition (26), it is clear that

$$\phi(z, \alpha, \beta) = \phi(z^{-1}, \beta, \alpha)$$

and comparing coefficients of z^{-n} on each side gives

$$c_{-n}(\alpha, \beta) = c_n(\beta, \alpha). \tag{32}$$

Combining (31) and (32) it follows that for any positive integer n,

$$\begin{aligned} c_{-n}(\alpha, \beta) &= c_n(\beta, \alpha) \\ &= \frac{(1-\beta)(q^2-\beta)\cdots(q^{2n-2}-\beta)}{(1-\alpha q^2)(1-\alpha q^4)\cdots(1-\alpha q^{2n})} q^n c_0(\beta, \alpha) \\ &= \frac{(1-\beta)(q^2-\beta)\cdots(q^{2n-2}-\beta)}{(1-\alpha q^2)(1-\alpha q^4)\cdots(1-\alpha q^{2n})} q^n c_0(\alpha, \beta), \end{aligned} \tag{33}$$

where the last step follows by taking $n = 0$ in (32). It remains to evaluate $c_0(\alpha, \beta)$. Again, it is easy to check from the definition (26) that

$$\phi(z, \alpha q^2, \beta) = (1 + \alpha q z)\phi(z, \alpha, \beta),$$

and both sides represent analytic functions of z for $|\beta q| < |z| < |\alpha q^3|^{-1}$. Equating the constant coefficients and using (33) we obtain

$$c_0(\alpha q^2, \beta) = c_0(\alpha, \beta) + \alpha q c_{-1}(\alpha, \beta) = \frac{1 - \alpha\beta q^2}{1 - \alpha q^2} c_0(\alpha, \beta).$$

Iterating gives

$$c_0(\alpha, \beta) = c_0(\alpha q^{2n}, \beta) \prod_{j=1}^{n} \frac{1 - \alpha q^{2j}}{1 - \alpha\beta q^{2j}},$$

and taking the limit as $n \to \infty$ gives

$$c_0(\alpha, \beta) = c_0(0, \beta) \prod_{j=1}^{\infty} \frac{1 - \alpha q^{2j}}{1 - \alpha\beta q^{2j}}. \tag{34}$$

By (32) and (34), we deduce further that

$$c_0(0, \beta) = c_0(\beta, 0) = c_0(0, 0) \prod_{j=1}^{\infty} (1 - \beta q^{2j}), \tag{35}$$

and substituting (35) into (34) gives

$$c_0(\alpha, \beta) = c_0(0, 0) \prod_{j=1}^{\infty} \frac{(1 - \alpha q^{2j})(1 - \beta q^{2j})}{1 - \alpha\beta q^{2j}}. \tag{36}$$

Now take $\alpha = \beta = 1$ and observe that $\phi(z, 1, 1) = 1$ identically. Hence, $c_0(1, 1) = 1$. Using this in (36) gives

$$1 = c_0(0, 0) \prod_{j=1}^{\infty} (1 - q^{2j})$$

and so

$$c_0(0, 0) = \prod_{j=1}^{\infty} \frac{1}{(1 - q^{2j})},$$

therefore (36) becomes

$$c_0(\alpha, \beta) = \prod_{j=1}^{\infty} \frac{(1 - \alpha q^{2j})(1 - \beta q^{2j})}{(1 - \alpha\beta q^{2j})(1 - q^{2j})}. \tag{37}$$

If we divide both sides of (28) by $c_0(\alpha, \beta)$ and use (31), (33) and (37), the final result is Ramanujan's ${}_1\psi_1$ summation formula:

$$\begin{aligned} &\prod_{n=1}^{\infty} \frac{(1 + q^{2n-1}z)(1 + q^{2n-1}z^{-1})(1 - \alpha\beta q^{2j})(1 - q^{2j})}{(1 + \alpha q^{2n-1}z)(1 + \beta q^{2n-1}z^{-1})(1 - \alpha q^{2j})(1 - \beta q^{2j})} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(1 - \alpha)(q^2 - \alpha) \cdots (q^{2n-2} - \alpha)}{(1 - \beta q^2)(1 - \beta q^4) \cdots (1 - \beta q^{2n})} \, q^n z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{(1 - \beta)(q^2 - \beta) \cdots (q^{2n-2} - \beta)}{(1 - \alpha q^2)(1 - \alpha q^4) \cdots (1 - \alpha q^{2n})} q^n z^{-n}. \end{aligned} \tag{38}$$

By the ratio test, the series involving positive powers of z in (38) converges for $|\alpha q z| < 1$, while the series involving negative powers of z converges for $|\beta q/z| < 1$. The two series will converge simultaneously and represent an analytic function of z when both conditions are satisfied, and this is precisely the condition given by (27).

It should be noted that the Jacobi triple product identity (24) is the limiting case $\alpha, \beta \to 0$ of (38). In this case, the annulus of convergence becomes $0 < |z| < \infty$.

For some historical background and references to other proofs of Ramanujan's ${}_1\psi_1$ summation formula (38), see [2, pp. 54–56] and [28].

5. The Jordan–Kronecker function [pp. 37–43]

The Jordan-Kronecker function is defined by

$$F(x,y) = \prod_{j=1}^{\infty} \frac{(1-q^{j-1}xy)(1-q^j x^{-1}y^{-1})(1-q^j)^2}{(1-q^{j-1}x)(1-q^j x^{-1})(1-q^{j-1}y)(1-q^j y^{-1})}. \tag{39}$$

It has two important properties. The first is its expansion as a series:

$$F(x,y) = \sum_{j=-\infty}^{\infty} \frac{x^j}{1-yq^j}, \quad \text{provided } |q| < |x| < 1. \tag{40}$$

This is an immediate consequence of Ramanujan's ${}_1\psi_1$ summation formula: set $(z, \alpha, \beta) = (-xy/q, 1/y, y)$ in (38), divide both sides by $1-y$, then replace q^2 with q.

The infinite product (39) is symmetric in x and y, but the series in (40) is not. Symmetry in the series can be restored by a calculation that is similar to (7). That is, consider the terms for which j is positive, negative or zero, and expand into a double series, to get

$$F(x,y) = \frac{1-xy}{(1-x)(1-y)} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q^{jk}(x^j y^k - x^{-j}y^{-k}). \tag{41}$$

This converges for $|q| < |x|, |y| < |q|^{-1}$, apart from simple poles at $x=1$, $y=1$. The series (41) can be manipulated to yield a series that converges for all x and y, apart from at the poles, by separating the terms in (41) into the three cases $j=k$, $j>k$ and $j<k$, and summing the geometric series. The final result is

$$\begin{aligned} F(x,y) = {} & \frac{1-xy}{(1-x)(1-y)} \\ & + \sum_{j=1}^{\infty} q^{j^2} x^j y^j \left(1 + \frac{q^j x}{1-q^j x} + \frac{q^j y}{1-q^j y}\right) \\ & - \sum_{j=1}^{\infty} q^{j^2} x^{-j} y^{-j} \left(1 + \frac{q^j x^{-1}}{1-q^j x^{-1}} + \frac{q^j y^{-1}}{1-q^j y^{-1}}\right). \end{aligned} \tag{42}$$

The formulas (39) and (41) were given by Kronecker[2] in lectures in July 1876, and then in a paper published in 1881 [30]. The formulas (39), (40) and (42) appear in Jordan's Cours d'Analyse [29, p. 507–511]. For these reasons, Venkatachaliengar refers to the function $F(x, y)$ as the Jordan-Kronecker function.

Kroencker's paper has a reference to the 1850 paper of Jacobi "Sur la rotation d'un corps", [25]. If eq. (3) in [25, p. 297] is multiplied by $i = \sqrt{-1}$ and the result subtracted from eq. (4), ibid., the result eventually simplifies to the identity (40). Thus, although (40) is implicit in Jacobi's work, it would be a bit of a stretch to attribute the identity to him. However, Jacobi's work is noteworthy because the genesis of the identity may be seen there, and because the identity arises in a physical application.

Kronecker's analysis has been examined and simplified by Weil [41, pp. 70–71]. Both Kronecker and Weil expand the product in (39) as a Laurent series and use Cauchy's theorem to compute the coefficients.

The second important property is the result which Venkatachaliengar calls the fundamental multiplicative identity:

$$F(x,t)F(y,t) = t\frac{\partial}{\partial t}F(xy,t) + F(xy,t)\left(\rho_1(x) + \rho_1(y)\right), \tag{43}$$

where ρ_1 is defined in (5). This can be proved by writing

$$F(x,t)F(y,t) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{x^m y^n}{(1-tq^m)(1-tq^n)},$$

breaking the sum into the cases $m = n$ and $m \neq n$, and using partial fractions in the latter case. The details have been reproduced in [18] and [21].

Two special cases of (43) deserve special mention. First, if we set $t = e^v$ in (43), expand both sides in powers of v, then equate coefficients of v^0, the result is Venkatachaliengar's identity (11). See [18, p. 75] for the details. Thus, (43) may be regarded as a generalization of Ramanujan's identity (1) that involves two extra parameters.

The other noteworthy special case is to take the limit as $y \to 1/x$ in (43). The result simplifies to

$$F(x,t)F(x^{-1},t) = t\frac{d}{dt}\rho_1(t) - x\frac{d}{dx}\rho_1(x). \tag{44}$$

This can be shown to be equivalent to the Weierstrassian identity [42, p. 451]

$$\frac{\sigma(\alpha+\beta)\sigma(\alpha-\beta)}{\sigma^2(\alpha)\sigma^2(\beta)} = \wp(\beta) - \wp(\alpha).$$

[2] Kronecker gives the identity in the form

$$\frac{\sum(-1)^{(\mu-1)/2}\mu q^{\mu^2/4}\sum(-1)^{(\nu-1)/2}q^{\nu^2/4}(x^\nu y^\nu - x^{-\nu}y^{-\nu})}{\sum(-q)^{m^2}x^{2m}\sum(-q)^{n^2}y^{2n}} = \sum\sum q^{\mu\nu/2}(x^\mu y^\nu - x^{-\mu}y^{-\nu})$$

where the sums are over positive odd integers μ and ν, and all integers m and n. Kronecker's form can be manipulated into the form we have given by converting each of the four sums in the quotient into infinite products via Jacobi's triple product identity and making a change of variable.

The identity (43) should be regarded as classical. It has been rediscovered and reproved many times. For, dividing by $F(xy,t)$ gives

$$\begin{aligned}\frac{F(x,t)F(y,t)}{F(xy,t)} &= t\frac{\partial}{\partial t}\log F(xy,t)+\rho_1(x)+\rho_1(y)\\ &= \rho_1(x)+\rho_1(y)+\rho_1(t)-\rho_1(xyt),\end{aligned}$$

and this can be shown to be equivalent to the Weierstrassian identity

$$\frac{\sigma(\alpha+\beta)\sigma(\alpha-\beta)\sigma(2\gamma)}{\sigma(\alpha+\gamma)\sigma(\alpha-\gamma)\sigma(\beta+\gamma)\sigma(\beta-\gamma)}=\zeta(\alpha+\gamma)-\zeta(\alpha-\gamma)+\zeta(\beta-\gamma)-\zeta(\beta+\gamma).$$

This was given by G.-H. Halphen [23, p. 187]. Further historical background for the identity (43) has been given in [2, p. 60].

6. Inversion for elliptic functions, and theories for alternative bases [pp. 52–56, 89–95]

At the end of Chapter 3, a novel approach to the connection between theta functions and hypergeometric functions is proposed. For $|q|<1$, define $x=x(q)$ by

$$x=\left(\sum_{n=-\infty}^{\infty}q^{\left(n+\frac{1}{2}\right)^2}\Big/\sum_{n=-\infty}^{\infty}q^{n^2}\right)^4. \tag{45}$$

It can be shown that as q increases from 0 to 1, x also increases from 0 to 1. The basic problem is to find a formula for the inverse function $q=q(x)$. The result is

$$q=\exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)}\right), \tag{46}$$

where

$${}_2F_1(a,b;c;x)=1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n$$

is the hypergeometric function, and

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1)$$

is the shifted factorial. These results were known to Jacobi in 1828, see, e.g., [24].

Venkatachaliengar's approach is to disregard (45) and begin instead by defining z and x implicitly by

$$Q(q)=z^4(1-x+x^2)\quad\text{and}\quad R(q)=z^6(1+x)(1-2x)(1-x/2). \tag{47}$$

The goal is to solve the equations in (47) for q to obtain (46). The main step in the argument is to use Ramanujan's differential equations (17) to show that z satisfies the following hypergeometric differential equation with respect to x:

$$\frac{d}{dx}\left(x(1-x)\frac{dz}{dx}\right) = \frac{z}{4}.$$

Venkatachaliengar's proof has been explained in more detail by S. Bhargava [9, pp. 315–318].

In Chapter 5, [pp. 89–95] Venkatachaliengar proposes to use the same approach to prove the analogous results for (what are now known as) Ramanujan's theories of elliptic functions to alternative bases. These are commonly referred to by their signatures, which are 3, 4, or 6. Jacobi's classical theory involving (45) and (46) is said to belong to the theory of signature 2. In the signature 3 theory Venkatachaliengar begins by defining z_3 and x_3 implicitly by

$$Q = z_3^4(1+8x_3), \quad R = z_3^6(1-20x_3-8x_3^2). \tag{48}$$

There are similar definitions in each of the theories for signatures 4 and 6.

Let us discuss some of the merits and objections to Venkatachaliengar's approach just described. The utility of Venkatachaliengar's method has been demonstrated by H. H. Chan and Y. L. Ong [17] in establishing a septic theory. The most obvious objection is that the formulas (47) and (48) are unmotivated and have to be known in advance. Also, z and x are multivalued and a selection has to be made. Moreover, this is almost certainly not the method Ramanujan would have used. Ramanujan in fact outlined his approach to this inversion problem for elliptic functions in the first few entries of Chapter 17 in his second notebook [34]. Full details of Ramanujan's method have been worked out by B. C. Berndt [3, pp. 87–102]. Ramanujan's method is original, and Berndt makes the comment [3, p. 99]: "Proofs in the latter half of the second notebook are very rare indeed". In contrast, Ramanujan left no clues as to how he discovered the results for signatures 3, 4 and 6. The reader is referred to [4, Chapter 33], [6,7,10–13,15], and [20] for more information about Ramanujan's theories of elliptic functions to alternative bases.

The reviewer [20] has proposed the following alternative starting point, which leads to a unified treatment of the classical and alternative theories. Let $r = 1, 2, 3$ or 4. Let $z_r = z_r(q)$ be defined by

$$z_r = \begin{cases} (Q(q))^{1/4} & \text{if } r = 1, \\ \left(\frac{rP(q^r)-P(q)}{r-1}\right)^{1/2} & \text{if } r = 2, 3 \text{ or } 4. \end{cases}$$

Let $x_r = x_r(q)$ be defined to be the solution of the initial value problem

$$q\frac{dx_r}{dq} = z_r^2 x_r(1-x_r), \qquad x_r(e^{-2\pi/\sqrt{r}}) = \frac{1}{2}.$$

It can be shown that as q increases from 0 to 1, x_r increases from 0 to 1. Therefore, the inverse function $q = q_r(x_r)$ exists. It is given explicitly by the formula

$$q = \exp\left(-\frac{2\pi}{\sqrt{r}} \frac{{}_2F_1\left(c_r, 1-c_r; 1; 1-x_r\right)}{{}_2F_1\left(c_r, 1-c_r; 1; x_r\right)}\right),$$

where $c_r = 1/6$, $1/4$, $1/3$ or $1/2$, according to whether $r = 1, 2, 3$ or 4, respectively. The quantity $1/c_r$ has come to be known as the signature. In the theory of modular forms, the parameter r is called the level. Thus, levels $r = 1, 2, 3$ and 4 correspond to the signatures 6, 4, 3 and 2, respectively. The formulas (45) and (46) are the results for level 4 (signature 2).

7. Halphen's differential equations [pp. 57–76]

Let

$$\Omega(\theta) = i\frac{d}{d\theta}\rho_1(e^{i\theta}).$$

From (7), we may obtain the explicit formula

$$\Omega(\theta) = \frac{1}{4}\csc^2\frac{\theta}{2} - 2\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}\cos j\theta. \tag{49}$$

It can be shown, e.g., [18, p. 69], that

$$\Omega(\theta) = \wp(\theta) + \frac{P}{12}, \tag{50}$$

where $\wp(z)$ is the Weierstrass elliptic function with periods 2π and $2\pi\tau$ and P is Ramanujan's Eisenstein series defined by (13).

Venkatachaliengar defines three functions u_1, u_2 and u_3 by

$$\begin{aligned} u_1 &= 4\Omega(\pi), \\ u_2 &= 4\Omega(\pi\tau), \\ u_3 &= 4\Omega(\pi+\pi\tau). \end{aligned}$$

Using (49), and setting $h = e^{i\pi\tau}$ so that $h^2 = q$, we find that

$$u_1 = 1 - 8\sum_{n=1}^{\infty}\frac{(-1)^n nh^{2n}}{1-h^{2n}}, \tag{51}$$

$$u_2 = -8\sum_{n=1}^{\infty}\frac{nh^n}{1-h^{2n}}, \tag{52}$$

$$u_3 = -8\sum_{n=1}^{\infty}\frac{(-1)^n nh^n}{1-h^{2n}}. \tag{53}$$

G.-H. Halphen [23, p. 330] proved that the functions given by (51)–(53) satisfy the system of differential equations

$$h\frac{d}{dh}(u_1+u_2)=u_1u_2, \tag{54}$$

$$h\frac{d}{dh}(u_2+u_3)=u_2u_3, \tag{55}$$

$$h\frac{d}{dh}(u_3+u_1)=u_3u_1. \tag{56}$$

Halphen noted that the system (54)–(56) is a special case of the differential equations

$$C(dA+dB)=B(dA+dC)=A(dB+dC),$$

posed by G. Darboux [22, p. 149] in 1878. More recently, the system (54)–(56) and other related systems of nonlinear differential equations have been studied by W. Zudilin [43].

By ingenious use of (11), Venkatachaliengar gave a simple proof that the functions defined by (51)–(53) are solutions of the system (54)–(56).

By series manipulations using (13) and (51)–(53), we find that

$$u_1+u_2+u_3=P.$$

Applying the differential operator $h\frac{d}{dh}=2q\frac{d}{dq}$, and using (54)–(56) on the left and (17) on the right, we find that

$$u_1^2+u_2^2+u_3^2-u_1u_2-u_2u_3-u_3u_1=Q.$$

Differentiating again using (54)–(56) and (17), and simplifying, it can be shown that

$$\frac{1}{2}(2u_1-u_2-u_3)(2u_2-u_3-u_1)(2u_3-u_1-u_2)=R.$$

The differences u_j-u_k have simple representations as infinite products. For example, subtracting (55) from (56) gives

$$h\frac{d}{dh}(u_1-u_2)=(u_1-u_2)u_3.$$

Divide by $(u_1-u_2)h$ and integrate with respect to h, to get

$$\begin{aligned}\log(u_1-u_2)&=-8\sum_{n=1}^{\infty}\int\frac{(-1)^n nh^{n-1}}{1-h^{2n}}\,dh\\&=4\sum_{n=1}^{\infty}(-1)^n\log\left(\frac{1-h^n}{1+h^n}\right)+c.\end{aligned}$$

Letting $h=0$ we find that the constant of integration is given by $c=0$. It follows that

$$u_1-u_2=\prod_{n=1}^{\infty}\frac{(1-h^{2n})^4(1+h^{2n-1})^4}{(1-h^{2n-1})^4(1+h^{2n})^4}.$$

A formula, for the number of representations of an integer as a sum of four squares as a divisor sum, can be deduced from this result. Infinite products for the differences $u_1 - u_3$ and $u_3 - u_2$ can be obtained by the same method.

Lastly, we note that by expanding each of (51)–(53) as double series and interchanging the order of summation, if follows that $u_1 > 0$, $u_2 < 0$ and $u_3 > 0$ for $0 < h < 1$.

8. Jacobian elliptic functions [pp. 97–114]

In Chapter 6 of [39], Venkatachaliengar uses the Jordan-Kronecker function (39) and the fundamental multiplicative identity (43) to give a new and efficient development of the main properties of Jacobian elliptic functions. An outline of this theory goes as follows. Let

$$f_1(\theta) = -iF(e^{i\pi}, e^{i\theta}), \tag{57}$$

$$f_2(\theta) = -ie^{i\theta/2}F(e^{i\pi\tau}, e^{i\theta}), \tag{58}$$

$$f_3(\theta) = -ie^{i\theta/2}F(e^{i\pi+i\pi\tau}, e^{i\theta}). \tag{59}$$

The factors $-i$ and $-ie^{i\theta/2}$ are included so that f_1, f_2 and f_3 will be real valued when θ is real. Infinite product expansions follow directly from the definition (39). The results are

$$f_1(\theta) = \frac{1}{2}\cot\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{(1+2q^n\cos\theta+q^{2n})(1-q^n)^2}{(1-2q^n\cos\theta+q^{2n})(1+q^n)^2}, \tag{60}$$

$$f_2(\theta) = \frac{1}{2}\csc\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{(1-2q^{n-\frac{1}{2}}\cos\theta+q^{2n-1})(1-q^n)^2}{(1-2q^n\cos\theta+q^{2n})(1-q^{n-\frac{1}{2}})^2}, \tag{61}$$

$$f_3(\theta) = \frac{1}{2}\csc\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{(1+2q^{n-\frac{1}{2}}\cos\theta+q^{2n-1})(1-q^n)^2}{(1-2q^n\cos\theta+q^{2n})(1+q^{n-\frac{1}{2}})^2}. \tag{62}$$

The Fourier expansions follow directly from (40) or (41), and using the symmetry property $F(x, y) = F(y, x)$. After simplification, we find that

$$f_1(\theta) = \frac{1}{2}\cot\frac{\theta}{2} - 2\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin n\theta, \tag{63}$$

$$f_2(\theta) = \frac{1}{2}\csc\frac{\theta}{2} + 2\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1-q^{n-\frac{1}{2}}}\sin\left(n-\frac{1}{2}\right)\theta, \tag{64}$$

$$f_3(\theta) = \frac{1}{2}\csc\frac{\theta}{2} - 2\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}}\sin\left(n-\frac{1}{2}\right)\theta. \tag{65}$$

The Fourier expansions for the squares of each function can be obtained from (44). For, taking $t = e^{i\theta}$ and successively setting $x = e^{i\pi}$, $e^{i\pi\tau}$ and $e^{i\pi+i\pi\tau}$ in (44) gives, respectively,

$$f_1^2(\theta) = \left[i\frac{d}{d\alpha}\rho_1(e^{i\alpha}) \right]_{\pi}^{\theta}, \tag{66}$$

$$f_2^2(\theta) = \left[i\frac{d}{d\alpha}\rho_1(e^{i\alpha}) \right]_{\pi\tau}^{\theta}, \tag{67}$$

$$f_3^2(\theta) = \left[i\frac{d}{d\alpha}\rho_1(e^{i\alpha}) \right]_{\pi+\pi\tau}^{\theta}, \tag{68}$$

and the Fourier expansions now follow by applying (49). For future reference, we note that (66)–(68) imply

$$\left[i\frac{d}{dz}\rho_1(e^{iz}) \right]_{\alpha}^{\beta} = [f_1^2(z)]_{\alpha}^{\beta} = [f_2^2(z)]_{\alpha}^{\beta} = [f_3^2(z)]_{\alpha}^{\beta}. \tag{69}$$

Comparison of the products (60)–(62) and series (63)–(65) with those in [42, pp. 508, 511–512], reveals that the functions f_1, f_2 and f_3 are, up to rescaling, the Jacobian elliptic functions cs, ns and ds, respectively.

From the infinite products, it is easy to show that

$$f_1(\theta + 2\pi m + 2\pi\tau n) = (-1)^n f_1(\theta), \tag{70}$$

$$f_2(\theta + 2\pi m + 2\pi\tau n) = (-1)^m f_2(\theta), \tag{71}$$

$$f_3(\theta + 2\pi m + 2\pi\tau n) = (-1)^{m+n} f_3(\theta). \tag{72}$$

Here m and n are integers. Thus f_1 is doubly periodic with periods 2π and $4\pi\tau$, f_2 is doubly periodic with periods 4π and $2\pi\tau$, while f_3 is doubly periodic with periods 4π and $2\pi + 2\pi\tau$.

The zeros and poles may be determined from the infinite products (60)–(62). The zeros of f_1, f_2 and f_3 are at $\theta = (2m+1)\pi + 2n\pi\tau$, $\theta = 2m\pi + (2n+1)\pi\tau$ and $\theta = (2m+1)\pi + (2n+1)\pi\tau$, respectively, where m and n are any integers. The poles of f_1, f_2 and f_3 all occur when $\theta = 2m\pi + 2n\pi\tau$.

The fundamental multiplicative identity (43) may be used to compute the derivatives of each of f_1, f_2 and f_3, as well as their addition formulas. We begin with the derivatives. In (43), let $t = e^{i\theta}$ to get

$$F(a, e^{i\theta})F(b, e^{i\theta}) = \frac{1}{i}\frac{\partial}{\partial\theta}F(ab, e^{i\theta}) + F(ab, e^{i\theta})(\rho_1(a) + \rho_1(b)). \tag{73}$$

Now let $a = e^{i\pi}$ and $b = e^{i\pi\tau}$. Noting that $\rho_1(e^{i\pi}) = 0$ and $\rho_1(e^{i\pi\tau}) = \frac{1}{2}$, the result (73) eventually simplifies to

$$f_3'(\theta) = -f_1(\theta)f_2(\theta). \tag{74}$$

Similarly, letting $a = e^{i\pi\tau}$, $b = e^{i\pi+i\pi\tau}$ and $a = e^{i\pi+i\pi\tau}$, $b = e^{i\pi}$ in (73) leads, respectively, to

$$f_1'(\theta) = -f_2(\theta)f_3(\theta), \tag{75}$$
$$f_2'(\theta) = -f_3(\theta)f_1(\theta). \tag{76}$$

In order to deduce the addition formulas, write the fundamental multiplicative identity (43) in the form

$$F(e^{i\alpha}, e^{i\theta})F(e^{i\beta}, e^{i\theta}) = \frac{1}{i}F(e^{i(\alpha+\beta)}, e^{i\theta}) + F(e^{i(\alpha+\beta)}, e^{i\theta})(\rho_1(e^{i\alpha}) + \rho_1(e^{i\beta})).$$

Apply $\partial/\partial\alpha - \partial/\partial\beta$ to both sides. The result is

$$\begin{aligned}&\frac{\partial}{\partial\alpha}F(e^{i\alpha}, e^{i\theta})F(e^{i\beta}, e^{i\theta}) - F(e^{i\alpha}, e^{i\theta})\frac{\partial}{\partial\beta}F(e^{i\beta}, e^{i\theta})\\ &\quad = F(e^{i(\alpha+\beta)}, e^{i\theta})\left(\frac{d}{d\alpha}\rho_1(e^{i\alpha}) - \frac{d}{d\beta}\rho_1(e^{i\beta})\right).\end{aligned}$$

Rearranging this and using (69) gives

$$F(e^{i(\alpha+\beta)}, e^{i\theta}) = \frac{i\left[\frac{\partial}{\partial\alpha}F(e^{i\alpha}, e^{i\theta})F(e^{i\beta}, e^{i\theta}) - F(e^{i\alpha}, e^{i\theta})\frac{\partial}{\partial\beta}F(e^{i\beta}, e^{i\theta})\right]}{f_j^2(\alpha) - f_j^2(\beta)}, \tag{77}$$

which holds for $j = 1, 2$ or 3. Let $\theta = \pi$, take $j = 1$ and apply (75) to get the addition formula for f_1:

$$f_1(\alpha+\beta) = \frac{f_1(\alpha)f_2(\beta)f_3(\beta) - f_1(\beta)f_2(\alpha)f_3(\alpha)}{f_1^2(\beta) - f_1^2(\alpha)}.$$

Similarly, letting $\theta = \pi\tau$ and $\theta = \pi + \pi\tau$ in (77) leads to

$$f_2(\alpha+\beta) = \frac{f_2(\alpha)f_3(\beta)f_1(\beta) - f_2(\beta)f_3(\alpha)f_1(\alpha)}{f_2^2(\beta) - f_2^2(\alpha)},$$

$$f_3(\alpha+\beta) = \frac{f_3(\alpha)f_1(\beta)f_2(\beta) - f_3(\beta)f_1(\alpha)f_2(\alpha)}{f_3^2(\beta) - f_3^2(\alpha)}.$$

In summary, this efficient approach to the Jacobian elliptic functions, due to Venkatachaliengar, is quite different from any other theory that has been presented before.

9. Conclusions

Many of Ramanujan's results relate to, or involve, elliptic functions. The usual way of studying these is to use the theory of complex variables. Yet, it is generally agreed that Ramanujan did not use the theory of complex variables. An interesting discussion on how Ramanujan may have studied elliptic functions has been given by B. C. Berndt [3, p. 2]. The fact will always remain, that because Ramanujan's notebooks generally do not contain proofs, we will never know for sure what his methods were.

The book by A. Weil [41] shows how the theory of elliptic functions may be developed, without the theory of complex variables, by a process called Eisenstein summation. Weil's book is based on the pioneering works of Eisenstein and Kronecker. Venkatachaliengar's book shows that it is also possible to use q-series to develop the theory of elliptic functions in an efficient, logical and systematic way. Thus, there are now two comprehensive answers, elucidated by Weil [41] and Venkatachaliengar [39], to the question of how the theory of elliptic functions can be developed by series manipulations and without using the theory of complex variables as the major tool.

Venkatachaliengar's methods frequently make maximal use of symmetry. The proof of Ramanujan's ${}_1\psi_1$ summation formula (38) is one example of this. Moreover, the statement of the ${}_1\psi_1$ identity is the same as the symmetric form given by Ramanujan in his second notebook. Most proofs of this identity prove the non-symmetric version in terms of the ${}_1\psi_1$ basic hypergeometric series, from which the symmetric form can be deduced by a change of variable. The quintuple product identity in [39, Appendix B] (not discussed in this review, but see [19]) is similarly presented in its most symmetric form.

Many proofs of Ramanujan's ${}_1\psi_1$ summation formula rely on functional equations like (29). Since $\phi(z, \alpha, \beta)$ and $\phi(q^2 z, \alpha, \beta)$ converge in different annuli, the usual approach is to make an additional assumption on the parameters to ensure that the two annuli of convergence overlap; the extra assumption is then removed at the end of the proof by analytic continuation. In Venkatachlaliengar's proof, the introduction of the function Ψ eliminates the need for any analytic continuation argument. The series manipulations in the proof hold for all values of the parameters for which the identity (38) is valid.

Unfortunately, the book was not carefully edited. There are many typographical errors. Sometimes topics are repeated unnecessarily. Some of the proofs are incomplete, but most can be (and have been) fixed.

Due to space limitations, several other topics in Venkatachaliengar's book have not been discussed in this review. These include the modular transformation $\tau \to -1/\tau$, Landen's transformation $\tau \to 2\tau$, Picard's theorem, modular equations, the quintuple product identity, and the addition theorem for elliptic integrals.

In summary, a high degree of originality is present throughout the book. The successive generalizations of Ramanujan's identity (1) given by (11) and then by (43) are wonderful achievements. The use of the Jordan-Kronecker function (39) along with the fundamental multiplicative identity (43) to study the Jacobian elliptic functions is also quite different from any other method that has been used to study these functions.

We end this review by mentioning three other published works on elliptic functions by Venkatachaliengar: the research article [37], the presidential address [38], and the survey [40].

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References

[1] R. P. Agarwal, Resonance of Ramanujan's mathematics, Vol. III, New Age International Publishers Limited, New Delhi (1999).
[2] G. E. Andrews and B. C. Berndt, Ramanujan's Lost Notebook, Part II, Springer, New York (2009).
[3] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York (1991).
[4] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York (1998).
[5] B. C. Berndt, Number theory in the spirit of Ramanujan, American Mathematical Society, Providence, RI (2006).
[6] B. C. Berndt, S. Bhargava and F. G. Garvan, Ramanujan's theories of elliptic functions to alternative bases, *Trans. Amer. Math. Soc.*, **347** (1995) 4163–4244.
[7] B. C. Berndt, H. H. Chan and W.-C. Liaw, On Ramanujan's quartic theory of elliptic functions, *J. Number Theory*, **88** (2001) 129–156.
[8] B. C. Berndt and A. J. Yee, Ramanujan's contributions to Eisenstein series, especially in his lost notebook. In: Number theoretic methods (Iizuka, 2001), 31–53, *Dev. Math.*, **8** Kluwer, Dordrecht (2002).
[9] S. Bhargava, A look at Ramanujan's work in elliptic function theory and further developments, *Proc. Nat. Acad. Sci. India, Sect. A*, **68** (1998) 299–347. Reprinted in [1, pp. 168–216].
[10] J. M. Borwein and P. B. Borwein, A remarkable cubic mean iteration. In: Computational methods and function theory (Valparaíso, 1989), 27–31, Lecture Notes in Math., 1435, Springer, Berlin (1990).
[11] J. M. Borwein and P. B. Borwein, A cubic counterpart of Jacobi's identity and the AGM, *Trans. Amer. Math. Soc.*, **323** (1991) 691–701.
[12] J. M. Borwein, P. B. Borwein and F. G. Garvan, Hypergeometric analogues of the arithmetic-geometric mean iteration, *Constr. Approx.*, **9** (1993) 509–523.
[13] J. M. Borwein, P. B. Borwein and F. G. Garvan, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.*, **343** (1994) 35–47.
[14] J. M. Burgers, Application of a model system to illustrate some points of the statistical theory of free turbulence, *Nederl. Akad. Wetensch., Proc.*, **43** (1940) 2–12.
[15] H. H. Chan, On Ramanujan's cubic transformation for ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$, *Math. Proc. Cambridge Phil. Soc.*, **124** (1998) 193–204.
[16] H. H. Chan, S. Cooper and P. C. Toh, Ramanujan's Eisenstein series and powers of Dedekind's eta-function, *J. London Math. Soc. (2)*, **75** (2007) 225–242.
[17] H. H. Chan and Y. L. Ong, On Eisenstein series and $\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}$, *Proc. Amer. Math. Soc.*, **127** (1999) 1735–1744.
[18] S. Cooper, The Development of Elliptic Functions According to Ramanujan and Venkatachaliengar, *Research Letters in the Information and Mathematical Sciences*, **1** (2000) 65–78. http://iims.massey.ac.nz/research/letters/volume1number1/index.html
[19] S. Cooper, The quintuple product identity, *Int. J. Number Theory*, **2** (2006) 115–161.
[20] S. Cooper, Inversion formulas for elliptic functions, *Proc. London Math. Soc.*, **99** (2009) 461–483.
[21] S. Cooper and H. Y. Lam, Sums of two, four, six and eight squares and triangular numbers: an elementary approach, *Indian J. Math.*, **44** (2002) 21–40.

[22] G. Darboux, Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux, *Annales scientifiques de l'École Normale Supérieure, Sér. 2*, **7** (1878) 101–150.
[23] G.-H. Halphen, Traité des fonctions elliptiques et de leurs applications, Part 1, Gauthier-Villars, Paris (1886).
[24] C. G. J. Jacobi, Note sur les fonctions elliptiques, *Journal für die reine und angewandte Mathematik*, **3** (1828) 192–195. Reprinted in [26, pp. 251–254].
[25] C. G. J. Jacobi, Sur la rotation d'un corps. Extrait d'une lettre adressée à l'académie des sciences de Paris, *Journal für die reine und angewandte Mathematik*, **39** (1850) 293–350. Reprinted in [27, pp. 289–352].
[26] C. G. J. Jacobi, Gesammelte Werke, vol. 1, Chelsea, New York (1969).
[27] C. G. J. Jacobi, Gesammelte Werke, vol. 2, Chelsea, New York (1969).
[28] W. P. Johnson, How Cauchy missed Ramanujan's ${}_1\psi_1$ summation, *Amer. Math. Monthly*, **111** (2004) 791–800.
[29] C. Jordan, Cours d'Analyse, vol. 2, 3rd edition, Gauthier-Villars, Paris, 1913. Reprinted by Jacques Gabay, Sceaux (1991).
[30] L. Kronecker, Zur Theorie der elliptischen Functionen, Monatsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin vom Jahre 1881, 1165–1172. Reprinted in [31, 311–318].
[31] L. Kronecker, Leopold Kronecker's Werke, vol. 4, Teubner, Leipzig, 1929. Reprinted by Chelsea, New York (1968).
[32] S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.*, **22** (1916) 159–184. Reprinted in [33, pp. 136–162].
[33] S. Ramanujan, Collected Papers, Cambridge, University Press, 1927. Reprinted by AMS Chelsea, Providence, RI (2000).
[34] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay (1957).
[35] S. Ramanujan, The lost notebook and other unpublished papers, Narosa, New Delhi (1988).
[36] B. van der Pol, On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications. I, II, *Nederl. Akad. Wetensch. Proc. Ser. A*, **54** (Same as: Indagationes Math., **13** (1951)) 261–271, 272–284.
[37] K. Venkatachaliengar, Elliptic modular functions and Picard's theorem, Hyperbolic complex analysis (Proc. All India Sem., Ramanujan Inst., Univ. Madras, Madras, 1977) pp. 21–41, Publ. Ramanujan Inst., 4, Univ. Madras, Madras (1979).
[38] K. Venkatachaliengar, Ramanujan manuscripts, II, *Math. Student*, **52** (1984) 215–233.
[39] K. Venkatachaliengar, Development of elliptic functions according to Ramanujan, Madurai Kamaraj University, Madurai (1988).
[40] K. Venkatachaliengar, Ramanujan's contributions to the theory of elliptic functions. In: Wazir Hasan Abdi (ed.), Toils and triumphs of Srinivasa Ramanujan the man and the mathematician, pp. 182–206. National Publishing House, Chaura Rasta, Jaipur (1992).
[41] A. Weil, Elliptic functions according to Eisenstein and Kronecker, Springer-Verlag, Berlin-New York (1976).
[42] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, 4th edition (1927).
[43] W. Zudilin, The hypergeometric equation and Ramanujan functions, *Ramanujan J.*, **7** (2003) 435–447.

List of Publications of K. Venkatachaliengar

[1] *Weierstrass' non-differentiable function*, J. Indian Math. Soc. **19** (1931), 54–58.

[2] *On the validity of the Raman-Banerjee analysis of the pianoforte hammer problem*, Proc. Indian Acad. **1** (1934), 60–66.

[3] *On series whose terms as well as the sumfunction are continuous in an inteval, and which converges non-uniformly in every sub-interval*, J. Indian Math. Soc. **20** (1934), 173–175.

[4] *Generalisation of Jacobi's θ-function formulae*, J. Indian Math. Soc. **1** (1934), 23–28.

[5] *On the reducibility of the general elliptic integral into logarithms*, J. Indian Math. Soc. **1** (1934), 59–65.

[6] *A note on Julia's Lemma*, Math. Student **2** (1934), 149–150.

[7] *The method of finding the class-number and the structure of the class group of any algebraic field*, Proc. Indian Acad. Sci., Sect. A, **1** (1935), 446–450.

[8] *A simple proof of the Jordan-curve theorem*, Quart. J. Math. (Oxford) **8** (1937), 241–244.

[9] *On a problem of the tetrahedron*, Proc. Indian Acad. Sci., Sect. A **7** (1938), 257–260.

[10] (with K. S. K. Iyengar) *On a problem relating to a tetrahedron*, Proc. Indian Acad. Sci. A **7** (1938), 305–311.

[11] (with B. S. Madhava Rao) *On an inequality concerning lattice sums*, Proc. Indian Acad. Sci. A **8** (1938), 524–528.

[12] *A note on Poncelet's problem*, J. Mysore Univ., Sect. B **3** (1942), 131–134.

[13] *A deepening of the binomial inequality*, J. Mysore Univ., Sect. B **3** (1942), 135–138.

[14] *Pairs of symmetric and skew matrices in an arbitrary field. I*, Proc. Indian Acad. Sci., Sect. A. **22** (1945), 243–264.

[15] (with B. S. Madhava Rao and V. R. Thiruvenkatachar) *Algebra related to elementary particles of spin $\frac{3}{2}$*, Proc. Roy. Soc. London, Ser. A. **187** (1946), 385–397.

[16] (with C. N. Srinivasiengar, H. Gupta, S. C. Chakrabarty, and V. R. Tiruvenkatachar) *On Kemmer's identity in combinatory functions*, Math. Student **15** (1947), 93–100.

[17] (with K. N. Srinivasa Rao) *The irreducible representations of the Lie-algebra of the orthogonal group with spin* $3/2$, Proc. Nat. Inst. Sci. India **20** (1954), 509–523.

[18] (with S. K. Lakshmana Rao) *On Turán's inequality for ultraspherical polynomials*, Proc. Amer. Math. Soc. **8** (1957), 1075–1087.

[19] *An elementary proof of Morley's theorem*, Amer. Math. Monthly **65** (1958), 612–613.

[20] *Elementary proofs of the infinite product for Sin Z and allied formulae*, Amer. Math. Monthly **69** (1962), 541–545.

[21] *The congruent reduction of a pencil of two skew symmetric matrices over a commutative field of characteristic* 2*, and the criteria for congruence of two such given pencils*, Math. Student **33** (1965), 33–35.

[22] (with T. A. Soundararajan) *A covering theorem for skew fields*, Nederl. Akad. Wetensch. Proc. Ser. A **72** = Indag. Math. **31** (1969), 441–442.

[23] (with T. A. Soundararajan) *Topologies for Galois groups of extension fields*, J. Reine Angew. Math. **242** (1970), 1–7.

[24] (with T. A. Soundararajan) *On the Krull Galois theory for non-algebraic extension fields*, Bull. Austral. Math. Soc. **4** (1971), 367–387.

[25] (with V. Ramamani) *On a partition theorem of Sylvester*, Michigan Math. J. **19** (1972), 137–140.

[26] *Elliptic modular functions and Picard's theorem*, in Hyperbolic Complex Analysis (Proc. All India Sem., Ramanujan Inst., Univ. Madras, Madras, 1977), pp. 21–41; Publ. Ramanujan Inst., 4, Univ. Madras, Madras, 1979.

[27] *Ramanujan manuscripts. II*, Math. Student **52** (1984), 215–233.

[28] *Development of Elliptic Functions According to Ramanujan*, Technical Report, No. 2, Department of Mathematics, Madurai Kamaraj University, Madurai, 1988.

[29] *Ramanujan's contributions to the theory of elliptic functions*, in *Toils and Triumphs of Srinivasa Ramanujan the Man and the Mathematician*, Wazir Hasan Abdi, ed., National Publishing House Chaura Rasta, Jaipur, 1992, pp. 182–206.

[30] (with T. Soundararajan) *A variety of applications of a theorem of B. H. Neumann on groups*, Acta Math. Vietnam **19** (1994), 55–60.

[31] (with B. C. Berndt) *On the transformation formula for the Dedekind eta-function*, in *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics*, F. G. Garvan and M. E. H. Ismail, eds., Kluwer, Dordrecht, 2001, pp. 73–77.

[32] *Some features of Brahmagupta's work in geometry*, bibliographic data unknown.

[33] *The development of mathematics in ancient India: role of Brahmagupta*, bibliographic data unknown.

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